

Title: The compositional structure of multipartite quantum entanglement

Date: Mar 29, 2010 04:00 PM

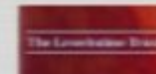
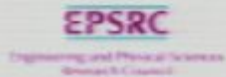
URL: <http://pirsa.org/10030114>

Abstract: Multipartite quantum states constitute a (if not the) key resource for quantum computations and protocols. However obtaining a generic, structural understanding of entanglement in N-qubit systems is still largely an open problem. Here we show that multipartite quantum entanglement admits a compositional structure. The two SLOCC-classes of genuinely entangled 3-qubit states, the GHZ-class and the W-class, exactly correspond with the two kinds of commutative Frobenius algebras on C^2 , namely 'special' ones and 'anti-special' ones. Within the graphical language of symmetric monoidal categories, the distinction between 'special' and 'anti-special' is purely topological, in terms of 'connected' vs. 'disconnected'. These GHZ and W Frobenius algebras form the primitives of a graphical calculus which is expressive enough to generate and reason about representatives of arbitrary N-qubit states.

This calculus induces a generalised graph state paradigm for measurement-based quantum computing, and refines the graphical calculus of complementary observables due to Duncan and one of the authors [ICALP'08], which has already shown itself to have many applications and admit automation.

References: Bob Coecke and Aleks Kissinger, <http://arxiv.org/abs/1002.2540>

The Compositional Structure of Multipartite Quantum Entanglement
Bob Coecke and Aleks Kissinger
Oxford University Computing Laboratory



SLOCC

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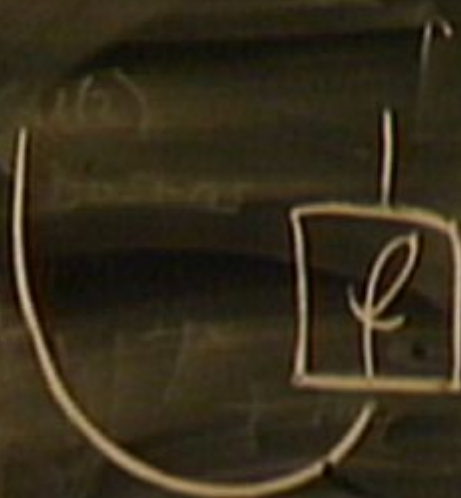
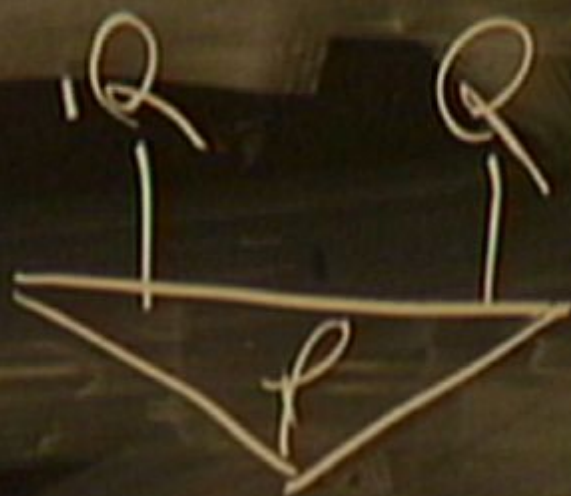


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Proof: Significantly less trivial.

GHZ-SLOCC-class representative:

$$GHZ = |000\rangle + |111\rangle$$

Many applications in quantum computing e.g. fault-tolerance; canonical witness of quantum non-locality.

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 - GHZ/W-duality more fundamental than complementarity?

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1) Symmetry

$$\text{Diagram 1} = \text{Diagram 2}$$

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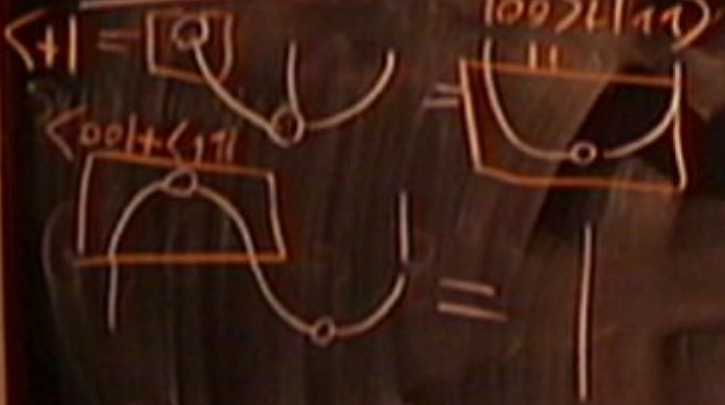
2) Reduce



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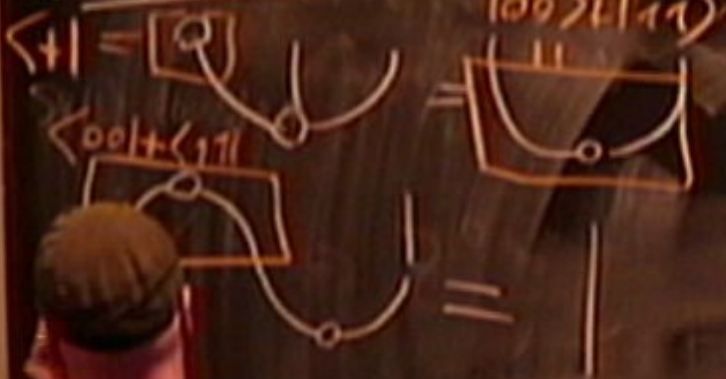
2) Reduce



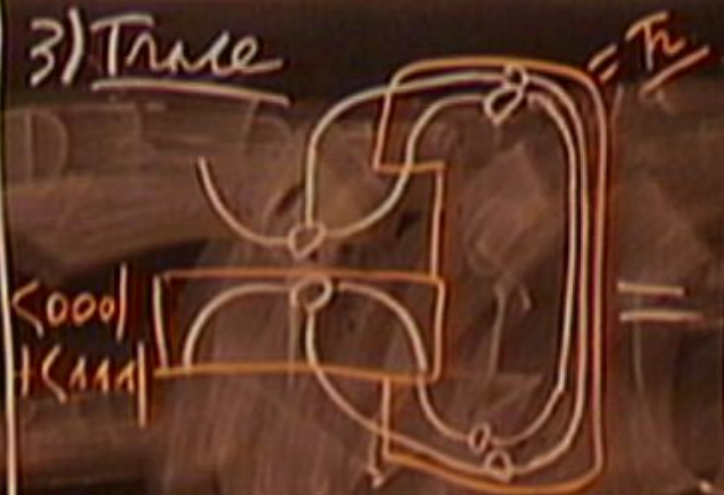
1) Symmetry



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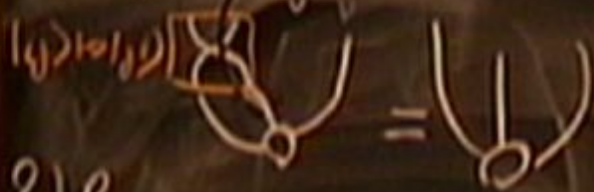
3) Trace



4) Flip



1) Symmetry



2) Reduce



COMPOSE



3) Trace



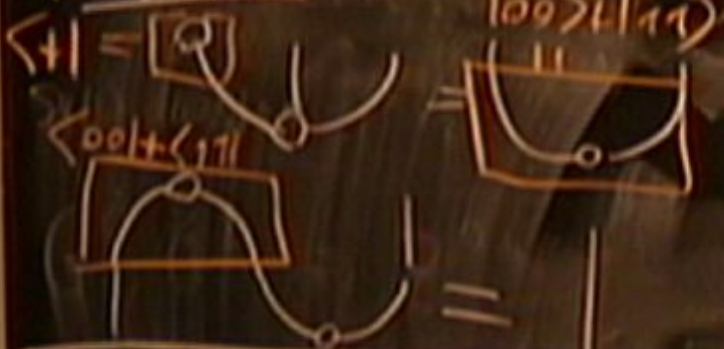
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1) the same

