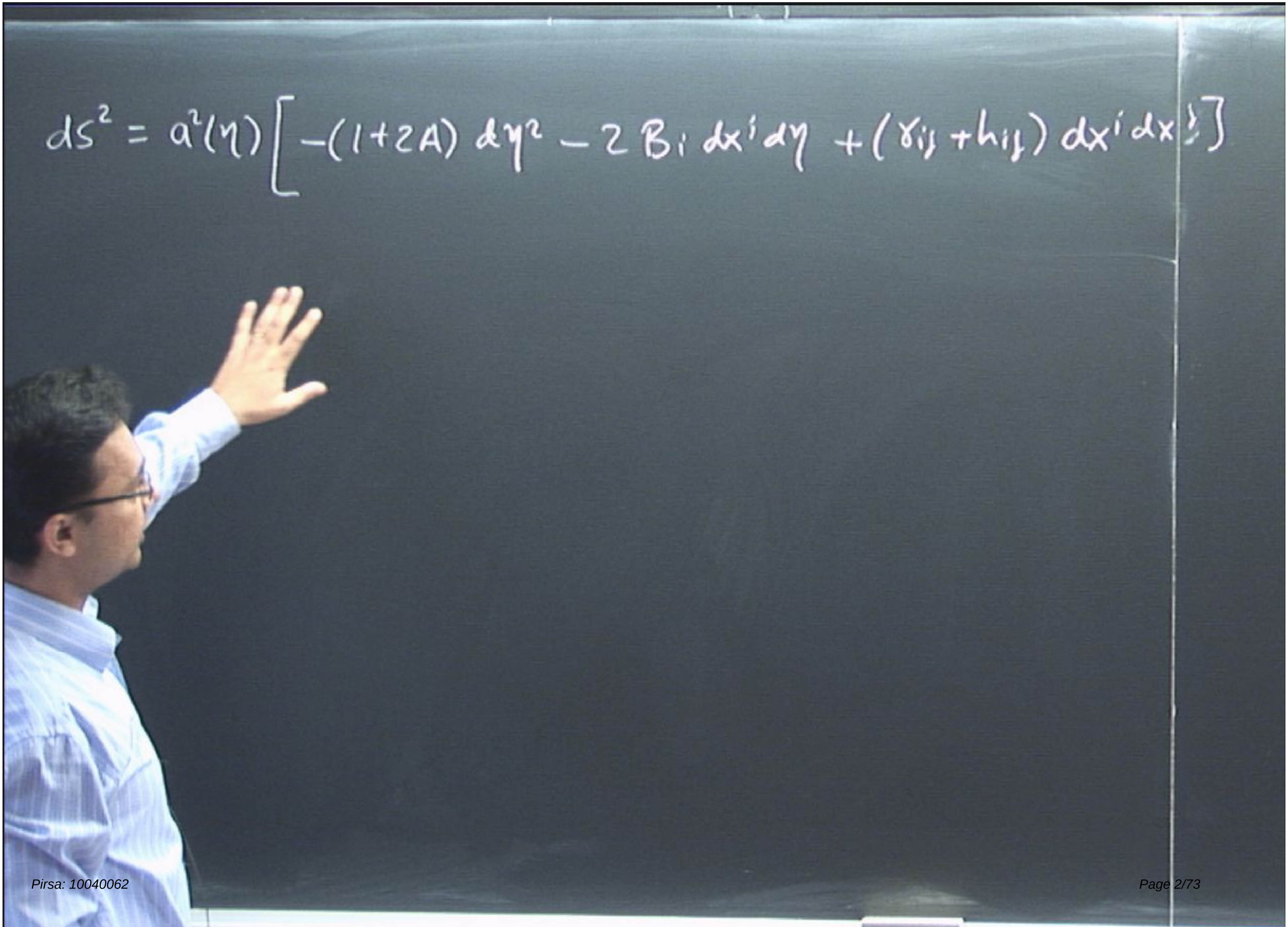
Title: Explorations in Theoretical Astrophysics (PHYS 7890) - Lecture 5

Date: Apr 09, 2010 09:30 AM

URL: http://pirsa.org/10040062

Abstract: <span>Gauge Invariant Cosmological Perturbation theory from 3+1 formulation of General Relativity. This course will aim to study in detail the 3+1 decomposition in General Relativity and use the formalism to derive Gauge invariant perturbation theory at the linear order. Some applications will be studied.</span>


$ds^2 = a^2(\eta)\left[-(1+2A)\,d\eta^2 - 2B_i\,dx^i d\eta + (\gamma_{ij} + h_{ij})\,dx^i dx^j\right]$

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d

e

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SVT deco... n :

Any vector field ... $= D^i B + D$

d
e

$$ds^2 = a^2(\eta)\left[-(1+2A)\,d\eta^2 - 2B_i\,dx^i d\eta + (\gamma_{ij}+h_{ij})\,dx^i dx^j\right]$$

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<u>SVT</u> decomposition :

Any vector field $\quad B^i = D^i B + \bar{B}^i \quad$ s.t. $D^i \bar{B}_i = 0$

d

e

$$ds^2 = a^2(\eta)\left[-(1+2A)\,d\eta^2 - 2B_i\,dx^i d\eta + (\gamma_{ij}+h_{ij})\,dx^i dx^j\right]$$

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## SVT decomp...

Any vector field ... $= D^i B + \bar{B}^i$ s.t. $D^i \bar{B}_i = 0$

A ra... $h_{ij} = 2$...

d
e

$$ds^2 = a^2(\eta)\left[-(1+2A)\,d\eta^2 - 2B_i\,dx^i d\eta + (\gamma_{ij}+h_{ij})\,dx^i dx^j\right]$$

$$ds^2 = a^2(\eta)\left[-d\eta^2 + \gamma_{ij}\,dx^i dx^j\right]$$

SVT decom... n :

Any vector field ... $= D^i B + \bar{B}^i$ s.t. $D^i \bar{B}_i = 0$

A ran...

$$h_{ij} = 2C\gamma_{ij} + 2D_iD_jE + 2D_{(i}\bar{E}_{j)} + 2$$

$$ds^2 = a^2(\eta)\left[-(1+2A)\,d\eta^2 - 2B_i\,dx^i d\eta + (\gamma_{ij}+h_{ij})\,dx^i dx^j\right]$$

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<u>SVT</u> decomposition :

Any vector field : $B^i = D^i B + \bar{B}^i$ s.t. $D^i \bar{B}_i = 0$

A rank 2 symmetric tensor : $h_{ij} = 2C\gamma_{ij} + 2D_i D_j E + 2D_{(i}\bar{E}_{j)} + 2\bar{E}_{ij}$

with $D_i \bar{E}^{ij} = 0$ , $\bar{E}^i_{\ i} = 0$

$$ds^2 = a^2(\eta)\left[-(1+2A)\,d\eta^2 - 2B_i\,dx^i d\eta + (\gamma_{ij}+h_{ij})\,dx^i dx^j\right]$$

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SVT decomposition:

Any vector field: $B^i = D^i B + \bar{B}^i$ s.t. $D^i \bar{B}_i = 0$

A rank 2 symmetric tensor: $h_{ij} = 2C\gamma_{ij} + 2D_i D_j E + 2D_{(i}\bar{E}_{j)} + 2\bar{E}_{ij}$

with $D_i \bar{E}^{ij} = 0$, $\bar{E}^i{}_i = 0$

]

$+2\bar{E}_{ij}$

4 scalar d.o.f : $A, B, C, E$

]

4 scalar perturbations : $A, B, C, E$ $\longrightarrow 4$

2 vector perturbations $\bar{B}^i$ , $\bar{E}^i$ $\longrightarrow 4$

1 Tensor perturbation : $\bar{E}^{ij}$ $\longrightarrow 2$

$+ 2\bar{E}_{ij}$

]

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4 scalar perturbations : $A, B, C, E \longrightarrow 4$

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Scalar perturbations of the metric :

$$\boxed{ds^2 = a^2(\eta)\left[-(1+2A)d\eta^2 - 2(D_iB)dx^i d\eta + ((1+2C)\gamma_{ij} + 2D_iD_jE)dx^i dx^j\right]}$$

]

$+2\bar{E}_{ij}$

$\bar{E}_i = 0$

4 scalar perturbations : $A, B, C, E$ $\longrightarrow 4$

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Scalar perturbations of the metric :

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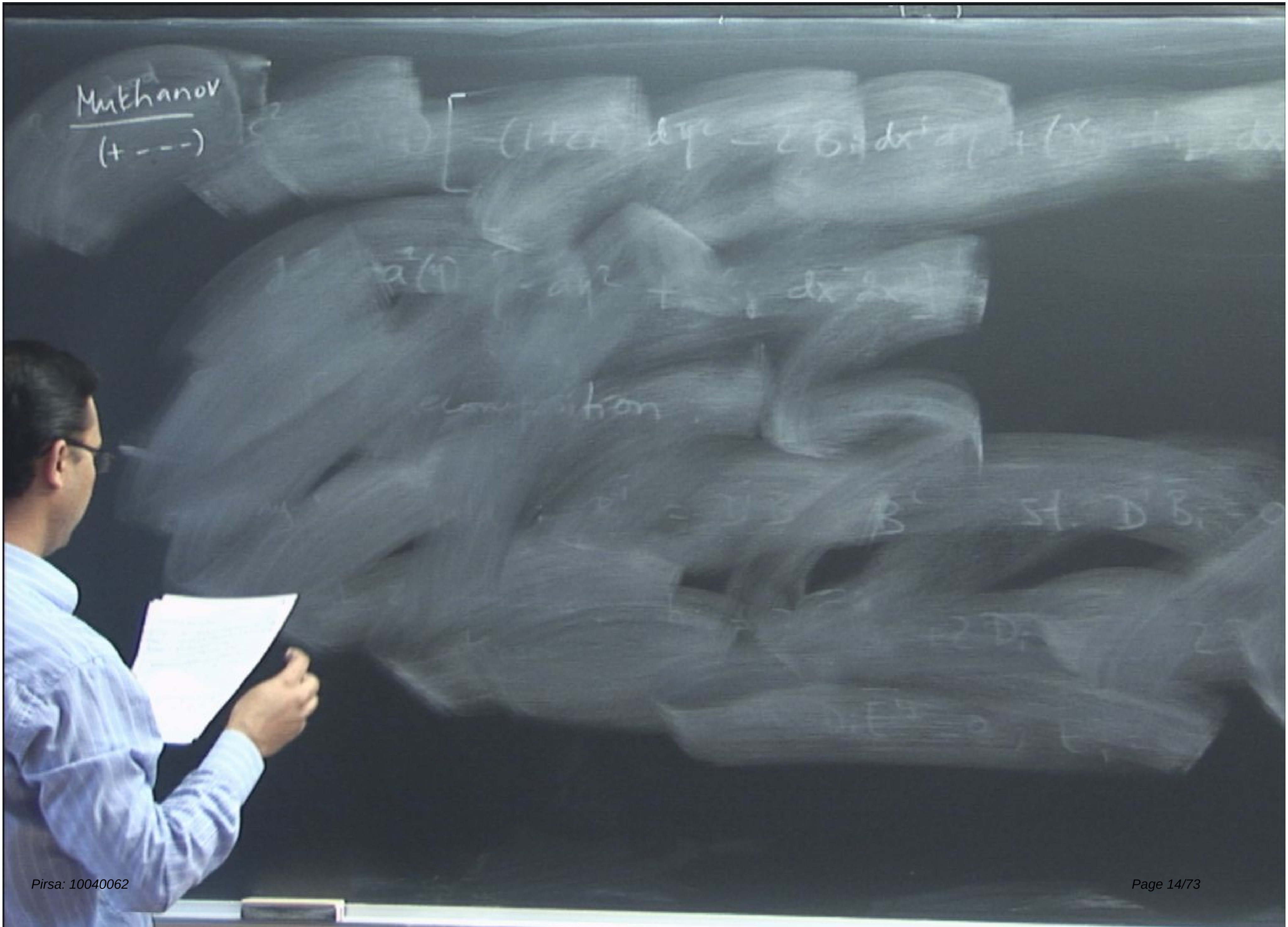
Mukhanov
(+ - - -)

