Title: Explorations in Theoretical Astrophysics (PHYS 7890) - Lecture 15

Date: Apr 23, 2010 09:30 AM

URL: http://pirsa.org/10040072

Abstract: Gauge Invariant Cosmological Perturbation theory from 3+1 formulation of General Relativity. This course will aim to study in detail the 3+1 decomposition in General Relativity and use the formalism to derive Gauge invariant perturbation theory at the linear order. Some applications will be studied.

$$\Pi = \frac{\Gamma}{a^2 P_0}$$

Zero shear gauge. $\chi = 0$.

$$\kappa = -3\dot{\varphi} + 3H\alpha$$

$$\Phi_H = \varphi = -\Psi \quad ; \quad \Phi_A = \alpha = \Phi$$

$$\kappa = 3\dot{\Psi} + 3H\Phi$$

Zero shear gauge. $\chi = 0$.

$$k = -3\dot{\varphi} + 3H\alpha$$

$$\Phi_H = \varphi = -\Psi \quad ; \quad \Phi_A = \alpha = \Phi$$

$$k = 3\dot{\Psi} + 3H\Phi$$

(44)
$$\dot{k} + 2Hk = -\left(\frac{\Delta}{a^2} + 3\dot{H}\right)\alpha + 4\pi G\left(\Sigma + 3\pi\right)$$

Zero shear gauge. $\chi = 0$. $\boxed{C = 0 \; ; \; \Gamma = 0 \; ; \; \Pi = 0} \Rightarrow \Psi = \Phi$

$$\kappa = -3\dot{\varphi} + 3H\alpha$$

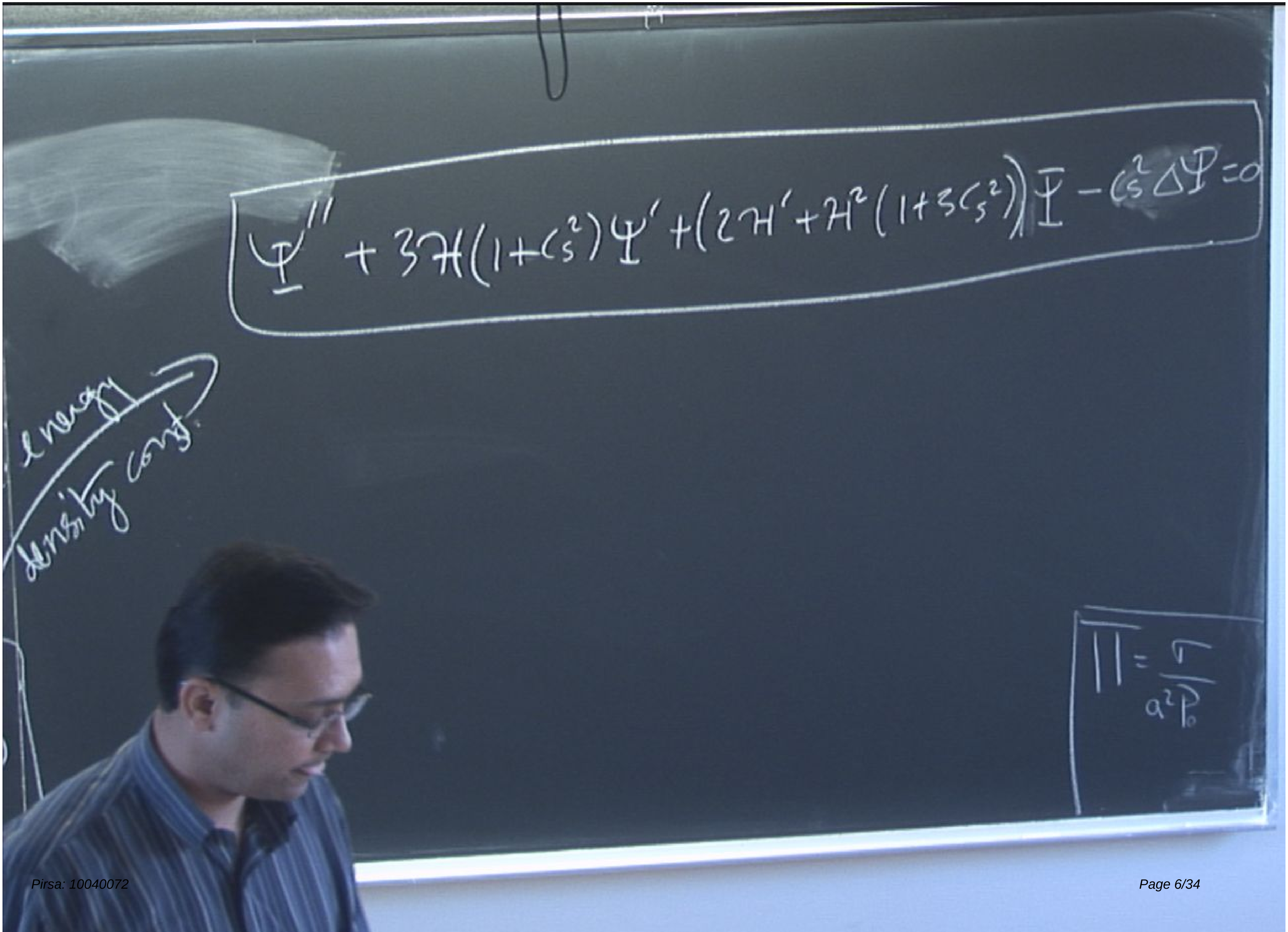
$$\Phi_H = \varphi = -\Psi \quad ; \quad \Phi_A = \alpha = \Phi$$

$$\kappa = 3\dot{\Psi} + 3H\Phi$$

$$\boxed{\dot{\kappa} + 2H\kappa = -\left(\frac{\Delta}{a^2} + 3\dot{H}\right)\alpha + 4\pi G(\varepsilon + 3\pi)} \quad (44)$$

use energy density const.

$$\boxed{\ddot{\Psi} + 3H\dot{\Psi} + 2\dot{H}\Psi + 2H^2\Psi + \frac{\Delta\Psi}{a^2} = \frac{4\pi G}{3}\delta\rho(1 + 3c_s^2)}$$


$\Psi'' + 3\mathcal{H}(1+c_s^2)\Psi' + (2\mathcal{H}' + \mathcal{H}^2(1+3c_s^2))\Psi - c_s^2\Delta\Psi = 0$
energy density const.
$\Pi = \frac{\pi}{a^2 p_0}$

$$\boxed{\Psi'' + 3\mathcal{H}(1+c_s^2)\Psi' + \left(2\mathcal{H}' + \mathcal{H}^2(1+3c_s^2)\right)\Psi - c_s^2\Delta\Psi}$$

$$= 4\pi G a^2 P \Gamma$$

energy density const.

