Title: Explorations in Numerical Relativity - Lecture 3

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PHY 3E7M THE 3+1 (ADM / HAMILTONIAN) FORM of GR

REFERENCES

(1) MTW CHAPTER 21

(2) YORK, "KINEMATICS & DYNAMICS of GR" IN "SOURCES of GRAV. RADIATION" (L. SMARR ed)

(3) WALD, APP. E.(?) & CHAPTER 10

(4) ARNOWITT, DESER & MISNER (1962) "THE DYNAMICS of GR" IN "GRAVITATION: AN INTRODUCTION TO CURRENT RESEARCH" (L. WITTEN ed)

- ADM MOTIVATION WAS, AGAIN, PREPARATION FOR QUANTIZATION of GR; FORMALISM TURNED OUT TO BE EXCELLENT BASIS FOR COMPUTATIONAL (NUMERICAL) ASSAULT ON EINSTEIN EQUATIONS

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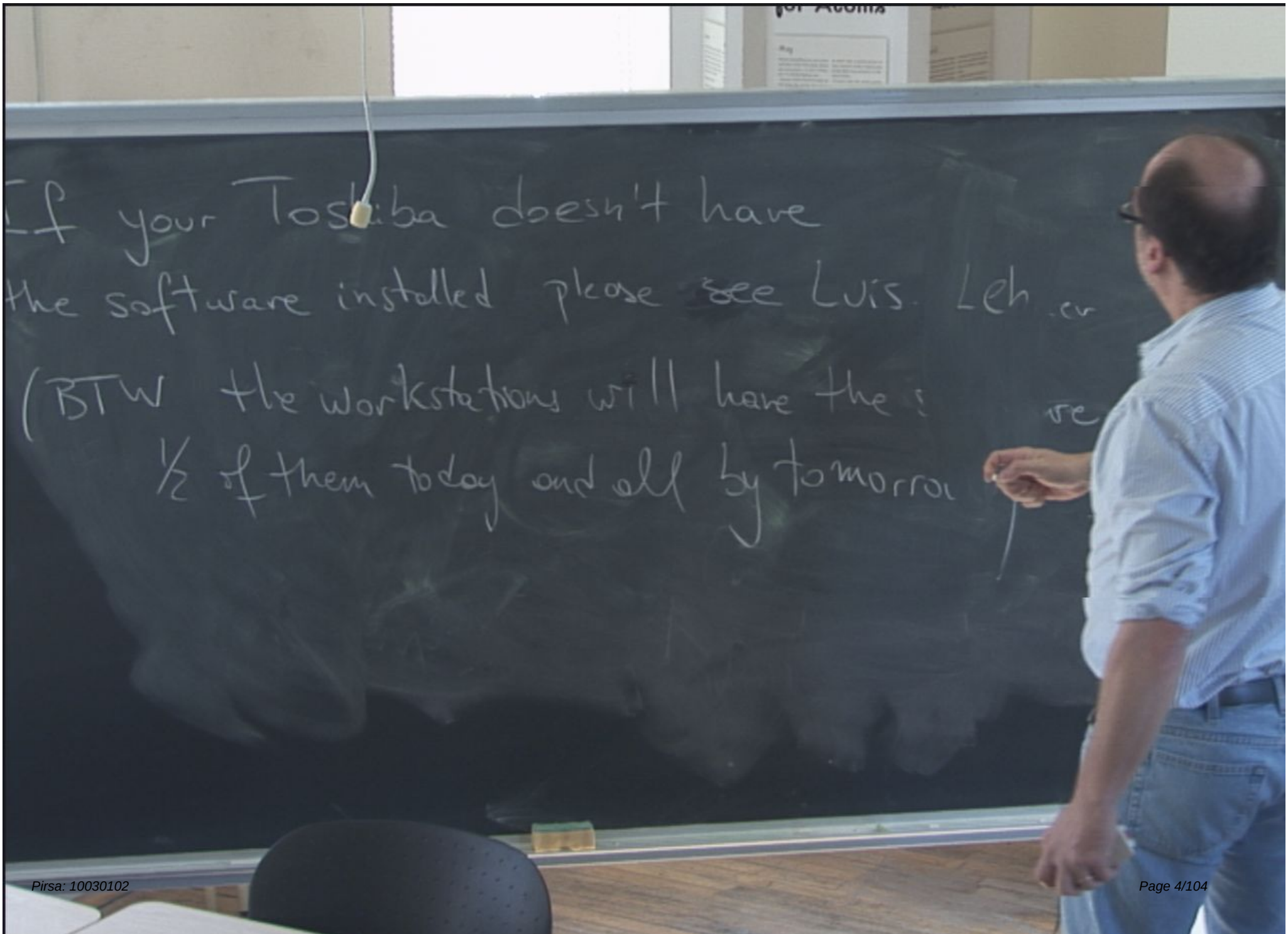
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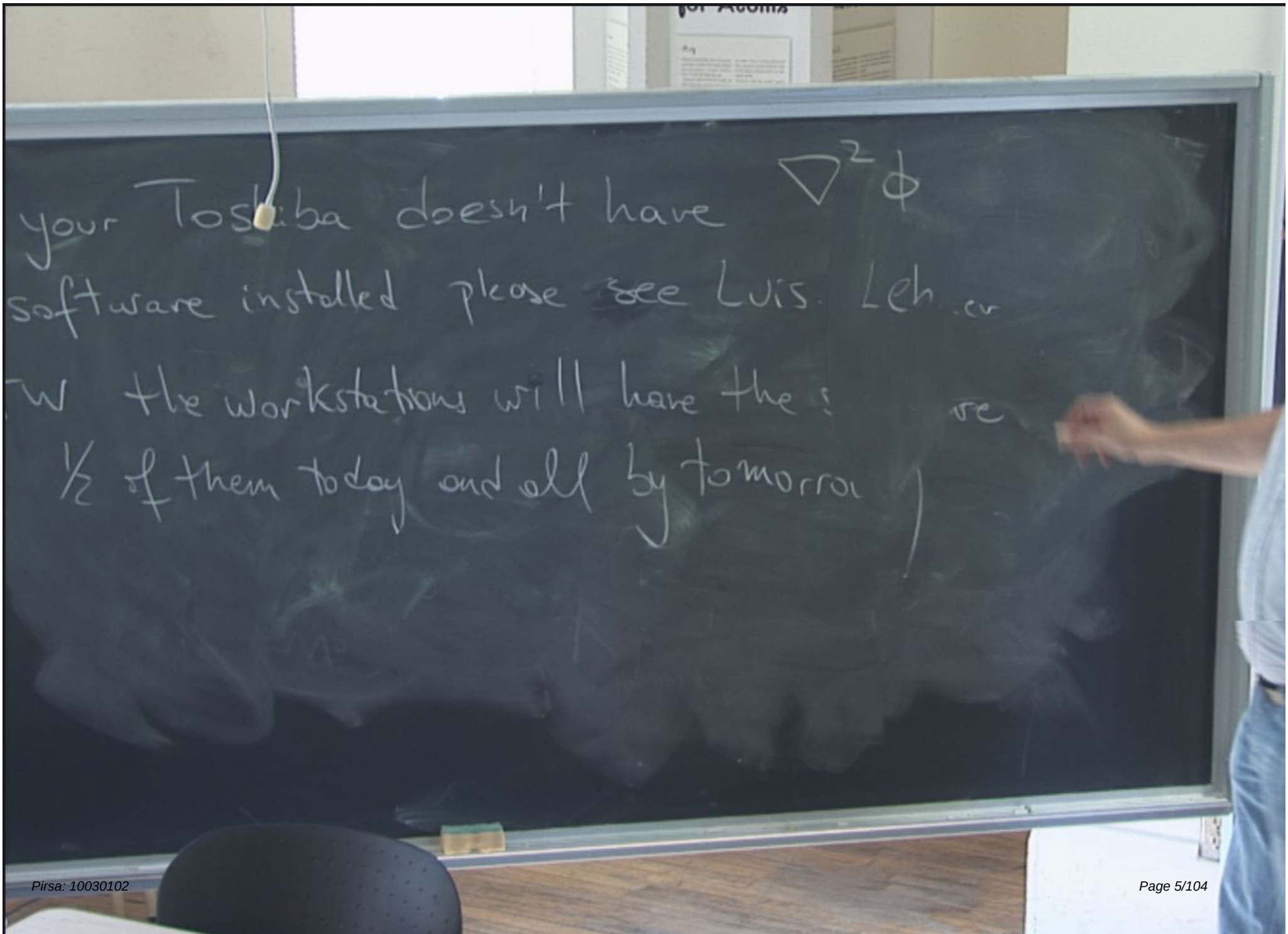
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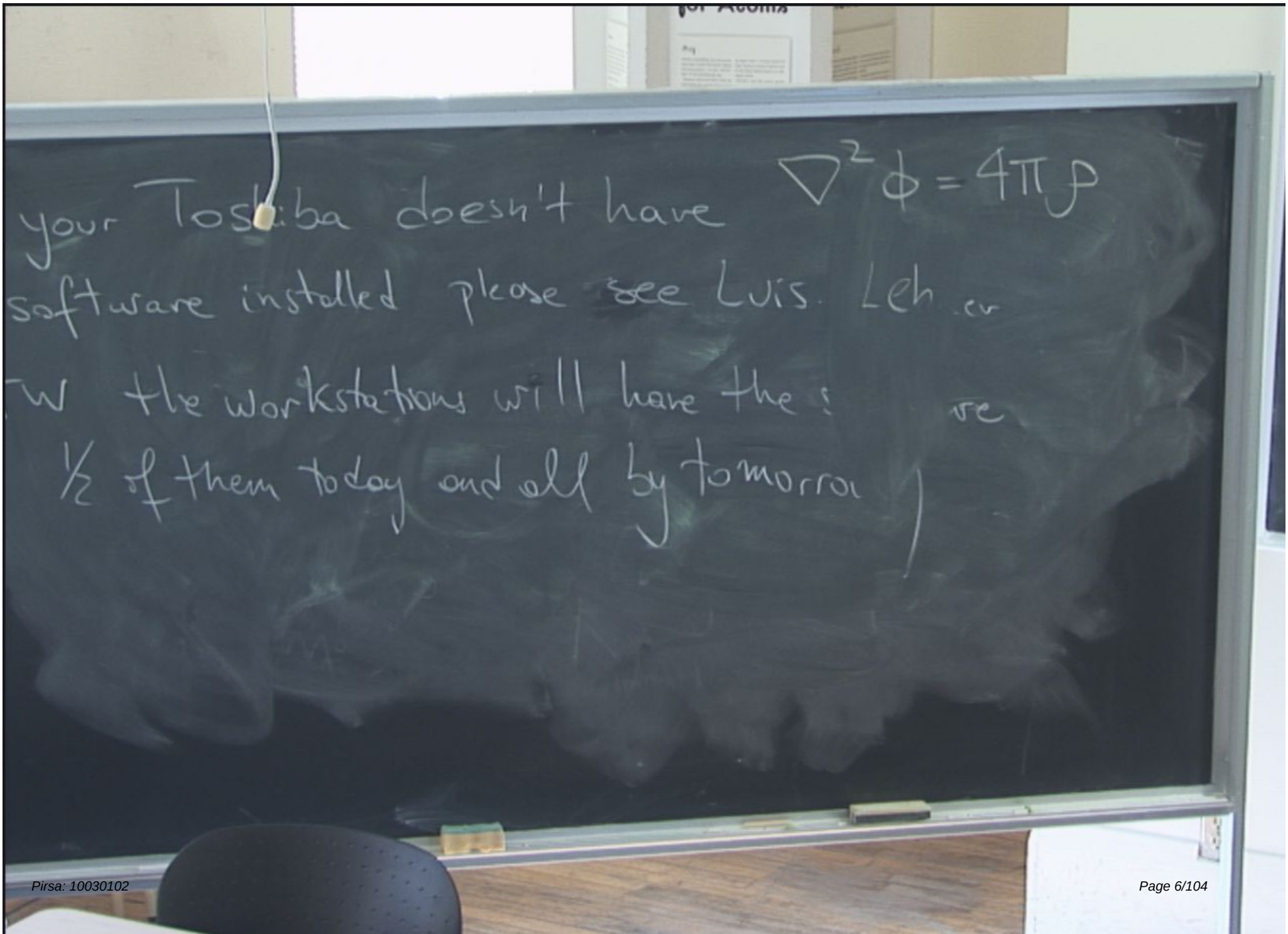
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ULTIMATE GOAL: REFORMULATE

