Title: Not altogether desperate: An exposition of Newton's Scholium Problem and the theoretical definition of duration

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Abstract: In the Scholium in Newton's Principia which contains the discussions about absolute space, time, and the bucket experiment, Newton also posed a problem that Julian Barbour has denoted the "Scholium problem". Newton writes there "But how are we to obtain the true motions from their causes, effects, and apparent differences, and the converse, shall be explained more at large in the following treatise. For to this end it was that I composed it". This problem was clearly considered very important by Newton who claims he wrote the Principia dedicated to this problem. Interestingly Newton never returned to the problem. In this talk we are going to give a mathematical precise formulation of the Scholium problem. A subpart of the Scholium problem consists of determining how accurate the observers clock is. We are going to start from that end and see that the problem of defining duration is inseparately intertwined with the full scholium problem.

Time and the Newton's Scholium Problem

Hans Westman

Sydney University, Australia
(In collaboration with Maki Takahashi)

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confound real quantities with their relations and sensible measures [vulgaribus mensuris].

XIV. It is indeed a matter of great difficulty to discover [cognoscere], and effectually to distinguish [actu discriminare], the true motions of particular bodies from the apparent; because the parts of that immovable space, in which those motions are performed, do by no means come under the observation of our senses [incurrent in sensus]. Yet the thing is not altogether desperate: for we have some arguments to guide us, partly from the apparent motions, which are the differences of the true motions; partly from the forces, which are the causes and effects of the true motions.

Newon then takes great pains to show that there are instances in which no purely relative theory of motion could account for certain forces.[2]

For instance, if two globes, kept at a given distance one from the other by means of a cord that connects them, were revolved about their common centre of gravity, we might, from the tension of the cord, discover the endeavor of the globes to recede from the axis of their motion, and from thence we might compute the quantity of their circular motions. And then if any equal forces should be impressed at once on the alternate faces of the globes to augment or diminish their circular motions, from the increase or decrease of the tension of the cord, we might infer the increment or decrement of their motions; and thence would be found on what faces those forces ought to be impressed, that compute the quantity of their circular motions. And then if any equal forces should be impressed at once on the alternate faces of the globes to augment or diminish their circular motions, from the increase or decrease of the tension of the cord, we might infer the increment or decrement of their motions; and thence would be found on what faces those forces ought to be impressed, that the motions of the globes might be most augmented; that is, we might discover their hindmost faces, or those which, in the circular motion, do follow. But the faces which follow being known, and consequently the opposite ones that precede, we should likewise know the determination of their motions. And thus we might find both the quantity and the determination of this circular motion, even in an immense vacuum, where there was nothing external or sensible with which the globes could be compared. But now, if in that space some remote bodies were placed that kept always a given position one to another, as the fixed stars do in our regions, we could not indeed determine from the relative translation of the globes among those bodies, whether the motion did belong to the globes or to the bodies. But if we observed the cord, and found that its tension was that very tension which the motions of the globes required, we might conclude the motion to be in the globes, and the bodies to be at rest; and then, lastly, from the translation of the globes among the bodies, we should find the determination of their motions. But how we are to obtain the true motions from their causes, effects, and apparent differences, and the converse, shall be explained more at large in the following treatise. For to this end it was that I composed it.

[2]In XII he claimed that "The effects which distinguish absolute from relative motion are, the forces of receding from the axis of circular motion. For there are no such forces in a circular motion purely relative, but in a true and absolute circular motion, they are greater or less, according to the quantity of the motion." He follows this with the 'bucket' experiment, showing that even in cases were there *is*

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of gravity, we might, from the tension of the cord, discover the endeavor of the globes to recede from the axis of their motion, and from thence we might compute the quantity of their circular motions. And then if any equal forces should be impressed at once on the alternate faces of the globes to augment or diminish their circular motions, from the increase or decrease of the tension of the cord, we might infer the increment or decrement of their motions; and thence would be found on what faces those forces ought to be impressed, that the motions of the globes might be most augmented; that is, we might discover their hindmost faces, or those which, in the circular motion, do follow. But the faces which follow being known, and consequently the opposite ones that precede, we should likewise know the determination of their motions. And thus we might find both the quantity and the determination of this circular motion, even in an immense vacuum, where there was nothing external or sensible with which the globes could be compared. But now, if in that space some remote bodies were placed that kept always a given position one to another, as the fixed stars do in our regions, we could not indeed determine from the relative translation of the globes among those bodies, whether the motion did belong to the globes or to the bodies. But if we observed the cord, and found that its tension was that very tension which the motions of the globes required, we might conclude the motion to be in the globes, and the bodies to be at rest; and then, lastly, from the translation of the globes among the bodies, we should find the determination of their motions. But how we are to obtain the true motions from their causes, effects, and apparent differences, and the converse, shall be explained more at large in the following treatise. For to this end it was that I composed it.

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of receding from the axis of circular motion. For there are no such forces in a circular motion purely relative, but in a true and absolute circular motion, they are greater or less, according to the quantity of the motion." He follows this with the 'bucket' experiment, showing that even in cases were there *is* relative motion, we cannot view the relative motion as responsible for the force, for the effect of the force—the endeavour to recede [centrifugal force]—is in existence when the water and bucket are at rest relative to one another.

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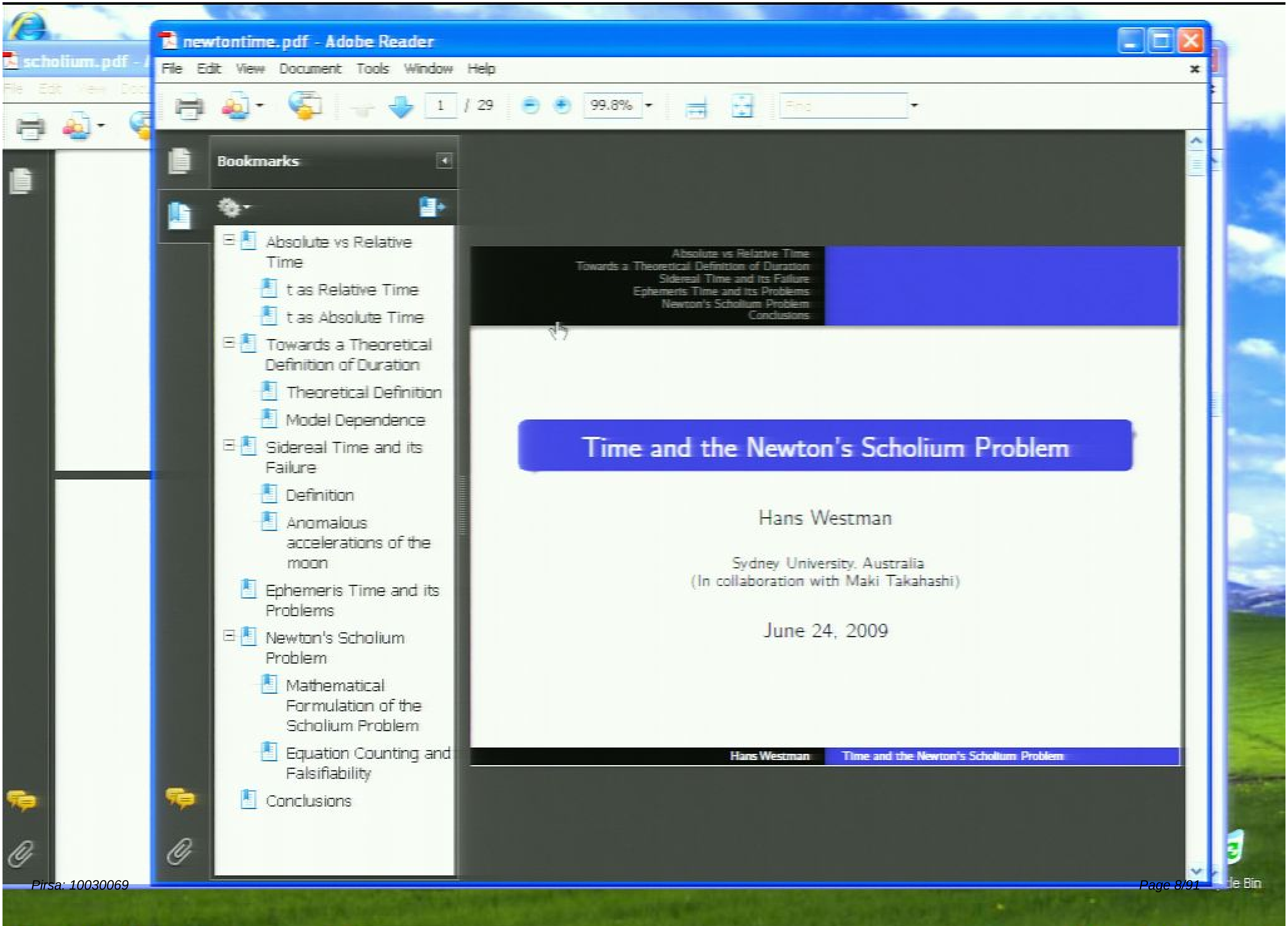
In this case, there is simply nothing else which the motion can be referred to: QED! So he has shown with his two thought experiments that (1) relative motion cannot be responsible for centrifugal forces because they can be present during relative rest; (2) relative motion cannot be responsible for centrifugal forces because they can be present when there are no other bodies are present.

The Scholium problem was interpreted (for example by mach and Reichenbach) as fuzzy metaphysics. However, this appears to have been a mistake.

3 To What Does t Refer?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> " Absolute, true, and mathematical time, of itself, and from its own nature flows equably without relation to anything external, and by an other name is called duration: relative, apparent, and common time, is some sensible and external (whether accurate or unequble) measure of duration by the means of motion, which is commonly used instead of true time; such as an hour, a day, a month, a year."

The Symbol t in Newton's Equations

- In the Newtonian equations of motion a symbol t appears

$$m_k \frac{d^2\mathbf{x}_k}{dt^2} = \mathbf{F}_k$$

- Does the t in Newton's equations of motion represent absolute or relative time?

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Laboratory Interpretation

- A moments reflection reveals that in a laboratory situations t is almost universally taken to represent an external physical clock, i.e. the t is taken to represent relative time.

$$\frac{d^2\theta}{dt^2} = \frac{g}{l}\cos\theta$$

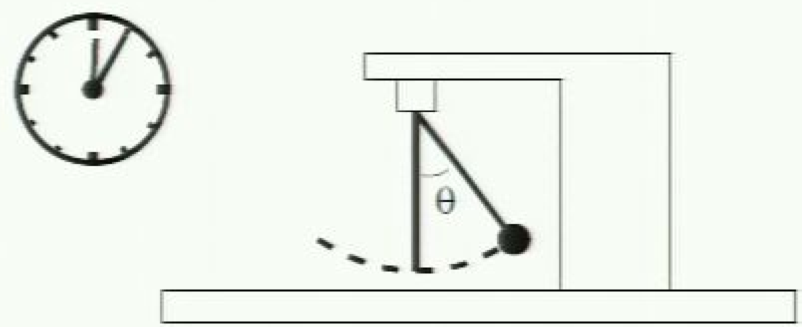

Figure: The "physicist" interpretation of the symbol t"

- This laboratory "physicist" interpretation is also the one promoted in Einstein'a 1905 paper on special relativity.

Hidden Assumption

- We have assumed that the laboratory clock represents an "accurate clock". After all would could not use any clock in order to test the predictions of our pendulum theory.
- Our theory for the pendulum contains a mathematical variable θ that describe position of the pendulum mass, and also a symbol t which we took to represent the external clock.
- However, it is clear that this theory does not faithfully model the external clock. After all, a clock is a complicated device with many internal degrees of freedom. In this pendulum theory it is represented by only one real number!
- The fact that the external clock was not faithfully modelled or represented within our simple pendulum theory means that it is not possible to justify this "accurate clock" assumption within this simple pendulum theory.

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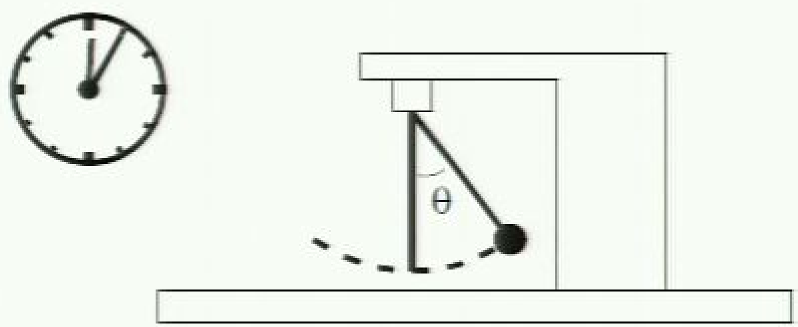

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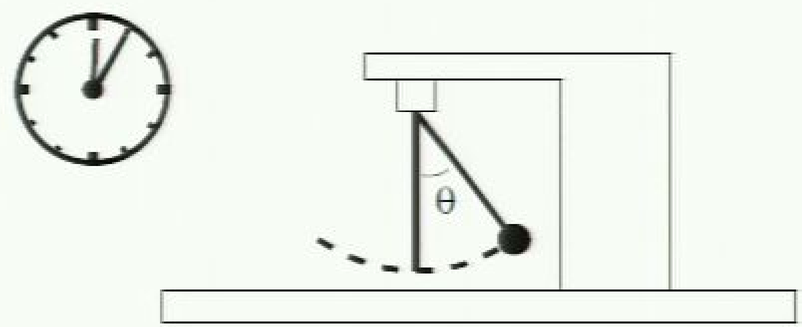

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Elementary Questions that you won't find in textbooks

- Thus, that the clock is accurate has to be taken on faith: perhaps we trust the manufacturer.
- But what do we even mean by "accurate clock" here?
- What is the theoretical definition of an "accurate clock"?
- Some clock are certainly more accurate than others (e.g. a pendulum clock vs a cesium clock) but how can we determine empirically which one is the better?
- Against what exactly do we compare a clock to see that it is a good clock?
- And if a clock is inaccurate, how can such inaccuracy be numerically quantified, or measured?
- As we shall see, these questions are inseparably intertwined with determination of masses, inertial frames etc..

t as Absolute Time

- As should be clear, these questions cannot be answered satisfactorily if t is taken to represent relative time.
- Instead we must, together with Newton, assume that the symbol t in the equations of motion represents absolute time which "flows without relation to anything external".
- If we "weed out" all metaphysics then Newton's absolute time is simply this: it is the symbol t in Newton's equations of motion. Nothing more nothing less.
- From the point of view of the equations of motion the distinction between relative and absolute time is very natural. We realize that there is a special symbol t in the equations of motion and that the readings of a physical clock need not accurately "march in step" with that t and so cannot be identified with it. This is the main point in the scholium.