$$\boxed{\Pi = \frac{\sigma}{a^2 P_0}}$$

$$\boxed{\Psi'' + 3\mathcal{H}(1+c_s^2)\Psi' + \left(2\mathcal{H}' + \mathcal{H}^2(1+3c_s^2)\right)\Psi - c_s^2\Delta\Psi} = 4\pi G a^2 P\Gamma$$

$$\boxed{\zeta_L := \frac{2}{3(1+w)}\left[\Psi + \frac{\Phi'}{\mathcal{H}}\right] + \Phi}$$

$$\boxed{\Pi = \frac{\sigma}{a^2 P_0}}$$

$$\boxed{\Psi'' + 3\mathcal{H}(1+c_s^2)\Psi' + \left(2\mathcal{H}' + \mathcal{H}^2(1+3c_s^2)\right)\Psi - c_s^2\Delta\Psi}$$

$$= 4\pi G a^2 P\Gamma$$

$$\boxed{\zeta_L := \frac{2}{3(1+w)}\left[\Psi + \frac{\Phi'}{\mathcal{H}}\right] + \Phi}$$

$$\zeta_L' = \frac{2\mathcal{H}}{3(1+w)}\left[4\pi G\frac{a^2}{\mathcal{H}^2}P\Gamma - c_s^2\frac{k^2}{\mathcal{H}^2}\Phi + \frac{C}{\mathcal{H}^2}\left(\frac{1+3w}{2\mathcal{H}}\Phi' + 3(c_s^2 - w)\Phi\right)\right]$$

Zero shear gauge. $\chi = 0$. $\boxed{C = 0 \; ; \; \Gamma = 0 \; ; \; \Pi = 0}$

(i) Perturbations are adiabatic ie. $\Gamma = 0$

$$\boxed{\Psi'' + 3\mathcal{H}(1+c_s^2)\Psi' + \left(2\mathcal{H}' + \mathcal{H}^2(1+3c_s^2)\right)\Psi - c_s^2\Delta\Psi} = 4\pi G a^2 P\Gamma$$

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$$\boxed{\Psi'' + 3\mathcal{H}(1+c_s^2)\Psi' + \left(2\mathcal{H}' + \mathcal{H}^2(1+3c_s^2)\right)\Psi - c_s^2\Delta\Psi = 4\pi G a^2 P\Gamma}$$

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Zero shear gauge. $\chi = 0$. $\boxed{\zeta = 0 \; ; \; \Gamma = 0 \; ; \; \Pi = 0}$

(i) Perturbations are adiabatic ie. $\Gamma = 0$

(ii) $\zeta = 0$

(iii) $\frac{k}{H} \ll 1$

Zero shear gauge. $\chi = 0$. $\boxed{C = 0 \; ; \; \Gamma = 0 \; ; \; \Pi = 0}$

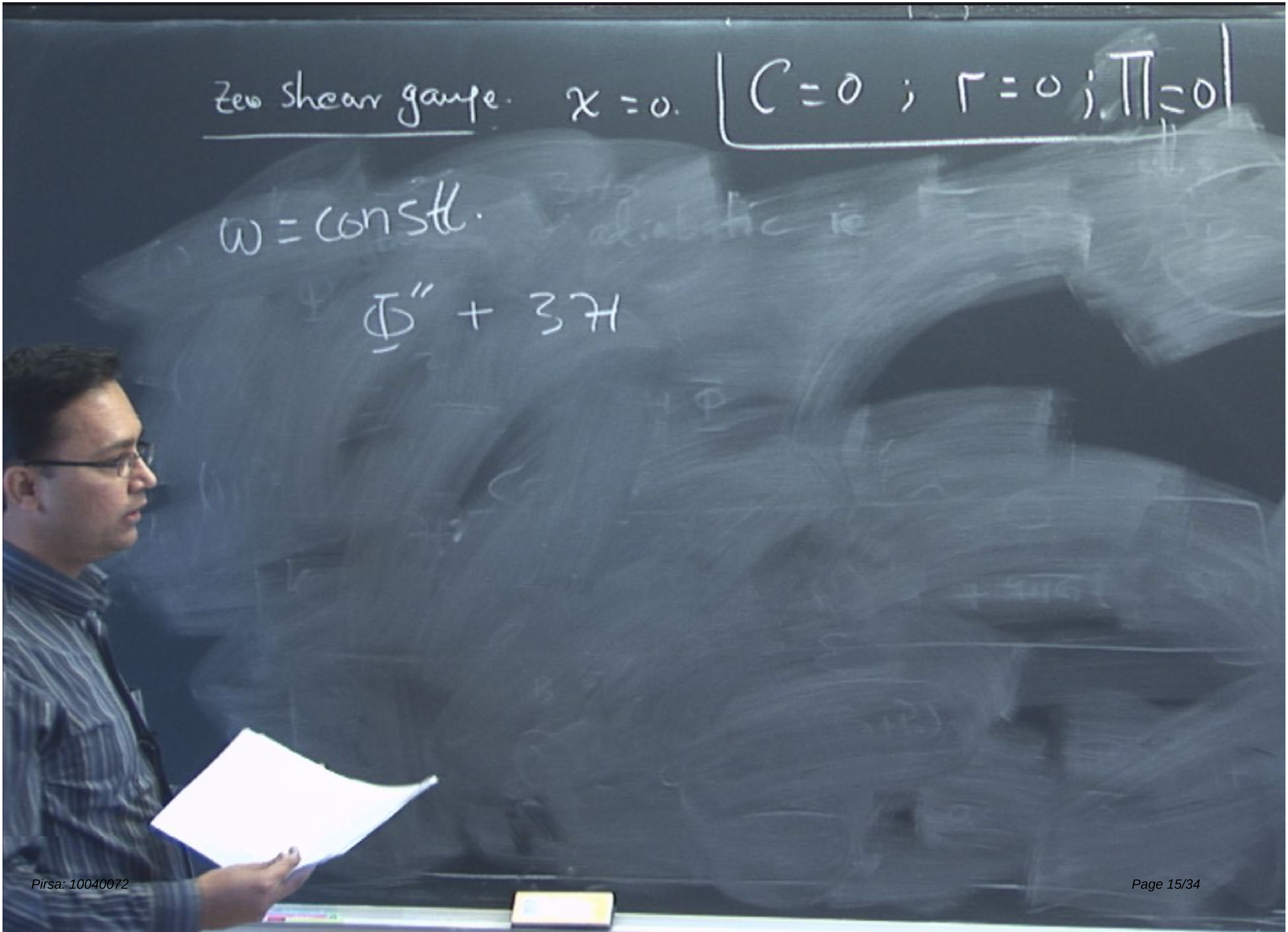
(i) Perturbations are adiabatic ie. $\Gamma = 0$