$$G_{ab} = 8\pi T_{ab}$$

AS SYSTEM OF FIRST-ORDER (IN TIME) PDE'S FOR THE GRAV. FIELD VBL'S, WHICH CAN THEN BE SOLVED AS AN "INITIAL-VALUE" OR "CAUCHY" PROBLEM

SHIFT IN PERSPECTIVE: UP TO NOW $G_{ab} = 8\pi T_{ab}$ DESCRIBED THE LINKAGE OF THE GEOMETRY of SPACETIME (4-D) TO THE DISTRIBUTION of MATTER-STRESS-ENERGY IN S.T.

NEW VIEW: GEOMETRY of S.T IS "TIME-HISTORY" (EVOLUTION) of GEOMETRY of A SPACELIKE HYPERSURFACE (3-D) $\Rightarrow$ GEOMETRODYNAMICS; "VIEW" NOT UNIQUE SINCE THERE ARE

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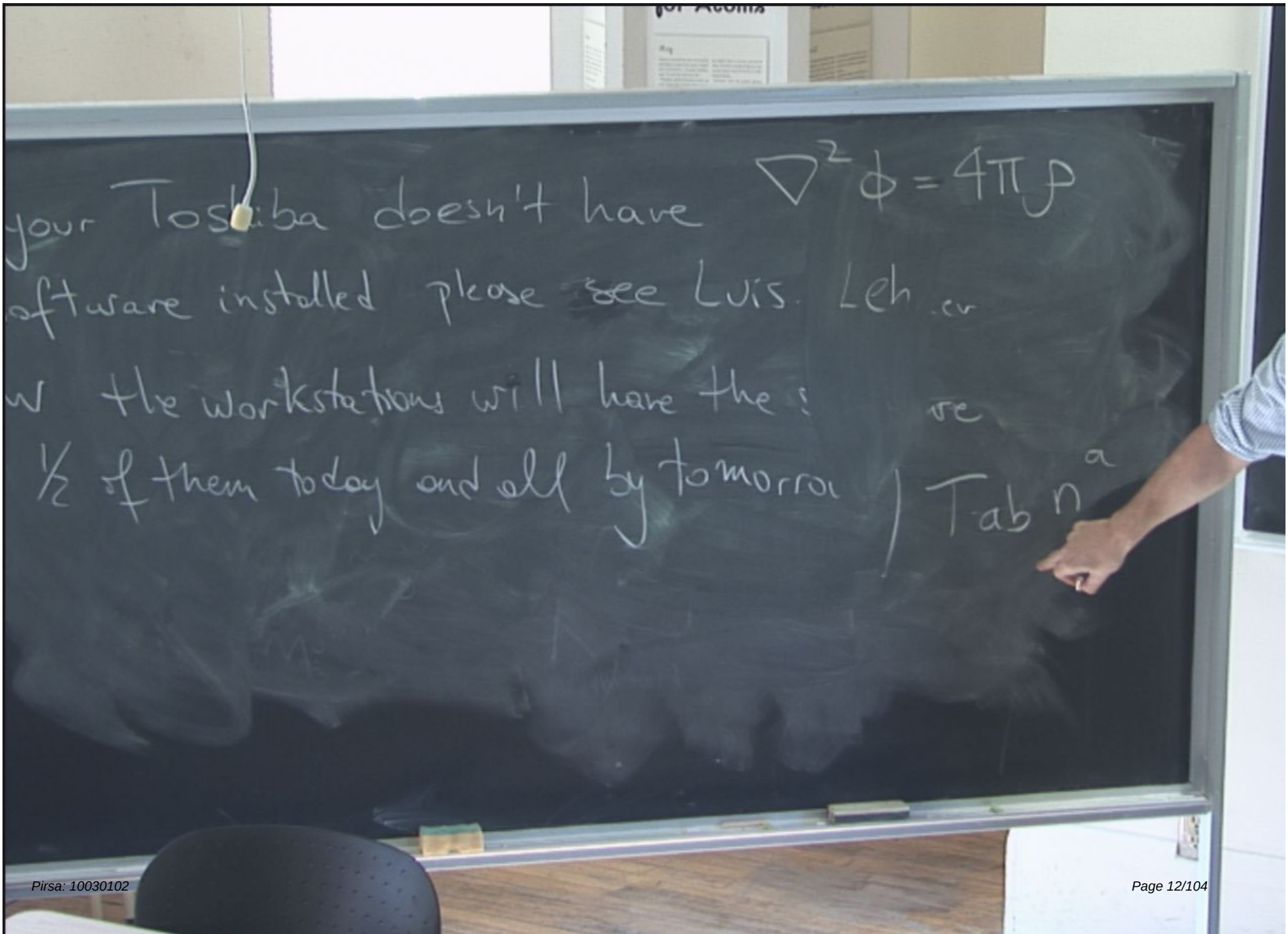
SPLITTING SPACETIME INTO SPACE-PLUS-TIME (THE 3+1 SPLIT)

- SPACETIME IS 4-DIMENSIONAL MANIFOLD M, WITH LORENTZIAN SIGNATURE METRIC g_{ab} $(-+++)$
- INTRODUCE COORDINATES $\{x^\mu\} = \{t, x^i\}$ (MAY NOT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NOTATION / CONVENTIONS:

GREEK INDICES: $(\mu, \nu$ etc.) $0, 1, 2, 3$

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"INTEGER" LATIN " (i, j, k, l, m, n) $1, 2, 3$ (SPATIAL)

WILL ADOPT USUAL EINSTEIN SUMMATION CONVENTION FOR BOTH TYPES!

- DEMAND THAT $t = \text{const}$ SURFACES (HYPERSURFACES), $\Sigma(t)$, ARE SPACELIKE; I.E. IF DISTINCT EVENTS P, P'

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PHY 7E7N THE 3+1 FORMULATION of GR

HAVE COORDS (t, x^i), $(t, x^{i'})$ THEN $ds^2(P, P') > 0$

• SAFEST TO VIEW t AS THE PARAMETER ; MAY NOT (IN FACT, IN GENERAL WILL NOT) HAVE ANY PARTIC. SIGNIFICANCE AS A PHYSICAL TIME - SUCH A NOTION IS LARGELY MEANINGLESS IN GR

• EACH $\Sigma(t)$ IS A DIFFERENTIABLE MANIFOLD IN ITS

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- EACH $\Sigma(t)$ IS A DIFFERENTIABLE MANIFOLD IN ITS OWN RIGHT, WITH A 3-METRIC ${}^{(3)}g_{ij} = {}^{(3)}g_{(ij)}$ WHICH IS INDUCED ON $\Sigma(t)$ BY THE 4-METRIC ${}^{(4)}g_{\mu\nu}$ OF THE ENVELOPING SPACETIME

- ANY GIVEN FOLIATION (= CHOICE OF TIME COORD = CHOICE OF SLICING) ALSO DEFINES A NATURAL, UNIT-NORM VECTOR FIELD, n^μ, WHICH IS NORMAL (ORTHOGONAL) TO THE SLICES, AND POINTS "TO THE FUTURE"

