Title: The Next-to-Simplest Quantum Field Theories

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Abstract: We apply newly-developed techniques for studying perturbative scattering amplitudes to gauge theories with matter. It is well known that the N=4 SYM theory has a very simple S-matrix; do other gauge theories see similar simplifications in their S-matrices? It turns out the one-loop gluon S-matrix simplifies if the matter representations satisfy some group theoretic constraints. In particular, these constraints can be expressed as linear Diophantine equations involving the higher order Indices (or higher-order Casimirs) of these representations. We solve these constraints to find examples of theories whose gluon scattering amplitudes are as simple as those of the N=4 theory. This class includes the N=2, SU(K) theory with a symmetric and anti-symmetric tensor hypermultiplet. Non-supersymmetric theories with appropriately tuned matter content can also see remarkable simplifications. We find an infinite class of non-supersymmetric amplitudes that are cut-constructible even though naive power counting would suggest the presence of rational remainders.

The Next-to-Simplest Quantum Field Theories

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30 March 2010

arXiv:0910.0930 – Shailesh Lal, S.R.
arXiv:1003.5264 – Shailesh Lal, S.R.

Setting

- Over the past few years we have learned that scattering amplitudes in gauge theories and gravity have fascinating properties.
- The study of these properties has led to the development of efficient techniques to calculate amplitudes that are useful at the LHC.
- More interestingly, we hope that this will provide us with a fresh perspective on quantum field theory itself.

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- The study of these properties has led to the development of efficient techniques to calculate amplitudes that are useful at the LHC.
- More interestingly, we hope that this will provide us with a fresh perspective on quantum field theory itself.
- Much of this work has focused on $\mathcal{N} = 4$ SYM. This seems to have the nicest and simplest amplitudes despite having a very complicated Lagrangian.

Less or no supersymmetry

- Perhaps $\mathcal{N} = 4$ SYM (and $\mathcal{N} = 8$ SUGRA) are "the simplest quantum field theories?"
- However, it would not be so much fun if this programme stopped after finding the S-matrix of planar $\mathcal{N} = 4$ SYM.
- If this way of looking at quantum field theories is useful, we should ask:

  What are the next-to-simplest quantum field theories?

- Can we generalize these S-matrix techniques to these theories?

Summary

- We will scan gauge theories coupled to matter to look for amplitudes that are as simple as $\mathcal{N} = 4$ SYM at one-loop.
- It turns out that the condition for the gluon S-matrix to simplify can be written as set of group-theoretic constraints on the matter-representations.

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- We will scan gauge theories coupled to matter to look for amplitudes that are as simple as $\mathcal{N} = 4$ SYM at one-loop.
- It turns out that the condition for the gluon S-matrix to simplify can be written as set of group-theoretic constraints on the matter-representations.
- We will find two theories where gluon amplitudes are as simple as those of $\mathcal{N} = 4$ SYM.
- We will also find several theories (including non-supersymmetric ones) that see simplifications in their S-matrices but to a lesser extent than $\mathcal{N} = 4$ SYM.

Outline

Motivation

Review

- Tree Amplitudes
- Loop Amplitudes

Deriving the group theoretic constraints

- Bubble and Triangle Coefficients
- Rational Terms
- The Constraints: Putting Everything Together

Indices: A detour into group theory

The next-to-simplest quantum field theories

- Only Boxes
- No Bubbles
- No Rational Terms

Summary

Parke-Taylor formula (How it all began)

- Consider a tree-level gluon scattering amplitude where 2 gluons have negative helicity and all others have positive helicity.
- One might suspect that this amplitude is a mess. If we have 100,000 gluons, then this amplitude is related to the 100,000 pt correlation function in YM theory. This is ugly even at tree-level!

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- Answer is very beautiful and very simple. Called The Parke-Taylor Formula (from 1986):

$$M^{--++\cdots} = \frac{\langle \lambda_1, \lambda_2 \rangle^4}{\langle \lambda_1, \lambda_2 \rangle \langle \lambda_2, \lambda_3 \rangle \ldots \langle \lambda_n, \lambda_1 \rangle}$$

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BCFW Relations

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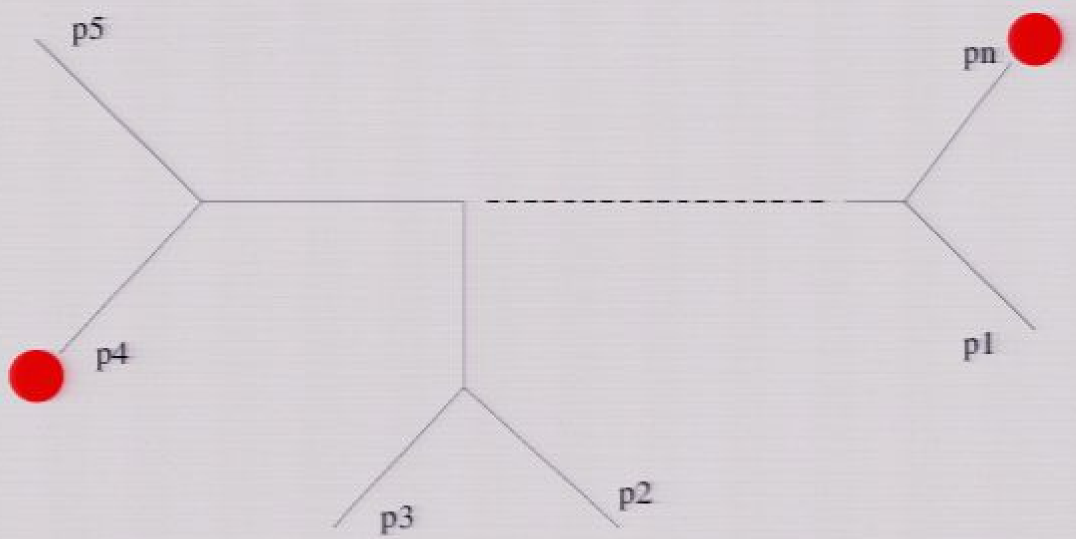
- Consider a n-point gluon amplitude.

Figure: BCFW EXTENSION



- Extend *any* two momenta on shell

$$p_4 \to p_4 + qz; \quad p_n \to p_n - qz$$
$$q^2 = q \cdot p_4 = q \cdot p_n = 0$$

- For each p, one of two gauge boson polarization vectors also grows as $O(z)$.