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Theoretical Model of an "Accurate Clock"

- By aid of Newton's absolute time, i.e. the symbol t in the equation of motion, we can now provide a definition of what an "accurate clock" is. The clock should "march in step" with t.
- In order to make this precise we first provide a theoretical model of the physical system representing the clock.
- Assume then that the clock is composed of N point particles $X = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N)$ interacting through some forces $\mathbf{F}_k$. (We can also allow for fluids, strings, etc. if we like but we stick to point-particles for simplicity.)
- Given inital data $X(t=0) = X_0$ and $\frac{dX}{dt}(t=0) = V_0$ we get a unique solution $X(t)$ to the equations of motion.
- We further assume that the clock reading T is a function of the dynamical variables, i.e. $T = T(X)$.

Theoretical Definition: "Accurate Clock"

Definition: *An* accurate clock *is a physical system described by some variables* $X(t)$ *such that the reading of the clock* T *satisfies*

$$T = T(X(t)) = a + bt$$

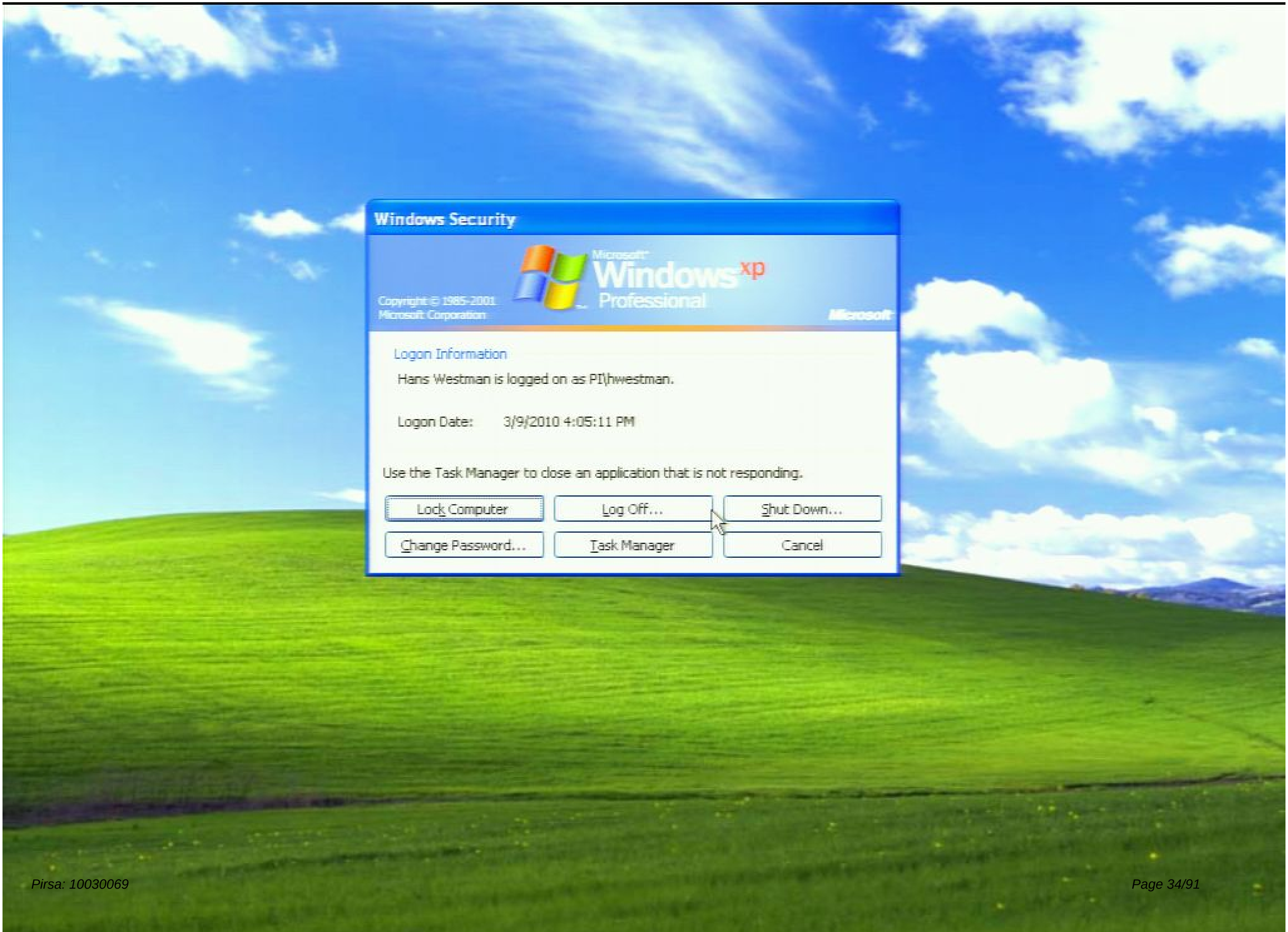
or

$$\Delta T = T(X(t_2)) - T(X(t_1)) = b(t_2 - t_1) = b\Delta t.$$

- Here a represents an irrelevant starting time and b just a choice of units.
- The statement the a good clock should "march in step" with t means mathematically just this: $\Delta T = b\Delta t$.

Model Dependence

- It is important to recognize that this definition of an accurate clock is model dependent.
- Indeed, how can we know that the physical clock was accurately modelled theoretically?
- How do we know that we did not neglect important friction terms, that we inerted the correct numerical values of the masses, and other constants, etc.?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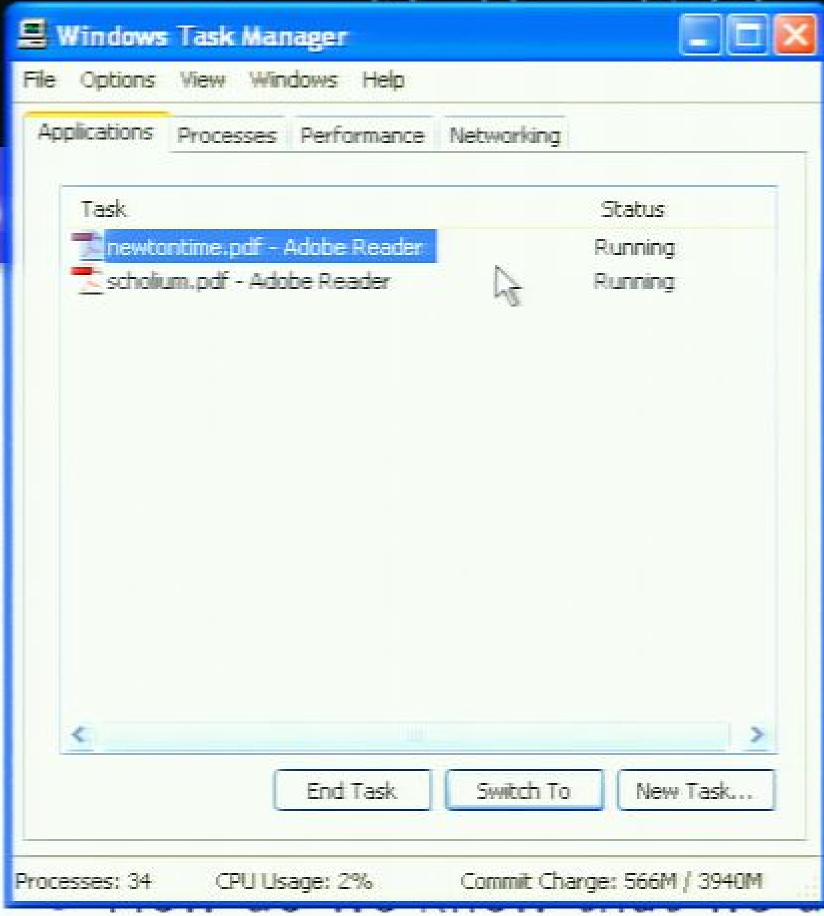

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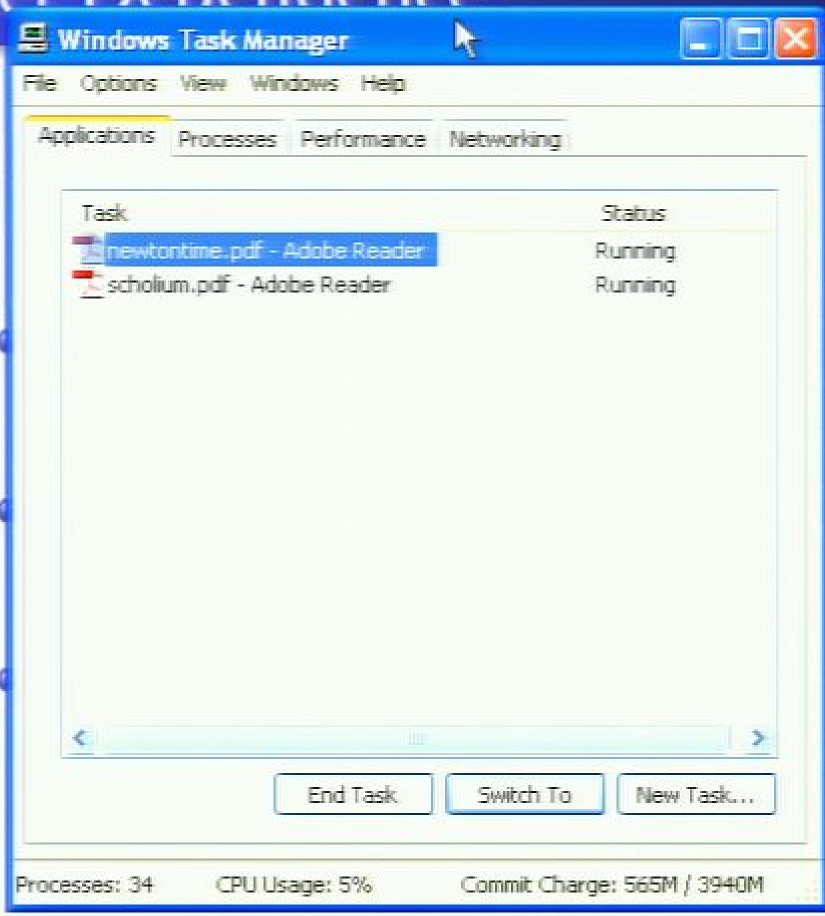

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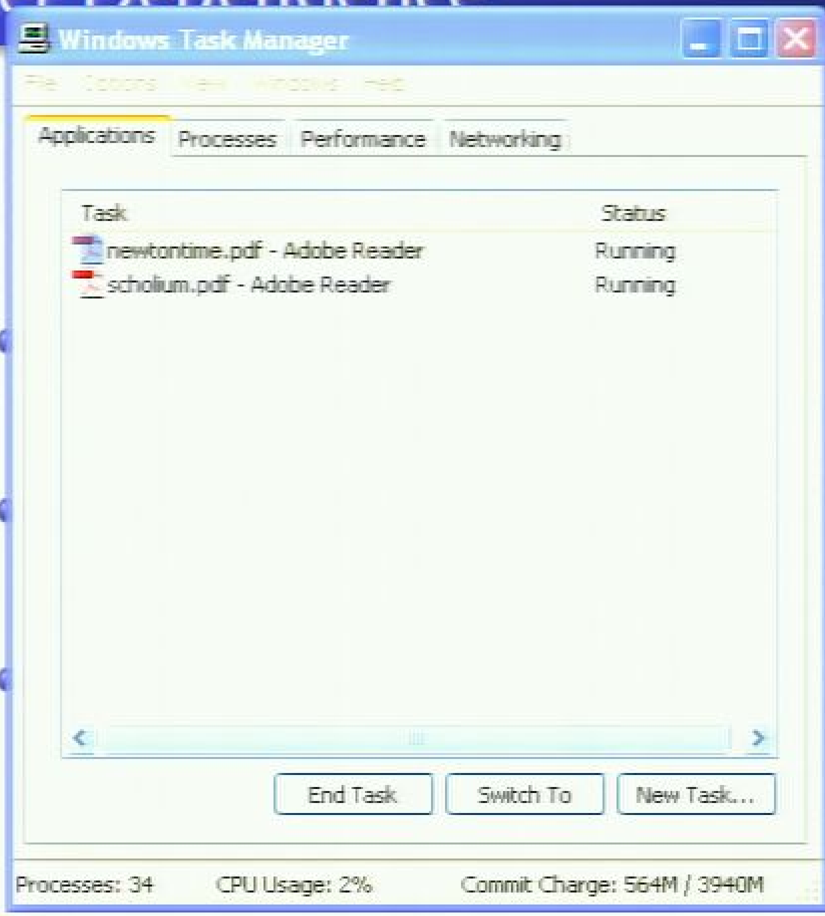

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Example: The failure of Sidereal Time

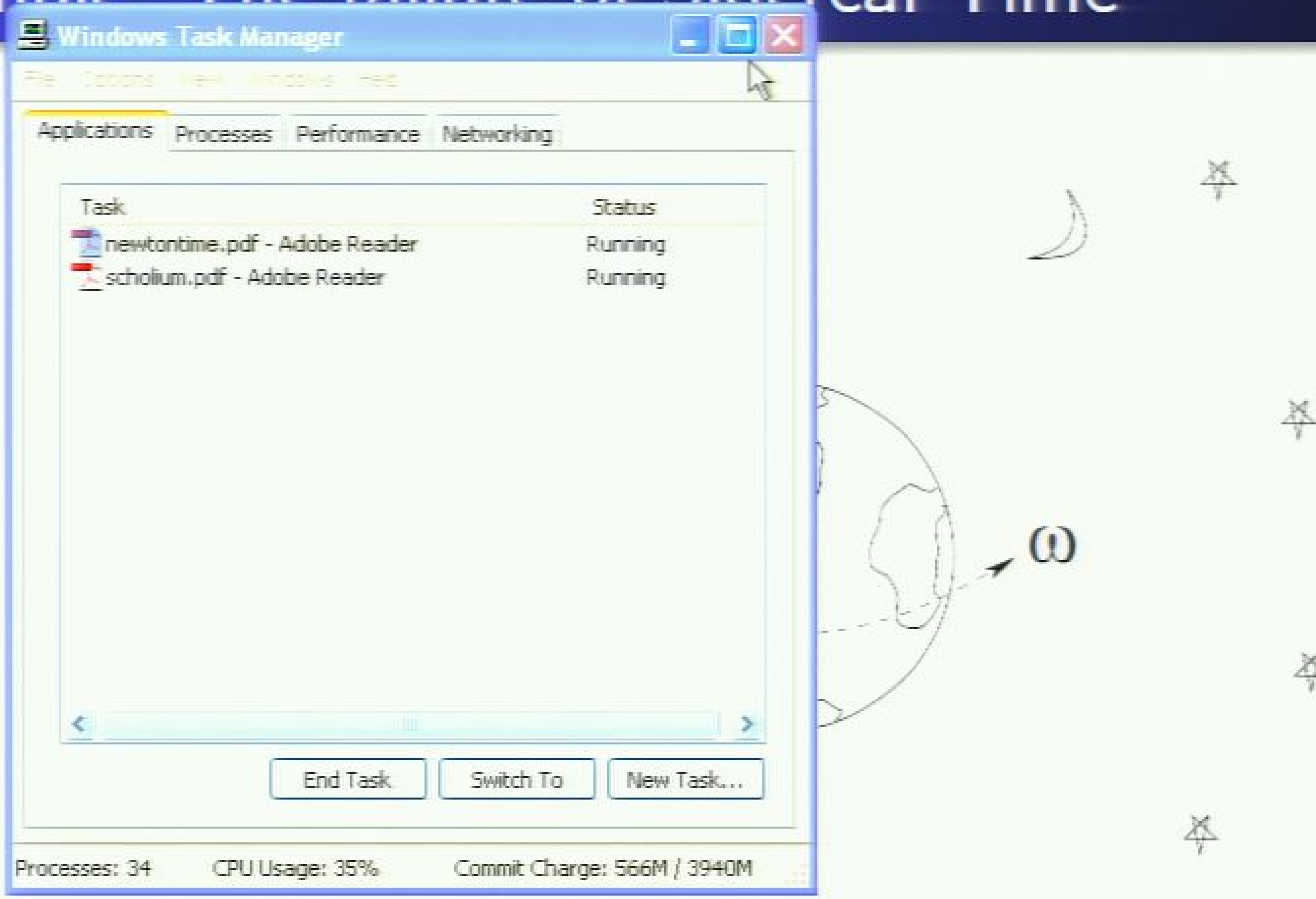

Figure: Cartoon illustration of sidereal time.