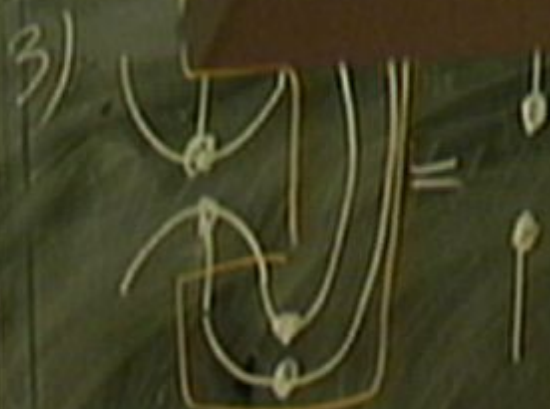


h) same

1) the same

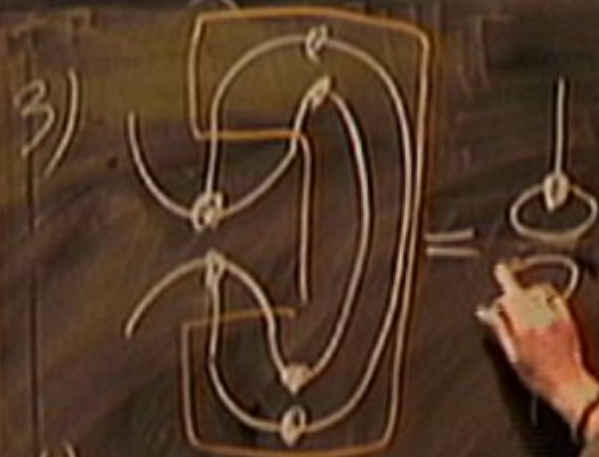
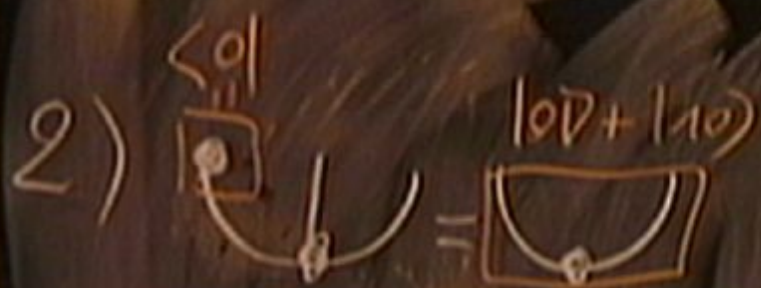
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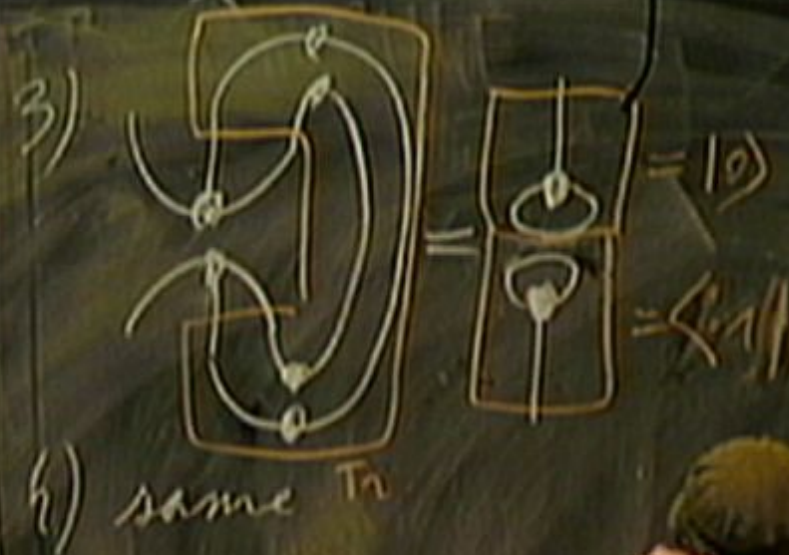
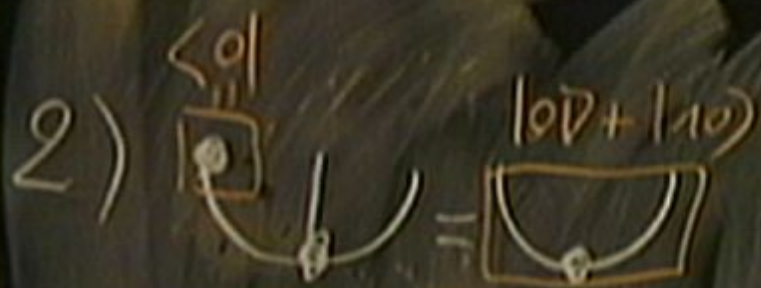
4) same T_2

1) the same

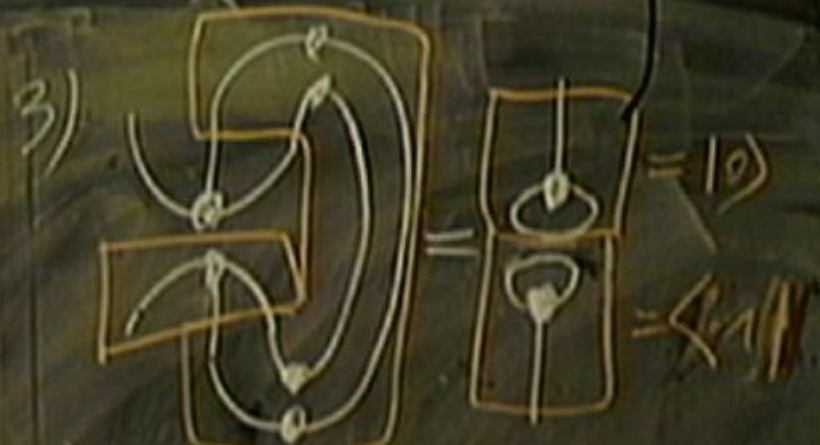
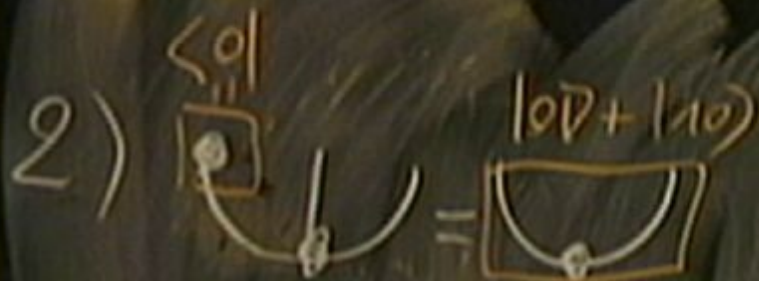


4) same T_2

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$$|10\rangle + |01\rangle + |00\rangle + |11\rangle$$

— *commutative Frobenius algebras* —

A **commutative monoid** is a set A with a binary map

$$\mu(-, -) : A \times A \rightarrow A$$

which is commutative, associative and unital i.e

$$\mu(\mu(a, b), c) = \mu(a, \mu(b, c)) \quad \mu(a, b) = \mu(b, a) \quad \mu(a, 1) = a$$

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$$\psi : \vdash \rightarrow 1$$

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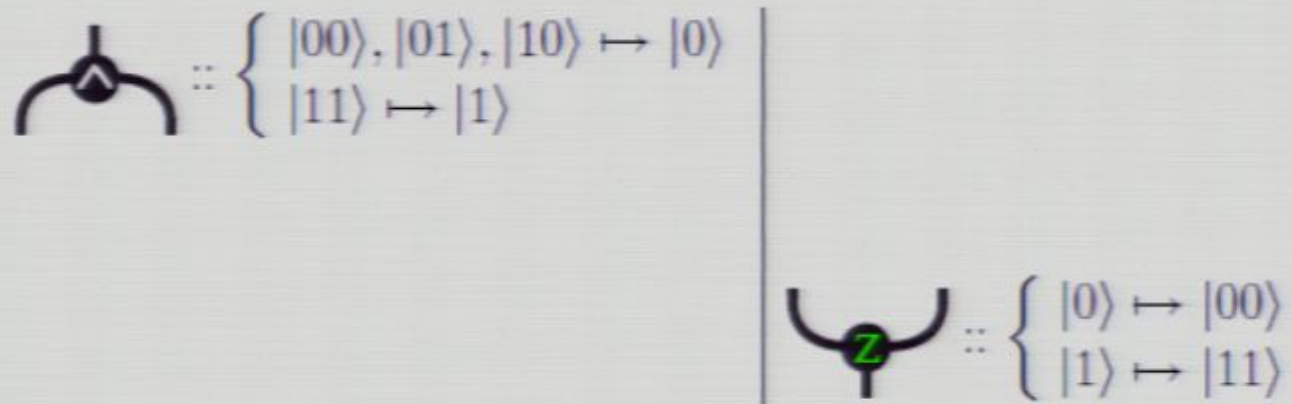
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FSet:



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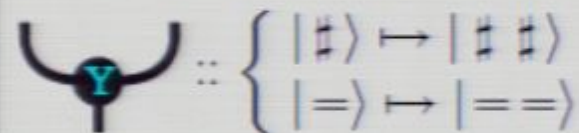
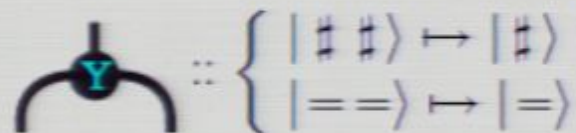
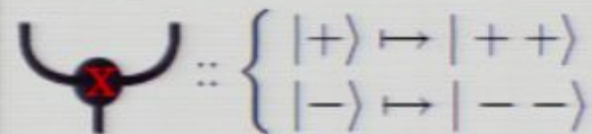
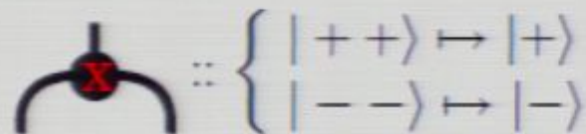
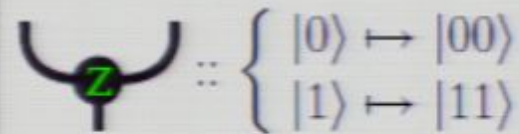
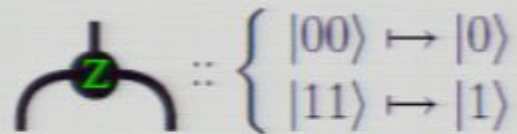
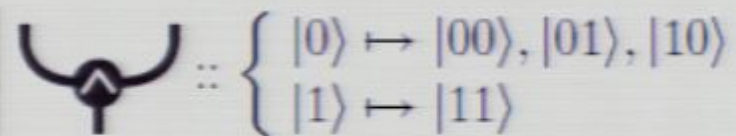
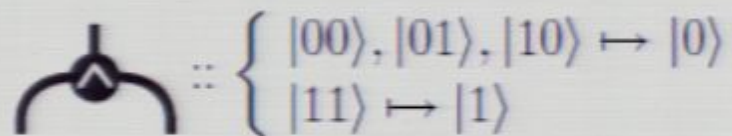
FSet:

$$\begin{array}{c} \text{---} \circ \text{---} \\ \text{---} \end{array} \vdots \begin{cases} |00\rangle, |01\rangle, |10\rangle \mapsto |0\rangle \\ |11\rangle \mapsto |1\rangle \end{cases}$$