$$n^\mu n_\mu = {}^{(4)}g_{\mu\nu} n^\mu n^\nu = -1$$

SPACETIME DISPLACEMENT IN THE 3+1 SPLIT

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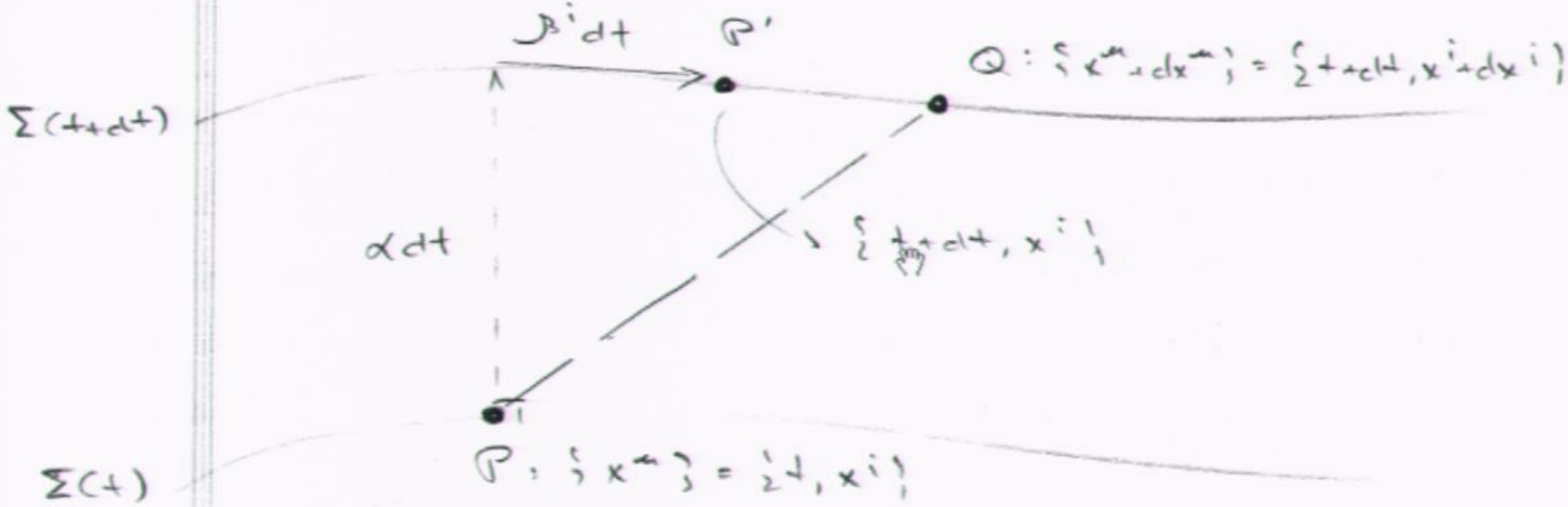
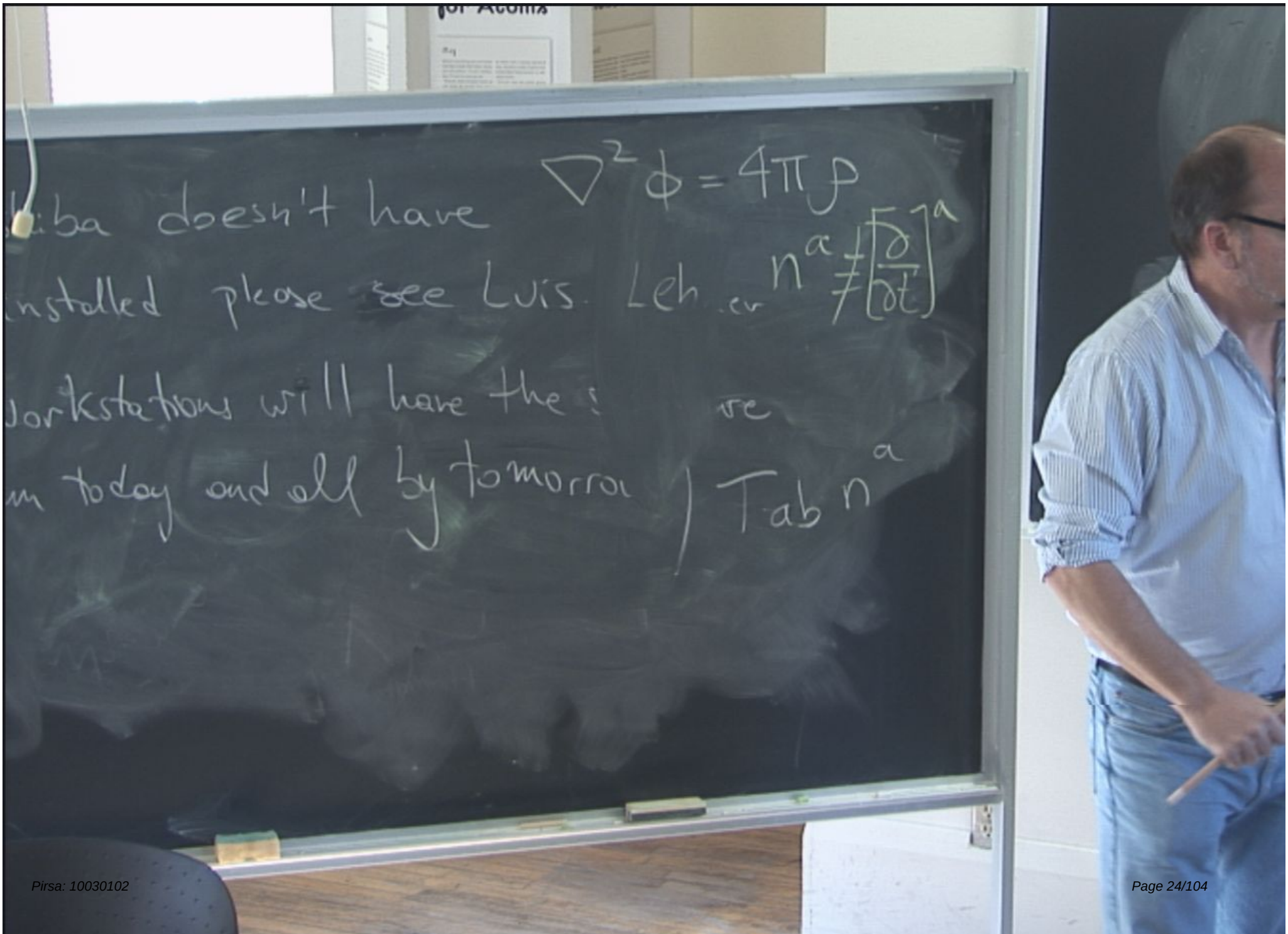


"SPACETIME PYTHAGOREAN THM" (lim $dt \to 0$) $\Rightarrow$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$\Sigma(t)$ $\{ x^\alpha \} = \{ t, x^i \}$

"SPACETIME PYTHAGOREAN THM" $(\lim dt \to 0) \Rightarrow$

$$\text{distance}(P,Q)^2 = {}^{(4)}ds^2$$

$$= -\alpha^2 dt^2 + {}^{(3)}g_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

$$= (-\alpha^2 + {}^{(3)}g_{ij}\beta^i\beta^j)\,dt^2 + 2\,{}^{(3)}g_{ij}\beta^i dx^j dt$$

$$+ {}^{(3)}g_{ij}\,dx^i dx^j \qquad (1)$$

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$$+ {}^{(3)}g_{ij}\,dx^i dx^j \qquad (1)$$

NOMENCLATURE:

$\alpha \equiv \alpha(t, x^i)$: LAPSE FUNCTION ("LAPSE") : GIVES LAPSE OF PROPER TIME PER UNIT COORDINATE TIME FOR AN OBSERVER

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$\beta^i \equiv \beta^i(t, x^i)$: SHIFT VECTOR ("SHIFT") : 3-VECTOR DESCRIBING SHIFTING OF SPATIAL COORDINATES RELATIVE TO "NORMAL PROPAGATION"

TOGETHER $\{\alpha, \beta^i\}$ CONSTITUTE 4-FOLD COORD. FREEDOM of GR

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DUAL VIEWS

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DUAL VIEWS

(1) α, β^i ARE ESSENTIALLY FREELY SPECIFIABLE FCNS (COORDINATE FREEDOM)

(2) SOME PRESCRIPTION FOR α, β^i MUST BE GIVEN

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DUAL VIEWS

(1) α, β^i ARE ESSENTIALLY FREELY SPECIFIABLE FCNS (COORDINATE FREEDOM)

(2) SOME PRESCRIPTION FOR α, β^i MUST BE GIVEN "FROM OUTSIDE" – I.E. E.O.M. (EINSTEIN EQN) ALONE WILL NOT, IN GENERAL, DETERMINE THEM ("GAUGE FIXING")

THE 3+1 FORMULATION of GR

(3)

NOTE: TENSORS SUCH AS β^i ARE DEFINED ON $\Sigma(t)$, AND ARE CALLED SPATIAL TENSORS

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CLEARLY, ${}^{(3)}g_{ij}$ IS A SPATIAL TENSOR; IT HAS AN ASSOCIATED INVERSE ${}^{(3)}g^{ij}$ SATISFYING

$$ {}^{(3)}g^{ij}\,{}^{(3)}g_{jk} = \delta^i{}_k $$

INDICES ON 3-TENSORS ARE RAISED / LOWERED WITH ${}^{(3)}g^{ij}$, ${}^{(3)}g_{ij}$; THUS, WE CAN REWRITE (1) AS

$$ {}^{(4)}ds^2 = (-\alpha^2 + \beta^i\beta_i)\,dt^2 + 2\beta_j\,dx^j\,dt + {}^{(3)}g_{ij}\,dx^i dx^j $$

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$$ {}^{(4)}ds^2 = (-\alpha^2 + \beta^i\beta_i)\,dt^2 + 2\beta_j\,dx^j\,dt + {}^{(3)}g_{ij}\,dx^i dx^j $$

SO, WE HAVE

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$$ {}^{(3)}g^{ij}\, {}^{(3)}g_{jk} = \delta^{i}{}_{k} $$

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$$ {}^{(4)}g_{\mu\nu} = \begin{bmatrix} {}^{(4)}g_{00} & {}^{(4)}g_{0j} \\ {}^{(4)}g_{i0} & {}^{(4)}g_{ij} \end{bmatrix} = \begin{bmatrix} -\alpha^2 + \beta^k \beta_k & \beta_j \\ \beta_i & {}^{(3)}g_{ij} \end{bmatrix} \qquad (2) $$

IN PARTICULAR, NOTE THAT

$$ {}^{(4)}g_{ij} = {}^{(3)}g_{ij} $$

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IN PARTICULAR, NOTE THAT

$$ {}^{(4)}g_{ij} = {}^{(3)}g_{ij} $$

I.E. SPATIAL COVARIANT COMPONENTS OF 4- AND 3-METRICS ARE IDENTICAL

- THIS IS A GENERAL RESULT; GIVEN ANY 4-TENSOR OF TYPE (0, k) (COVARIANT TENSOR), THE SPATIAL COMPONENTS OF THAT TENSOR CAN BE IDENTIFIED AS THE SPATIAL

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$$ {}^{(4)}g_{\mu\nu} = \begin{bmatrix} {}^{(4)}g_{00} & {}^{(4)}g_{0j} \\ {}^{(4)}g_{i0} & {}^{(4)}g_{ij} \end{bmatrix} = \begin{bmatrix} -\alpha^2 + \beta^k \beta_k & \beta_j \\ \beta_i & {}^{(3)}g_{ij} \end{bmatrix} \qquad (2) $$

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WHY? RECALL THAT COVARIANT TENSOR COMPONENTS CAN BE DEFINED IN TERMS OF THE ACTION OF THE TENSOR ON THE COORDINATE BASIS VECTORS ${}^{(a)}\vec{e}_\mu$, $\mu = 0, 1, 2, 3$

E.G. ${}^{(a)}t_{\mu\nu} = {}^{(a)}t\left({}^{(a)}\vec{e}_\mu, {}^{(a)}\vec{e}_\nu\right)$

AND ${}^{(a)}t_{ij} = {}^{(a)}t\left({}^{(a)}\vec{e}_i, {}^{(a)}\vec{e}_j\right)$