(ii) $C = 0$

(iii) $\frac{k}{\mathcal{H}} \ll 1$, $\zeta_L' \simeq 0$

$$\zeta_B = \varphi + \frac{\varepsilon}{3(\mathcal{E}_0 + P_0)}$$

$$\mathcal{R} = \varphi + \frac{\mathcal{H}\psi}{(\mathcal{E}_0 + P_0)}$$


Zero shear gauge. $\chi = 0$. $\boxed{C = 0 \; ; \; \Gamma = 0 \; ; \; \Pi = 0}$
$\omega = \text{const.}$
$\Phi'' + 3\mathcal{H}$

Zero shear gauge. $\chi = 0$. $\boxed{C = 0 \; ; \; \Gamma = 0 \; ; \; \Pi = 0}$

$w = \text{const}$.

$$\Phi'' + 3\mathcal{H}(1+w)\Phi' + k^2 c_s^2 \Phi = 0.$$

$$f = x^\nu \Phi \;, \quad x = k\eta \quad ; \quad a \propto \eta^\nu \; ; \; \boxed{\nu = \frac{2}{1+\dots}}$$

$w = \frac{P}{\rho}$

$$\frac{d^2 f}{dx^2} + \frac{2}{x}\frac{df}{dx} + \left[w - \frac{\nu(\nu+1)}{x^2}\right] f = 0$$

$$\Phi = -\frac{3\nu^2}{2}$$

Zero shear gauge. $\chi = 0.$ $\boxed{C = 0 \; ; \; \Gamma = 0 \; ; \; \Pi = 0}$

$w = \text{const}.$

$$\Phi'' + 3\mathcal{H}(1+w)\Phi' + k^2 c_s^2 \Phi = 0.$$

$$f \equiv x^{\nu}\Phi \; , \quad x = k\eta \quad ; \quad a \propto \eta^{\nu} \; ; \quad \nu = \ldots$$

$$\frac{d^2 f}{dx^2} + \frac{2}{x}\frac{df}{dx} + \left[w - \frac{\nu(\nu+1)}{x^2} \right] f = 0$$

$$\Phi = -\frac{3\nu^2}{2} x^{-\nu} \left[A \, j_{\nu}(c_s x) + B \, n_{\nu}(c_s x) \right]$$

related to Bessel of 1st kind ... Neumann

Zero shear gauge. $\chi = 0$. $\boxed{C = 0 \; ; \; \Gamma = 0 \; ; \; \Pi = 0}$

$w = \text{const}$.

$$\Phi'' + 3\mathcal{H}(1+w)\Phi' + k^2 c_s^2 \Phi = 0.$$

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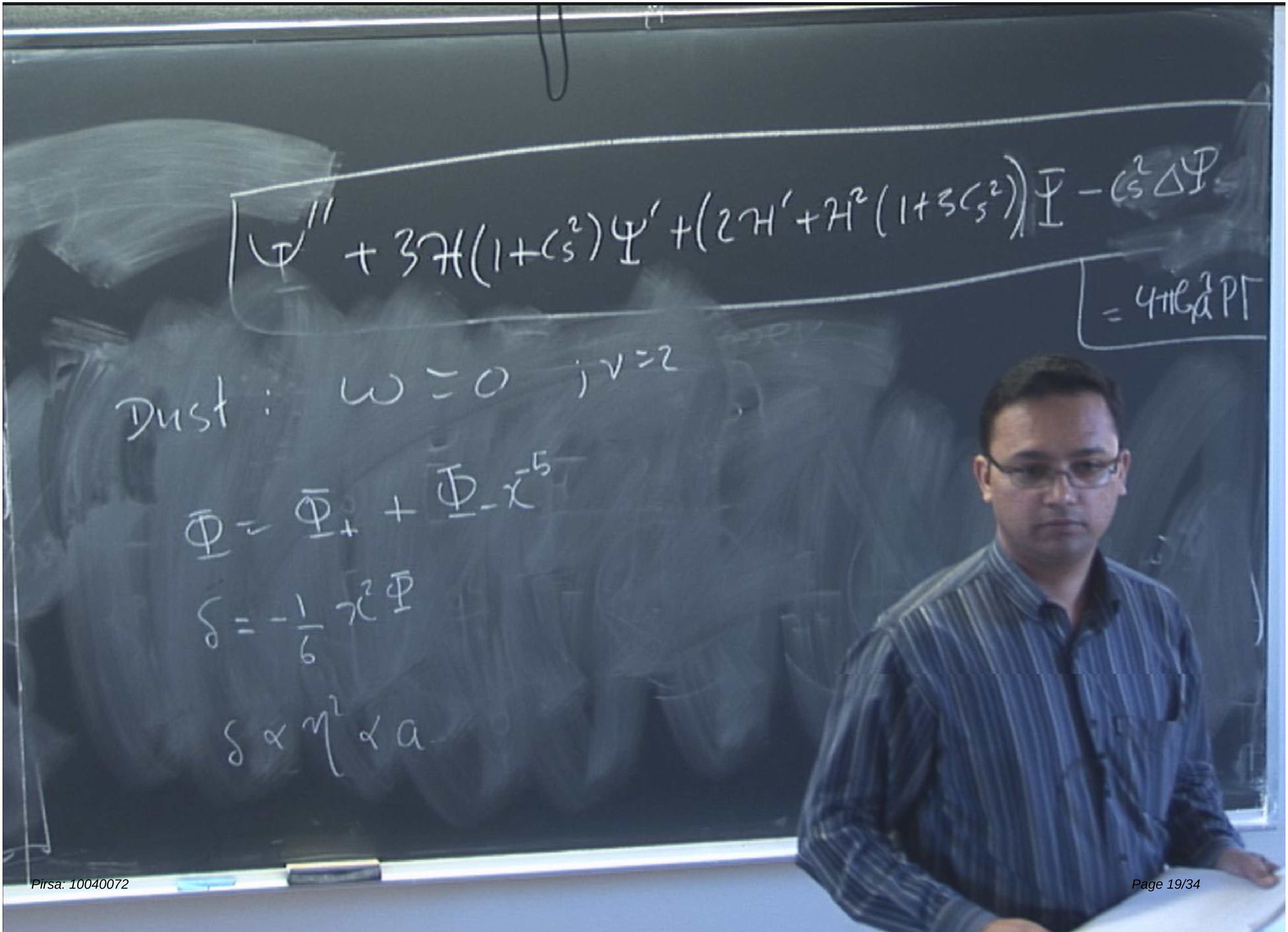
$\Psi'' + 3\mathcal{H}(1+c_s^2)\Psi' + (2\mathcal{H}' + \mathcal{H}^2(1+3c_s^2))\Phi - c_s^2 \Delta\Psi$
$= 4\pi G a^2 P \Gamma$
Dust : $\omega = 0$; $\nu = 2$
$\Phi = \Phi_+ + \Phi_- x^{-5}$
$\delta = -\frac{1}{6} x^2 \Phi$
$\delta \propto \eta^2 \propto a$

