Title: Explorations in Numerical Relativity (PHYS 642) - Lecture 5

Date: Mar 19, 2010 11:20 AM

URL: http://pirsa.org/10030104

Abstract:

Stability Analysis

- Will use the property captured by (95) as working definition of stability.
- In particular, if you believe (95) is true for the wave equation, then you believe the wave equation is stable.
- Fundamentally, if FDA approximation *converges*, then expect the same behaviour for the difference solution:

$$\|u_j^n\| \sim \|u_j^0\| \,. \tag{97}$$

- FD solution constructed by *iterating in time*, generating

$$u_j^0,\; u_j^1,\; u_j^2,\; u_j^3,\; u_j^4,\; \cdots$$

in succession, using the FD equation

$$u_j^{n+1} = 2u_j^n - u_j^{n-1} + \lambda^2 \left(u_{j+1}^n - 2u_j^n + u_{j-1}^n \right) \,.$$

Stability Analysis

- One of the most frustrating/fascinating features of FD solutions of time dependent problems: discrete solutions often "blow up"—e.g. floating-point overflows are generated at some point in the evolution
- 'Blow-ups" can sometimes be caused by legitimate (!) "bugs"—i.e. an incorrect implementation—at other times it is simply the *nature of the FD scheme* which causes problems.
- Are thus lead to consider the *stability* of solutions of difference equations
- Again consider the 1-d wave equation, $u_{tt} = u_{xx}$
- Note that it is a *linear, non-dispersive* wave equation
- Thus the "size" of the solution does *not* change with time:

$$\|u(x,t)\| \sim \|u(x,0)\| \, , \qquad (95)$$

where $\|\cdot\|$ is an suitable norm, such as the L_2 norm:

$$\|u(x,t)\| \equiv \left(\int^{1} u(x,t)^2 \, dx \right)^{1/2} . \qquad (96)$$

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Stability Analysis

- *Not* guaranteed that (97) holds for all values of $\lambda \equiv \triangle t \,/\, \triangle x$.
- For certain λ, have

$$\|u_j^n\| \gg \|u_j^0\| \,,$$

  and for those λ, $\|u^n\|$ *diverges* from u, even (especially!) as $h \to 0$—that is, the difference scheme is *unstable*.
- For many wave problems (including all linear problems), given that a FD scheme is *consistent* (i.e. so that $\hat{\tau} \to 0$ as $h \to 0$), *stability is the necessary and sufficient condition for convergence* (Lax's theorem).

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Heuristic Stability Analysis

- Write general time-dependent FDA in the form

$$\mathbf{u}^{n+1} = \mathbf{G}[\mathbf{u}^n]\,, \tag{98}$$

- $\mathbf{G}$ is some *update operator* (linear in our example problem)
- $\mathbf{u}$ is a column vector containing sufficient unknowns to write the problem in first-order-in-time form.
- Example: introduce new, auxiliary set of unknowns, v_j^n, defined by

$$v_j^n = u_j^{n-1}\,,$$

then can rewrite differenced-wave-equation (16) as

$$u_j^{n+1} = 2u_j^n - v_j^n + \lambda^2\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)\,, \tag{99}$$

$$v_j^{n+1} = u_j^n\,, \tag{100}$$

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Heuristic Stability Analysis

- Thus with

$$\mathbf{u}^n = [u_1^n, v_1^n,\ u_2^n, v_2^n, \cdots u_J^n, v_J^n] ,$$

(for example), (99-100) is of the form (98).

- Equation (98) provides compact way of describing the FDA solution.

- Given initial data, $\mathbf{u}^0$, solution after n time-steps is

$$\mathbf{u}^n = \mathbf{G}^n \mathbf{u}^0 , \quad (101)$$

where $\mathbf{G}^n$ is the n-th power of the matrix $\mathbf{G}$.

- Assume that $\mathbf{G}$ has a complete set of orthonormal eigenvectors

$$\mathbf{e}_k, \quad k = 1, 2, \cdots J ,$$

and corresponding eigenvalues

$$\mu_k, \quad k = 1, 2, \cdots J ,$$

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Heuristic Stability Analysis

- Thus have

$$\mathbf{G}\,\mathbf{e}_k = \mu_k\,\mathbf{e}_k, \quad k = 1, 2, \cdots J\,.$$

- Can then write initial data as (spectral decomposition):

$$\mathbf{u}^0 = \sum_{k=1}^{J} c_k^0\,\mathbf{e}_k\,,$$

where the c_k^0 are coefficients.

- Using (101), solution at time-step n is

$$\mathbf{u}^n = \mathbf{G}^n \left(\sum_{k=1}^{J} c_k^0\,\mathbf{e}_k \right) \qquad (102)$$

$$= \sum_{k=1}^{J} c_k^0\,(\mu_k)^n\,\mathbf{e}_k\,. \qquad (103)$$

Heuristic Stability Analysis

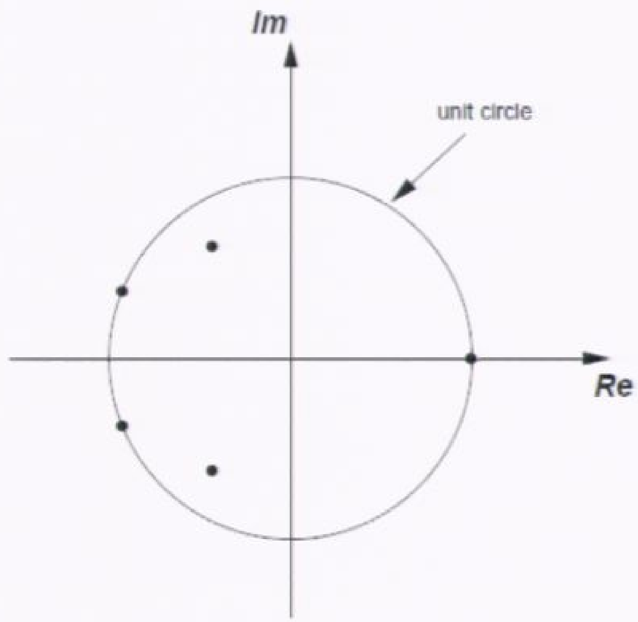
- If difference scheme is to be stable, must have

$$|\mu_k| \le 1 \qquad k = 1, 2, \cdots J \tag{104}$$

(Note: μ_k will be complex in general, so $|\mu|$ denotes the complex modulus, $|\mu| \equiv \sqrt{\mu\mu^\star}$).

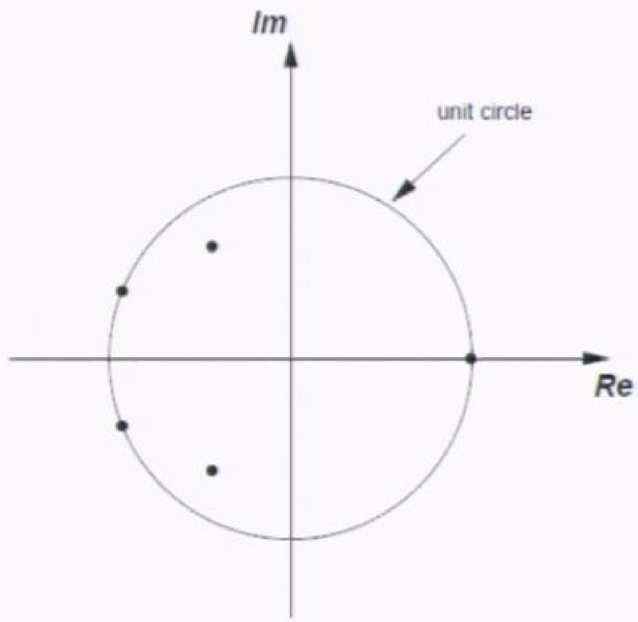
- Geometric interpretation: eigenvalues of the update matrix must lie on or within the unit circle

Heuristic Stability Analysis



- Schematic illustration of location in complex plane of eigenvalues of update matrix **G**.
- In this case, all eigenvalues (dots) lie on or within the unit circle, indicating that the corresponding finite difference scheme is stable.