BUT CLEARLY THE $\{{}^{(a)}\vec{e}_i\}$ ARE PRECISELY THE COORDINATE BASIS VECTORS $\{{}^{(2)}\vec{e}_i\}$ FOR $\Sigma(t)$ WITH COORDINATES $\{x^i\}$; THUS INTERPRETING ${}^{(a)}t(\ldots)$ AS A 3-TENSOR ${}^{(2)}t(\ldots)$, WE NECESSARILY HAVE

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WHY? RECALL THAT COVARIANT TENSOR COMPONENTS CAN BE DEFINED IN TERMS OF THE ACTION OF THE TENSOR ON THE COORDINATE BASIS VECTORS ${}^{(a)}\vec{e}_\mu$, $\mu = 0, 1, 2, 3$

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E.G. $\quad {}^{(4)}t_{\mu\nu} = {}^{(4)}t\left({}^{(4)}\vec{e}_\mu, {}^{(4)}\vec{e}_\nu\right)$

AND $\quad {}^{(4)}t_{ij} = {}^{(4)}t\left({}^{(4)}\vec{e}_i, {}^{(4)}\vec{e}_j\right)$

BUT CLEARLY THE $\{{}^{(4)}\vec{e}_i\}$ ARE PRECISELY THE COORDINATE TRIAD VECTORS $\{{}^{(3)}\vec{e}_i\}$ FOR $\Sigma(t)$ WITH COORDINATES $\{x^i\}$; THUS INTERPRETING ${}^{(4)}t(\ldots)$ AS A 3-TENSOR ${}^{(3)}t(\ldots)$, WE NECESSARILY HAVE

$$ {}^{(3)}t_{ij} = {}^{(4)}t_{ij} $$

ON THE OTHER HAND, THE SPATIAL MEMBERS $\{{}^{(4)}\underline{\omega}^i\}$ OF THE DUAL BASIS $\{{}^{(4)}\underline{\omega}^\mu\}$, WILL <u>NOT</u>, IN GENERAL COINCIDE WITH $\{{}^{(3)}\underline{\omega}^i\}$ — THE DUAL BASIS ON $\Sigma(t)$ DEPENDS ON HOW $\Sigma(t)$ IS EMBEDDED IN THE S.T.

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$$ {}^{(3)}t_{ij} = {}^{(4)}t_{ij} $$

ON THE OTHER HAND, THE SPATIAL MEMBERS $\{ {}^{(4)}\underline{\omega}^{i} \}$ OF THE DUAL BASIS $\{ {}^{(4)}\underline{\omega}^{m} \}$, WILL NOT, IN GENERAL COINCIDE WITH $\{ {}^{(3)}\underline{\omega}^{i} \}$ – THE DUAL BASIS ON $\Sigma(t)$ DEPENDS ON HOW $\Sigma(t)$ IS EMBEDDED IN THE S.T.

• THUS, IN GENERAL

$$ {}^{(3)}t^{ij} \neq {}^{(4)}t^{ij} $$

EXAMPLE: CONSIDER THE INVERSE 4-METRIC COMPONENTS FROM (2)

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THE DUAL BASIS $\{ {}^{(4)}\underline{\omega}^{\mu} \}$, WILL NOT, IN GENERAL COINCIDE WITH $\{ {}^{(3)}\underline{\omega}^{i} \}$ — THE DUAL BASIS ON $\Sigma(t)$ DEPENDS ON HOW $\Sigma(t)$ IS EMBEDDED IN THE S.T.

THUS, IN GENERAL

$$ {}^{(3)}t^{ij} \neq {}^{(4)}t^{ij} $$

EXAMPLE: CONSIDER THE INVERSE 4-METRIC COMPONENTS FROM (2)

$$ {}^{(4)}g^{\mu\nu} = \begin{bmatrix} -1/\alpha^2 & \beta^j/\alpha^2 \\ \beta^i/\alpha^2 & {}^{(3)}g^{ij} - \beta^i\beta^j/\alpha^2 \end{bmatrix} \qquad (3) $$

(EXERCISE: VERIFY)

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COORDINATES $\{x^i\}$; THUS INTERPRETING ${}^{(a)}t(\ldots)$ AS A 3-TENSOR ${}^{(3)}t(\ldots)$, WE NECESSARILY HAVE

$$ {}^{(3)}t_{ij} = {}^{(a)}t_{ij} $$

ON THE OTHER HAND, THE SPATIAL TENSORS $\{{}^{(a)}\underline{\omega}^i\}$ OF THE DUAL BASIS $\{{}^{(a)}\underline{\omega}^m\}$, WILL NOT, IN GENERAL COINCIDE WITH $\{{}^{(3)}\underline{\omega}^i\}$ — THE DUAL BASIS ON $\Sigma(t)$ DEPENDS ON HOW $\Sigma(t)$ IS EMBEDDED IN THE S.T.

• THUS, IN GENERAL

$$ {}^{(3)}t^{ij} \neq {}^{(a)}t^{ij} $$

EXAMPLE: CONSIDER THE INVERSE 4-TENSOR COMPONENTS

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COORDINATES $\{x^i\}$; THUS INTERPRETING ${}^{(a)}t(\ldots)$ AS A 3-TENSOR ${}^{(3)}t(\ldots)$, WE NECESSARILY HAVE

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• THUS, IN GENERAL

$$ {}^{(3)}t^{ij} \neq {}^{(a)}t^{ij} $$

EXAMPLE: CONSIDER THE INVERSE 4-TENSOR COMPONENTS

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AS A 3-TENSOR $t(\ldots)$, WE NECESSARILY HAVE

$$ {}^{(3)}t_{ij} = {}^{(4)}t_{ij} $$

ON THE OTHER HAND, THE SPATIAL MEMBERS $\{ {}^{(4)}\underline{\omega}^{i} \}$ OF THE DUAL BASIS $\{ {}^{(4)}\underline{\omega}^{m} \}$, WILL NOT, IN GENERAL COINCIDE WITH $\{ {}^{(3)}\underline{\omega}^{i} \}$ – THE DUAL BASIS ON $\Sigma(t)$ DEPENDS ON HOW $\Sigma(t)$ IS EMBEDDED IN THE S.T.

THUS, IN GENERAL

$$ {}^{(3)}t^{ij} \neq {}^{(4)}t^{ij} $$

EXAMPLE: CONSIDER THE INVERSE 4-METRIC COMPONENTS g^{ab} (2)

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THE DUAL BASIS $\{ {}^{(4)}\underline{\omega}^{\mu} \}$, WILL NOT, IN GENERAL COINCIDE WITH $\{ {}^{(3)}\underline{\omega}^{i} \}$ — THE DUAL BASIS ON $\Sigma(t)$ DEPENDS ON HOW $\Sigma(t)$ IS EMBEDDED IN THE S.T.

THUS, IN GENERAL

$$ {}^{(3)}t^{ij} \neq {}^{(4)}t^{ij} $$

EXAMPLE: CONSIDER THE INVERSE 4-METRIC COMPONENTS FROM (2)

$$ {}^{(4)}g^{\mu\nu} = \begin{bmatrix} -1/\alpha^2 & \beta^j/\alpha^2 \\ \beta^i/\alpha^2 & {}^{(3)}g^{ij} - \beta^i\beta^j/\alpha^2 \end{bmatrix} \qquad (3) $$

(EXERCISE: VERIFY)

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$$ {}^{(3)}t^{ij} \neq {}^{(4)}t^{ij} $$

EXAMPLE: CONSIDER THE INVERSE 4-METRIC COMPONENTS FROM (2)

$$ {}^{(4)}g^{\mu\nu} = \begin{bmatrix} -1/\alpha^2 & \beta^j/\alpha^2 \\ \beta^i/\alpha^2 & {}^{(3)}g^{ij} - \beta^i\beta^j/\alpha^2 \end{bmatrix} \qquad (3) $$

(EXERCISE: VERIFY)

- FROM (2) WE CAN ALSO COMPUTE THE USEFUL RESULT

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$$ {}^{(3)}t^{ij} \neq {}^{(4)}t^{ij} $$

EXAMPLE: CONSIDER THE INVERSE 4-METRIC COMPONENTS FROM (2)

$$ {}^{(4)}g^{\mu\nu} = \begin{bmatrix} -1/\alpha^2 & \beta^j/\alpha^2 \\ \beta^i/\alpha^2 & {}^{(3)}g^{ij} - \beta^i\beta^j/\alpha^2 \end{bmatrix} \qquad (3) $$

(EXERCISE: VERIFY)

WHY BOTHER THE 3+1 FORMULATION of GR

- FROM (2) WE CAN ALSO COMPUTE THE USEFUL RESULT

$$ \sqrt{{}^{(4)}\ldots} \qquad \sqrt{{}^{(3)}\ldots} $$

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PHY 387M THE 3+1 FORMULATION of GR

- FROM (2) WE CAN ALSO COMPUTE THE USEFUL RESULT

$$\sqrt{-{}^{(4)}g} = \alpha\sqrt{{}^{(3)}g} \qquad (4)$$

THE NORMAL VECTOR FIELD n^μ

- EASIEST TO START WITH ASSOC. DUAL-VECTOR (ONE-FORM) FIELD, n_μ

GEOM. INTERP of DUAL-VECTOR FIELD: LEVEL SURFACES of SCALAR FUNCTION → DUAL MOTION TO

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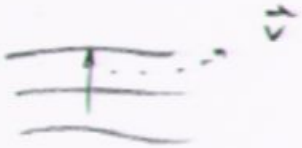
THE NORMAL VECTOR FIELD n^μ

- EASIEST TO START WITH ASSOC. DUAL-VECTOR (ONE-FORM) FIELD, n_μ

GEOM. INTERP of DUAL VECTOR FIELD: LEVEL SURFACES of SCALAR FUNCTION → DUAL MOTION TO "INFINITESIMAL DISPLACEMENT" (VECTOR)

$\underline{d}f$  $\vec{v}$ $2\,\underline{d}f$

$\langle \vec{v}, \underline{d}f \rangle = v^\mu (df)_\mu$ = # of level surfaces "pierced" by $\vec{v}$

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$\underline{d}t \qquad 2\,\underline{d}t$

$$\langle \vec{v}, \underline{d}f \rangle = v^{\mu} (df)_{\mu} = \#\text{ of level surfaces "pierced" by } \vec{v}$$