$$\boxed{\Psi'' + 3\mathcal{H}(1+c_s^2)\Psi' + \left(2\mathcal{H}' + \mathcal{H}^2(1+3c_s^2)\right)\Phi - c_s^2 \Delta\Psi}$$

$$\boxed{= 4\pi G a^2 P \Gamma}$$

Dust : $\omega = 0 \; ; \nu = 2$

$$\Phi = \Phi_+ + \Phi_- x^{-5}$$

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$$\delta \propto \eta^2 \propto a$$

Zero shear gauge. $\chi = 0.$ $\boxed{C = 0\ ;\ \Gamma = 0\ ;\ \Pi = 0}$

$w = \text{const.}$

$$\Phi'' + 3\mathcal{H}(1+w)\,\Phi' + k^2 c_s^2\,\Phi = 0.$$

$$\mathcal{H} \sim \frac{1}{\eta}$$

$$f \equiv x^{\nu}\,\Phi\ ,\quad x = k\eta\ ;\quad a \propto \eta^{\nu}\ ;\quad \boxed{\nu = \frac{\ }{1+}}$$

$$\frac{d^2 f}{dx^2} + \frac{2}{x}\frac{df}{dx} + \left[\omega - \frac{\nu(\nu+1)}{x^2}\right] f = 0$$

$$\Phi = -\frac{3\nu^2}{2}\,x^{-\nu}\left[A\,\underbrace{j_\nu(c_s x)}_{\text{related to Bessel of 1st kind}} + B\,\underbrace{n_\nu(c_s x)}_{\text{2nd Neumann}}\right]$$

$$\boxed{\Psi'' + 3\mathcal{H}(1+c_s^2)\Psi' + \left(2\mathcal{H}' + \mathcal{H}^2(1+3c_s^2)\right)\Phi - c_s^2\Delta\Psi} = 4\pi G a^2 P\Gamma$$

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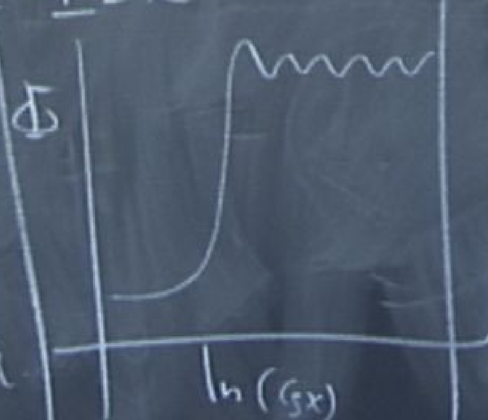