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— *commutative Frobenius algebras* —

FdHilb:



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Thm. In FHilb **special CFAs**, i.e



Frobenius

special

exactly correspond with (arbitrary i.e. non-ONB) **bases** on the underlying Hilbert space via the correspondence:

$$\{|i\rangle\}_i \longleftrightarrow |i\rangle \mapsto |ii\rangle$$

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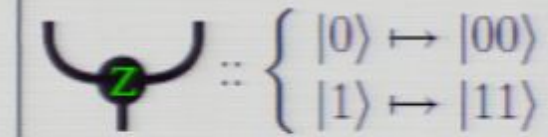
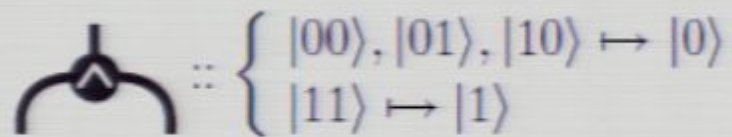
$$\text{comultiplication} : A \rightarrow A \otimes A \quad \text{unit} : A \rightarrow I$$

s.t.

$$\text{comultiplication} \circ \text{comultiplication} = \text{comultiplication} \circ \text{comultiplication} \quad \text{comultiplication} \circ \text{comultiplication} = \text{comultiplication} \circ \text{comultiplication} \quad \text{unit} = \text{comultiplication} \circ \text{comultiplication}$$

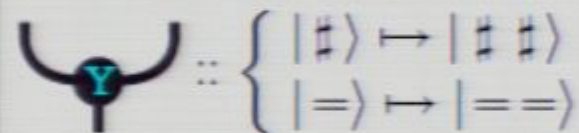
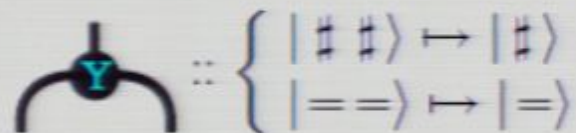
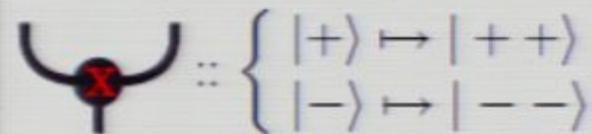
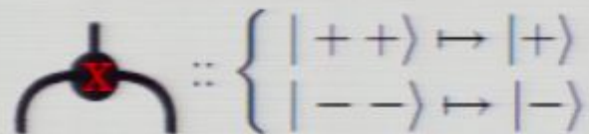
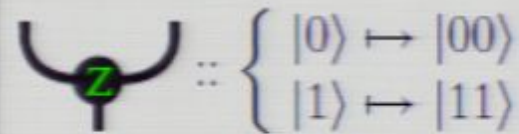
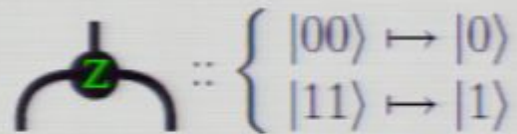
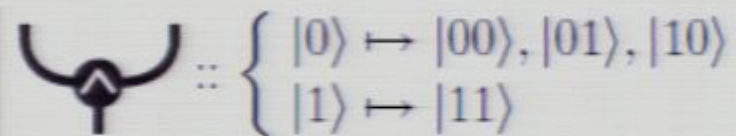
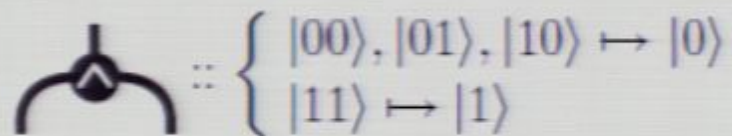
— commutative Frobenius algebras —

FSet:



— *commutative Frobenius algebras* —

FdHilb:



— *commutative Frobenius algebras* —

Thm. In FHilb **special CFAs**, i.e



Frobenius

special

exactly correspond with (arbitrary i.e. non-ONB) **bases** on the underlying Hilbert space via the correspondence:

$$\{|i\rangle\}_i \longleftrightarrow |i\rangle \mapsto |ii\rangle$$

$V(E) = \{e_0, e_1\}$

$(e_0 e_0 e_0) + (e_1 e_1 e_1) \vdash_A A : A \rightarrow A$

$e : \{*\} \rightarrow A$

$\vdash \vdash \rightarrow 1$

**GHZ AND W:
ALGEBRAIC SIMILARITY
and
TOPOLOGICAL DIFFERENCE**

— *commutative Frobenius algebras* —

Thm. In FHilb **special CFAs**, i.e



Frobenius

special

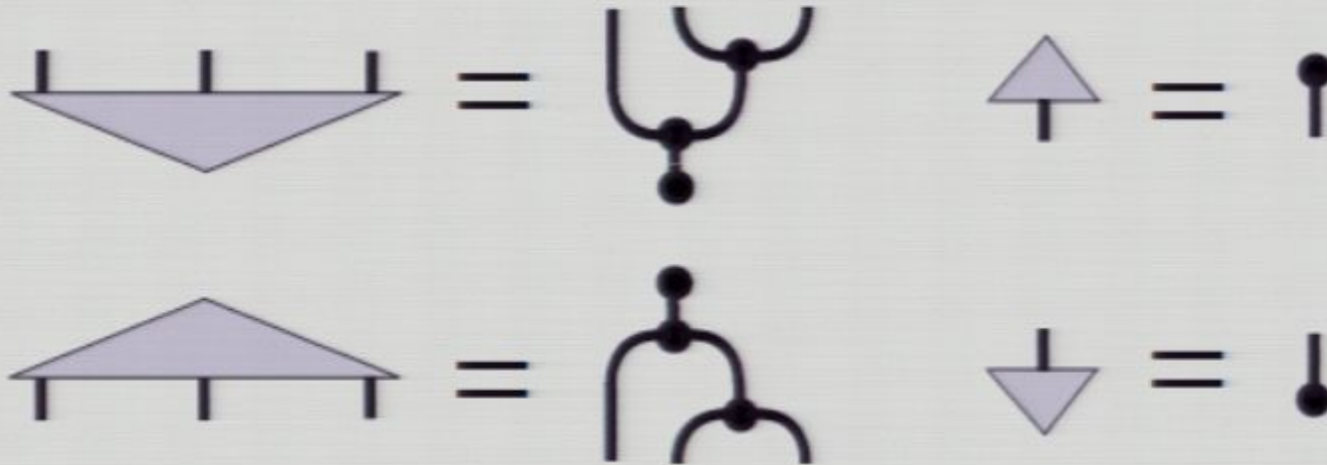
exactly correspond with (arbitrary i.e. non-ONB) **bases**
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$$\{|i\rangle\}_i \longleftrightarrow |i\rangle \mapsto |ii\rangle$$

**GHZ AND W:
ALGEBRAIC SIMILARITY
and
TOPOLOGICAL DIFFERENCE**

— $C(F)As \longleftrightarrow \text{tripartite states}$ —

From (co)monoid and (co)unit we build tripartite (co)state:



— $C(F)As \longleftrightarrow \text{tripartite states}$ —

From tripartite state and unit we build:



and via transposition we obtain comonoid.

Proposition. A **special CFA** on \mathbb{C}^2 , i.e.