Heuristic Stability Analysis



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Von-Neumann (Fourier) Stability Analysis (Summary)

- Von-Neumann (VN) stability analysis based on the ideas sketched above
- Assumes that difference equation is linear with constant coefficients, periodic boundary conditions boundary conditions are periodic
- Can then use Fourier analysis: difference operators in real-space variable $x \longrightarrow$ algebraic operations in Fourier-space variable k
- VN applied to wave-equation example shows that must have

$$\lambda \equiv \frac{\triangle t}{\triangle x} \le 1 \,,$$

  for stability of scheme (16).

- Condition is often called the CFL condition—after Courant, Friedrichs and Lewy who derived it in 1928
- This type of instability has "physical" interpretation, often summarized by the statement *the numerical domain of dependence of an explicit difference scheme*

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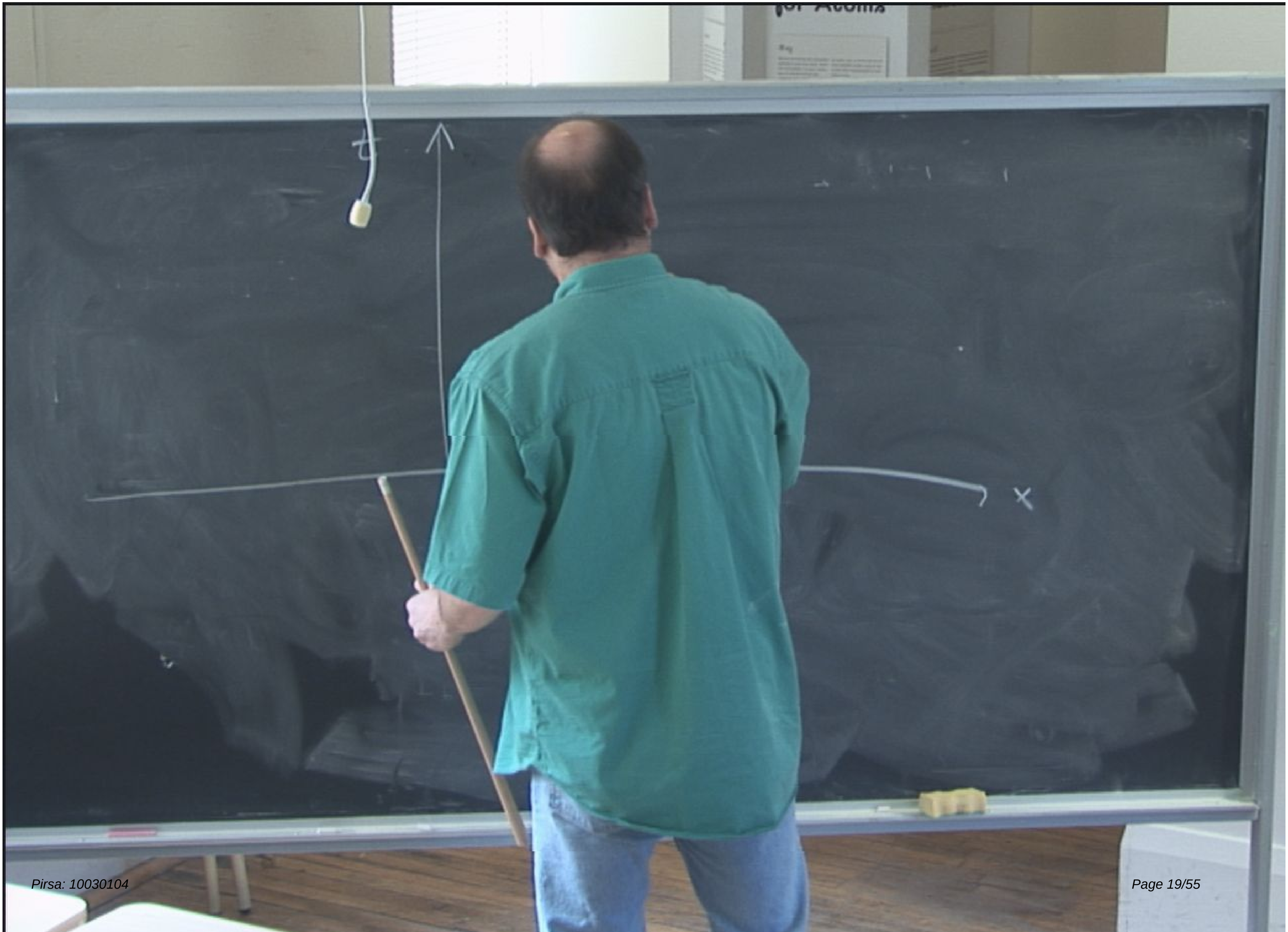
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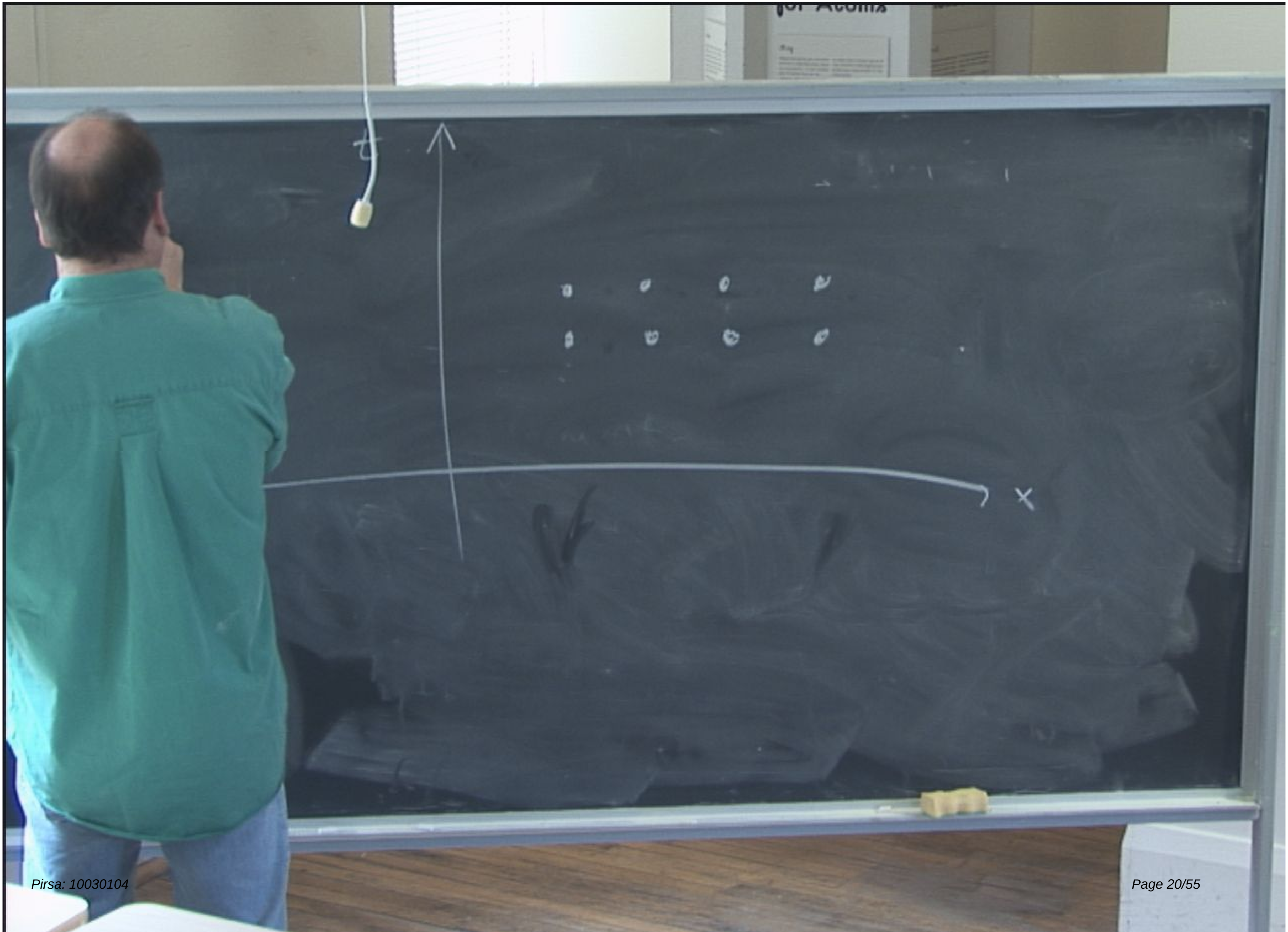
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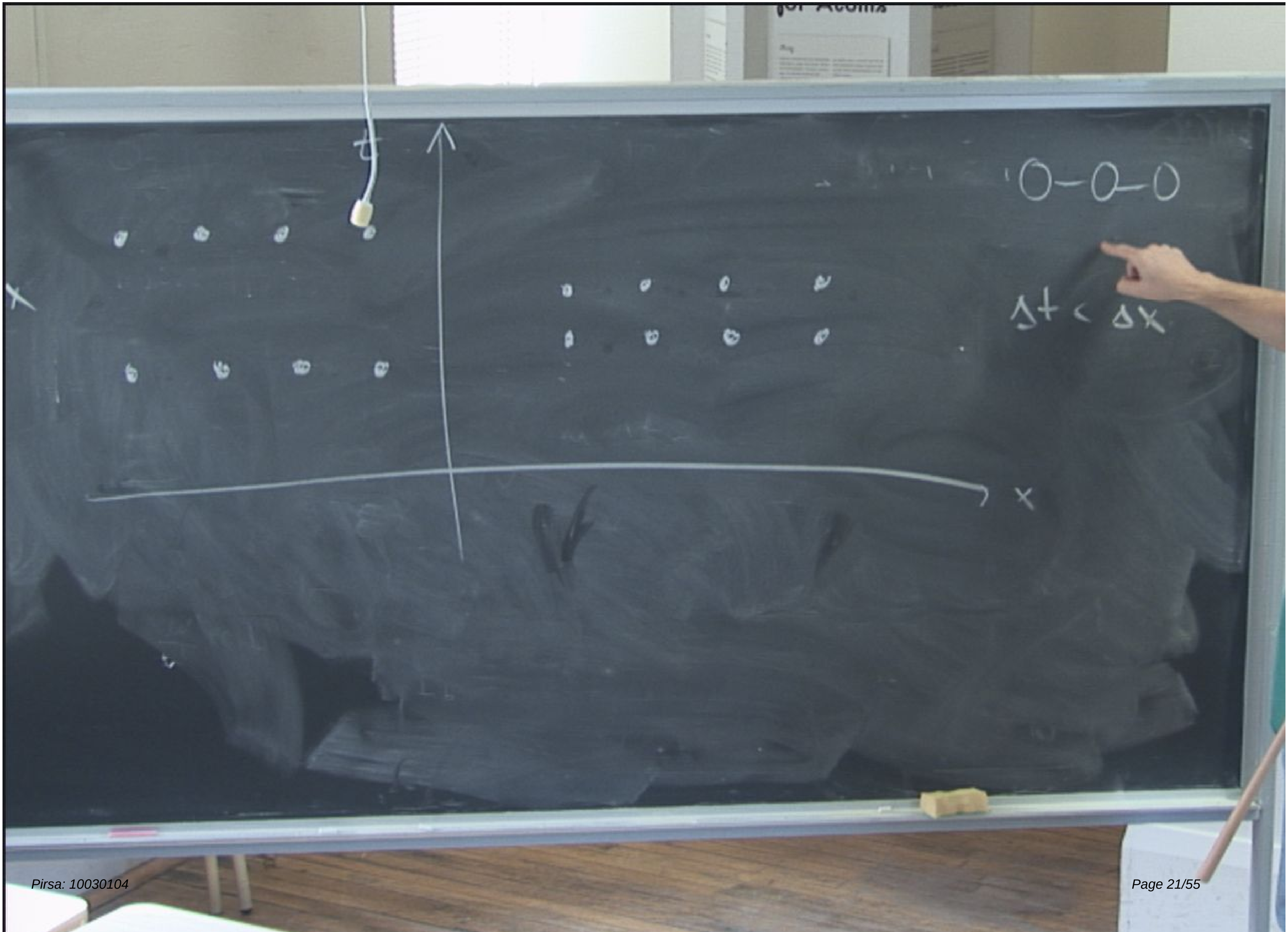
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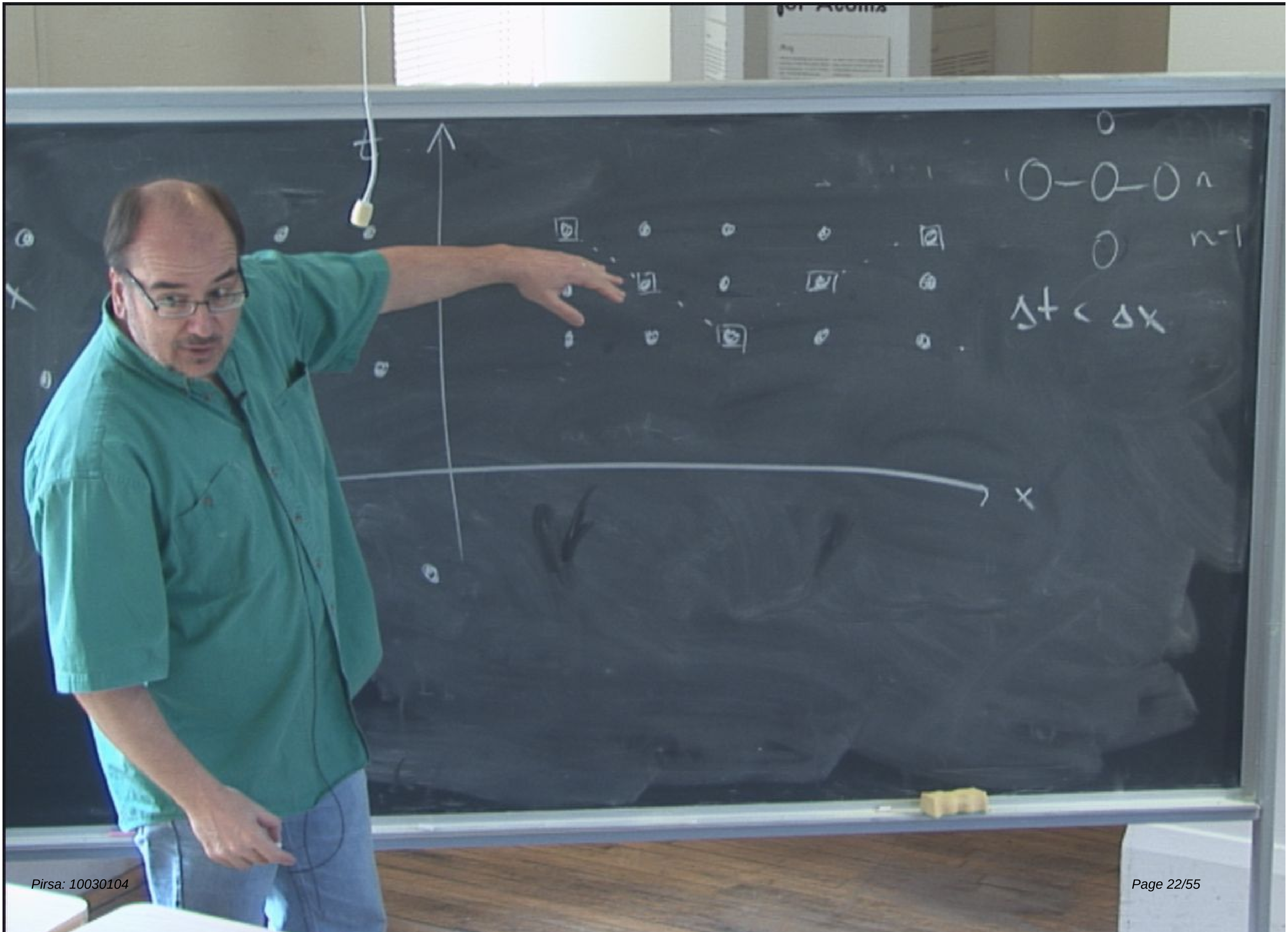