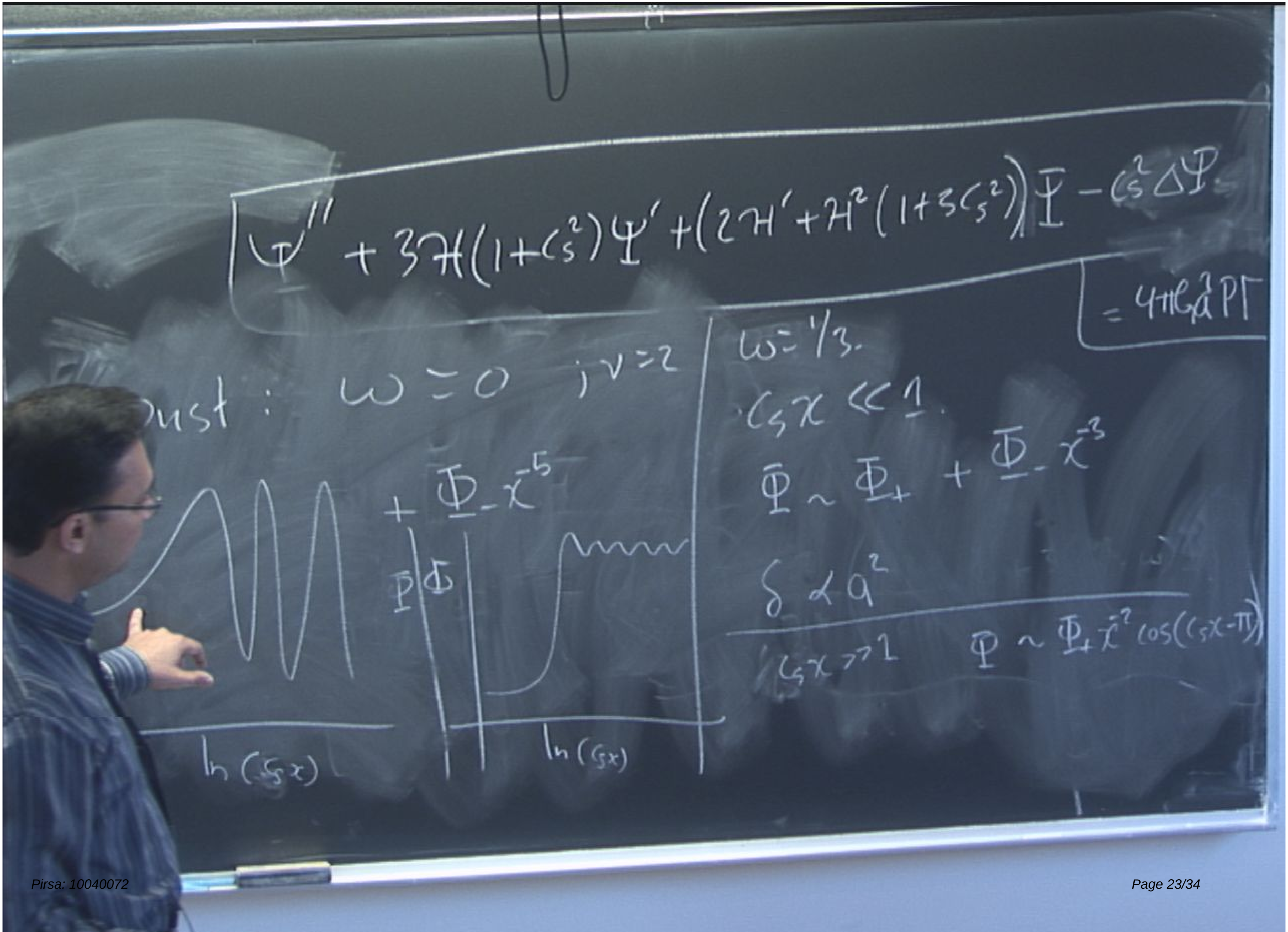
$\omega = 1/3.$

$c_s x \ll 1.$

$$\Phi \sim \Phi_+ + \Phi_- x^{-3}$$

$$\delta \propto a^2$$

$c_s x \gg 1 \qquad \Phi \sim \Phi_+ x^{-2} \cos(c_s x - \pi)$


$\Psi'' + 3\mathcal{H}(1+c_s^2)\Psi' + (2\mathcal{H}' + \mathcal{H}^2(1+3c_s^2))\Phi - c_s^2\Delta\Psi$
$= 4\pi G a^2 P \Gamma$