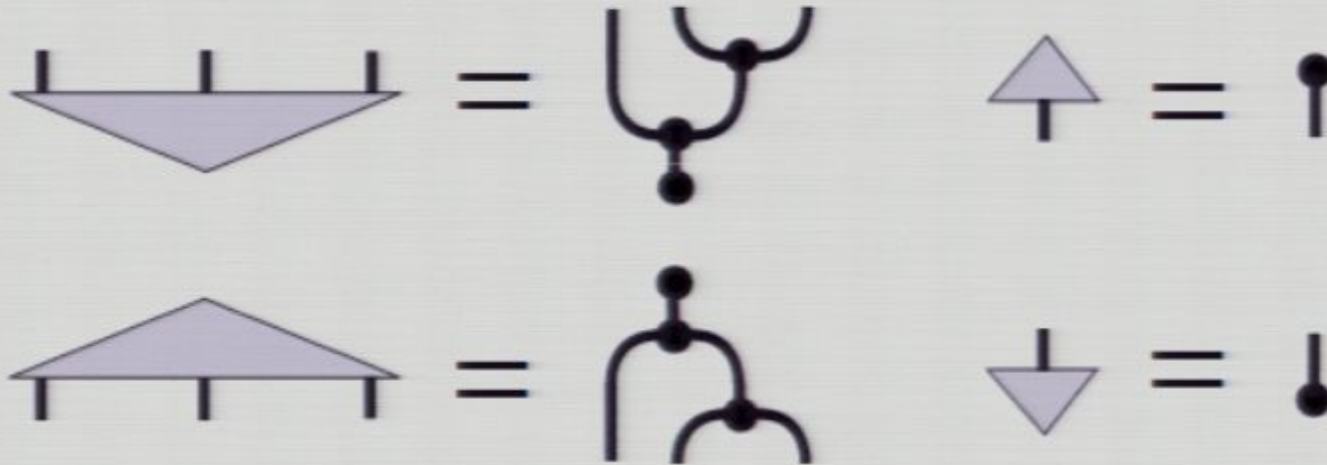
$$\begin{array}{c} \text{Diagram 1} \end{array} = \begin{array}{c} \text{Diagram 2} \end{array} \quad \boxed{\begin{array}{c} \text{Diagram 3} \end{array} = \begin{array}{c} \text{Diagram 4} \end{array}}$$

The image shows two equations of string diagrams. The first equation shows a diagram with two vertical lines connected by a horizontal line with a dot, equal to a diagram with two vertical lines connected by a horizontal line with a dot. The second equation, enclosed in a pink box, shows a diagram with a circle and a vertical line, equal to a diagram with a vertical line.

induces a symmetric **GHZ-class state**, and vice versa.

— $C(F)As \longleftrightarrow \text{tripartite states}$ —

From (co)monoid and (co)unit we build tripartite (co)state:



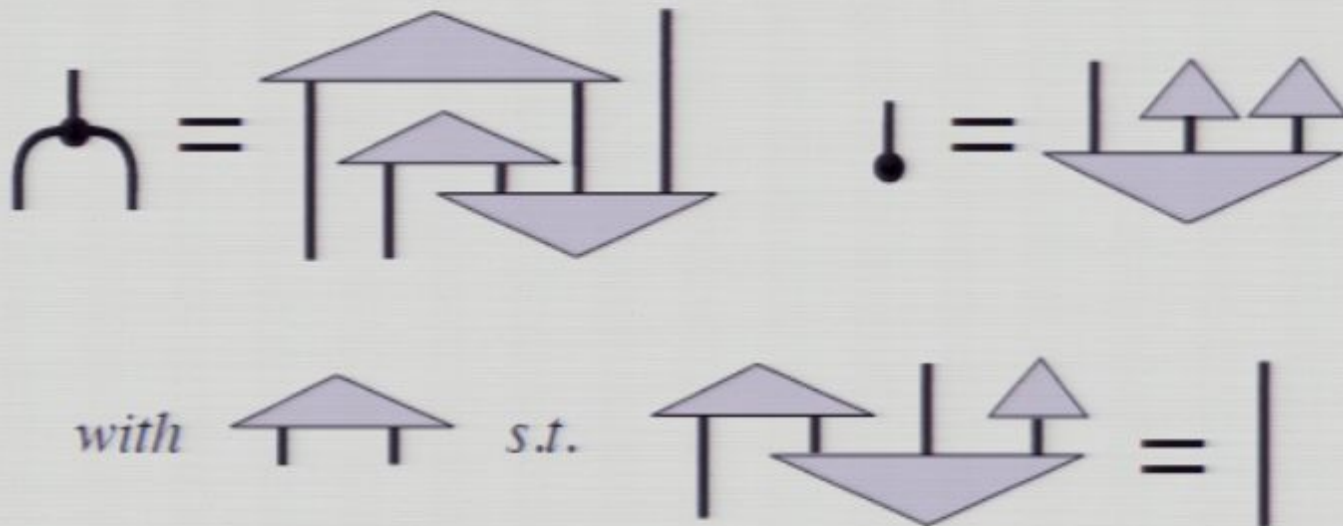
Proposition. A **special CFA** on \mathbb{C}^2 , i.e.

$$\begin{array}{c} \text{Diagram 1: A line with a dot at the top, a loop, and a dot at the bottom.} \end{array} = \begin{array}{c} \text{Diagram 2: Two lines meeting at a dot in the middle.} \end{array} \quad \boxed{\begin{array}{c} \text{Diagram 3: A circle with a dot at the top and a dot at the bottom.} \end{array} = \begin{array}{c} \text{Diagram 4: A vertical line.} \end{array}}$$

induces a symmetric **GHZ-class state**, and vice versa.

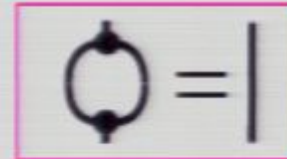
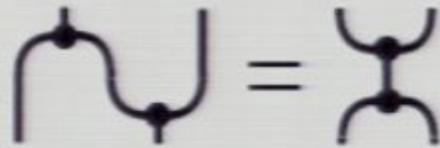
— $C(F)As \longleftrightarrow \text{tripartite states}$ —

From tripartite state and unit we build:



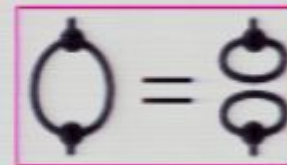
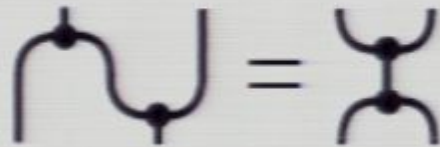
and via transposition we obtain comonoid.

Proposition. A **special CFA** on \mathbb{C}^2 , i.e.



induces a symmetric **GHZ-class state**, and vice versa.

Proposition. An **anti-special CFA** on \mathbb{C}^2 , i.e.



induces a symmetric **W-class state**, and vice versa.

— *CFA axioms* —

$$\begin{array}{c} \text{U} \text{ with a dot on the top line} \\ \text{U} \text{ with a dot on the bottom line} \end{array} = \begin{array}{c} \text{U} \text{ with a dot on the top line} \\ \text{U} \text{ with a dot on the bottom line} \end{array} \quad \begin{array}{c} \text{U} \\ \text{U} \text{ with a dot on the bottom line} \end{array} = \begin{array}{c} \text{U} \\ \text{U} \text{ with a dot on the bottom line} \end{array}$$

$$\begin{array}{c} | \\ | \end{array} = \begin{array}{c} \text{U} \text{ with a dot on the top line} \\ \text{U} \text{ with a dot on the bottom line} \end{array} \quad \begin{array}{c} \text{U} \text{ with a dot on the top line} \\ \text{U} \text{ with a dot on the bottom line} \end{array} = \begin{array}{c} \text{U} \text{ with a dot on the top line} \\ \text{U} \text{ with a dot on the bottom line} \end{array}$$

— *CFA axioms* —

$$\begin{array}{c} \text{U} \text{ with two dots} = \text{U with one dot} \end{array} \quad \begin{array}{c} \text{U} = \text{loop with one dot} \end{array}$$

$$\begin{array}{c} | = \text{U with one dot} \end{array} \quad \begin{array}{c} \text{loop with two dots} = \text{X with two dots} \end{array}$$

$$\begin{array}{c} \text{X with two dots and two triangles} \end{array} = \begin{array}{c} \text{X with two dots} \end{array} = |$$

GRAPHICAL REASONING WITH SCFAs and ACFAs

— *CFA axioms* —

$$\begin{array}{c} \text{U} \text{ with a dot on the top line} \end{array} = \begin{array}{c} \text{U} \text{ with a dot on the bottom line} \end{array} \quad \begin{array}{c} \text{U} \end{array} = \begin{array}{c} \text{U with a crossing} \end{array}$$

$$\begin{array}{c} | \end{array} = \begin{array}{c} \text{U with a dot on the top line} \end{array} \quad \begin{array}{c} \text{U with a dot on the top line and a crossing} \end{array} = \begin{array}{c} \text{U with a dot on the bottom line and a crossing} \end{array}$$

Proposition. A **special CFA** on \mathbb{C}^2 , i.e.

$$\text{Diagram 1} = \text{Diagram 2} \quad \boxed{\text{Diagram 3} = \text{Diagram 4}}$$

induces a symmetric **GHZ-class state**, and vice versa.

Proposition. An **anti-special CFA** on \mathbb{C}^2 , i.e.

$$\text{Diagram 1} = \text{Diagram 2} \quad \boxed{\text{Diagram 3} = \text{Diagram 4}}$$

induces a symmetric **W-class state**, and vice versa.

Proposition. Every CFA on \mathbb{C}^2 is either special or anti-special; every monoid on \mathbb{C}^2 extends to an CFA.