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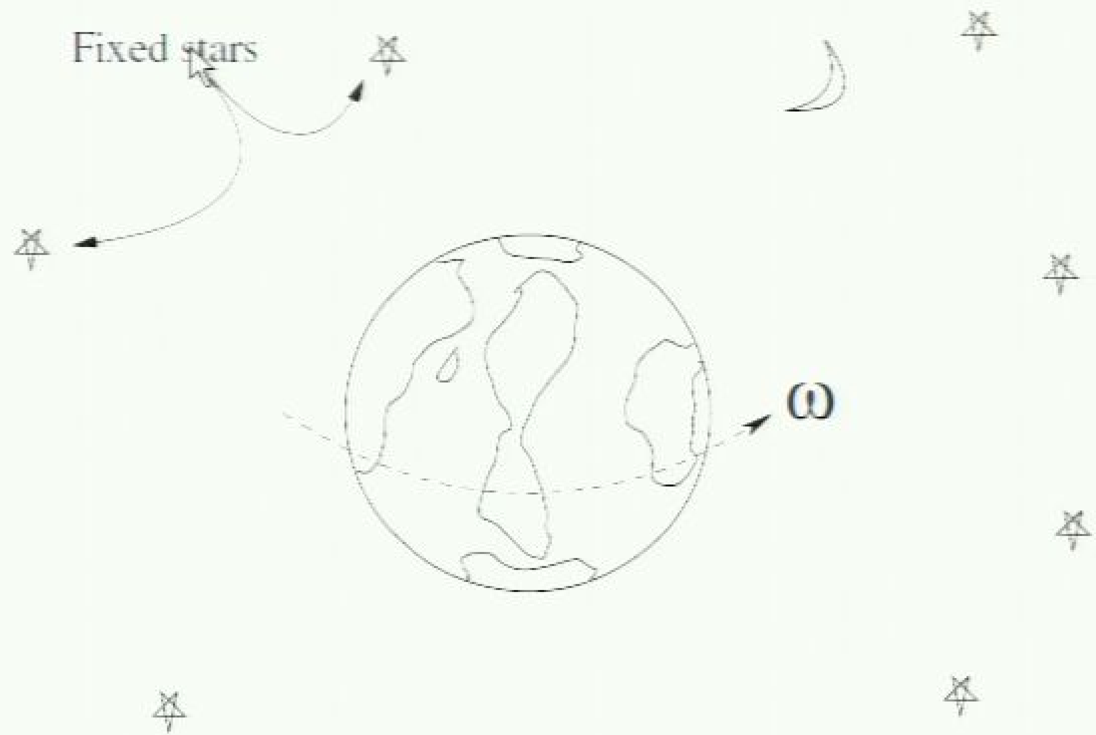

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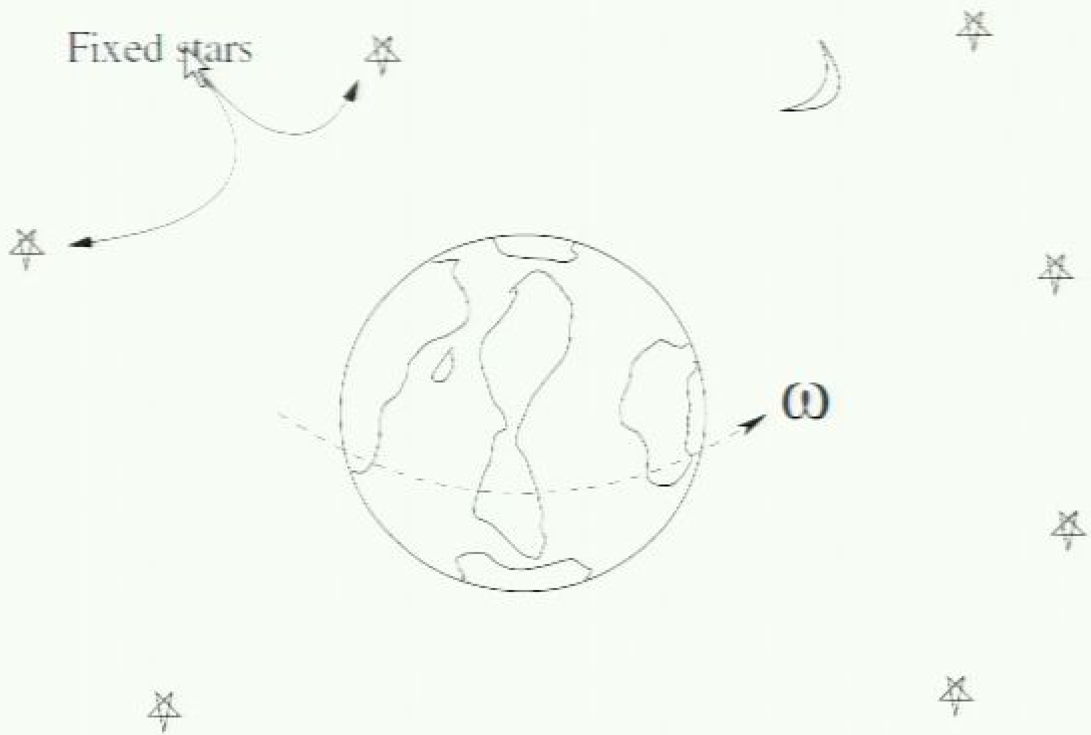

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Simple Theoretical Model of Sidereal Time

- To provide a simple theoretical model of sidereal time we assume that earth is a perfect spherical solid with uniform density throughout its interior.
- The earth's rotation can now be completely described by it's Euler angles (α, β, γ).
- Newton's equations immediately yields:

$$\alpha(t) = \alpha_0 \qquad \beta(t) = \beta_0 \qquad \gamma(t) = \gamma_0 + \omega t$$

- Thus, the third Euler angle γ can be regarded as the reading T of our "sidereal" clock, the rotating earth.

Anomalous Accelerations of the moon

- However, in the end of the 19'th century, astromomers observed puzzling "accelerations" of the moon.
- The moon was observed to follow the right Newtonian trajectory through space, but with the wrong speed. The moon was observed to speed up and slow down along the Newtonian trajectory.
- In order to explain these anomalous accelerations unobserved celestial bodies were conjectured to exist.
- However, these bodies should have modified the trajectory and not just the speed along the trajectory.

Anomalous Accelerations of the moon

- The resolution was to propose that sidereal time does not provide an accurate measure of duration (Newton's absolute time).
- Once the accuracy of sidereal time is questioned it is easy to understand these anomalous accelerations of the moon: Earth is not a perfect spherical solid body and its rotation will not be uniform.
- In reality earth is ellipic and more importantly it's surface consists of water which is influenced by the moon, and its interior is not a solid but rather fluid magma.

More Accurate Model of Sidereal Time

- If we construct a more accurate model taking into account the elliptical shape, the fluid interior and the tidal effects we would find when solving Newton's equations of motion that γ is not linear in t. Only approximately.
- A Taylor expansion of γ would yield

$$\gamma(t) = \gamma_0 + \omega t + \delta \frac{t^2}{2} + \dots .$$

- Here δ provides a good numerical measure of how accurate sideral time is.

Ephemeris Time

- As a better notion of time astronomers came up with "ephemeris time". It make use of the displacements $d\mathbf{x}_k$ of the planets in the solar system to calculate the time that has lapsed.
- The basic idea is to make use of energy conservation.

$$E = \sum_k \frac{1}{2} m_k \left(\frac{d\mathbf{x}_k}{dt} \right)^2 + V(X)$$

- We can now readily solve for dt yielding

$$dt = \sqrt{\frac{\sum_k \frac{1}{2} m_k \left(d\mathbf{x}_k \right)^2}{E - V}}$$

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- Here a represents an irrelevant starting time and b just a choice of units.
- The statement the a good clock should "march in step" with t means mathematically just this: $\Delta T = b\Delta t$.

Theoretical Model of an "Accurate Clock"

- By aid of Newton's absolute time, i.e. the symbol t in the equation of motion, we can now provide a definition of what an "accurate clock" is. The clock should "march in step" with t.
- In order to make this precise we first provide a theoretical model of the physical system representing the clock.
- Assume then that the clock is composed of N point particles $X = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N)$ interacting through some forces $\mathbf{F}_k$. (We can also allow for fluids, strings, etc. if we like but we stick to point-particles for simplicity.)
- Given inital data $X(t=0) = X_0$ and $\frac{dX}{dt}(t=0) = V_0$ we get a unique solution $X(t)$ to the equations of motion.
- We further assume that the clock reading T is a function of the dynamical variables, i.e. $T = T(X)$.

Simple Theoretical Model of Sidereal Time

- To provide a simple theoretical model of sidereal time we assume that earth is a perfect spherical solid with uniform density throughout its interior.
- The earth's rotation can now be completely described by it's Euler angles (α, β, γ).
- Newton's equations immediately yields:

$$\alpha(t) = \alpha_0 \qquad \beta(t) = \beta_0 \qquad \gamma(t) = \gamma_0 + \omega t$$

- Thus, the third Euler angle γ can be regarded as the reading T of our "sidereal" clock, the rotating earth.

Anomalous Accelerations of the moon

- However, in the end of the 19'th century, astromomers observed puzzling “accelerations” of the moon.
- The moon was observed to follow the right Newtonian trajectory through space, but with the wrong speed. The moon was observed to speed up and slow down along the Newtonian trajectory.
- In order to explain these anomalous accelerations unobserved celestial bodies were conjectured to exist.
- However, these bodies should have modified the trajectory and not just the speed along the trajectory.

More Accurate Model of Sidereal Time

- If we construct a more accurate model taking into account the elliptical shape, the fluid interior and the tidal effects we would find when solving Newton's equations of motion that γ is not linear in t. Only approximately.
- A Taylor expansion of γ would yield

$$\gamma(t) = \gamma_0 + \omega t + \delta \frac{t^2}{2} + \dots .$$

- Here δ provides a good numerical measure of how accurate sideral time is.

Ephemeris Time

- As a better notion of time astronomers came up with "ephemeris time". It make use of the displacements $d\mathbf{x}_k$ of the planets in the solar system to calculate the time that has lapsed.
- The basic idea is to make use of energy conservation.

$$E = \sum_k \frac{1}{2} m_k \left(\frac{d\mathbf{x}_k}{dt} \right)^2 + V(X)$$

- We can now readily solve for dt yielding

$$dt = \sqrt{\frac{\sum_k \frac{1}{2} m_k \left(d\mathbf{x}_k \right)^2}{E - V}}$$

Ephemeris Time

- Or in integrated form

$$t = \int \sqrt{\frac{\sum_k \frac{1}{2} m_k \left(d\mathbf{x}_k \right)^2}{E - V}} = \int_{\lambda_1}^{\lambda_2} d\lambda \sqrt{\frac{\sum_k \frac{1}{2} m_k \left(\frac{d\mathbf{x}_k}{d\lambda} \right)^2}{E - V}}$$

- Here we have introduced an arbitrary parameter. It could be sidereal time or some other inaccurate clock.
- In order to determine t we first empirically determine the displacements $|d\mathbf{x}_k|^2$ of the planets in the solar system and then we plug this data into the above expression for ephemeris time.
- For about 30 years, before the introduction of the atomic clock, astronomers made use of ephemeris time as a more accurate measure of duration (i.e. absolute time).

Problems with Ephemeris Time

- At first sight we might think that ephemeris time provides us with the ultimate solution to our problem: an operational definition of duration. However, this is not true.
- In order to correctly calculate the ephemeris time from the empirically measured displacements $d\mathbf{x}_k$ we need to know *à priori* the masses m_k and other constant appearing in the potential V, and an inertial reference frame according to which the displacements are empirically determined.
- But how did the astromomers know which frames are inertial frames? After all the earth is rotating and hurling around the sun, which is itself alse hurling around the milky way.
- To operationally identify inertial frames and to operationally determine an accurate measure of duration (Newton's absolute time) is the "Scholium Problem".

Problems with Ephemeris Time

- Newton writes in the scholium of the Principia

  > "True motion is neither generated nor altered, but by some force impressed upon the body moved; but relative motion may be generated or altered without any force impressed upon the body... But how are we to obtain the true motions from their causes, effects, and apparent differences, and the converse, shall be explained more at large i the following treatise. For to this end it was that I composed it."

- Interestingly, Newton never returned to this "Scholium Problem" and to this day it has been largely ignored.

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Newton's Scholium Problem

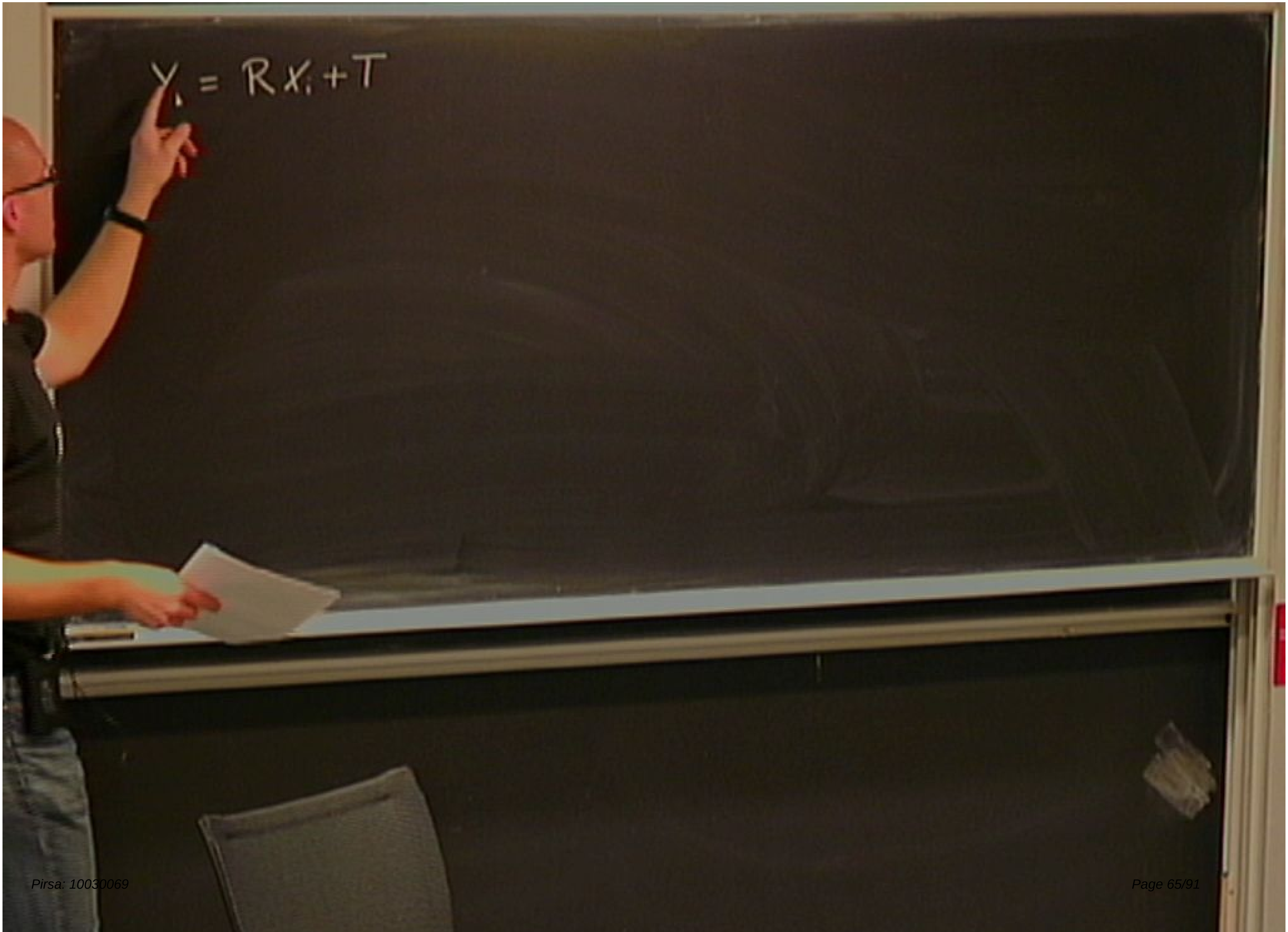
- Broadly speaking, Newton's scholium problem consists of the simultaneous determination of duration, inertial reference frames, masses and other constants, from measurable quantities.
- We shall now put fourth a mathematical formulation of Newton's Scholium problem, but before we continue we need one more observation.
- All we can *directly* observe from earth are the angular coordinates of the planets in the sky. The distances could perhaps be determined by a light signal, but in order to determine the distance we need to be able to accurately measure the time it tooks, which presupposes an accurate clock, which we don't have yet...

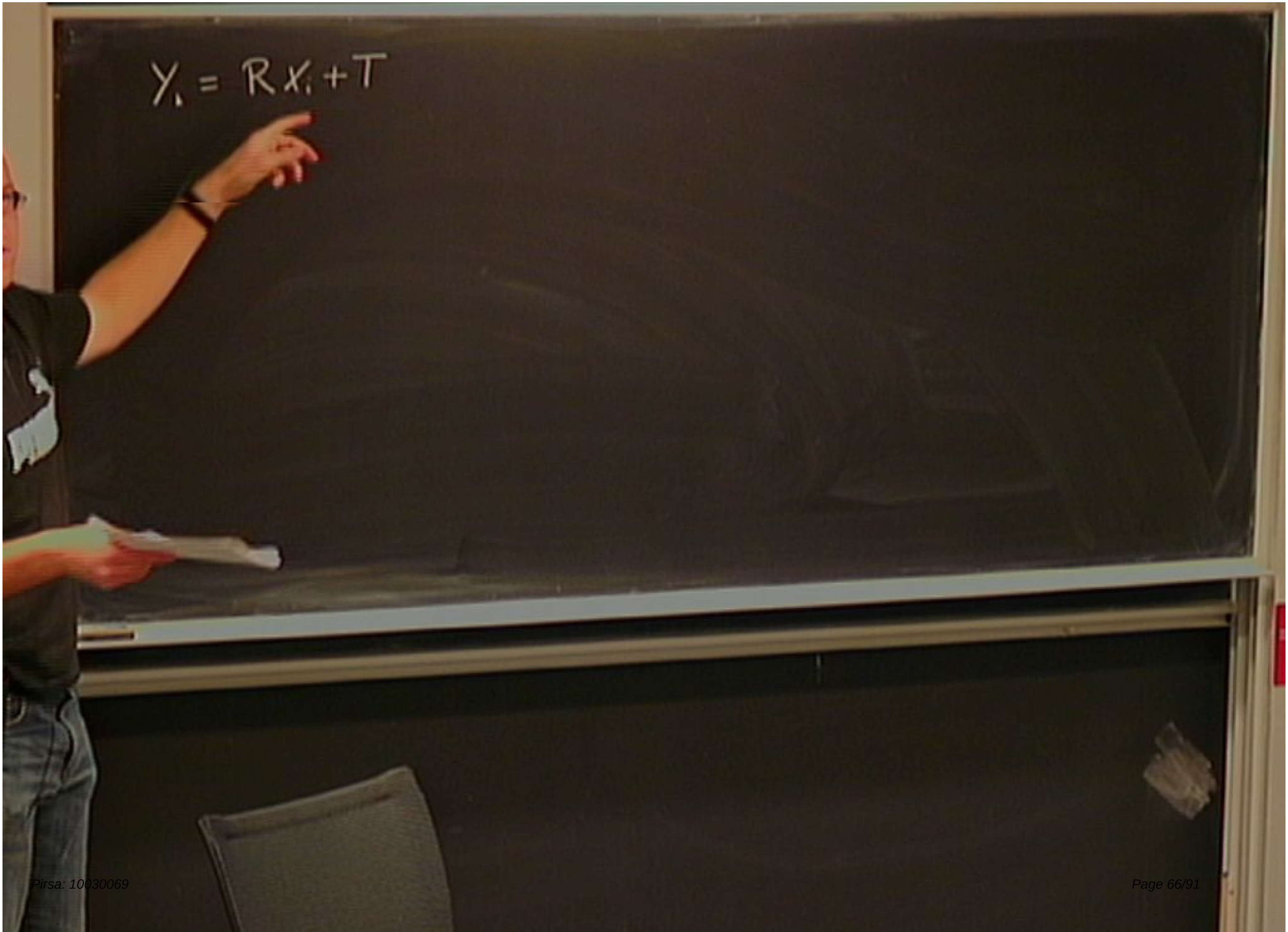
Identification of the Unknowns

- Therefore, we shall assume that the empirical data only consists of the angular coordinates the we observe on the sky. The distances would have to be determined indirectly from the angular data.
- Mathematically this means the follwing. Let $X = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N)$ be the positions of all the planets in some Euclidean reference frame (be it inertial or not).
- We assume the $\mathbf{x}_3$ represents the position of earth so that $\mathbf{x}_3 = 0$ at all times and earth would define the origin of our Euclidean reference frame.
- Furthermore, we express the positions of the planets $\mathbf{x}_k$ in spherical polar coordinates $(r_k, \theta_k, \varphi_k)$. The latter two are the angular coordinates that we can directly observe from earth.

Identification of the Unknowns

- In order to simplify the problem we are going to assume that the speed of light is infinite and that no bending of light rays occur. These two assumptions are of course false but are justifiable since we are interested in attacking the scholium problem *within* Newtonian mechanics.
- Furtermore, the terrestial Euclidean reference frame is not an inertial frame. Its mathematical relation to an inertial reference frame is unknown *à priori*.
- More precisely, let $Y = (\mathbf{y}_1 . \mathbf{y}_2 . \dots . \mathbf{y}_N)$ be an inertial reference frame. Then the Euclidean frame X is related to the inertial frame Y by some unknown and possibly time-dependent action of the Euclidean group, i.e. a translation and a rotation: $\mathbf{x}_k = (r \circ \tau)(\mathbf{y}_k)$.


$Y_i = R x_i + T$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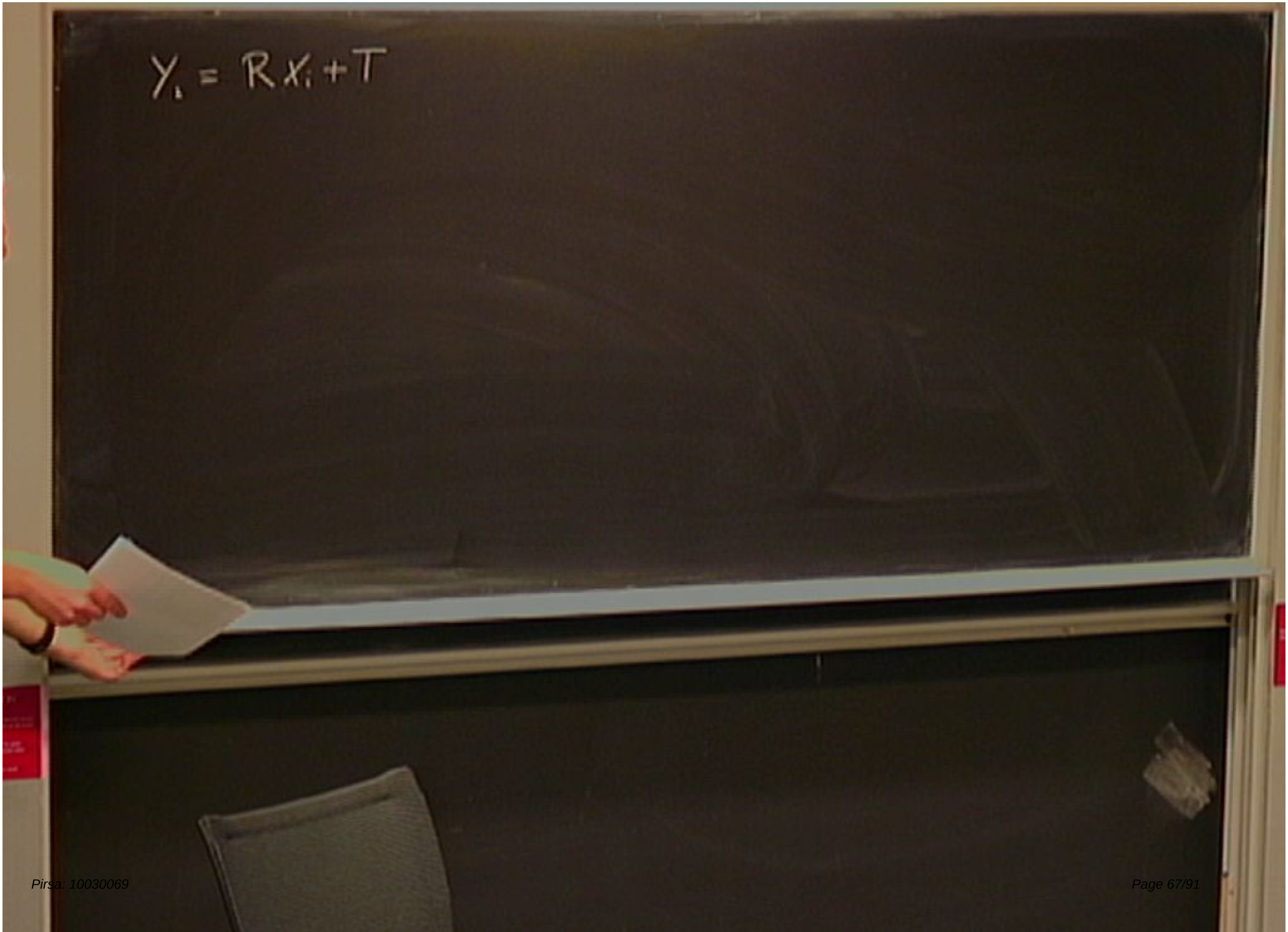
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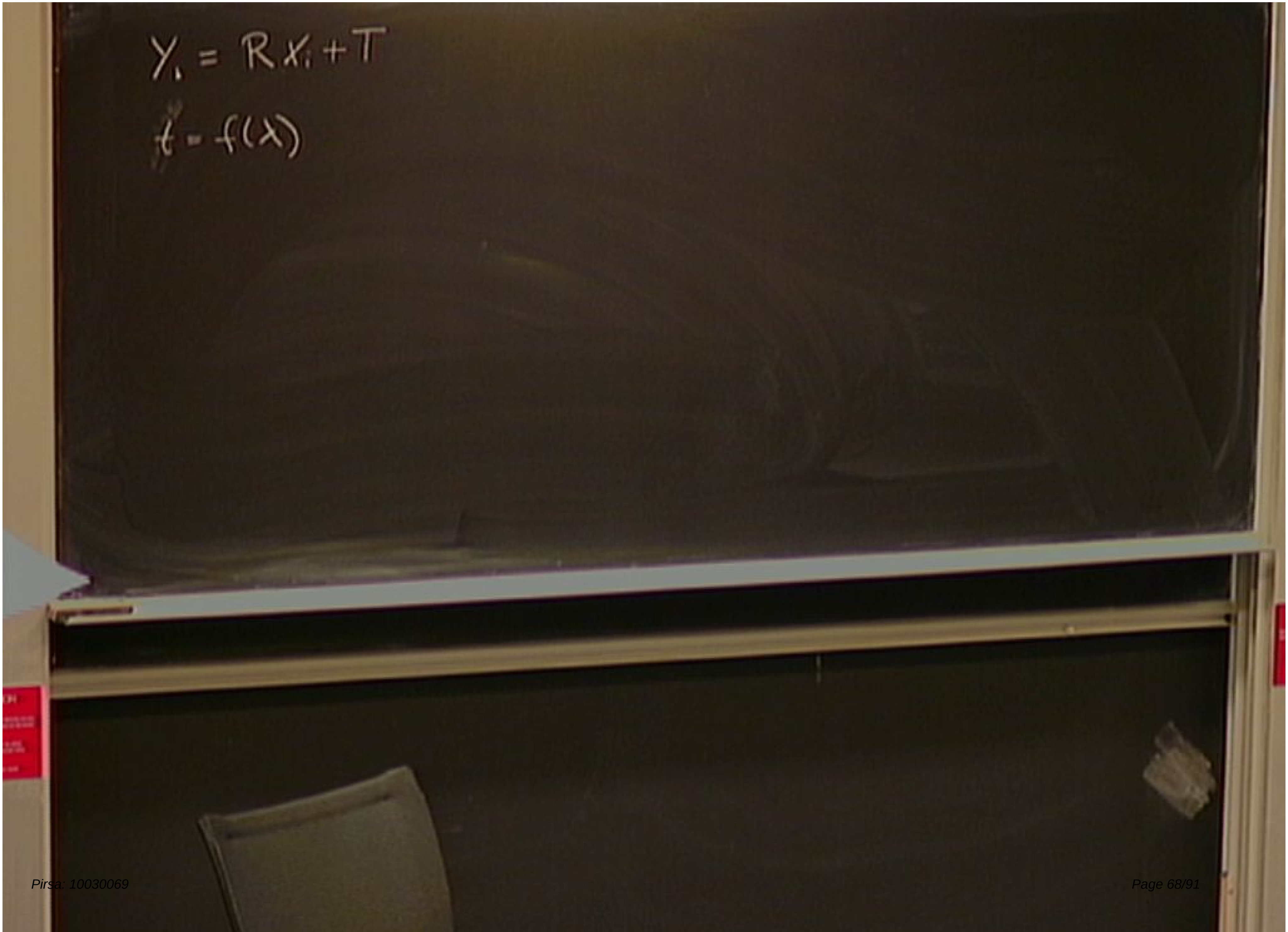
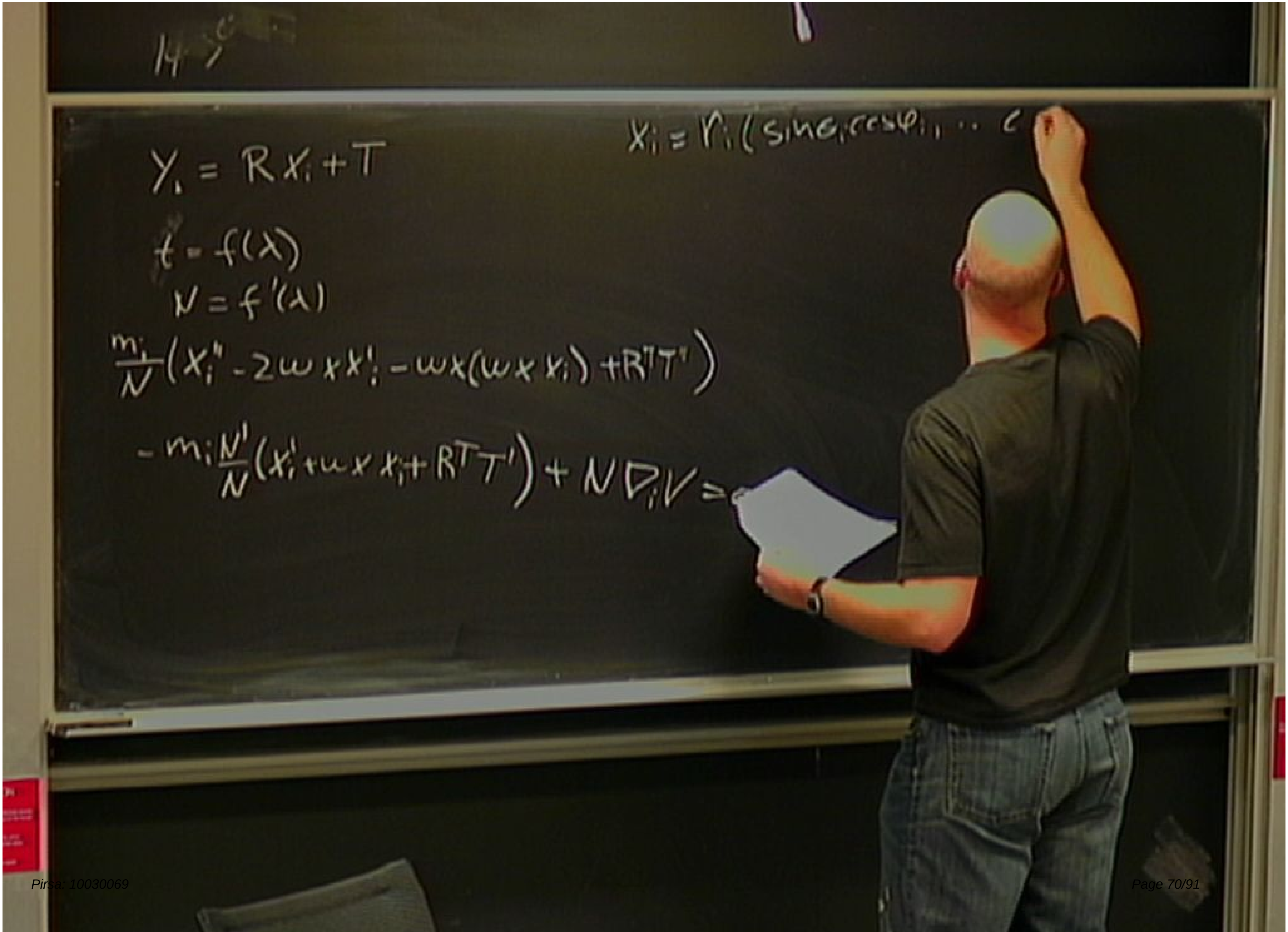

Yi = R Xi + T
t = f(λ)

$$Y_i = R X_i + T$$

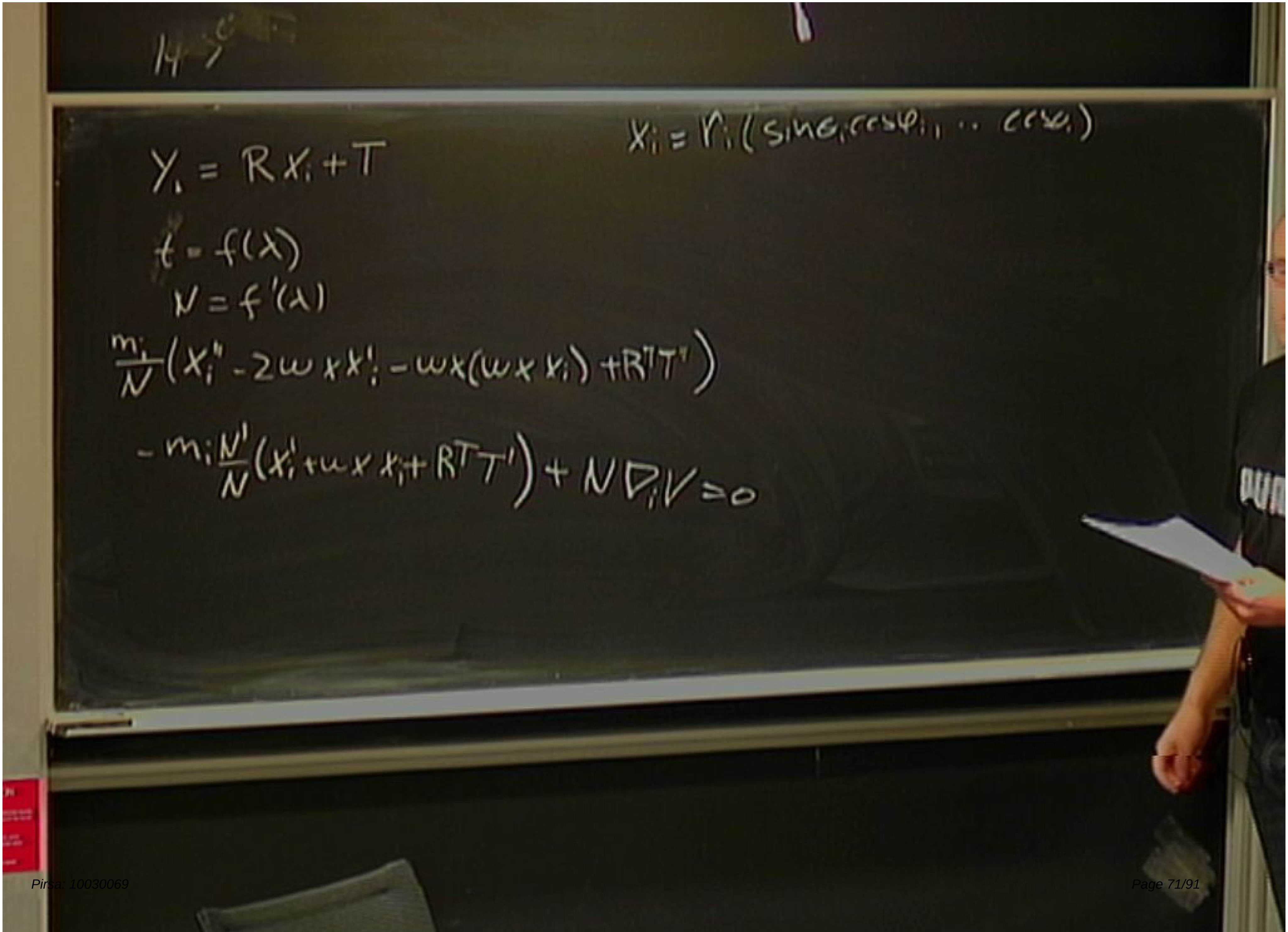
$$t = f(\lambda)$$

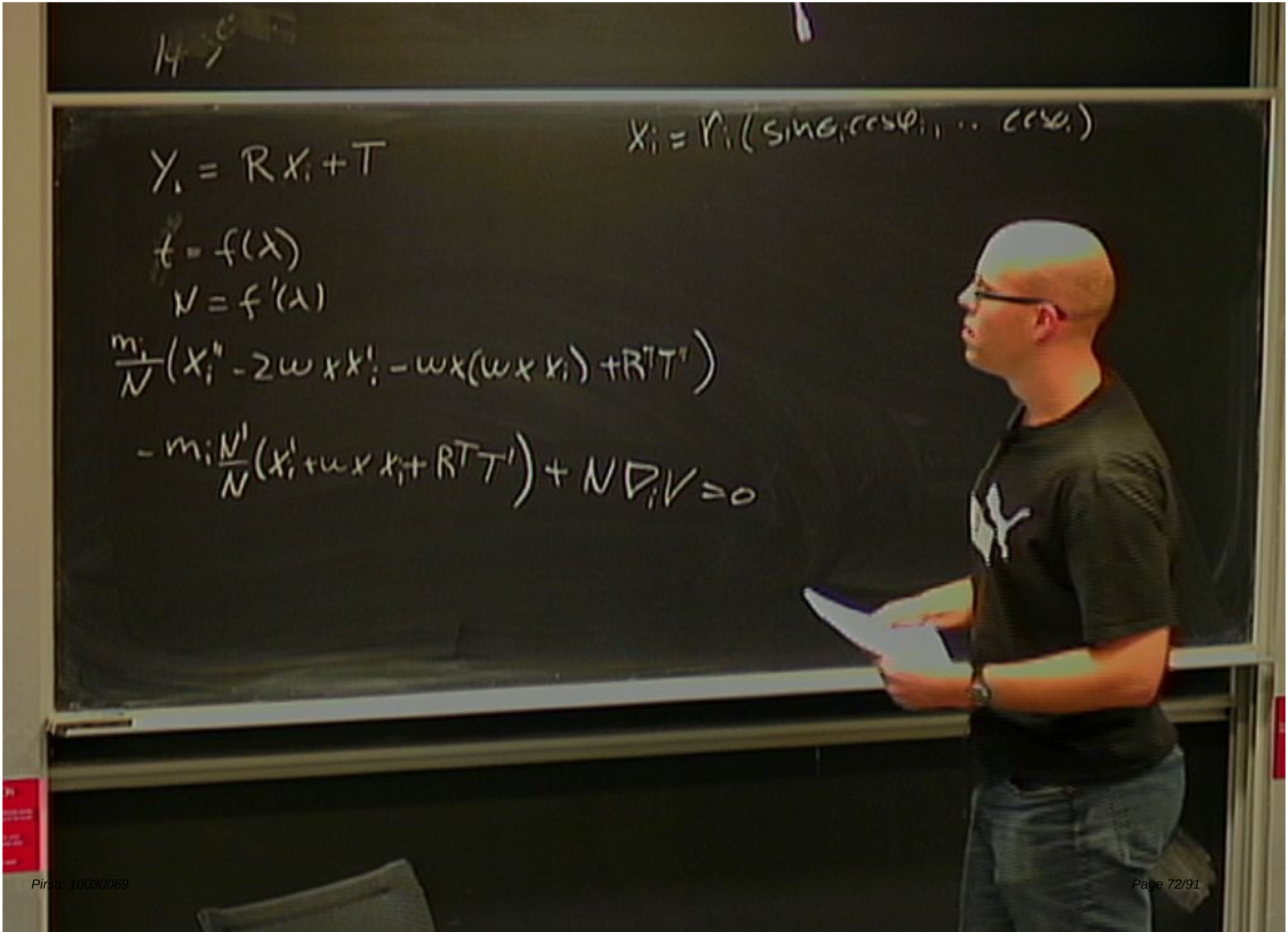
$$\frac{m_i}{N}\left(X_i'' - 2\omega \times X_i' - \omega \times (\omega \times X_i) + R^T T''\right)$$

$$- m_i \frac{N'}{N}\left(X_i' + \omega \times X_i + R^T T'\right) + N \nabla_i V = 0$$



$Y_i = R X_i + T$
$t = f(\lambda)$
$N = f'(\lambda)$
$\frac{m_i}{N}\left(X_i'' - 2\omega \times X_i' - \omega \times (\omega \times X_i) + R^T T''\right)$
$- m_i \frac{N'}{N}\left(X_i' + \omega \times X_i + R^T T'\right) + N \nabla_i V =$


x_i = r_i (sinθ_i cosφ_i1 … cosθ_i)
Y_i = R x_i + T
t = f(λ)
N = f'(λ)
m_i/N (x_i'' - 2ω × x_i' - ω × (ω × x_i) + R^T T'')
- m_i N'/N (x_i' + ω × x_i + R^T T') + N ∇_i V = 0