HERE, OUR SCALAR FUNCTION IS THE TIME COORDINATE t, WITH ASSOCIATED DUAL-VECTOR FIELD $\underline{d}t$, THEN

$$\underline{n} \propto \underline{d}t$$

OR IN COMPONENT FORM

$$n_{\mu} = (n_0, 0, 0, 0)$$

THEN, FROM

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HERE, OUR SCALAR FUNCTION IS THE TIME COORDINATE t, WITH ASSOCIATED DUAL-VECTOR FIELD $\underline{dt}$, THEN

$$\underline{n} \propto \underline{dt}$$

OR IN COMPONENT FORM

$$n_\mu = (n_0, 0, 0, 0)$$

THEN, FROM

$$g^{\mu\nu} n_\mu n_\nu = -1 \qquad (4)$$

WE HAVE

SIGN CHOSEN SO THAT n^μ IS "FUTURE-DIRECTED"

$$n_\mu = (-\alpha, 0, 0, 0) \qquad (5)$$

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, WITH ASSOCIATED DUAL-VECTOR FIELD $\underline{dt}$, THEN

$$\underline{n} \propto \underline{dt}$$

OR IN COMPONENT FORM

$$n_\mu = (n_0, 0, 0, 0)$$

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$$\langle \vec{v}, \underline{d}f \rangle = v^{\mu} (df)_{\mu} = \#\text{ of level surfaces "pierced" by } \vec{v}$$

- HERE, OUR SCALAR FUNCTION IS THE TIME COORDINATE t, WITH ASSOCIATED DUAL-VECTOR FIELD $\underline{dt}$, THEN

$$\underline{n} \propto \underline{dt}$$

OR IN COMPONENT FORM

$$n_{\mu} = (n_0, 0, 0, 0)$$

THEN, FROM

$$(A) \quad g^{\mu\nu} n_{\mu} n_{\nu} = -1$$

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$\underline{d}f$ $2\,\underline{d}f$

$$\langle \vec{v}, \underline{d}f \rangle = v^{\mu} (df)_{\mu} = \#\text{ of level surfaces "pierced" by } \vec{v}$$

HERE, OUR SCALAR FUNCTION IS THE TIME COORDINATE t, WITH ASSOCIATED DUAL-VECTOR FIELD $\underline{d}t$, THEN

$$\underline{n} \propto \underline{d}t$$

OR IN COMPONENT FORM

$$n_{\mu} = (n_0, 0, 0, 0)$$

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WE HAVE

SIGN CHOSEN SO THAT n^μ IS "FUTURE-DIRECTED"

$$n_\mu = (-\alpha, 0, 0, 0) \qquad (5)$$

AND THEN

$$n^\mu = {}^{(4)}g^{\mu\nu} n_\nu = \left(\frac{1}{\alpha}, -\frac{\beta^i}{\alpha} \right) \qquad (6)$$

EXTRINSIC CURVATURE

- THE INTRINSIC GEOMETRY OF $\Sigma(t)$ IS DESCRIBED BY ${}^{(3)}g_{ij}$ WHICH ENCODES ALL GEOM. INFO. WHICH MAY BE DEDUCED BY MAKING MEASUREMENTS ON $\Sigma(t)$ ALONE

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OR IN COMPONENT FORM

$$n_\mu = (n_0, 0, 0, 0)$$

THEN, FROM

$$\overset{(A)}{g^{\mu\nu}} n_\mu n_\nu = -1$$

WE HAVE

SIGN CHOSEN SO THAT n^μ IS "FUTURE-DIRECTED"

$$n_\mu = (-\alpha, 0, 0, 0) \qquad (5)$$

AND THEN

$$n^\mu \overset{(A)}{=} g^{\mu\nu} n_\nu = \left(\frac{1}{\alpha} \quad -\frac{\beta^i}{\alpha} \right) \qquad (6)$$

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$$g^{\mu\nu}\, n_\mu n_\nu = -1$$

WE HAVE

SIGN CHOSEN SO THAT n^μ IS "FUTURE-DIRECTED"

$$n_\mu = (-\alpha, 0, 0, 0) \qquad (5)$$

AND THEN

$$n^\mu = {}^{(4)}g^{\mu\nu} n_\nu = \left(\frac{1}{\alpha}, -\frac{\beta^i}{\alpha} \right) \qquad (6)$$

EXTRINSIC CURVATURE

- THE INTRINSIC GEOMETRY OF $\Sigma(t)$ IS DESCRIBED BY ${}^{(3)}g_{ij}$ WHICH ENCODES ALL GEOM. INFO. WHICH MAY BE DEDUCED BY TAKING MEASUREMENTS ON $\Sigma(t)$ ALONE

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EXTRINSIC CURVATURE

- THE INTRINSIC GEOMETRY OF $\Sigma(t)$ IS DESCRIBED BY g_{ij} WHICH ENCODES ALL GEOM. INFO. WHICH MAY BE DEDUCED BY MAKING MEASUREMENTS ON $\Sigma(t)$ ALONE

- HOWEVER, A GIVEN 3-GEOMETRY (SLICE, HYPERSURFACE) MAY BE EMBEDDED IN SPACETIME IN INFINITELY MANY DISTINGUISHABLE (BY 4-D MEASUREMENTS) WAYS

EXAMPLE: 2D EMBEDDED IN 3D; FLAT SURFACE EMBEDDED WITH / WITHOUT EXTRINSIC CURVATURE

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- THE MANNER IN WHICH $\Sigma(t)$ IS EMBEDDED CAN BE CHARACTERIZED BY INVESTIGATING THE CHANGE IN THE DIRECTION OF THE NORMAL FIELD AS A FCN OF POS ON $\Sigma(t)$ – THIS DEFINES THE EXTRINSIC CURVATURE TENSOR (AKA THE SECOND FUNDAMENTAL FORM)

$$K_{ij} = -\nabla_i n_j = -\nabla_{(i} n_{j)} \qquad (7)$$

ANY 3+1 THE 3+1 FORMULATION OF GR

5

$$\nabla_i n_j = \partial_i n_j - \Gamma^{\mu}{}_{ij} n_{\mu}$$

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THE SECOND FUNDAMENTAL FORM

$$K_{ij} = -\nabla_i n_j = -\nabla_{(i} n_{j)} \qquad (7)$$

WHY BEGIN THE 3+1 FORMULATION of GR

5

$$\nabla_i n_j = \partial_i n_j - \Gamma^{\mu}{}_{ij} n_{\mu}$$

BUT $n_\mu = (-\alpha, 0, 0, 0)$ SO

$$\nabla_i n_j = \alpha^{(4)}\Gamma^{0}{}_{ij}$$

$$= \alpha\left({}^{(4)}g^{0\nu}\,{}^{(4)}\Gamma_{\nu ij} + {}^{(4)}g^{0\ell}\,{}^{(4)}\Gamma_{\ell ij}\right)$$

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THE SECOND FUNDAMENTAL FORM

$$K_{ij} = -\nabla_i n_j = -\nabla_{(i} n_{j)} \qquad (7)$$

PART 3 THE 3+1 FORMULATION of GR

(5)

$$\nabla_i n_j = \partial_i n_j - \Gamma^{\mu}{}_{ij} n_{\mu}$$

BUT $n_\mu = (-\alpha, 0, 0, 0)$ SO

$$\nabla_i n_j = \alpha^{(4)}\Gamma^{0}{}_{ij}$$

$$= \alpha\left({}^{(4)}g^{00}\,{}^{(4)}\Gamma_{0ij} + {}^{(4)}g^{0l}\,{}^{(4)}\Gamma_{lij}\right)$$

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THE 3+1 FORMULATION of GR

(5)

$$\nabla_i n_j = \partial_i n_j - \Gamma^{\mu}{}_{ij} n_\mu$$

BUT $n_\mu = (-\alpha, 0, 0, 0)$ SO

$$\nabla_i n_j = \alpha \, {}^{(4)}\Gamma^{0}{}_{ij}$$

$$= \alpha \left({}^{(4)}g^{0\nu} \, {}^{(4)}\Gamma_{\nu ij} + {}^{(4)}g^{0\ell} \, {}^{(4)}\Gamma_{\ell ij} \right)$$

$$\Gamma_{0ij} = \frac{1}{2} \left({}^{(4)}g_{0i,j} + {}^{(4)}g_{0j,i} - \underline{{}^{(4)}g_{ij,0}} \right)$$

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PART 3 THE 3+1 FORMULATION of GR

$$\nabla_i n_j = \partial_i n_j - \Gamma^{\mu}{}_{ij} n_\mu$$

BUT $n_\mu = (-\alpha, 0, 0, 0)$ SO

$$\nabla_i n_j = \alpha \, {}^{(4)}\Gamma^{0}{}_{ij}$$

$$= \alpha \left({}^{(4)}g^{0\nu} \, {}^{(4)}\Gamma_{\nu ij} + {}^{(4)}g^{0\ell} \, {}^{(4)}\Gamma_{\ell ij} \right)$$

$$\Gamma_{0ij} = \frac{1}{2} \left({}^{(4)}g_{0i,j} + {}^{(4)}g_{0j,i} - {}^{(4)}g_{ij,0} \right)$$

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$$= \alpha \left({}^{(4)}g^{0\alpha}\, {}^{(4)}\Gamma_{0ij} + {}^{(4)}g^{0\ell}\, {}^{(4)}\Gamma_{\ell ij} \right)$$

$$\Gamma_{0ij} = \frac{1}{2}\left({}^{(4)}g_{0i,j} + {}^{(4)}g_{0j,i} - {}^{(4)}g_{ij,0} \right)$$

THUS

$$K_{ij} = -\frac{1}{2\alpha}\frac{\partial\, {}^{(3)}g_{ij}}{\partial t} + \ldots$$

SO WE CAN VIEW THE EXTRINSIC CURVATURE AS THE "VELOCITY" OF THE 3-METRIC ${}^{(3)}g_{ij}$ ("CONJUGATE MOMENTA")

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THE 3+1 EINSTEIN EQUATIONS

These notes rework the calculation of the 3+1 equations as presented in *Kinematics and Dynamics of General Relativity*, by J. W. York, Jr., which itself is contained in the volume *Sources of Gravitational Radiation*, edited by L. Smarr. Many calculational details omitted from that source are included here.

(**Note:** A colleague of mine, who will remain anonymous this week, has used these notes to demonstrate the power of his symbolic manipulation software and has found at least one error amongst the many "intermediate" results that are derived below. There is a $20 reward for the PSI student who first identifies such an error and who submits a hand-written explanation of the gaffe, along with the corrected expression, to the tutor.)