hanov

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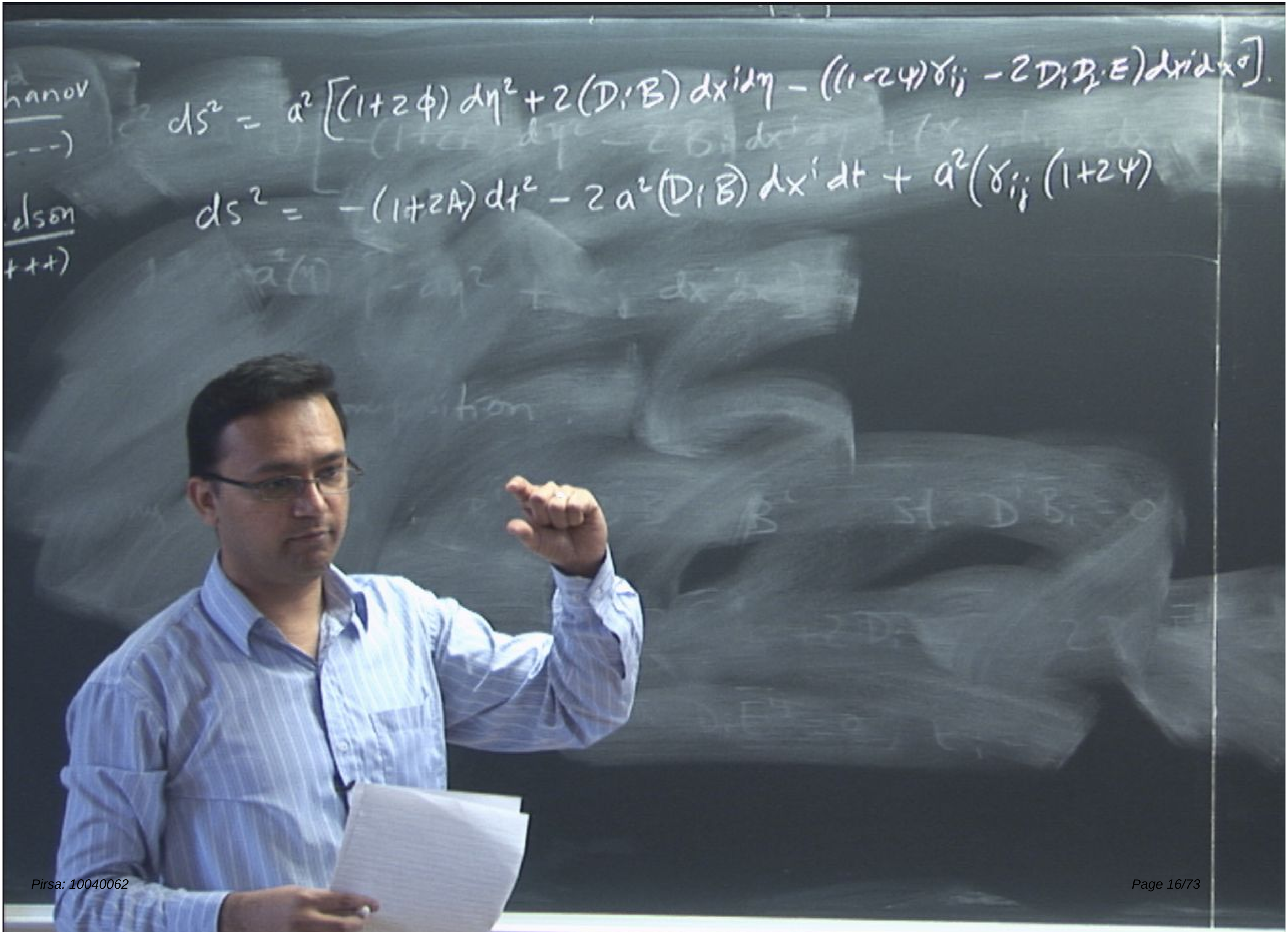
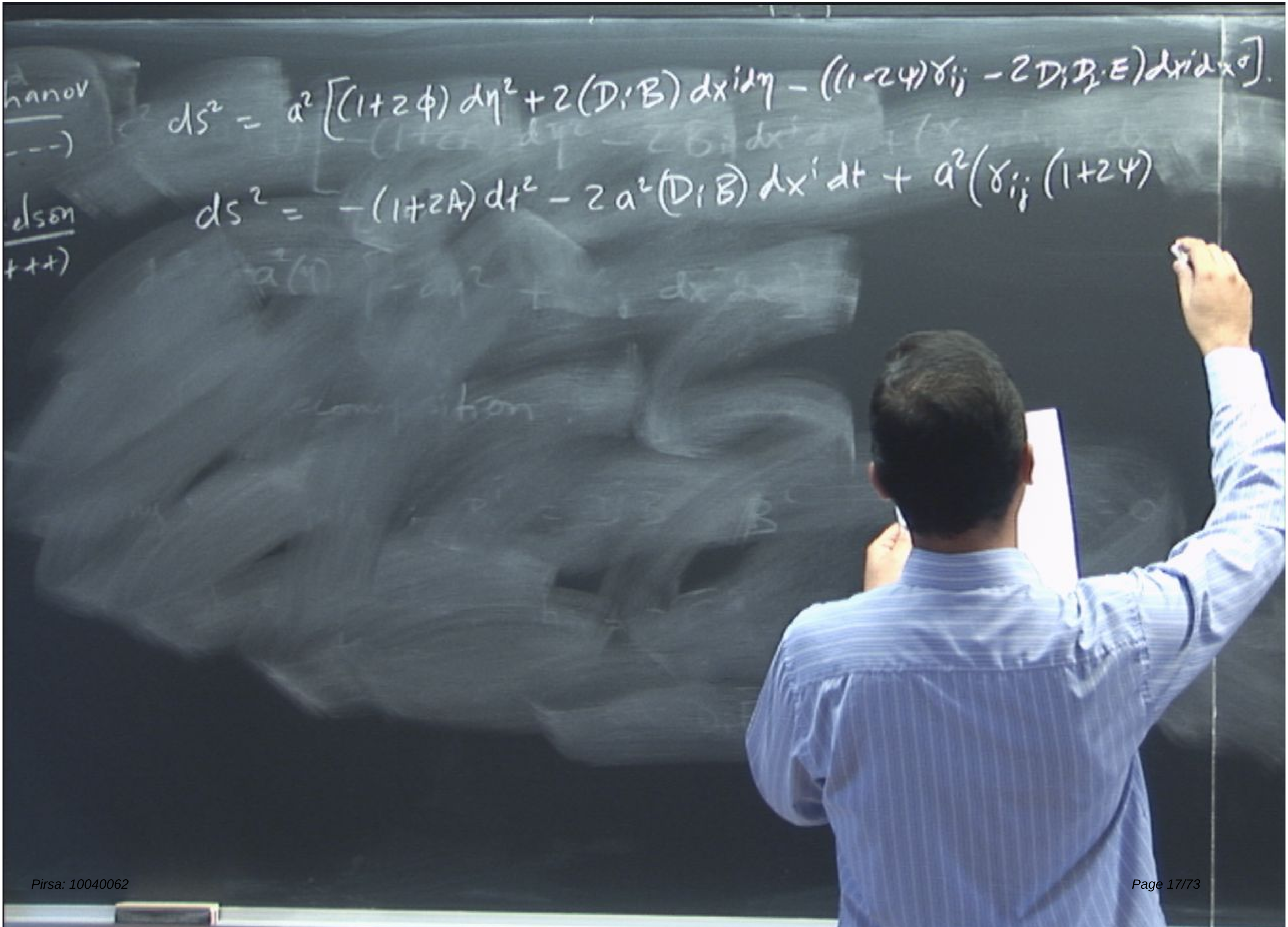
$$ds^2 = a^2\left[(1+2\phi)\,d\eta^2 + 2(D_iB)\,dx^i d\eta - \left((1-2\psi)\gamma_{ij} - 2D_iD_jE\right)dx^i dx^j\right].$$


ds² = a²[(1+2φ) dη² + 2(D_iB) dx^i dη − ((1−2ψ)γ_ij − 2D_iD_jE) dx^i dx^j].
ds² = −(1+2A) dt² − 2a²(D_iB) dx^i dt + a²(γ_ij (1+2ψ)


hanov
(- - - -)
elson
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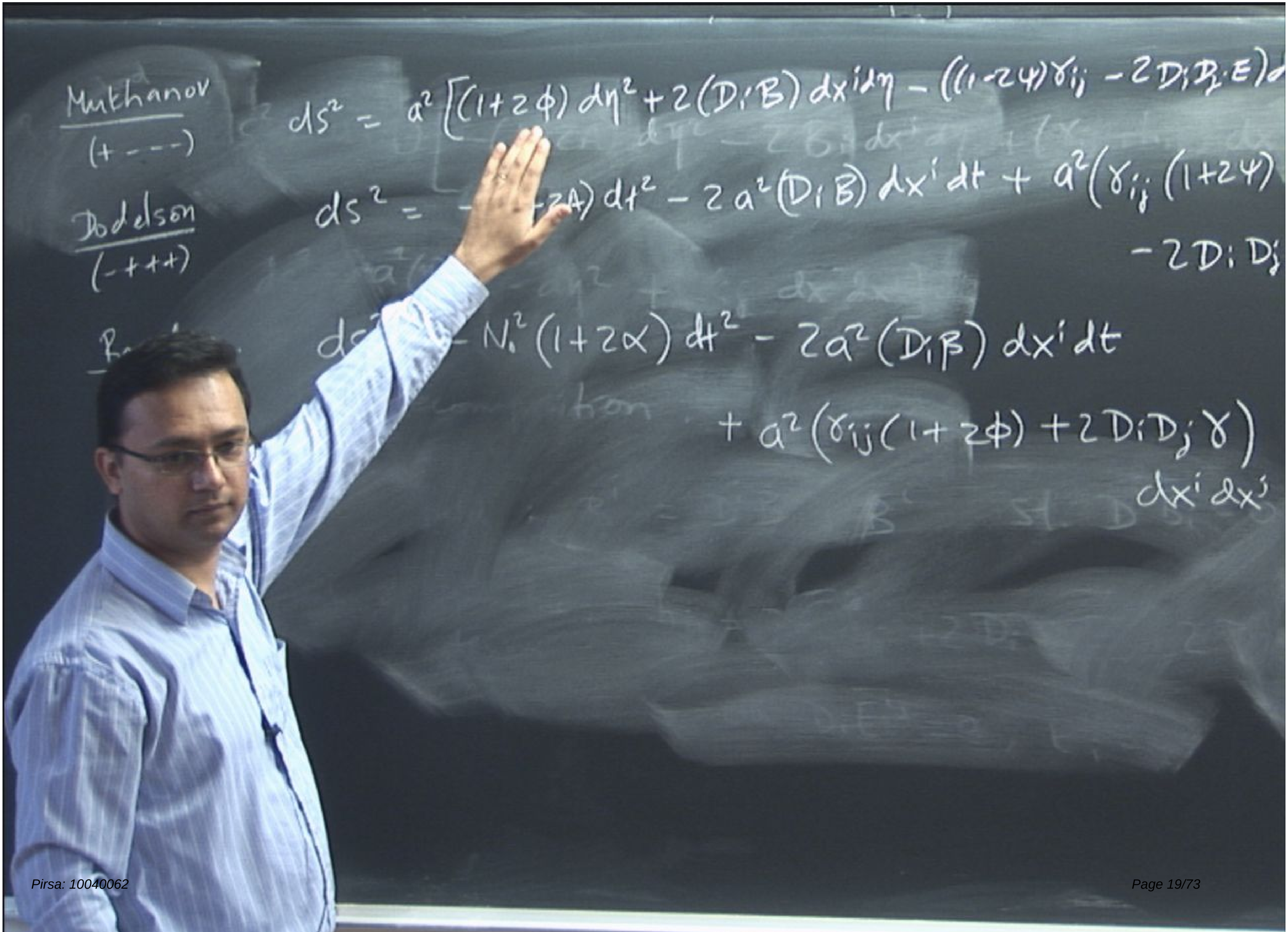
$ds^2 = a^2\left[(1+2\phi)\,d\eta^2 + 2(D_iB)\,dx^i d\eta - \left((1-2\psi)\gamma_{ij} - 2D_iD_jE\right)dx^i dx^j\right].$
$ds^2 = -(1+2A)\,dt^2 - 2a^2(D_iB)\,dx^i dt + a^2\left(\gamma_{ij}(1+2\psi)\right.$

$$ds^2 = a^2\left[(1+2\phi)\,d\eta^2 + 2(D_iB)\,dx^i d\eta - \left((1-2\psi)\gamma_{ij} - 2D_iD_jE\right)dx^i dx^j\right].$$

$$ds^2 = -(1+2A)\,dt^2 - 2a^2(D_iB)\,dx^i dt + a^2\Big(\gamma_{ij}(1+2\psi) - 2D_iD_jE\Big)\,dx^i dx^j\Big]$$


Mukhanov
(+ - - -)
$ds^2 = a^2[(1+2\phi)\,d\eta^2 + 2(D_iB)\,dx^i d\eta - ((1-2\psi)\gamma_{ij} - 2D_iD_jE)d$
Dodelson
(- + + +)
$ds^2 = \ldots A)\,dt^2 - 2a^2(D_iB)\,dx^i dt + a^2(\gamma_{ij}(1+2\psi)$
$-2D_iD_j$
$-N_0^2(1+2\alpha)\,dt^2 - 2a^2(D_i\beta)\,dx^i dt$
$+\,a^2(\gamma_{ij}(1+2\phi) + 2D_iD_j\gamma)$
$dx^i dx^j$

Mukhanov

$(+---)$

$$ds^2 = a^2\left[(1+2\phi)\,d\eta^2 + 2(D_iB)\,dx^i d\eta - ((1-2\psi)\gamma_{ij} - 2D_iD_jE)\,d\right.$$