$X_i = r_i(\sin\theta_i \cos\varphi_i, \ldots \cos\theta_i)$
$Y_i = RX_i + T$
$t = f(\lambda)$
$N = f'(\lambda)$
$\frac{m_i}{N}(X_i'' - 2\omega \times X_i' - \omega \times (\omega \times X_i) + R^T T'')$
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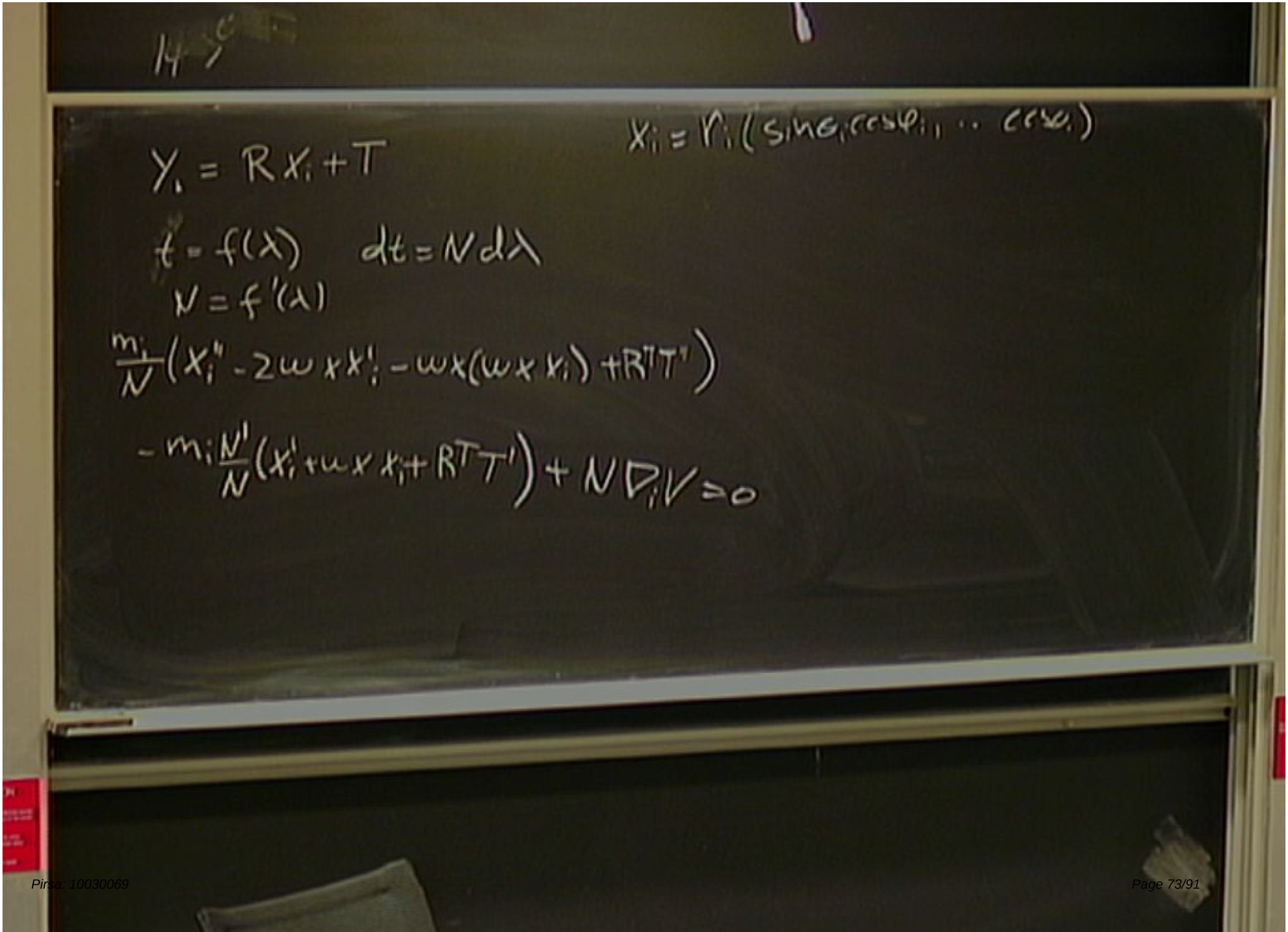
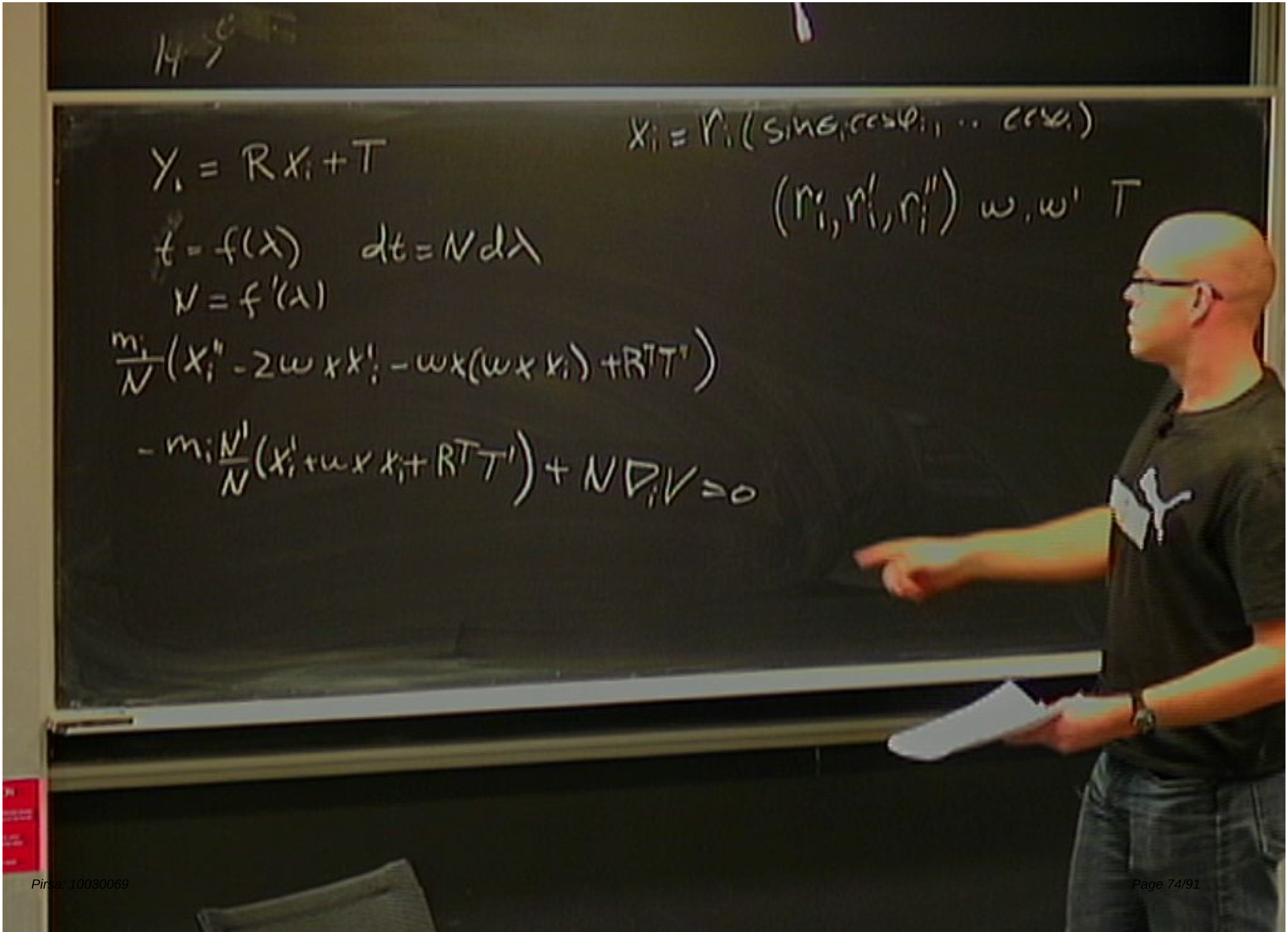

Xi = ri (sinθi cosφi,1 ·· cosθi)
Yi = R Xi + T
t = f(λ)   dt = N dλ
N = f'(λ)
mi/N (Xi'' - 2ω × X'i - ω×(ω × Xi) + RᵀT'')
- mi N'/N (X'i + ω × Xi + RᵀT') + N ∇i V = 0


Y_i = R x_i + T
x_i = r_i (sin θ_i cos φ_i , … cos θ_i)
(r_i, r_i', r_i'') ω, ω' T
t = f(λ) dt = N dλ
N = f'(λ)
m_i/N (x_i'' - 2ω × x_i' - ω × (ω × x_i) + R^T T'')
- m_i N'/N (x_i' + ω × x_i + R^T T') + N ∇_i V = 0

$$X_i = r_i(\sin\theta_i\cos\varphi_i, \ldots \cos\theta_i)$$

$$Y_i = R X_i + T$$

$$(r_i, r_i', r_i'') \quad \omega, \omega' \quad T$$

$$t = f(\lambda) \qquad dt = N\,d\lambda$$

$$N = f'(\lambda)$$

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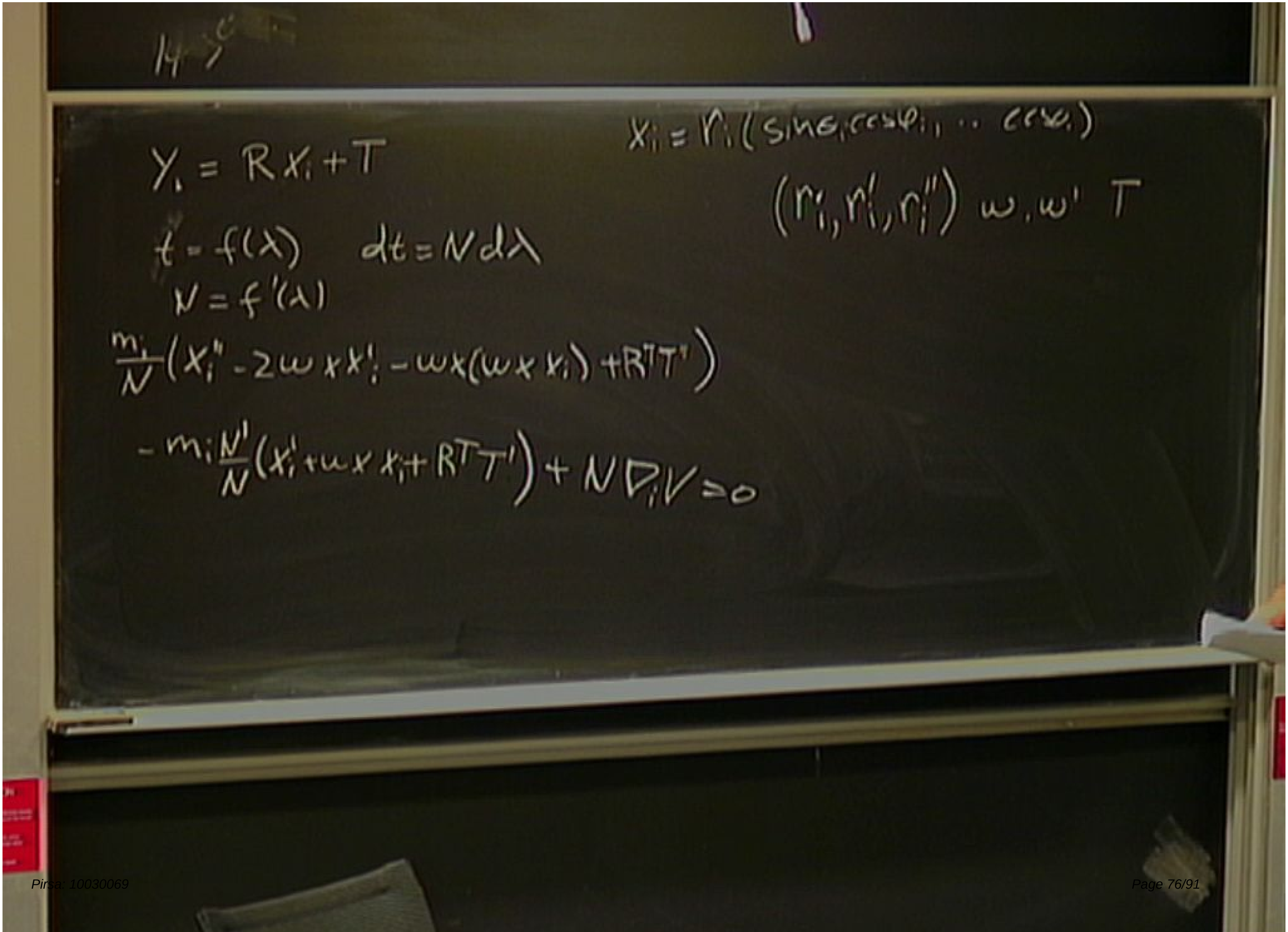
Y_i = R X_i + T
x_i = r_i (sinθ_i cosφ_i ... cosω_i)
(r_i, r_i', r_i'') ω, ω' T
t = f(λ)   dt = N dλ
N = f'(λ)
m_i/N (X_i'' - 2ω × X_i' - ω × (ω × X_i) + R^T T'')
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$$t = f(\lambda) \qquad dt = N d\lambda$$

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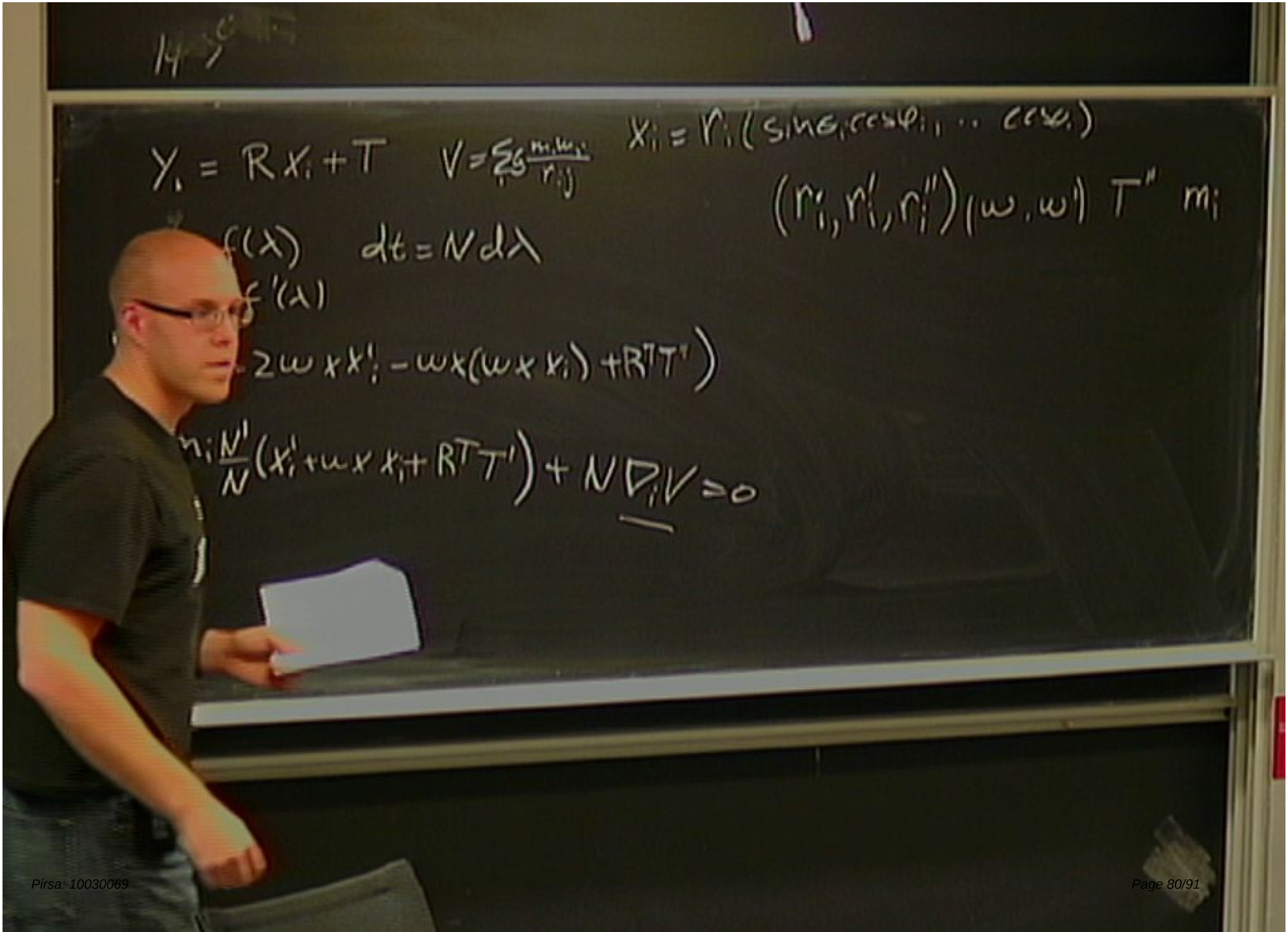