Next-to-Simplest QFTs

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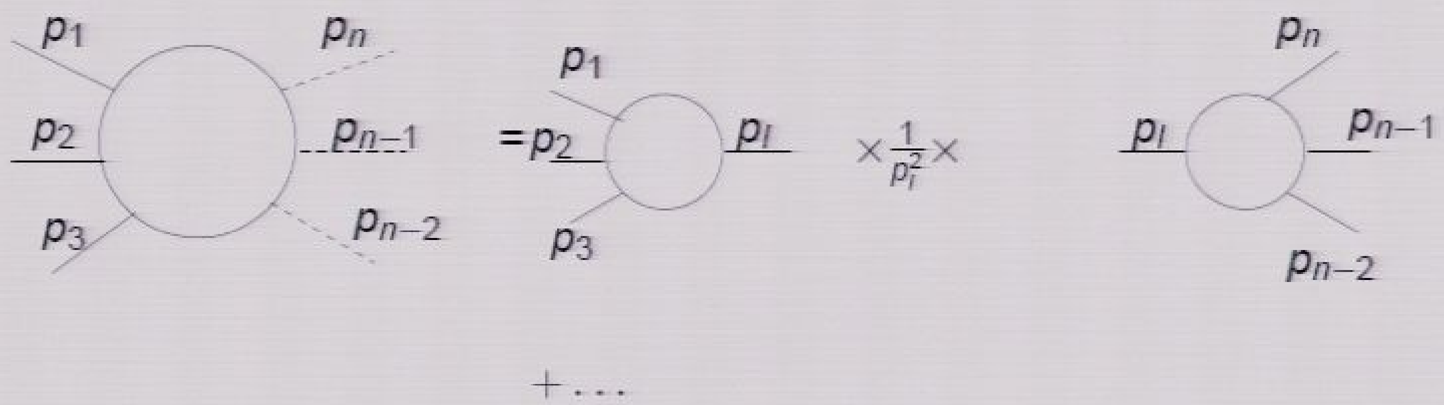
- Poles in the amplitude occur when an internal line goes on shell. Residues are lower pt on-shell amplitudes.
- So,

$$M(z) \sim \sum_{\text{partitions}} M_{\text{left}} \frac{1}{P_L^2(z)} M_{\text{right}}$$

Schematic BCFW

Figure: Recursion Relations



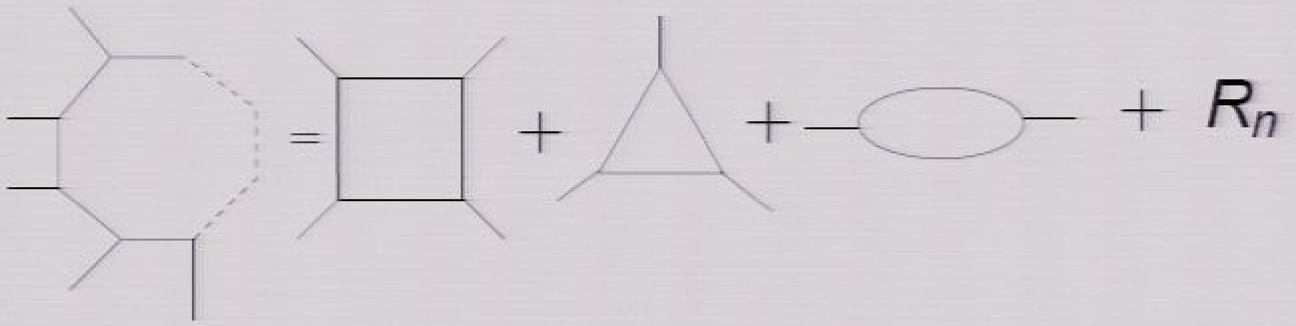
Gravity, $\mathcal{N} = 4$ SYM and $\mathcal{N} = 8$ SUGRA

- For gravity, these recursion relations are even more useful. Perturbative gravity is a mess!. It has an infinite set of interaction vertices and already 2850 terms in the 4-pt interaction (DeWitt '67)
- Here, everything comes from a 3-pt on-shell function that is determined by Lorentz invariance.
- These recursion relations can also be generalized to $\mathcal{N} = 4$ SYM and $\mathcal{N} = 8$ SUGRA. (Brandhuber et al. '08, Arkani-Hamed et al. '08)
- $\mathcal{N} = 4$ SYM and $\mathcal{N} = 8$ SUGRA amplitudes are even nicer than pure YM/pure gravity.
- In the latter, one needs to worry about the external polarizations to ensure the BCFW deformed amplitude is well behaved at ∞. For $\mathcal{N} = 4$ and $\mathcal{N} = 8$ all tree-amplitudes behave well at ∞ under the modified BCFW extension.

On-shell techniques at one-loop

- ANY one loop amplitude in any quantum field theory can be written as a sum of scalar boxes, triangles and bubbles with rational coefficients and a possible rational remainder. (Passarino, Veltman 1979)

Figure: ONE LOOP DECOMPOSITION



- What is new is that we have learned to efficiently calculate these rational coefficients.(Forde '08, Arkani-Hamed et al. '08, Badger '08)
- Surprisingly, triangle and bubble coefficients are calculated through products of BCFW deformed tree-amplitudes!

The simplest quantum field theories

- Recall that at tree-level, in the $\mathcal{N} = 4$ theory (and the $\mathcal{N} = 8$ theory), all amplitudes die off at large z die off under an appropriate BCFW extension.
- Using this, one can show that the $\mathcal{N} = 4$ theory and $\mathcal{N} = 8$ theory have only boxes at one-loop. Called the no-triangle property.
- In most theories, we expect all terms – boxes, triangles, bubbles and rational terms to be present. So, the scattering amplitudes of these two theories have the simplest possible analytic structure.
- The "leading singularity conjecture" states that this property persists to all loops.
- Further, it seems possible to calculate the leading singularity of a n pt. amplitude with total helicity $n - 2k$ as contour integrals in the space of k-planes in n dimensions. (Arkani-Hamed et al. '09)

A question?

- Are there other amplitudes that, at one-loop, can be written only in terms of boxes.

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- Are there other amplitudes that, at one-loop, can be written only in terms of boxes. (we know one set: photon amplitudes in QED with more than 8 external photons (Badger et al. '08) Are there others?)
- Are there amplitudes that have triangles but no bubbles or rational terms?
- Are there non-supersymmetric amplitudes that have no rational terms?

Gauge theories with matter

- We will study gluon amplitudes in gauge theories with matter.
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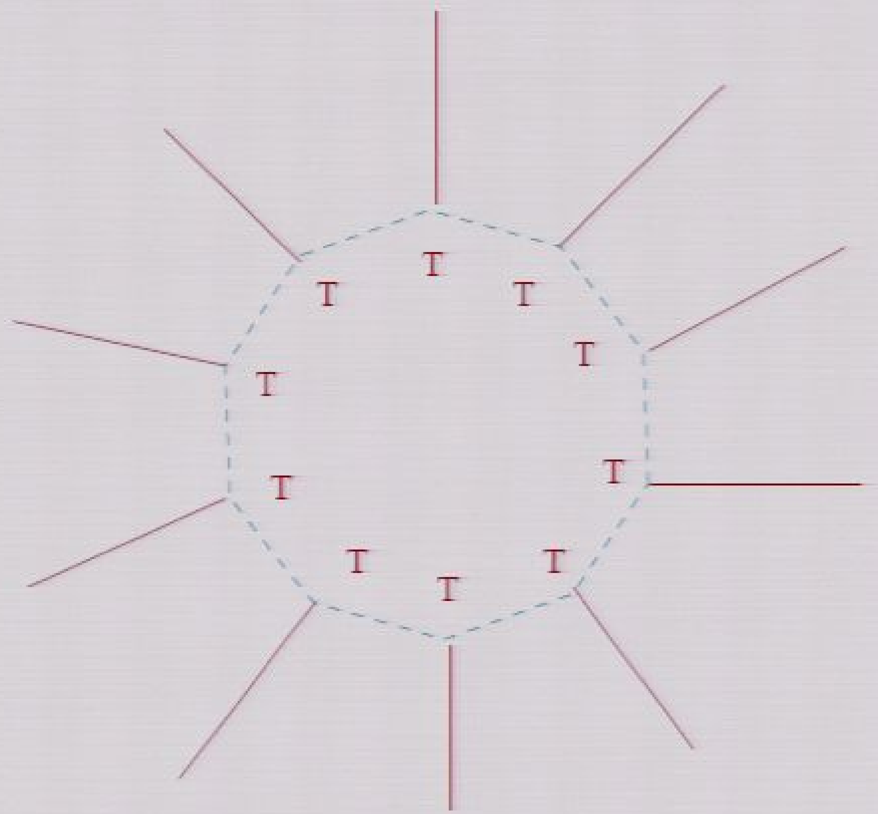
Gauge theories with matter