1) Foliations and Normals

As before, we consider a spacetime M with metric g_{ab} which is sliced into a foliation $\{\Sigma\}$ defined by the isosurfaces of a scalar field τ (the time parameter). Then the spacelike hypersurfaces are, at least locally, described by a *closed* one-form (dual vector field), Ω_a:

$$\Omega_a = \nabla_a \tau \,. \tag{1}$$

Note that since Ω_a is the gradient of a scalar function, and ∇_a is torsion-free, we have

$$\nabla_{[a}\Omega_{b]} = \nabla_{[a}\nabla_{b]}\tau = 0 \,. \tag{2}$$

The norm of Ω_a is given by

$$g^{ab}\Omega_a\Omega_b = -\alpha^{-2} \,, \tag{3}$$

where α is the lapse function, as previously. Thus we can construct the unit-norm dual-vector field,

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$$n_a = -\alpha\Omega_a = -\alpha\nabla_a\tau \,, \tag{4}$$

where the sign is chosen so that the associated unit-norm, hypersurface-orthogonal vector field, n^a,

$$n^a = g^{ab}n_b \,, \tag{5}$$

is future-directed. Note that we can view n^a as the 4-velocity field of a congruence of observers moving orthogonally to the slices (not necessarily coordinate-stationary). Such observers will have a 4-acceleration, a^b given by

$$a^b = n^a\nabla_a n^b \,. \tag{6}$$

2) The Projection Tensor and the Spatial Metric

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2) The Projection Tensor and the Spatial Metric

In the derivation of the 3+1 form of the Einstein equations, we will necessarily be interested in decomposing various spacetime tensors into hypersurface-tangential ("spatial") and hypersurface-orthogonal ("temporal") pieces. Determining the "temporal" part of a tensor is straightforward, we simply contract with n^a. York throws in a slight twist by introducing a relative minus depending on whether a vector or covector index is being projected. Thus, for a vector field, W^a, we define

$$W^{\hat{n}} = -W^a n_a\,, \tag{7}$$

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$$W^{\hat{n}} = -W^a n_a \,, \tag{7}$$

and, in general, any upstairs $\hat{n}$ index denotes that the original tensor index has been contracted with $-n_a$. On the other hand, for a dual vector field, W_a, we define

$$W_{\hat{n}} = +W_a n^a \,, \tag{8}$$

and then any downstairs $\hat{n}$ index denotes contraction with $+n^a$. To determine the spatial parts of tensors, it is convenient to introduce the notion of a *projection tensor* which, as the name suggests, projects tensors onto the hypersurface. The mixed form of this two-rank tensor is denoted $\perp^a{}_b$ and is defined by

$$\perp^a{}_b \equiv \delta^a{}_b + n^a n_b \,. \tag{9}$$

Note the relative "+" between the identity tensor and $n^a n_b$ which follows from the Lorenztian signature on spacetime. By construction, we have

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$$\perp n^a \equiv \perp^a{}_b n^b = (\delta^a{}_b + n^a n_b)\, n^b = n^a - n^a = 0 \,, \tag{10}$$

where we have also introduced the notation that a $\perp$ with no indices, operating on an arbitrary tensor expression, means apply the projection tensor to very free tensor index in the expression. Thus, for example

$$\perp S^a{}_{bc} \equiv \perp^a{}_d \perp^e{}_b \perp^f{}_c S^d{}_{ef} \,. \tag{11}$$

Any tensor which has had all its free indices projected in this manner is called a *spatial tensor*. It is worth emphasizing the rather obvious point that $\perp$ applied to any tensor expression of the form "tensor product of n^a (or n_a) and something else", vanishes. We will use this fact many times in the following.

If we apply the projection tensor to the spacetime metric g_{ab} itself (which is clearly the same

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If we apply the projection tensor to the spacetime metric g_{ab} itself (which is clearly the same thing as lowering an index on the projection tensor) we get the (spatial) metric, γ_{ab}, on the hypersurfaces:

$$\gamma_{ab} = g_{ab} + n_a n_b \,. \tag{12}$$

Similarly, the contravariant form of the spatial metric is given by

$$\gamma^{ab} = g^{ac} g^{bd} \gamma_{cd} = g^{ab} + n^a n^b \,. \tag{13}$$

Note that *all* tensor indices continue to be raised and lowered with the *spacetime* metric, g_{ab}, and that γ_{ab} and γ^{ab} are *not* inverses. (In fact, of course, the mixed form, $\gamma^a{}_b$, of the spatial metric is just the projection tensor $\perp^a{}_b$.) We also have

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$$\mathrm{Tr} \perp \equiv \perp^a{}_a = \delta^a{}_a + n^a n_a = 4 - 1 = 3 \,. \tag{14}$$

Note also, however, that *spatial* tensors can equally well have their indices raised and lowered with γ_{ab}.

3) The Spatial Derivative Operator and Curvature Tensor

We can also use the projection tensor to define a natural derivative operator, D_a, for spatial tensors. Formally, we define

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$$D_a \equiv \perp \nabla_a\,, \tag{15}$$

so that for a scalar field ψ, for example, we have

$$D_a\psi \equiv \perp \nabla_a \psi = \perp^b{}_a \nabla_b \psi\,. \tag{16}$$

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while for a (spatial) vector field, W^a

$$D_a W^b \equiv \perp \nabla_a W^b = \perp^c{}_a \perp^b{}_d \nabla_c W^d . \tag{17}$$

The action of D_a on an arbitrary spatial tensor is then defined in the obvious fashion. D_a is the natural derivative operator for spatial tensors since it is compatible with the spatial metric, i.e.

$$D_a \gamma_{bc} = \perp \nabla_a \gamma_{bc} = \perp \nabla_a (g_{bc} + n_b n_c) = \perp \nabla_a (n_b n_c) = \perp (n_c \nabla_a n_b + n_b \nabla_a n_c) = 0 . \tag{18}$$

$D_a \gamma^{bc} = 0$ follows from an exactly parallel computation, or, more directly, simply by raising indices (using either metric!) on the above expression.

The intrinsic curvature of the three-dimensional hypersurfaces is given by the Riemann tensor associated with the spatial metric and is denoted $\mathcal{R}_{abc}{}^d$. It may be defined via its action on an arbitrary spatial dual-vector, W_a:

$$(D_a D_b - D_b D_a) W_c = \mathcal{R}_{abc}{}^d W_d . \tag{19}$$

$\mathcal{R}_{abc}{}^d$ is, of course, a spatial-tensor itself, and hence satisfies

$$\mathcal{R}_{abc}{}^d n^a = \mathcal{R}_{abc}{}^d n^b = \mathcal{R}_{abc}{}^d n^c = \mathcal{R}_{abc}{}^d n_d = 0 . \tag{20}$$

In addition, $\mathcal{R}_{abcd}$ has the usual symmetries:

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4) The Extrinsic Curvature Tensor

The embedding of the slices in the spacetime is described by the *extrinsic curvature* tensor. Before defining this tensor and discussing its properties, we establish two useful results concerning derivatives of the normal vector field. The first of these is

$$\perp \nabla_{[a} n_{b]} = 0 \,. \tag{26}$$

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To see this, start from

$$\begin{aligned}
\perp\nabla_a n_b &= (\delta^c{}_a + n^c n_a)(\delta^d{}_b + n^d n_b)\nabla_c n_d \\
&= \nabla_a n_b + n_a n^c \nabla_c n_b + n_b n^d \nabla_a n_d + n_a n^c n_b n^d \nabla_c n_d \\
&= \nabla_a n_b + n_a n^c \nabla_c n_b \qquad (27) \\
&= \nabla_a n_b + n_a a_b\,, \qquad (28)
\end{aligned}$$

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where we have used

$$n^d \nabla_a n_d = n_d \nabla_a n^d = \frac{1}{2} \nabla_a \left(n^d n_d \right) = \frac{1}{2} \nabla_a (-1) = 0 \,, \tag{29}$$

to eliminate the last two terms in the second line. We now consider each of the two terms of (27) in turn. Using (4), we have

$$\nabla_a n_b = -\nabla_a (\alpha \Omega_b) = -(\nabla_a \alpha)\, \Omega_b - \alpha (\nabla_a \Omega_b) = -(\nabla_a \alpha)\, \Omega_b - \alpha (\nabla_a \nabla_b \tau) \,, \tag{30}$$

so that

$$\nabla_{[a} n_{b]} = -(\nabla_{[a} \alpha)\, \Omega_{b]} \,. \tag{31}$$

Next, again using (4), we have

$$\begin{aligned} n_a n^c \nabla_c n_b &= \alpha^2 \Omega_a \Omega^c \left(-(\nabla_c \alpha)\, \Omega_b - \alpha (\nabla_c \Omega_b) \right) \\ &= -\alpha^2 \Omega^c (\nabla_c \alpha)\, \Omega_a \Omega_b - \alpha^3 \Omega_a \Omega^c \nabla_c \Omega_b \\ &= -\alpha^2 \Omega^c (\nabla_c \alpha)\, \Omega_a \Omega_b - \alpha^3 \Omega_a \Omega^c \nabla_b \Omega_c \\ &= -\alpha^2 \Omega^c (\nabla_c \alpha)\, \Omega_a \Omega_b - \frac{\alpha^3}{2} \Omega_a \nabla_b \left(-\alpha^{-2} \right) \\ &= -\alpha^2 \Omega^c (\nabla_c \alpha)\, \Omega_a \Omega_b - (\nabla_b \alpha)\, \Omega_a \,, \end{aligned} \tag{32}$$

where we have used (2) in going from the second to third line, and (3) in going from the third to fourth. Thus,

$$n_{[a} n^c \nabla_c n_{b]} = -(\nabla_{[b} \alpha)\, \Omega_{a]} = +(\nabla_{[a} \alpha)\, \Omega_{b]} \,, \tag{33}$$

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$$
\begin{aligned}
n_a n^c \nabla_c n_b &= \alpha^2 \Omega_a \Omega^c \left(-\left(\nabla_c \alpha\right) \Omega_b - \alpha \left(\nabla_c \Omega_b\right) \right) \\
&= -\alpha^2 \Omega^c \left(\nabla_c \alpha\right) \Omega_a \Omega_b - \alpha^3 \Omega_a \Omega^c \nabla_c \Omega_b \\
&= -\alpha^2 \Omega^c \left(\nabla_c \alpha\right) \Omega_a \Omega_b - \alpha^3 \Omega_a \Omega^c \nabla_b \Omega_c \\
&= -\alpha^2 \Omega^c \left(\nabla_c \alpha\right) \Omega_a \Omega_b - \frac{\alpha^3}{2} \Omega_a \nabla_b \left(-\alpha^{-2}\right) \\
&= -\alpha^2 \Omega^c \left(\nabla_c \alpha\right) \Omega_a \Omega_b - \left(\nabla_b \alpha\right) \Omega_a \,, \qquad (32)
\end{aligned}
$$

where we have used (2) in going from the second to third line, and (3) in going from the third to fourth. Thus,

$$
n_{[a} n^c \nabla_c n_{b]} = -\left(\nabla_{[b} \alpha\right) \Omega_{a]} = +\left(\nabla_{[a} \alpha\right) \Omega_{b]} \,, \qquad (33)
$$

which, when combined with (31), immediately establishes (26) from (27).