Dodelson

$(-+++)$

$$ds^2 = -(1+2A)\,dt^2 - 2a^2(D_iB)\,dx^i dt + a^2(\gamma_{ij}(1+2\psi) - 2D_iD_j$$

Bardeen

$(-+++)$

$$ds^2 = -N_0^2(1+2\alpha)\,dt^2 - 2a^2(D_i\beta)\,dx^i dt + a^2\left(\gamma_{ij}(1+2\phi) + 2D_iD_j\gamma\right)dx^i dx^j$$

Mukhanov

$(+---)$

$$ds^2 = a^2\left[(1+2\phi)\,d\eta^2 + 2(D_iB)\,dx^i d\eta - ((1-2\psi)\gamma_{ij} - 2D_iD_jE)\,d\ldots\right.$$

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$$ds^2 = -(1+2A)\,dt^2 - 2a^2(D_iB)\,dx^i dt + a^2(\gamma_{ij}(1+2\psi) - 2D_iD_j\ldots$$

Bardeen

$(-+++)$

$$ds^2 = -N_0^2(1+2\alpha)\,dt^2 - 2a^2(D_i\beta)\,dx^i dt + a^2(\gamma_{ij}(1+2\phi) + 2D_iD_j\gamma)\,dx^i dx^j$$

| Bardeen | Mukhanov | Dodelson |
|---|---|---|
| $\alpha$ | $\phi$ | $A$ |
| $\beta$ | $B$ | $B$ |
| $\phi$ | $-\psi$ | $\psi$ |
| $\gamma$ | $-E$ | $-E$ |

$dx^i dx^j]$

4 scalar perturbations : $A, B, C, E$ $\longrightarrow 4$

2 vector perturbation $\bar{B}^i$ , $\bar{E}^i$ $\longrightarrow 4$

$\longrightarrow 2$

1 Tensor perturbation : $\bar{E}^{ij}$

...een's Potential $\Phi_H = \phi - H\chi$ ; $\Phi_A = \alpha - \frac{\dot{\chi}}{N_0}$

$dx^i dx^j]$

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| | Mukhanov | Dodelson |
|---|---|---|
| | $-\Psi$ | |

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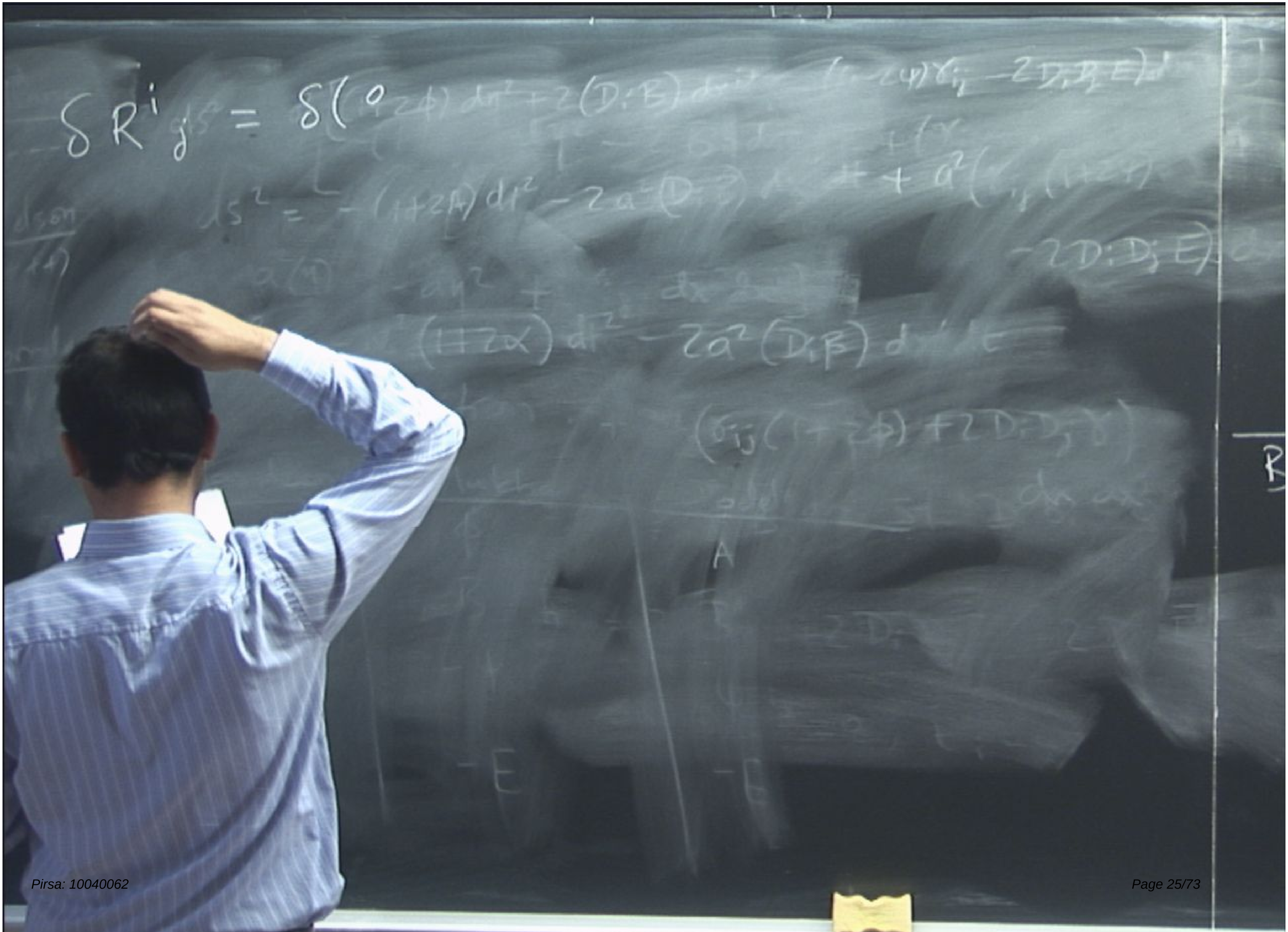
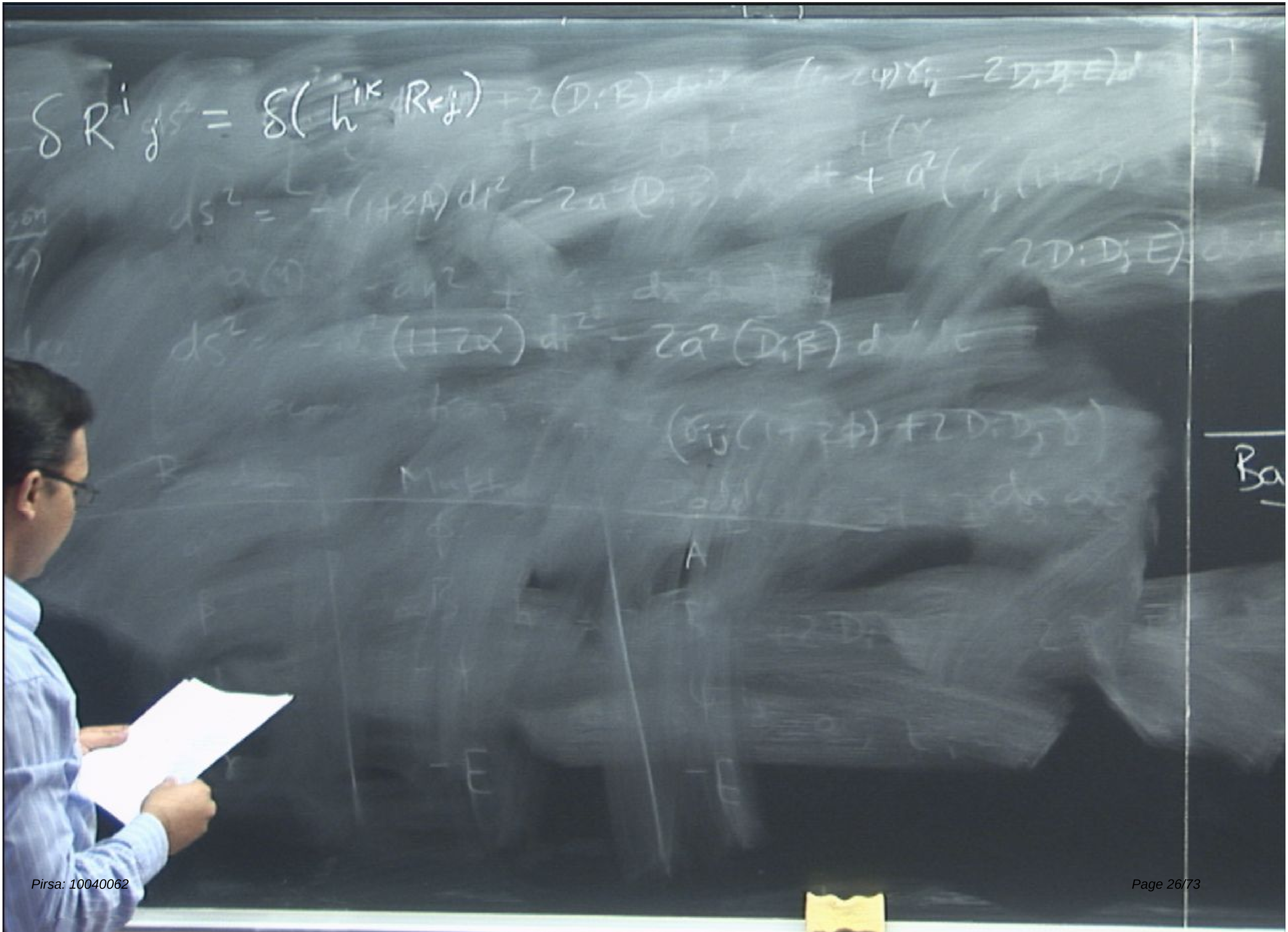
| Bardeen | Mukhanov | Dodelson |
|---|---|---|
| $\Phi_H$ | $-\Psi$ | $-\Phi$ |
| $\Phi_A$ | $\Phi$ | $\Psi$ |


δ R^i_j = δ(


δ R^i_j = δ( h^ik R_kj )

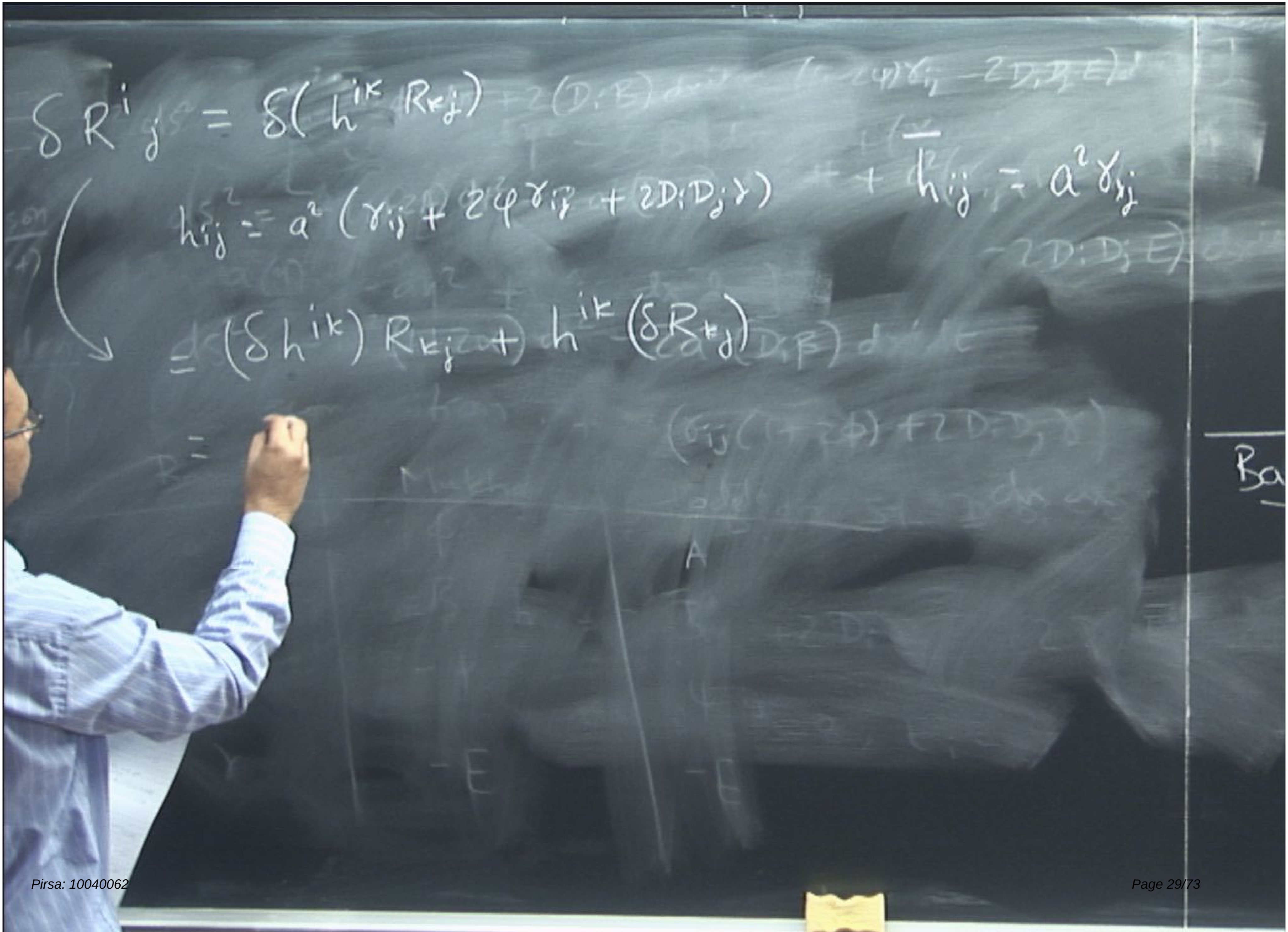
$$\delta R^i{}_j = \delta( h^{ik} R_{kj})$$