- We will study gluon amplitudes in gauge theories with matter.
- It turns out that each of the conditions above can be represented as group-theoretic constraints on the matter-representations.
- No Triangles $\supset$ No Bubbles $\supset$ No Rational Terms.
- We find two theories where gluon amplitudes have no triangles, many theories with triangles but no bubbles and an infinite class of non-supersymmetric theories with no rational terms.

Outline

Matter contribution at one loop



$\sim \mathrm{Tr}\left(T^{10}\right)$

Figure: Are Matter Contributions Very Complicated?

Matter contribution is simpler than expected

- At first sight, the contribution of matter seems very complicated. A 100-gluon amplitude can get a contribution from 100 generators.

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- At first sight, the contribution of matter seems very complicated. A 100-gluon amplitude can get a contribution from 100 generators.
- So, if we replace the adjoint matter of the $\mathcal{N} = 4$ theory with matter in a different representation how can we hope to retain the simplicity of the $\mathcal{N} = 4$ S-matrix?
- Turns out that matter contributions at one-loop are not so complicated. Triangles, bubbles and rational terms are not sensitive to the entire character of the matter-representation; just to a few of its invariants.

Idea Behind the Simplification

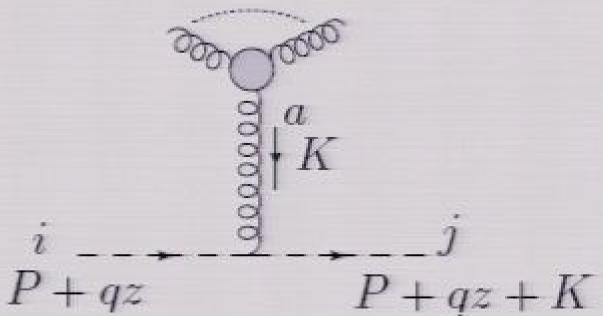
- The contribution of matter to bubble coefficients can be calculated in terms of the product of two BCFW extended tree-amplitudes (similar story for triangles.)

$$C = \int d\Omega \oint_{z=\infty} \frac{dz}{2\pi i z} \sum \mathcal{A}^{\rm t}_{\rm left}(p_1 + qz, -p_2 - qz, \ldots)$$
$$\mathcal{A}^{\rm t}_{\rm right}(-p_1 - qz, p_2 + qz, \ldots).$$

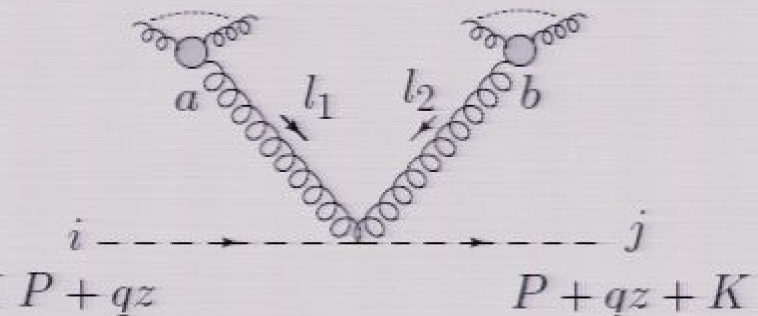
- Each of these tree-amplitudes involves two matter particles and an arbitrary number of gluons.
- Under a BCFW extension, this amplitude simplifies.

Many gluons and two BCFW-extended scalars

- Choose q-lightcone gauge, $q \cdot A = 0$.



(a) O(z) Contribution



(b) O(1) contribution

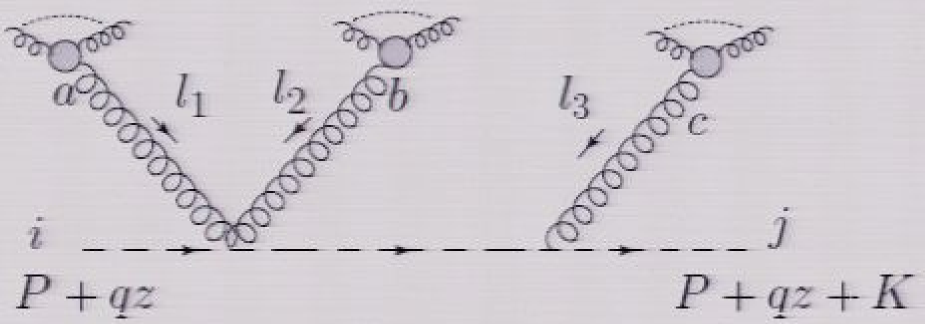


Figure: O($\frac{1}{z}$) Contribution

- More generators come with more powers of $\frac{1}{z}$.

Many gluons and 2 matter particles

- We can prove that

$$\mathcal{A}(s^-, s^+, \ldots) = \sum_{p=1}^{n} \frac{c_{a_1 \ldots a_p}}{z^{p-2}} \left[T^{(a_1} \ldots T^{a_p)} \right]_{ji} .$$

where $s^\pm$ are the two scalars, $\ldots$ denotes the external gluons and for large z, $c \to O(1)$.

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  where $s^{\pm}$ are the two scalars, $\ldots$ denotes the external gluons and for large z, $c \to \mathrm{O}(1)$.
- An identical result holds for fermions.
- Moreover, in supersymmetric theories a remarkable simplification occurs:

$$c^s - c^f = [\frac{1}{z}].$$

Contribution of matter to bubbles: non-susy

- We can now put these results together. Extracting the z^0 coefficient in the product of two tree-amplitudes tells us

$$C_{s/f} = \sum_{n=2,4} \omega_{a_1 \ldots a_n} \mathrm{Tr}_{R_s/R_f}(T^{(a_1} \ldots T^{a_n)}),$$

- The ω are independent of the representation.
- We can never get more than the symmetrized trace of four generators.