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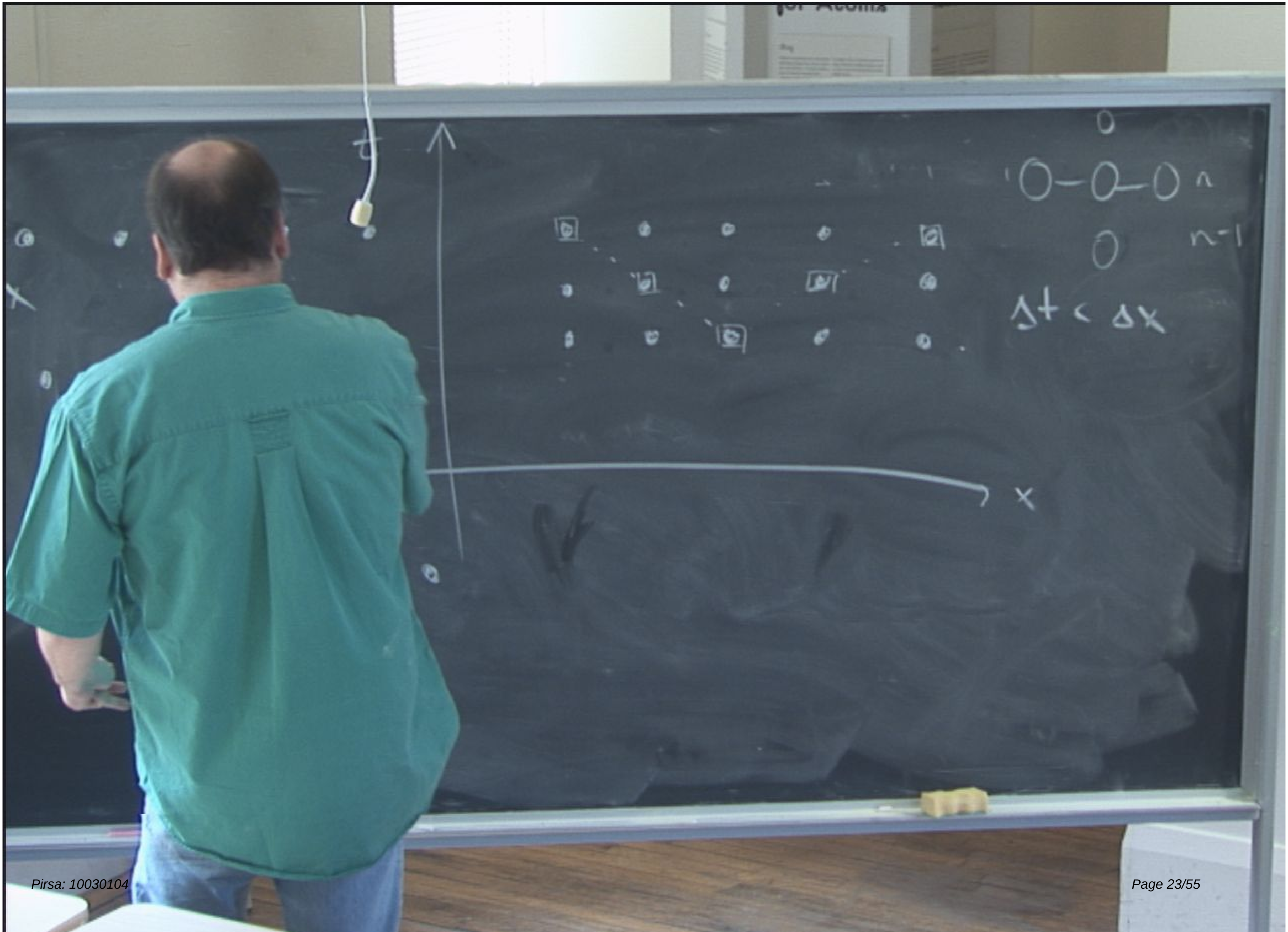
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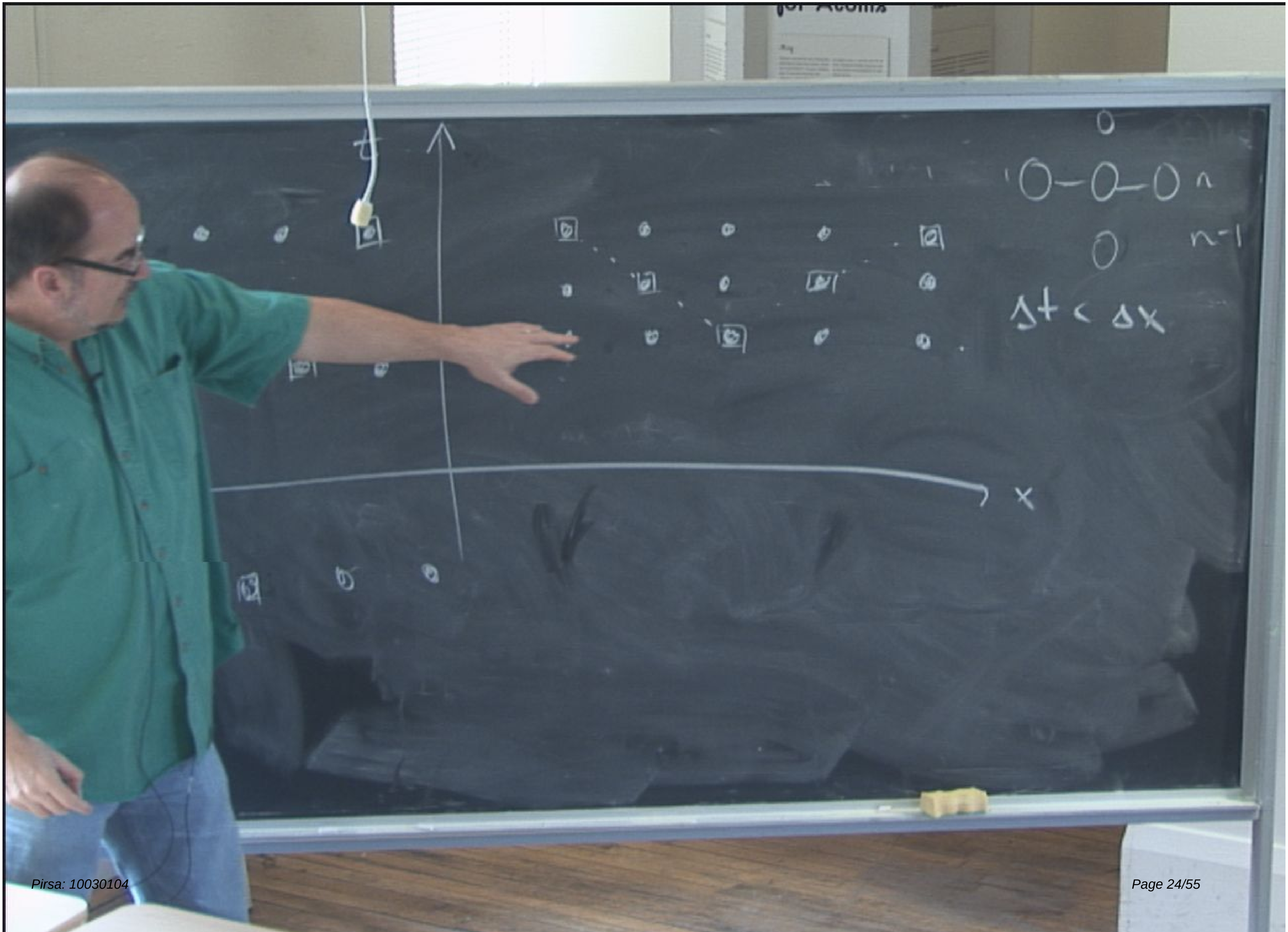
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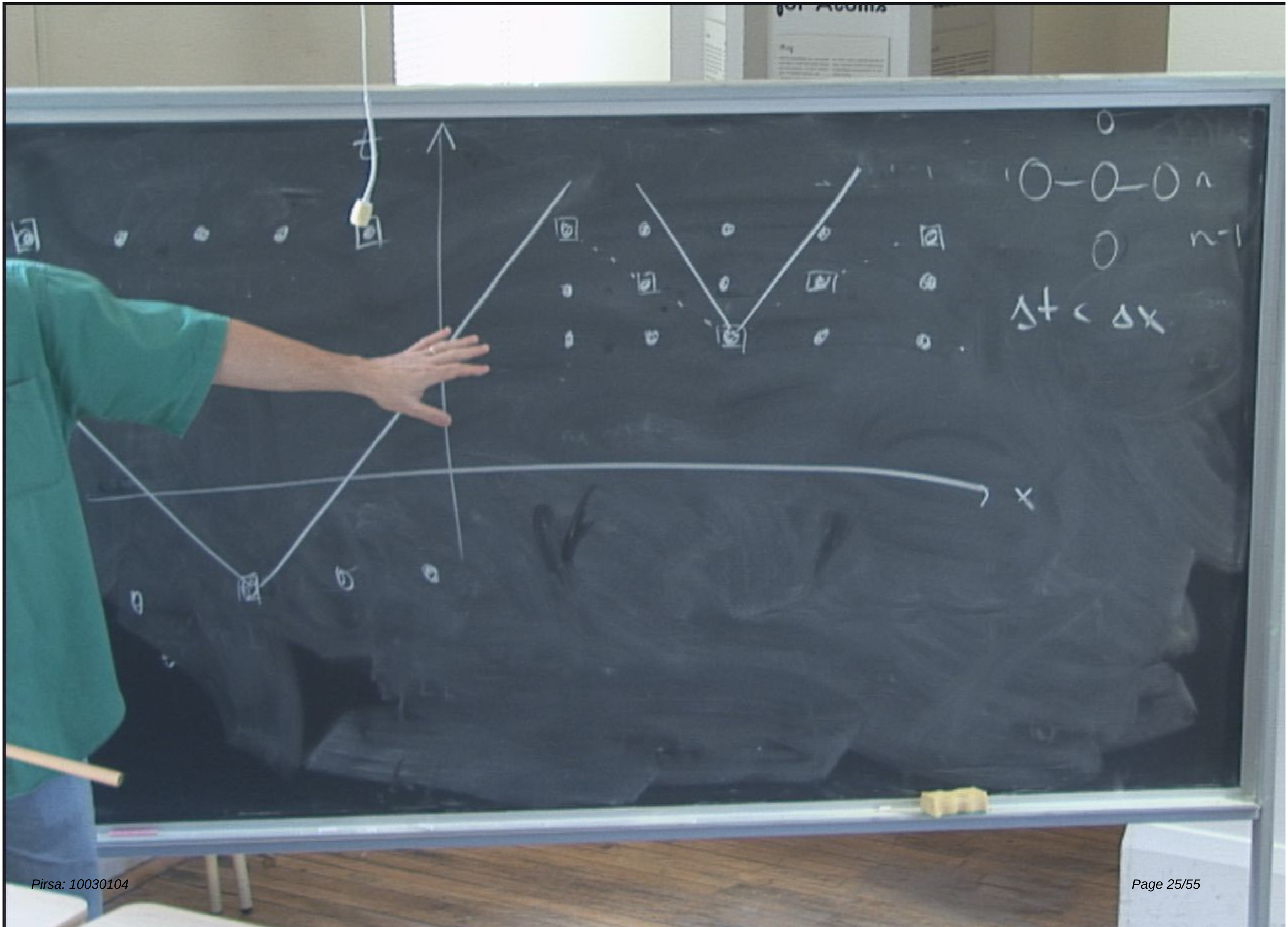
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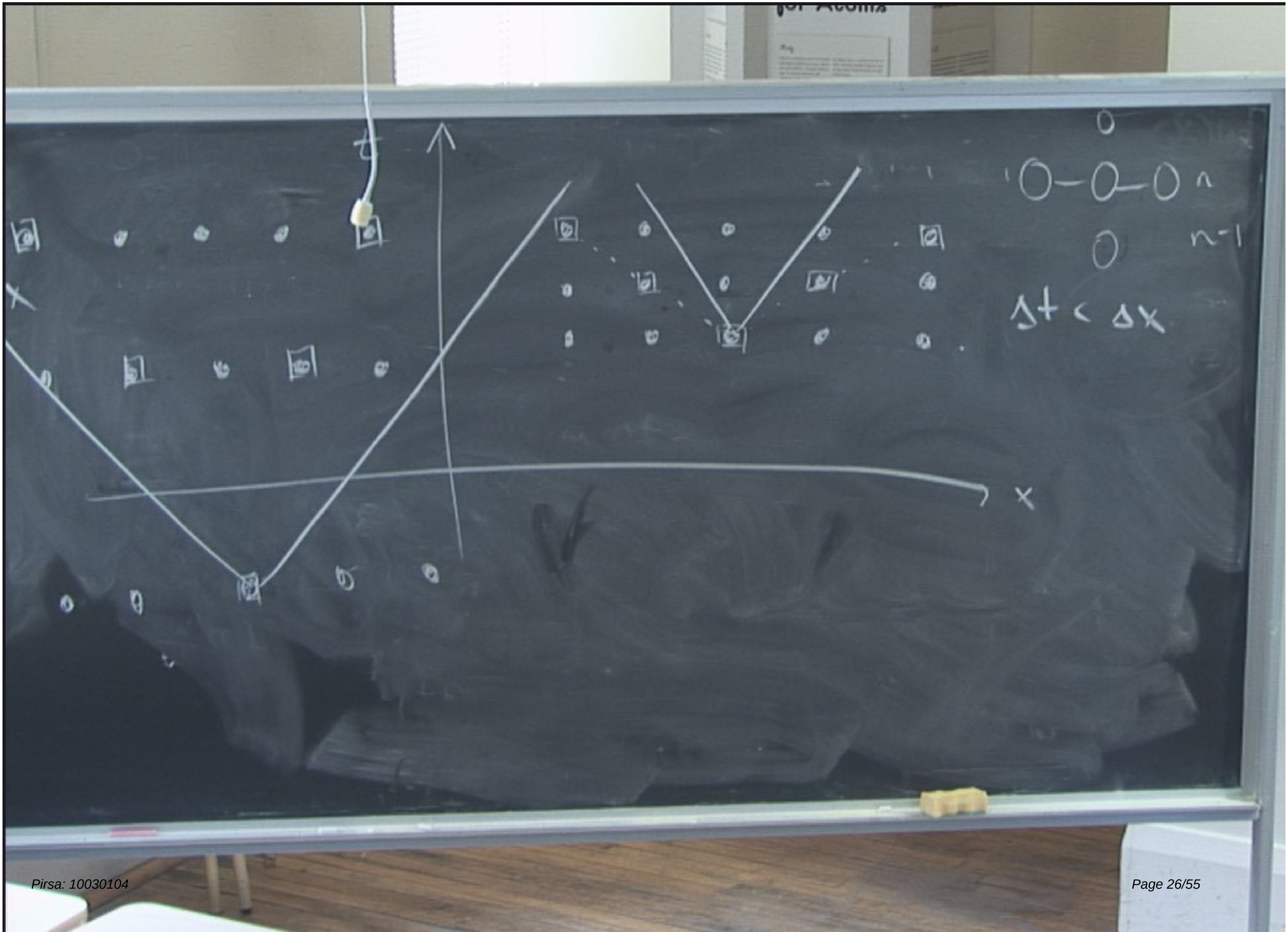
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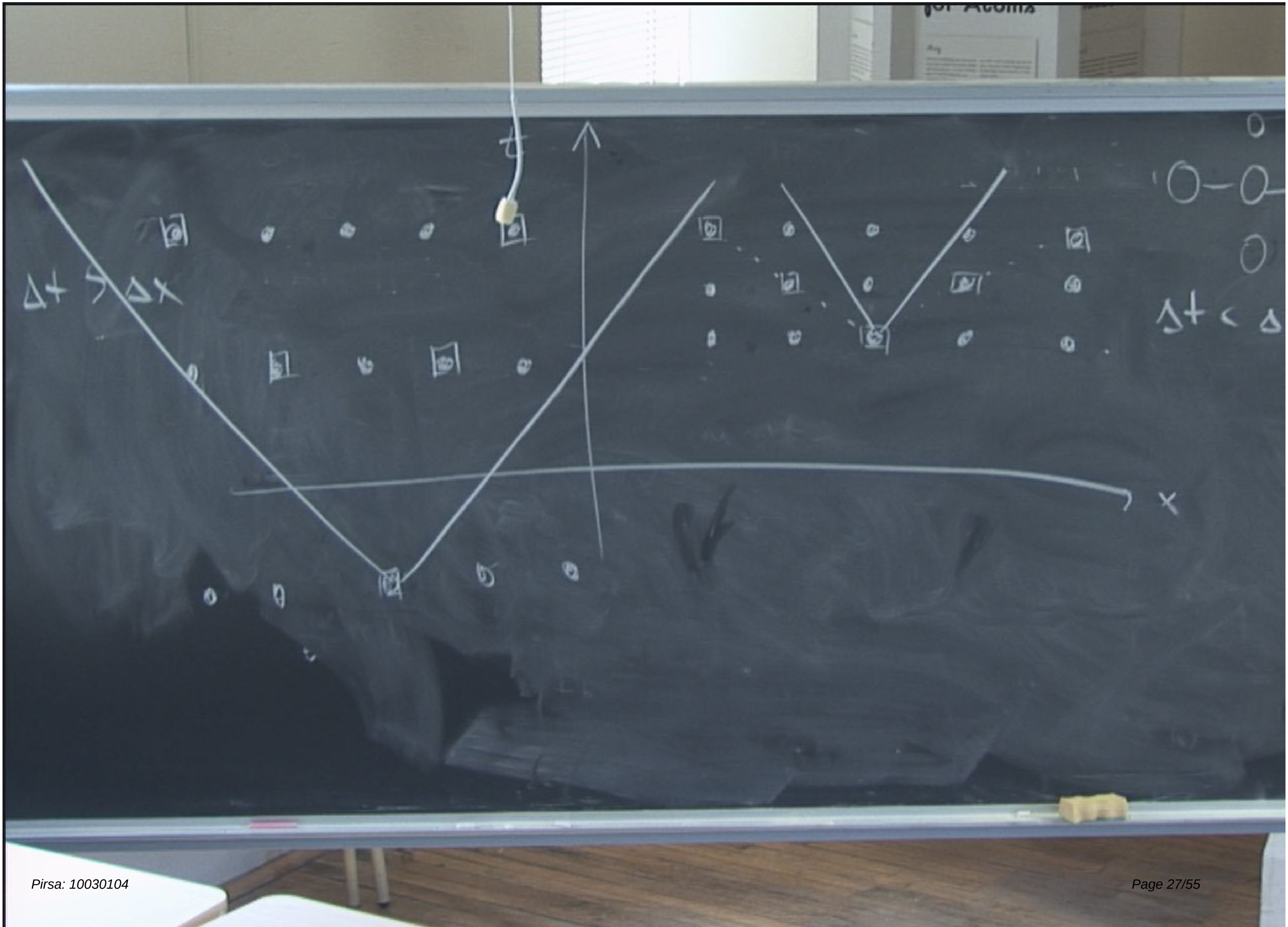