$$h_{ij} = a^2 ( \gamma_{ij} + 2\varphi \gamma_{ij} + 2 D_i D_j \gamma )$$

$$\delta R^i{}_j = \delta\left(h^{ik} R_{kj}\right)$$

$$h_{ij} = a^2\left(\gamma_{ij} + 2\varphi\gamma_{ij} + 2D_iD_j\gamma\right)$$

$$= (\delta h^{ik}) R_{kj} + h^{ik}(\delta R_{kj})$$



$\delta R^i{}_j = \delta(h^{ik} R_{kj})$
$h_{ij} = a^2(\gamma_{ij} + 2\varphi\gamma_{ij} + 2D_iD_j\gamma)$
$\bar{h}_{ij} = a^2\gamma_{ij}$
$= (\delta h^{ik}) R_{kj} + h^{ik}(\delta R_{kj})$
$=$

$$\delta R^i{}_j = \delta( h^{ik} R_{kj})$$

$$h_{ij} = a^2 (\gamma_{ij} + 2\varphi \gamma_{ij} + 2 D_i D_j \gamma) \qquad \bar{h}_{ij} = a^2 \gamma_{ij}$$

$$= (\delta h^{ik}) R_{kj} + h^{ik} (\delta R_{kj})$$

$$= (\delta h^{ik}) \bar{R}_{kj} + \bar{h}^{ik} \delta R_{kj}$$

Ba

$$\delta R^i{}_j = \delta(h^{ik} R_{kj})$$

$$h_{ij} = a^2(\gamma_{ij} + 2\varphi\gamma_{ij} + 2D_iD_j\gamma) \qquad + \quad \bar{h}_{ij} = a^2\gamma_{ij}$$

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Bardeen's Potential $\quad \Phi_H = \phi - H\chi \; ; \quad \Phi_A = \alpha - \dfrac{\dot{\chi}}{N_0}$

| Bardeen | Mukhanov | Dodelson | Straumann |
|---|---|---|---|
| $\Phi_H$ | $-\Psi$ | $-\Phi$ | |
| $\Phi_A$ | $\Phi$ | $\Psi$ | |

$$\delta R^i{}_j = \delta(h^{ik} R_{kj})$$

$$h_{ij} = a^2(\gamma_{ij} + 2\varphi\gamma_{ij} + 2D_iD_j\gamma)$$

$$\bar{h}_{ij} = a^2\gamma_{ij}$$

$$= (\delta h^{ik}) R_{kj} + h^{ik}(\delta R_{kj})$$

Palatini Identity :

$$\boxed{\delta R_{\mu\nu} = (\delta\Gamma^\alpha{}_{\mu\nu})_{;\alpha} - (\delta\Gamma^\alpha{}_{\mu\alpha})_{;\nu}}$$

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$$R_{\mu\nu} = \partial_\alpha\Gamma^\alpha_{\nu\mu} - \partial_\nu\Gamma^\alpha_{\alpha\mu} + \Gamma^\alpha_{\alpha\lambda}\Gamma^\lambda_{\nu\mu} - \Gamma^\alpha_{\nu\lambda}\Gamma^\lambda_{\alpha\mu}$$

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$$\delta R_{\mu\nu} = \partial_\alpha\delta\Gamma^\alpha_{\nu\mu} - \partial_\nu\delta\Gamma^\alpha_{\alpha\mu} + (\delta\Gamma^\alpha_{\alpha\lambda})\Gamma^\lambda_{\nu\mu} + \Gamma^\alpha_{\alpha\lambda}\delta\Gamma^\lambda_{\nu\mu} - \ldots$$

$$\delta R^i{}_j = \delta(h^{ik} R_{kj})$$

$$h_{ij} = a^2(\gamma_{ij} + 2\varphi\gamma_{ij} + 2D_iD_j\gamma) \qquad \bar{h}_{ij} = a^2\gamma_{ij}$$

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Palatini Identity :

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$$\delta R_{\mu\nu} = \partial_\alpha\delta\Gamma^\alpha{}_{\nu\mu} - \partial_\nu\delta\Gamma^\alpha{}_{\alpha\mu} + (\delta\Gamma^\alpha{}_{\alpha\lambda})\Gamma^\lambda{}_{\nu\mu} + \Gamma^\alpha{}_{\alpha\lambda}\delta\Gamma^\lambda{}_{\nu\mu} - \dots$$

$$\nabla_\alpha\delta\Gamma^\alpha{}_{\nu\mu} = \partial_\alpha\delta\Gamma^\alpha{}_{\nu\mu} - (\delta\Gamma^\alpha{}_{\lambda\mu})\Gamma^\lambda{}_{\nu\alpha} + \Gamma^\alpha{}_{\nu\lambda}\delta\Gamma^\lambda{}_{\alpha\mu} - \delta\Gamma^\alpha{}_{\alpha\lambda}\Gamma^\lambda{}_{\nu\mu}$$

4 scalar perturbations : $A, B, C, E$ $\longrightarrow 4$

2 vector perturbation $\bar{B}^i$ , $\bar{E}^i$ $\longrightarrow 4$

1 Tensor perturbation : $\bar{E}^{ij}$ $\longrightarrow 2$

$$\boxed{\delta R_{\mu\nu} = \nabla_\alpha \delta \Gamma^\alpha_{\nu\mu} - \nabla_\nu \delta \Gamma^\alpha_{\alpha\mu}}$$

$$\delta \Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} \left[ (\delta g_{\nu\alpha})_{;\beta} + (\delta g_{\nu\beta})_{;\alpha} - (\delta g_{\alpha\beta})_{;\nu} \right]$$

$$\delta R^i{}_j = \delta(h^{ik} R_{kj})$$

$$\delta R_{ij} = \frac{1}{2} h^{ab} \left[ \delta h \right.$$

Palatini Identity:

$$\boxed{\delta R_{\mu\nu} = (\delta \Gamma^\alpha_{\mu\nu})_{;\alpha} \cdots )_{;\nu}}$$

$$R_{\mu\nu} = \partial_\alpha \Gamma^\alpha_{\nu\mu} - \partial_\nu \Gamma^\alpha_{\alpha\mu} + \Gamma^\alpha_{\alpha\lambda} \Gamma^\lambda_{\nu\mu} - \Gamma^\alpha \cdots$$

$$\delta R_{\mu\nu} = \partial_\alpha \delta\Gamma^\alpha_{\nu\mu} - \partial_\nu \delta\Gamma^\alpha_{\alpha\mu} + (\delta\Gamma^\alpha_{\alpha\lambda}) \Gamma \cdots {}^\lambda_{\nu\mu}$$

$$\nabla_\alpha \delta\Gamma^\alpha_{\nu\mu} = \partial_\alpha \delta\Gamma^\alpha_{\nu\mu} - (\delta\Gamma^\alpha_{\lambda\mu}) \cdots {}_{\nu\mu}$$

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$$\boxed{\delta R_{\mu\nu} = \nabla_\alpha \delta\Gamma^\alpha_{\nu\mu} - \nabla_\nu \delta\Gamma^\alpha_{\alpha\mu}}$$

$$\delta\Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\nu}\left[(\delta g_{\nu\alpha})_{;\beta} + (\delta g_{\nu\beta})_{;\alpha} - (\delta g_{\alpha\beta})_{;\nu}\right]$$

$$\delta h_{ij} = 2\varphi a^2 \gamma_{ij} + 2a^2 D_i D_j \gamma$$

$$R^i{}_j = \delta(h^{ik} R_{kj})$$

$$\delta R_{ij} = \frac{1}{2} h^{ab}\left[\delta h_{bi|ja} - \delta h_{ab|ij} + \delta h_{bj|ia} - \delta h_{ij|ab}\right]$$

identity :

$$\boxed{\delta R_{\mu\nu} = (\delta\Gamma^{\alpha}_{\mu\nu})_{;\alpha} - (\delta\Gamma^{\alpha}_{\mu\alpha})_{;\nu}}$$

$$= \partial_\alpha \Gamma^{\alpha}_{\nu\mu} - \partial_\nu \Gamma^{\alpha}_{\alpha\mu} + \Gamma^{\alpha}_{\alpha\lambda}\Gamma^{\lambda}_{\nu\mu} - \Gamma^{\alpha}_{\nu\lambda}\Gamma^{\lambda}_{\alpha\mu}$$

$$- \partial_\nu \delta\Gamma^{\alpha}_{\alpha\mu} + (\delta\Gamma^{\alpha}_{\alpha\lambda})\Gamma^{\lambda}_{\nu\mu} + \Gamma^{\alpha}_{\alpha\lambda}\delta\Gamma^{\lambda}_{\nu\mu} - \dots$$

$$= \partial_\alpha \delta\Gamma^{\alpha}_{\nu\mu} - (\delta\Gamma^{\alpha}_{\lambda\mu})\Gamma^{\lambda}_{\nu\alpha} + \Gamma^{\alpha}_{\nu\lambda}\delta\Gamma^{\lambda}_{\alpha\mu} - \delta\Gamma^{\alpha}_{\alpha\lambda}\Gamma^{\lambda}_{\nu\mu}$$

$$\delta R^i{}_j = \delta( h^{ik} R_{kj}) = (\delta h^{ik}) R_{kj} + h^{ik} \delta R_{kj}$$

$$\delta R_{ij} = \frac{1}{2} h^{ab} \Big[ \underset{(i)}{\delta h_{bi||ja}} - \underset{(ii)}{\delta h_{ab|ij}} + \underset{(iii)}{\delta h_{bj||ia}} - \underset{(iv)}{\delta h_{ij|ab}} \Big]$$

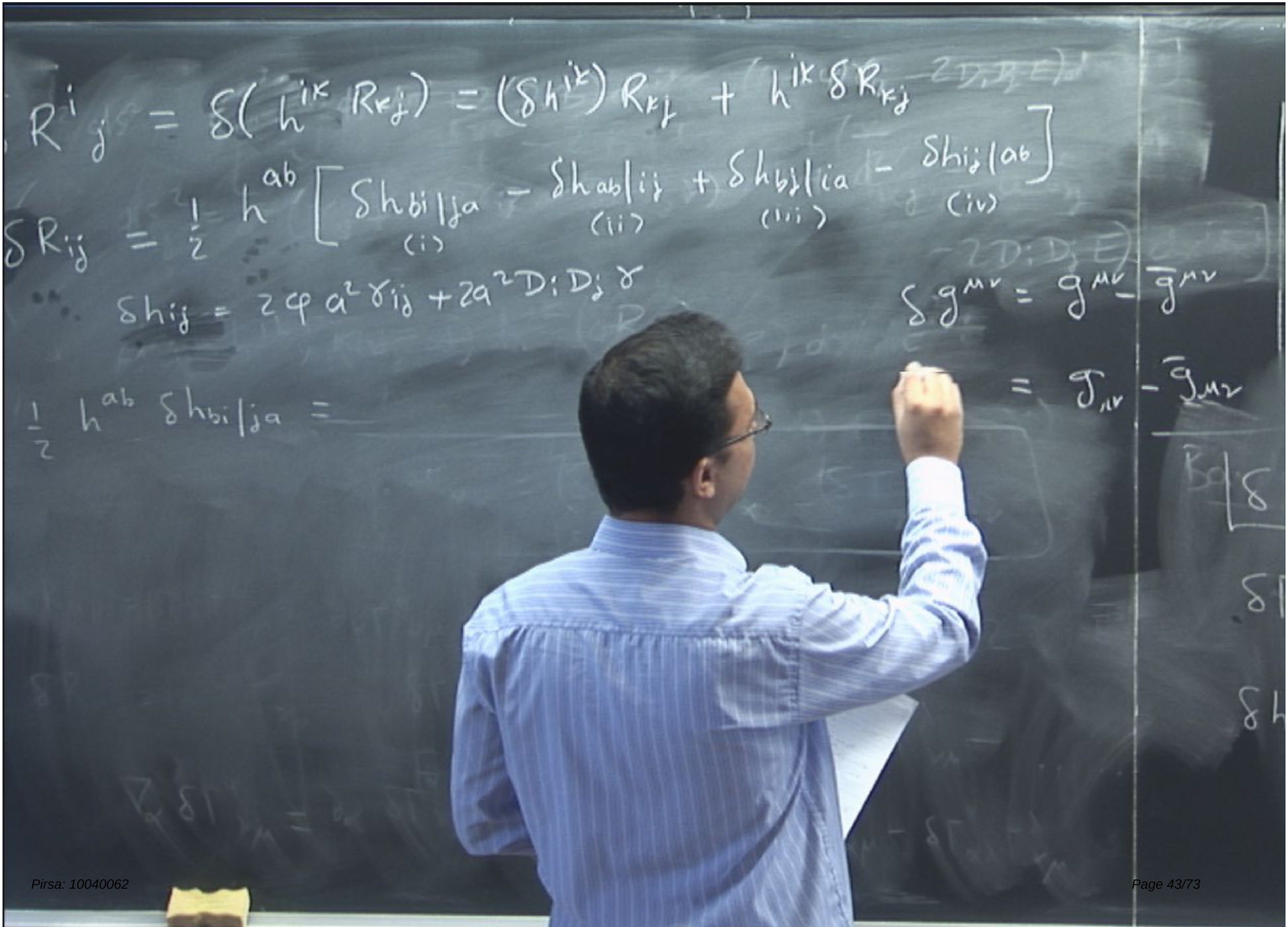
$$\delta h_{ij} = 2\varphi a^2 \gamma_{ij} + 2a^2 D_i D_j \gamma$$