Contribution of matter to bubbles: susy

- In supersymmetric theories, this result simplifies.
- This is because, since the fermion and scalar BCFW extended tree-amplitudes are the same to leading order, the leading contribution to the bubble coefficient cancels.
- A chiral multiplet only contributes to the bubble coefficient through the trace of two generators.

$$C_\chi = \frac{\omega_{a_1 a_2}}{2} \mathrm{Tr}_{R_\chi}(\{T^{a_1}, T^{a_2}\})$$

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Contribution of matter to triangles

- The triangle coefficient depends on the product of three tree-amplitudes.

$$B = \sum_{\pm} \oint_{z=\infty} \frac{dz}{4\pi\sqrt{r^2 - z^2}} A^{\mathrm{t}}(p_1^{\pm}(z), -p_2^{\pm}(z), \ldots)$$
$$A^{\mathrm{t}}(-p_1^{\pm}(z), p_3^{\pm}(z), \ldots) A^{\mathrm{t}}(p_2^{\pm}(z), -p_3^{\pm}(z), \ldots).$$

- The dependence on z is not quite the BCFW deformation but at large z the difference is unimportant.
- Recall,

$$\mathcal{A}(s^-, s^+, \ldots) = \sum_{p=1}^{n} \frac{c_{a_1 \ldots a_p}}{z^{p-2}} \left[T^{(a_1} \ldots T^{a_p)} \right]_{ji}.$$

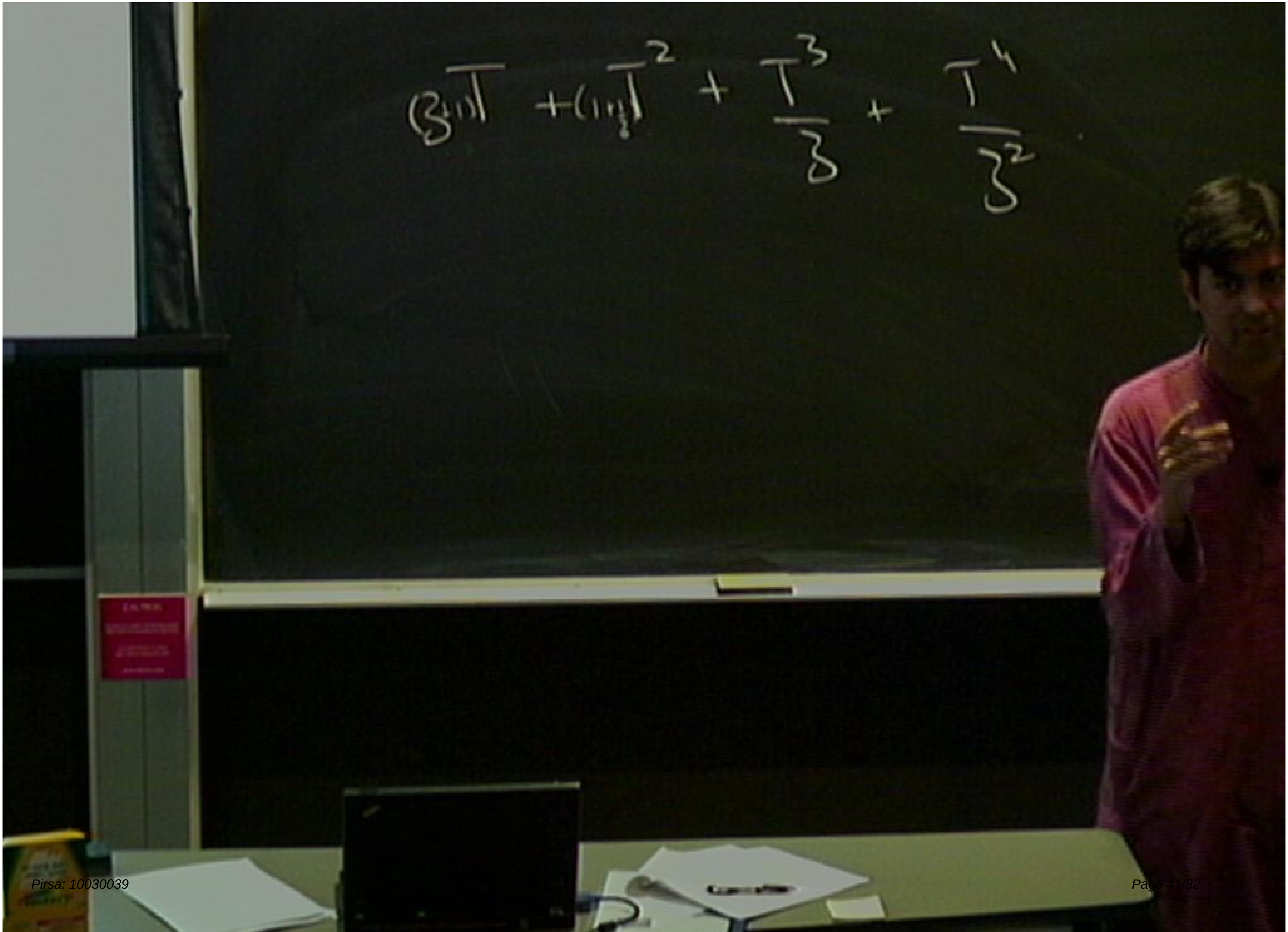
Contribution of matter to triangles

- It is clear that for non-susy theories, the triangle coefficient can depend on a symmetrized trace of up to six generators.

$$B_{s/f} = \sum_{n=2,4,5,6} \omega_{a_1 \ldots a_n} \mathrm{Tr}_{R_s/R_f}(T^{(a_1} \ldots T^{a_n)}).$$

- For susy theories, the triangle coefficient can depend on a symmetrized trace of up to five generators.

$$B_{\chi} = \sum_{n=2,4,5} \omega_{a_1 \ldots a_n} \mathrm{Tr}_{R_\chi} \left(T^{(a_1} \ldots T^{a_n)} \right)$$


$(3+1)T + (1+\frac{1}{3})T^2 + \frac{T^3}{3} + \frac{T^4}{3^2}$

Contribution of matter to rational terms

- Supersymmetric amplitudes do not have rational remainders.
- For non-supersymmetric amplitudes, the rational terms are notoriously hard to extract.
- However, recently a clever way has been developed to calculate these terms (Badger '08).

Rational terms via a fake mass

- Note, that since susy amplitudes have no rational terms, we only need the contribution of scalars (the contribution of fermions is the negative of this).
-
  1. To extract the rational term, assign the scalar in the loop a fake mass μ.
  2. Now, calculate the box (A), triangle (B) and bubble (C) coefficients as usual.
  3. Consider the large mass limit of these coefficients.
  4. The rational terms are given by

  $$\mathcal{R} = \#A|_{\mu^4} + \#B|_{\mu^2} + \#C|_{\mu^2}$$

  where # are some coefficients.
- The key for us is the "large mass" bit. Once again, tree amplitudes with two very massive scalars and many gluons simplify.