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$$\Pi = \frac{a}{\alpha}\left(\dot{\phi} - \beta \phi'\right)$$

$$\rightarrow \quad \dot{\phi} = \frac{\alpha}{a}\Pi + \beta\phi' = \frac{\alpha}{a}\Pi + \beta\Phi$$

- BUT $\phi' = \Phi$, SO

$$\dot{\Phi} = \left(\beta\Phi + \frac{\alpha}{a}\Pi\right)' \qquad (32)$$

- TO FIND $\dot{\Pi}$ EQN., RECALL THAT

$$\Box\phi = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\, g^{\mu\nu}\partial_\nu\phi\right)$$

$$\Box\phi = 0 \rightarrow \partial_\mu\left(\sqrt{-g}\, g^{\mu\nu}\partial_\nu\phi\right) = 0$$

$$\rightarrow \partial_t\left(\sqrt{-g}\, g^{t t}\partial_t\phi\right) + \partial_r\left(\sqrt{-g}\, g^{r r}\partial_r\phi\right) \ldots$$

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PHY 387M — SPHERICAL SYMMETRY (6)

CHARACTERISTIC ANALYSIS of THE SCALAR FIELD

REF: COURANT & HILBERT "METHODS of MATH. PHYS", VOL II, ch 5

- EQNS (37)-(38) ARE A 1ST-ORDER, QUASI-LINEAR SYS FOR OUR RADIATION FIELD. DEFINING

$$u \equiv (\Phi, \Pi)^T$$

WE CAN WRITE

$$u_t + A u_x = B \qquad (39)$$

$$A = -\begin{pmatrix} \beta & \alpha/a \\ \alpha/a & \beta \end{pmatrix} \qquad (40)$$