— *CFA axioms* —

$$\begin{array}{c} \text{U} \\ \text{U} \end{array} = \begin{array}{c} \text{U} \\ \text{U} \end{array} \quad \begin{array}{c} \text{U} \\ \text{U} \end{array} = \begin{array}{c} \text{U} \\ \text{U} \end{array}$$

$$\text{I} = \begin{array}{c} \text{U} \\ \text{U} \end{array} \quad \begin{array}{c} \text{U} \\ \text{U} \end{array} = \begin{array}{c} \text{U} \\ \text{U} \end{array}$$

— *CFA axioms* —

$$\begin{array}{c} \text{U} \text{ with two dots} = \text{U with one dot} \end{array} \quad \begin{array}{c} \text{U} = \text{U with one dot} \end{array}$$

$$\begin{array}{c} | = \text{U with one dot} \end{array} \quad \begin{array}{c} \text{U with two dots} = \text{U with one dot} \end{array}$$

$$\begin{array}{c} \text{U with two dots} \end{array} = \begin{array}{c} \text{U with one dot} \end{array} = |$$

GRAPHICAL REASONING WITH SCFAs and ACFAs

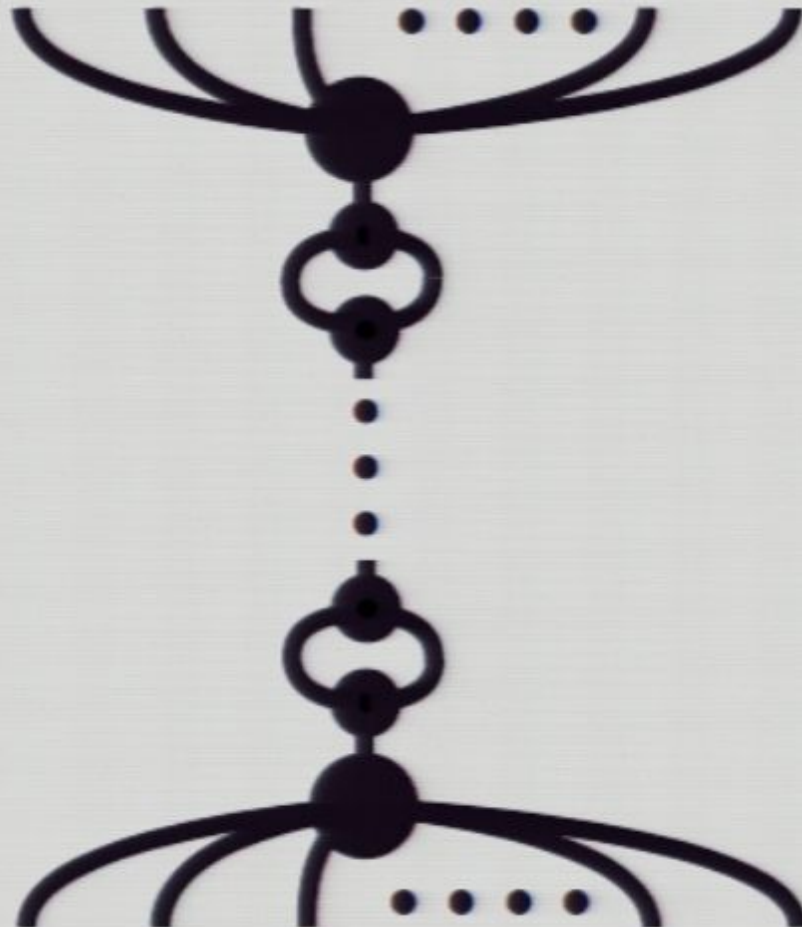
— *CFA axioms* —

$$\begin{array}{c} \text{U} \end{array} = \begin{array}{c} \text{U} \end{array} \quad \begin{array}{c} \text{U} \end{array} = \begin{array}{c} \text{X} \end{array}$$

$$\begin{array}{c} | \end{array} = \begin{array}{c} \text{U} \end{array} \quad \begin{array}{c} \text{U} \end{array} = \begin{array}{c} \text{X} \end{array}$$

$$\begin{array}{c} \text{O} \end{array} = \begin{array}{c} | \end{array} \quad \text{or} \quad \begin{array}{c} \text{O} \end{array} = \begin{array}{c} \text{O} \end{array}$$

— *CFA normal form* —



— *CFA axioms* —

$$\begin{array}{c} \text{U} \end{array} = \begin{array}{c} \text{U} \end{array} \quad \begin{array}{c} \text{U} \end{array} = \begin{array}{c} \text{X} \end{array}$$

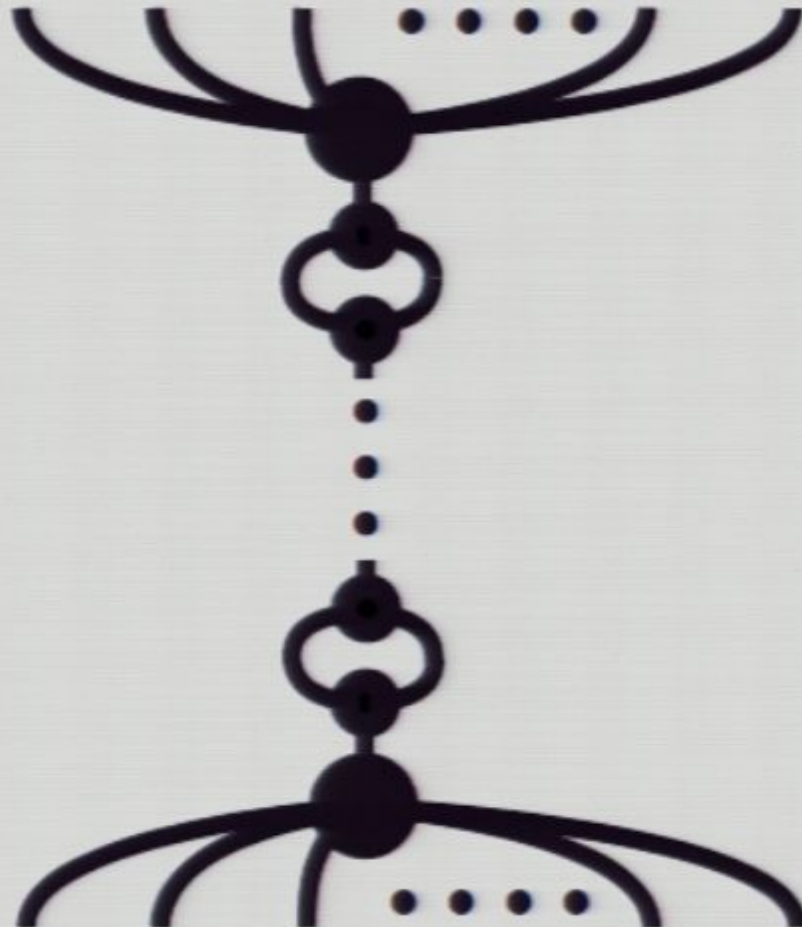
$$\begin{array}{c} | \end{array} = \begin{array}{c} \text{U} \end{array} \quad \begin{array}{c} \text{U} \end{array} = \begin{array}{c} \text{X} \end{array}$$

$\begin{array}{c} \text{O} \end{array} = \begin{array}{c} | \end{array}$

or

$\begin{array}{c} \text{O} \end{array} = \begin{array}{c} \text{O} \end{array}$

— *CFA normal form* —



— *CFA axioms* —

$$\begin{array}{c} \text{U} \end{array} = \begin{array}{c} \text{U} \end{array} \quad \begin{array}{c} \text{U} \end{array} = \begin{array}{c} \text{X} \end{array}$$

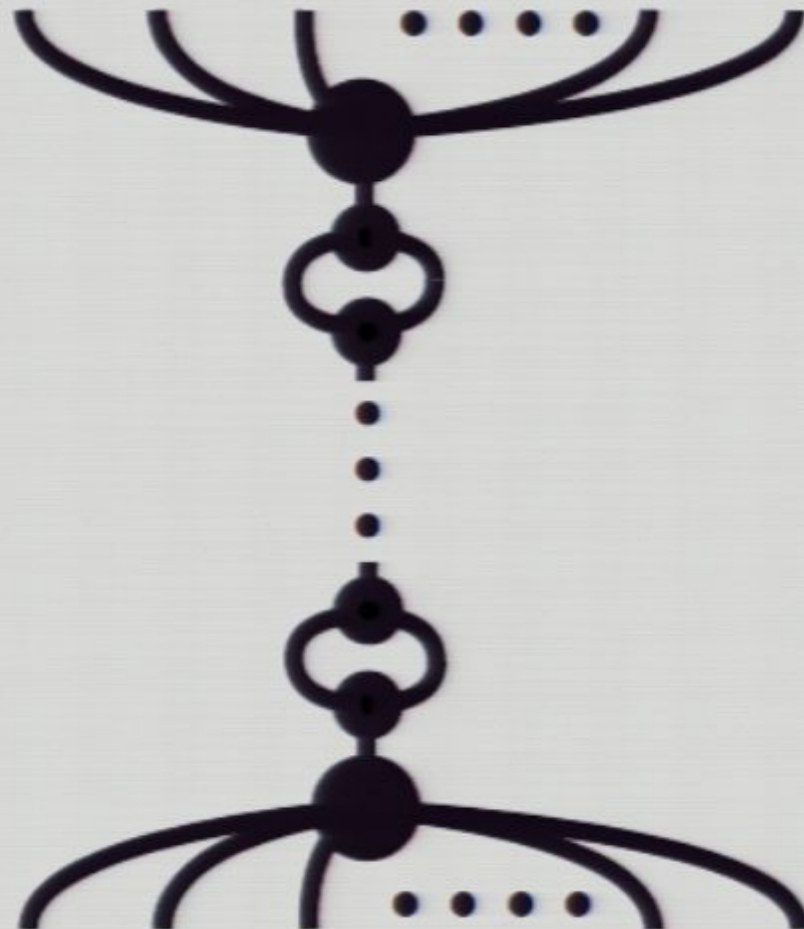
$$\begin{array}{c} | \end{array} = \begin{array}{c} \text{U} \end{array} \quad \begin{array}{c} \text{U} \end{array} = \begin{array}{c} \text{X} \end{array}$$