Dust : $\omega = 0$; $\nu = 2$
$+ \Phi_- x^{-5}$
Φ
$\ln(c_s x)$
$\ln(c_s x)$
$\omega = 1/3$.
$c_s x \ll 1$.
$\Phi \sim \Phi_+ + \Phi_- x^{-3}$
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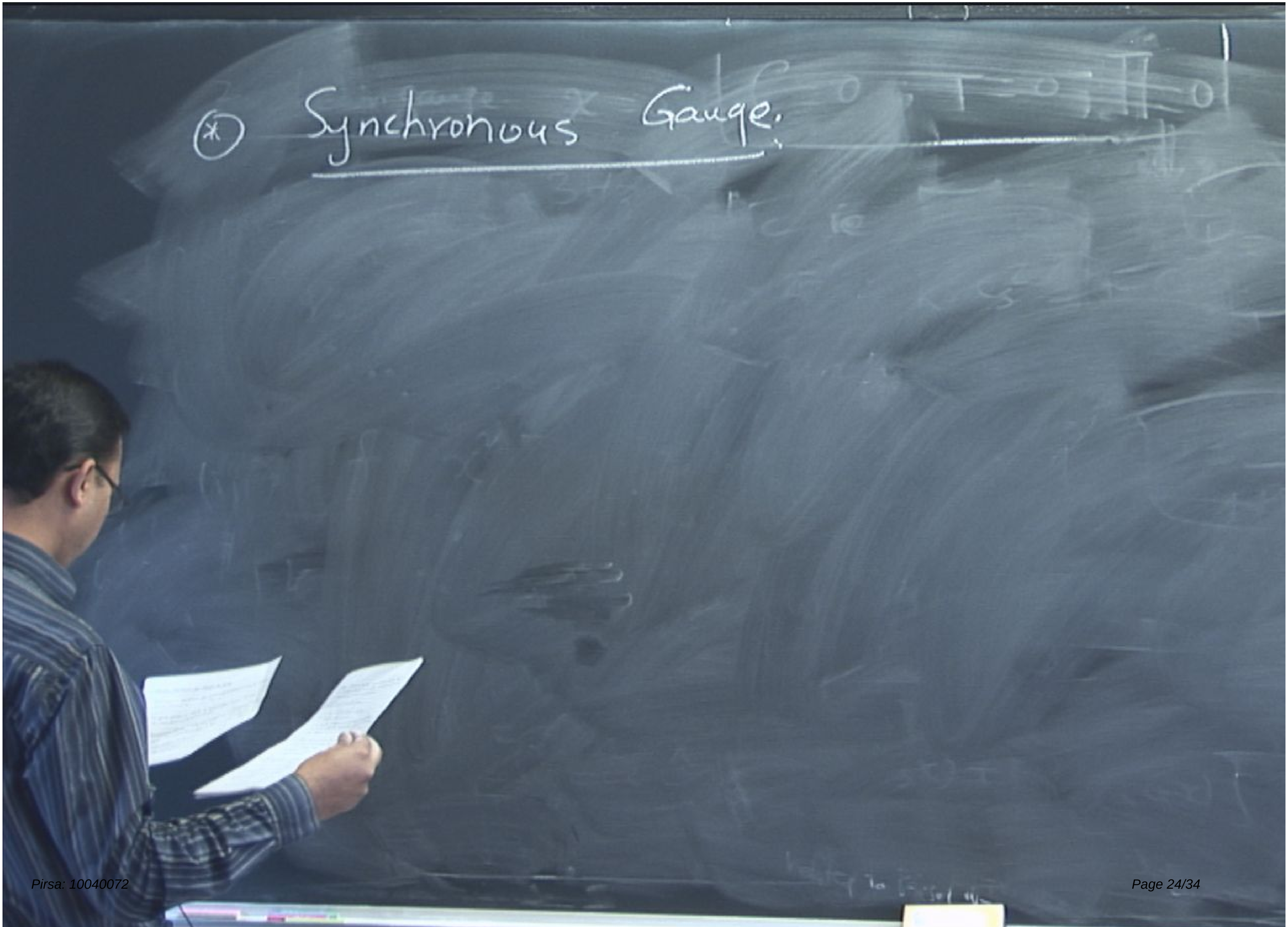
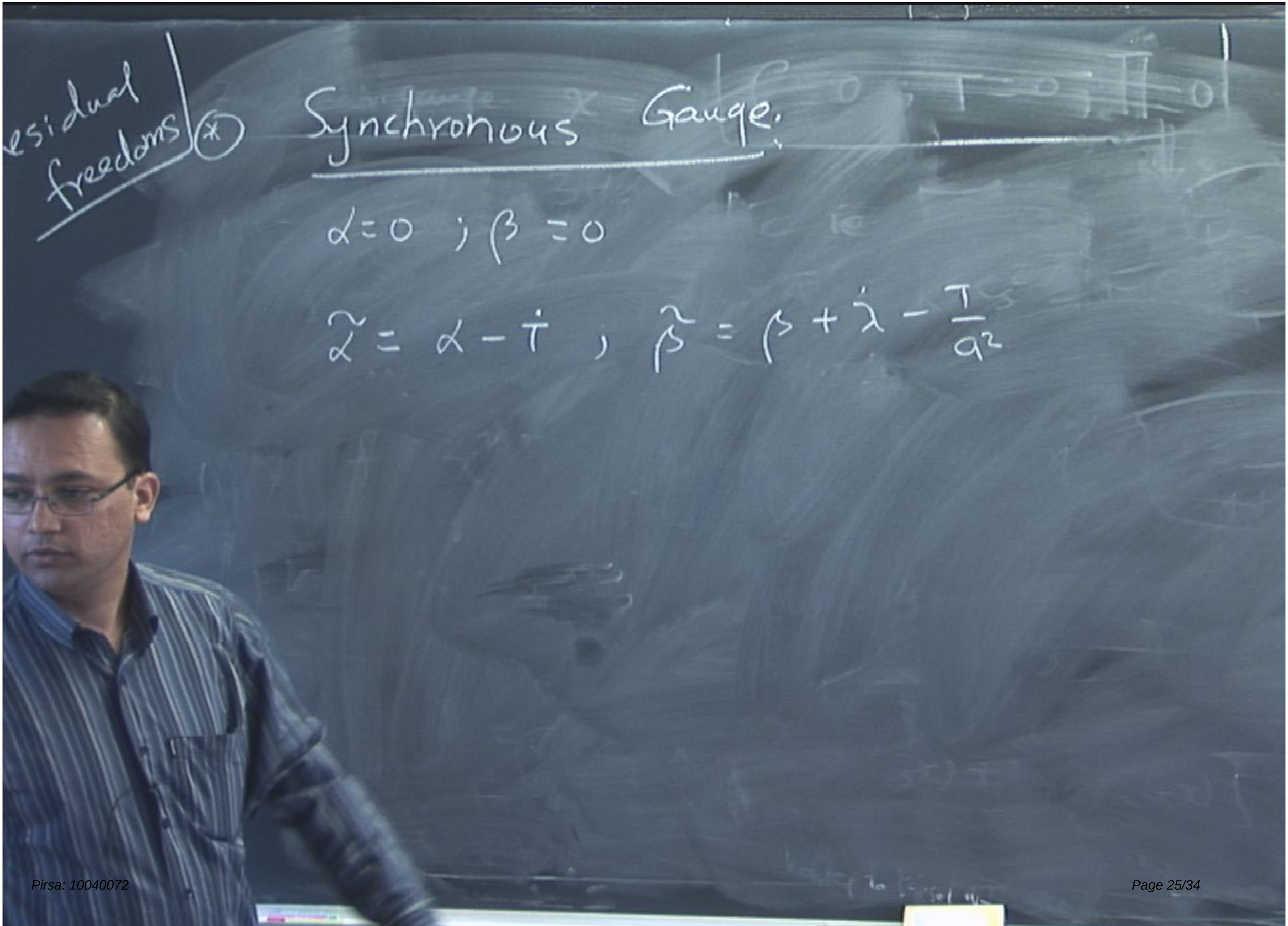
$c_s x \gg 1$
$\Phi \sim \Phi_+ x^{-2} \cos(c_s x - \pi)$