Our second useful preliminary result relates the 4-acceleration a_b to the derivative of the lapse function. Specifically, we have

$$
a_b = D_b \ln \alpha \,. \qquad (34)
$$

To see this, we reexpress the right and left hand sides to show that they are indeed equal. On the one hand we have from (32), and again using (4),

$$
a_b = n^c \nabla_c n_b = \alpha \Omega^c \left(\nabla_c \alpha\right) \Omega_b + \alpha^{-1} \nabla_b \alpha \,, \qquad (35)
$$

while on the other we have

$$
D_b \ln \alpha = \perp^c{}_b \nabla_c \ln \alpha = \left(\delta^c{}_b + n^c n_b\right) \left(\alpha^{-1} \nabla_c \alpha\right) = \alpha \Omega^c \left(\nabla_c \alpha\right) \Omega_b + \alpha^{-1} \nabla_b \alpha \,. \qquad (36)
$$

Recalling our first preliminary result (26), we now define the extrinsic curvature tensor, K_{ab}:

$$
K_{ab} = K_{(ab)} = -\perp \nabla_{(a} n_{b)} = -\perp \nabla_a n_b \,. \qquad (37)
$$

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$$
\begin{aligned}
&= -\alpha^2\Omega^c\left(\nabla_c\alpha\right)\Omega_a\Omega_b - \alpha^3\Omega_a\Omega^c\nabla_b\Omega_c \\
&= -\alpha^2\Omega^c\left(\nabla_c\alpha\right)\Omega_a\Omega_b - \frac{\alpha^3}{2}\Omega_a\nabla_b\left(-\alpha^{-2}\right) \\
&= -\alpha^2\Omega^c\left(\nabla_c\alpha\right)\Omega_a\Omega_b - \left(\nabla_b\alpha\right)\Omega_a\,, \qquad (32)
\end{aligned}
$$

where we have used (2) in going from the second to third line, and (3) in going from the third to fourth. Thus,

$$n_{[a}n^c\nabla_c n_{b]} = -\left(\nabla_{[b}\alpha\right)\Omega_{a]} = +\left(\nabla_{[a}\alpha\right)\Omega_{b]}\,, \qquad (33)$$

which, when combined with (31), immediately establishes (26) from (27).

Our second useful preliminary result relates the 4-acceleration a_b to the derivative of the lapse function. Specifically, we have

$$a_b = D_b \ln\alpha\,. \qquad (34)$$

To see this, we reexpress the right and left hand sides to show that they are indeed equal. On the one hand we have from (32), and again using (4),

$$a_b = n^c\nabla_c n_b = \alpha\Omega^c\left(\nabla_c\alpha\right)\Omega_b + \alpha^{-1}\nabla_b\alpha\,, \qquad (35)$$

while on the other we have

$$D_b\ln\alpha = \perp^c{}_b\nabla_c\ln\alpha = \left(\delta^c{}_b + n^c n_b\right)\left(\alpha^{-1}\nabla_c\alpha\right) = \alpha\Omega^c\left(\nabla_c\alpha\right)\Omega_b + \alpha^{-1}\nabla_b\alpha\,. \qquad (36)$$

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Using this definition and (28), we have

$$\nabla_a n_b = -K_{ab} - n_a a_b \qquad (38)$$

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$$D_b \ln \alpha = \perp^c{}_b \nabla_c \ln \alpha = (\delta^c{}_b + n^c n_b) \left(\alpha^{-1} \nabla_c \alpha \right) = \alpha \Omega^c (\nabla_c \alpha) \Omega_b + \alpha^{-1} \nabla_b \alpha \,. \tag{36}$$

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which rather explicitly displays the decomposition of the derivative of the normal field into a hypersurface-tangential piece—the extrinsic curvature, and a hypersurface-orthogonal piece—the 4-acceleration.

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Equations (52) and (54) are known as the *Gauss-Codazzi* equations.

6) The Constraint Equations

We are now nearly ready to derive the constraint equations. We begin by noting that, as is easily verified, a generic type (0,2) symmetric tensor, $\sigma_{ab} = \sigma_{(ab)}$ has the following 3+1 decomposition:

$$\sigma_{ab} = \perp\sigma_{ab} - 2n_{(a}\perp\sigma_{b)\hat{n}} + n_a n_b \sigma_{\hat{n}\hat{n}} \,. \tag{55}$$

We define the following projections of the stress tensor, T_{ab}

$$\rho \equiv T_{\hat{n}\hat{n}} = T_{ab}n^a n^b \,, \tag{56}$$

$$j^a \equiv \perp T^{a\hat{n}} = -\perp(T^{ab}n_b) \,, \tag{57}$$

$$S_{ab} \equiv \perp T_{ab} \,. \tag{58}$$

ρ, j^a and S_{ab} may be interpreted as the local energy density, momentum density and spatial stress tensor, respectively, as measured by observers moving orthogonally to the slices.

We now consider

$$\begin{aligned} \perp R_{ab} &= \perp(g^{cd}R_{acbd}) \\ &= \perp(\gamma^{cd}R_{acbd}) - \perp(n^c n^d R_{acbd}) \\ &= \perp(\gamma^{cd}R_{acbd}) - \perp R_{a\hat{n}b\hat{n}} \,. \end{aligned} \tag{59}$$

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where

$$K \equiv K^a{}_a\,, \tag{73}$$

is the trace of the extrinsic curvature tensor. Thus, we find from (71) and (72)

$$\mathcal{R} + K^2 - K^a{}_b K^b{}_a = 16\pi\rho\,, \tag{74}$$

which is known as the *Hamiltonian constraint*.

We now consider the Einstein equation in the form

$$G^{ab} = R^{ab} - \frac{1}{2} g^{ab} R = 8\pi T^{ab}\,, \tag{75}$$

and contract one index with $-n_a$ (recall the convention given by equation (7)):

$$G^{a\hat{n}} = R^{a\hat{n}} + \frac{1}{2} n^a R = 8\pi T^{a\hat{n}}\,. \tag{76}$$

Projecting the remaining index onto the hypersurface and using the definition (57) of the momentum density, we have

$$\perp G^{a\hat{n}} = \perp R^{a\hat{n}} = 8\pi \perp T^{a\hat{n}} = 8\pi j^a\,. \tag{77}$$

Following a development precisely analogous to (61)–(62), we find

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$$\perp R_{a\hat{n}} = g^{cd} \perp R_{ac\hat{n}d} - \perp R_{a\hat{n}\hat{n}\hat{n}} = -g^{cd} \perp R_{acd\hat{n}}\,. \tag{78}$$

Using the second of the Gauss-Codazzi equations (54), this becomes

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Using the second of the Gauss-Codazzi equations (54), this becomes

$$\perp R_{a\dot{n}} = -g^{cd} \left(D_c K_{ad} - D_a K_{cd} \right) = D_a K - D^b K_{ab} \,. \tag{79}$$

Raising the remaining free index we have (again recalling (7))

Thus, we find

$$D_b K^{ab} - D^a K = 8\pi j^a \,, \tag{81}$$

which is known as the *momentum constraint*.

—The crucial feature of the constraint equations (74) and (81), is that they involve *only* spatial tensors (including spatial derivatives of spatial tensors)—in particular, they do *not* involve *explicit time derivatives* of spatial tensors. Thus, these equations *are* equations of constraint which must be satisfied by the fundamental 3+1 variables, γ_{ab} and K_{ab} at all times (i.e. on all slices).

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$$\perp R^{a\hat{n}} = \perp G^{a\hat{n}} = D_b K^{ab} - D^a K \,. \tag{80}$$

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7) Time and Time Derivatives

In this section we establish some results concerning certain vector fields and Lie derivatives along these vector fields. The idea here is to introduce sufficiently general notions of "time" and "time derivatives while maintaining a geometric approach.

We first prove two results concerning Lie derivatives along the normal vector field, n^a, of *spatial* type $(0,l)$ tensors (spatial covariant tensors). The first result states that if $S_{a_1 a_2 \cdots a_l}$ is a spatial tensor, so that

$$n^{a_i} S_{a_1 \cdots a_l} = 0 \qquad i = 1, 2, \cdots, l, \tag{82}$$

then $\pounds_n S_{a_1 \cdots a_l}$ is also a spatial tensor. Denoting the general type $(0,l)$ spatial tensor by $\mathbf{S}$ we can thus write

$$\perp \pounds_n \mathbf{S} = \pounds_n \mathbf{S}\,. \tag{83}$$

The proof is straightforward. We have

$$\pounds_n S_{a_1 \cdots a_l} \;=\; n^c \nabla_c S_{a_1 \cdots a_l} + \sum_{i=1}^{l} (\nabla_{a_i} n^c)\, S_{a_1 \cdots c \cdots a_l}\,. \tag{84}$$

Now contract the ith index with n^a:

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$$= f\left(n^c \nabla_c S_{a_1\cdots a_l} + \sum_{i=1}^{l} (\nabla_{a_i} n^c)\, S_{a_1\cdots c\cdots a_l}\right)$$

$$= f\,\pounds_n S_{a_1 a_2\cdots a_l}\,. \tag{88}$$

Below we will argue that the vector field, N^a defined by

$$N^a = \alpha\, n^a \tag{89}$$

is a *natural* orthogonal vector field with which to Lie-differentiate tensors in computing general time derivatives. This vector field has the important property that

$$\pounds_N \perp^a{}_b = 0 \tag{90}$$

which implies that if $\mathbf{S}$ is *any* spatial tensor (not necessarily type $(0,l)$), then $\pounds_N \mathbf{S}$ is also spatial:

$$\perp \pounds_N \mathbf{S} = \pounds_N \mathbf{S} \tag{91}$$

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To see that (91) follows from (90), note that for a general type (k,l) spatial tensor we have

$$S^{a_1 a_2 \cdots a_k}{}_{b_1 b_2 \cdots b_l} = \perp^{a_1}{}_{c_1} \perp^{a_2}{}_{c_2} \cdots \perp^{a_k}{}_{c_k} \perp^{d_1}{}_{b_1} \perp^{d_2}{}_{b_2} \cdots \perp^{d_l}{}_{b_l} S^{c_1 c_2 \cdots c_k}{}_{d_1 d_2 \cdots d_l} \tag{92}$$