$\begin{array}{c} \text{O} \end{array} = \begin{array}{c} | \end{array}$

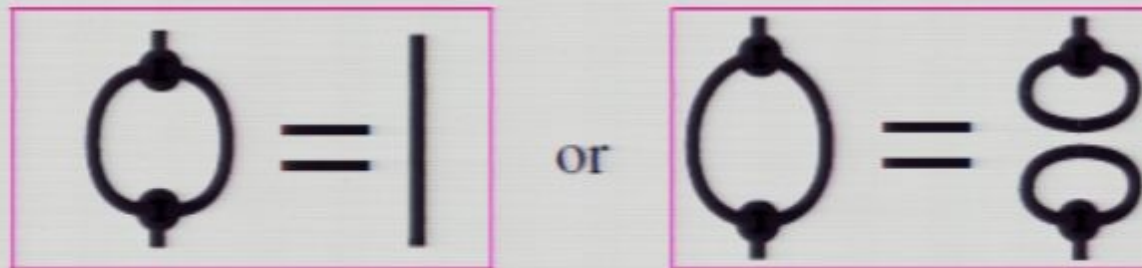
or

$\begin{array}{c} \text{O} \end{array} = \begin{array}{c} \text{O} \end{array}$

— *CFA normal form* —

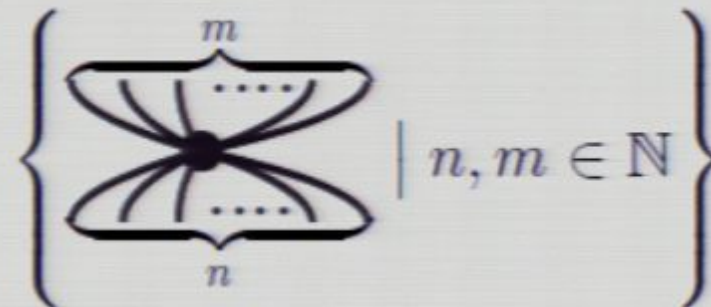


— *CFA axioms* —

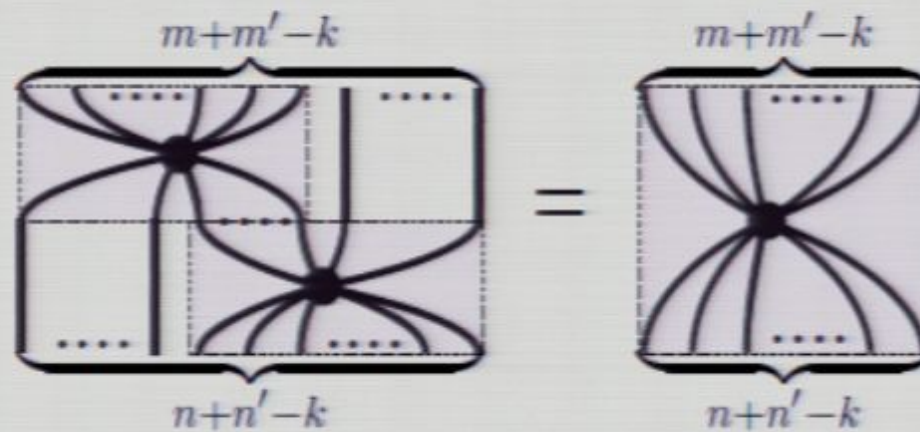


— GHZ-spiders —

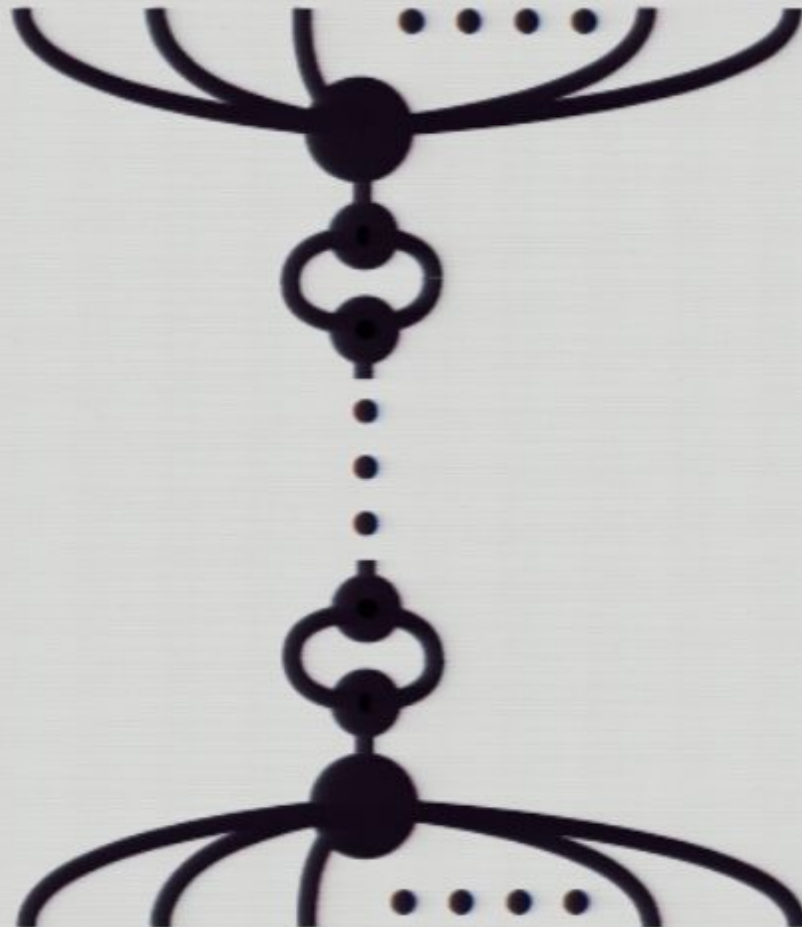
Data:



Rules:



— *CFA normal form* —



— *CFA axioms* —

$$\begin{array}{c} \text{U} \end{array} = \begin{array}{c} \text{U} \end{array} \quad \begin{array}{c} \text{U} \end{array} = \begin{array}{c} \text{X} \end{array}$$

$$\begin{array}{c} | \end{array} = \begin{array}{c} \text{U} \end{array} \quad \begin{array}{c} \text{U} \end{array} = \begin{array}{c} \text{X} \end{array}$$

$$\begin{array}{c} \text{O} \end{array} = \begin{array}{c} | \end{array} \quad \text{or} \quad \begin{array}{c} \text{O} \end{array} = \begin{array}{c} \text{O} \end{array}$$

— *W-spiders* —

Data:

$$\left\{ \begin{array}{c} m \\ \text{Diagram of a spider with } m \text{ top legs and } n \text{ bottom legs} \\ n \end{array} , \text{Diagram of a loop with a dot} , \text{Diagram of a loop with a dot and a tail} \mid n, m \in \mathbb{N} \right\}$$

Rules:

$$\begin{array}{c} m+m'-1 \\ \text{Diagram of a spider with } m+m'-1 \text{ top legs and } n+n'-1 \text{ bottom legs, split into two parts} \\ n+n'-1 \end{array} = \begin{array}{c} m+m'-1 \\ \text{Diagram of a spider with } m+m'-1 \text{ top legs and } n+n'-1 \text{ bottom legs} \\ n+n'-1 \end{array}$$

— *W-spiders* —

Data:

$$\left\{ \begin{array}{c} m \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ n \end{array} \right\}, \text{---}, \text{---} \mid n, m \in \mathbb{N}$$

Rules:

$$\begin{array}{c} m+m'-2 \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ n+n'-2 \end{array} = \begin{array}{c} m+m'-2 \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ n+n'-2 \end{array}$$

— *CFA axioms* —

$$\begin{array}{c} \text{U} \end{array} = \begin{array}{c} \text{U} \end{array} \quad \begin{array}{c} \text{U} \end{array} = \begin{array}{c} \text{X} \end{array}$$

$$\begin{array}{c} | \end{array} = \begin{array}{c} \text{U} \end{array} \quad \begin{array}{c} \text{U} \end{array} = \begin{array}{c} \text{X} \end{array}$$