Synchronous Gauge.


esidual freedoms
(*) Synchronous Gauge.
$\alpha = 0 \; ; \; \beta = 0$
$\tilde{\alpha} = \alpha - \dot{T} \; ; \; \tilde{\beta} = \beta + \dot{\lambda} - \frac{T}{a^2}$

esidual freedoms ⊛ Synchronous Gauge.

$$\alpha = 0 \; ; \; \beta = 0$$

$$\tilde{\alpha} = \alpha - \dot{T} \; ; \; \tilde{\beta} = \beta + \dot{\lambda} - \frac{T}{a^2}$$

We go to synchronous gauge : $\tilde{\alpha} = \tilde{\beta} = 0$

$$\dot{T} = \alpha \; ; \; \dot{\lambda} = \beta - \frac{T}{a^2}$$

$$T = \int \alpha \, dt + C_1$$

$$\lambda = -\int \beta \, dt + \int \frac{T}{a^2} dt + C_2 .$$

esidual freedoms

(*) Synchronous Gauge.

$\alpha = 0 \; ; \; \beta = 0$

$\tilde{\alpha} = \alpha - \dot{T} \; , \; \tilde{\beta} = \beta + \dot{\lambda} - \frac{T}{a^2}$

We go to synchronous gauge : $\tilde{\alpha} = \tilde{\beta} = 0$

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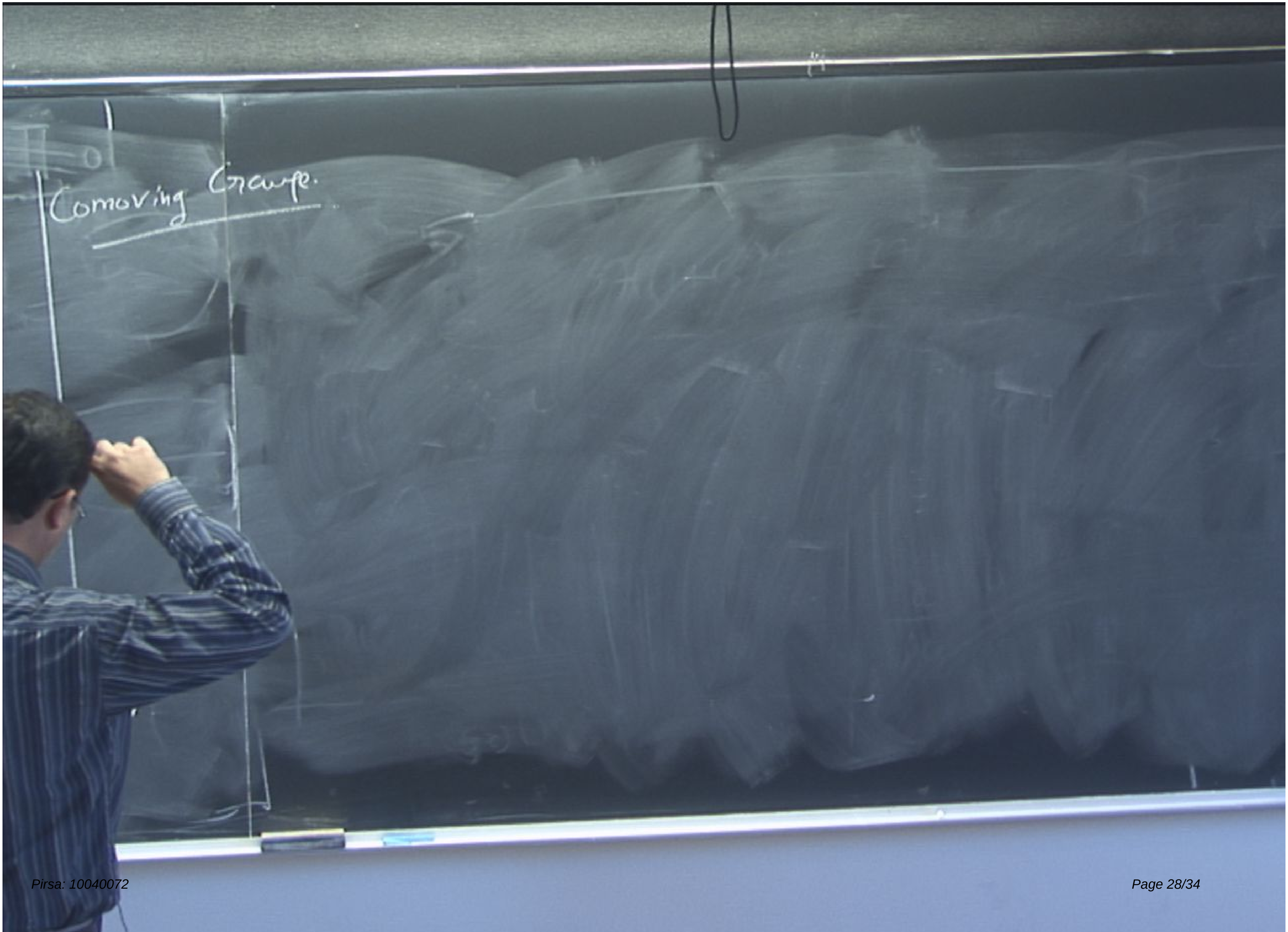
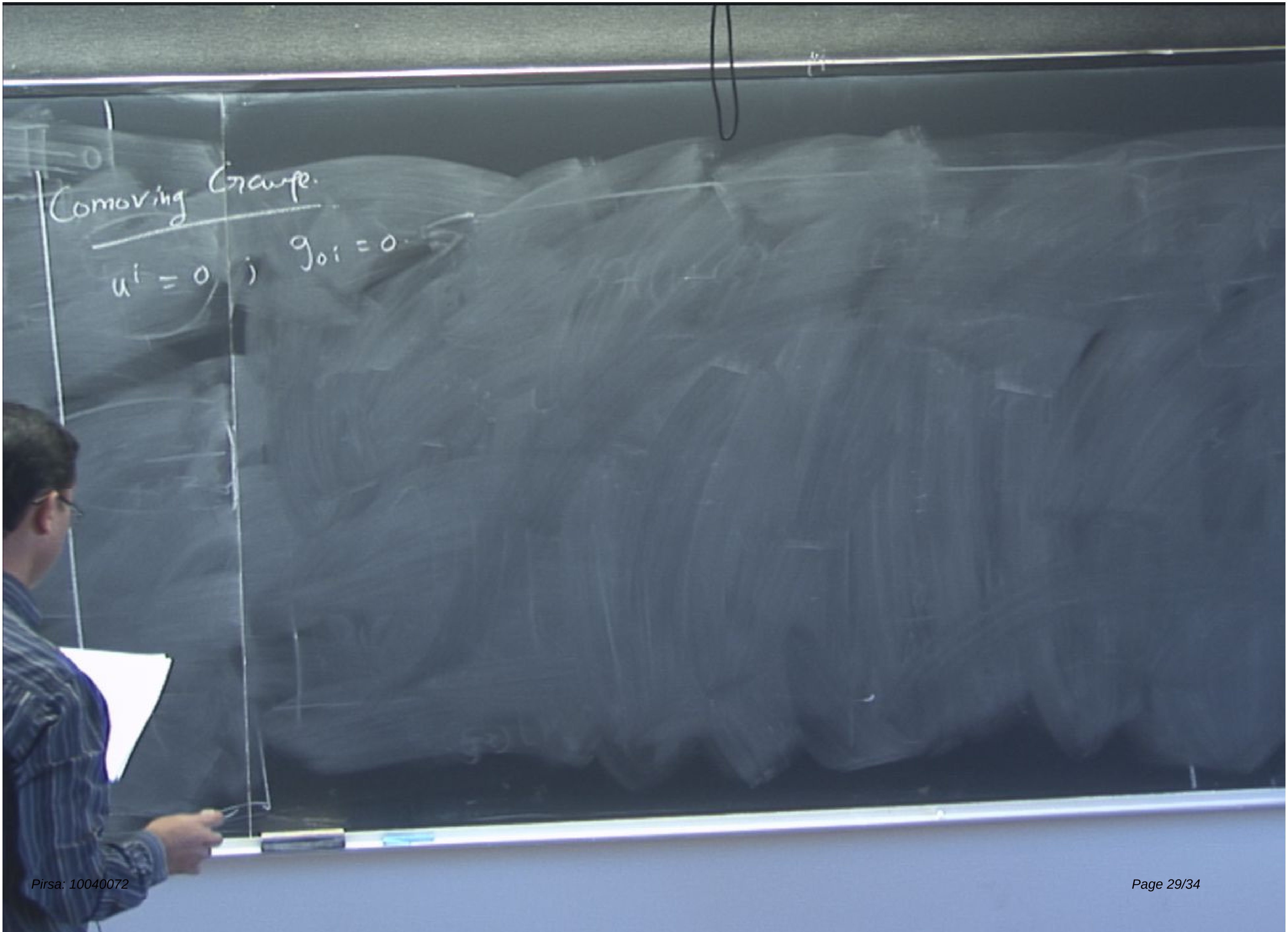
$\lambda = -\int \beta \, dt + \int \frac{T}{a^2} \, dt + C_2.$


Comoving Gauge.


Comoving Gauge.
$u^i = 0$; $g_{0i} = 0$.

Comoving Gauge.

$u^i = 0 \; ; \quad g_{0i} = 0.$

$\Rightarrow \; \psi = 0, \quad \beta = 0$

$\tilde{\Psi} = \Psi + (\mathcal{E}_0 + P_0)\, T$

Comoving Gauge.

$u^i = 0$; $g_{0i} = 0$.

$\Rightarrow$ $\psi = 0$, $\beta = 0$

$$\tilde{\psi} = \psi + (\varepsilon_0 + P_0) T$$

$$\tilde{\beta} = \beta + \dot{\lambda} - \frac{T}{a^2}$$

$\tilde{\psi} = \tilde{\beta} = 0$; $T = \dfrac{\psi}{(\varepsilon_0 + P_0)}$

$$\dot{\lambda} = -\beta + \frac{T}{a^2} = -\beta + \frac{\psi}{(\varepsilon_0 + P_0)a^2}$$

$$\lambda = -\int \beta \, dt + \int \frac{\psi}{(\varepsilon_0 + P_0)a^2} dt + C_1$$

Comoving Gauge.