Contribution to rational terms through boxes

- For box coefficients, this simplification is particularly easy to see. The cut momenta, at large mass look like

$$p = p_0(\mu) + |\mu|\,\chi$$

- Here, $\chi^2 = 1$, but that is unimportant; for all purposes, this behaves exactly like a BCFW extension at large μ.
- So, the tree-amplitude goes like

$$\mathcal{A} = \sum_{k=1}^{n} \frac{c_{a_1 \ldots a_k} T^{(a_1} \ldots T^{a_k)}}{|\mu|^{k-2}},$$

Contribution of scalars to rational terms

- The box coefficient depends on the product of four tree amplitudes.
- So the box-contribution to rational terms can only depend on the symmetrized trace of at most four generators.

$$\mathcal{R} = \omega_{i_1 i_2} \mathrm{Tr}_R \left(T^{i_1} T^{i_2} \right) + \omega_{i_1 i_2 i_3 i_4} \mathrm{Tr}_R \left(T^{(i_1} T^{i_2} T^{i_3} T^{i_4)} \right) .$$

- We can repeat this analysis for the triangle- and bubble-contribution to rational terms. This does not change the result.

A summary of matter contributions

- To summarize, the contribution of matter to triangles, bubbles and rational terms only depends on the trace of a small number of generators of the matter representation. This is true for arbitrary numbers of external legs.

A summary of matter contributions

- For triangles:

$$B_{s/f} = \sum_{n=2,4,5,6} \omega^{B}_{a_1 \ldots a_n} \mathrm{Tr}_{R_s/R_f}(T^{(a_1} \ldots T^{a_n)}), \quad \text{non-susy}$$

$$B_{\chi} = \sum_{n=2,4,5} \omega^{B}_{a_1 \ldots a_n} \mathrm{Tr}_{R_\chi}(T^{(a_1} \ldots T^{a_n)}, \text{susy}$$

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- For bubbles:

$$C_{s/f} = \sum_{n=2,4} \omega^{C}_{a_1 \ldots a_n} \mathrm{Tr}_{R_s/R_f}(T^{(a_1} \ldots T^{a_n)}), \quad \text{non-susy}$$

$$C_{\chi} = \omega^{C}_{a_1 a_2} \mathrm{Tr}_{R_\chi}(T^{a_1} T^{a_2}), \text{susy}$$

- For rational terms:

$$\mathcal{R}_{s/f} = \sum_{n=2,4} \omega^{\mathcal{R}}_{a_1 \ldots a_n} \mathrm{Tr}_{R_s/R_f}(T^{(a_1} \ldots T^{a_n)}), \quad \text{non-susy},$$

$$\mathcal{R}_{\chi} = 0, \quad \text{susy}$$

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- The $\mathcal{N} = 4$ theory can also be thought of as a gauge theory with matter in the adjoint representation.
- What if we find a representation where the trace of a small number of generators mimics the same trace in the adjoint?
- Since triangle and bubble coefficients and rational terms are sensitive only to these traces, such a theory would have a simple S-matrix as well!

Conditions for the S-matrix to Simplify

Condition (**C**): $\mathrm{Tr}_{\mathrm{R}}(\Pi_{i=1}^{n} T^{a_i}) = m\,\mathrm{Tr}_{\mathrm{adj}}(\Pi_{i=1}^{n} T^{a_i}),\ n \leq p$		
Non-susy theories have	only boxes	no bubbles
if R_f satisfies **C** with	p=6, m=4	p=4,m=4
and R_s satisfies **C** with	p=6, m=6	p=4,m=6.
Susy theories have	only boxes	no bubbles
if R_χ satisfies **C** with	p=5, m=3	p=2,m=3.

Table: Conditions for triangles and/or bubbles to vanish

- Furthermore, non-susy theories have no rational terms if

$$\mathrm{Tr}_{R_f}\left(T^{(a_1}\dots T^{a_n)}\right) - \frac{1}{2}\mathrm{Tr}_{R_s}\left(T^{(a_1}\dots T^{a_n)}\right)$$
$$= \mathrm{Tr}_{\mathrm{adj}}\left(T^{(a_1}\dots T^{a_n)}\right), n = 2, 4.$$

- Susy theories never have rational terms

$$C: \mathrm{Tr}_R\left(\prod_{i=1}^{n} T^{a_i}\right) = m\,\mathrm{Tr}_{adj}\left(\prod_{i=1}^{n} T^{a_i}\right),\quad n \leq$$

Non-susy

only boxes: R_f satisfies C with $p=6, m=4$.