$\begin{array}{c} \text{O} \end{array} = \begin{array}{c} | \end{array}$

or

$\begin{array}{c} \text{O} \end{array} = \begin{array}{c} \text{O} \end{array}$

— GHZ-spiders —

Data:

$$\left\{ \begin{array}{c} m \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ n \end{array} \right\} \mid n, m \in \mathbb{N}$$

Rules:

$$\begin{array}{c} m+m'-k \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ n+n'-k \end{array} = \begin{array}{c} m+m'-k \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ n+n'-k \end{array}$$

— *W-spiders* —

Data:

$$\left\{ \begin{array}{c} m \\ \text{Diagram of a spider with } m \text{ top legs and } n \text{ bottom legs} \\ n \end{array} , \text{Diagram of a loop} , \text{Diagram of a loop with a dot} \mid n, m \in \mathbb{N} \right\}$$

Rules:

$$\begin{array}{c} m+m'-1 \\ \text{Diagram of two spiders side-by-side} \\ n+n'-1 \end{array} = \begin{array}{c} m+m'-1 \\ \text{Diagram of a single spider with combined legs} \\ n+n'-1 \end{array}$$

— *W-spiders* —

Data:

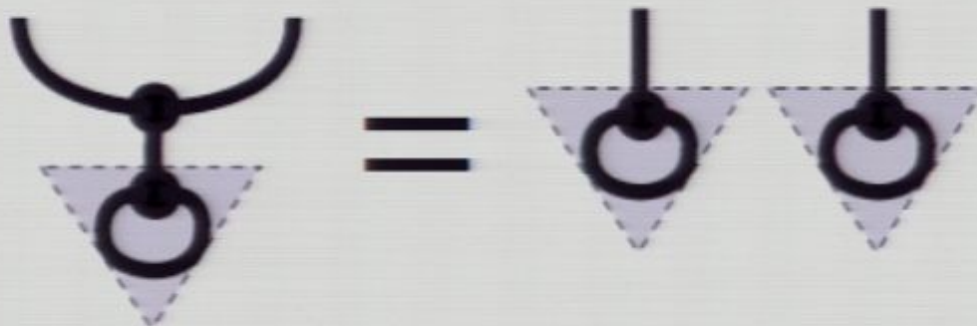
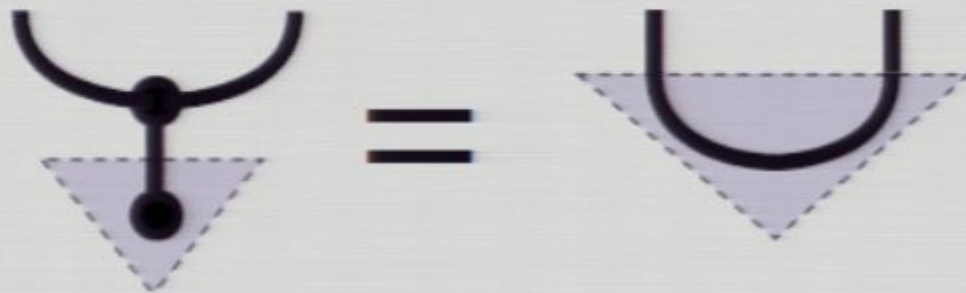
$$\left\{ \begin{array}{c} m \\ \text{[Diagram of a spider with } m \text{ top legs and } n \text{ bottom legs]} \\ n \end{array} , \text{ [Diagram of a circle with a dot]} , \text{ [Diagram of a circle with a dot and a loop]} \mid n, m \in \mathbb{N} \right\}$$

Rules:

$$\begin{array}{c} m+m'-2 \\ \text{[Diagram of a large spider with } m+m'-2 \text{ top legs and } n+n'-2 \text{ bottom legs]} \\ n+n'-2 \end{array} = \begin{array}{c} m+m'-2 \\ \text{[Diagram of a row of circles with dots]} \\ n+n'-2 \end{array}$$

— *W-spider example* —

Examples:



COMPOSING W AND GHZ

— *composition of structures* —

Examples:

- GHZ -states \Rightarrow multi-qubit GHZ
- W -states \Rightarrow multi-qubit W

COMPOSING W AND GHZ

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Examples:

- GHZ -states \Rightarrow multi-qubit GHZ
- W -states \Rightarrow multi-qubit W

— *composition of different structures* —

Examples:

- $+ \ \& \ \times \Rightarrow$ general polynomials.
 - Interact via distributive law
 - Interconvert via exponential

— *composition of different structures* —

Examples:

- $+$ & $\times \Rightarrow$ general polynomials.
 - Interact via distributive law
 - Interconvert via exponential
- $\{|0\rangle, |1\rangle\}$ - & $\{|+\rangle, |-\rangle\}$ -bases \Rightarrow graph states.
 - Interact via bialgebra-like laws
 - Interconvert via Hadamard gate

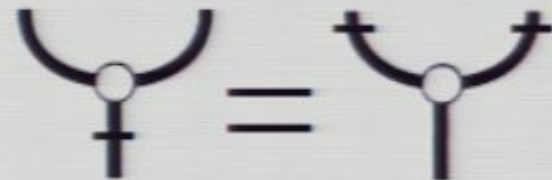
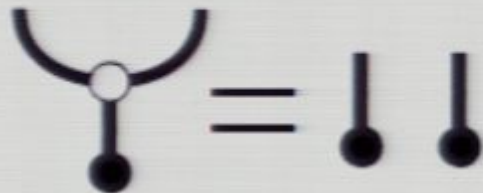
— *composition of different structures* —

Examples:

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 - Interact via Hopf-like law
 - Interconvert via Hadamard gate
- GHZ - & W -states \Rightarrow ???
 - Interact via ???
 - Interconvert via ???

— *composition of W- and GHZ-CFAs* —

Interaction rules:



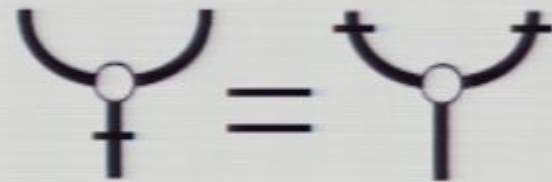
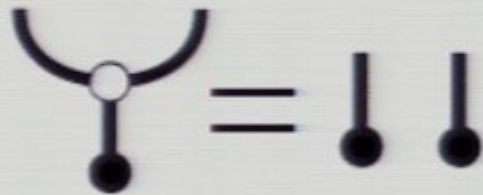
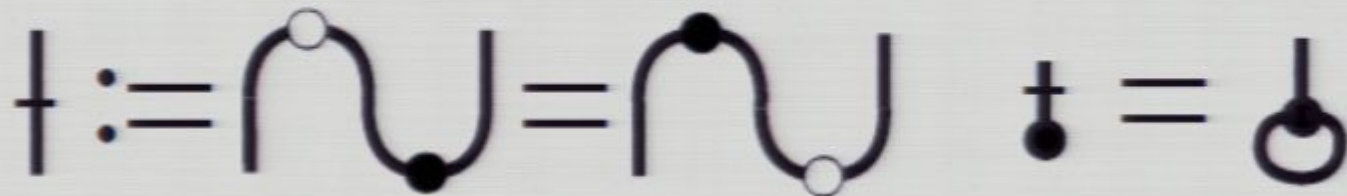
— *composition of different structures* —

Examples:

- $+ \ \& \ \times \Rightarrow$ general polynomials.
 - Interact via distributive law
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- GHZ - & W -states \Rightarrow ???
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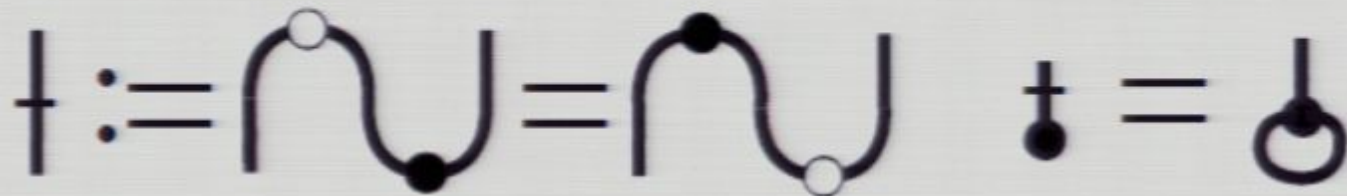
— *composition of W- and GHZ-CFAs* —

Interaction rules:



— *composition of W- and GHZ-CFAs* —

Interaction rules:



Correspondence:

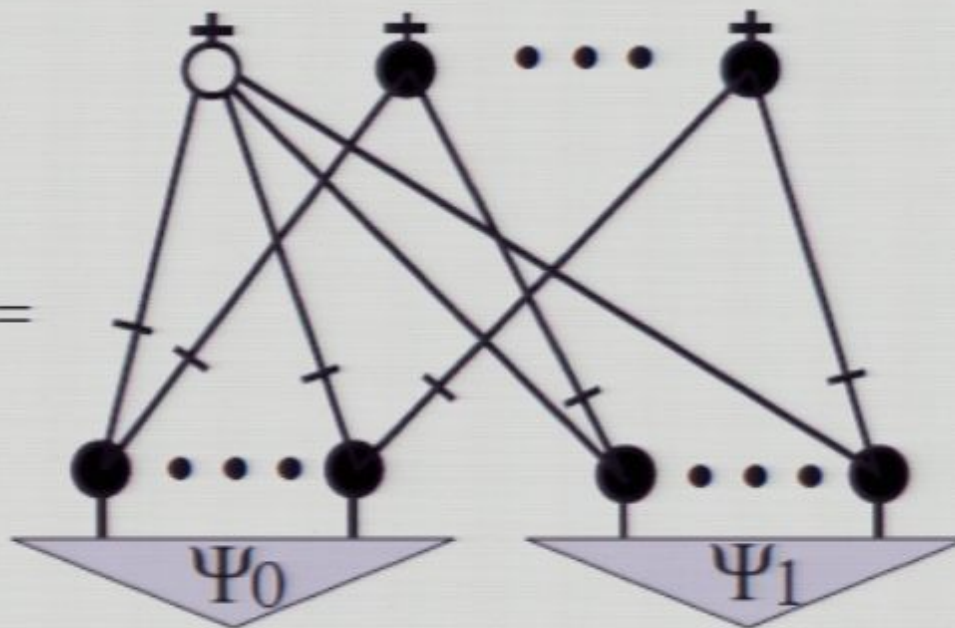
- **Structural points** of W are **copyable points** of GHZ

— *composition of W- and GHZ-CFAs* —

Generating power:

Emulating SLOCC-superclass generation:

$$|0\Psi_0\rangle + |1\Psi_1\rangle =$$



— *composition of W- and GHZ-CFAs* —

Interaction rules:



Correspondence:

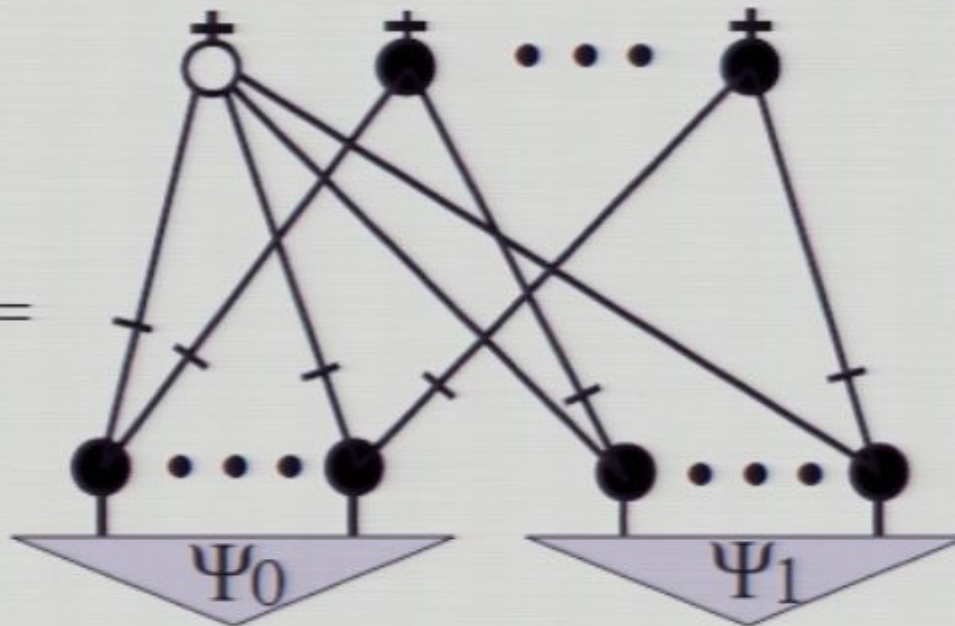
- **Structural points** of W are **copyable points** of GHZ

— *composition of W- and GHZ-CFAs* —

Generating power:

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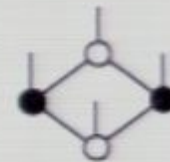
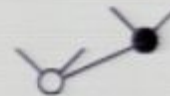
$$|0\Psi_0\rangle + |1\Psi_1\rangle =$$



— *composition of W- and GHZ-CFAs* —

Generating power:

Some four qubit SLOCC-superclass representatives:



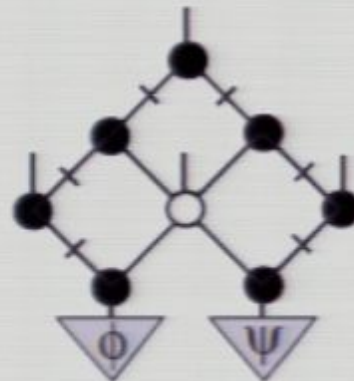
— *composition of W- and GHZ-CFAs* —

Generating power:

Some four qubit SLOCC-superclass representatives:

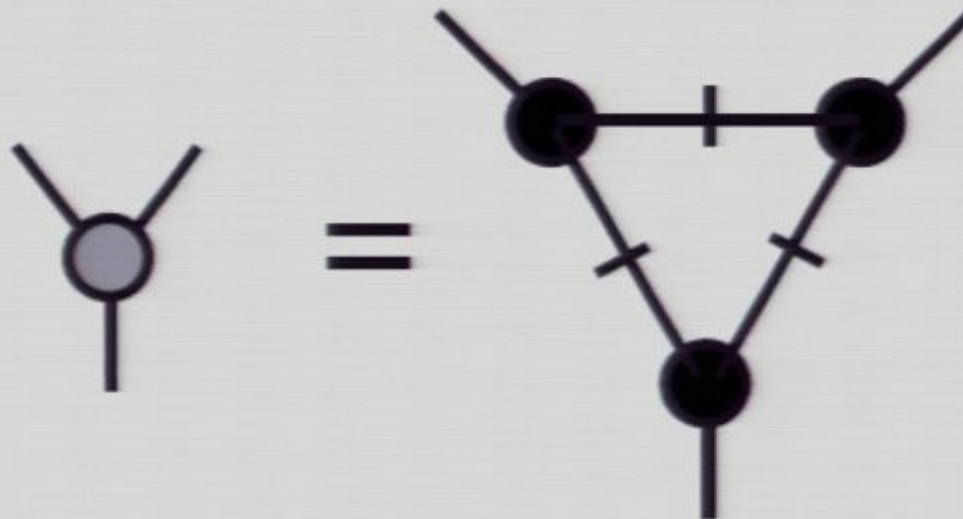


A four qubit continuous SLOCC-superclass:



COMPLEMENTARY IS SUBSUMED

Thm. Under the assumption that one-input operations are determined by their action of W structural points,



together with the GHZ CFA define a pair of complementary observables.

— *this talk* —

- The **algebraic similarity** and **purely topological** difference for GHZ and W SLOCC-class states.
- A **compositional** paradigm for multipartite quantum entanglement, with GHZ and W as generators.
 - Discreteness supports automation e.g. protocol design
- **Quantum structural paradigm** and **graphical calculus** which subsumes complementary observables:
 - GHZ/W-duality more fundamental than complementarity?

Ref: B. Coecke and A. Kissinger (2010) *The compositional structure of multipartite quantum entanglement*. arXiv:1002.2540

— *CFA axioms* —

$$\begin{array}{c} \text{U} \text{ with a dot on the top line} \end{array} = \begin{array}{c} \text{U} \text{ with a dot on the bottom line} \end{array} \quad \begin{array}{c} \text{U} \end{array} = \begin{array}{c} \text{U with a crossing} \end{array}$$

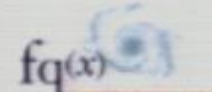
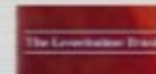
$$\begin{array}{c} | \end{array} = \begin{array}{c} \text{U with a dot on the top line} \end{array} \quad \begin{array}{c} \text{U with a dot on the top line and a crossing} \end{array} = \begin{array}{c} \text{U with a dot on the bottom line and a crossing} \end{array}$$

Classifying entanglement: Two multipartite quantum states **compare** if by (possibly probabilistic) either local or classical means one can be turned into the other.

The Compositional Structure of Multipartite Quantum Entanglement

Bob Coecke and Aleks Kissinger

Oxford University Computing Laboratory



SLOCC

— *commutative Frobenius algebras* —

A **commutative monoid** is object A with morphisms

$$\text{multiplication} : A \otimes A \rightarrow A \quad \text{comultiplication} : I \rightarrow A$$

s.t.

$$\text{multiplication} \circ \text{multiplication} = \text{multiplication} \circ \text{multiplication} \quad \text{multiplication} \circ \text{multiplication} = \text{multiplication} \circ \text{multiplication} \quad \text{multiplication} = \text{multiplication}$$

A **cocommutative comonoid** is object A with morphisms

$$\text{comultiplication} : A \rightarrow A \otimes A \quad \text{multiplication} : A \rightarrow I$$

s.t.

$$\text{comultiplication} \circ \text{comultiplication} = \text{comultiplication} \circ \text{comultiplication} \quad \text{comultiplication} \circ \text{comultiplication} = \text{comultiplication} \circ \text{comultiplication} \quad \text{comultiplication} = \text{comultiplication}$$

— *composition of structures* —

Examples:

- GHZ -states \Rightarrow multi-qubit GHZ
- W -states \Rightarrow multi-qubit W

— *W-spider example* —

Examples:

