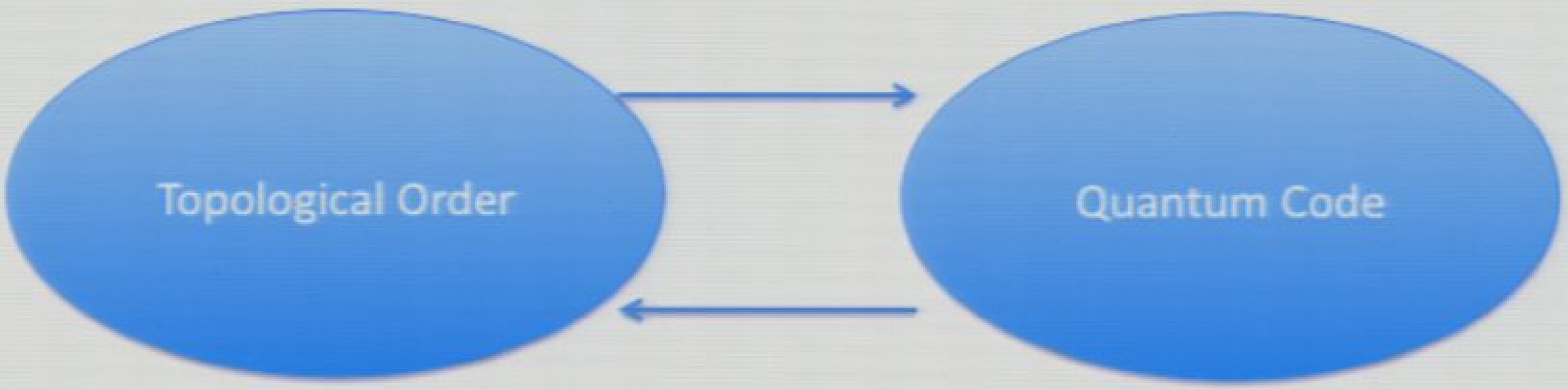
Title: Quantum code with translation and scale symmetries

Date: Mar 17, 2010 04:00 PM

URL: http://pirsa.org/10030033

Abstract: Topological phases in spin systems are exciting frontiers of research with intimate connections to quantum coding theory. However, there is a disconnection between quantum codes and the idea of topology, in the absence of geometry and physical realizability. Here, we introduce a toy model, in which quantum codes are constrained to not only have a local geometric description, but also have translation and scale symmetries. These additional physical constraints enable us to assign topologically invariant properties to geometric shapes of logical operators of the code. Topological phases of the model are analyzed by geometrically classifying logical operators. The classification scheme also has topologically universal properties which are invariant under local unitary transformations and local perturbations, and may explain how global symmetries of a system Hamiltonian give rise to topological phases in correlated spin systems.

Quantum code with translation and scale symmetries

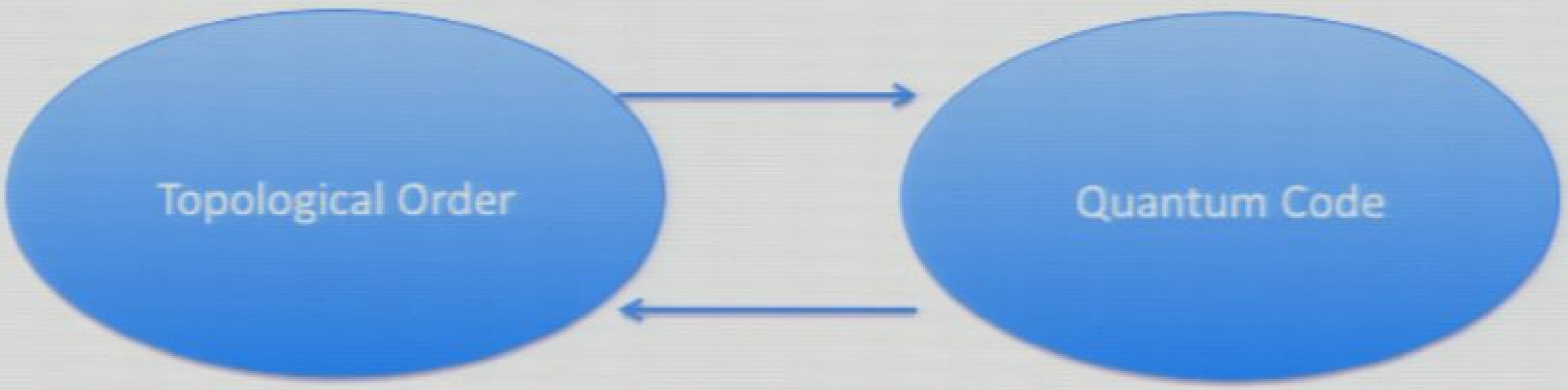


Beni Yoshida (Dept of Physics, MIT)

In collaboration with Isaac. L. Chuang

March 17, Perimeter Institute, PIqudos

Quantum code with translation and scale symmetries



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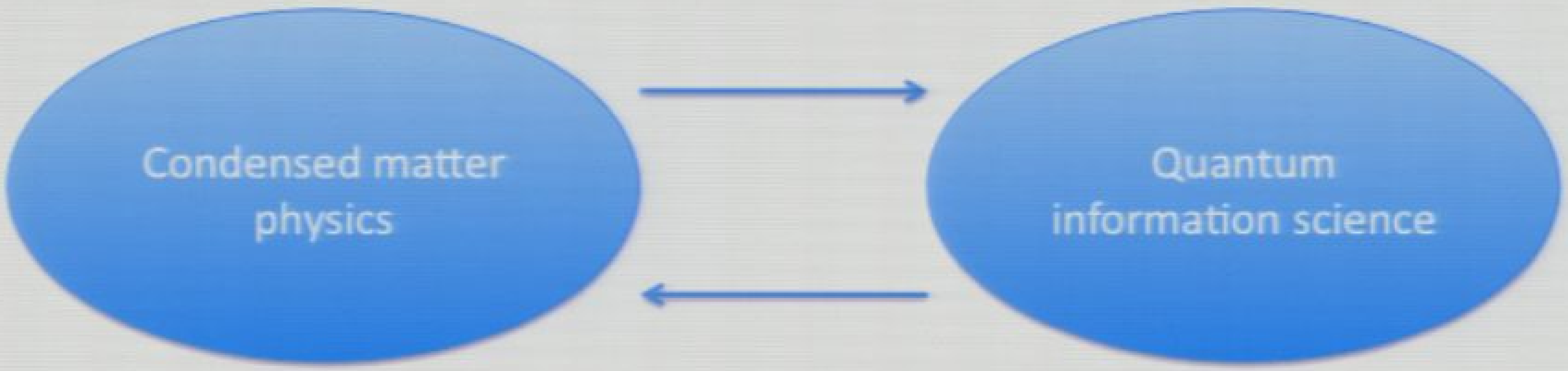
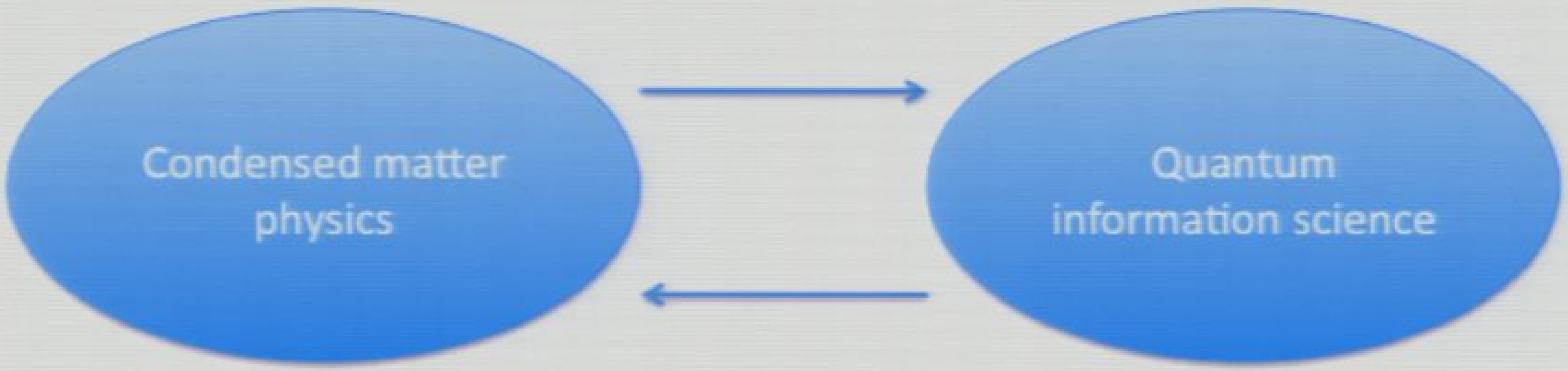
Introduction 1



Introduction 1



Open questions

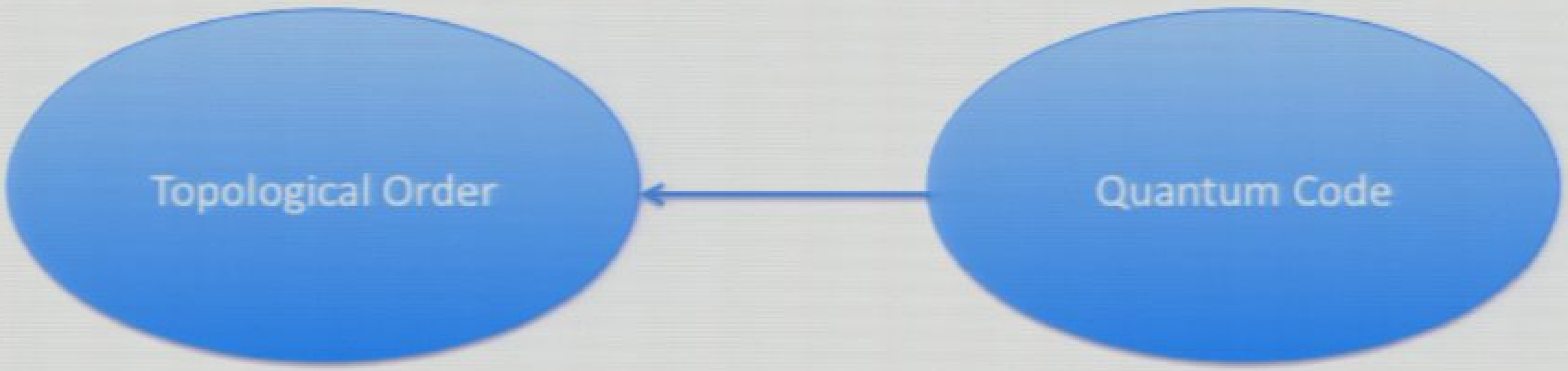
- High Tc
- Novel quantum order
- Numerical algorithm
- Resource for Q computation and communication

etc

Quantum information theoretical techniques

(MPS, TPS, entanglement entropy etc)

Introduction 2



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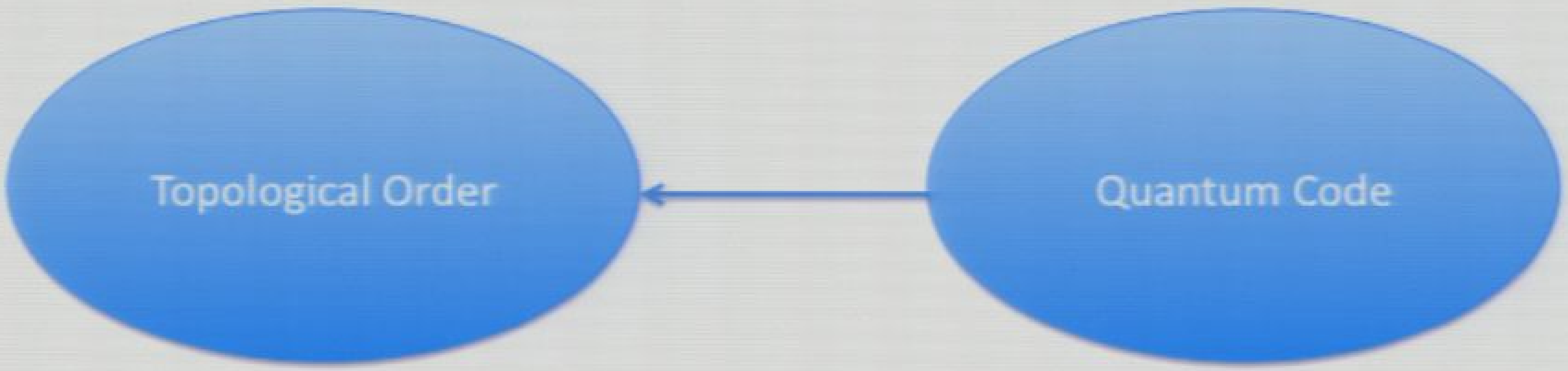


$$[H, \ell] = 0$$

Symmetry Operators : global symmetries

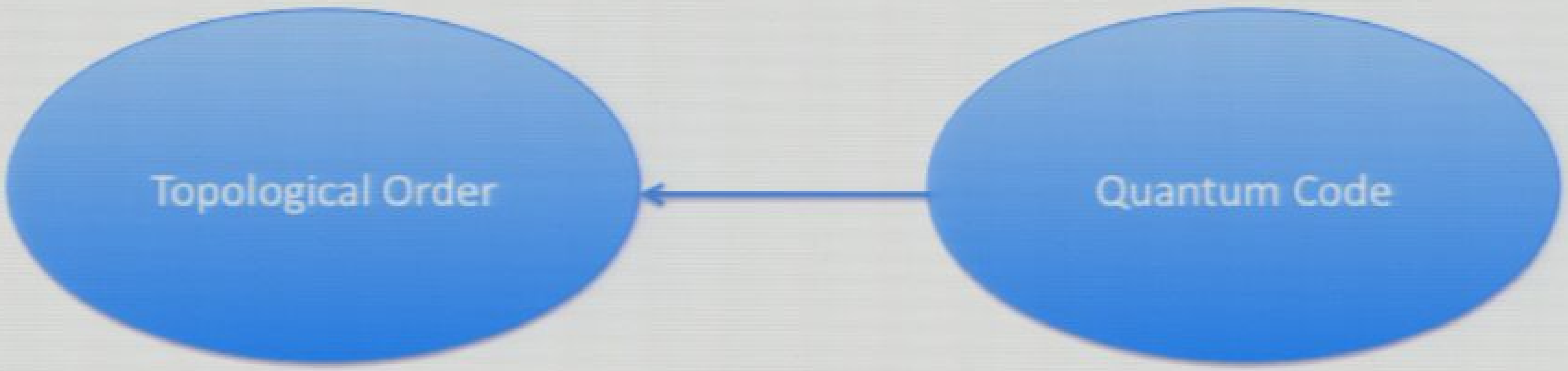
Topological degeneracy

$$[H, \ell] = 0$$

Logical Operator : global entanglement

Logical qubit

Introduction 2



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Symmetry Operators : global symmetries

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Logical qubit

<u>Quantum coding theoretical techniques</u>

Operator algebra based on a finite group

Introduction 2

Topological Order ← Quantum Code

$$[H, \ell] = 0$$

Symmetry Operators : global symmetries

Topological degeneracy

$$[H, \ell] = 0$$

Logical Operator : global entanglement

Logical qubit

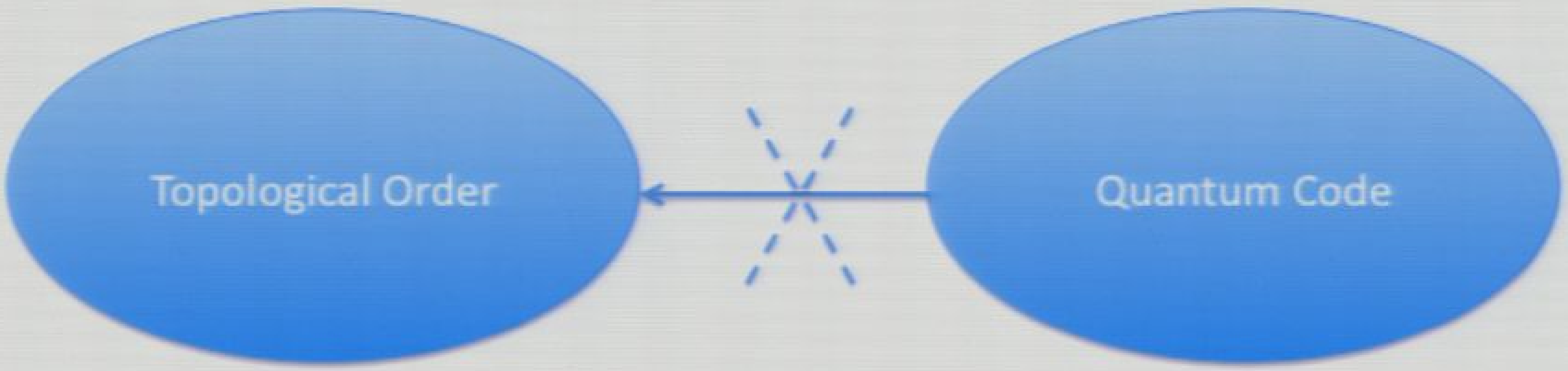
Mysteries in topological order

- Topological order at finite temperature (self-correcting memory)
- Topological phase transition
- Classification of topological order

Quantum coding theoretical techniques

Operator algebra based on a finite group → (Mysteries in topological order)

Introduction 3



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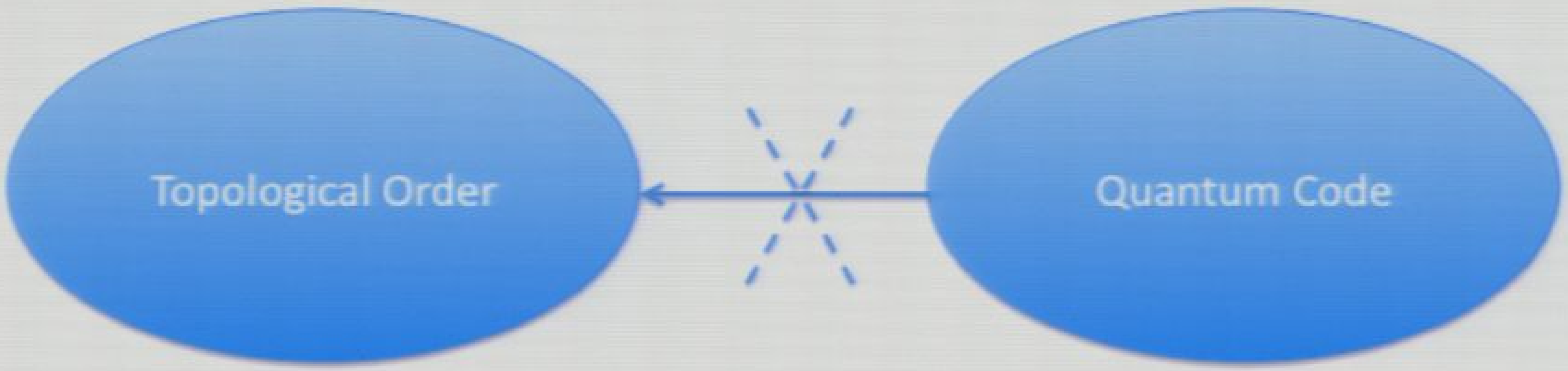
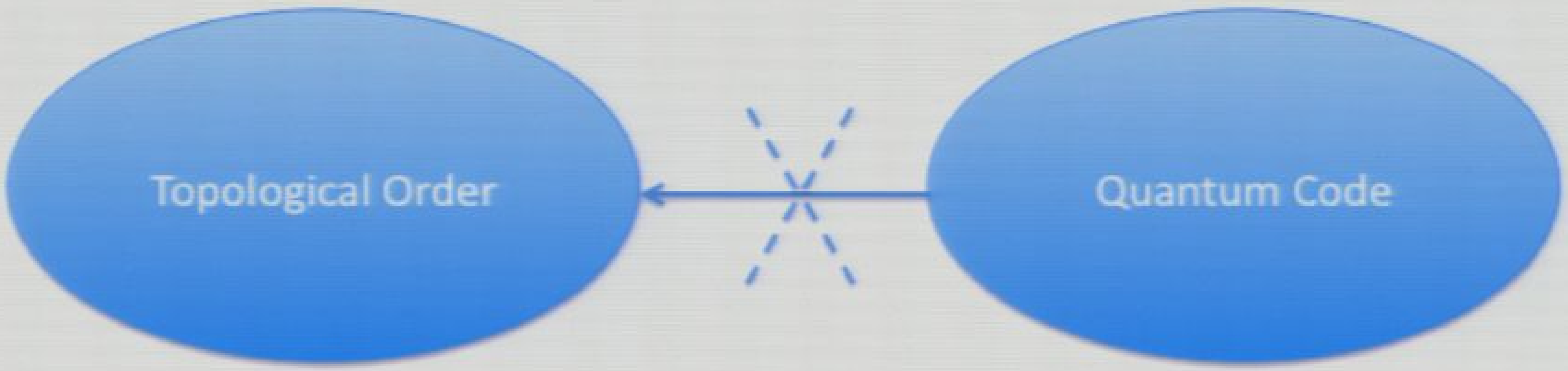


Quantum code needs

- Hamiltonian → Stabilizer formalism (Gottesman 96)
- Geometry (metric) → Local interactions on a lattice (Bravyi & Terhal 09)
- Physical realizability

Introduction 3

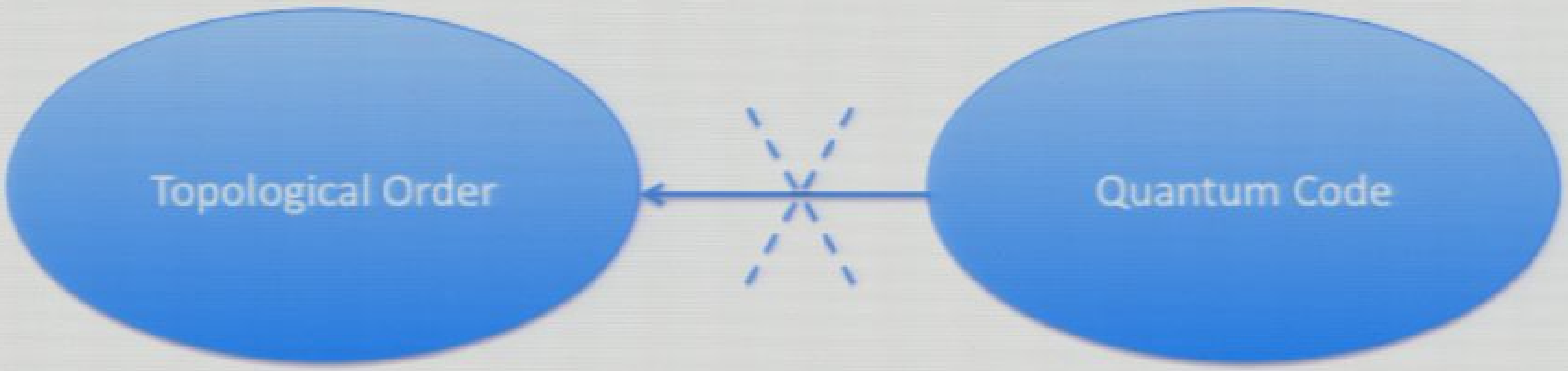


Quantum code needs

- Hamiltonian → Stabilizer formalism (Gottesman 96)
- Geometry (metric) → Local interactions on a lattice (Bravyi & Terhal 09)
- Physical realizability → Translation and Scale symmetries

→ Toy Model

Introduction 3



Quantum code needs

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→ Toy Model

Goals : Questions in topological order

- Self-correcting quantum memory
- Topological quantum phase transition

←

- Geometric classification of logical operators
- Topological order and coding properties

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Tool 1

Stabilizer code in a bi-partition

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Tool 1

Stabilizer code in a bi-partition

$$g_A + g_B = 2k$$

Model

Toy Model

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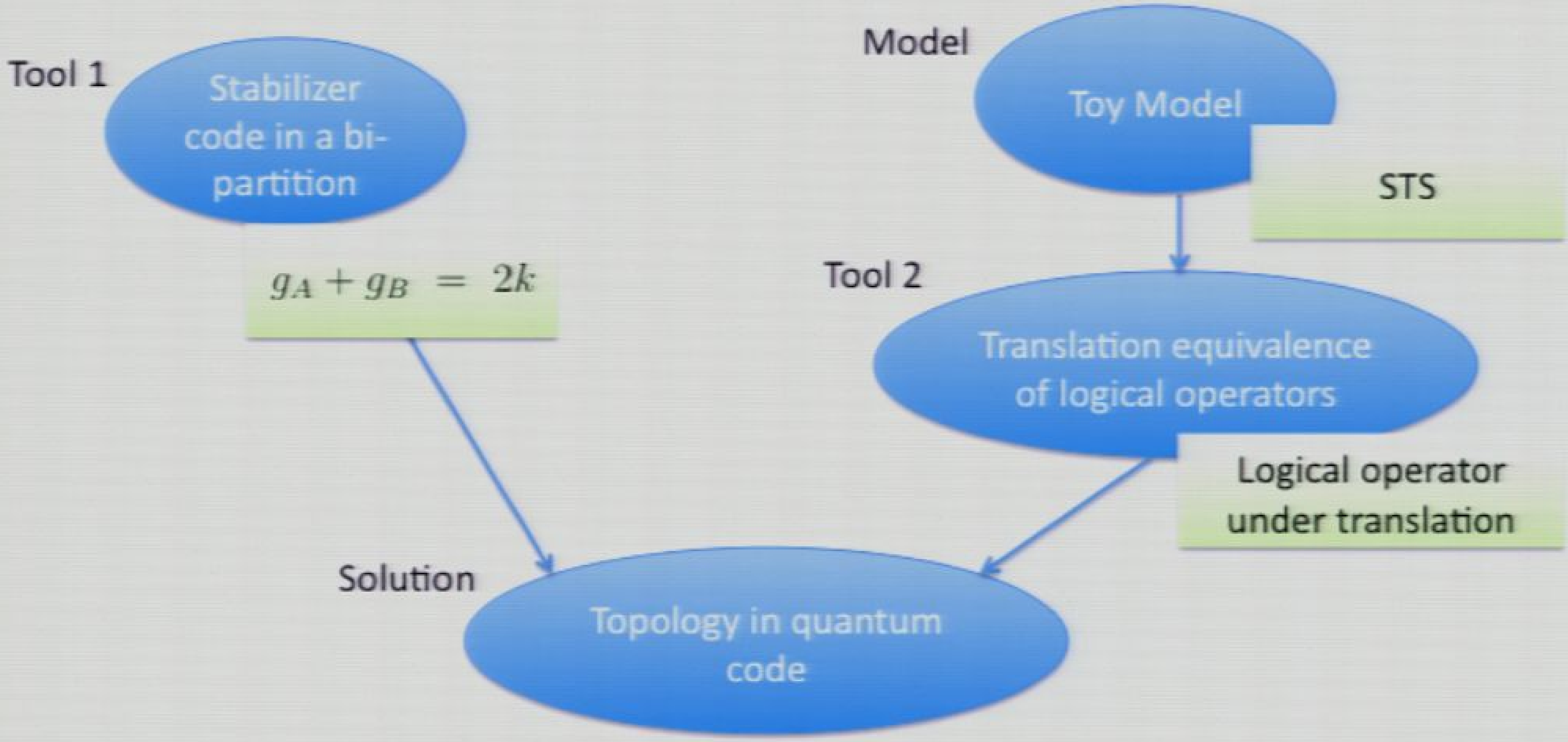

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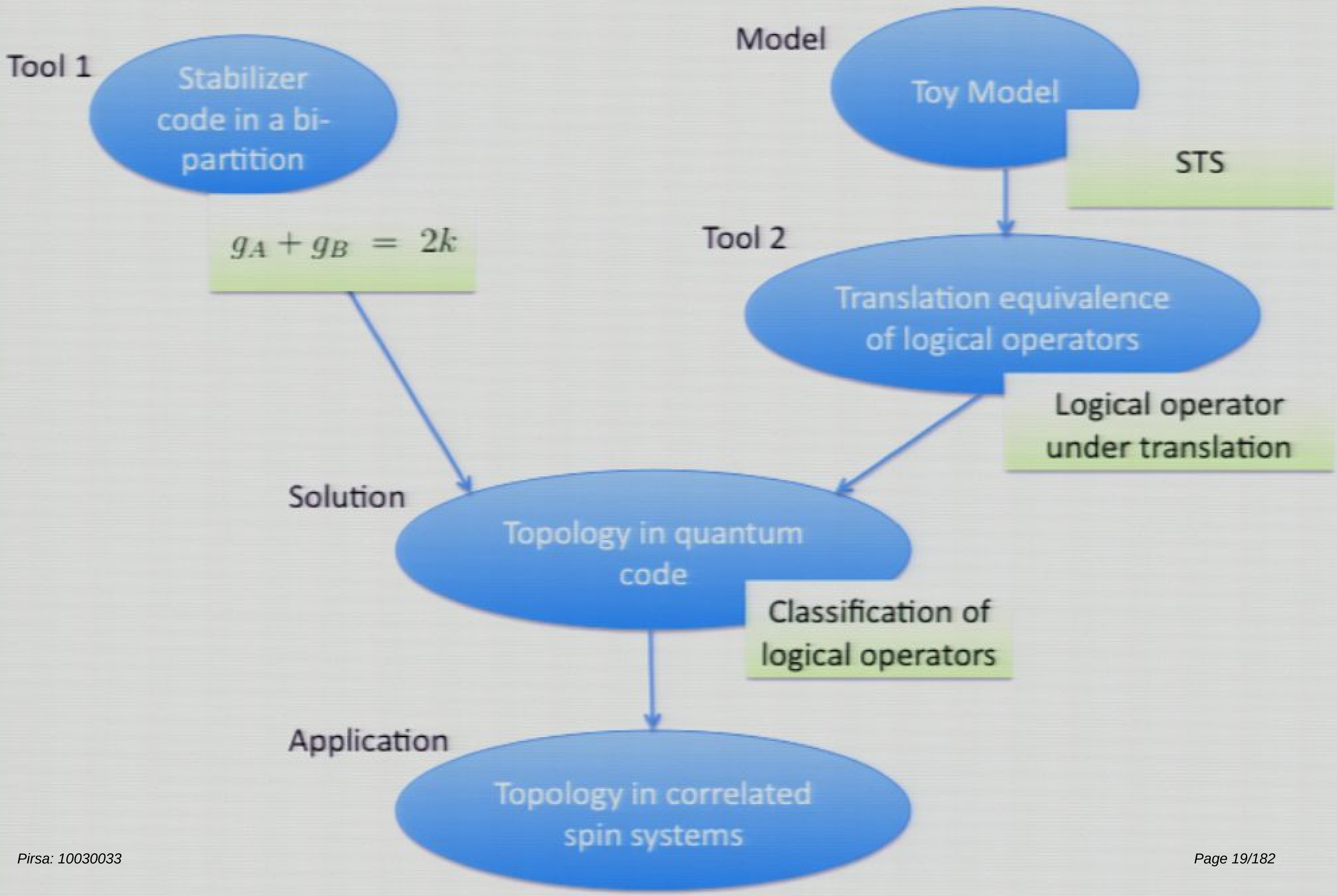

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Review of the stabilizer formalism

Review of stabilizer code 1

$$\mathcal{P} = \langle Z_1, X_1, \cdots, Z_N, X_N \rangle$$: Pauli operator group

$$H = -\sum_j S_j$$: System Hamiltonian

$$[S_i, S_j] = 0$$

$$S_j|\psi\rangle = |\psi\rangle$$

$$\mathcal{S} = \langle S_1, \cdots, S_{N-k} \rangle \in \mathcal{P}$$: The stabilizer group

stabilizers

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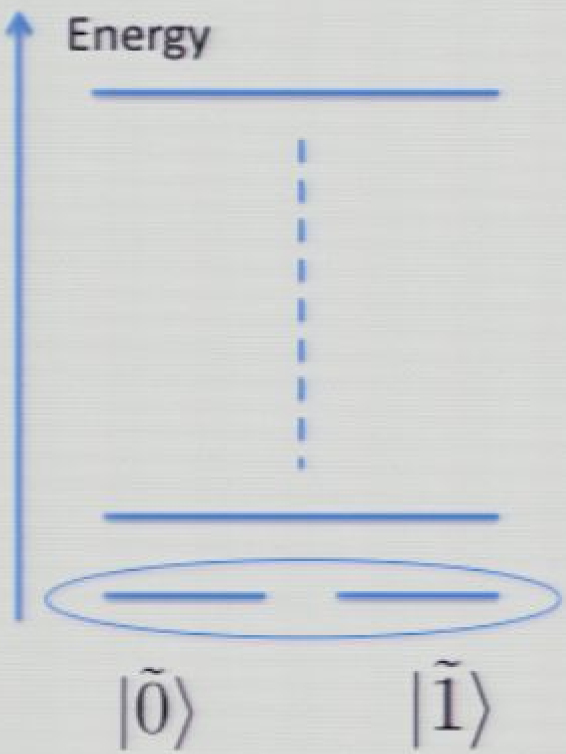

Ground state space = Codeword space

2^k Ground states = k logical qubits

Review of stabilizer code 2

$\ell \in \mathcal{P}$ s.t $[\ell, H] = 0$ but $\ell \notin \mathcal{S}$: Logical operators

$\mathcal{C} = \left\langle U \in \mathcal{P} : [U, S_j] = 0, \forall j \right\rangle$: Centralizer group

$\mathcal{C} = \langle S_1, \cdots, S_{N-k}, \ell_1, \cdots, \ell_k, r_1, \cdots, r_k \rangle$ → Logical operators

$[\ell_i, \ell_j] = 0 \quad [r_i, r_j] = 0 \quad [\ell_i, r_j] = 0$ for $(i \neq j)$ $\quad \{\ell_i, r_i\} = 0$

Review of stabilizer code 2

$\ell \in \mathcal{P} \quad \text{s.t} \quad [\ell, H] = 0 \quad \text{but} \quad \ell \notin \mathcal{S}$: Logical operators

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$\mathcal{C} = \langle S_1, \cdots, S_{N-k}, \ell_1, \cdots, \ell_k, r_1, \cdots, r_k \rangle$ → Logical operators ($\ell_1, \cdots, \ell_k, r_1, \cdots, r_k$)

$[\ell_i, \ell_j] = 0 \qquad [r_i, r_j] = 0 \qquad [\ell_i, r_j] = 0 \ \text{for} \ (i \neq j) \qquad \{\ell_i, r_i\} = 0$

$$\mathcal{C} = \left\langle \begin{matrix} S_1, & \cdots, & S_{N-k}, & \ell_1, & \cdots, & \ell_k \\ & & & r_1, & \cdots, & r_k \end{matrix} \right\rangle$$

- Operators in the same column anti-commute with each other
- Operators in the different columns commute with each other

Review of stabilizer code 2

$\ell \in \mathcal{P}$ s.t $[\ell, H] = 0$ but $\ell \notin \mathcal{S}$: Logical operators

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$$|\psi\rangle = \bigotimes_{i=1}^{k} \left(\alpha_i|\tilde{0}\rangle_i + \beta_i|\tilde{1}\rangle_i \right)$$

$\ell \sim \ell' \quad \ell\ell' \in \mathcal{S}$: equivalent logical operators

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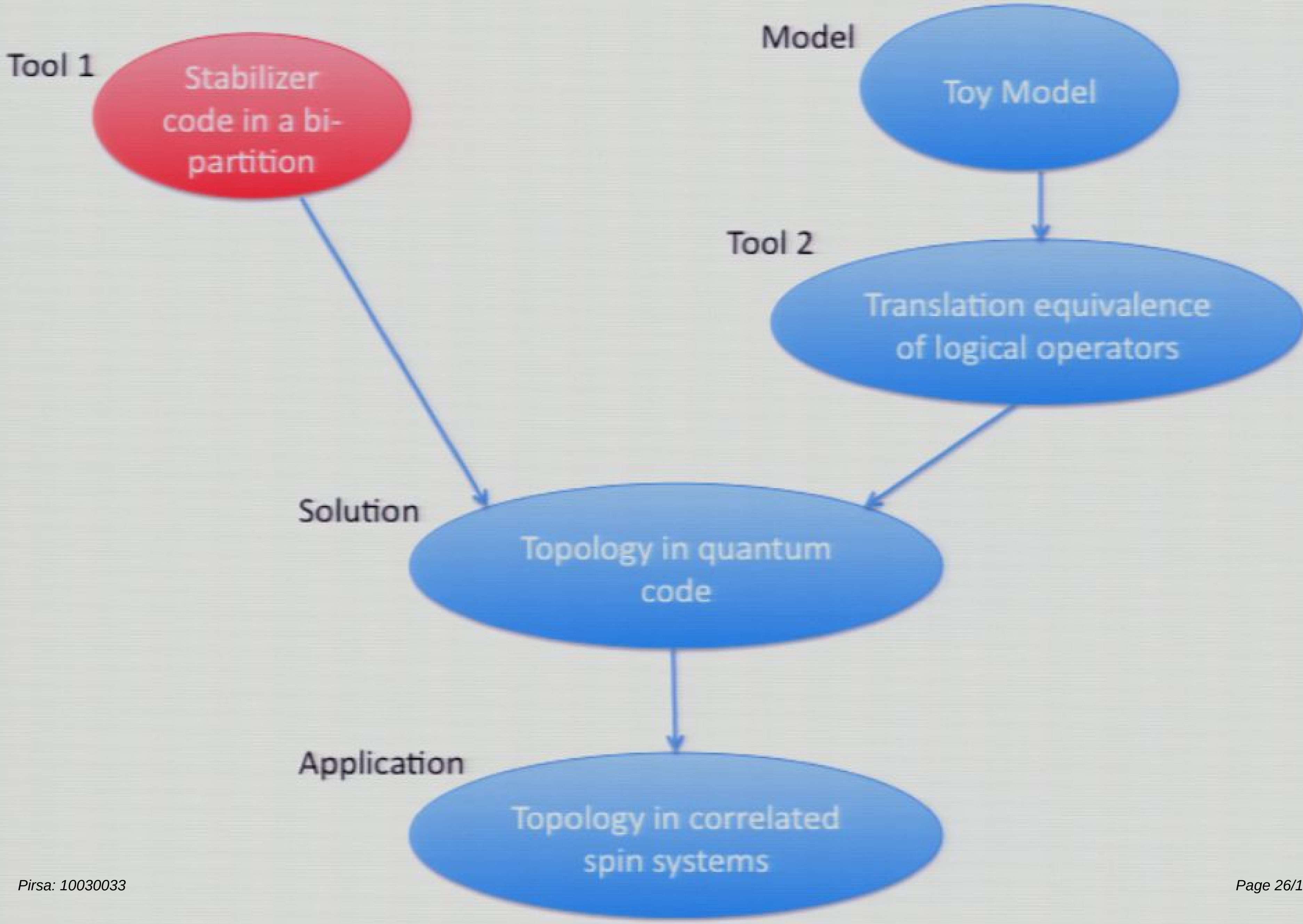
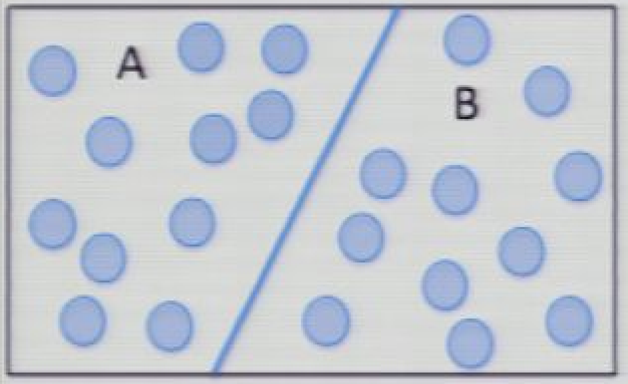
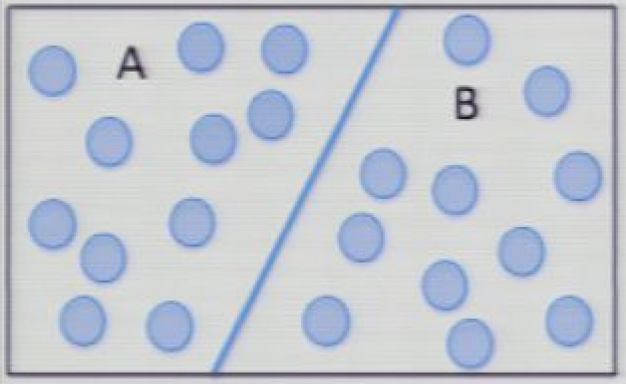


Stabilizer code in a bi-partition 1



Bi-partitioning a system into A and B

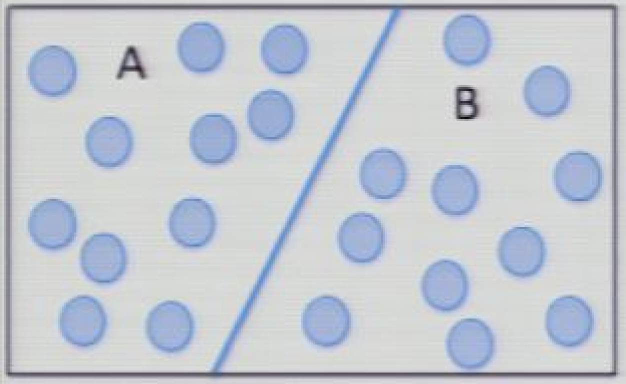
Stabilizer code in a bi-partition 1



Bi-partitioning a system into A and B

Logical operators defined non-locally over A and B are responsible for non-local correlations and entanglement over A and B.

Stabilizer code in a bi-partition 1



Bi-partitioning a system into A and B

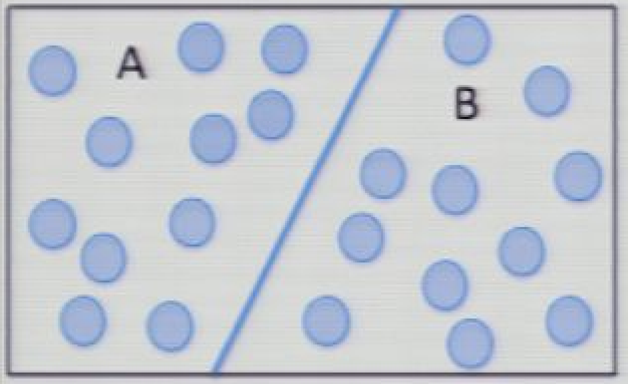
Logical operators defined non-locally over A and B are responsible for non-local correlations and entanglement over A and B.

Start with locally defined logical operators

Def : A logical operator can be defined locally inside A

if a logical operator has an equivalent representation which can be supported only inside A.

Stabilizer code in a bi-partition 1



Bi-partitioning a system into A and B

Logical operators defined non-locally over A and B are responsible for non-local correlations and entanglement over A and B.

Start with locally defined logical operators

Def : A logical operator can be defined locally inside A

if a logical operator has an equivalent representation which can be supported only inside A.

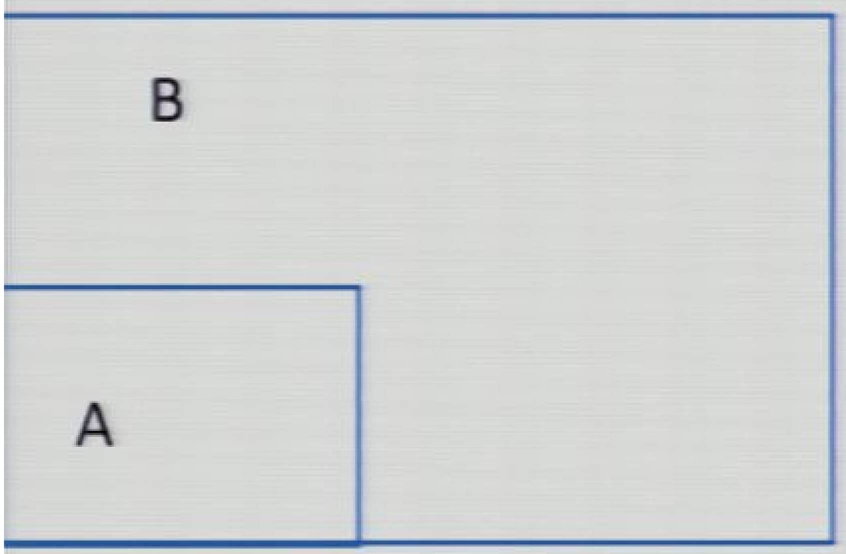
Def : A logical operator is non-locally defined over A and B

if a logical operator cannot be defined either inside A or B

Stabilizer code in a bi-partition 2

Stabilizer code in a bi-partition 2

How many logical operators can be defined inside A locally ?

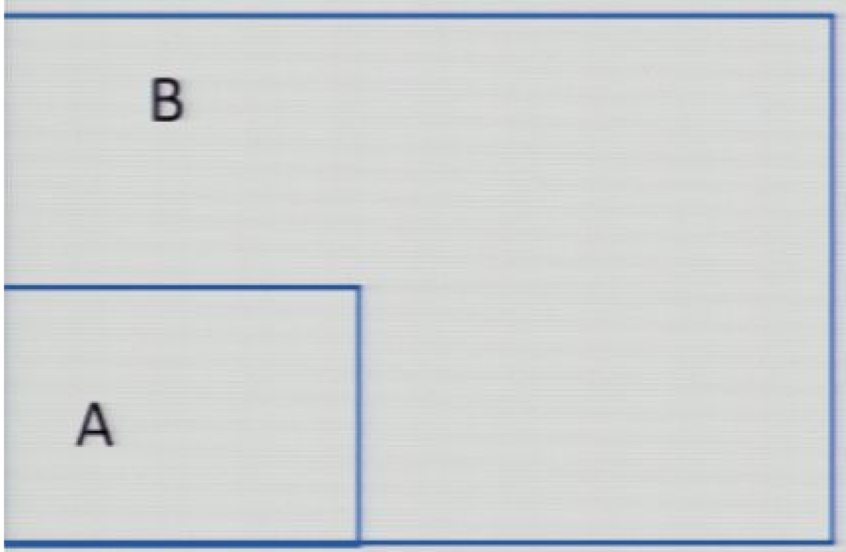


Stabilizer code in a bi-partition 2

How many logical operators can be defined inside A locally ?

Def

g_A # of independent logical operators defined inside A



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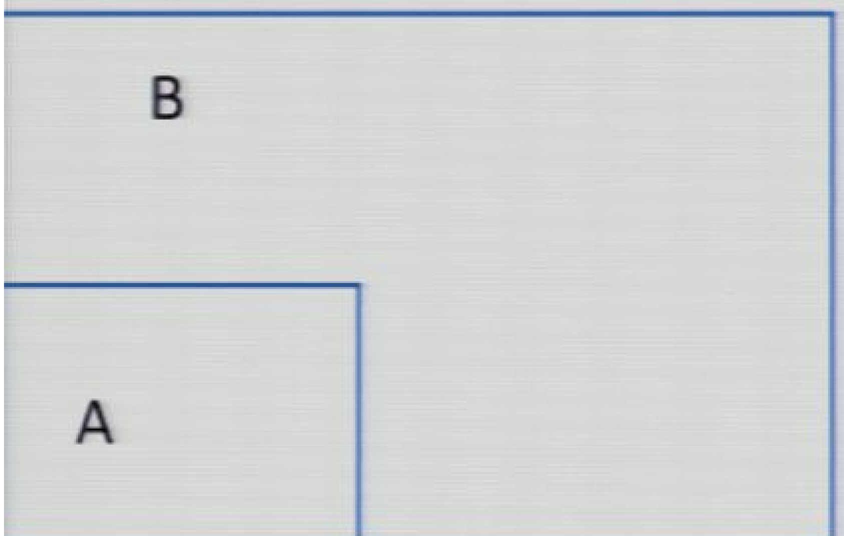
Duality in a bi-partition

$$g_A + g_B = 2k$$



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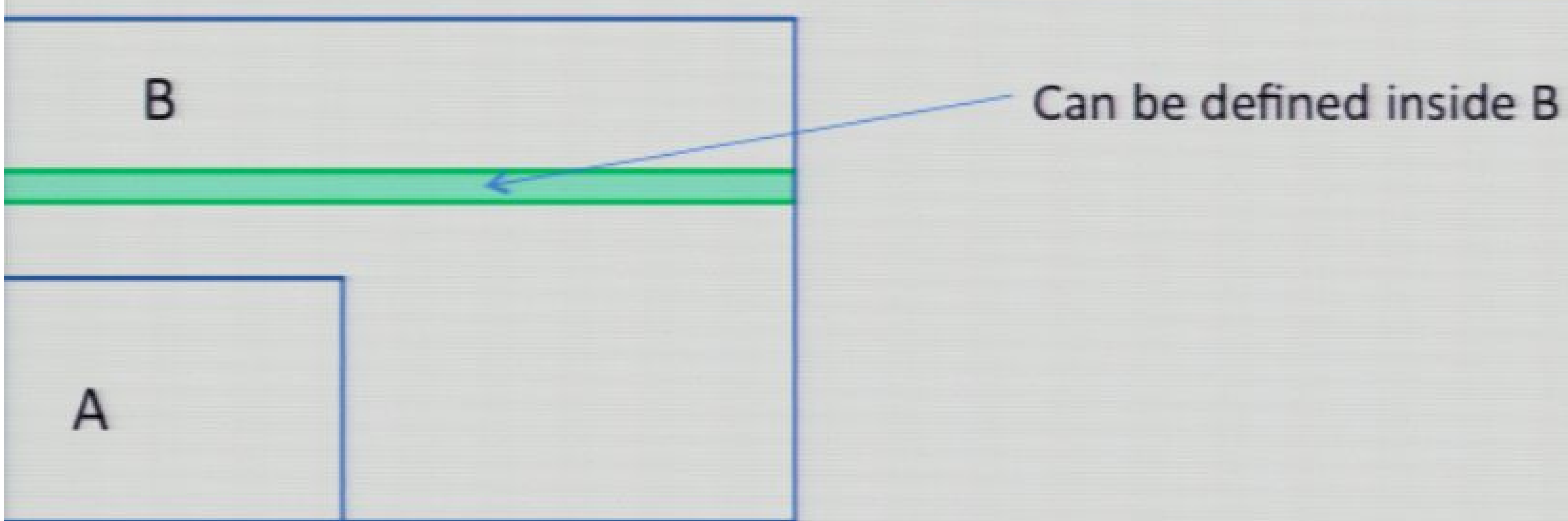

Stabilizer code in a bi-partition 2

How many logical operators can be defined inside A locally ?

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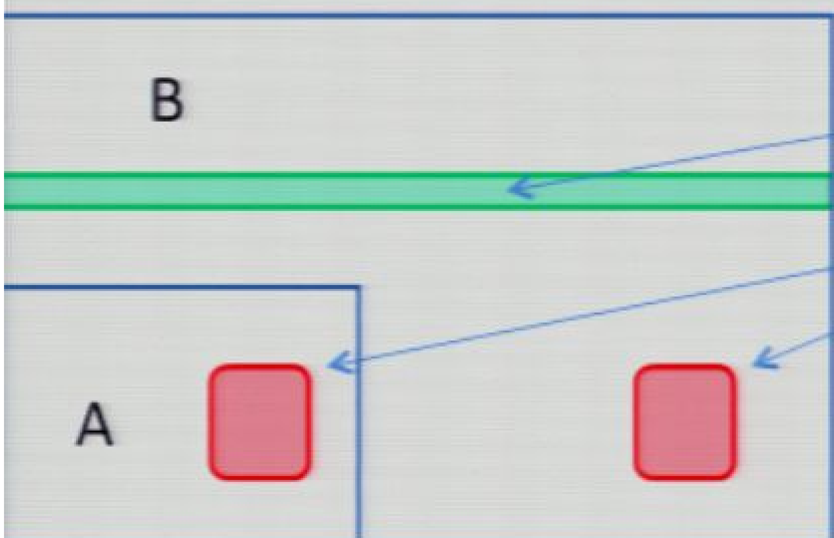

Can be defined inside B

Can be defined both on A and B

Stabilizer code in a bi-partition 2

How many logical operators can be defined inside A locally ?

Def

g_A # of independent logical operators defined inside A

Duality in a bi-partition

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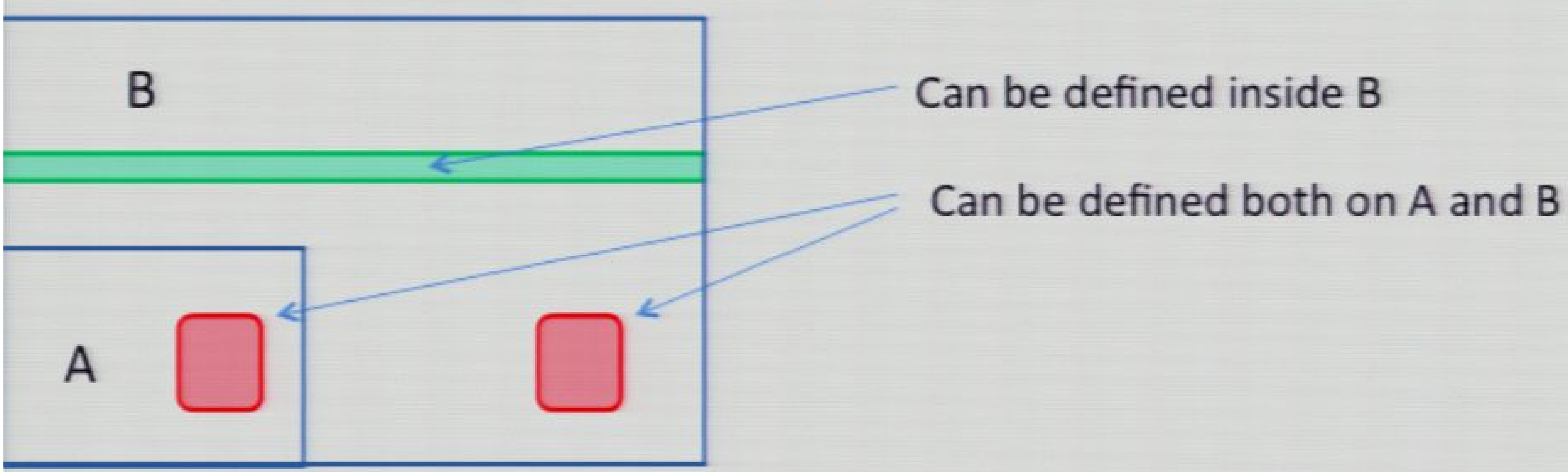
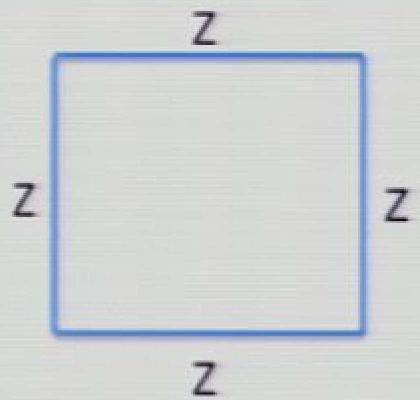
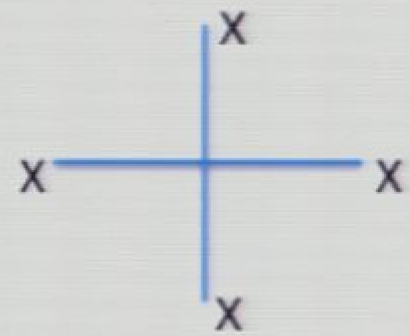


See my recent paper ...,
(Beni Yoshida, Isaac L. Chuang)

Stabilizer code in a bi-partition 3

<u>Geometry</u> in
Toric Code (k =2)





$$g_A + g_B \ = \ 2k$$

Stabilizer code in a bi-partition 3

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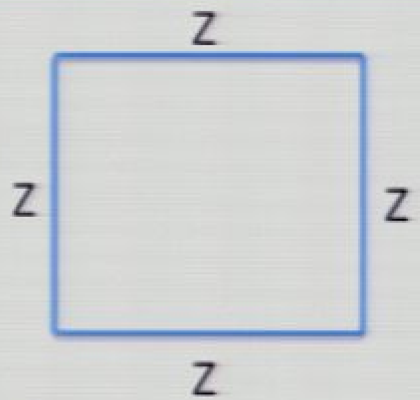
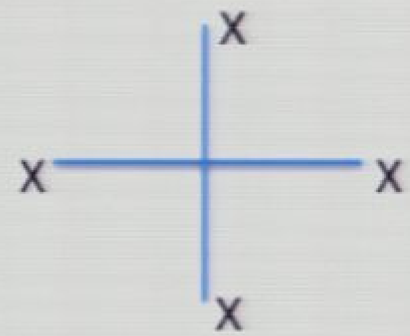
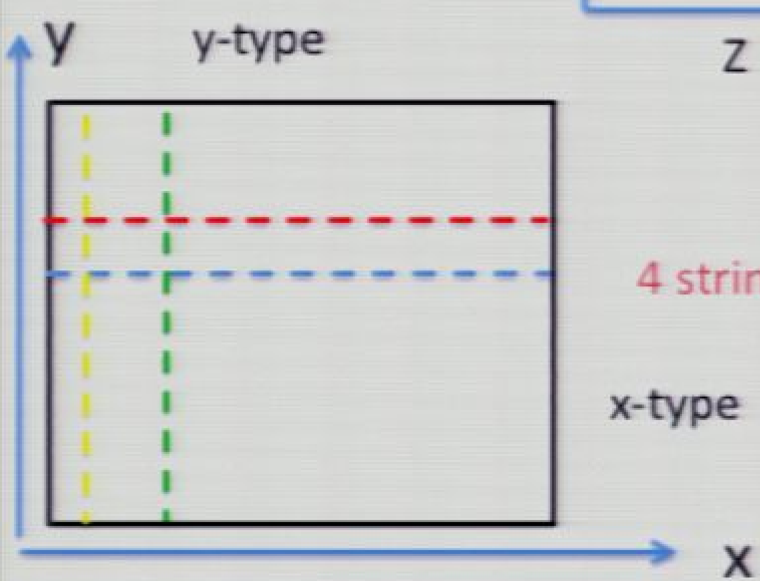




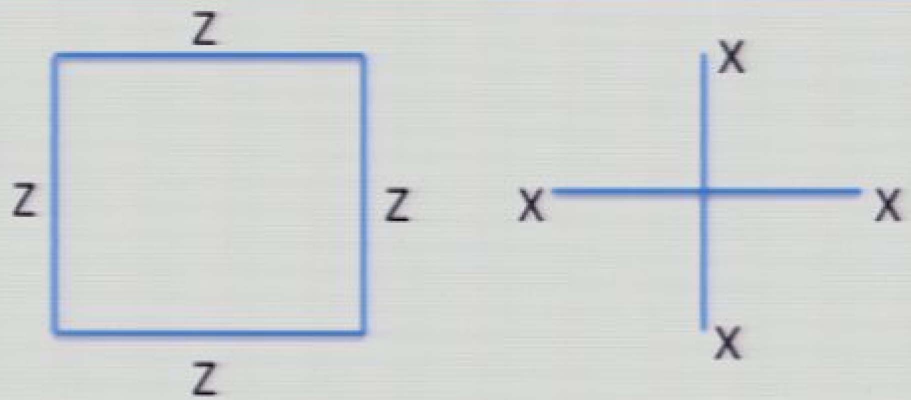
$$g_A + g_B = 2k$$



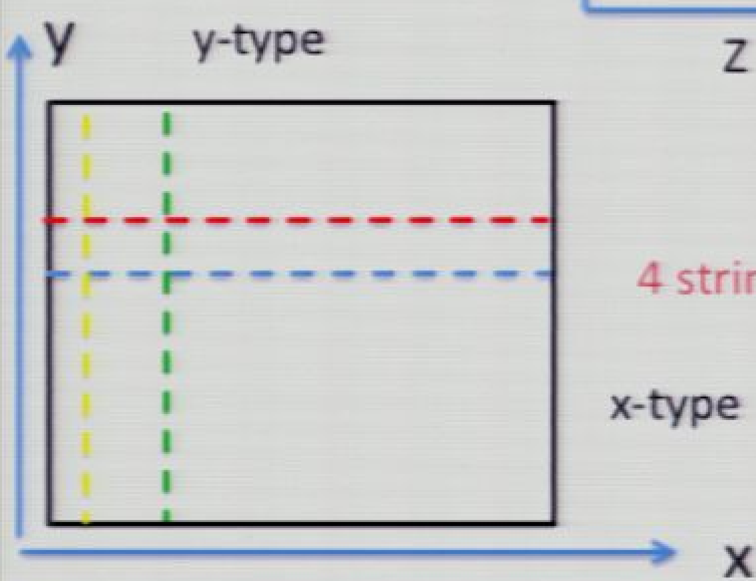
4 string operators

Stabilizer code in a bi-partition 3

Geometry in Toric Code (k =2)



$$g_A + g_B \ = \ 2k$$



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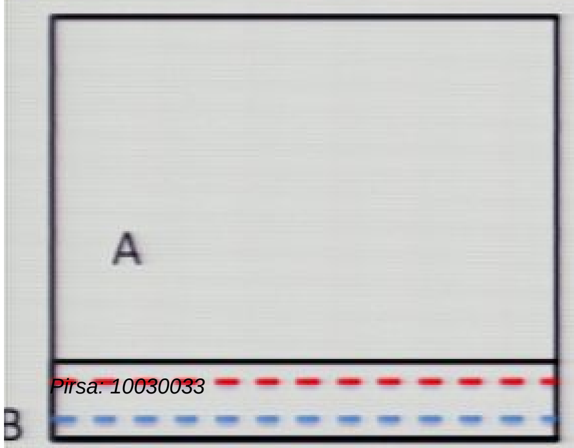
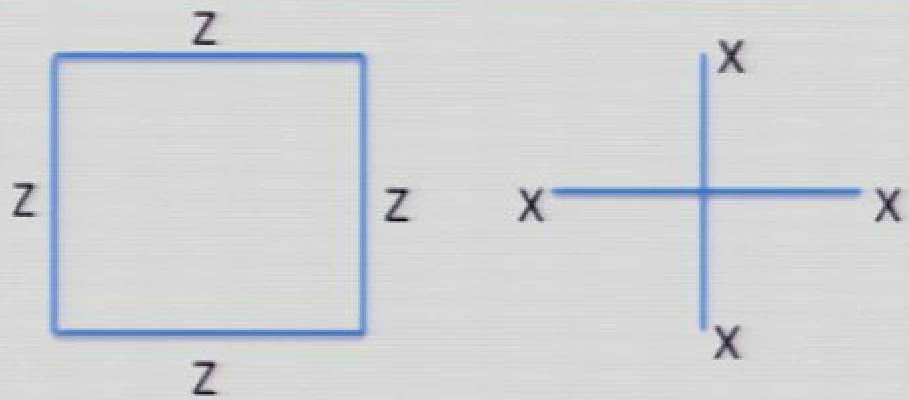


Stabilizer code in a bi-partition 3

Geometry in
Toric Code (k =2)



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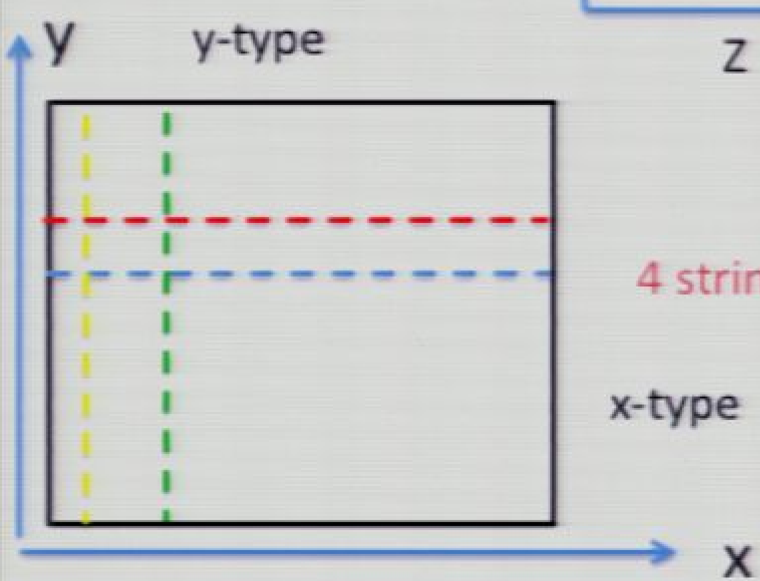

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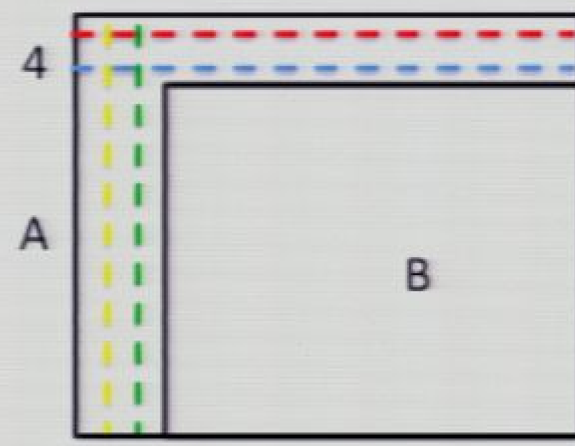

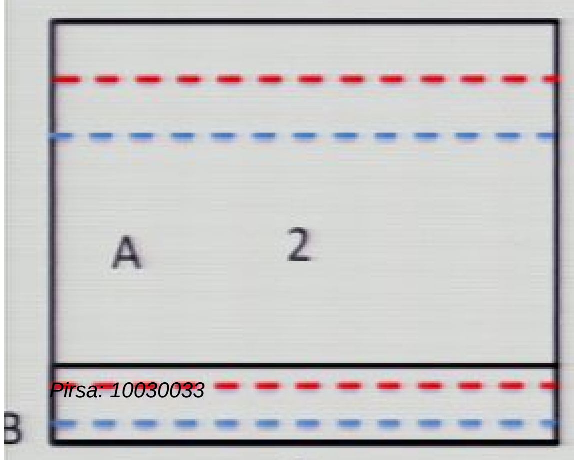

y-type logical operators
cannot be defined either
inside A or B

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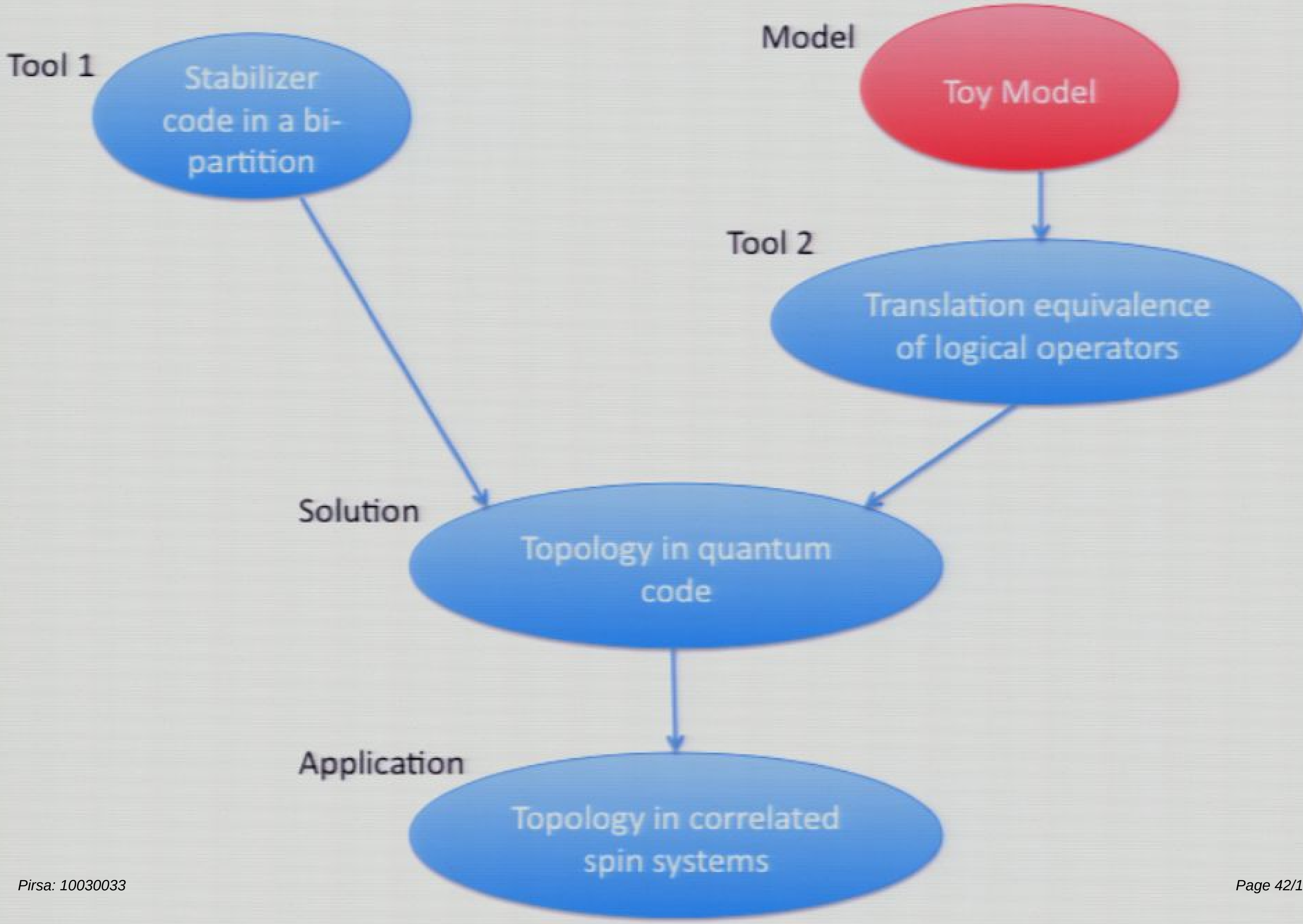
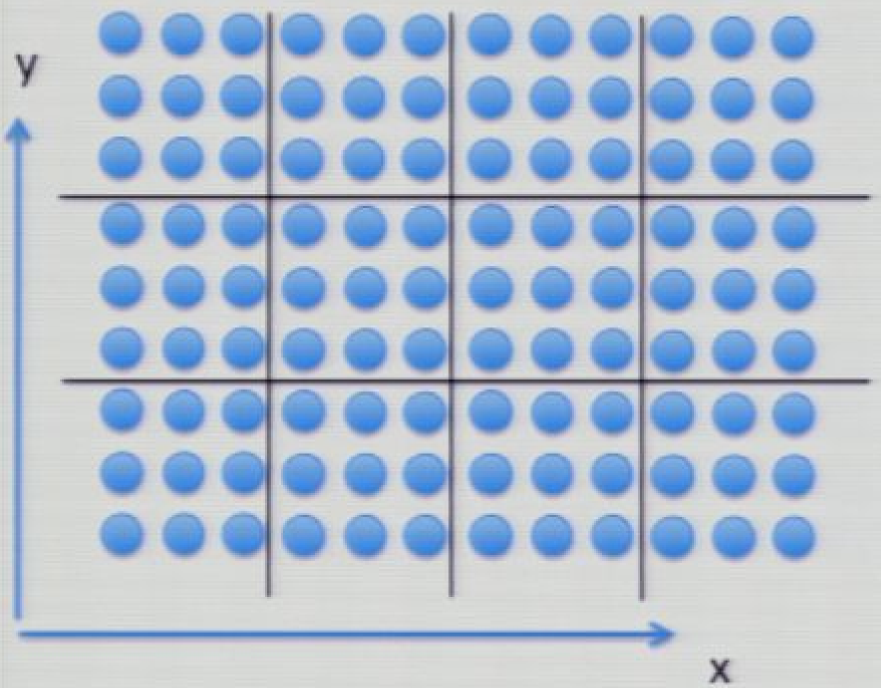


Stabilizer code with translation and scale symmetries

- Translation symmetries
- Local interactions
- Scale symmetries

Stabilizer code with translation and scale symmetries 1

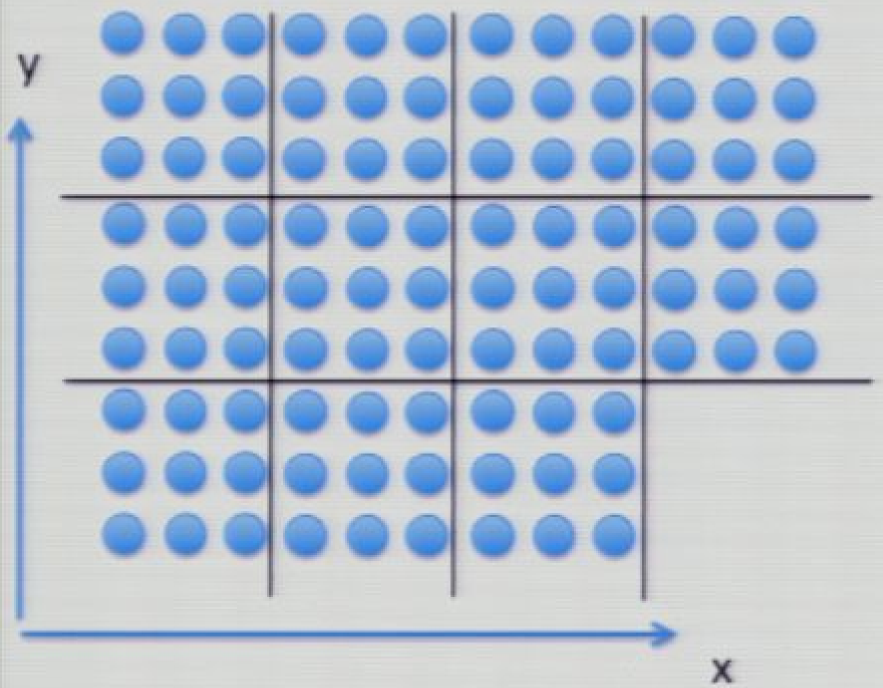


<u>Translation symmetries</u>

Stabilizer code defined on

- D dimensional square lattice of qubits
- System Hamiltonian is invariant under finite translations of qubits
- Periodic boundary conditions

Stabilizer code with translation and scale symmetries 1



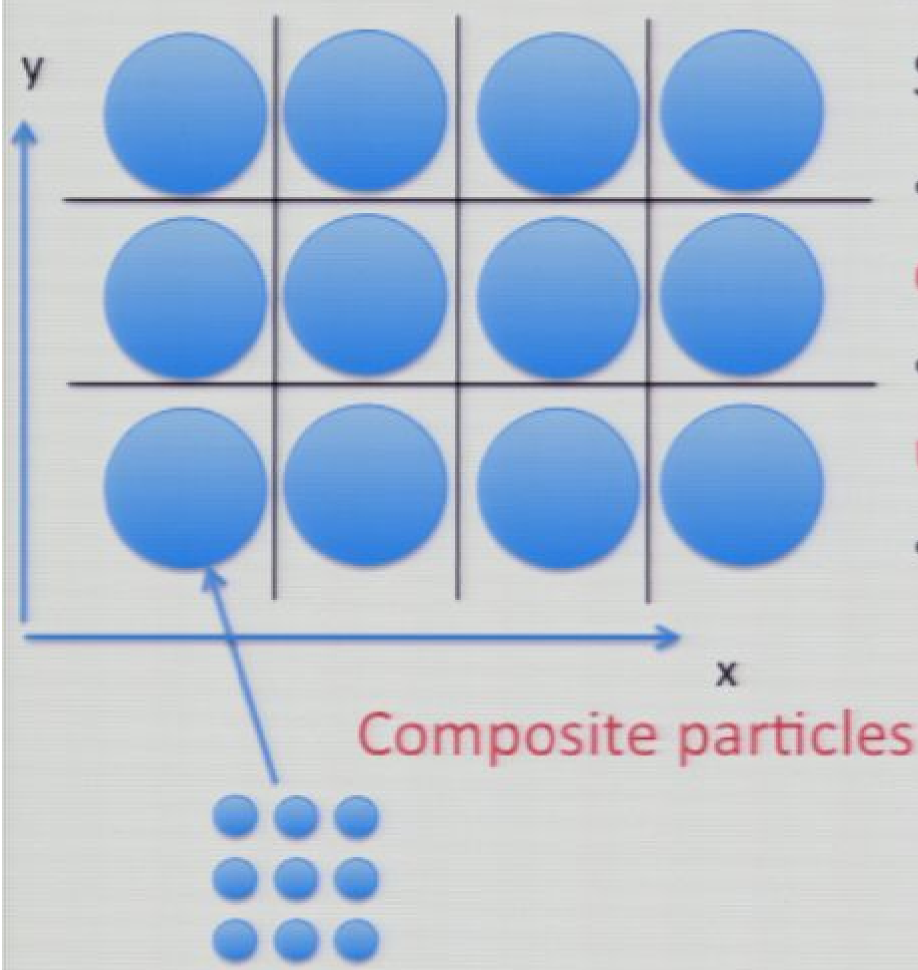
Translation symmetries

Stabilizer code defined on

- D dimensional square lattice of qubits
- System Hamiltonian is invariant under finite translations of qubits
- Periodic boundary conditions

In this example, the stabilizer code is invariant under translations by 3 qubits.

Stabilizer code with translation and scale symmetries 2

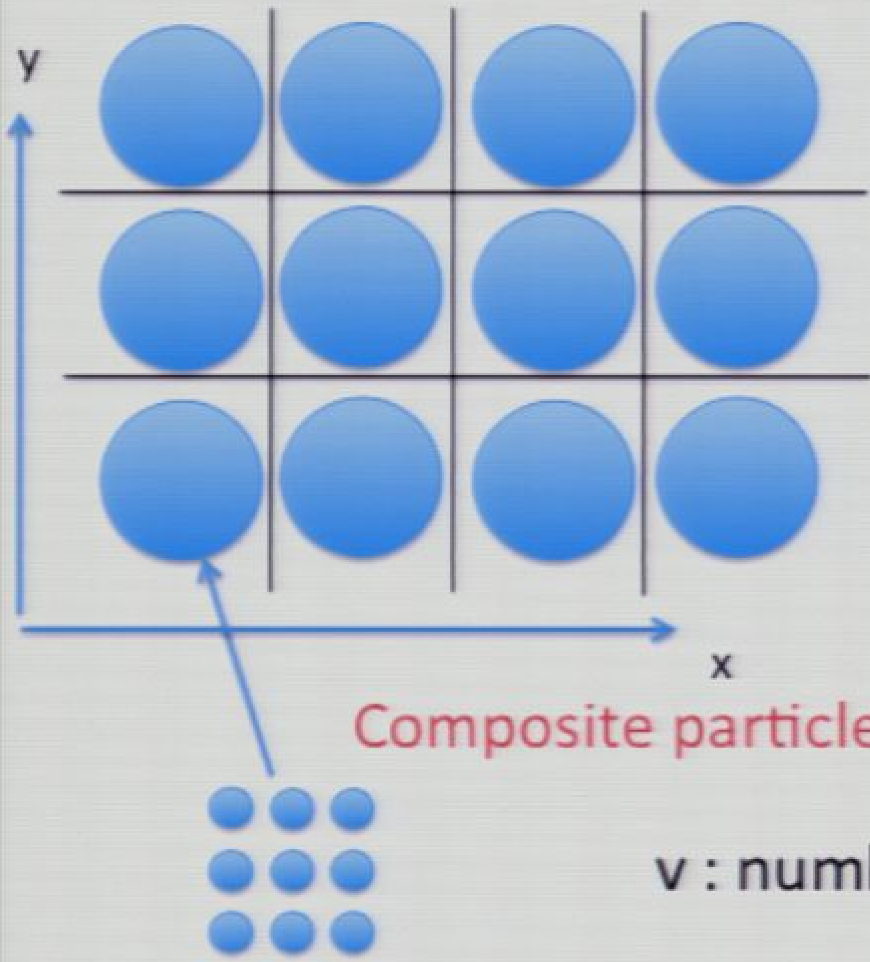


<u>Translation symmetries</u>

Stabilizer code is defined on

- D dimensional square lattice of composite particles
- System Hamiltonian is invariant under unit translations of composite particles
- Periodic boundary conditions

Stabilizer code with translation and scale symmetries 2



<u>Translation symmetries</u>

Stabilizer code is defined on

- D dimensional square lattice of composite particles
- System Hamiltonian is invariant under unit translations of composite particles
- Periodic boundary conditions

Composite particles $\mathbb{C}^2 \quad \rightarrow \quad \mathbb{C}^{2^v}$

v : number of qubits inside a composite particle

$$\left\{ \begin{array}{ccc} X_1, & \cdots, & X_v \\ Z_1, & \cdots, & Z_v \end{array} \right\}$$

Coarse-graining of the system

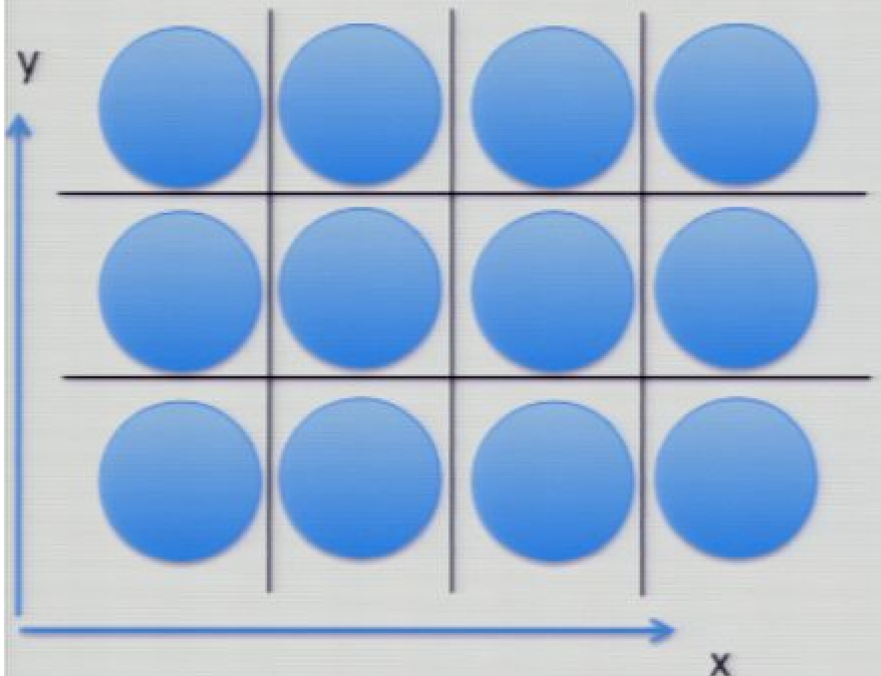
Stabilizer code with translation and scale symmetries

- Translation symmetries
- Local interactions
- Scale symmetries

→ Stabilizer code with translation and scale symmetries (STS)

Stabilizer code with translation and scale symmetries 3

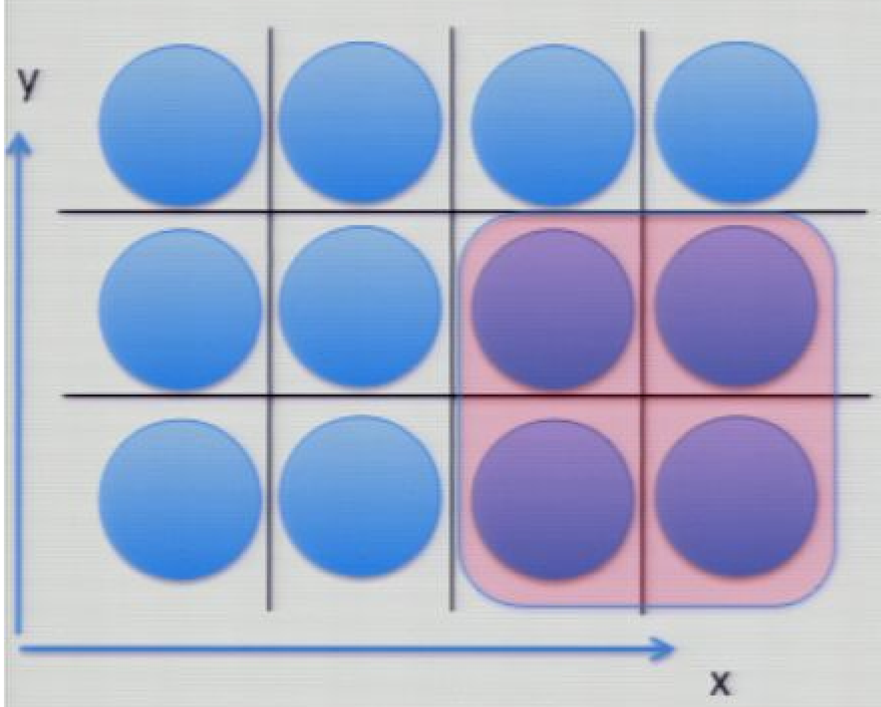
Local interactions



Stabilizer code with translation and scale symmetries 3

Local interactions

Interaction terms (stabilizers) are defined inside a region with $2 \times \cdots \times 2$ composite particles.



Stabilizer code with translation and scale symmetries

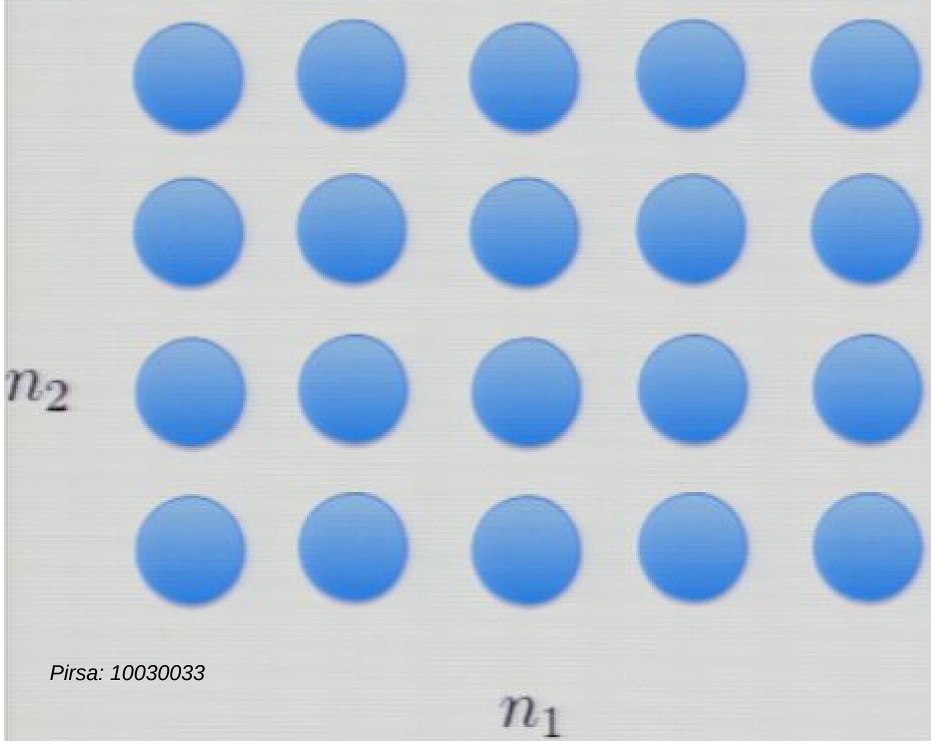
- Translation symmetries
- Local interactions
- Scale symmetries

→ Stabilizer code with translation and scale symmetries (STS)

Stabilizer code with translation and scale symmetries 4

Scale symmetries → Change of system sizes

$n_1 \times \cdots \times n_D$



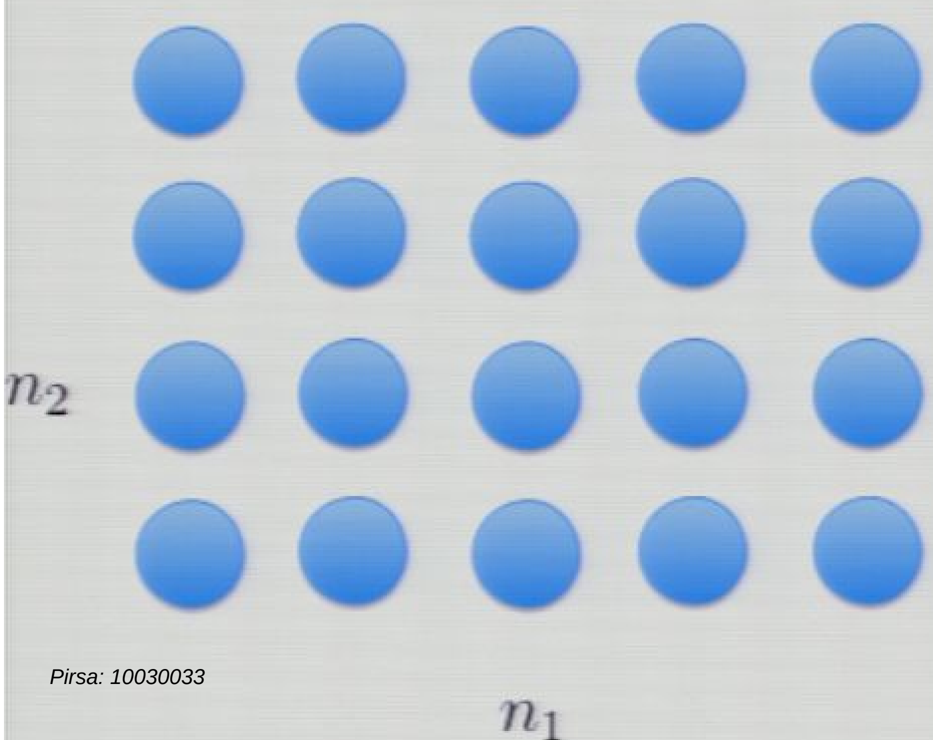
Stabilizer code with translation and scale symmetries 4

Scale symmetries ⟶ Change of system sizes

$\vec{n} = (n_1, \cdots, n_D)$ $k_{\vec{n}}$: number of logical qubits

$$k_{\vec{n}} = k \qquad \text{for all } \vec{n}$$

$n_1 \times \cdots \times n_D$



The number of logical qubits does not depend on the system size.

Stabilizer code with translation and scale symmetries 6

- Translation symmetries
- Local interaction
- Scale symmetries

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Intuition of solving the model

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Stabilizer code with translation and scale symmetries 6

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- We are interested in properties at "thermodynamic limit" where n is infinitely large.
- Due to the scale symmetries, there exist some universal properties regardless of n.
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Stabilizer code with translation and scale symmetries 6

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→ Effective reduction of dimensions

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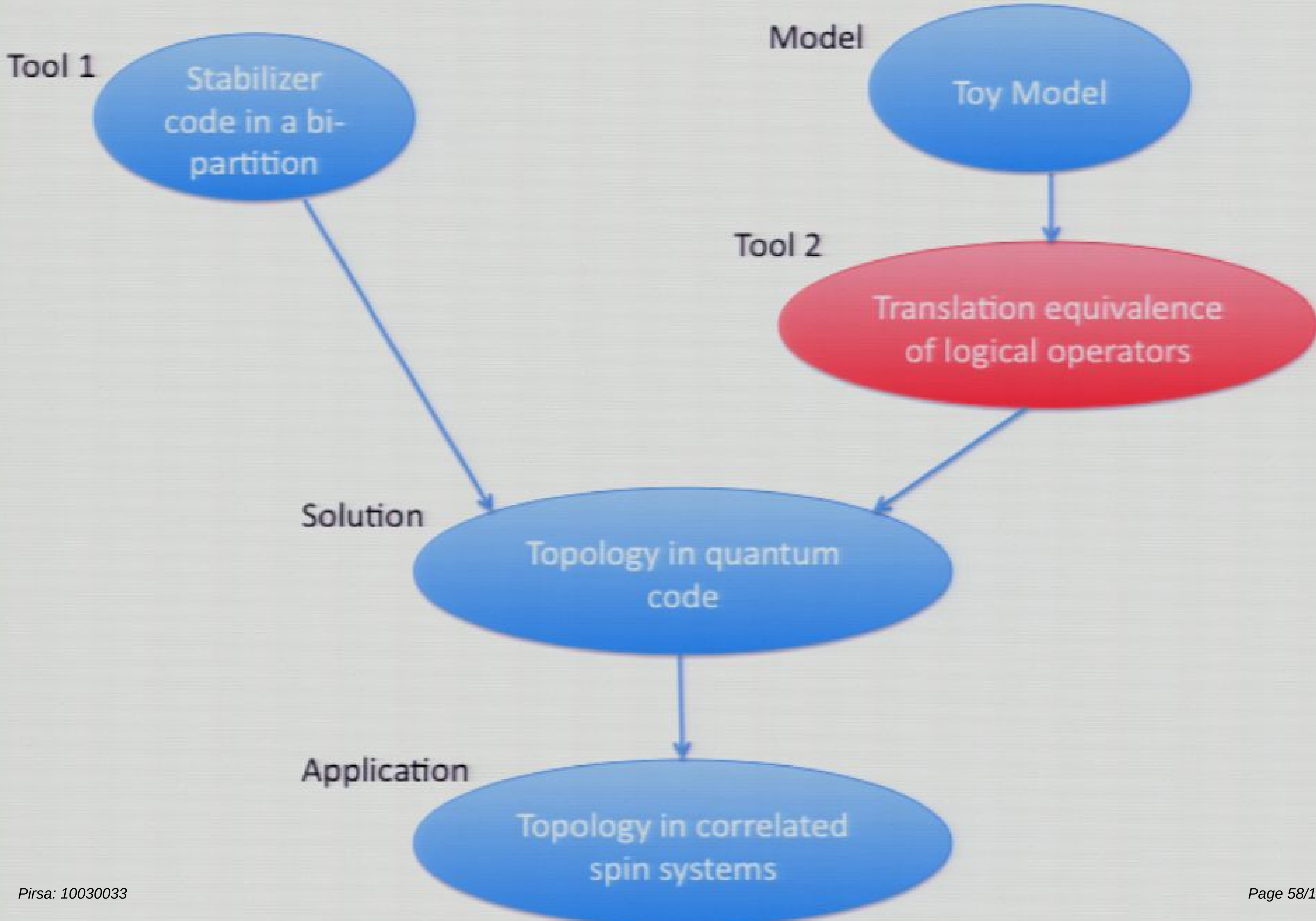
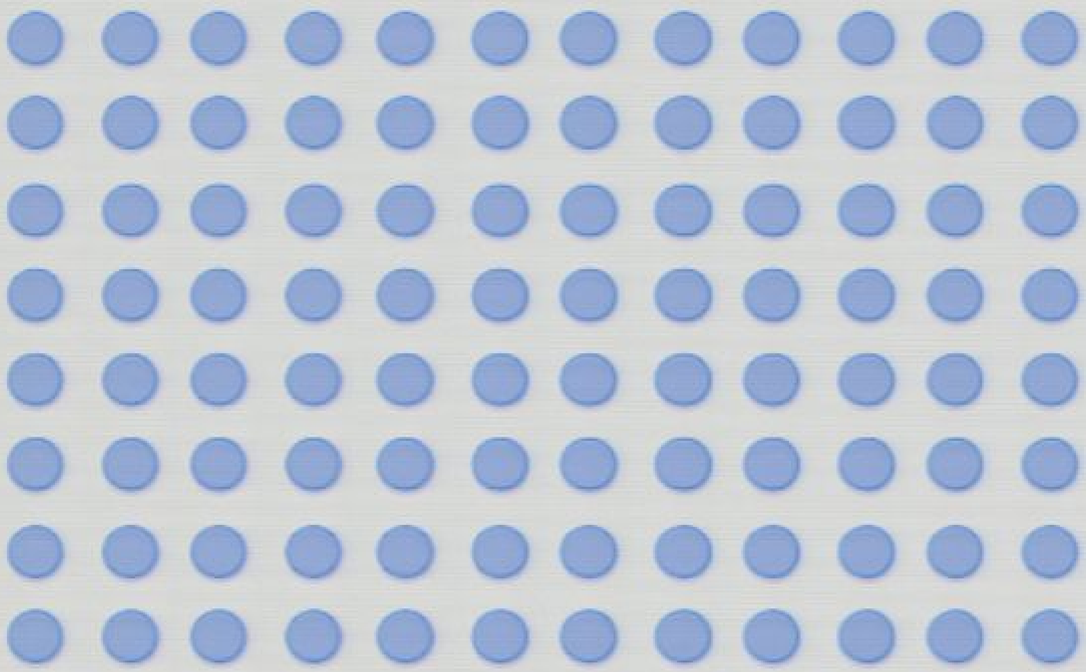


Translation equivalence of logical operators 1

Translation and scale symmetries : How can we simplify the problem ?



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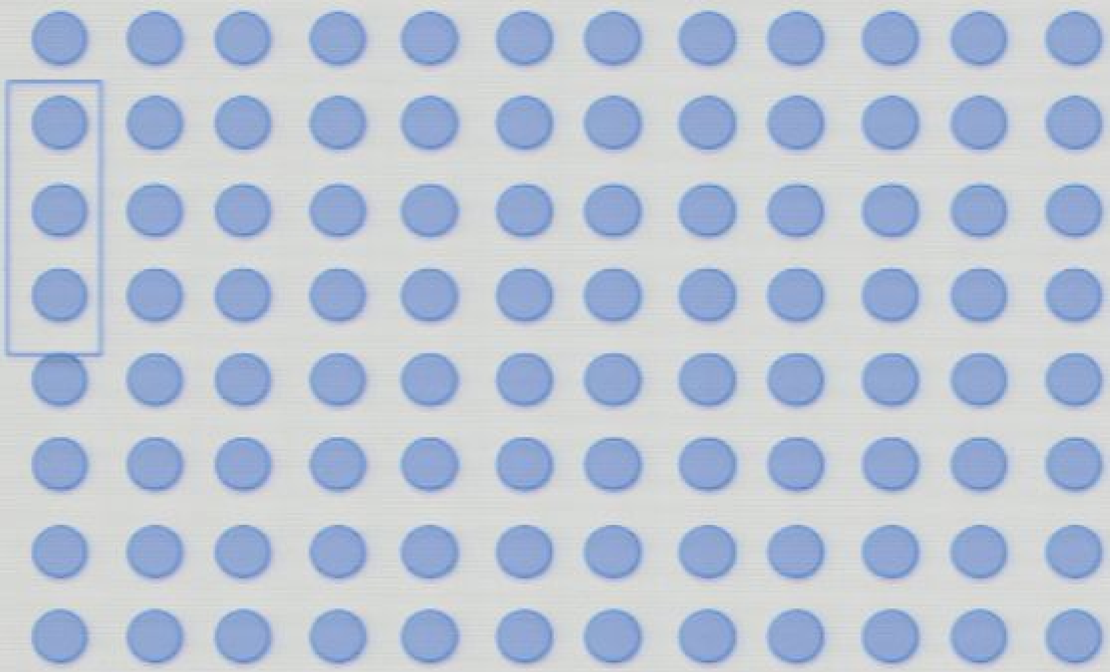
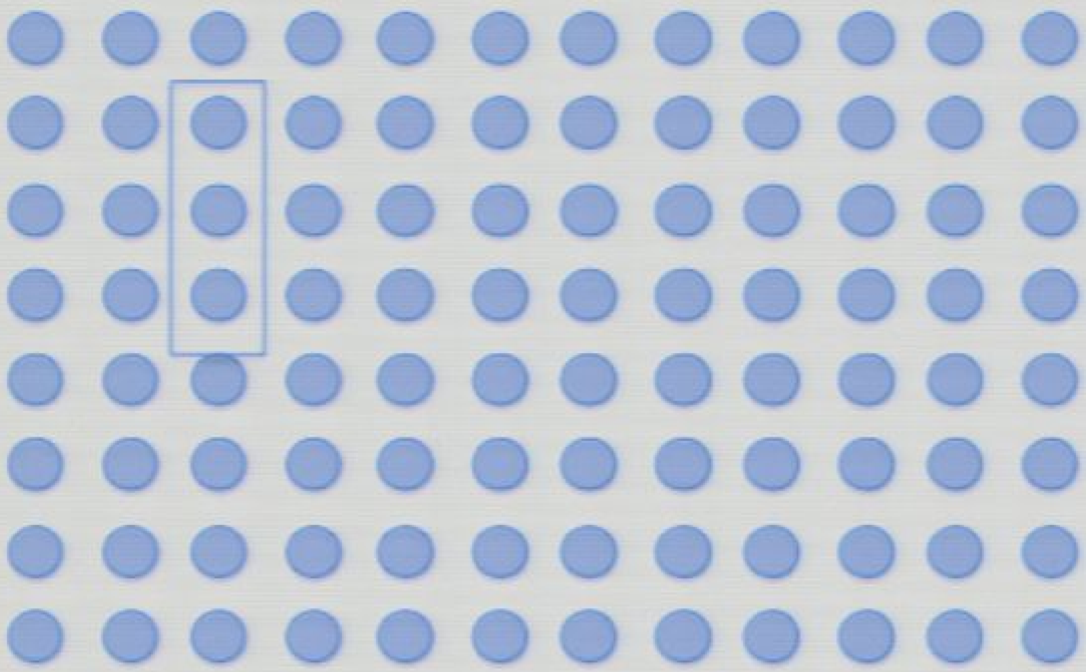


Translations of logical operators are logical operators

Translation equivalence of logical operators 1

Translation and scale symmetries : How can we simplify the problem ?



Translations of logical operators are logical operators

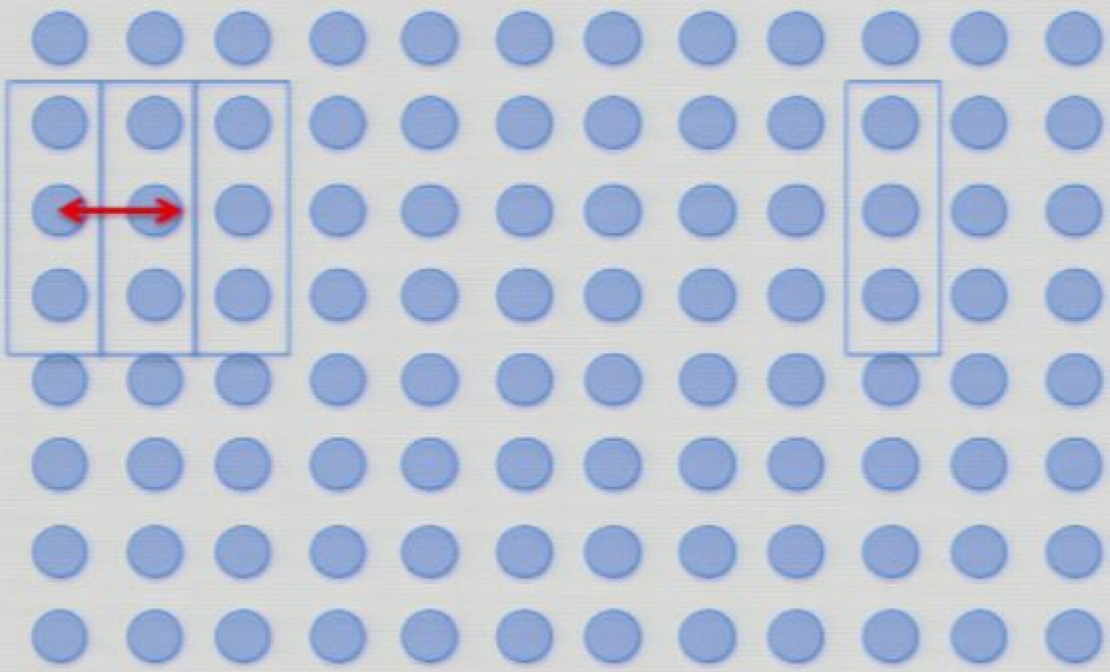
There are small number of independent logical operators

There exists a finite translation which keep logical operators equivalent

Translation equivalence of logical operators 2

In fact….

All the logical operators remain equivalent under unit translations with respect to composite particles.



Physical meanings

Ground states are invariant under unit translations of composite particles

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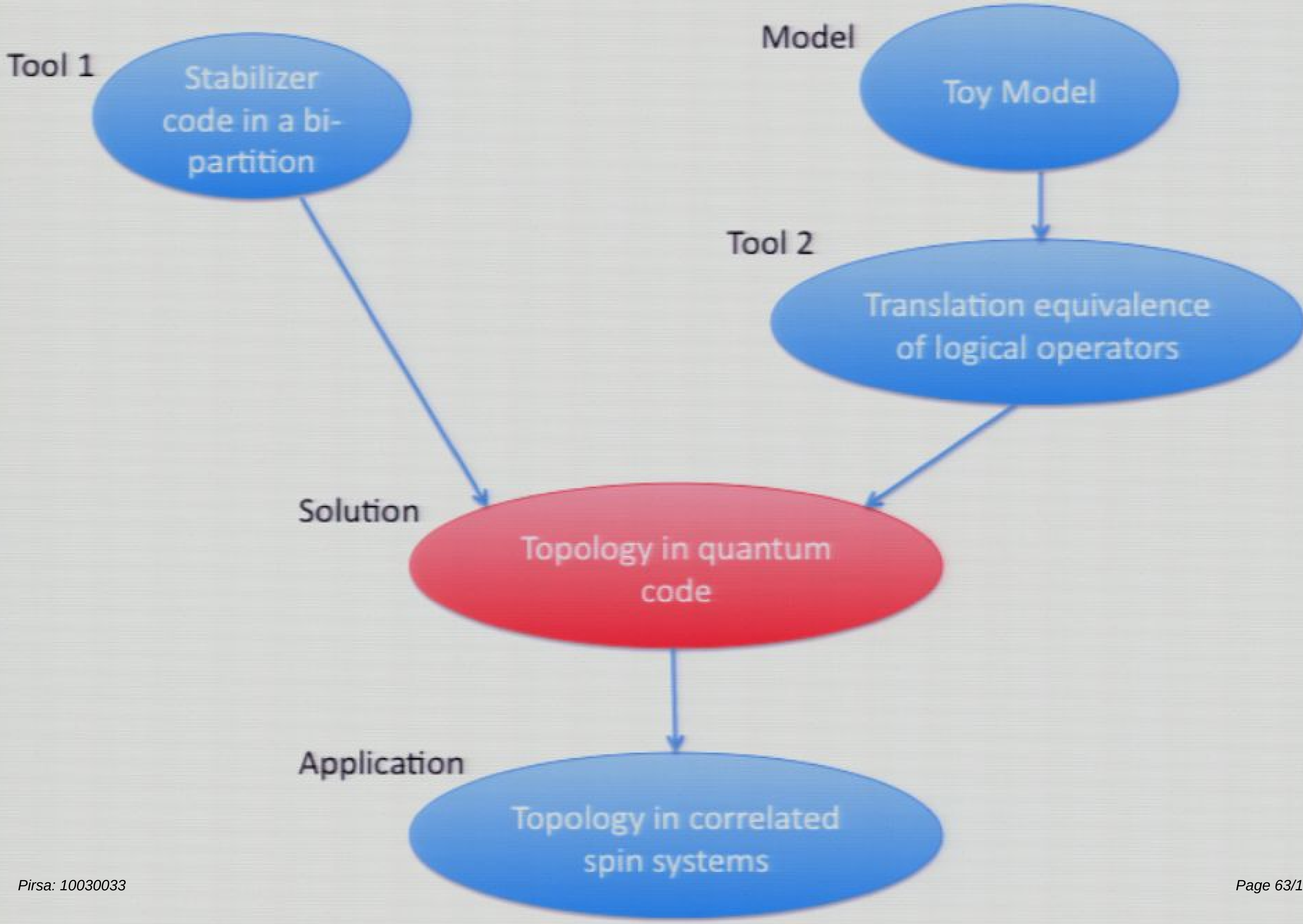
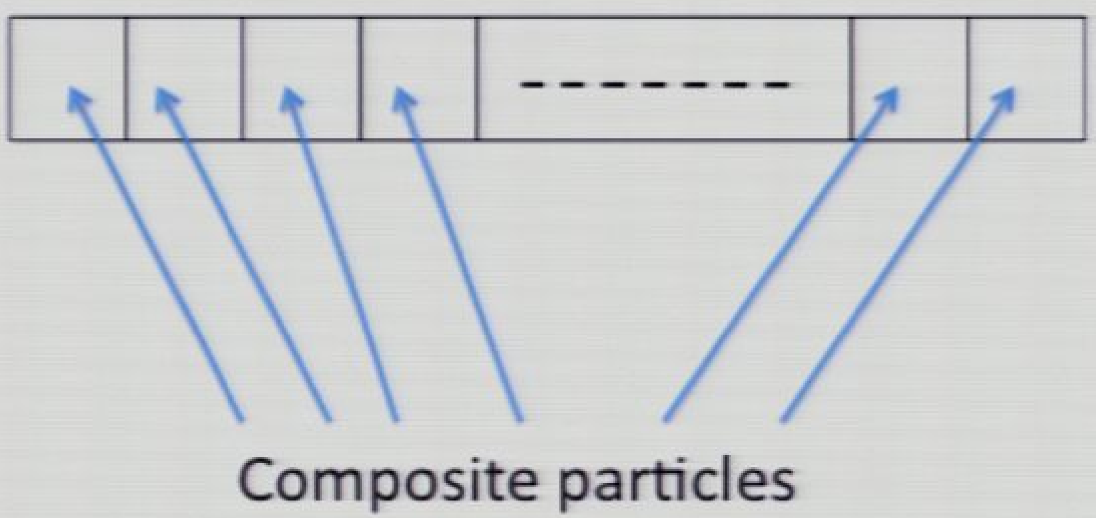


Logical operators in 1 dim

(For simplicity, k =1 first)

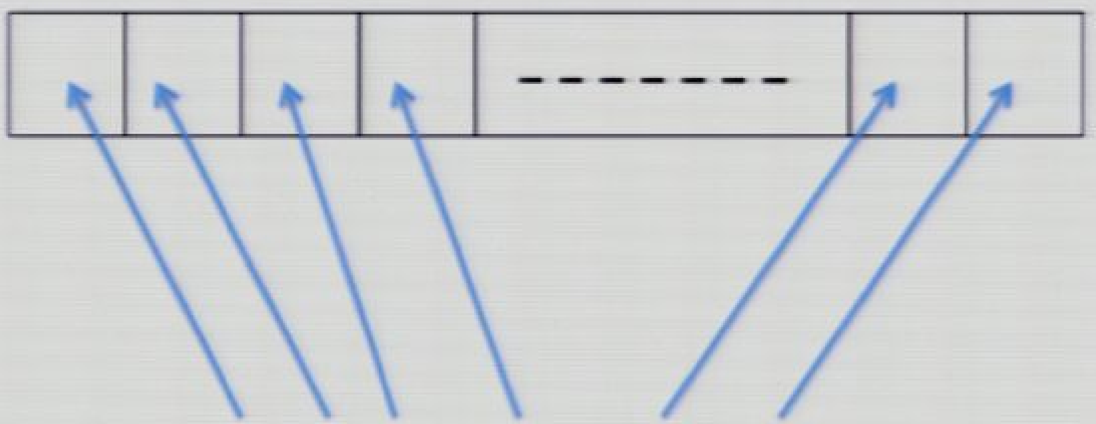


Logical operators in 1 dim

(For simplicity, k =1 first)

$$\{\bar{X}, \bar{Z}\} = 0$$

ℓ	$\bar{X}$	$\bar{X}$	$\bar{X}$	$\bar{X}$	--------	$\bar{X}$	$\bar{X}$
r	$\bar{Z}$				--------		



Composite particles

Logical operators in 1 dim

(Now for general k)

$$\left\{ \begin{array}{ccc} \bar{X}_1, & \cdots, & \bar{X}_k \\ \bar{Z}_1, & \cdots, & \bar{Z}_k \end{array} \right\}$$

ℓ_i	$\bar{X}_i$	$\bar{X}_i$	$\bar{X}_i$	$\bar{X}_i$	--------	$\bar{X}_i$	$\bar{X}_i$
r_i	$\bar{Z}_i$				--------		

Logical operators in 1 dim

(Now for general k)

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- Code distance is at most v (number of qubits in a composite particle)
- GHZ-like entanglement

$$|\psi\rangle = |\psi_0\rangle + |\psi_1\rangle \qquad |\psi_0\rangle = \ell|\psi_1\rangle$$

Logical operators in 1 dim

(Now for general k)

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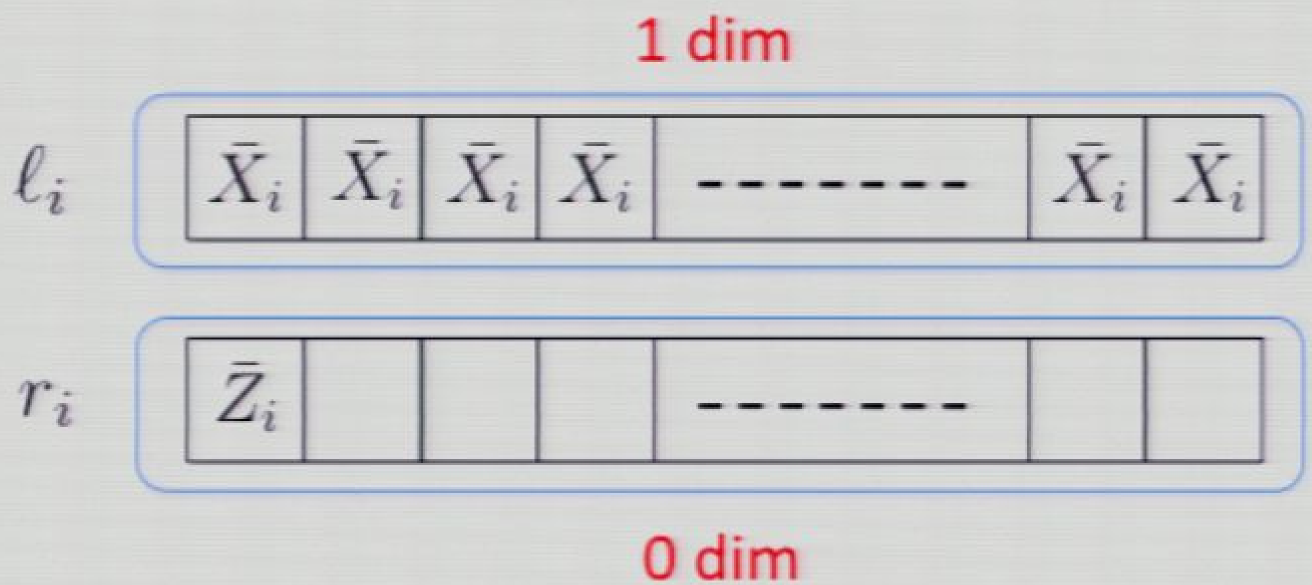

- Code distance is at most v (number of qubits in a composite particle)
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$$|\psi\rangle = |\psi_0\rangle + |\psi_1\rangle \qquad |\psi_0\rangle = \ell|\psi_1\rangle$$

0-dim logical operator and 1-dim logical operator form a pair

Logical operators in 1 dim

However, logical operators have many equivalent representations.

Is this classification universal ?

Logical operators in 1 dim

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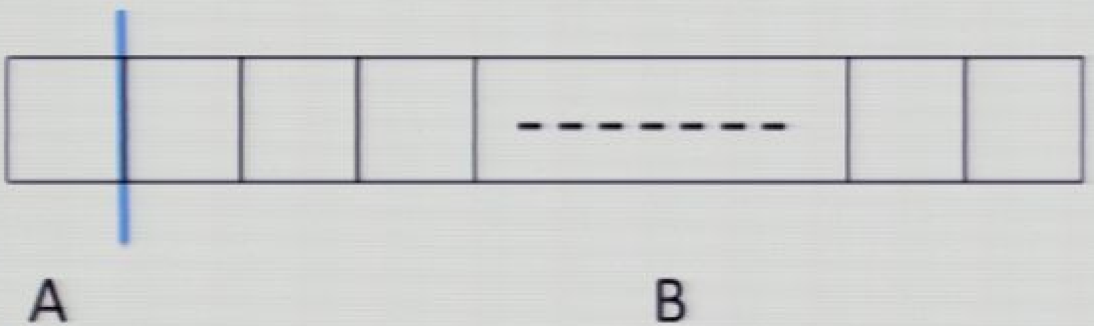
Is this classification universal ?



Logical operators in 1 dim

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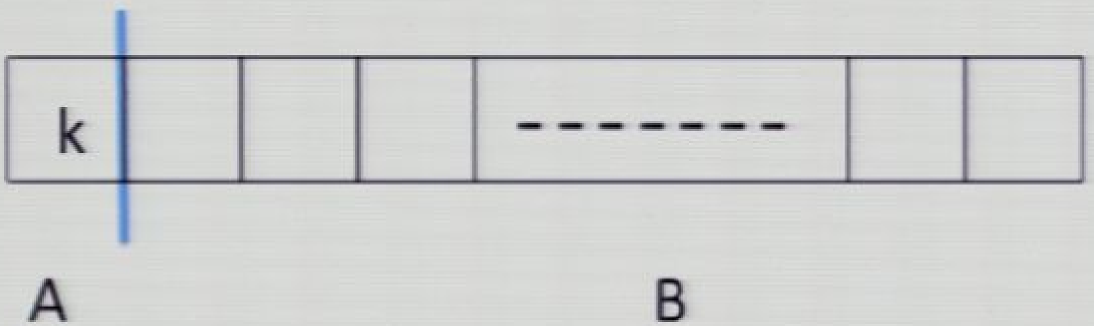


$$g_A + g_B = 2k$$

$$\downarrow$$

$$g_A = g_B = k$$

B can support only 0-dim logical operators

Logical operators in 2 dim

Logical operators in 2 dim

ℓ_i

$\bar{Z}_i$				

r_i

$\bar{X}_i$	$\bar{X}_i$	$\bar{X}_i$	$\bar{X}_i$	$\bar{X}_i$
$\bar{X}_i$	$\bar{X}_i$	$\bar{X}_i$	$\bar{X}_i$	$\bar{X}_i$
$\bar{X}_i$	$\bar{X}_i$	$\bar{X}_i$	$\bar{X}_i$	$\bar{X}_i$
$\bar{X}_i$	$\bar{X}_i$	$\bar{X}_i$	$\bar{X}_i$	$\bar{X}_i$
$\bar{X}_i$	$\bar{X}_i$	$\bar{X}_i$	$\bar{X}_i$	$\bar{X}_i$

Logical operators in 2 dim

ℓ_i

$\bar{Z}_i$				

ℓ_i

$\bar{Z}_i$				
$\bar{Z}_i$				
$\bar{Z}_i$				
$\bar{Z}_i$				
$\bar{Z}_i$				

0 dim

r_i

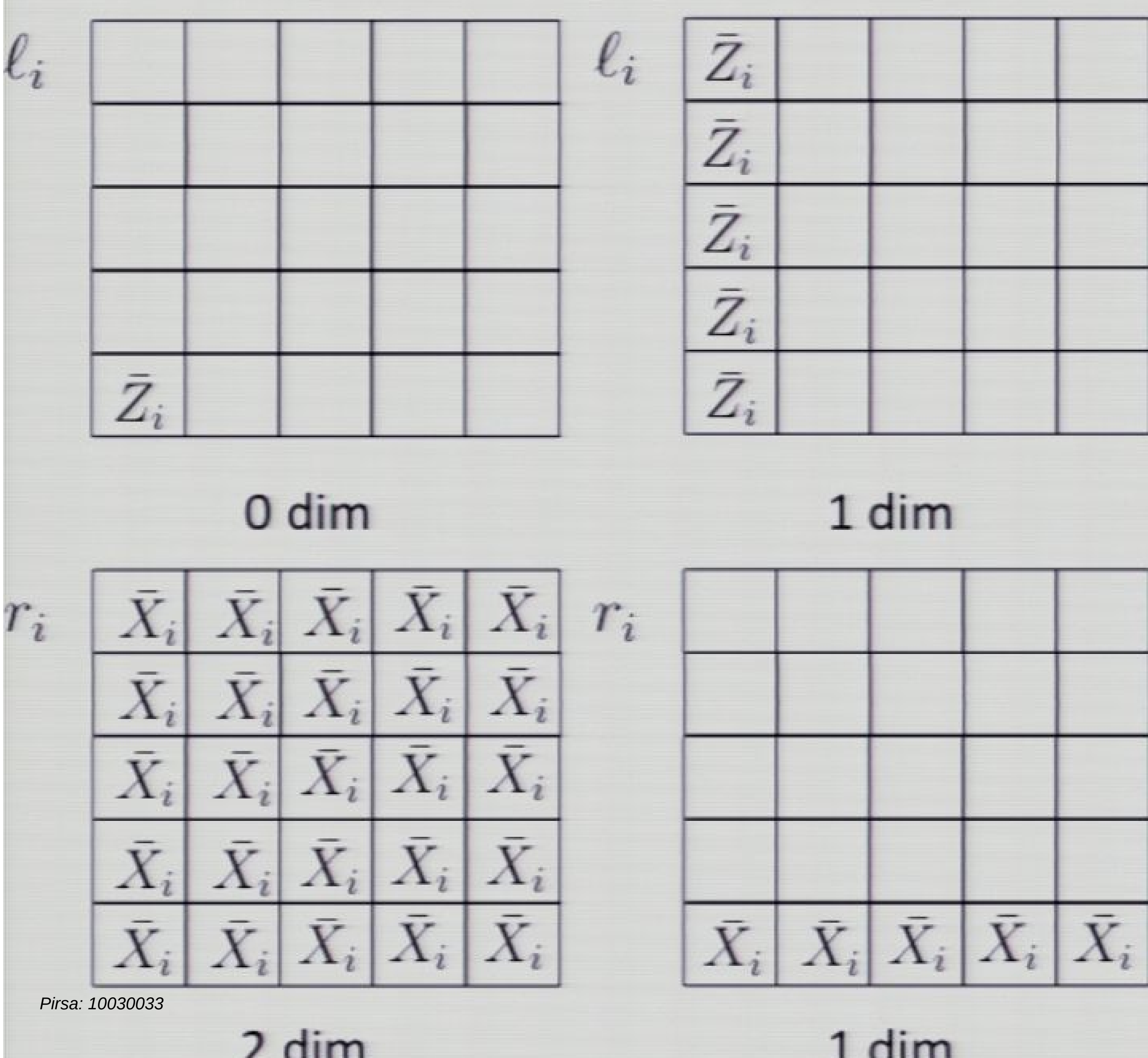
$\bar{X}_i$	$\bar{X}_i$	$\bar{X}_i$	$\bar{X}_i$	$\bar{X}_i$
$\bar{X}_i$	$\bar{X}_i$	$\bar{X}_i$	$\bar{X}_i$	$\bar{X}_i$
$\bar{X}_i$	$\bar{X}_i$	$\bar{X}_i$	$\bar{X}_i$	$\bar{X}_i$
$\bar{X}_i$	$\bar{X}_i$	$\bar{X}_i$	$\bar{X}_i$	$\bar{X}_i$
$\bar{X}_i$	$\bar{X}_i$	$\bar{X}_i$	$\bar{X}_i$	$\bar{X}_i$

r_i

$\bar{X}_i$	$\bar{X}_i$	$\bar{X}_i$	$\bar{X}_i$	$\bar{X}_i$

2 dim

Logical operators in 2 dim



Sum of dimensions

D = 2

Logical operators in 2 dim

ℓ_i

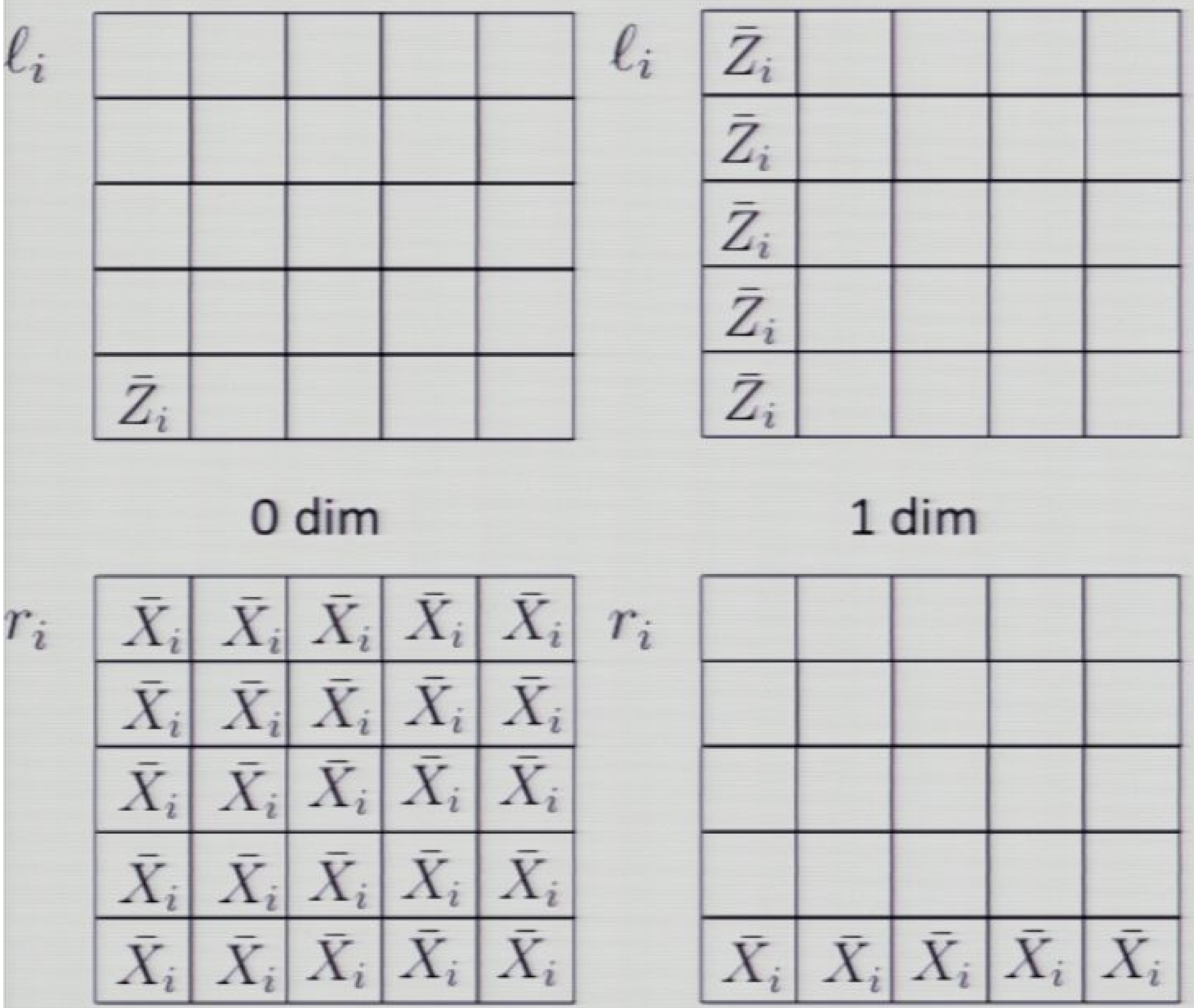
$\bar{Z}_i$				
$\bar{Z}_i$				
$\bar{Z}_i$				
$\bar{Z}_i$				
$\bar{Z}_i$				

r_i

$\bar{X}_i$	$\bar{X}_i$	$\bar{X}_i$	$\bar{X}_i$	$\bar{X}_i$

Physical properties

Logical operators in 2 dim



0 dim

1 dim

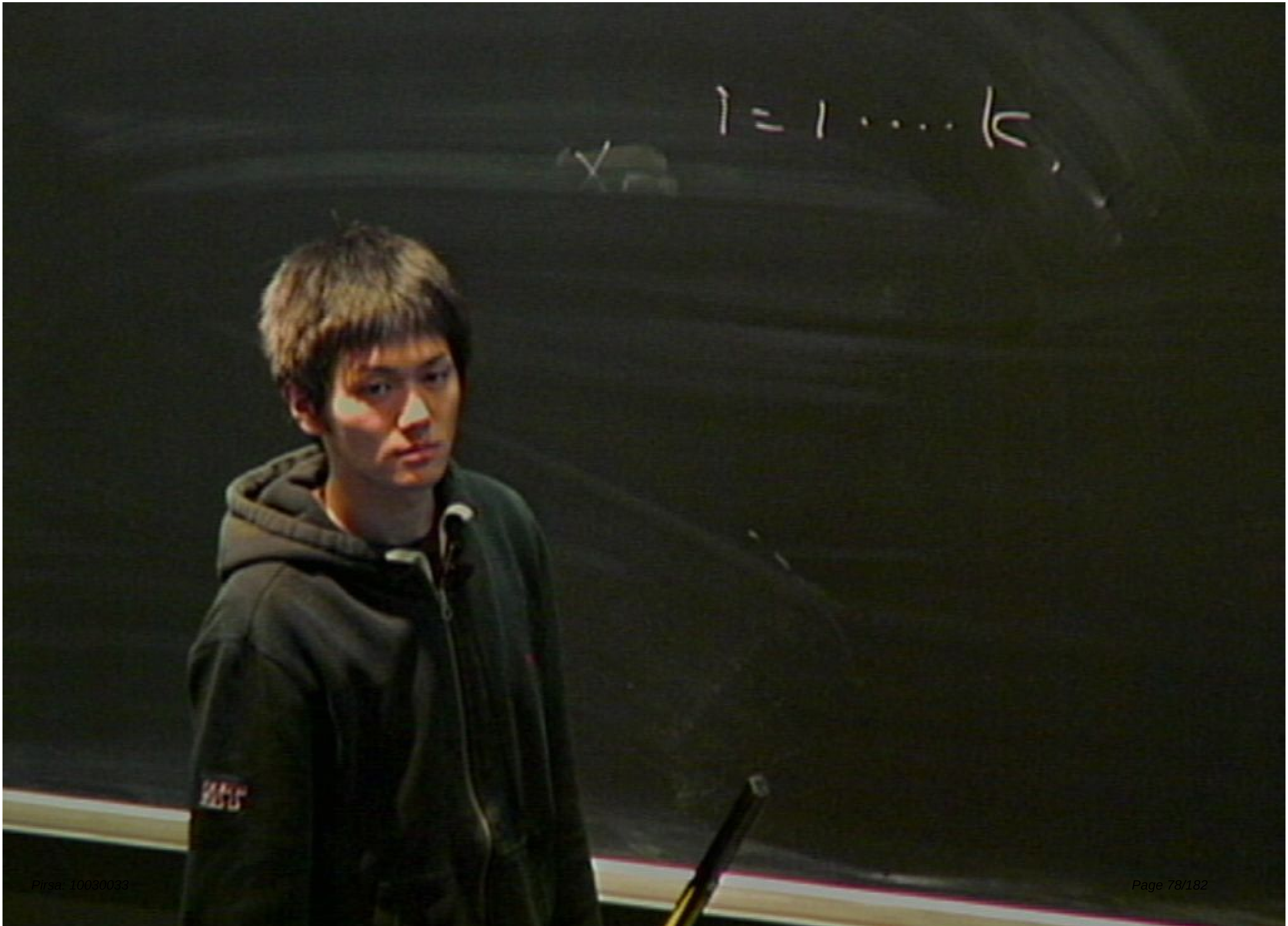
Sum of dimensions

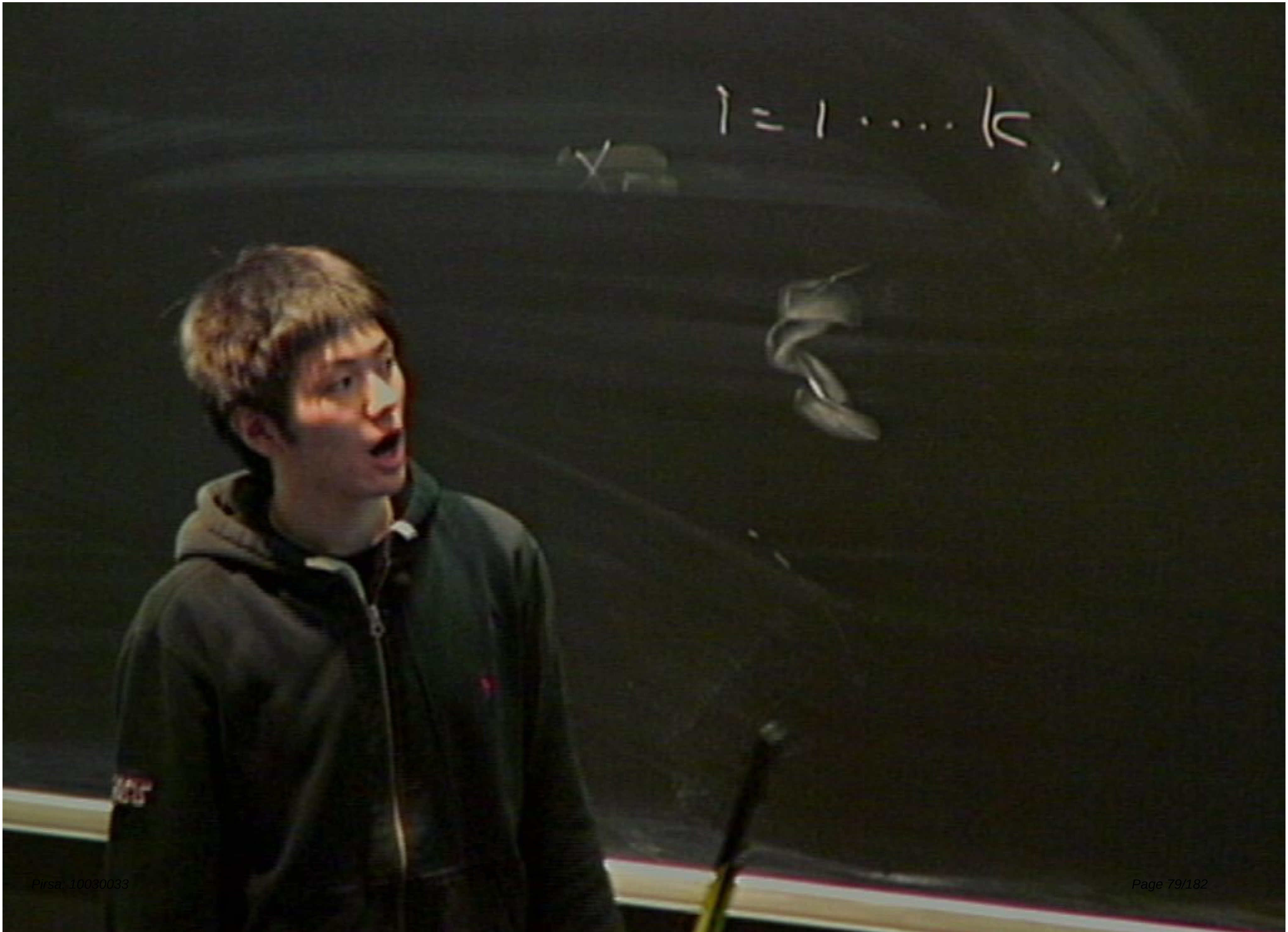
↓

D = 2

2 dim

1 dim


i = 1 . . . k ,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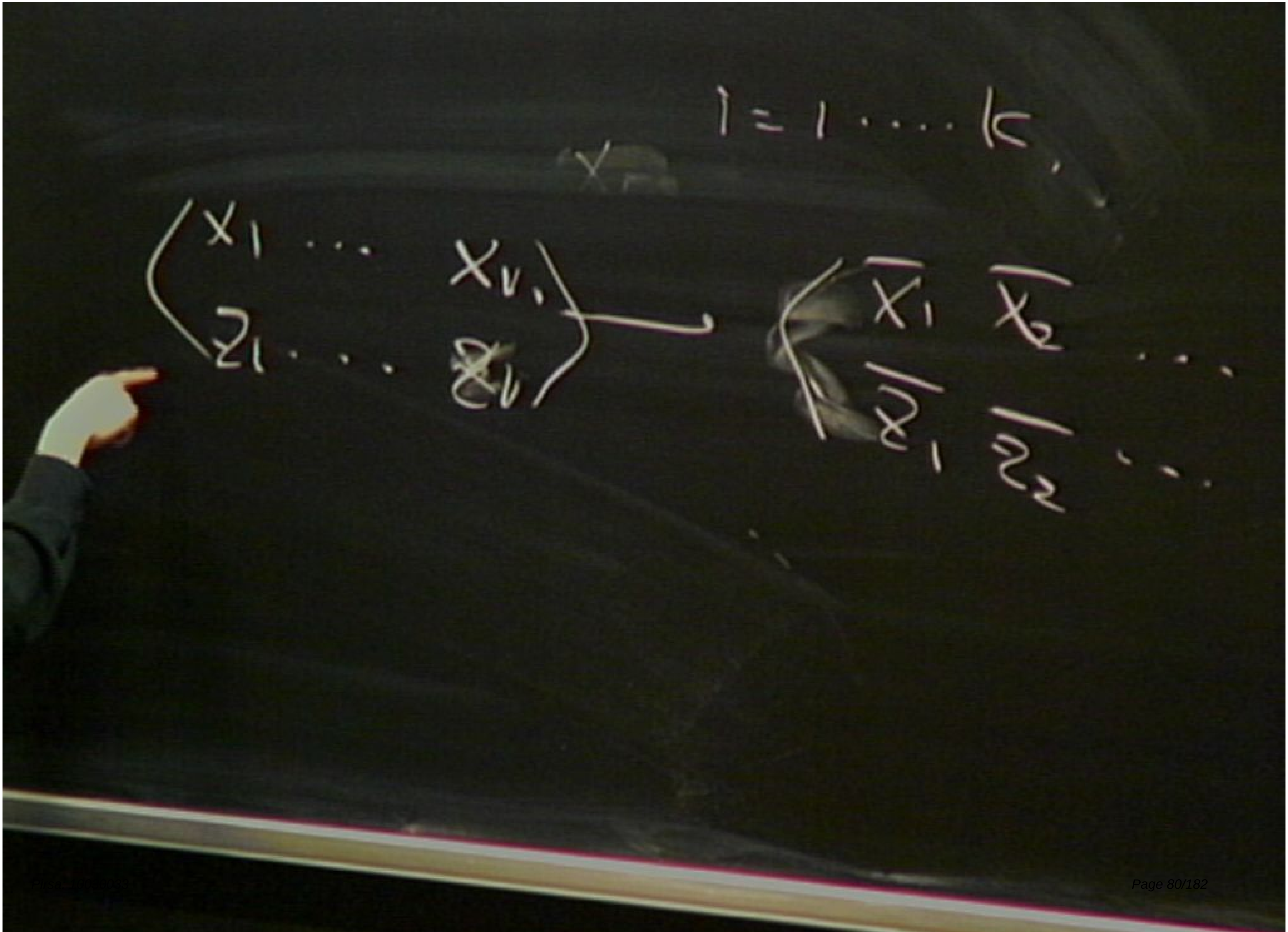
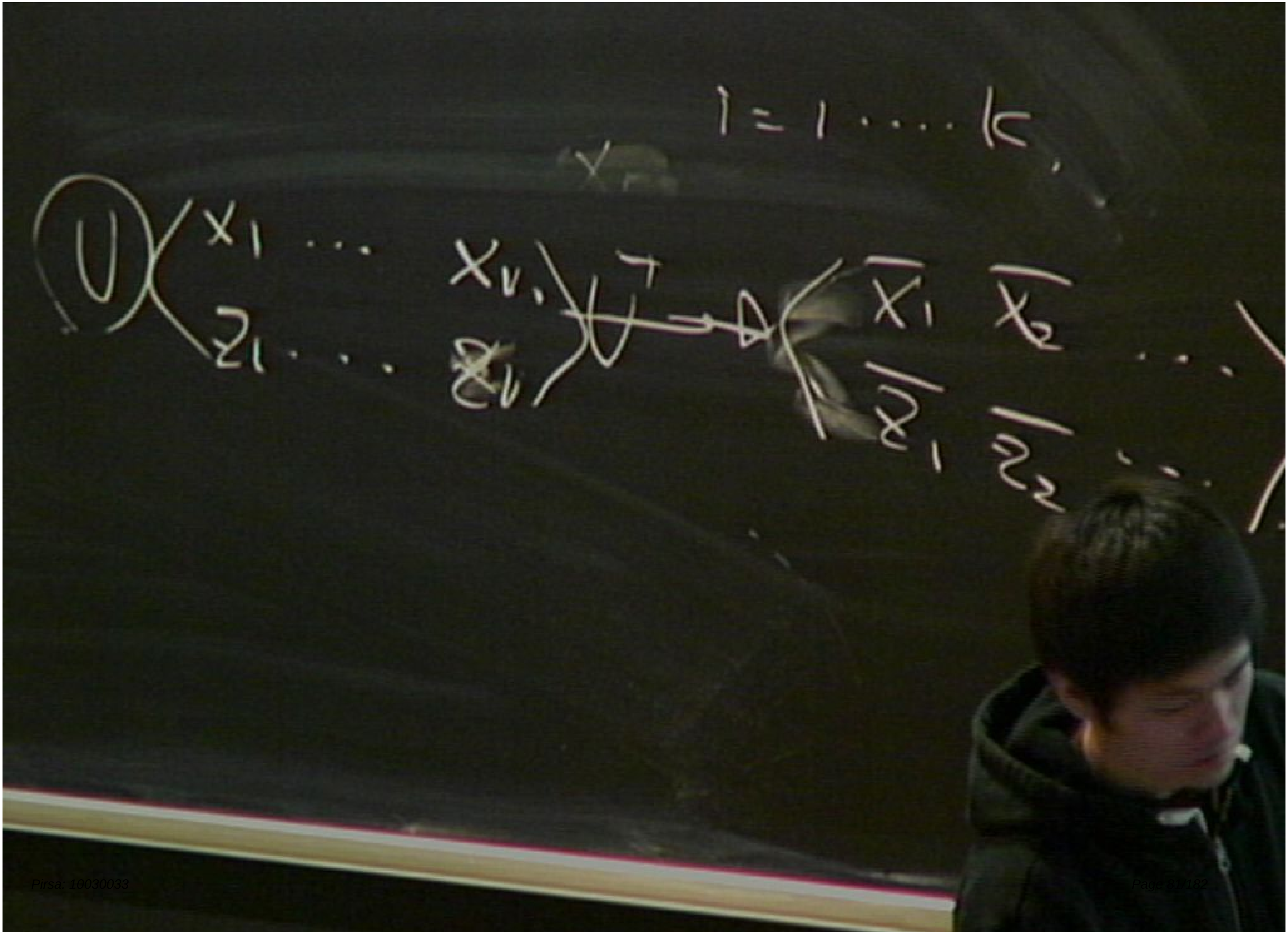

i = 1 .... k,


i = 1 . . . k,
U
x1 . . . xN,
z1 . . . zN,

Logical operators in 2 dim

<u>Physical properties</u>

ℓ_i

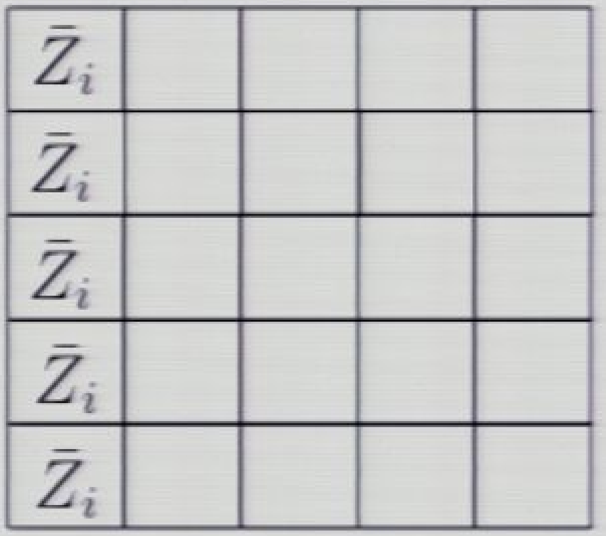
$\bar{Z}_i$				
$\bar{Z}_i$				
$\bar{Z}_i$				
$\bar{Z}_i$				
$\bar{Z}_i$				

r_i

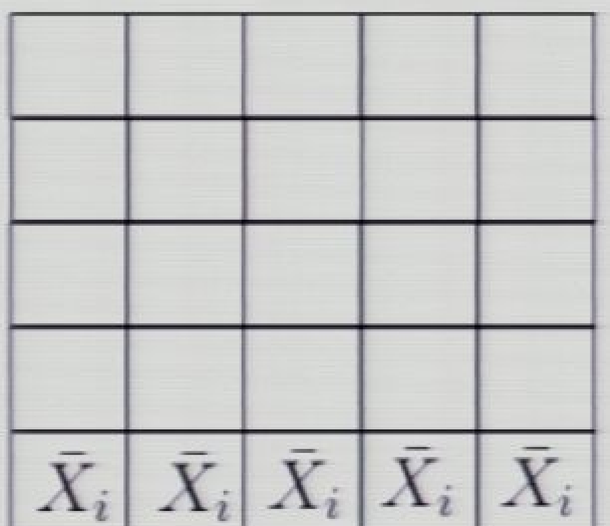
$\bar{X}_i$	$\bar{X}_i$	$\bar{X}_i$	$\bar{X}_i$	$\bar{X}_i$

Logical operators in 2 dim

ℓ_i



r_i

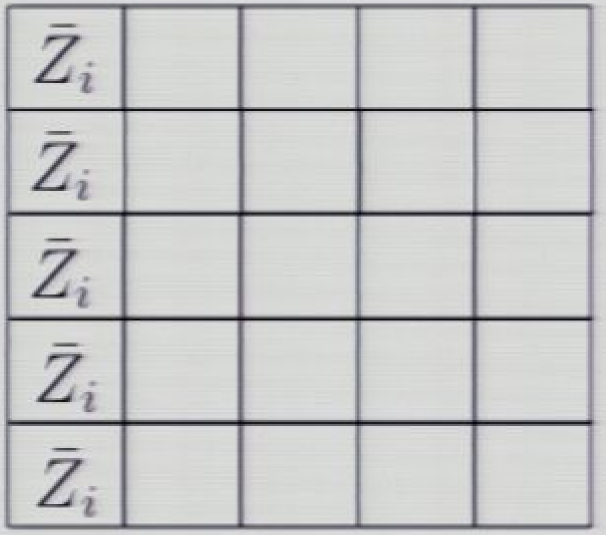


Physical properties

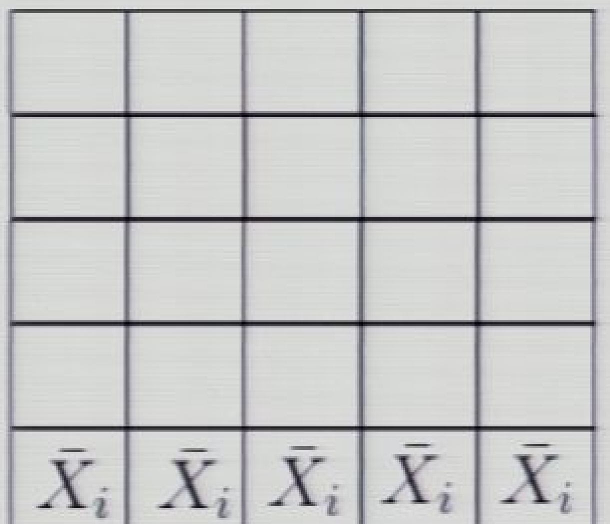
Endpoints of string operators can be viewed as anyonic excitations

Logical operators in 2 dim

ℓ_i



r_i



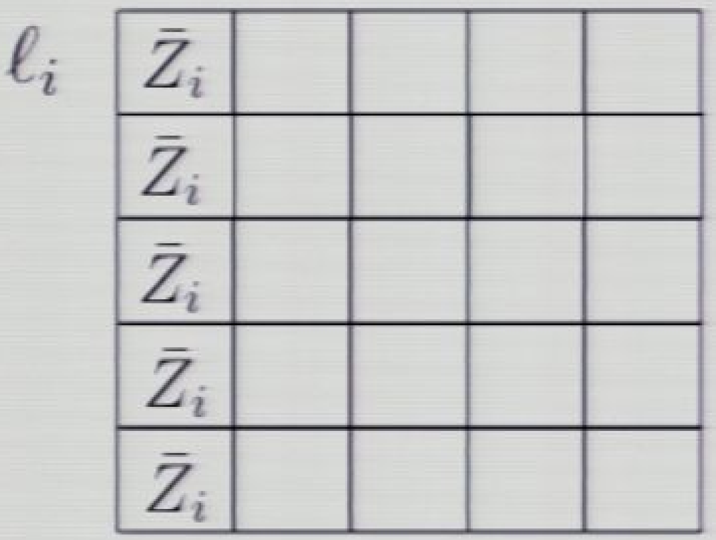
<u>Physical properties</u>

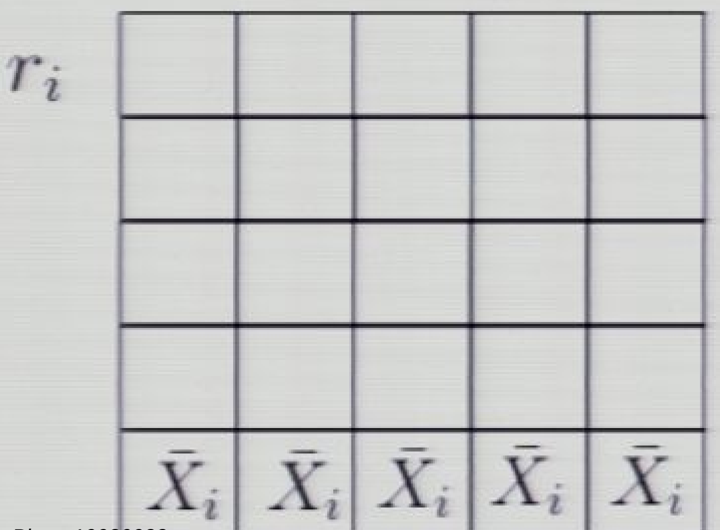
Endpoints of string operators can be viewed as anyonic excitations

Braiding group

$$\mathbb{Z}_2 \otimes \cdots \otimes \mathbb{Z}_2$$

Logical operators in 2 dim





Physical properties

Endpoints of string operators can be viewed as anyonic excitations

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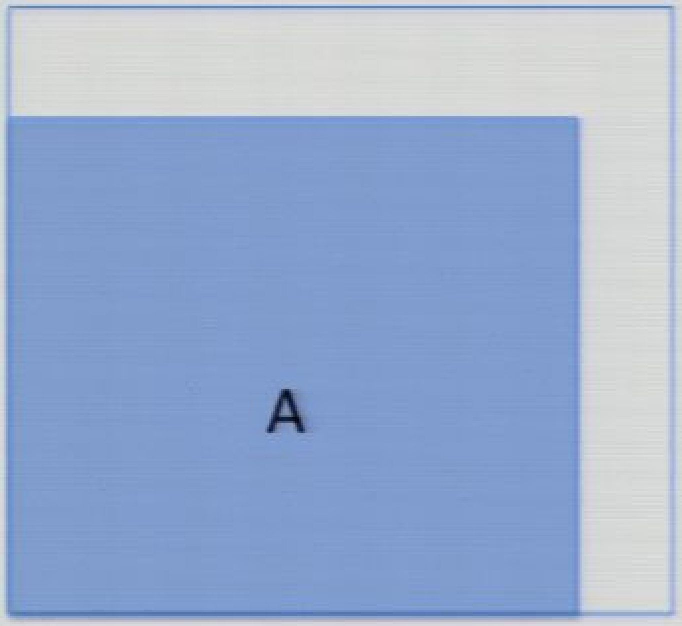
Topological order resulting from 1 dim logical operators

Logical operators in 2 dim

Universality of classification

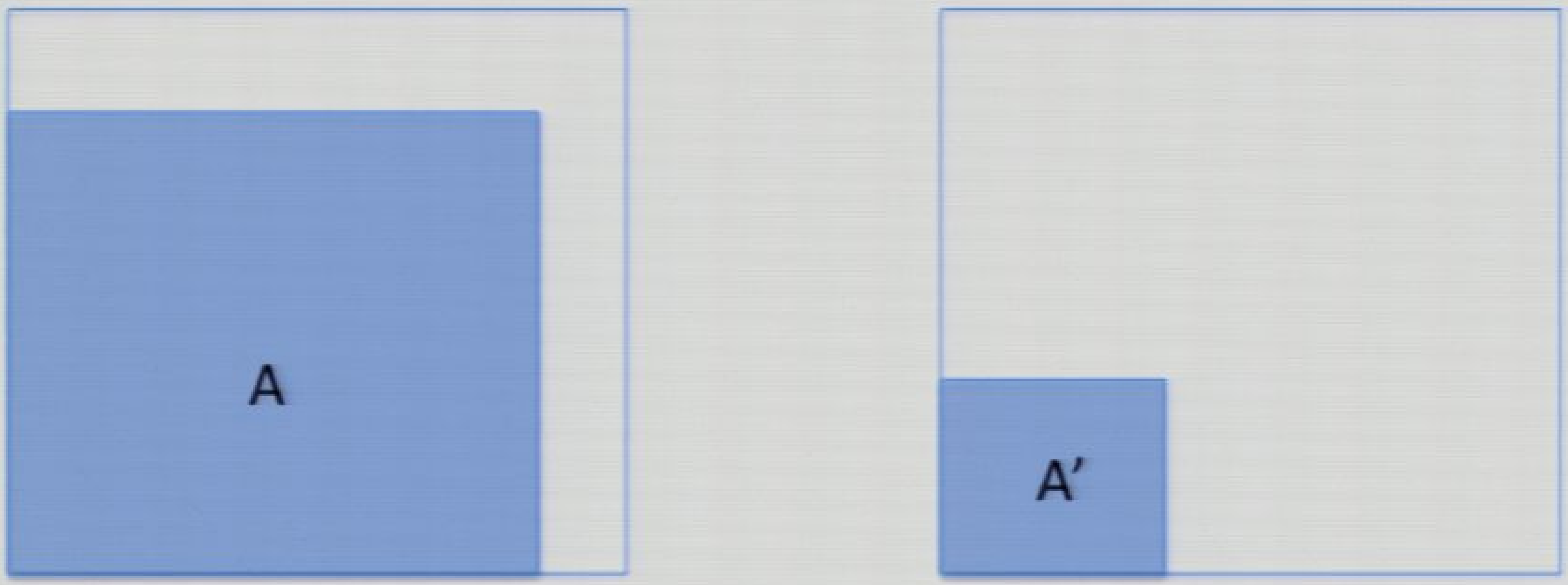
Logical operators in 2 dim

Universality of classification



Logical operators in 2 dim

Universality of classification



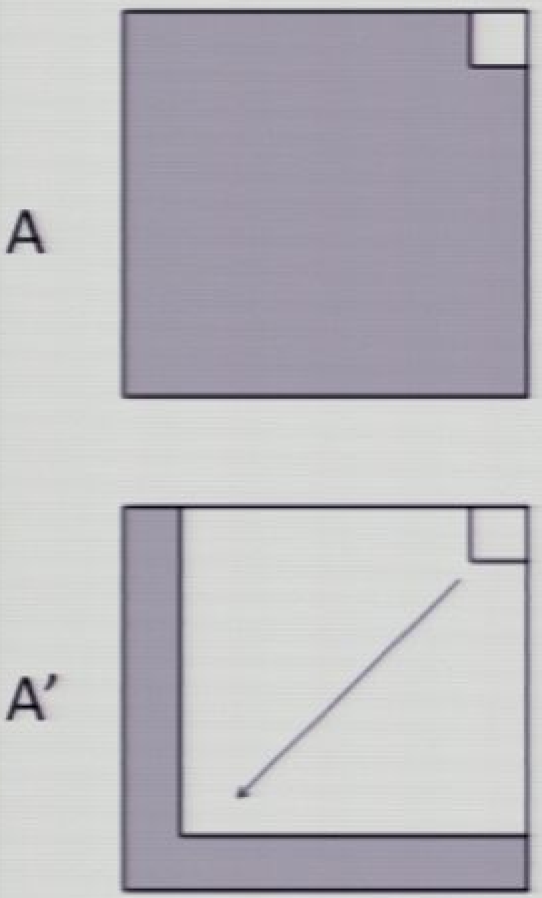
Logical operators in 2 dim

Universality of classification

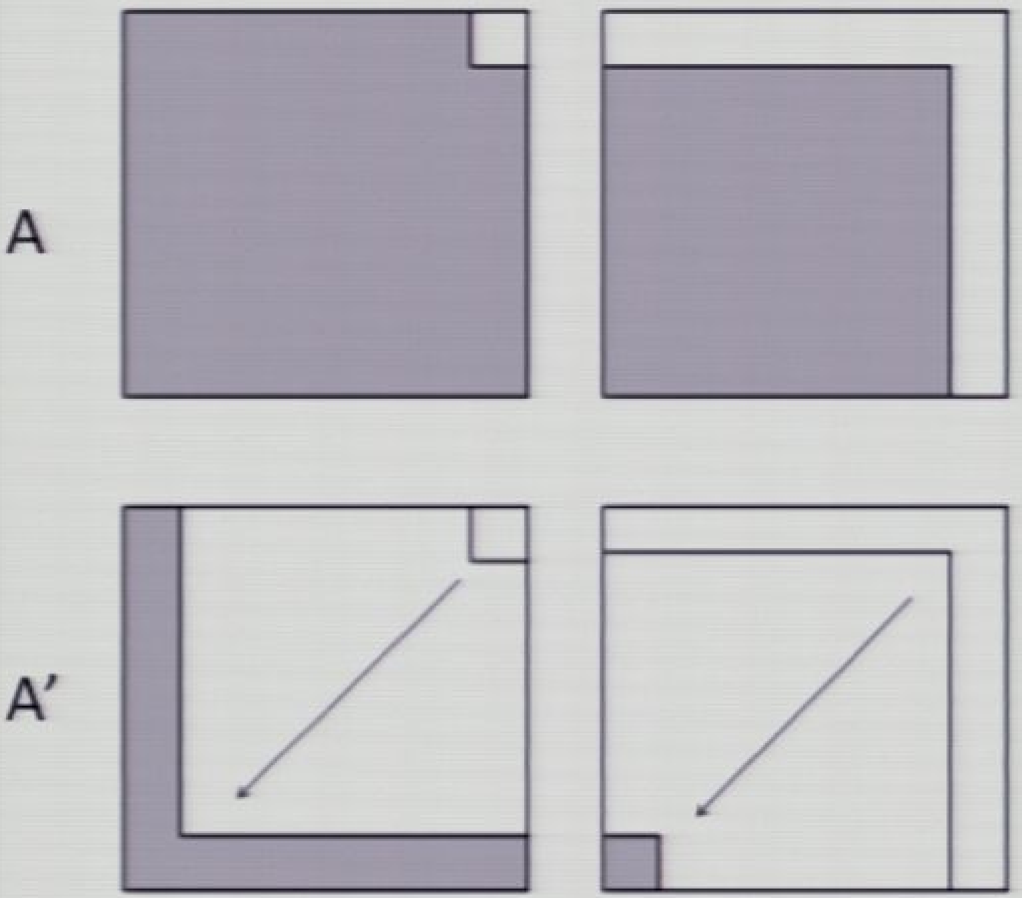
$$g_A = g_{A'}$$

If we are given a logical operator defined inside A, we can change its geometric shape to A'.

Logical operators in 2 dim



Logical operators in 2 dim



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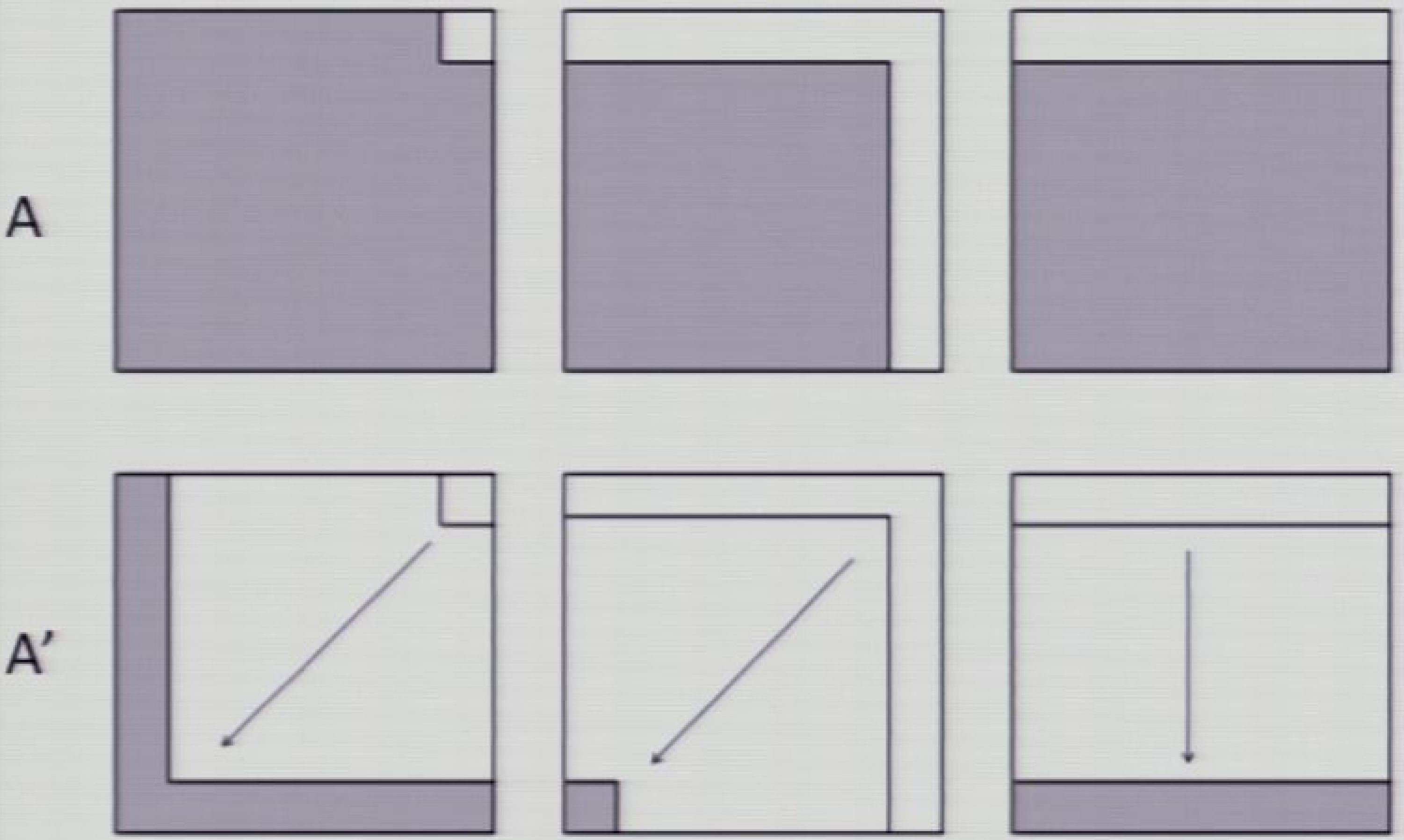
Logical operators in 2 dim



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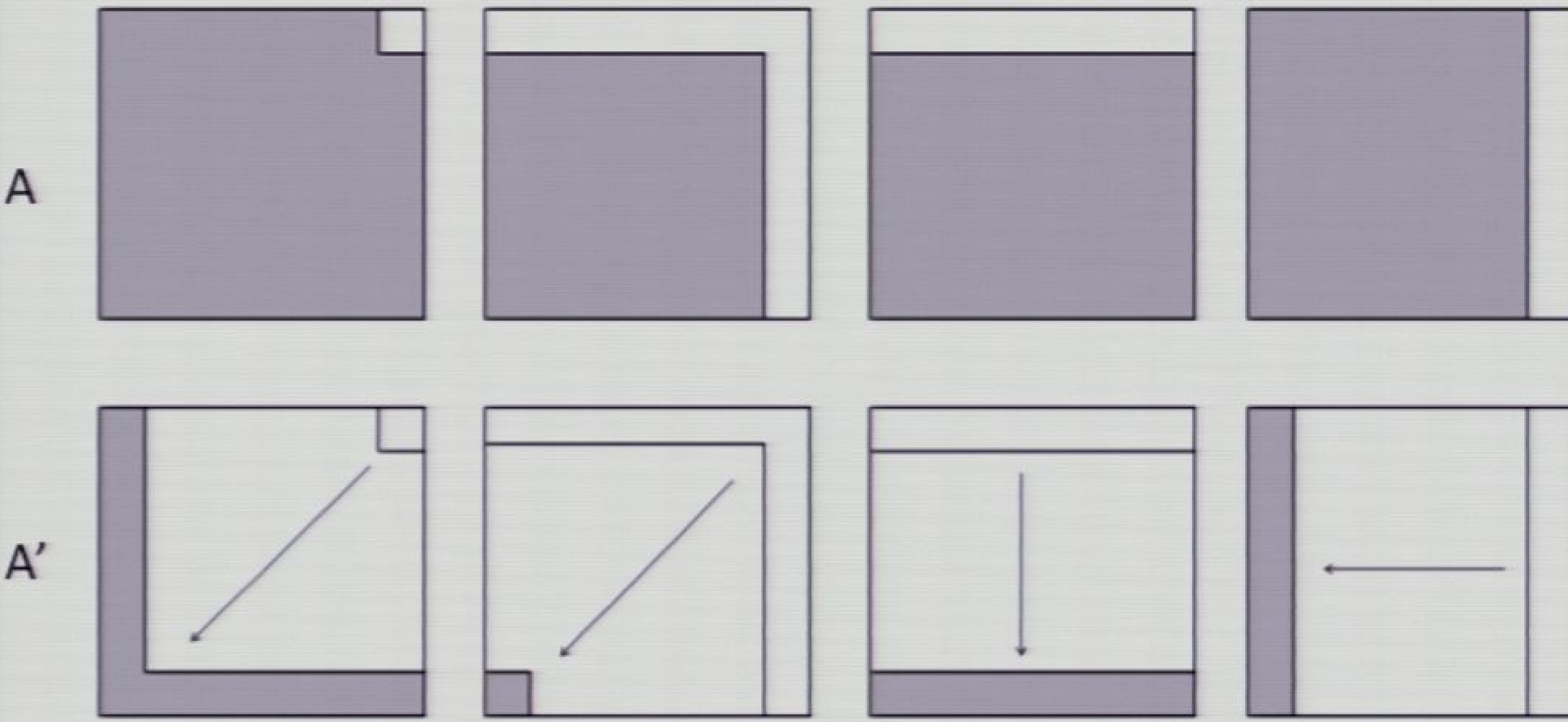
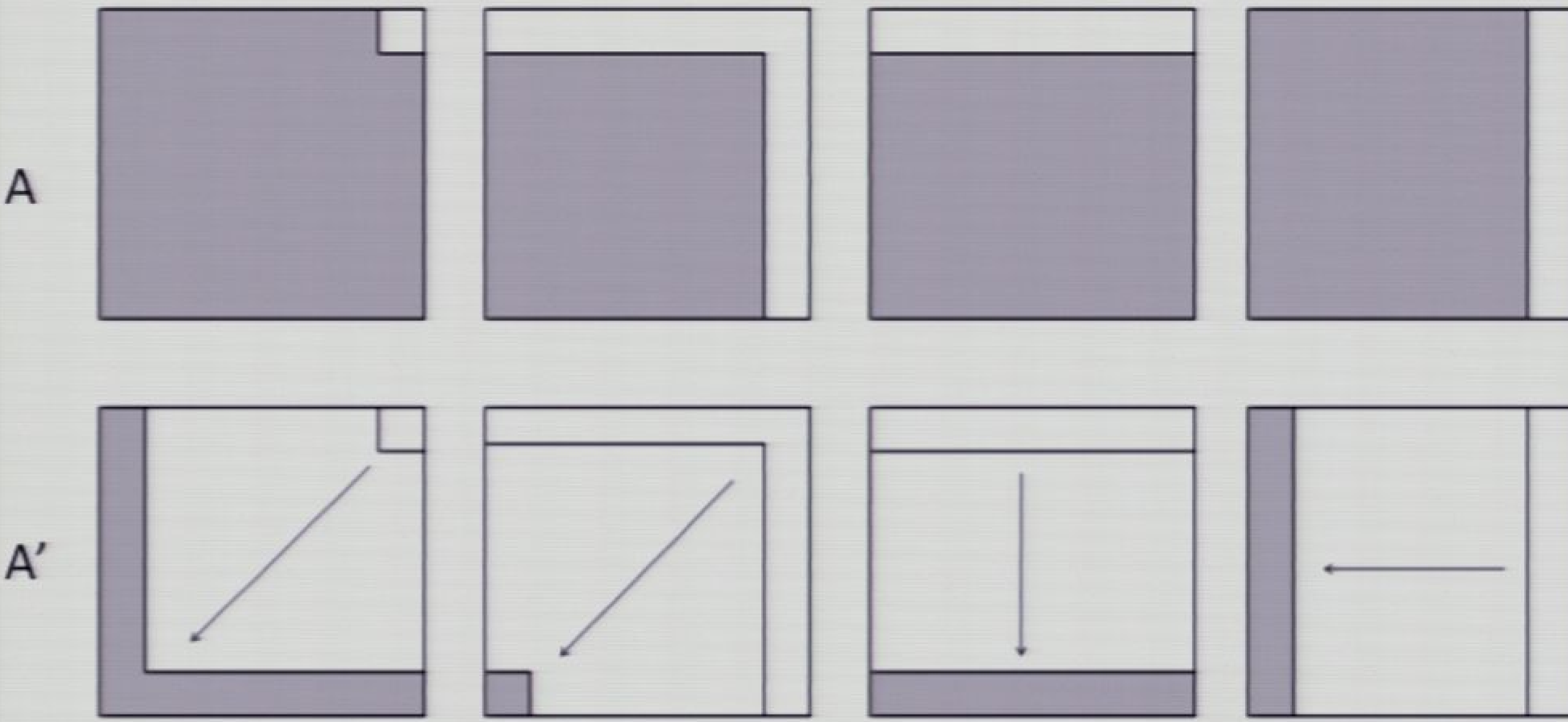
Logical operators in 2 dim



$$g_A = g_{A'}$$

Logical operators in 2 dim

We are allowed to deform geometric shapes of logical operators continuously.

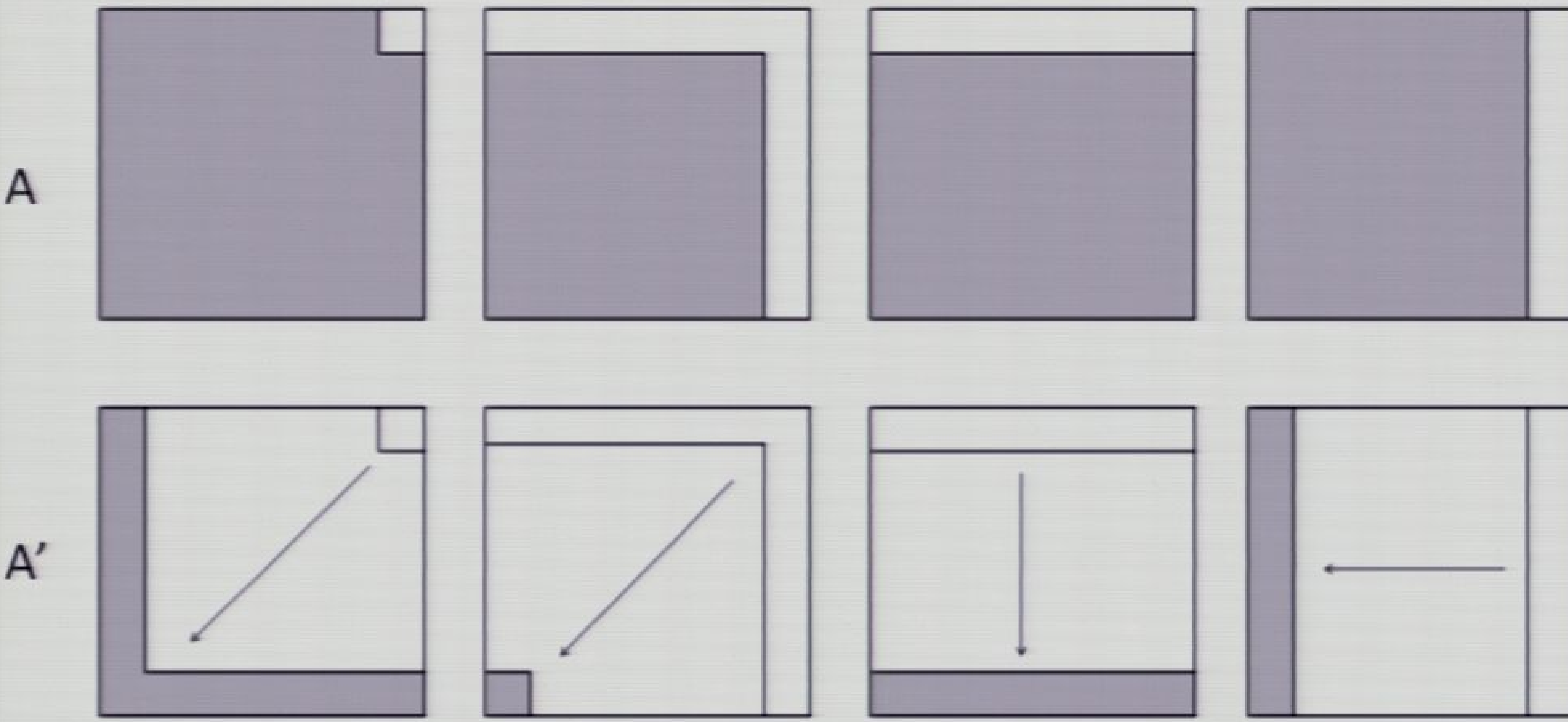


$$g_A = g_{A'}$$

Logical operators in 2 dim

We are allowed to deform geometric shapes of logical operators continuously.

Topological deformation of logical operators



$$g_A = g_{A'}$$

Logical operators in D dim

Logical operators in D dim

0 - dim	**D** - dim
1 - dim	**D-1** - dim
⋮	
D/2 - dim	**D/2** - dim

D : even

Logical operators in D dim

D : even		D : odd	
0 - dim	**D** - dim	**0** - dim	**D** - dim
1 - dim	**D-1** - dim	**1** - dim	**D-1** - dim
⋮		⋮	
D/2 - dim	**D/2** - dim	**(D-1)/2** - dim	**(D+1)/2** - dim

Logical operators in D dim

D : even		D : odd	
0 - dim	**D** - dim	**0** - dim	**D** - dim
1 - dim	**D-1** - dim	**1** - dim	**D-1** - dim
⋮		⋮	
D/2 - dim	**D/2** - dim	**(D-1)/2** - dim	**(D+1)/2** - dim

D : even

D : odd

Dimensional duality in logical operators

Logical operators in D dim

D : even		D : odd	
0 - dim	**D** - dim	**0** - dim	**D** - dim
1 - dim	**D-1** - dim	**1** - dim	**D-1** - dim
⋮		⋮	
D/2 - dim	**D/2** - dim	**(D-1)/2** - dim	**(D+1)/2** - dim

D : even

D : odd

Dimensional duality in logical operators

Topological deformation of logical operator holds in D dimensions too.

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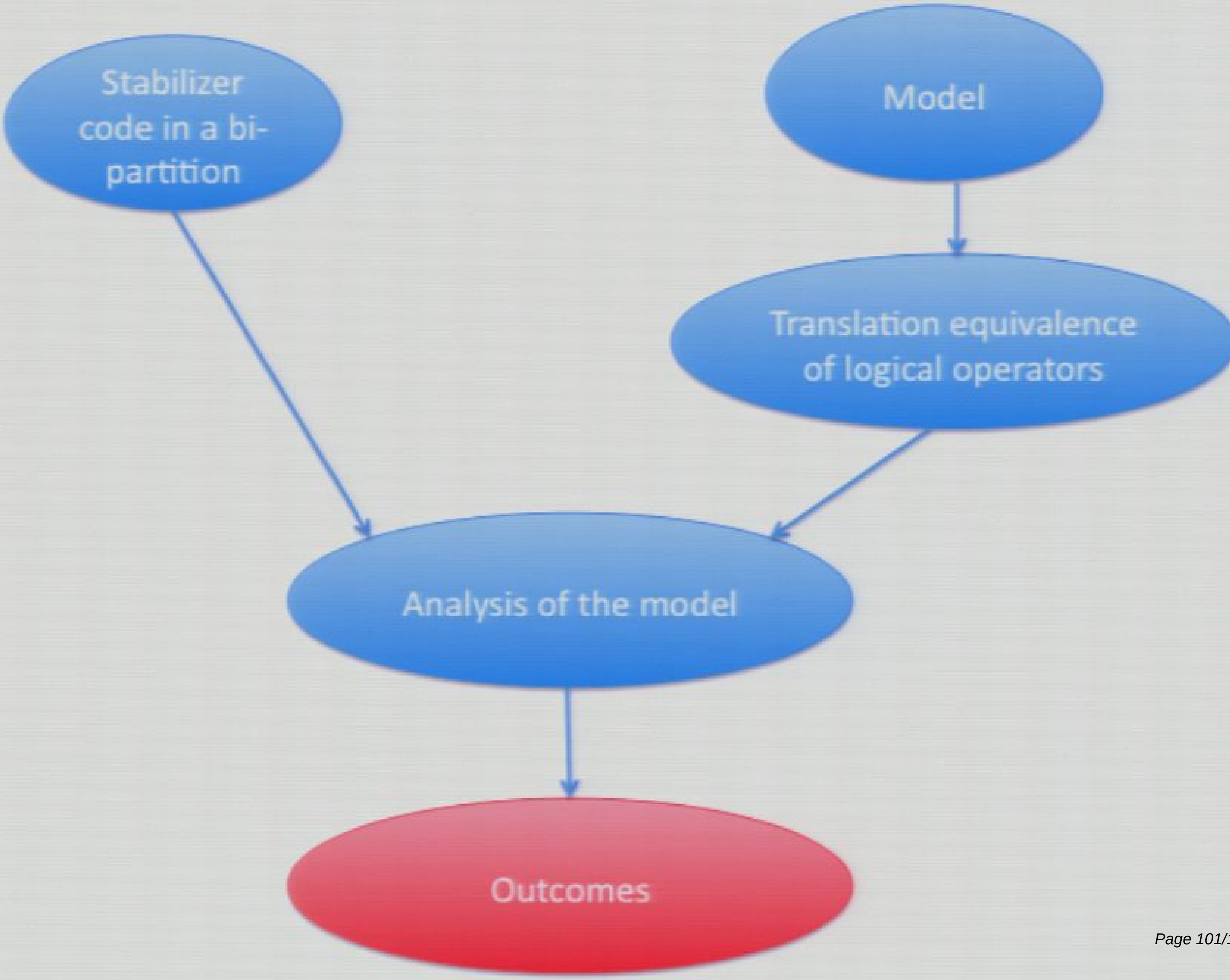
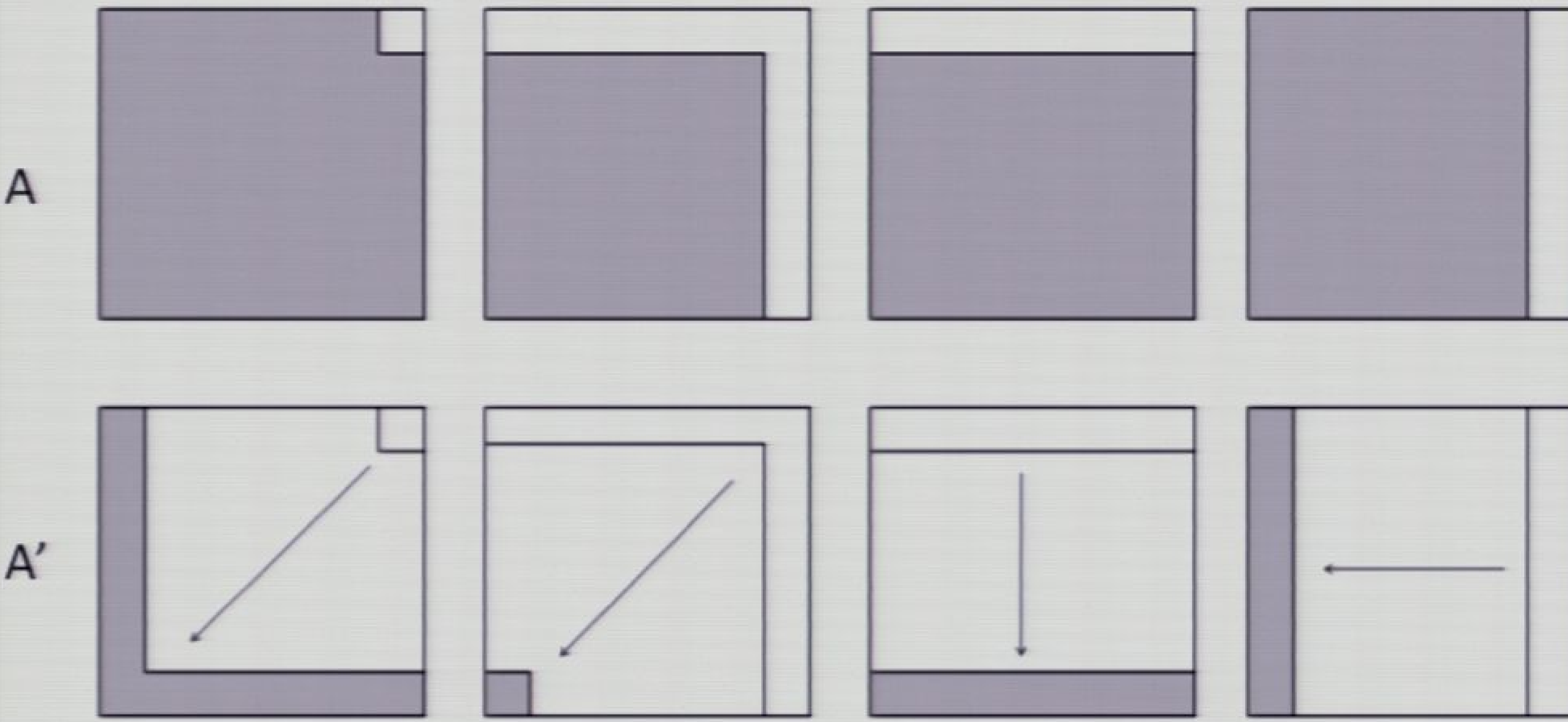


Logical operators in 2 dim

We are allowed to deform geometric shapes of logical operators continuously.

Topological deformation of logical operators



$$g_A = g_{A'}$$

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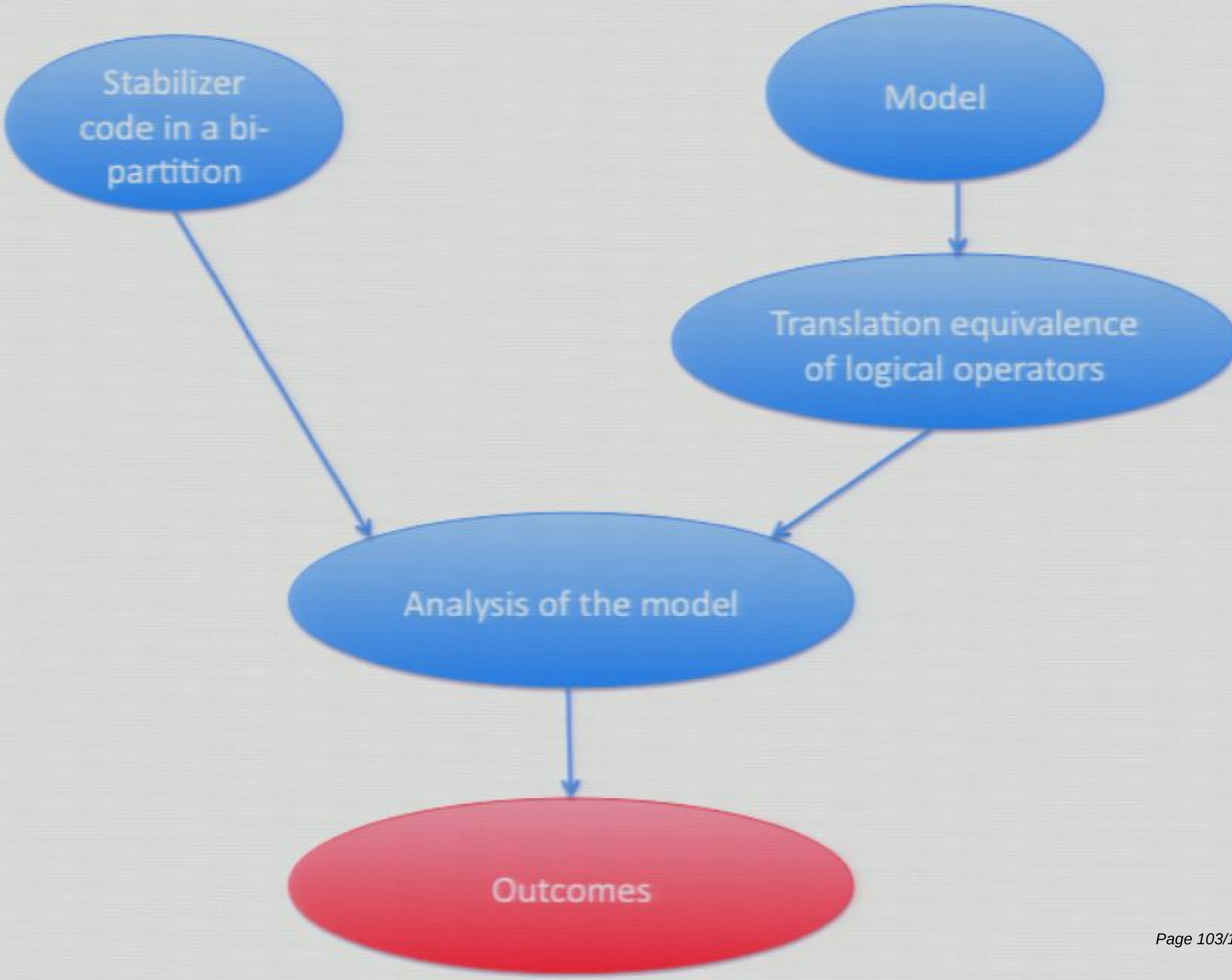

Application

- Feasibility of self-correcting memory
- Topological Quantum Phase Transitions

Application

- Feasibility of self-correcting memory
- Topological Quantum Phase Transitions

Application (self-correcting memory)

Open Question 1

Code distance = robustness of the code

Upper bound on code distance of local stabilizer codes

Application (self-correcting memory)

Open Question 1

Code distance = robustness of the code

Upper bound on code distance of local stabilizer codes

- Toric code (D-dimensional lattice)

$d = O(L^{D/2})$	D = even
$d = O(L^{(D-1)/2})$	D = odd

$d = O(1)$ (1-dim) $\quad$ $d = O(L)$ (2, 3-dim)

Application (self-correcting memory)

Open Question 1

Code distance = robustness of the code

Upper bound on code distance of local stabilizer codes

- Toric code (D-dimensional lattice)

$d = O(L^{D/2})$ D = even

$d = O(L^{(D-1)/2})$ D = odd

$d = O(1)$ (1-dim) $d = O(L)$ (2, 3-dim)

- Code distance (Terhal and Bravyi)

$d \leq O(L^{D-1})$ D=1 and 2

⇓

What is tight bound for **D>2**?

Application (self-correcting memory)

Open Question 2

Feasibility of self-correcting memory

Application (self-correcting memory)

Open Question 2

Feasibility of self-correcting memory

- **Self-Correcting Memory** : corrects errors by itself in the presence of large energy barrier

Application (self-correcting memory)

Open Question 2

Feasibility of self-correcting memory

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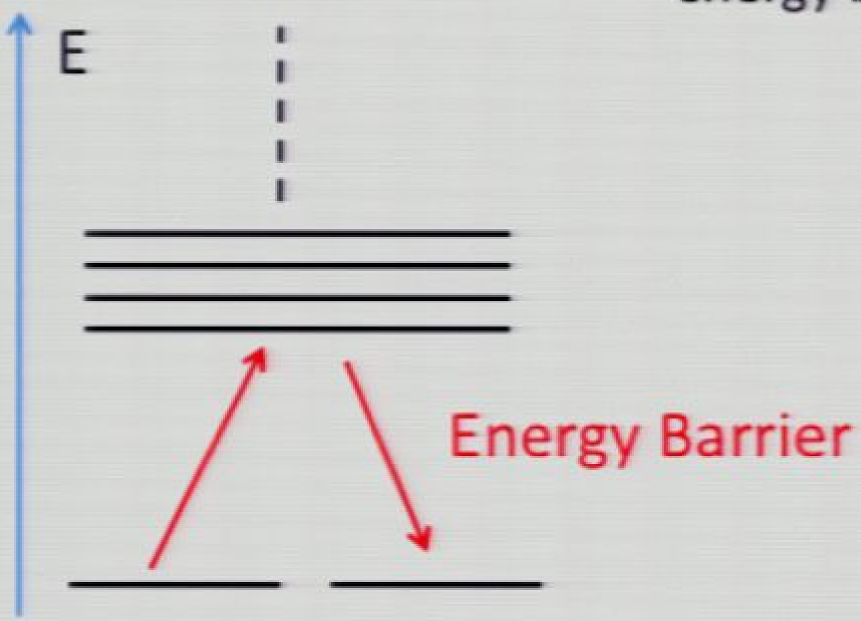
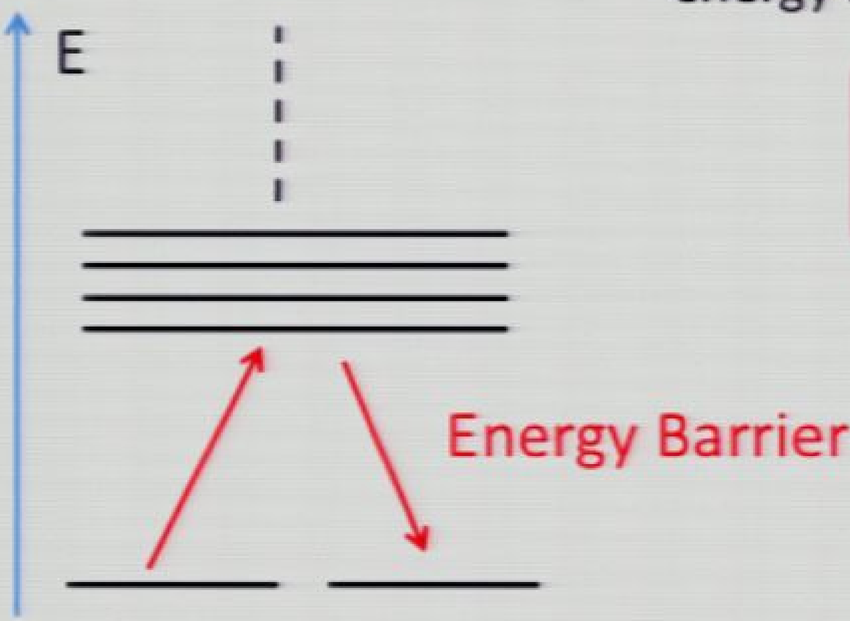


Open Question 2

Feasibility of self-correcting memory

- **Self-Correcting Memory** : corrects errors by itself in the presence of large energy barrier



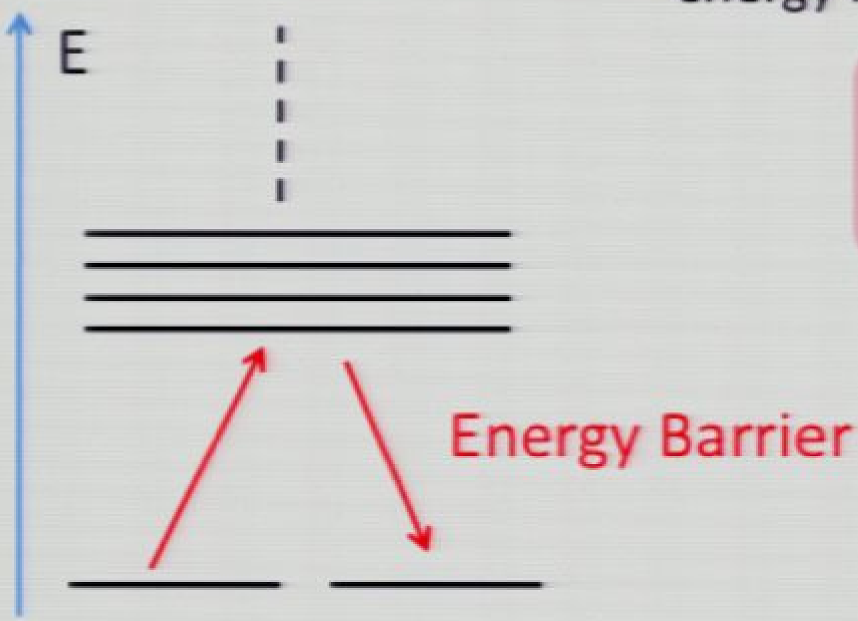
- Many proposals in 4-dim (4-dim Toric code)
- No 2D self-correcting memory (stabilizer)

Application (self-correcting memory)

Open Question 2

Feasibility of self-correcting memory

- **Self-Correcting Memory** : corrects errors by itself in the presence of large energy barrier



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⇓

3D self-correcting memory?

Open Question 2

Feasibility of self-correcting memory

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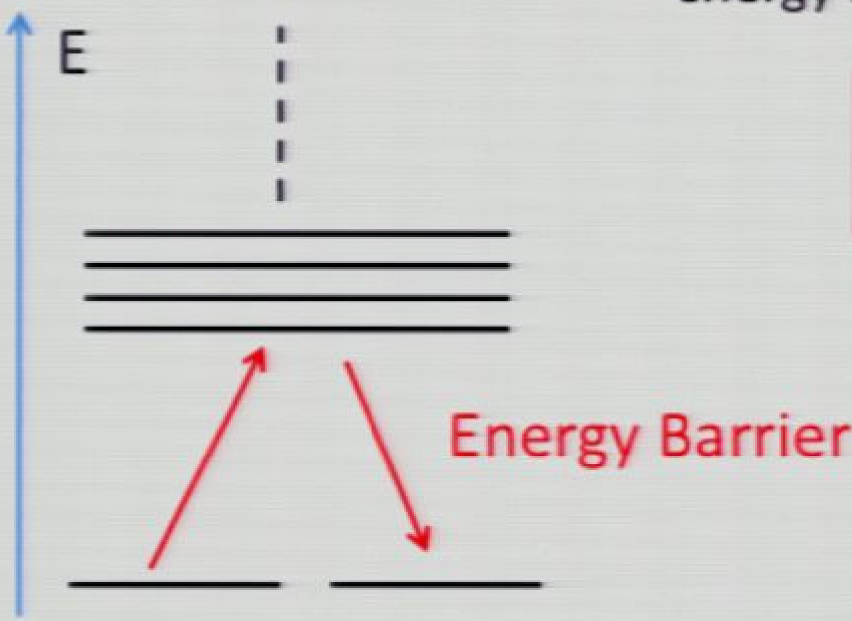

- Many proposals in 4-dim (4-dim Toric code)
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3D self-correcting memory?

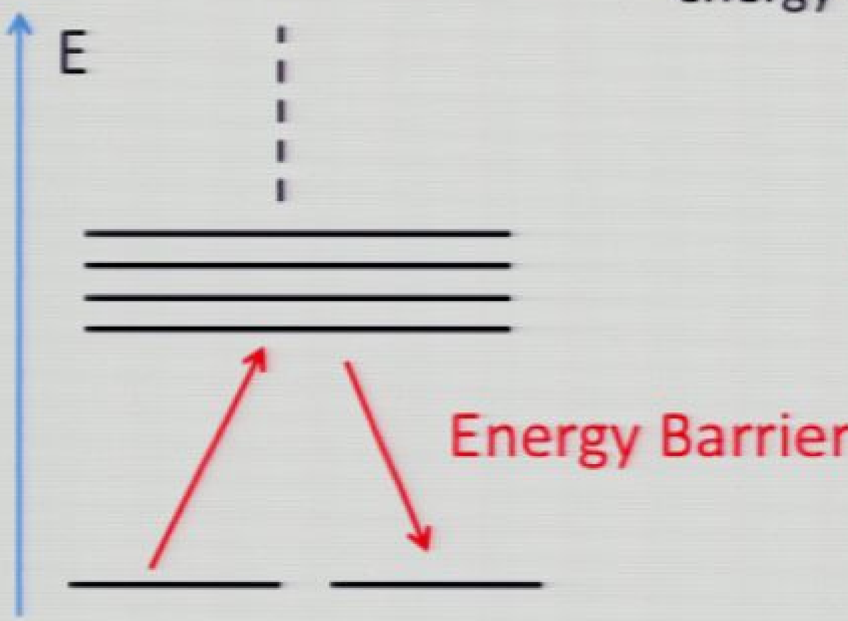
Topological order at finite temperature

Application (self-correcting memory)

Open Question 2

Feasibility of self-correcting memory

- Self-Correcting Memory : corrects errors by itself in the presence of large energy barrier



- Many proposals in 4-dim (4-dim Toric code)
- No 2D self-correcting memory (stabilizer)

3D self-correcting memory?

Topological order at finite temperature

→ Topological Classical Phase Transition

Application (self-correcting memory)

Open Question 2

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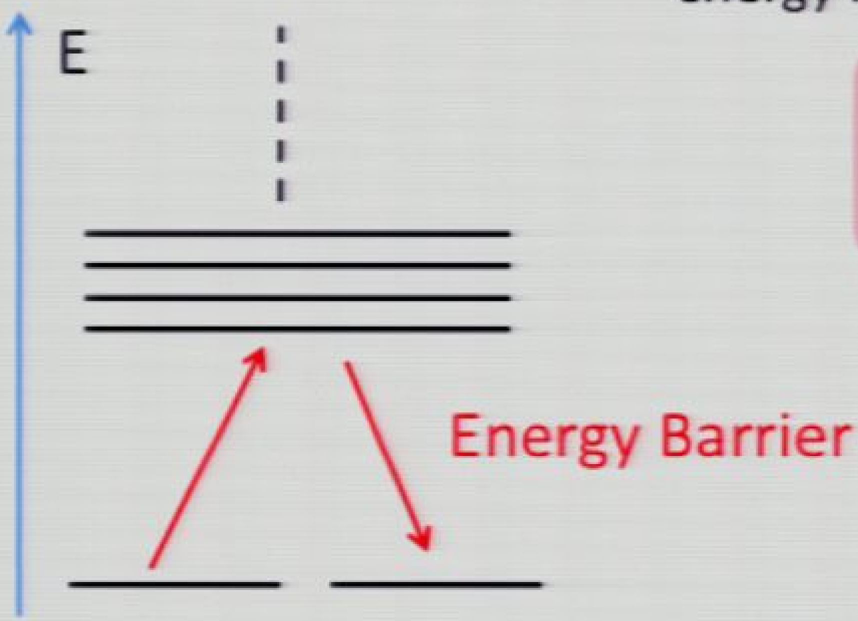

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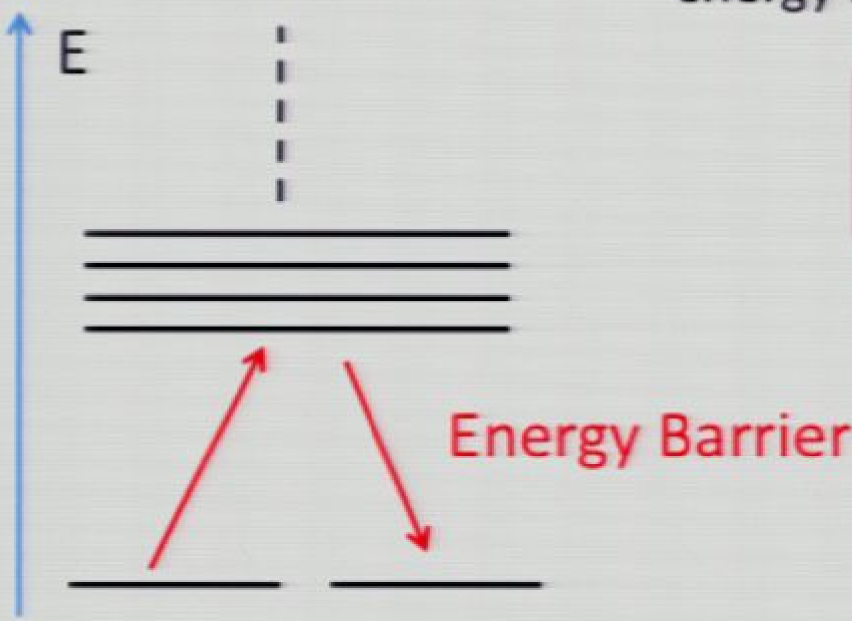

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Partial answers for two open questions

Application

- Feasibility of self-correcting memory
- Topological Quantum Phase Transitions

Applications (QPT)

In realistic physical systems

$$H = H(\epsilon) \qquad \epsilon \text{ : external parameters}$$

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→ Geometry of logical operators

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Logical operator characterizes global symmetries of the system Hamiltonian.

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<u>Assume</u>

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where H_1 and H_2 are <u>different</u> STSs.

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Expected phase diagram

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since local unitary transformations do not change the geometric shapes of logical operators. (no QPT or, 1st order)

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This model is dual to 2 + 1 dim Ising model. We have 2^{nd} order TQPT

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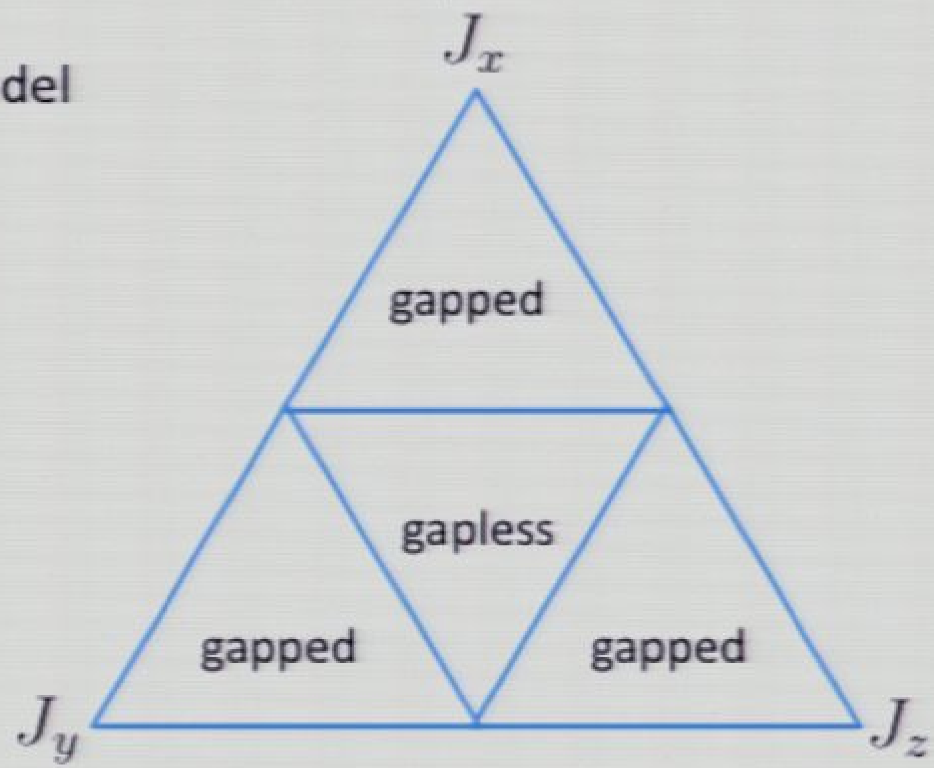

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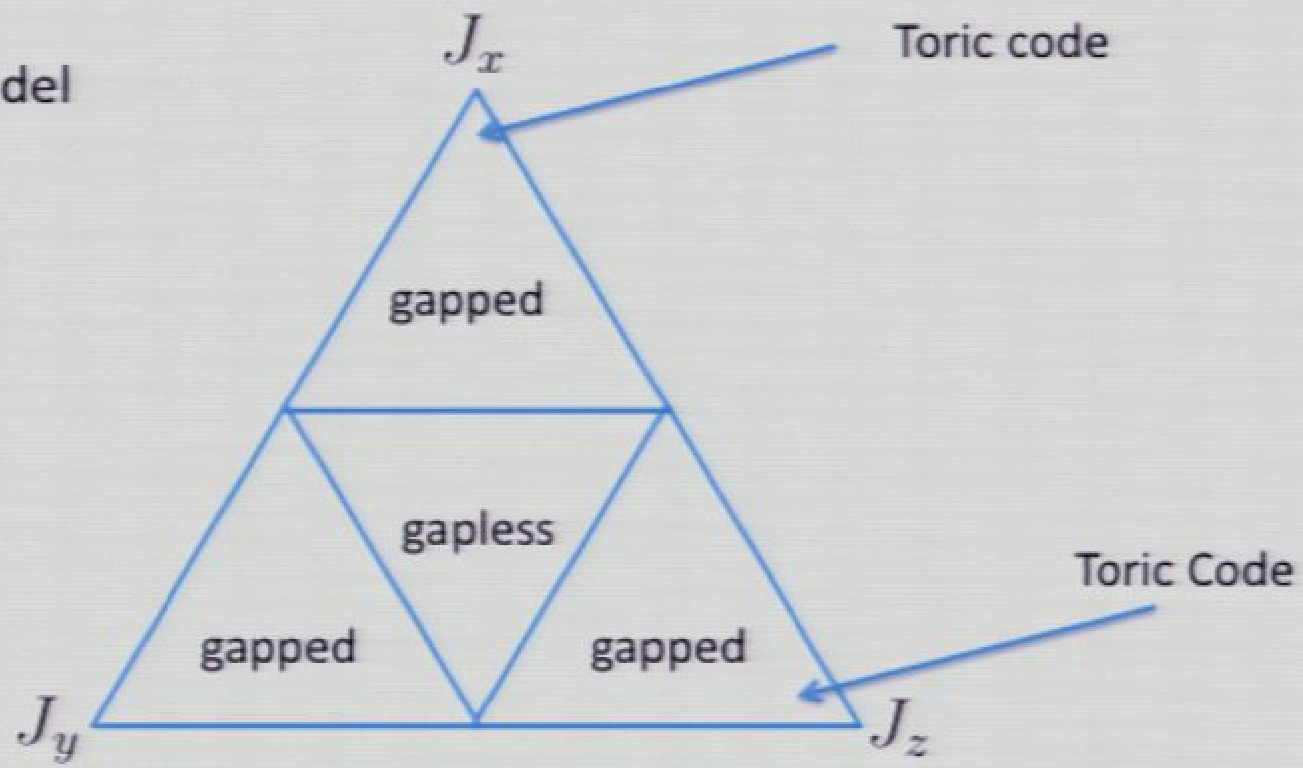

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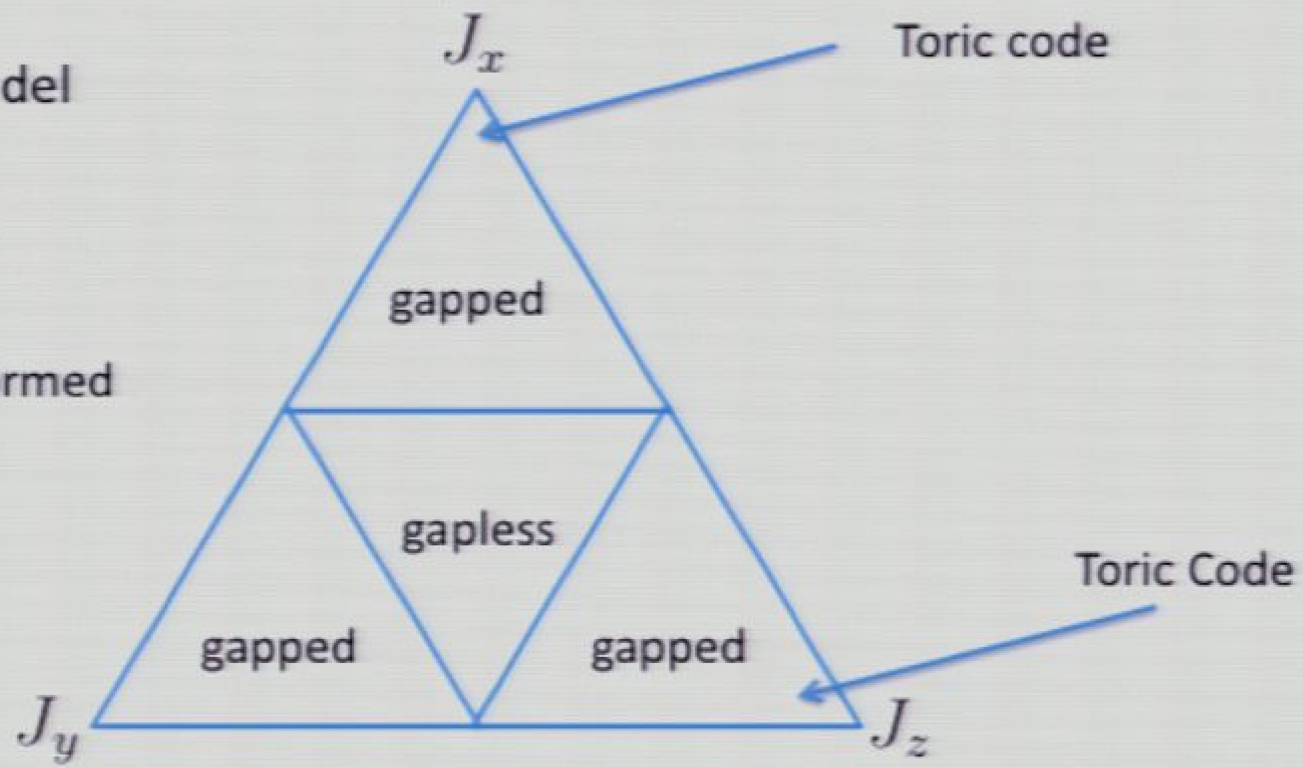

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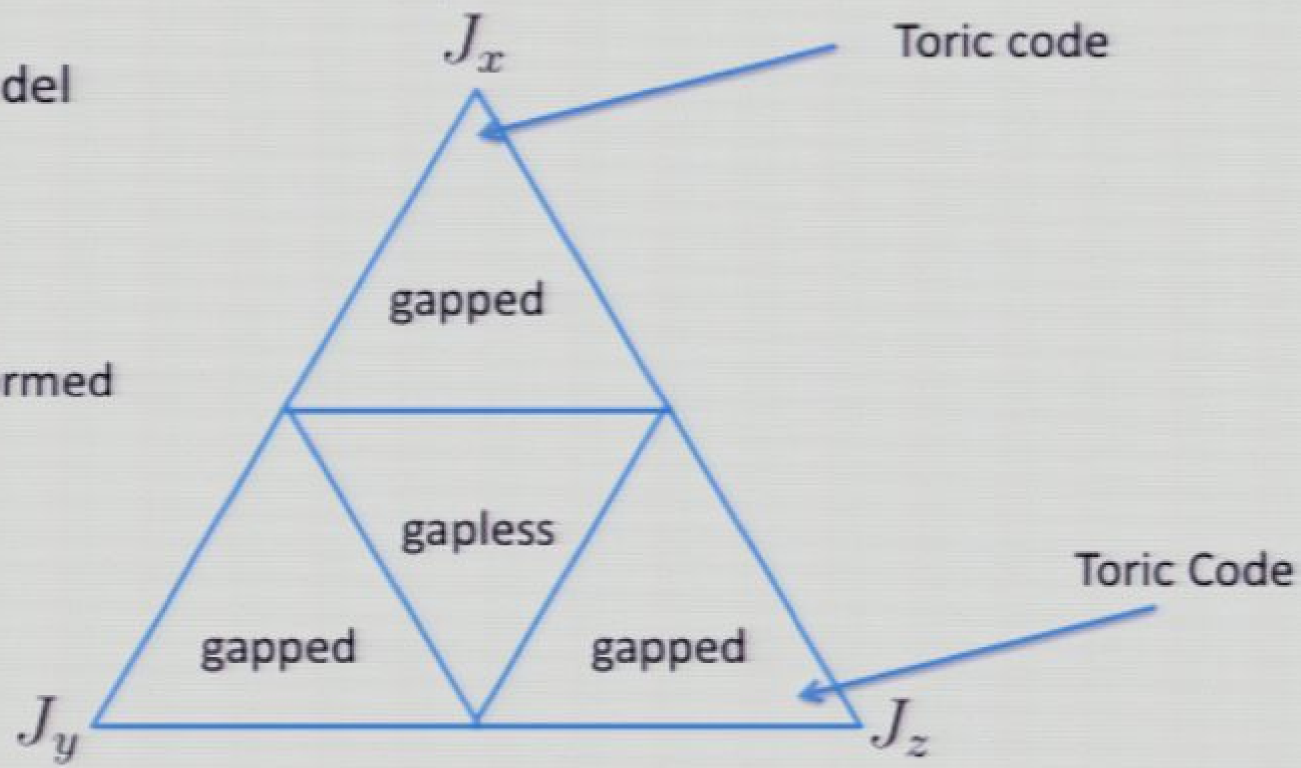

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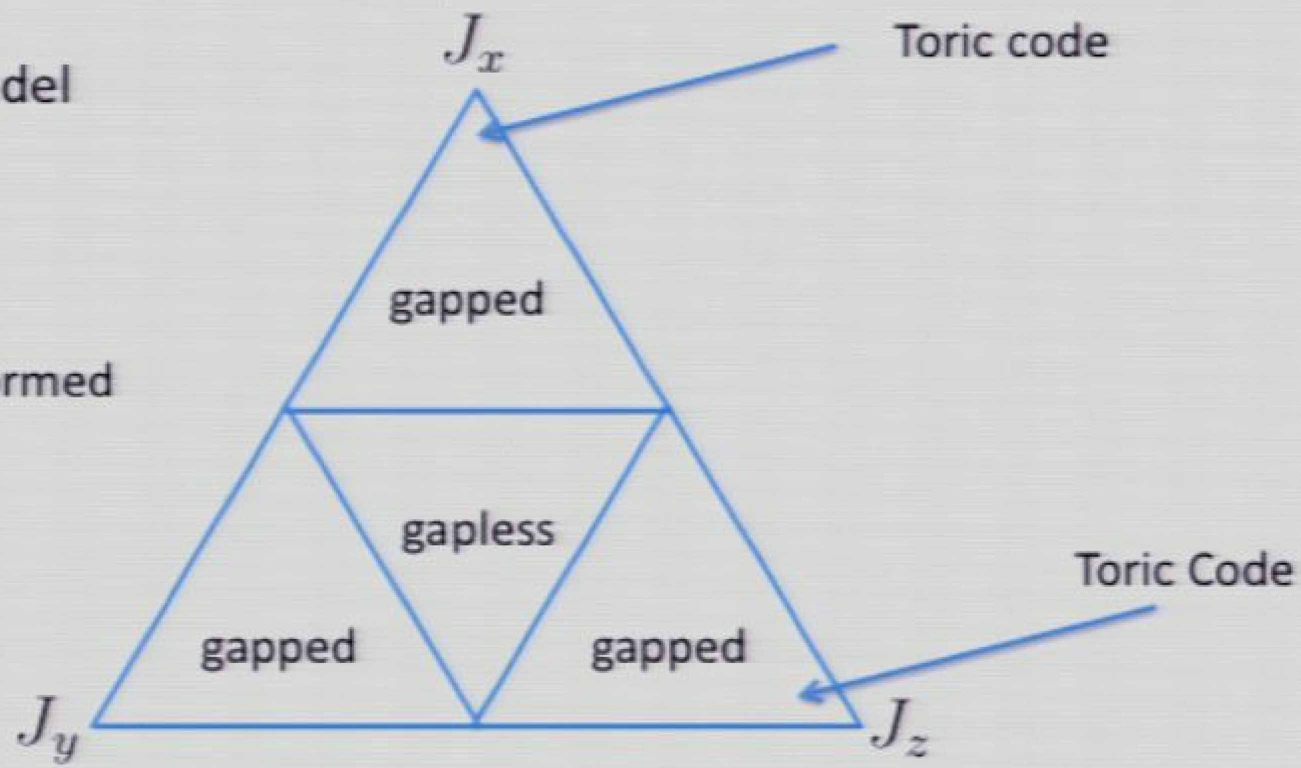

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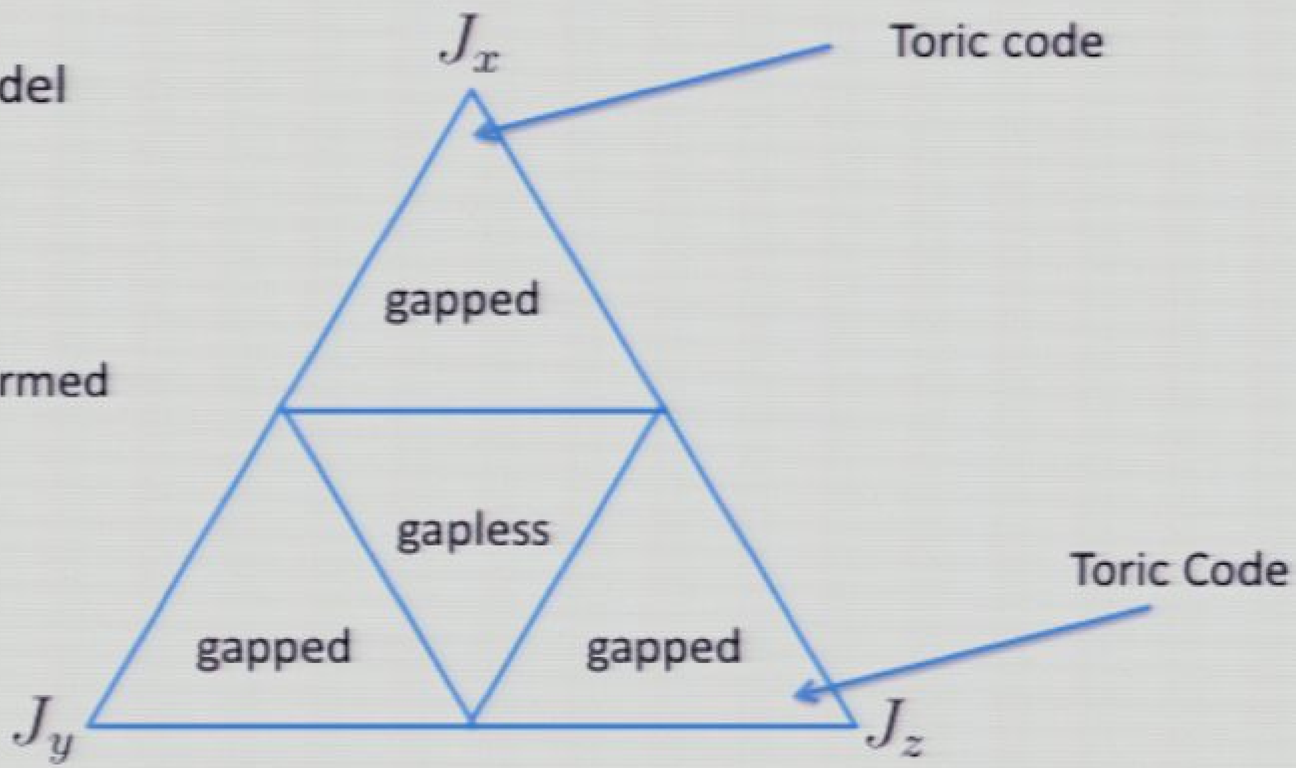

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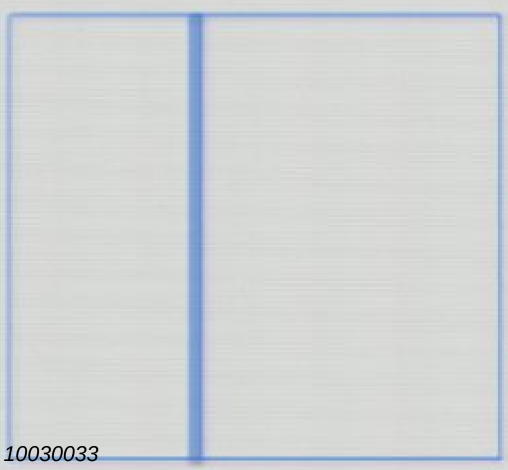

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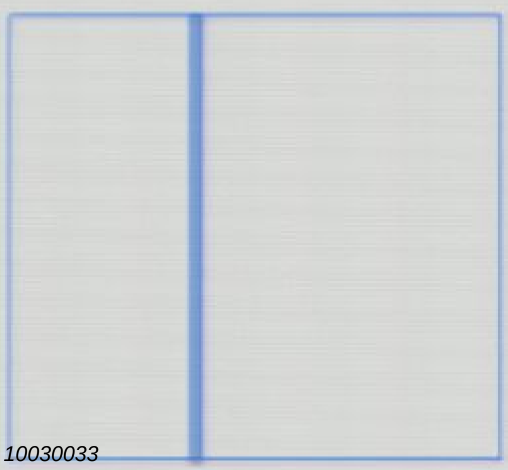

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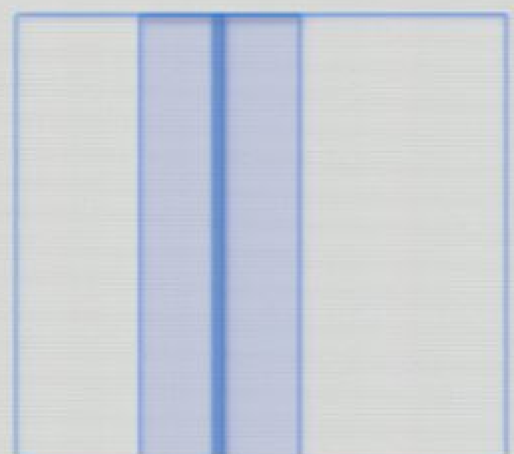
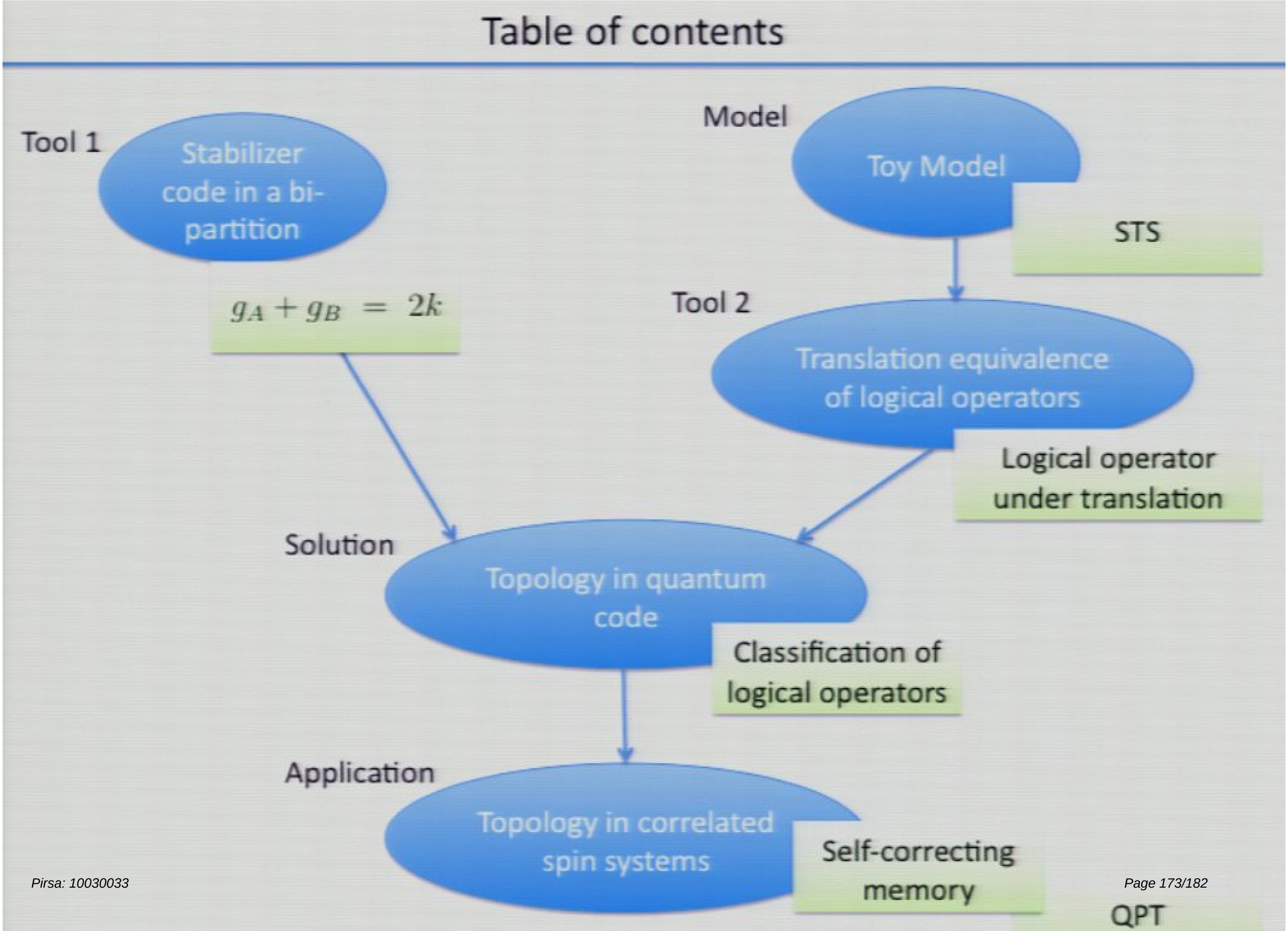


If t = O(1)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Summary and outlook

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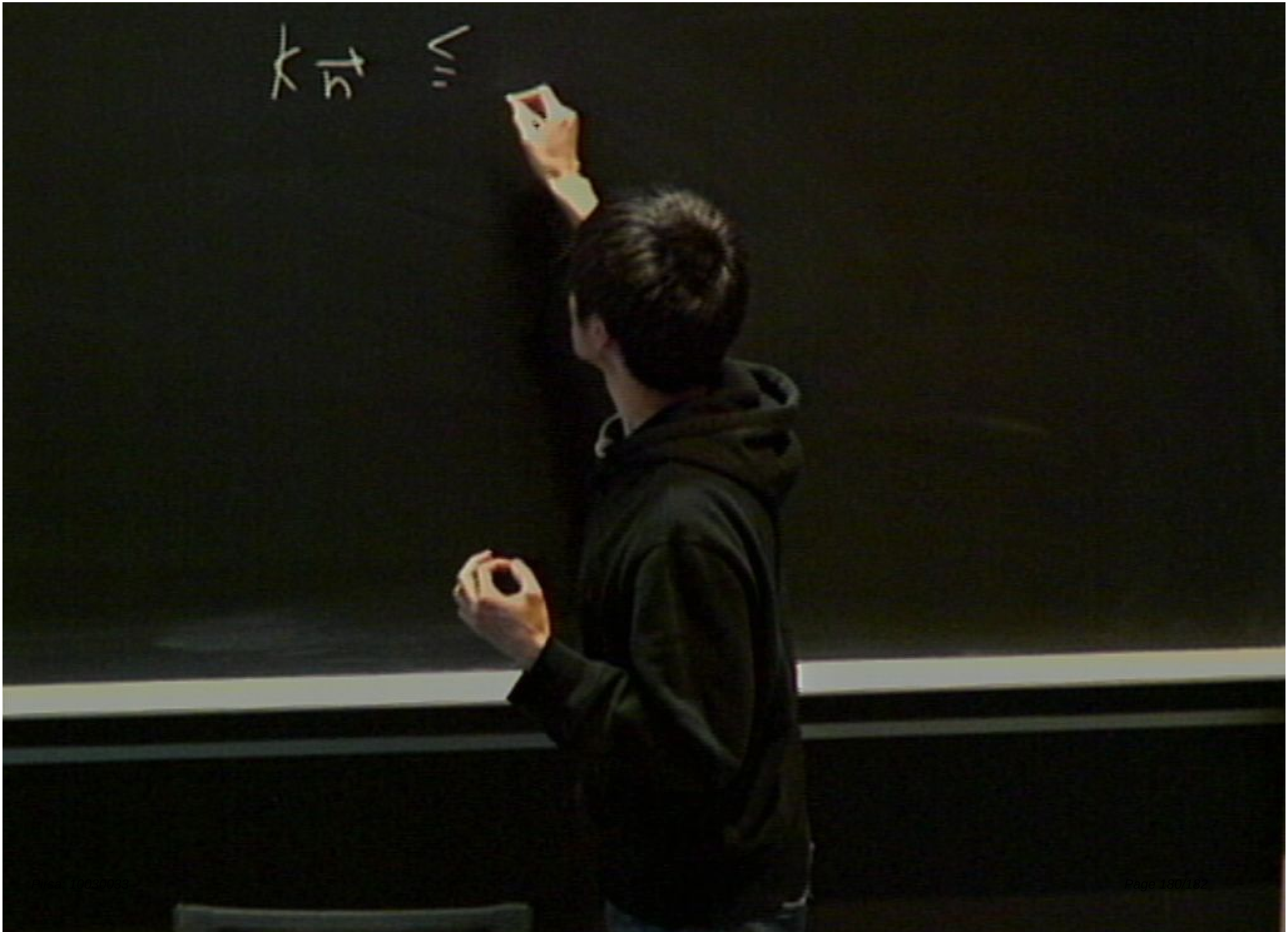
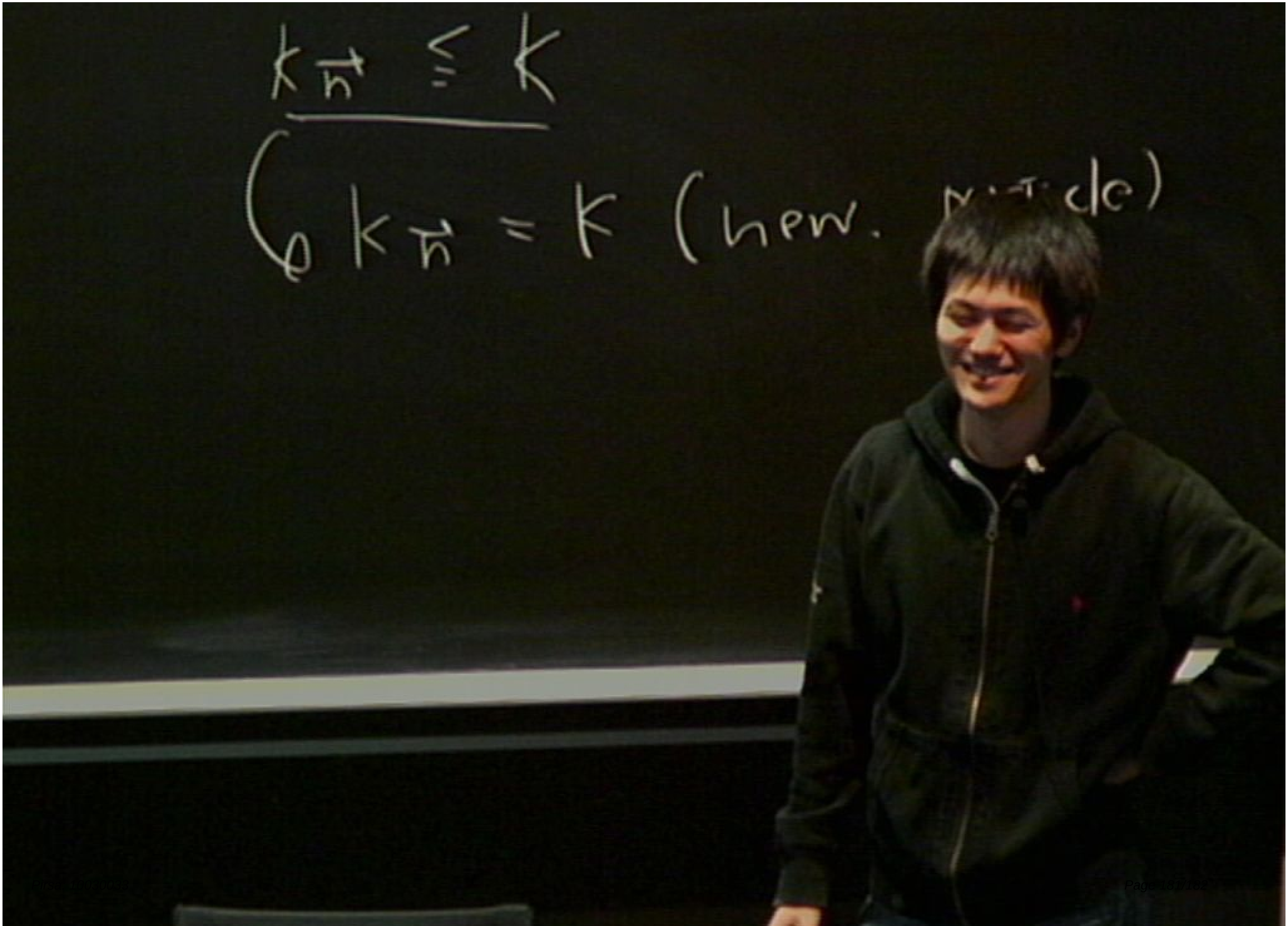
Topological quantum phase transition

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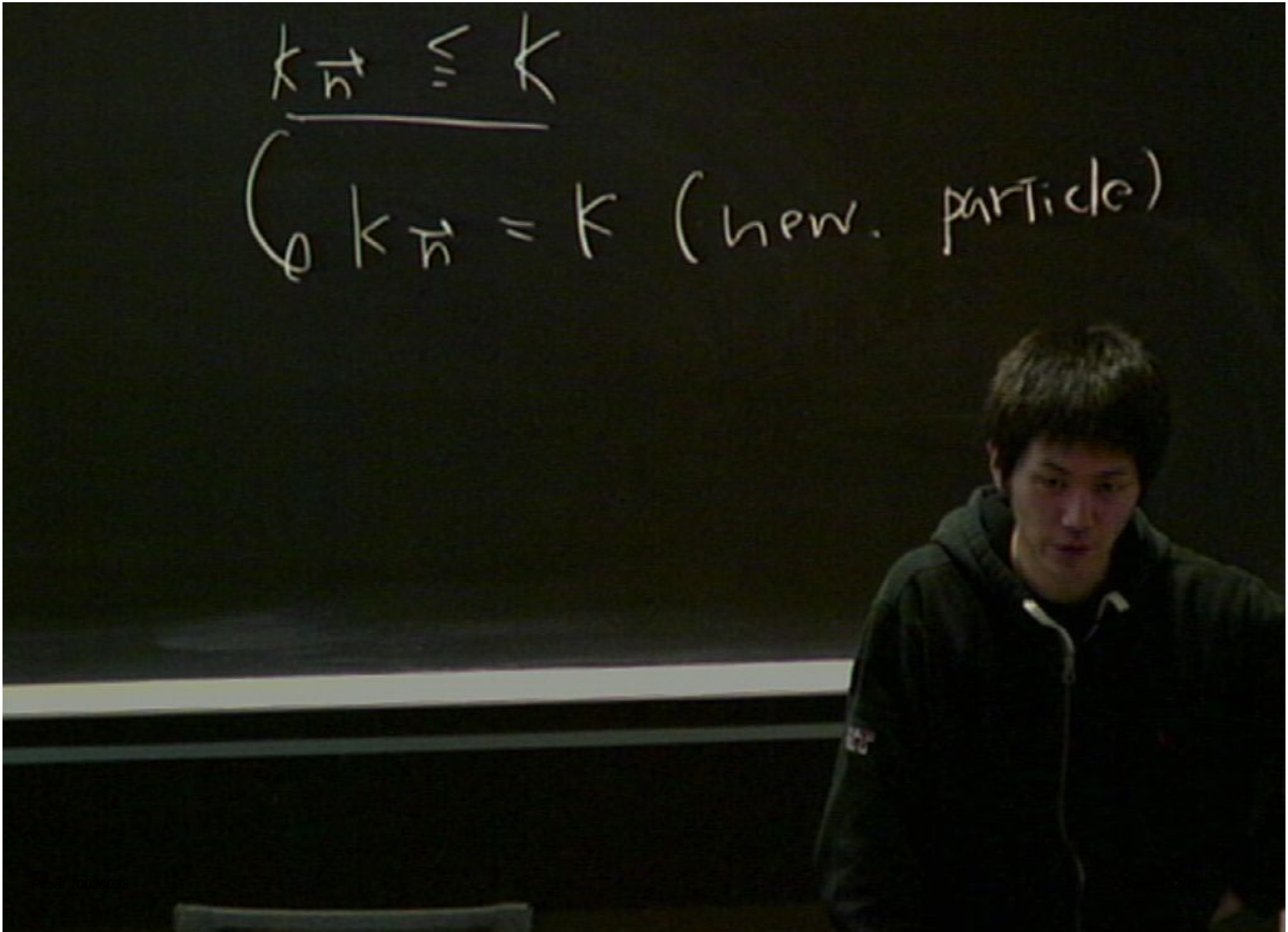
$k_{\vec{n}} \leq K$
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