$u^i = 0$; $g_{0i} = 0$

$\Rightarrow \psi = 0, \ \beta = 0$

$\tilde{\psi} = \psi + (E_0 + P_0) T$

$\tilde{\beta} = \beta + \dot{\lambda} - \frac{T}{a^2}$

$\tilde{\psi} = \tilde{\beta} = 0$; $T = \frac{\psi}{(E_0 + P_0)}$

$\dot{\lambda} = -\beta + \frac{T}{a^2} = -\beta + \frac{\psi}{(E_0 + P_0) a^2}$

$\lambda = -\int \beta \, dt + \int \frac{\psi}{(E_0 + P_0) a^2} dt + C$

Newtonian gauge.

$\gamma = 0$; $\beta = 0$

$\tilde{\chi} = \chi - N_0 T$

$\tilde{\chi} = 0 \longrightarrow T = \frac{\chi}{N_0}$

$\tilde{\gamma} = \gamma - \lambda$

$\tilde{\gamma} = 0 \longrightarrow \lambda = \gamma$

Comoving Gauge.

$u^i = 0$; $g_{0i} = 0$

$\Rightarrow \psi = 0, \quad \beta = 0$

$\tilde{\psi} = \psi + (\mathcal{E}_0 + P_0) T$

$\tilde{\beta} = \beta + \dot{\lambda} - \frac{T}{a^2}$

$\tilde{\psi} = \tilde{\beta} = 0$; $T = \frac{\psi}{(\mathcal{E}_0 + P_0)}$

$\dot{\lambda} = -\beta + \frac{T}{a^2} = -\beta + \frac{\psi}{(\mathcal{E}_0 + P_0) a^2}$

$\lambda = -\int \beta \, dt + \int \frac{\psi}{(\mathcal{E}_0 + P_0) a^2} dt + C$

Newtonian gauge.

$\gamma = 0$; $\beta = 0$

$\tilde{\chi} = \chi - N_0 T$

$\tilde{\chi} = 0 \longrightarrow T = \frac{\chi}{N_0}$

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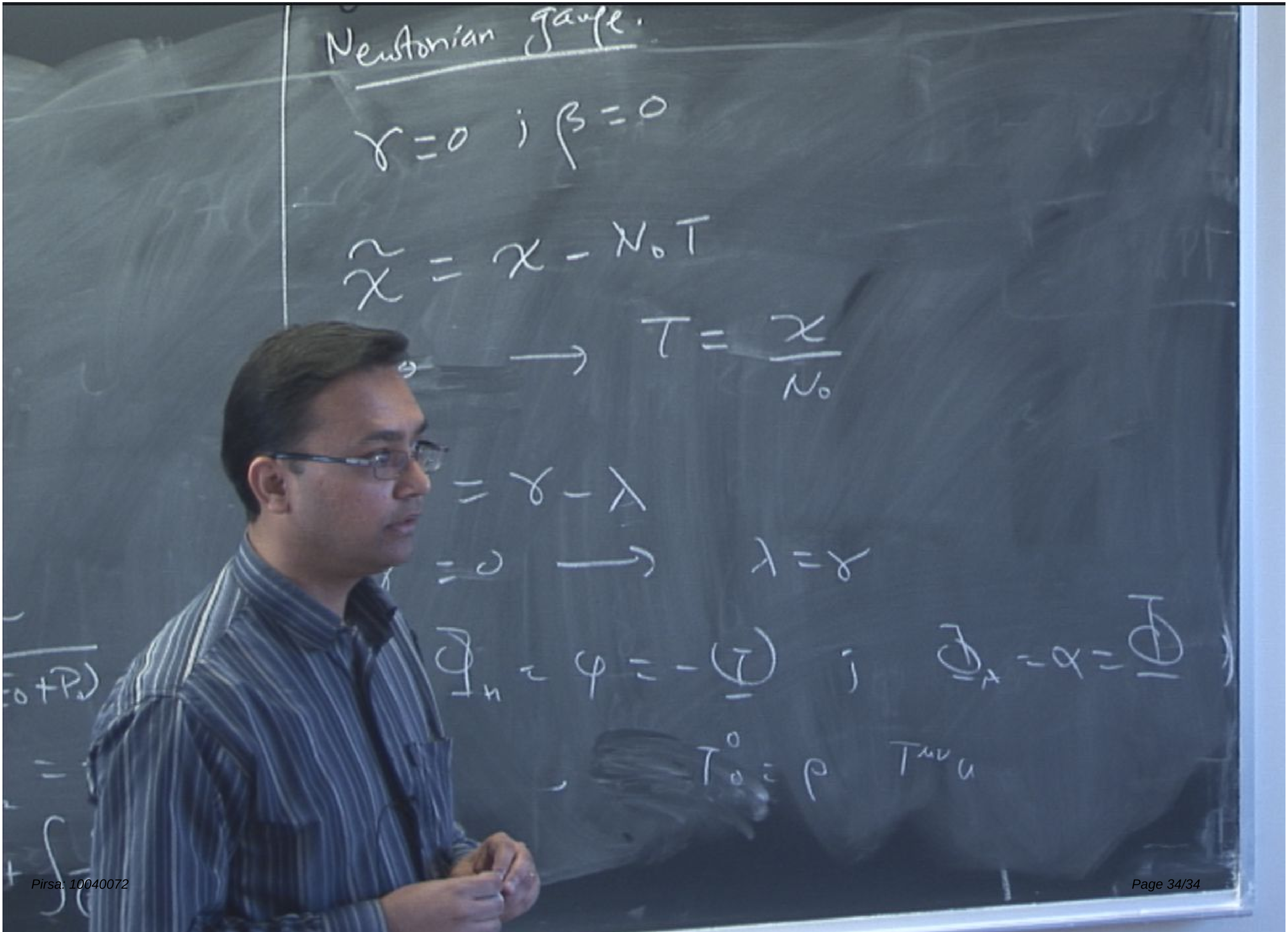
$\tilde{\gamma} = 0 \longrightarrow \lambda = \gamma$

$\Phi_n = \varphi = -(\Phi)$; $\Phi_A = \alpha =$


Newtonian gauge.
γ = 0 ; β = 0
x̃ = x − N₀T
→ T = x / N₀
= γ − λ
= 0 → λ = γ
Φ_N = φ = −(Ψ) ; Φ_A − α = Φ
T⁰₀ = ρ   T^μν u