R_s " " " $p=6, m=6$

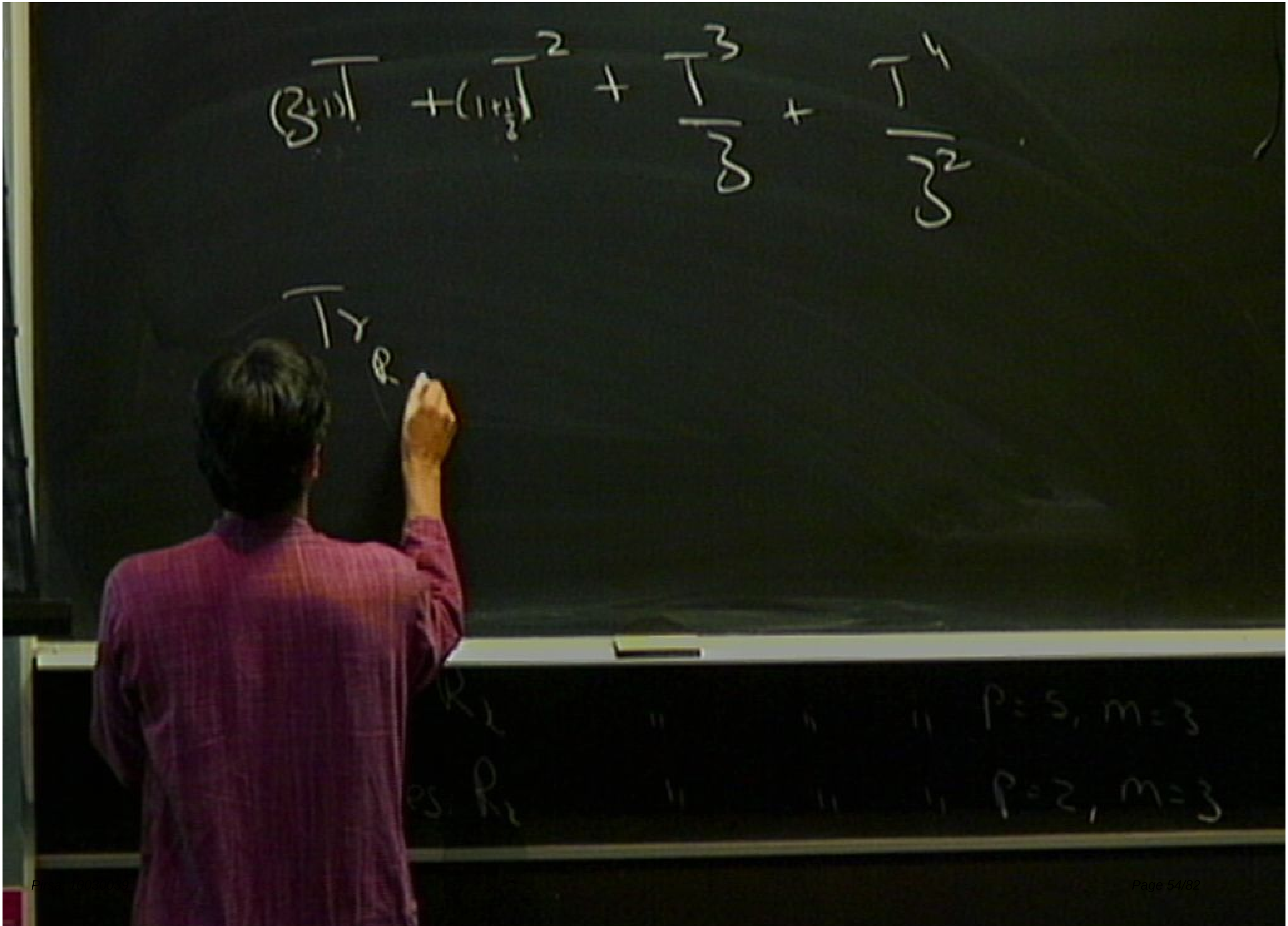
no bubbles: R_f " " " $p=4, m=4$

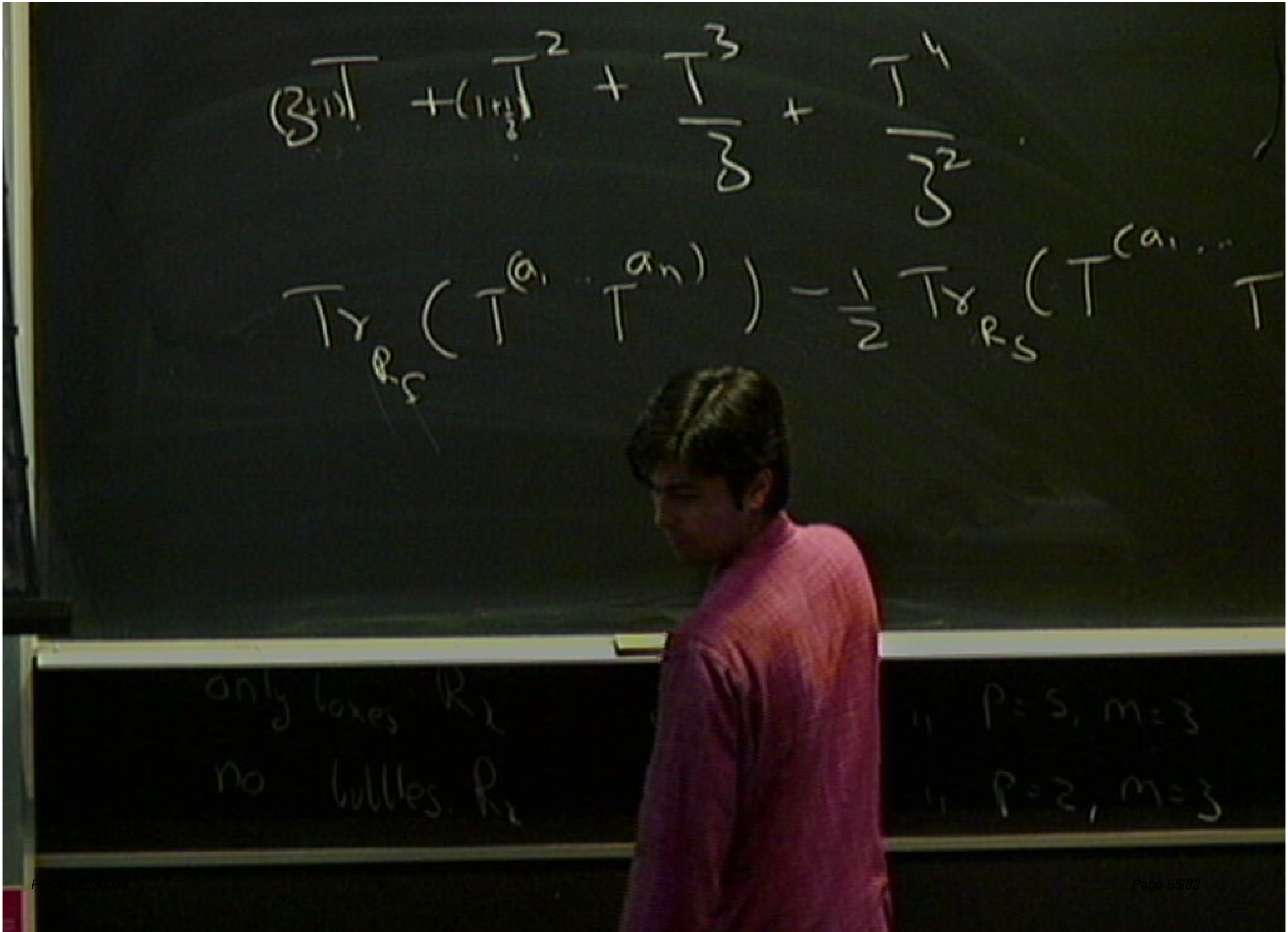
R_s " " " $p=4, m=6$

Susy

only boxes: R_χ " " " $p=5, m=3$

no bubbles: R_χ " " " $p=2, m=3$


p=5, m=3
p=2, m=3


only boxes R_2
no bubbles R_2
$p=5,\ m=3$
$p=2,\ m=3$

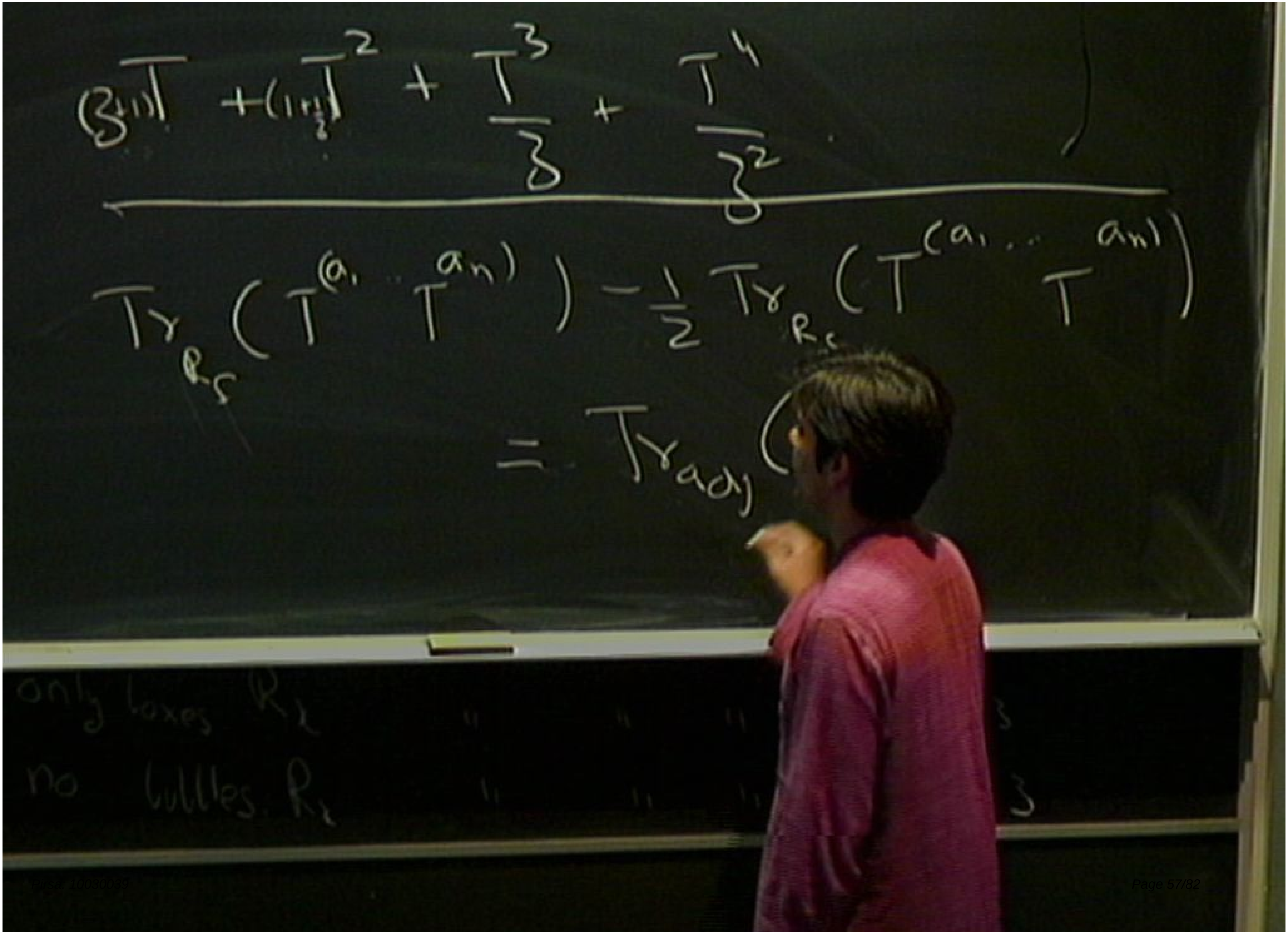
$$(\xi+1)\bar{T} + \left(1+\tfrac{1}{\xi}\right)\bar{T}^2 + \frac{\bar{T}^3}{\xi} + \frac{\bar{T}^4}{\xi^2} \Big)$$