$$\delta R^i{}_j = \delta(h^{ik} R_{kj}) = (\delta h^{ik}) R_{kj} + h^{ik}\,\delta R_{kj}$$

$$\delta R_{ij} = \frac{1}{2} h^{ab}\Big[\underbrace{\delta h_{bi|ja}}_{(i)} - \underbrace{\delta h_{ab|ij}}_{(ii)} + \underbrace{\delta h_{bj|ia}}_{(iii)} - \underbrace{\delta h_{ij|ab}}_{(iv)}\Big]$$

$$\delta h_{ij} = 2\varphi a^2 \gamma_{ij} + 2a^2 D_i D_j \gamma$$

$$\frac{1}{2} h^{ab}\,\delta h_{bi|ja} =$$



$\delta R^i{}_j = \delta(h^{ik} R_{kj}) = (\delta h^{ik}) R_{kj} + h^{ik} \delta R_{kj}$
$\delta R_{ij} = \frac{1}{2} h^{ab} [ \delta h_{bi|ja} - \delta h_{ab|ij} + \delta h_{bj|ia} - \delta h_{ij|ab} ]$
(i) (ii) (iii) (iv)
$\delta h_{ij} = 2\varphi a^2 \gamma_{ij} + 2a^2 D_i D_j \gamma$
$\delta g^{\mu\nu} = g^{\mu\nu} - \bar{g}^{\mu\nu}$
$= g_{\mu\nu} - \bar{g}_{\mu\nu}$
$\frac{1}{2} h^{ab} \delta h_{bi|ja} =$

$$\delta R^i{}_j = \delta(h^{ik} R_{kj}) = (\delta h^{ik}) R_{kj} + h^{ik}\delta R_{kj}$$

$$\delta R_{ij} = \frac{1}{2} h^{ab}\Big[\underbrace{\delta h_{bi|ja}}_{(i)} - \underbrace{\delta h_{ab|ij}}_{(ii)} + \underbrace{\delta h_{bj|ia}}_{(iii)} - \underbrace{\delta h_{ij|ab}}_{(iv)}\Big]$$

$$\delta h_{ij} = 2\varphi a^2 \gamma_{ij} + 2a^2 D_i D_j \gamma$$

$$\delta g^{\mu\nu} = g^{\mu\nu} - \bar{g}^{\mu\nu}$$

$$\uparrow$$

$$\delta g_{\mu\nu} = g_{\mu\nu} - \bar{g}_{\mu\nu}$$

$$\frac{1}{2} h^{ab}\,\delta h_{bi|ja} =$$

$$\delta(h^{ik} R_{kj}) = (\delta h^{ik}) R_{kj} + h^{ik}\,\delta R_{kj}$$

$$h^{ab}\Big[\underset{(i)}{\delta h_{bi|ja}} - \underset{(ii)}{\delta h_{ab|ij}} + \underset{(iii)}{\delta h_{bj|ia}} - \underset{(iv)}{\delta h_{ij|ab}}\Big]$$

$$2\varphi a^2 \gamma_{ij} + 2a^2 D_i D_j \gamma$$

$$|ja = \frac{1}{2}$$

$$\delta g^{\mu\nu} = g^{\mu\nu} - \bar{g}^{\mu\nu}$$

$$\uparrow$$

$$\delta g_{\mu\nu} = g_{\mu\nu} - \bar{g}_{\mu\nu}$$

4 scalar

2 vector

1 Tensor

$$\boxed{\delta R_{\mu\nu} = \nabla_\alpha}$$

$$\delta \Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu}$$

$$h_{ij} = 2\varphi a^2$$

$$\delta(h^{ik} R_{kj}) = (\delta h^{ik}) R_{kj} + h^{ik}\,\delta R_{kj}$$

$$h^{ab}\Big[\underset{(i)}{\delta h_{bi||ja}} - \underset{(ii)}{\delta h_{ab||ij}} + \underset{(iii)}{\delta h_{bj||ia}} - \underset{(iv)}{\delta h_{ij||ab}}\Big]$$

$$2\varphi a^2 \gamma_{ij} + 2a^2 D_i D_j \gamma$$

$$\delta g^{\mu\nu} = g^{\mu\nu} - \bar{g}^{\mu\nu}$$

$$\uparrow$$

$$\delta g_{\mu\nu} = g_{\mu\nu} - \bar{g}_{\mu\nu}$$

$$= \frac{1}{2}$$

4 scalar

2 vector

1 Tensor

$$\boxed{\delta R_{\mu\nu} = \nabla_\alpha}$$

$$\delta \Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu}$$

$$h_{ij} = a^2\gamma_{ij} + 2\varphi a^2$$

$$\delta R^i{}_j = \delta(h^{ik} R_{kj}) = (\delta h^{ik}) R_{kj} + h^{ik} \delta R_{kj}$$

$$\delta R_{ij} = \frac{1}{2} h^{ab} \Big[ \underset{(i)}{\delta h_{bi|ja}} - \underset{(ii)}{\delta h_{ab|ij}} + \underset{(iii)}{\delta h_{bj|ia}} - \underset{(iv)}{\delta h_{ij|ab}} \Big]$$

$$\delta h_{ij} = 2\varphi a^2 \gamma_{ij} + 2a^2 D_i D_j \gamma$$

$$\delta g^{\mu\nu} = g^{\mu\nu} - \bar{g}^{\mu\nu}$$

$$\delta g_{\mu\nu} \overset{!}{=} g_{\mu\nu} - \bar{g}_{\mu\nu}$$

$$\text{(i)} \quad \frac{1}{2} h^{ab} \delta h_{bi|ja} = \frac{\cancel{2}}{\cancel{2}} \frac{\gamma^{ab}}{a^2} \left( a^2 \varphi \gamma_{bi} + a^2 D_b D_i \gamma \right)_{|ja}$$

$h_{ij}$

$$\delta R^i{}_j = \delta(h^{ik} R_{kj}) = (\delta h^{ik}) R_{kj} + h^{ik}\delta R_{kj}$$

$$\delta R_{ij} = \frac{1}{2} h^{ab}\Big[\underset{(i)}{\delta h_{bi|ja}} - \underset{(ii)}{\delta h_{ab|ij}} + \underset{(iii)}{\delta h_{bj|ia}} - \underset{(iv)}{\delta h_{ij|ab}}\Big]$$

$$\delta h_{ij} = 2\varphi a^2 \gamma_{ij} + 2a^2 D_i D_j \gamma$$

$$\delta g^{\mu\nu} = g^{\mu\nu} - \bar{g}^{\mu\nu}$$

$$\delta g_{\mu\nu} = g_{\mu\nu} - \bar{g}_{\mu\nu}$$

$$(i)\ \frac{1}{2} h^{ab}\,\delta h_{bi|ja} = \frac{2}{2} \ldots \varphi\gamma_{bi} + a^2 D_b D_i \gamma\Big)_{|ja}$$

$$= \gamma^{ab}\gamma_{bi} D \ldots \gamma^{ab} D_a D_j$$

$h_{ij} =$

$$\delta R^i{}_j = \delta(h^{ik} R_{kj}) = (\delta h^{ik}) R_{kj} + h^{ik} \delta R_{kj}$$

$$\delta R_{ij} = \frac{1}{2} h^{ab} \Big[ \underset{(i)}{\delta h_{bi|ja}} - \underset{(ii)}{\delta h_{ab|ij}} + \underset{(iii)}{\delta h_{bj|ia}} - \underset{(iv)}{\delta h_{ij|ab}} \Big]$$

$$\delta h_{ij} = 2\varphi a^2 \gamma_{ij} + 2a^2 D_i D_j \gamma$$

$$\delta g^{\mu\nu} = g^{\mu\nu} - \bar{g}^{\mu\nu}$$

$$\uparrow$$

$$\delta g_{\mu\nu} = g_{\mu\nu} - \bar{g}_{\mu\nu}$$

$$\text{(i)} \quad \frac{1}{2} h^{ab} \delta h_{bi|ja} = \frac{2}{2} \frac{\gamma^{ab}}{a^2} \left( a^2 \varphi \gamma_{bi} + a^2 D_b D_i \gamma \right)_{|ja}$$

$$= \gamma^{ab} \gamma_{bi} D_a D_j \varphi + \gamma^{ab} D_a D_j D_b D_i \gamma$$

$h_{ij}$

$$\delta R^i{}_j = \delta(h^{ik} R_{kj}) = (\delta h^{ik}) R_{kj} + h^{ik}\,\delta R_{kj}$$

$$\delta R_{ij} = \frac{1}{2} h^{ab}\Big[\underbrace{\delta h_{bi|ja}}_{(i)} - \underbrace{\delta h_{ab|ij}}_{(ii)} + \underbrace{\delta h_{bj|ia}}_{(iii)} - \underbrace{\delta h_{ij|ab}}_{(iv)}\Big]$$

$$\delta h_{ij} = 2\varphi a^2 \gamma_{ij} + 2a^2 D_i D_j \gamma$$

$$\delta g^{\mu\nu} = g^{\mu\nu} - \bar{g}^{\mu\nu}$$

$$\delta g_{\mu\nu} = g_{\mu\nu} - \bar{g}_{\mu\nu}$$

(i) $$\frac{1}{2} h^{ab}\,\delta h_{bi|ja} = \frac{2}{2}\frac{\gamma^{ab}}{a^2}\left(a^2\varphi\gamma_{bi} + a^2 D_b D_i \gamma\right)_{|ja}$$

$$= \gamma^{ab}\gamma_{bi} D_a D_j \varphi + \gamma^{ab} D_a D_j D_b D_i \gamma$$

(ii) $$-\frac{1}{2} h^{ab}\,\delta h_{ab|ij} = -3 D_i D_j \varphi - D_j D_i \Delta\gamma \quad ; \; \Delta \equiv D^i D_i$$

$h_{ij}$

$$\delta R^i{}_j = \delta( h^{ik} R_{kj}) = (\delta h^{ik}) R_{kj} + h^{ik} \delta R_{kj}$$

$$\delta R_{ij} = \frac{1}{2} h^{ab} \Big[ \underbrace{\delta h_{bi|ja}}_{(i)} - \underbrace{\delta h_{ab|ij}}_{(ii)} + \underbrace{\delta h_{bj|ia}}_{(iii)} - \underbrace{\delta h_{ij|ab}}_{(iv)} \Big]$$

$$\delta h_{ij} = 2\varphi a^2 \gamma_{ij} + 2a^2 D_i D_j \gamma$$

$$\delta g^{\mu\nu} = g^{\mu\nu} - \bar{g}^{\mu\nu}$$

$$\delta g_{\mu\nu} = g_{\mu\nu} - \bar{g}_{\mu\nu}$$

(i) $\frac{1}{2} h^{ab} \delta h_{bi|ja} = \frac{\cancel{2}}{\cancel{2}} \frac{\gamma^{ab}}{a^2} \left( a^2 \varphi \gamma_{bi} + a^2 D_b D_i \gamma \right)_{|ja}$

$$= \gamma^{ab} \gamma_{bi} D_a D_j \varphi + \gamma^{ab} D_a D_j D_b D_i \gamma$$

(ii) $-\frac{1}{2} h^{ab} \delta h_{ab|ij} = -3 D_i D_j \varphi - D_j D_i \Delta \gamma \quad ; \Delta \equiv D^i D_i$