Y_i = R X_i + T
dt = N dλ
f'(λ)
2ω × X'_i − ω × (ω × X_i) + R^T T'
m_i N'/N (X'_i + ω × X_i + R^T T') + N ∇_i V = 0
(r_i, r'_i, r''_i) (ω, ω') T'' m_i

$$Y_i = R X_i + T \qquad V = \sum_{i<j} G \frac{m_i m_j}{r_{ij}} \qquad X_i = r_i (\sin\theta_i \cos\varphi_{i,1} \cdots \cos\varphi_i)$$

$$(r_i, r_i', r_i'') \; (\omega, \omega') \; T'' \; m_i \; G$$

$$t = f(\lambda) \qquad dt = N d\lambda$$

$$N = f'(\lambda)$$

$$\frac{m_i}{N}\left(X_i'' - 2\omega \times X_i' - \omega \times (\omega \times X_i) + R^T T''\right)$$

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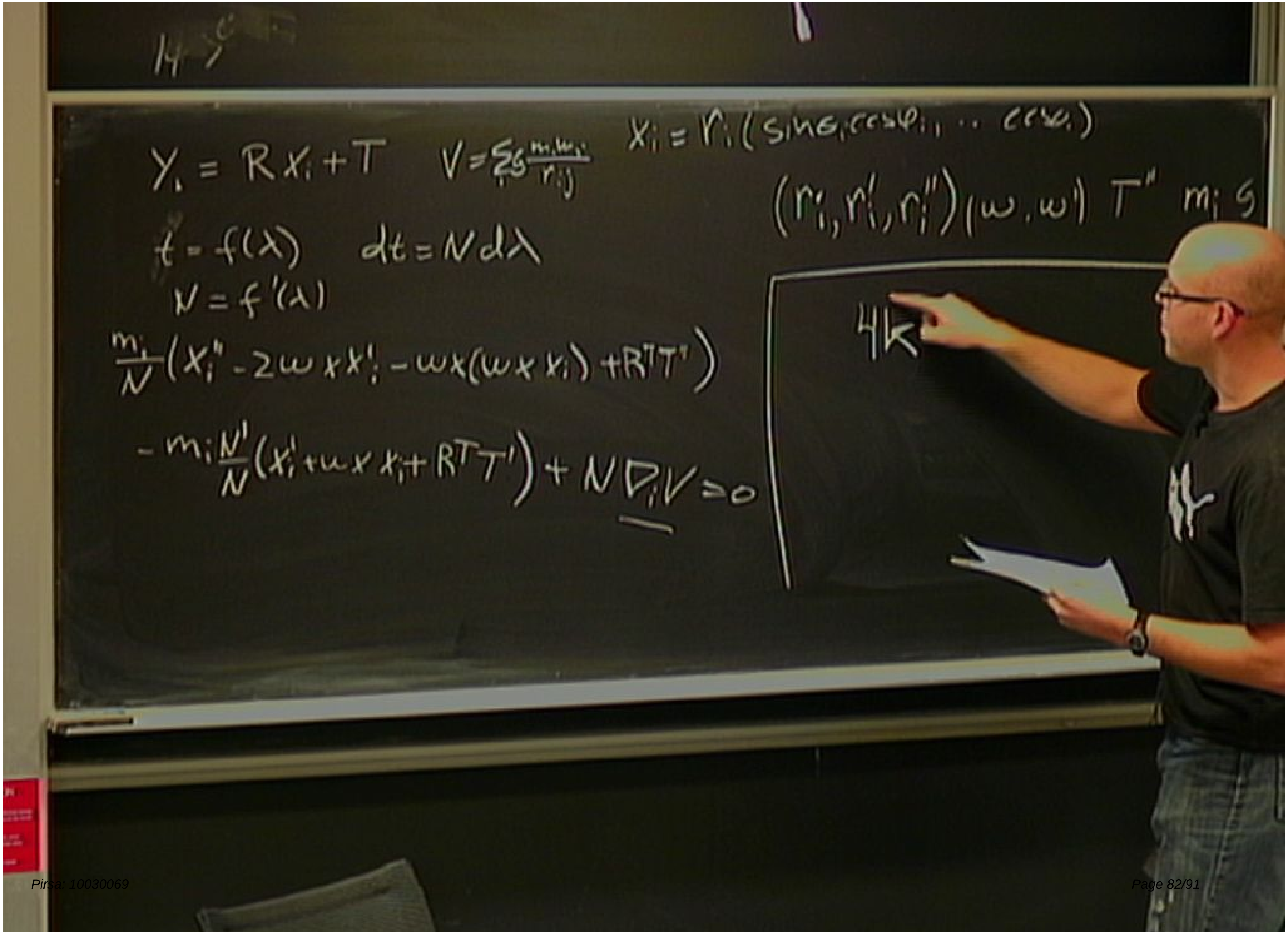

$Y_i = R X_i + T$
$V = \sum G \frac{m_i m_j}{r_{ij}}$
$X_i = r_i(\sin\theta_i \cos\varphi_i, \ldots \cos\theta_i)$
$(r_i, r_i', r_i'')\,(\omega, \omega')\; T''\; m_i\; G$
$t = f(\lambda)$
$dt = N d\lambda$
$N = f'(\lambda)$
$\frac{m_i}{N}(X_i'' - 2\omega \times X_i' - \omega \times (\omega \times X_i) + R^T T'')$
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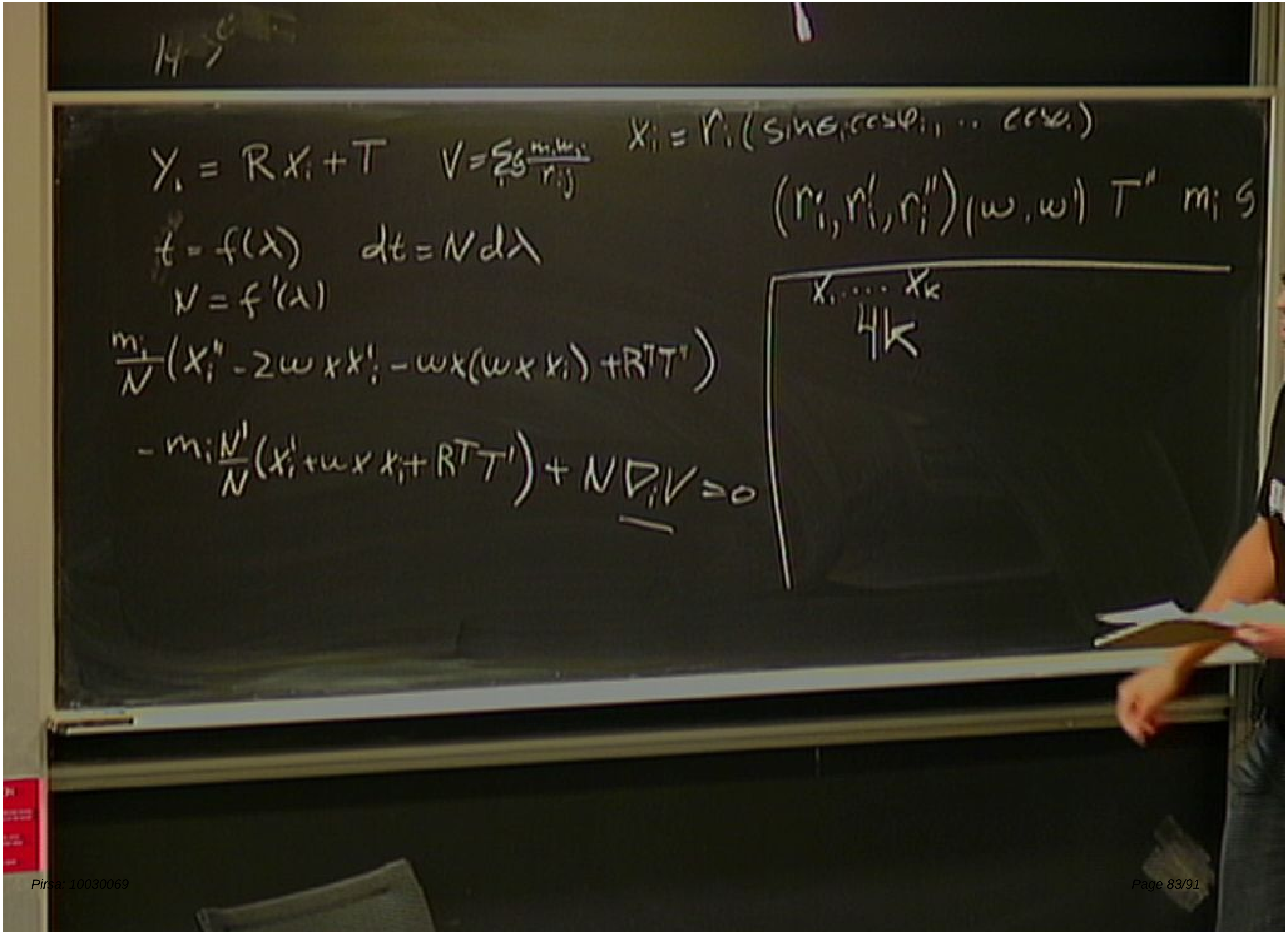
4k


$Y_i = R X_i + T$
$V = \sum_{ij} G \frac{m_i m_j}{r_{ij}}$
$X_i = r_i (\sin\theta_i \cos\varphi_i, \ldots \cos\theta_i)$
(r_i, r_i', r_i'') (ω, ω') T'' m_i G
$t = f(\lambda)$
$dt = N d\lambda$
$N = f'(\lambda)$
$\frac{m_i}{N}(X_i'' - 2\omega \times X_i' - \omega \times (\omega \times X_i) + R^T T'')$
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$X_1 \ldots X_k$
4k

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$4k$

$14 \to$

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$$\underline{X} = (X_1, \dots X_k)$$

$$4k + 9$$

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$(r_i, r_i', r_i'')(\omega, \omega') \; T'' \quad m_i \; G$