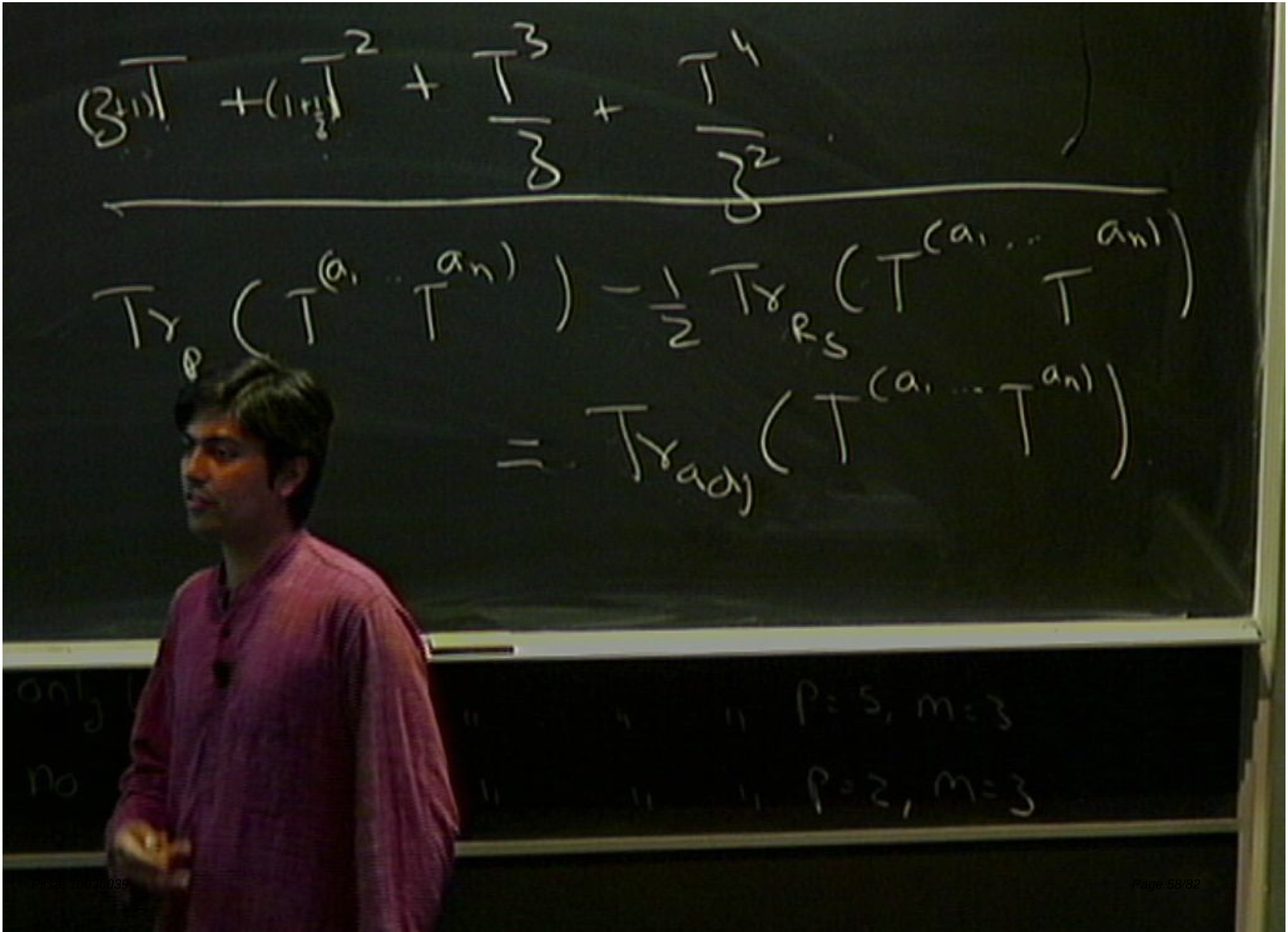
$$\mathrm{Tr}_{R_S}\left(T^{(a_1}\cdots T^{a_n)}\right) - \frac{1}{2}\mathrm{Tr}_{R_S}\left(T^{(a_1}\cdots T^{a_n)}\right)$$

$$= \mathrm{Tr}_{adj}\Big($$

only boxes R_ℓ 〃 〃 〃 $p=5,\ m=3$

no bubbles R_ℓ 〃 〃 〃 $p=2,\ m=3$


Tr_{R_S}(T^{(a_1} ... T^{a_n)}) - 1/2 Tr_{R_S}(T^{(a_1} ... T^{a_n)})
= Tr_adj (
only boxes R_2
no bubbles R_2