(iii) $\frac{1}{2} h^{ab} \delta h_{bj|ia} = \gamma^{ab} \gamma_{bj} D_a D_i \varphi + D_a D_i D^a D_j \gamma$

$$\delta R^i{}_j = \delta(h^{ik} R_{kj}) = (\delta h^{ik}) R_{kj} + h^{ik}\delta R_{kj}$$

$$\delta R_{ij} = \frac{1}{2} h^{ab}\Big[\underbrace{\delta h_{bi|ja}}_{(i)} - \underbrace{\delta h_{ab|ij}}_{(ii)} + \underbrace{\delta h_{bj|ia}}_{(iii)} - \underbrace{\delta h_{ij|ab}}_{(iv)}\Big]$$

$$\delta h_{ij} = 2\varphi a^2 \gamma_{ij} + 2a^2 D_i D_j \gamma$$

$$\delta g^{\mu\nu} = g^{\mu\nu} - \bar{g}^{\mu\nu}$$

$$\delta g_{\mu\nu} = g_{\mu\nu} - \bar{g}_{\mu\nu}$$

(i) $\frac{1}{2} h^{ab}\delta h_{bi|ja} = \frac{\cancel{2}}{\cancel{2}}\frac{\gamma^{ab}}{a^2}\left(a^2\varphi\gamma_{bi} + a^2 D_b D_i \gamma\right)_{|ja}$

$$= \boxed{\gamma^{ab}\gamma_{bi} D_a D_j \varphi + \gamma^{ab} D_a D_j D_b D_i \gamma}$$

(ii) $-\frac{1}{2} h^{ab}\delta h_{ab|ij} = -3 D_i D_j \varphi - D_j D_i \Delta\gamma \quad ; \Delta \equiv D^i D_i$

(iii) $\frac{1}{2} h^{ab}\delta h_{bj|ia} = \gamma^{ab}\gamma_{bj} D_a D_i \varphi + D_a D_i D^a D_j \gamma$

(iv) $-\frac{1}{2} g^{ab}\delta g_{ij|ab} = -a^2 \Delta\varphi\gamma_{ij} - \Delta D_i D_j \gamma$

$h_{ij} =$

$$\delta R^i{}_j = \delta(h^{ik} R_{kj}) = (\delta h^{ik}) R_{kj} + h^{ik}\delta R_{kj}$$

$$\delta R_{ij} = \frac{1}{2} h^{ab}\Big[\underset{(i)}{\delta h_{bi|ja}} - \underset{(ii)}{\delta h_{ab|ij}} + \underset{(iii)}{\delta h_{bj|ia}} - \underset{(iv)}{\delta h_{ij|ab}}\Big]$$

$$\delta h_{ij} = 2\varphi a^2 \gamma_{ij} + 2a^2 D_i D_j \gamma$$

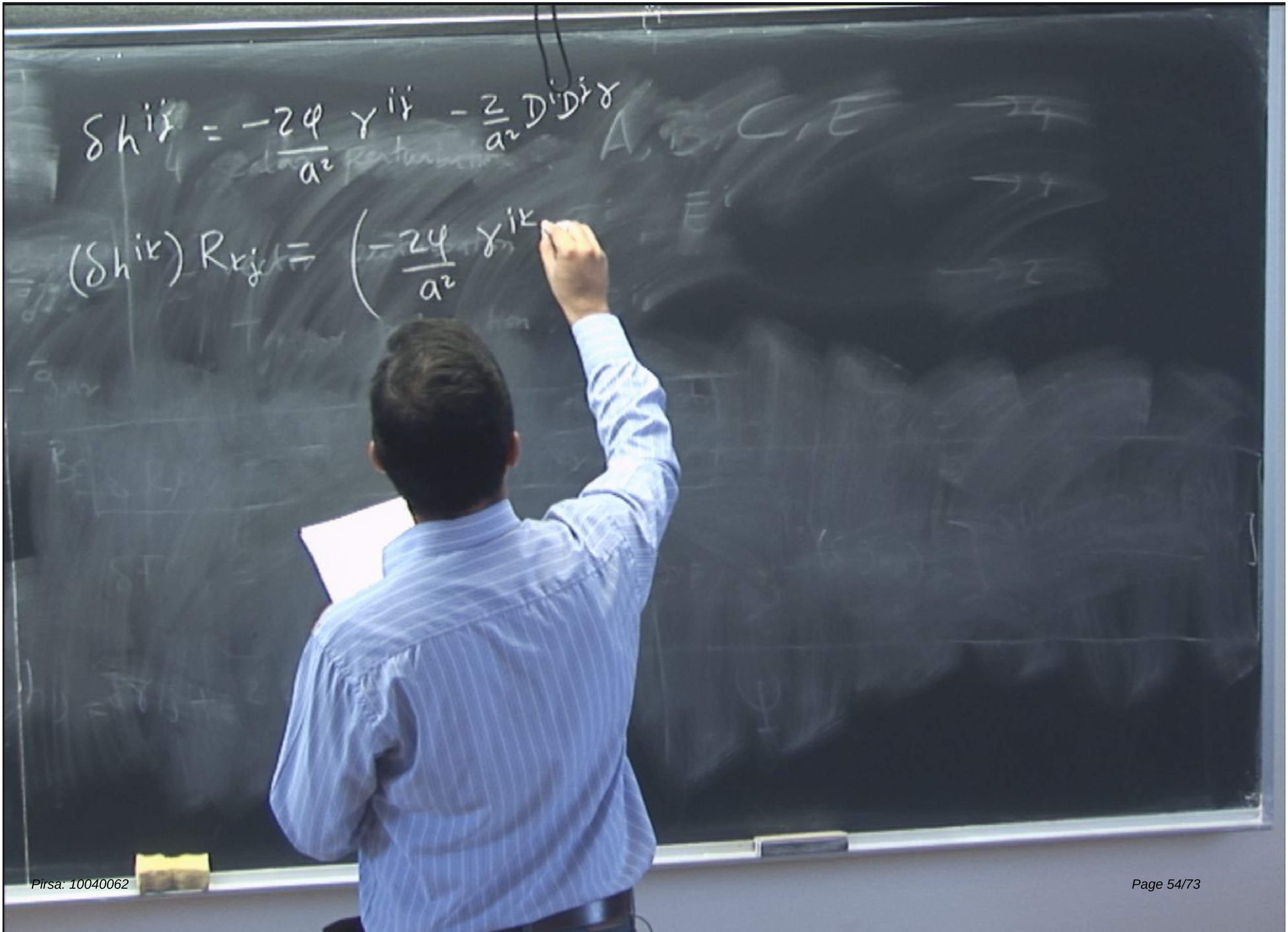
$$\text{(i)}\quad \frac{1}{2} h^{ab}\,\delta h_{bi|ja} = \frac{2}{2}\frac{\gamma^{ab}}{a^2}\left(a^2\varphi\gamma_{bi} + a^2 D_b D_i \gamma\right)\Big|_{ja}$$

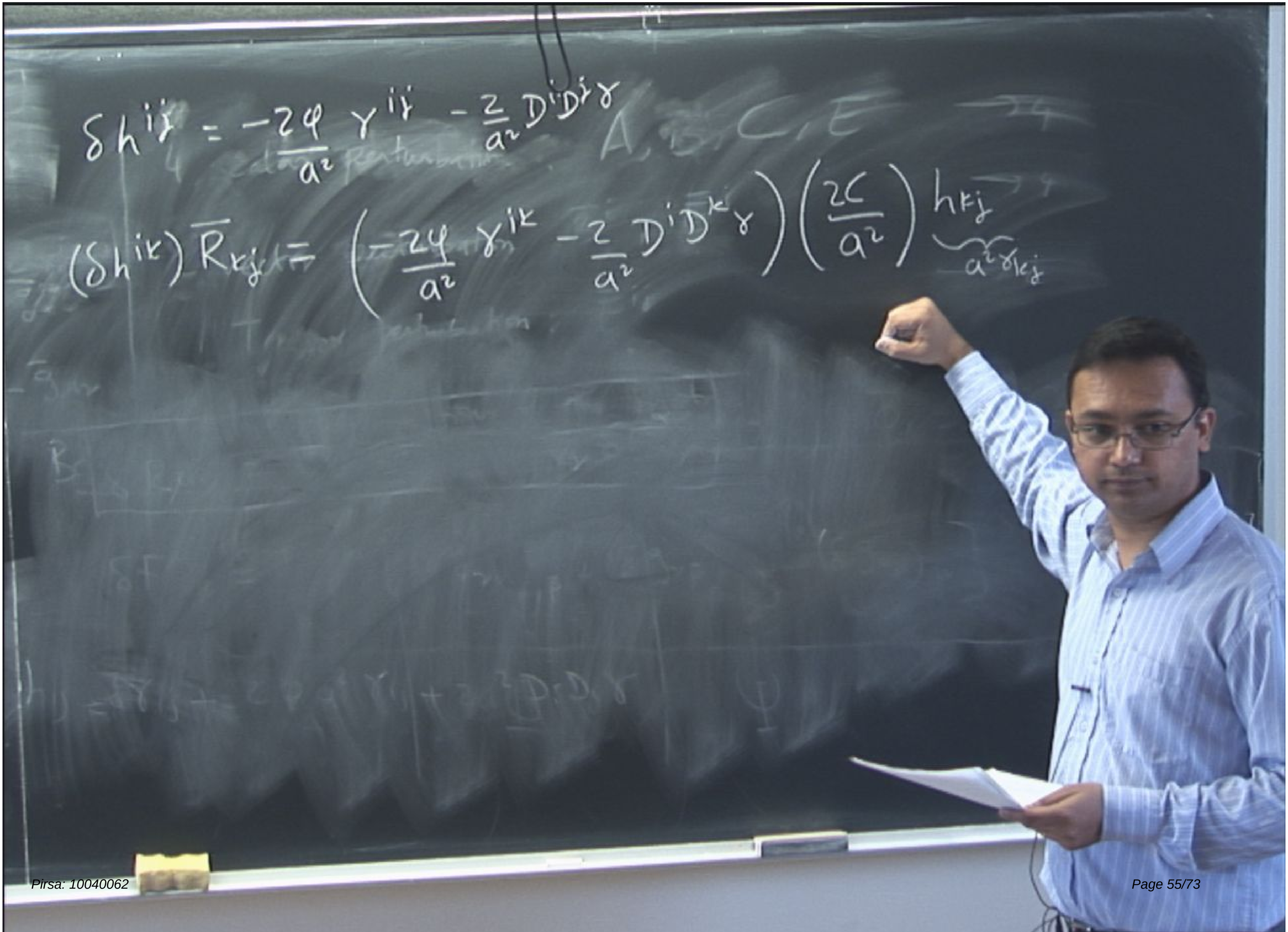
$$= \gamma^{ab}\gamma_{bi} D_a D_j \varphi + \gamma^{ab} D_a D_j D_b D_i \gamma$$

$$\text{(ii)}\quad -\frac{1}{2} h^{ab}\,\delta h_{ab|ij} = -3 D_i D_j \varphi - D_j D_i \Delta\gamma \quad ; \Delta \equiv D^i D_i$$

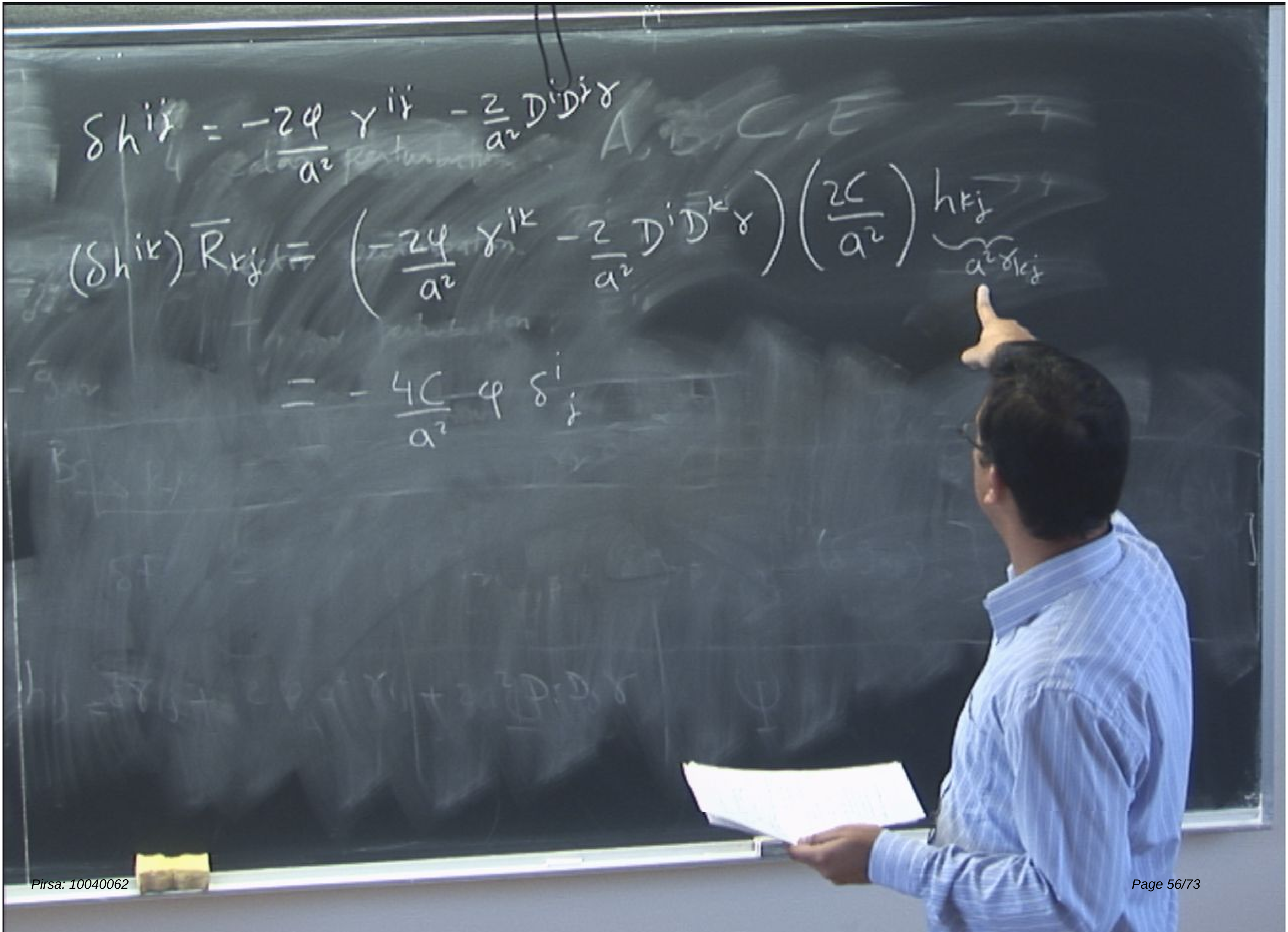
$$\text{(iii)}\quad \frac{1}{2} h^{ab}\,\delta h_{bj|ia} = \gamma^{ab}\gamma_{bj} D_a D_i \varphi + D_a D_i D^a D_j \gamma$$