Now, applying $\pounds_N$ to both sides of this expression, and using the Liebnitz rule, we easily see that, given (90), the only term which survives is the one where the Lie derivative acts on $S^{a_1 a_2 \cdots a_k}{}_{b_1 b_2 \cdots b_l}$:

$$\pounds_N S^{a_1 a_2 \cdots a_k}{}_{b_1 b_2 \cdots b_l} = \perp^{a_1}{}_{c_1} \perp^{a_2}{}_{c_2} \cdots \perp^{a_k}{}_{c_k} \perp^{d_1}{}_{b_1} \perp^{d_2}{}_{b_2} \cdots \perp^{d_l}{}_{b_l} \pounds_N S^{c_1 c_2 \cdots c_k}{}_{d_1 d_2 \cdots d_l}\,, \tag{93}$$

and this is precisely (91). It remains to show that (90) is true. We have

$$\begin{aligned}
\pounds_N \perp^a{}_b &= N^c \nabla_c \perp^a{}_b - \perp^c{}_b \nabla_c N^a + \perp^a{}_c \nabla_b N^c \\
&= (\alpha n^c) \nabla_c (\delta^a{}_b + n^a n_b) - (\delta^c{}_b + n^c n_b) \nabla_c (\alpha n^a) + (\delta^a{}_c + n^a n_c) \nabla_b (\alpha n^c) \\
&= \alpha n^a n^c \nabla_c n_b + \alpha n_b n^c \nabla_c n^a - \nabla_b (\alpha n^a) - n_b n^c \nabla_c (\alpha n^a) + \nabla_b (\alpha n^a) + n^a n_c \nabla_b (\alpha n^c) \\
&= \alpha n^a a_b + \alpha n_b a^a - \alpha n_b a^a - n_b n^a n^c \nabla_c \alpha + \alpha n^a n_c \nabla_b n^c - n^a \nabla_b \alpha \\
&= \alpha n^a \left(a_b - \alpha^{-1} \left(\nabla_b \alpha + n_b n^c \nabla_c \alpha \right) \right) \\
&= \alpha n^a \left(a_b - \alpha^{-1} D_b \alpha \right) = \alpha n^a \left(a_b - D_b \ln \alpha \right) = 0\,.
\end{aligned} \tag{94}$$

where we have used (9), (89), (29) and (34).

Now, recall that our foliation is defined by a closed one-form (dual-vector field), Ω_a:

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\pounds_N \perp^a{}_b &= N^c \nabla_c \perp^a{}_b - \perp^c{}_b \nabla_c N^a + \perp^a{}_c \nabla_b N^c \\
&= (\alpha n^c) \nabla_c (\delta^a{}_b + n^a n_b) - (\delta^c{}_b + n^c n_b) \nabla_c (\alpha n^a) + (\delta^a{}_c + n^a n_c) \nabla_b (\alpha n^c) \\
&= \alpha n^a n^c \nabla_c n_b + \alpha n_b n^c \nabla_c n^a - \nabla_b (\alpha n^a) - n_b n^c \nabla_c (\alpha n^a) + \nabla_b (\alpha n^a) + n^a n_c \nabla_b (\alpha n^c) \\
&= \alpha n^a a_b + \alpha n_b a^a - \alpha n_b a^a - n_b n^a n^c \nabla_c \alpha + \alpha n^a n_c \nabla_b n^c - n^a \nabla_b \alpha \\
&= \alpha n^a \left(a_b - \alpha^{-1} \left(\nabla_b \alpha + n_b n^c \nabla_c \alpha \right) \right) \\
&= \alpha n^a \left(a_b - \alpha^{-1} D_b \alpha \right) = \alpha n^a \left(a_b - D_b \ln \alpha \right) = 0\,.
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$$\Omega_a = \nabla_a \tau\,. \tag{95}$$

Since $n^a = -\alpha \Omega^a$ and $\Omega^a \Omega_a = -\alpha^{-2}$, we have

$$N^a \Omega_a = 1\,. \tag{96}$$

It is this normalization which makes N^a the *natural orthogonal* vector field to use in computing "time derivatives" (i.e. for use in Lie differentiation). However, there is no justification for restricting attention only to "normal time derivatives" and, in fact, we can and will consider Lie differentiation

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$$t^a\Omega_a = 1 \,, \tag{97}$$

by adding to N^a an arbitrary *spatial* vector β^a (which is just the *shift vector* we have previously discussed):

$$t^a = N^a + \beta^a = \alpha n^a + \beta^a \,, \tag{98}$$

$$= -\beta^a n_a = 0 \,. \tag{99}$$

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Now contract the jth index with n^a:

$$n^{a_j} \pounds_n S_{a_1 \cdots a_l} = n^{a_j} n^c \nabla_c S_{a_1 \cdots a_l} + \sum_{i=1}^{l} (\nabla_{a_i} n^c) n^{a_j} S_{a_1 \cdots c \cdots a_l}. \tag{85}$$

Now, because $S_{a_1 \cdots c \cdots a_l}$ is spatial, all of the terms in the sum (the "correction terms") vanish except when $i = j$ (i.e. when the jth index of $S_{a_1 \cdots c \cdots a_l}$ is being corrected. Also, we can use $n^{a_i} S_{a_1 \cdots a_l} = 0$ to throw the derivative in the first term onto n^{a_j}. Thus, we have

$$n^{a_j} \pounds_n S \quad = \quad -n^c \nabla_c n^{a_j} S \quad + n^c \nabla_c n^{a_j} S \quad = 0 \tag{86}$$

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we have

$$\begin{aligned} \pounds_t \gamma_{ab} &= \pounds_N \gamma_{ab} + \pounds_\beta \gamma_{ab} \\ &= \alpha \pounds_n \gamma_{ab} + \pounds_\beta \gamma_{ab} \,. \end{aligned} \tag{109}$$

or

$$\pounds_t \gamma_{ab} = -2\alpha K_{ab} + \pounds_\beta \gamma_{ab} \,. \tag{110}$$

8b) The Evolution Equations for the Extrinsic Curvature

We first observe that the Einstein equations

$$G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R = 8\pi T_{ab} \,, \tag{111}$$

may be contracted to yield

$$G = -R = 8\pi T \,. \tag{112}$$

Thus, the field equations may be rewritten as

$$R_{ab} = 8\pi T_{ab} + \frac{1}{2} g_{ab} R = 8\pi \left(T_{ab} - \frac{1}{2} g_{ab} T \right) \,. \tag{113}$$

Projecting onto the hypersurface, we have

$$\perp R_{ab} = 8\pi \left(\perp T_{ab} - \frac{1}{2} \gamma_{ab} T \right) \,. \tag{114}$$

Now from definitions (56)–(58), as well as our expression (55) for the 3+1 decomposition of a general, symmetric, type $(0,2)$ tensor, we find

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$$\perp T_{ab} \;\equiv\; S_{ab} \;=\; T_{ab} + 2n_{(a}\perp T_{b)\dot{n}} - n_a n_b T_{\dot{n}\dot{n}}\,. \tag{115}$$

Contracting, we get

$$S = T + T_{\dot{n}\dot{n}}\,, \tag{116}$$

or, using (56)

$$T = S - \rho\,. \tag{117}$$

Thus, (114) becomes

$$\perp R_{ab} = 8\pi\left(S_{ab} - \frac{1}{2}\gamma_{ab}\,(S-\rho)\right) \tag{118}$$

Now, from (62), we have

$$\perp R_{ab} = -\perp R_{a\dot{n}b\dot{n}} + g^{cd}\perp R_{acbd}\,. \tag{119}$$

Using (52) and (105), this becomes

$$\begin{aligned}\perp R_{ab} &= -\left(\alpha^{-1}\pounds_N K_{ab} + K_{ac}K^c{}_b + \alpha^{-1}D_aD_b\alpha\right) + g^{cd}\left(\mathcal{R}_{abcd} + K_{ab}K_{cd} - K_{ad}K_{cb}\right)\\ &= -\alpha^{-1}\pounds_N K_{ab} - 2K_{ac}K^c{}_b - \alpha^{-1}D_aD_b\alpha + \mathcal{R}_{ab} + KK_{ab}\,.\end{aligned} \tag{120}$$

Equating (118) and (120), and using

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we solve for $\pounds_t K_{ab}$ to get our evolution equations for the extrinsic curvature:

$$\pounds_t K_{ab} = \pounds_\beta K_{ab} - D_aD_b\alpha + \alpha\left(\mathcal{R}_{ab} + KK_{ab} - 2K_{ac}K^c{}_b - 8\pi\left(S_{ab} - \frac{1}{2}\gamma_{ab}\,(S-\rho)\right)\right)\,. \tag{122}$$

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we have

$$\begin{aligned} \pounds_t \gamma_{ab} &= \pounds_N \gamma_{ab} + \pounds_\beta \gamma_{ab} \\ &= \alpha \pounds_n \gamma_{ab} + \pounds_\beta \gamma_{ab} \,. \end{aligned} \tag{109}$$

or

$$\pounds_t \gamma_{ab} = -2\alpha K_{ab} + \pounds_\beta \gamma_{ab} \,. \tag{110}$$

8b) The Evolution Equations for the Extrinsic Curvature

We first observe that the Einstein equations

$$G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R = 8\pi T_{ab} \,, \tag{111}$$

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$$\pounds_t K_{ab} = \pounds_\beta K_{ab} - D_a D_b \alpha + \alpha \left(\mathcal{R}_{ab} + K K_{ab} - 2 K_{ac} K^c{}_b - 8\pi \left(S_{ab} - \frac{1}{2} \gamma_{ab} (S - \rho) \right) \right) . \quad (122)$$

We can derive an alternate version of this evolution equation which involves the "mixed" form, $K^a{}_b$ of the extrinsic curvature, which has been used by several researchers in the past, and which we will tend to use in the course. We start from

$$\alpha^{-1} \pounds_N K_{ab} - 2 K_{ac} K^c{}_b - \alpha^{-1} D_a D_b \alpha + \mathcal{R}_{ab} + K K_{ab} = 8\pi \left(S_{ab} - \frac{1}{2} \gamma_{ab} (S - \rho) \right) , \quad (123)$$

and note that because all of the tensors appearing in this expression are spatial, we can raise indices with γ^{ab} to get:

$$\alpha^{-1} \gamma^{ac} \pounds_N K_{ab} - 2 K^{ac} K_{cb} - \alpha^{-1} D^a D_b \alpha + R^a{}_b + K K^a{}_b = 8\pi \left(S^a{}_b - \frac{1}{2} \delta^a{}_b + n^a n_b (S - \rho) \right) . \quad (124)$$

Now

$$\begin{aligned} \pounds_N K^a{}_b &= \pounds_N (\gamma^{ac} K_{cb}) \\ &= K_{cb} \pounds_N \gamma^{ac} + \gamma^{ac} \pounds_N K_{cb} \end{aligned}$$