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THESE ARE THE "LOCAL SIGNAL SPEEDS" FOR THE SCALAR

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- MASSLESS SCALAR FIELD – WEAK (NON-SELF GRAVITATING) SIGNALS TRAVEL ALONG NULL GEODESICS → ALTERNATE DERIVATION of (A1)

$$ds^2 = -\alpha^2 dt^2 + a^2 (dr + \beta dt)^2 = 0$$

REGULARITY / LOCAL FLATNESS AT $r=0$

- OUR CAUCHY PROBLEM FOR THE EMKG MODEL IS TO BE SOLVED ON

$$t \geq 0 , \quad r \geq 0$$

- BOUNDARY CONDITIONS AS $r \to \infty$ WILL FOLLOW FROM ASYMPTOTIC FLATNESS, NO INCOMING RADIATION; $r=0$ NOT A REAL BOUNDARY, BUT COMPUTATIONALLY (I.E. WHEN FINITE DIFFERENCING) IS EFFECTIVELY ONE

- GET CONDITIONS AT $r=0$ BY DEMANDING THAT SCALAR

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TRAPPED SURFACES / APPARENT HORIZONS

- WANT TO STUDY BLACK HOLE FORMATION; BH CHARACTERIZED BY AN EVENT HORIZON WHICH CAN ONLY BE DETERMINED ONCE COMPLETE S.T. HAS BEEN CONSTRUCTED

- USEFUL TO BE ABLE TO COMPUTE "INSTANTANEOUS" APPROX. (I.E. ON ANY HYPERSURFACE $\Sigma(t)$) TO EH. — PROVIDED BY APPARENT HORIZON $\equiv$ OUTERMOST MARGINALLY TRAPPED SURFACE

PHYSICAL SPHERICAL SYMMETRY (E)

- TRAPPED SURFACE: 2-SURFACE WITH TOPOLOGY S^2 SUCH THAT DIVERGENCE OF OUTGOING NULL GEODESICS EMANATING FROM SURFACE < 0

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PHYS 530N SPHERICAL SYMMETRY (E)

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PAINLEVE SPHERICAL SYMMETRY (E)

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PHYS 530N SPHERICAL SYMMETRY (E)

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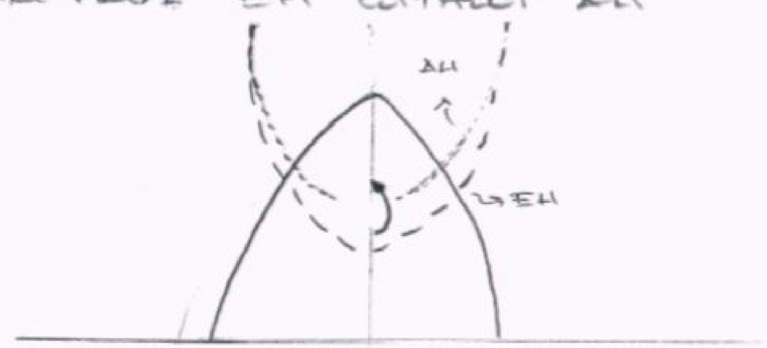
- MARGINALLY TRAPPED SURFACE: "< 0" → "$= 0$"

- MODULO COSMIC CENSORSHIP (NO NAKED SINGULARITIES) EXISTENCE of AH ⟹ EXISTENCE of EH; HOWEVER CAN HAVE EH WITHOUT AH



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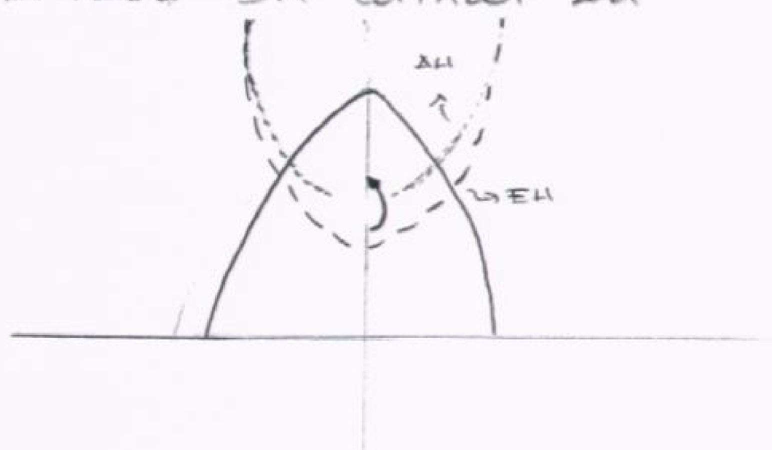
- MARGINALLY TRAPPED SURFACE: "< 0" → "$= 0$"

- MODULO COSMIC CENSORSHIP (NO NAKED SINGULARITIES) EXISTENCE OF AH ⇒ EXISTENCE OF EH; HOWEVER CAN HAVE EH WITHOUT AH



- WILL TEND TO USE TERMS AH, TS ; MTS INTER-CHANGEABLY, BUT SHOULD BE AWARE OF DISTINCTIONS

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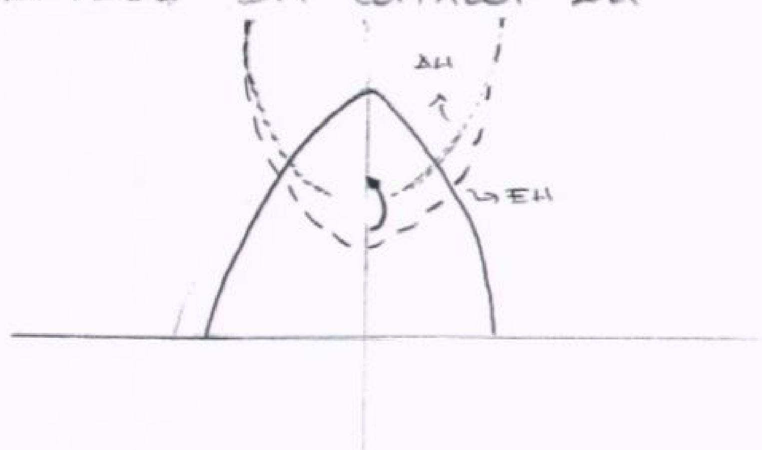

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(MARGINALLY) TRAPPED SURFACE EQN (AH EQN)

- CONSIDER A 2-SURFACE WITH OUTGOING NULL TANGENT u^a WHICH IS MARGINALLY TRAPPED, THEN

$$\nabla_a u^a = 0$$

NOW, CAN WRITE u^a AS

$$u^a = s^a + n^a$$

(n^a) ↳ UNIT FUTURE-DIRECTED TIMELIKE NORMAL TO Σ

(s^a) ↳ UNIT OUTWARDS-POINTING SPACELIKE NORMAL TO 2-SURF.