$$\text{(iv)}\quad -\frac{1}{2} g^{ab}\,\delta g_{ij|ab} = -a^2\Delta\varphi\gamma_{ij} - \Delta D_i D_j \gamma$$


$\delta h^{ij} = -\frac{2\varphi}{a^2}\gamma^{ij} - \frac{2}{a^2}D^iD^j\gamma$
$(\delta h^{ik})R_{kj} = \left(-\frac{2\varphi}{a^2}\gamma^{ik}\right.$



$\delta h^{ij} = -\frac{2\psi}{a^2}\gamma^{ij} - \frac{2}{a^2}D^iD^j\gamma$
$(\delta h^{ik})\bar{R}_{kj} = \left(-\frac{2\psi}{a^2}\gamma^{ik} - \frac{2}{a^2}D^iD^k\gamma\right)\left(\frac{2C}{a^2}\right)\underbrace{h_{kj}}_{a^2\gamma_{kj}}$


$\delta h^{ij} = -\frac{2\varphi}{a^2}\gamma^{ij} - \frac{2}{a^2}D^iD^j\gamma$
$(\delta h^{ik})\bar{R}_{kj} = \left(-\frac{2\varphi}{a^2}\gamma^{ik} - \frac{2}{a^2}D^iD^k\gamma\right)\left(\frac{2C}{a^2}\right)\underbrace{h_{kj}}_{a^2\gamma_{kj}}$
$= -\frac{4C}{a^2}\varphi\,\delta^i{}_j$

$$\delta h^{ij} = -\frac{2\varphi}{a^2}\gamma^{ij} - \frac{2}{a^2}D^iD^j\gamma$$

$$(\delta h^{ik})\bar{R}_{kj} = \left(-\frac{2\varphi}{a^2}\gamma^{ik} - \frac{2}{a^2}D^iD^k\gamma\right)\left(\frac{2C}{a^2}\right)\underbrace{h_{kj}}_{a^2\gamma_{kj}}$$

$$\boxed{(\delta h^{ik})\bar{R}_{kj} = -\frac{4C}{a^2}\varphi\,\delta^i_j - \frac{4C}{a^2}D^iD_j\gamma}$$

$$\delta h^{ij} = -\frac{2\varphi}{a^2}\gamma^{ij} - \frac{2}{a^2}D^iD^j\gamma$$

$$(\delta h^{ik})\bar{R}_{kj} = \left(-\frac{2\varphi}{a^2}\gamma^{ik} - \frac{2}{a^2}D^iD^k\gamma\right)\left(\frac{2C}{a^2}\right)\underbrace{h_{kj}}_{a^2\gamma_{kj}}$$

$$\boxed{(\delta h^{ik})\bar{R}_{kj} = -\frac{4C}{a^2}\varphi\,\delta^i_j - \frac{4C}{a^2}D^iD_j\gamma}$$

$$\delta R_{ij} = -D_iD_j\varphi - a^2\Delta\varphi\,\gamma_{ij} + D_aD_jD^aD_i\gamma - D_j$$

$$\delta h^{ij} = -\frac{2\varphi}{a^2}\gamma^{ij} - \frac{2}{a^2}D^i D^j \gamma$$

$$(\delta h^{ik})\bar{R}_{kj} = \left(-\frac{2\varphi}{a^2}\gamma^{ik} - \frac{2}{a^2}D^i D^k \gamma\right)\left(\frac{2C}{a^2}\right)\underbrace{h_{kj}}_{a^2\gamma_{kj}}$$

$$\boxed{(\delta h^{ik})\bar{R}_{kj} = -\frac{4C}{a^2}\varphi\,\delta^i_j - \frac{4C}{a^2}D^i D_j \gamma}$$

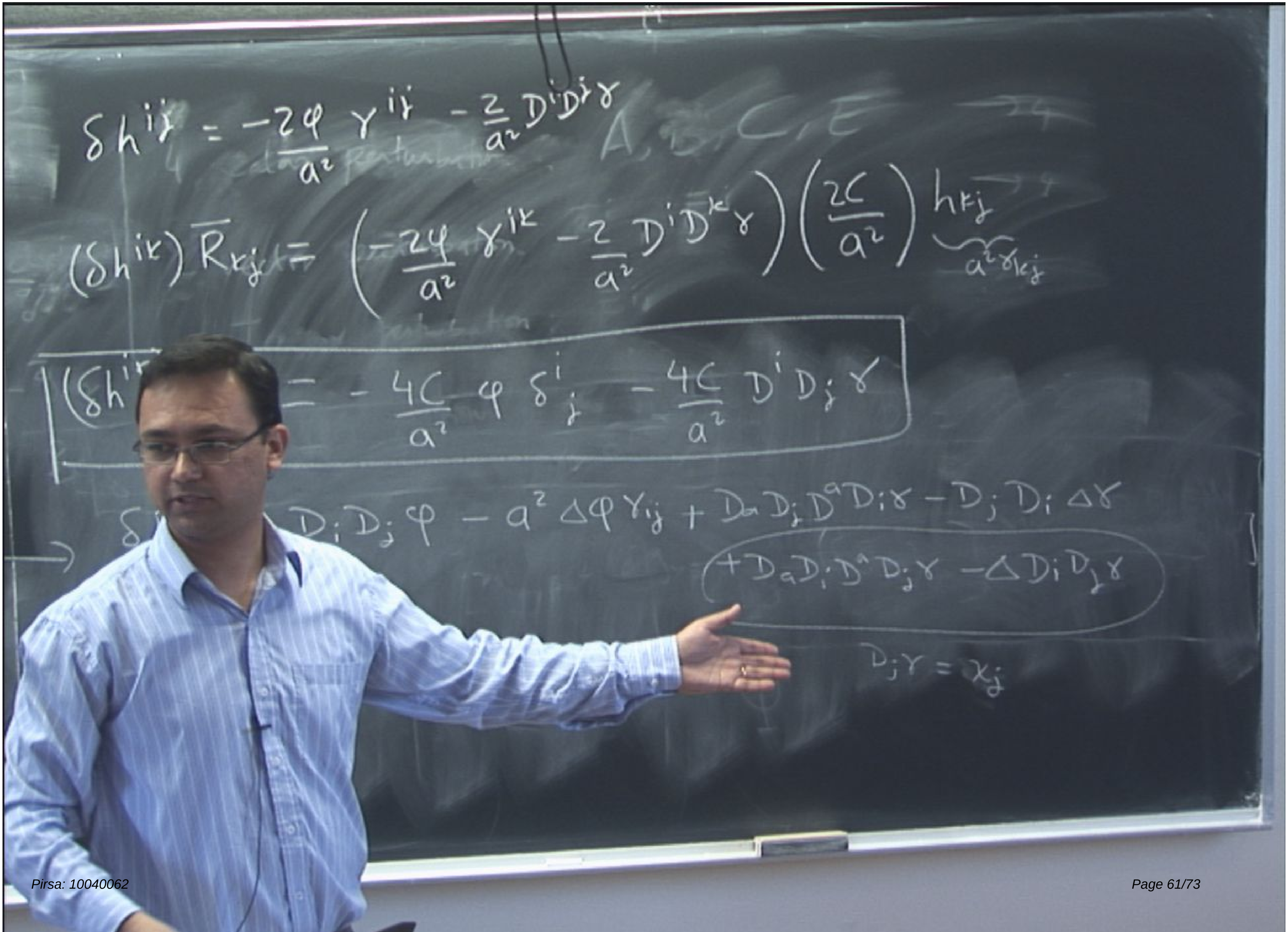
$$\delta R_{ij} = -D_i D_j \varphi - a^2 \Delta\varphi\,\gamma_{ij} + D_a D_j D^a D_i \gamma - D_j D_i \Delta\gamma$$
$$+ D_a D_i D^a D_j \gamma - \Delta$$

$$\delta h^{ij} = -\frac{2\varphi}{a^2}\gamma^{ij} - \frac{2}{a^2} D^i D^j \gamma$$

$$(\delta h^{ik})\,\bar{R}_{kj} = \left(-\frac{2\varphi}{a^2}\gamma^{ik} - \frac{2}{a^2} D^i D^k \gamma\right)\left(\frac{2C}{a^2}\right)\underbrace{h_{kj}}_{a^2\gamma_{kj}}$$

$$\boxed{(\delta h^{ik})\,\bar{R}_{kj} = -\frac{4C}{a^2}\varphi\,\delta^i_j - \frac{4C}{a^2} D^i D_j \gamma}$$

$$\delta R_{ij} = -D_i D_j \varphi - a^2 \Delta\varphi\,\gamma_{ij} + D_a D_j D^a D_i \gamma - D_j D_i \Delta \gamma$$
$$+ D_a D_i D^a D_j \gamma - \Delta D_i D_j \gamma$$



$\delta h^{ij} = -\frac{2\varphi}{a^2}\gamma^{ij} - \frac{2}{a^2} D^i D^j \gamma$
A, B, C, E
$(\delta h^{ik})\bar{R}_{kj} = \left(-\frac{2\varphi}{a^2}\gamma^{ik} - \frac{2}{a^2} D^i D^k \gamma\right)\left(\frac{2C}{a^2}\right)\underbrace{h_{kj}}_{a^2\gamma_{kj}}$
$= -\frac{4C}{a^2}\varphi\,\delta^i_j - \frac{4C}{a^2} D^i D_j \gamma$
$D_i D_j \varphi - a^2 \Delta\varphi\,\gamma_{ij} + D_a D_j D^a D_i \gamma - D_j D_i \Delta\gamma$
$+ D_a D_i D^a D_j \gamma - \Delta D_i D_j \gamma$
$D_j \gamma = x_j$

$$\delta h^{ij} = -\frac{2\varphi}{a^2}\gamma^{ij} - \frac{2}{a^2}D^iD^j\gamma$$

$$(\delta h^{ik})\bar{R}_{kj} = \left(-\frac{2\varphi}{a^2}\gamma^{ik} - \frac{2}{a^2}D^iD^k\gamma\right)\left(\frac{2C}{a^2}\right)\underbrace{h_{kj}}_{a^2\gamma_{kj}}$$

$$\boxed{(\delta h^{ik})\bar{R}_{kj} = -\frac{4C}{a^2}\varphi\,\delta^i_j - \frac{4C}{a^2}D^iD_j\gamma}$$

$$\delta R_{ij} = -D_iD_j\varphi - a^2\Delta\varphi\,\gamma_{ij} + D_aD_jD^aD_i\gamma - D_jD_i\Delta\gamma$$

$$+D_aD_iD^aD_j\gamma - \underset{\substack{\downarrow \\ D_aD^a}}{\Delta} D_iD_j\gamma$$

$$D_j\gamma = \chi_j$$

$$D_a(D_iD^a - D^aD_i)\chi_j$$

$$\delta h^{ij} = -\frac{2\varphi}{a^2}\gamma^{ij} - \frac{2}{a^2} D^i D^j \gamma$$

$$(\delta h^{ik})\bar{R}_{kj} = \left(-\frac{2\varphi}{a^2}\gamma^{ik} - \frac{2}{a^2} D^i D^k \gamma\right)\left(\frac{2C}{a^2}\right)\underbrace{h_{kj}}_{a^2\gamma_{kj}}$$

$$\boxed{(\delta h^{ik})\bar{R}_{kj} = -\frac{4C}{a^2} \cdots - \frac{4C}{a^2} D^i D_j \gamma}$$

$$\delta R_{ij} = -D_i D_j \varphi \cdots \varphi\gamma_{ij} + D_a D_j D^a D_i \gamma - D_j D_i \Delta\gamma$$

$$D_i D^a D_j \gamma - \underset{\substack{\downarrow \\ D_a D^a}}{\Delta} D_i D_j \gamma$$

$$D_j \gamma = \chi_j$$

$$D_a\left(D_i D^a - D^a D_i\right)\chi_j$$

$$\delta h^{ij} = -\frac{2\varphi}{a^2}\gamma^{ij} - \frac{2}{a^2}D^i D^j \gamma$$

$$(\delta h^{ik})\bar{R}_{kj} = \left(-\frac{2\varphi}{a^2}\gamma^{ik} - \frac{2}{a^2}D^i D^k \gamma\right)\left(\frac{2C}{a^2}\right)\underbrace{h_{kj}}_{a^2\gamma_{kj}}$$

$$\boxed{(\delta h^{ik})\bar{R}_{kj} = -\frac{4C}{a^2}\varphi\,\delta^i_j - \frac{4C}{a^2}D^i D_j \gamma}$$

$$\delta R_{ij} = -D_i D_j \varphi - a^2 \Delta\varphi\,\gamma_{ij} + \{D_a D_j D^a D_i \gamma - D_j D_i \Delta\gamma + D_a D_i D^a D_j \gamma - \underset{\downarrow\, D_a D^a}{\Delta} D_i D_j \gamma\}$$

$$= -D_i D_j \varphi - a^2\Delta\varphi\,\gamma_{ij} + \frac{4C}{a^2}D_i D_j \gamma$$

$$D_j \gamma = \chi_j$$

$$D_a(D_i D^a - D^a D_i)\chi_j$$

$$\delta h^{ij} = -\frac{2\varphi}{a^2}\gamma^{ij} - \frac{2}{a^2}D^iD^j\gamma$$

$$(\delta h^{ik})\bar{R}_{kj} = \left(-\frac{2\varphi}{a^2}\gamma^{ik} - \frac{2}{a^2}D^iD^k\gamma\right)\left(\frac{2C}{a^2}\right)\underbrace{h_{kj}}_{a^2\gamma_{kj}}$$

$$\boxed{(\delta h^{ik})\bar{R}_{kj} = -\frac{4C}{a^2}\varphi\,\delta^i_j - \frac{4C}{a^2}D^iD_j\gamma}$$

$$\delta R_{ij} = -D_iD_j\varphi - a^2\Delta\varphi\,\gamma_{ij} + \{D_aD_jD^aD_i\gamma - D_jD_i\Delta\gamma + D_aD_iD^aD_j\gamma - \underset{\downarrow\, D_aD^a}{\Delta}D_iD_j\gamma\}$$

$$= -D_iD_j\varphi - a^2\Delta\varphi\,\gamma_{ij} + \frac{4C}{a^2}D_iD_j\gamma$$

$$D_j\gamma = \chi_j$$

$$D_a(D_iD^a - D^aD_i)\chi_j$$

$$\delta R^i{}_j = \delta(h^{ik} R_{kj}) = (\delta h^{ik}) R_{kj} + h^{ik}\,\delta R_{kj}$$

$$\delta R_{ij} = \frac{1}{2} h^{ab}\Big[\underset{(i)}{\delta h_{bi|ja}} - \underset{(ii)}{\delta h_{ab|ij}} + \underset{(iii)}{\delta h_{bj|ia}} - \underset{(iv)}{\delta h_{ij|ab}}\Big]$$

$$(\delta R^i{}_j) = h^{ik}\,\delta R_{kj} + (\delta h^{ik}) R_{kj}$$

$$=$$

$$h^{ab}\,\delta h_{ab|ij} = -3 D_i D_j \varphi - D_j D_i \Delta \gamma \quad ; \ \Delta \equiv D^i D_i$$

$$h^{ab}\,\delta h_{bj|ia} = \gamma^{ab}\gamma_{bj} D_a D_i \varphi + D_a D_i D^a D_j \gamma$$

$$\frac{1}{2} g^{ab}\,\delta g_{ij|ab} = -a^2 \Delta \varphi \gamma_{ij} - \Delta D_i D_j \gamma$$

$$\delta R^i{}_j = \delta(h^{ik} R_{kj}) = (\delta h^{ik}) R_{kj} + h^{ik}\,\delta R_{kj}$$

$$\delta R_{ij} = \frac{1}{2} h^{ab}\Big[\underset{(i)}{\delta h_{bi|ja}} - \underset{(ii)}{\delta h_{ab|ij}} + \underset{(iii)}{\delta h_{bj|ia}} - \underset{(iv)}{\delta h_{ij|ab}}\Big]$$

$$(\delta R^i{}_j) = h^{ik}\,\delta R_{kj} + (\delta h^{ik}) R_{kj}$$

$$\ldots \varphi - \Delta\varphi\,\delta^i{}_j + \frac{4c}{a^2} D^i D_j \gamma$$

(ii) $-\frac{1}{2} h^{ab} \ldots = -3 D_i D_j \varphi - D_j D_i \Delta\gamma \quad ; \Delta \equiv D^i D_i$

(iii) $\ldots \gamma^{ab}\gamma_{bj} D_a D_i \varphi + D_a D_i D^a D_j \gamma$

(iv) $\ldots -a^2 \Delta\varphi \gamma_{ij} - \Delta D_i D_j \gamma$

$$\delta R^i{}_j = \delta(h^{ik} R_{kj}) = (\delta h^{ik}) R_{kj} + h^{ik}\delta R_{kj}$$

$$\delta R_{ij} = \frac{1}{2} h^{ab}\Big[\underbrace{\delta h_{bi|ja}}_{(i)} - \underbrace{\delta h_{ab|ij}}_{(ii)} + \underbrace{\delta h_{bj|ia}}_{(iii)} - \underbrace{\delta h_{ij|ab}}_{(iv)}\Big]$$

$$(\delta R^i{}_j) = h^{ik}\delta R_{kj} + (\delta h^{ik}) R_{kj}$$

$$= -D^i D_j \varphi - \Delta\varphi\, \delta^i_j + \frac{4c}{a^2} D^i D_j \gamma$$

(ii) $-\frac{1}{2} h^{ab}\, \delta h_{ab|ij} = -3 D_i D_j \varphi - D_j D_i \Delta\gamma \quad ; \Delta \equiv D^i D_i$

(iii) $\frac{1}{2} h^{ab}\, \delta h_{bj|ia} = \gamma^{ab}\gamma_{bj} D_a D_i \varphi + D_a D_i D^a D_j \gamma$

(iv) $-\frac{1}{2} g^{ab}\, \delta g_{ij|ab} = -a^2 \Delta\varphi\gamma_{ij} - \Delta D_i D_j \gamma$

$$\delta R^i{}_j = \delta(h^{ik} R_{kj}) = (\delta h^{ik}) R_{kj} + h^{ik}\delta R_{kj}$$

$$\delta R_{ij} = \frac{1}{2} h^{ab}\Big[\underbrace{\delta h_{bi|ja}}_{(i)} - \underbrace{\delta h_{ab|ij}}_{(ii)} + \underbrace{\delta h_{bj|ia}}_{(iii)} - \underbrace{\delta h_{ij|ab}}_{(iv)}\Big]$$

$$(\delta R^i{}_j) = h^{ik}\delta R_{kj} + (\delta h^{ik}) R_{kj}$$

$$= -D^i D_j \varphi - \Delta\varphi\,\delta^i_j + \cancel{\frac{4c}{a^2} D^i D_j \gamma}$$

$$-\frac{4c}{a^2}\varphi\,\delta^i_j - \cancel{\frac{4c}{a^2} D^i D_j \gamma}$$

$$\boxed{\delta R^i{}_j = -D^i D_j \varphi - \frac{4c}{a^2}\varphi\,\delta^i_j - \Delta\varphi\,\delta^i_j}$$

$$(iv)\quad -\frac{1}{2} g^{ab}\,\delta g_{ij|ab} = -a^2\Delta\varphi\,\gamma_{ij} - \Delta D_i D_j \gamma$$

$$\delta R^i{}_j = \delta(h^{ik} R_{kj}) = (\delta h^{ik}) R_{kj} + h^{ik}\delta R_{kj}$$

$$\delta R_{ij} = \frac{1}{2} h^{ab}\Big[\underset{(i)}{\delta h_{bi|ja}} - \underset{(ii)}{\delta h_{ab|ij}} + \underset{(iii)}{\delta h_{bj|ia}} - \underset{(iv)}{\delta h_{ij|ab}}\Big]$$

$$(\delta R^i{}_j) = h^{ik}\delta R_{kj} + (\delta h^{ik}) R_{kj}$$

$$= -D^i D_j \varphi - \Delta\varphi\, \delta^i_j + \frac{4c}{a^2}\cancel{D^i D_j \gamma}$$

$$\frac{4c}{a^2}\varphi\,\delta^i_j - \frac{4c}{a^2}\cancel{D^i D_j \gamma}$$

$$\boxed{-D^i D_j \varphi - \frac{4c}{a^2}\varphi\,\delta^i_j - \Delta\varphi\,\delta^i_j}$$

$$R^i{}_j + \delta R$$

$$\delta R^i{}_j = \delta(h^{ik} R_{kj}) = (\delta h^{ik}) R_{kj} + h^{ik}\delta R_{kj}$$

$$\delta R_{ij} = \frac{1}{2} h^{ab}\Big[\underset{(i)}{\delta h_{bi||ja}} - \underset{(ii)}{\delta h_{ab||ij}} + \underset{(iii)}{\delta h_{bj||ia}} - \underset{(iv)}{\delta h_{ij||ab}}\Big]$$

$$(\delta R^i{}_j) = h^{ik}\delta R_{kj} + (\delta h^{ik}) R_{kj}$$

$$= -D^i D_j \varphi - \Delta\varphi\, \delta^i_j + \cancel{\frac{4c}{a^2} D^i D_j \gamma}$$

$$-\frac{4c}{a^2}\varphi\,\delta^i_j - \cancel{\frac{4c}{a^2} D^i D_j \gamma}$$

$$\boxed{\delta R^i{}_j = -D^i D_j \varphi - \frac{4c}{a^2}\varphi\,\delta^i_j - \Delta\varphi\,\delta^i_j}$$

$$R^i{}_j = \bar{R}^i{}_j + \delta R^i_j = \frac{2c}{a^2} h^i{}_j$$

$$\delta h^{ij} = -\frac{2\varphi}{a^2}\gamma^{ij} - \frac{2}{a^2}D^iD^j\gamma$$

$$(\delta h^{ik})\bar{R}_{kj} = \left(-\frac{2\varphi}{a^2}\gamma^{ik} - \frac{2}{a^2}D^iD^k\gamma\right)\left(\frac{2C}{a^2}\right)\underbrace{h_{kj}}_{a^2\gamma_{kj}}$$

$$\boxed{(\delta h^{ik})\bar{R}_{kj} = -\frac{4C}{a^2}\varphi\,\delta^i_j - \frac{4C}{a^2}D^iD_j\gamma}$$

$$\delta R_{ij} = -D_iD_j\varphi - a^2\Delta\varphi\,\gamma_{ij} + \left\{D_aD_jD^aD_i\gamma - D_jD_i\Delta\gamma + D_aD_iD^aD_j\gamma - \Delta D_iD_j\gamma\right\}$$

$$\Delta \to D_aD^a$$

$$= -D_iD_j\varphi - a^2\Delta\varphi\,\gamma_{ij} + \frac{4C}{a^2}D_iD_j\gamma$$

$$D_j\gamma = \chi_j$$

$$D_a(D_iD^a - D^aD_i)\chi_j$$

$$h_{ij} = a^2\gamma_{ij} + 2a^2\varphi\gamma_{ij} + 2a^2D_iD_j\gamma$$

$$\delta(R^i{}_j) = h^{ik}\,\delta R_{kj} + (\delta h^{ik})\,R_{kj}$$

$$= -D^i D_j \varphi - \Delta\varphi\,\delta^i_j + \cancel{\frac{4c}{a^2} D^i D_j \gamma}$$

$$-\frac{4c}{a^2}\varphi\,\delta^i_j - \cancel{\frac{4c}{a^2} D^i D_j \gamma}$$

$$\boxed{\delta R^i{}_j = -D^i D_j \varphi - \frac{4c}{a^2}\varphi\,\delta^i_j - \Delta\varphi\,\delta^i_j}$$

$$R^i{}_j = \bar{R}^i{}_j + \delta R^i{}_j = \frac{2c}{a^2}\delta^i_j - D^i D_j \varphi - \frac{4c}{a^2}\varphi\,\delta^i_j -$$