$= \mathrm{Tr}_{adj}(T^{(a_1} \cdots T^{a_n)})$
$p=5, m=3$
$p=2, m=3$

$$(3+1)\,\overline{T} + \left(1+\tfrac{1}{3}\right)\overline{T}^2 + \frac{\overline{T}^3}{3} + \frac{\overline{T}^4}{3^2}$$

$$\mathrm{Tr}_{R_S}\left(T^{(a_1}\cdots T^{a_n)}\right) - \frac{1}{2}\,\mathrm{Tr}_{R_S}\left(T^{(a_1}\cdots T^{a_n)}\right)$$

$$= \mathrm{Tr}_{adj}\left(T^{(a_1}\cdots T^{a_n)}\right)$$

$$n \leq 4$$

$$\mathrm{Tr}_R\left(\prod_{i=1}^{n} T^{a_i}\right) = m\,\mathrm{Tr}_{adj}\left(\prod_{i=1}^{n} T^{a_i}\right), \quad n \leq 8$$

Solving these equations

- At first sight, an equation of the sort:

$$\mathrm{Tr}(\Pi_{i=1}^{4} T^{a_i}) = 3\,\mathrm{Tr}_{\mathrm{adj}}(\Pi_{i=1}^{4} T^{a_i})$$

  might seem very difficult to solve.
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- Similarly, for $SU(N)$, we can define

$$\frac{1}{2}\mathrm{Tr}_R\left(T^a\{T^b, T^c\}\right) = I_3(R)d^{abc}$$

I_3 is called the anomaly.

- At higher orders also,

$$\mathrm{Tr}_R\left(T^{(a}T^b T^c T^{d)}\right) = I_4(R)d^{abcd} + I_{2,2}(R)\kappa^{(ab}\kappa^{cd)}$$

- In general,

$$\mathrm{Tr}_R\left(T^{(a_1}T^{a_2}\dots T^{a_n)}\right) = I_n(R)d^{a_1a_2\dots a_n} + \text{products of lower tensors.}$$

- There are as many independent indices as the rank of the algebra

Diophantine equations

- Since, any representation can be reduced into irreducible representations,

$$R = \bigoplus n_i R_i,$$

  the group theoretic constraints lead to linear Diophantine equations in the n_i involving the first few indices!
- For example, the condition for no-triangles in susy theories

$$n_i I_\alpha(R_i) = 3 I_\alpha(\mathrm{adj})$$

  where I_α runs over $I_2, I_{2,2}$ and I_4.
- All the other constraints can be written in this way.

Solving the Diophantine equations

- Indices were studied in detail by Okubo and Patera in the early 80s and more recently by Ritbergen et al. '98. They can be calculated explicitly or extracted from the character of the representation.
- The condition for triangles to vanish is most restrictive.
- The condition for only bubbles to vanish is less restrictive.
- There are an infinite number of solutions to the condition for rational terms to vanish.

Theories with only boxes at large N

- At large N, there are several solutions to the conditions for triangles and bubbles to vanish.
- For example, the $SU(N)$ theory with $2N$ fundamental hypermultiplets.
- As a check, all orbifolds of $\mathcal{N}=4$ also satisfy this condition at large N (this had to be the case by planar equivalence)
- We found two examples, where these properties persisted to all N

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Theories with only boxes at all N

- $\mathcal{N}=2$, $SU(N)$ (for $N \geq 3$) theory with a symmetric tensor hypermultiplet and an antisymmetric tensor hypermultiplet.
- This is an orientifold of $\mathcal{N}=4$ (Ennes et al. 2000). However, the simplicity of these amplitudes at all N goes beyond planar equivalence.
- A theory based on the gauge-group G_2. (recall the adjoint has dimension **14**.) The $\mathcal{N}=1$ theory, with a chiral multiplet in the representation

$$R_\chi = 3 \cdot [\mathbf{7}] \oplus [\mathbf{27}],$$

also has only boxes.

Theories without bubbles

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- Can also find non-supersymmetric examples of theories that do not have bubbles.
- Example: $SU(N)$ theory with scalar content of a symmetric and anti-symmetric hypermultiplet but 4 adjoint fermions.
- Another Example: Consider $SU(2)$ theory with 7 complex scalar doublets, a pseudo-real scalar in the representation **4** and 4 adjoint fermions.

Theories without rational terms

- There is an infinite class of theories without rational terms.
- This is because any anomaly free solution to

$$n_i I_\alpha(R_i) = 0, \quad n_i \in \mathcal{Z},$$

  where I_α runs over the set $I_2, I_{2,2}, I_4$ leads to a theory without rational terms [all the positive n_i define the fermionic rep, all the negative n_i define the scalar rep]
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Examples of theories without rational terms

Table: Solutions to $n_i I_\alpha(R_i) = 0$

Group	Representations
SU(2)	-3[1]+2[2]-5[3]+4[4]-[5]
SU(3)	-[2,0]* +[2,1]* -[3,0]*
SU(4)	[1,0,0]* -3[2,0,0]* +3[1,1,0]* -6[0,1,0] - 2[0,2,0]
SU(5)	10[1,0,0,0]* +3[2,0,0,0]* –3[1,1,0,0]* +[0,2,0,0]*

- * means the conjugate rep. appears with the same multiplicity.
- Each line leads to a theory without rational terms. Eg. for $SU(3)$, we have

$$R_s = 2([2,0] + [0,2] + [3,0] + [0,3]),$$
$$R_f = [2,1] + [1,2] + [1,1].$$

Summary: Background

- Over the past few years, we have learned many exciting things about amplitudes in gauge theories and gravity.
- Amplitudes in $\mathcal{N} = 4$ SYM and $\mathcal{N} = 8$ SUGRA seem to be the nicest of all theories. It is believed that these are completely determined by their leading singularities.
- These simplifications (especially those in $\mathcal{N} = 8$ SUGRA) are very difficult to see via Feynman diagrams and come about because of non-trivial cancellations between terms.

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Summary: Question?

- We asked whether other gauge theories (coupled to matter) with less or no supersymmetry saw similar simplifications in their S-matrices?
- Can we tune the matter representations to bring about such simplifications?

Summary: Answer

- The answer is "Yes"!
- Triangles, bubbles and rational terms for gluon amplitudes depend on only a small number of indices of the matter representation.
- The conditions for the amplitude to have a simple analytic structure are equivalent to a set of linear Diophantine equations involving these indices.
- These conditions are hierarchical: no triangles $\supset$ no bubbles $\supset$ no rational terms.

Summary: Examples

- At large N, many theories have only boxes including the $\mathcal{N} = 2$, SU(N) theory with 2 N fundamental hypers.
- At finite N, the $\mathcal{N} = 2$ SU(N) theory with a symmetric and anti-symmetric tensor hypermultiplet has only boxes. Also, a G_2 theory with a specific representation content has this property.
- There are many examples of theories without bubbles.
- There are an infinite number of non-supersymmetric theories without rational terms.

Open Questions

- Several interesting questions remain open.
- In the $\mathcal{N} = 2$ SU(N) theory with a symm. + anti-symm. hypermultiplet, is matter-scattering also simple?
- Do these simplifications persist to higher-loops?
- More ambitiously: can one generalize the Grassmannian conjecture to this theory? (Requires us to understand non-planar $\mathcal{N} = 4$ SYM first.)
- Is there a similar story for gravity?