IN // TO 3+1 DECOMPOSITION, METRIC h_{ab}, h^{ab} IS INDUCED ON THE 2-SURFACE BY PROJECTION

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IN ~ TO 3+1 DECOMPOSITION, METRIC h_{ab}, h^{ab} IS INDUCED ON THE 2-SURFACE BY PROJECTION

$$h^{ab} = \gamma^{ab} - s^a s^b = g^{ab} + n^a n^b - s^a s^b$$

CAN SHOW (EXERCISE) THAT $\nabla_a u^a$ IS A "2-SCALAR"; I.E. IS INTRINSIC TO 2-SURFACE; I.E. DOES NOT DEPEND ON HOW 2-SURFACE IS EMBEDDED IN Σ, THEN

$$\nabla_a u^a = g^{ab} \nabla_a u_b = h^{ab} \nabla_a u_b$$

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$$\nabla_a u^a = g^{ab} \nabla_a u_b = h^{ab} \nabla_a u_b$$

$$= h^{ab} \nabla_a (s_b + n_b)$$

$$= h^{ab} \frac{1}{\ldots} \nabla_a (s_b + n_b) \qquad h^{ab} \text{ PROJECTS ONTO}$$

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$$h^{ab} = \gamma^{ab} - s^a s^b = g^{ab} + n^a n^b - s^a s^b$$

CAN SHOW (EXERCISE) THAT $\nabla_a u^a$ IS A "2-TENSOR"; I.E IS INTRINSIC TO 2-SURFACE; I.E. DOES NOT DEPEND ON HOW 2-SURFACE IS EMBEDDED IN Σ, THEN

$$\nabla_a u^a = g^{ab} \nabla_a u_b = h^{ab} \nabla_a u_b$$

$$= h^{ab} \nabla_a (s_b + n_b)$$

$$= h^{ab} \perp \nabla_a (s_b + n_b)$$

h_{ab} PROJECTS ONTO 2-SURFACE, SO CAN FIRST PROJECT ONTO Σ

$$= h^{ab} (D_a s_b + \perp \nabla_a n_b)$$

$$= h^{ab} (D_a s_b - K_{ab})$$

$$= (g^{ab} - s^a s^b)(D_a s_b - K_{ab})$$

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HOW 2-SURFACE IS EMBEDDED IN Σ, THEN

$$\nabla_a u^a = g^{ab} \nabla_a u_b = h^{ab} \nabla_a u_b$$

$$= h^{ab} \nabla_a (s_b + n_b)$$

$$= h^{ab} \perp \nabla_a (s_b + n_b)$$

$$= h^{ab} (D_a s_b + \perp \nabla_a n_b)$$

$$= h^{ab} (D_a s_b - K_{ab})$$

$$= (g^{ab} - s^a s^b)(D_a s_b - K_{ab})$$

$$= D^a s_a - K + s^a s^b K_{ab}$$

h_{ab} PROJECTS ONTO 2-SURFACE, SO CAN FIRST PROJECT ONTO Σ

$$\left(s^b D_a s_b = \tfrac{1}{2} D_a (s^b s_b) = \tfrac{1}{2} D_a (1) = 0 \right)$$

HOW 2-SURFACE IS EMBEDDED IN Σ, THEN

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$$= h^{ab}(D_a s_b + \perp\nabla_a n_b)$$

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$$\left(s^b D_a s_b = \tfrac{1}{2} D_a(s^b s_b) = \tfrac{1}{2} D_a(1) = 0 \right)$$

THUS, OUR TRAPPED SURFACE (AH) EQUATION IS

$$D^a s_a - K + s^a s^b K_{ab} = 0 \qquad (47)$$

AND ARGUING AS WE DID FOR THE 3+1 EQNS WE GET A VALID COMPONENT FORM OF THIS EQUATION BY TAKING $a \to i$, $b \to j$

$$D^i s_i - K + s^i s^j K_{ij} = 0 \qquad (48)$$

• SPECIALIZING NOW TO SPHERICAL SYMMETRY

$$ds^2 = a^2 dr^2 + r^2 b^2 d\Omega^2$$

$$\gamma_{ij} s^i s^j = 1 \to s^i = (a^{-1}, 0, 0)$$

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AND ARGUING AS WE DID FOR THE 3+1 EQNS WE GET A VALID COVARIANT FORM OF THIS EQUATION BY TAKING $a \to i,\ b \to j$

$$\boxed{D^i s_i - K + s^i s^j K_{ij} = 0} \quad (45)$$

• SPECIALIZING NOW TO SPHERICAL SYMMETRY

$$ds^2 = a^2 dr^2 + r^2 b^2 d\Omega^2$$

$$\gamma_{ij} s^i s^j = 1 \to s^i = (a^{-1}, 0, 0)$$

$$D_i s^i = \gamma^{-\frac{1}{2}} \partial_i (\gamma^{\frac{1}{2}} s^i) \qquad \gamma^{\frac{1}{2}} = a r^2 b^2$$

$$= \frac{1}{a r^2 b^2} (r^2 b^2)' = \frac{2 (rb)'}{a r b}$$

THUS, (45) BECOMES

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$$\gamma_{ij} s^i s^j = 1 \rightarrow s^i = (a^{-1}, 0, 0)$$

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$$= \frac{1}{a r^2 b^2} (r^2 b^2)' = \frac{2(rb)'}{arb}$$

THUS, (46) BECOMES

$$\frac{2(rb)'}{arb} - (K^r{}_r + 2K^\theta{}_\theta) + a^{-2} K_{rr} = 0$$

$\hookrightarrow K^r{}_r$

$$\boxed{(rb)' = arb\, K^\theta{}_\theta} \qquad (49)$$

NOW, RECALL EVOL. EQN (34) FOR b

$$\dot{b} = -\alpha b K^\theta{}_\theta + \beta (rb)'$$

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$$r_{ij} s^i s^j = 1 \rightarrow s^i = (a^{-2}, 0, 0)$$

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$$= \frac{1}{a r^2 b^2} (r^2 b^2)' = \frac{2 (rb)'}{a r b}$$

THUS, (42) BECOMES

$$\frac{2(rb)'}{arb} - (K^r{}_r + 2K^\theta{}_\theta) + a^{-2} K_{rr} = 0$$

↳ $K^r{}_r$

$$\boxed{(rb)' = a r b \, K^\theta{}_\theta} \qquad (49)$$

NOW, RECALL EVOL. EQN (34) FOR b

$$\dot b = -\alpha b K^\theta{}_\theta + \frac{\beta}{r} (rb)'$$

$$\Rightarrow K^\theta{}_\theta = -(\alpha b)^{-1} \left(\dot b - \frac{\beta}{r} (rb)' \right)$$

NOW, RECALL EVOL. EQN (34) FOR b

$$\dot{b} = -\alpha b K^\theta{}_\theta + \frac{\beta}{r}(rb)'$$

$$\Rightarrow K^\theta{}_\theta = -(\alpha b)^{-1}\left(\dot{b} - \frac{\beta}{r}(rb)'\right)$$

$$rK^\theta{}_\theta = -(\alpha b)^{-1}\left((r\dot{b}) - \beta(rb)'\right)$$

SO (49) CAN BE REWRITTEN AS

$$(r\dot{b}) + \left(\frac{\alpha}{a} - \beta\right)(rb)' = 0 \qquad (50)$$

WHICH SAYS THAT THE SURFACE OF CONSTANT AREAL RADIUS $R = rb$ IS OUTGOING NULL AT THE MARGINALLY TRAPPED SURFACE IN ACCORD WITH OUR PHYSICAL PICTURE

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NOW, RECALL EVOL. EQN (34) FOR b

$$\dot{b} = -\alpha b K^{\theta}{}_{\theta} + \frac{\beta}{r}(rb)'$$

$$\Rightarrow K^{\theta}{}_{\theta} = -(\alpha b)^{-1}\left(\dot{b} - \frac{\beta}{r}(rb)'\right)$$

$$r K^{\theta}{}_{\theta} = -(\alpha b)^{-1}\left((r\dot{b}) - \beta (rb)'\right)$$

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