$t = f(\lambda) \quad dt = N d\lambda$

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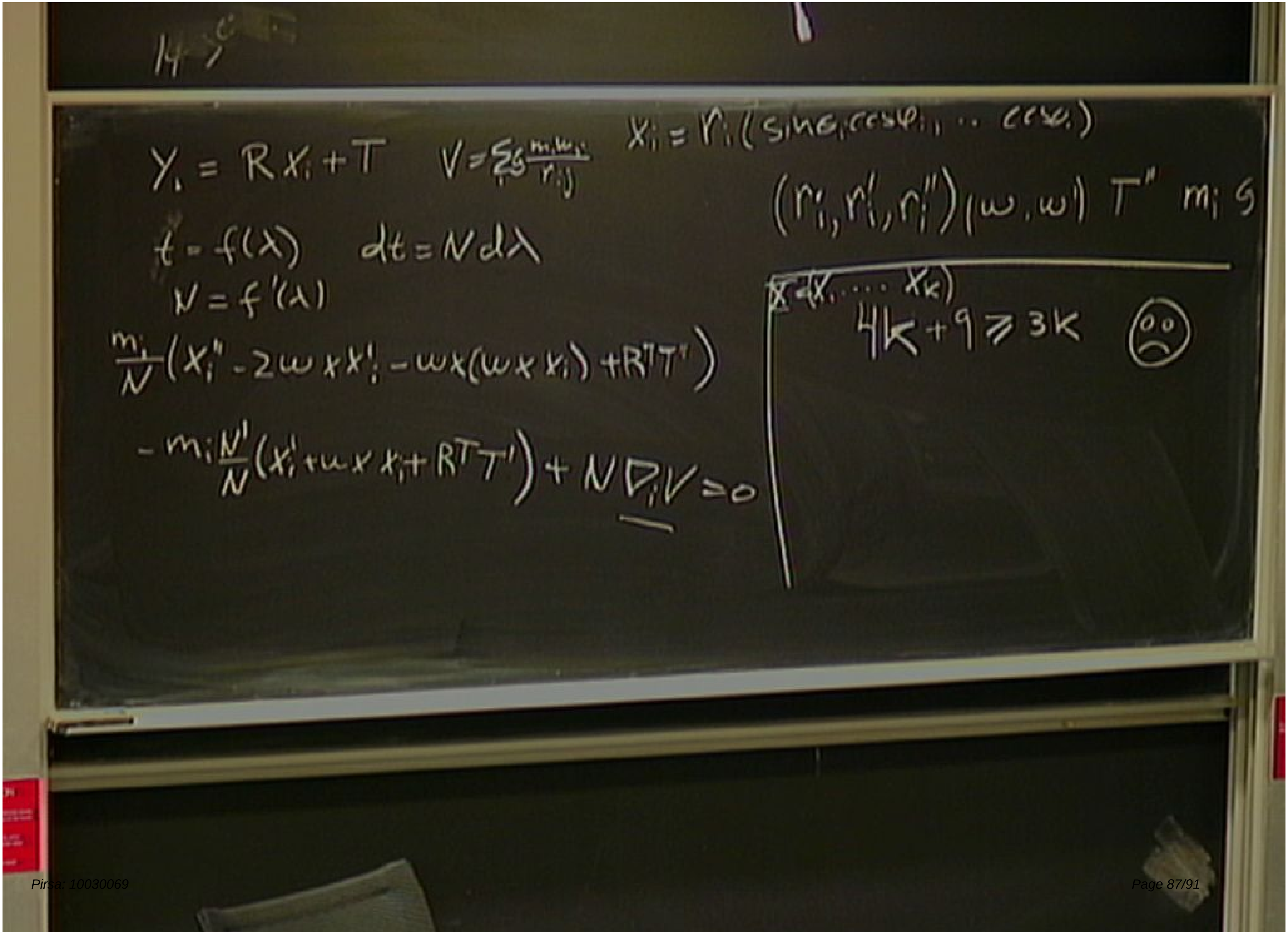
$4k + 9 \geq 3k$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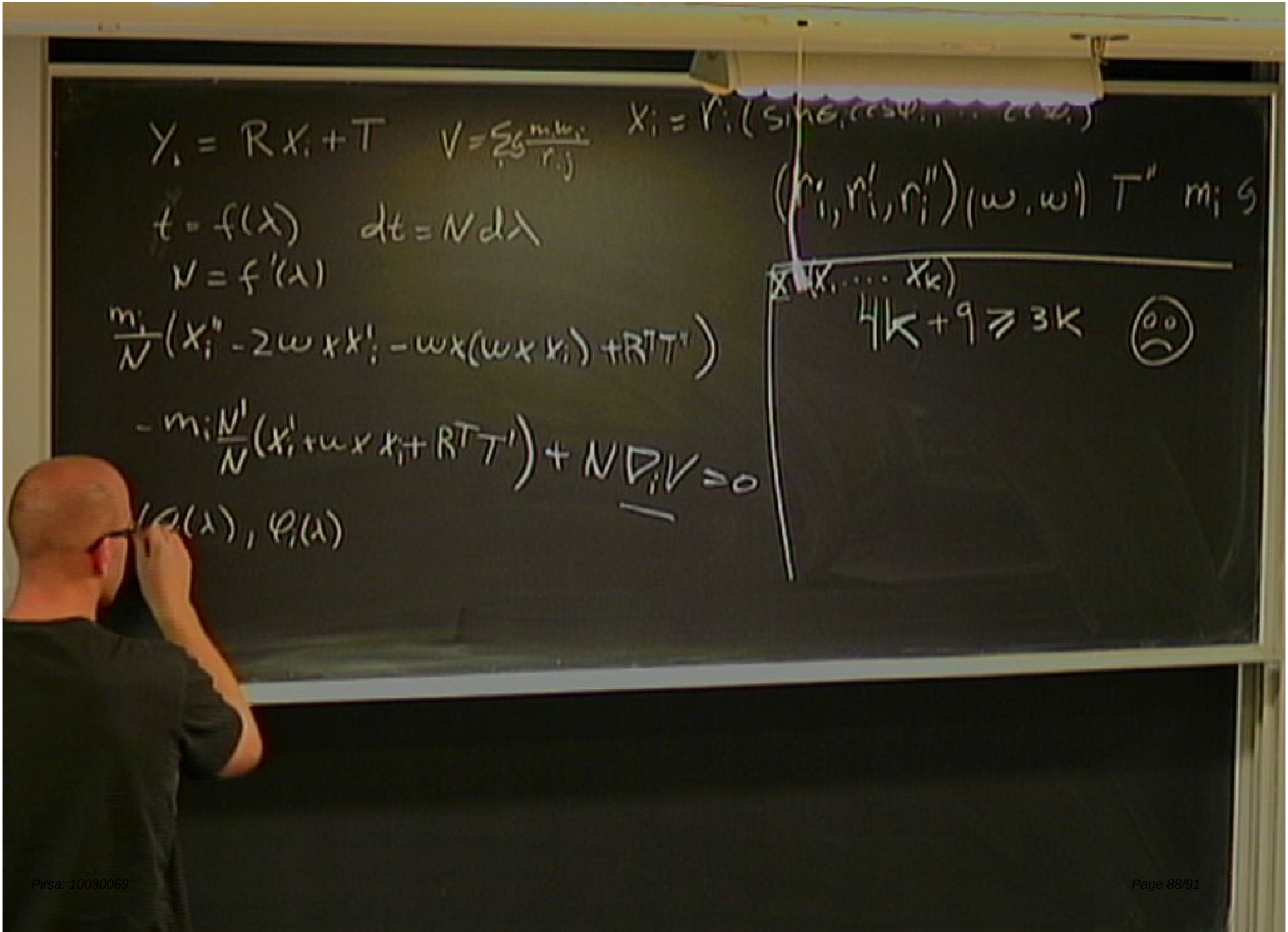
$- m_i \frac{N'}{N^2}(x_i' + \omega \times x_i + R^T T') + N \nabla_i V = 0$
$x = (x_1 \dots x_k)$
$4k + 9 \geq 3k$


Y_i = R X_i + T
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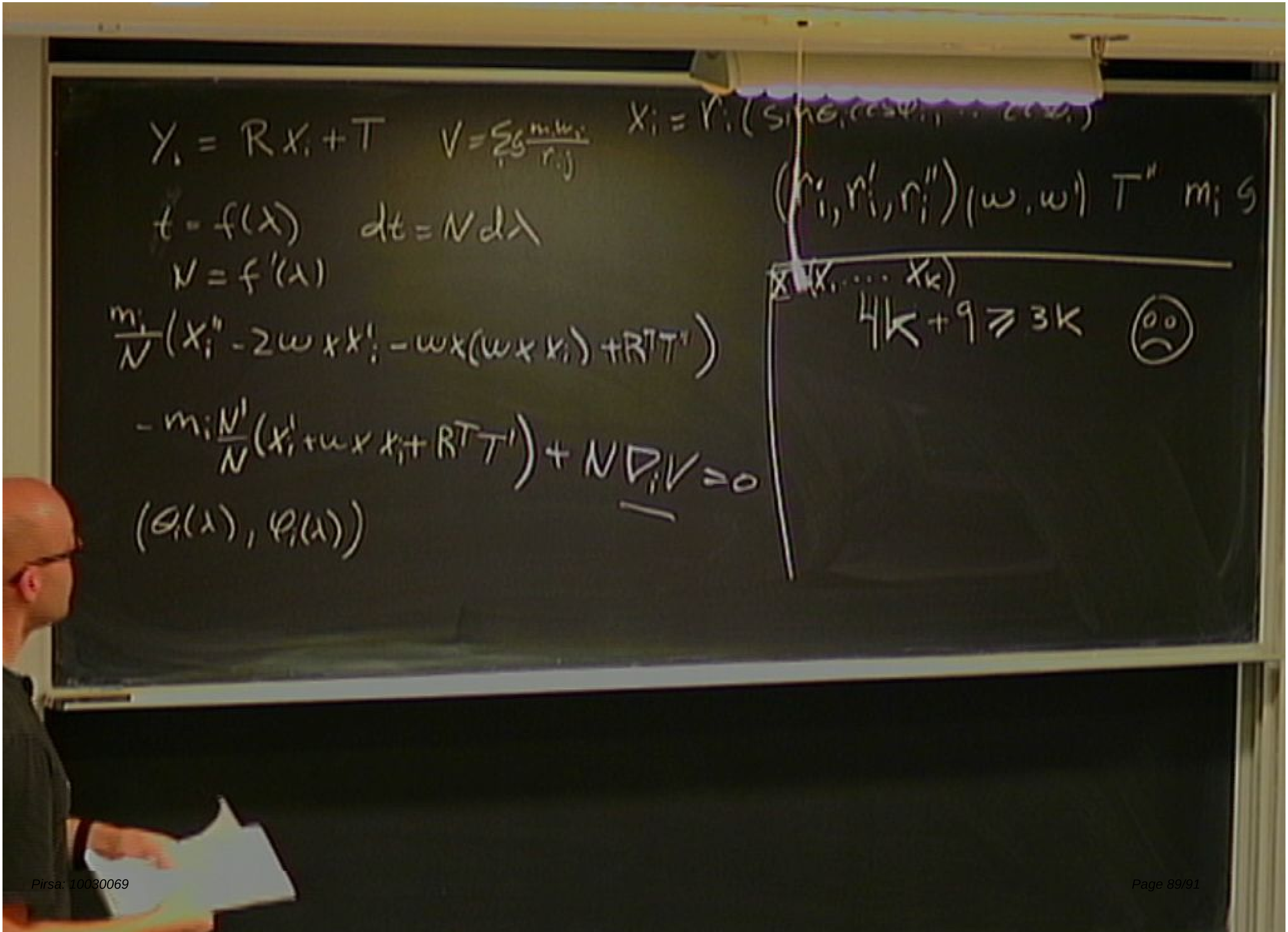
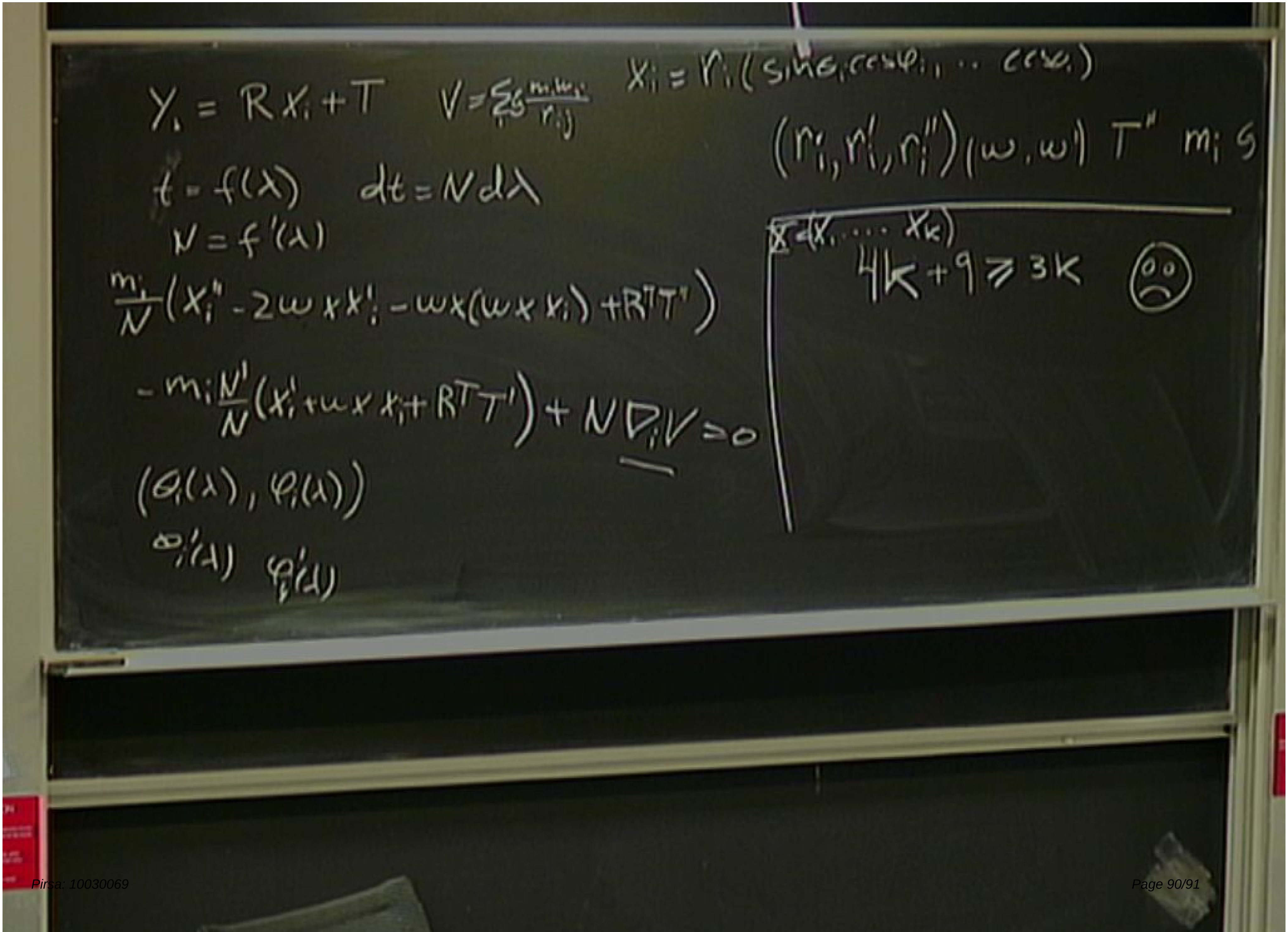
N = f'(λ)
(r_i, r'_i, r''_i) (ω, ω') T'' m_i G
4k + 9 ≥ 3k


Y_i = R X_i + T
V = Σ G m_i m_j / r_ij
X_i = r_i (sin θ_i cos φ_i, ...)
(r_i, r_i', r_i'') (ω, ω') T'' m_i G
t = f(λ)
dt = N dλ
N = f'(λ)
m_i/N (X_i'' - 2ω × X_i' - ω × (ω × X_i) + R^T T'')
- m_i N'/N (X_i' + ω × X_i + R^T T') + N ∇_i V = 0
(θ_i(λ), φ_i(λ))
X (X_1 ... X_k)
4k + 9 ≥ 3k


Y_i = R X_i + T
V = Σ G m_i m_j / r_ij
X_i = r_i (sin θ_i cos φ_i, … cos θ_i)
(r_i, r_i', r_i'') (ω, ω') T'' m_i G
t = f(λ)   dt = N dλ
N = f'(λ)
m_i/N (X_i'' − 2ω × X_i' − ω × (ω × X_i) + R^T T'')
− m_i N'/N (X_i' + ω × X_i + R^T T') + N ∇_i V = 0
(θ_i(λ), φ_i(λ))
θ_i'(λ)   φ_i'(λ)
X = (X_1 … X_k)
4k + 9 ≥ 3k

$Y_i = R X_i + T \quad V = \sum_{ij} G \frac{m_i m_j}{r_{ij}} \quad X_i = r_i (\sin\theta_i \cos\varphi_i, \ldots \cos\theta_i)$

$(r_i, r_i', r_i'')\ (\omega, \omega')\ T''\ m_i\ G$

$t = f(\lambda) \quad dt = N d\lambda$

$N = f'(\lambda)$

$\frac{m_i}{N}\left(X_i'' - 2\omega \times X_i' - \omega \times (\omega \times X_i) + R^T T''\right)$

$- m_i \frac{N'}{N}\left(X_i' + \omega \times X_i + R^T T'\right) + N \nabla_i V = 0$

$(\theta_i(\lambda), \varphi_i(\lambda))$

$\theta_i'(\lambda) \quad \varphi_i'(\lambda)$

$X = (X_1 \ldots X_k)$

$4k + 9 \geq 3k$ ☹

$2k(D - \frac{1}{2}) \geq 16$