Title: Entanglement entropy in the O(N) model

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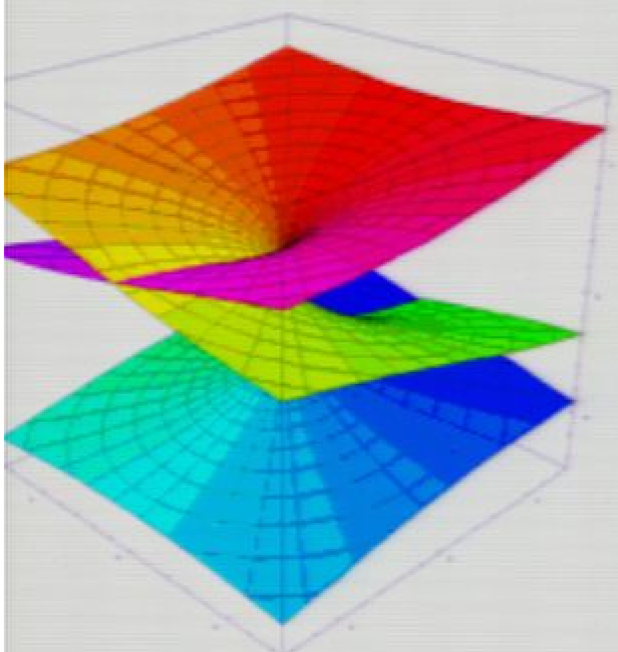
Abstract: In recent years the characterization of many-body ground states via the entanglement of their wave-function has attracted a lot of attention. One useful measure of entanglement is provided by the entanglement entropy S.
The interest in this quantity has been sparked, in part, by the result that at one dimensional quantum critical points (QCPs) S scales logarithmically with the subsystem size with a universal coefficient related to the central charge of the conformal field theory describing the QCP. On the other hand, in spatial dimension d > 1 the leading contribution to the entanglement entropy scales as the area of the boundary of the subsystem. The coefficient of this "area law" is non-universal. However, in the neighbourhood of a QCP, S is believed to possess subleading universal corrections. In this talk, I will present the first field-theoretic study of entanglement entropy in dimension d > 1 at a stable interacting QCP - the quantum O(N) model. Our results confirm the presence of universal corrections to the entanglement entropy and exhibit a number of surprises such as different epsilon -> 0 limits of the Wilson-Fisher and Gaussian fixed points, violation of large N counting and subtle dependence on replica index.

Entanglement entropy in the O(N) model



Max Metlitski

Carlos A. Fuertes

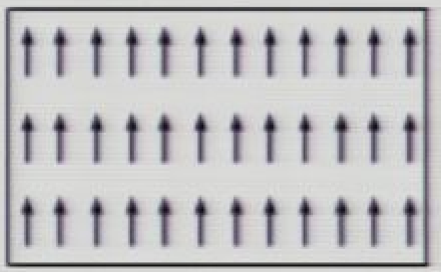
Subir Sachdev

Phys. Rev. B **80**, 115122 (2009)

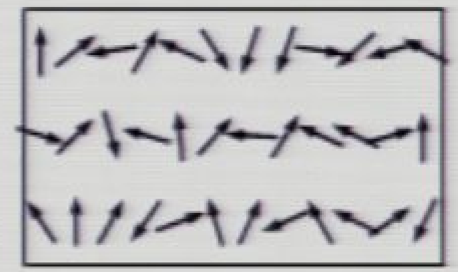
Perimeter Institute, March 3, 2010

The venerable Landau-Ginzburg-Wilson paradigm

- Classical theory of phase transitions



ordered



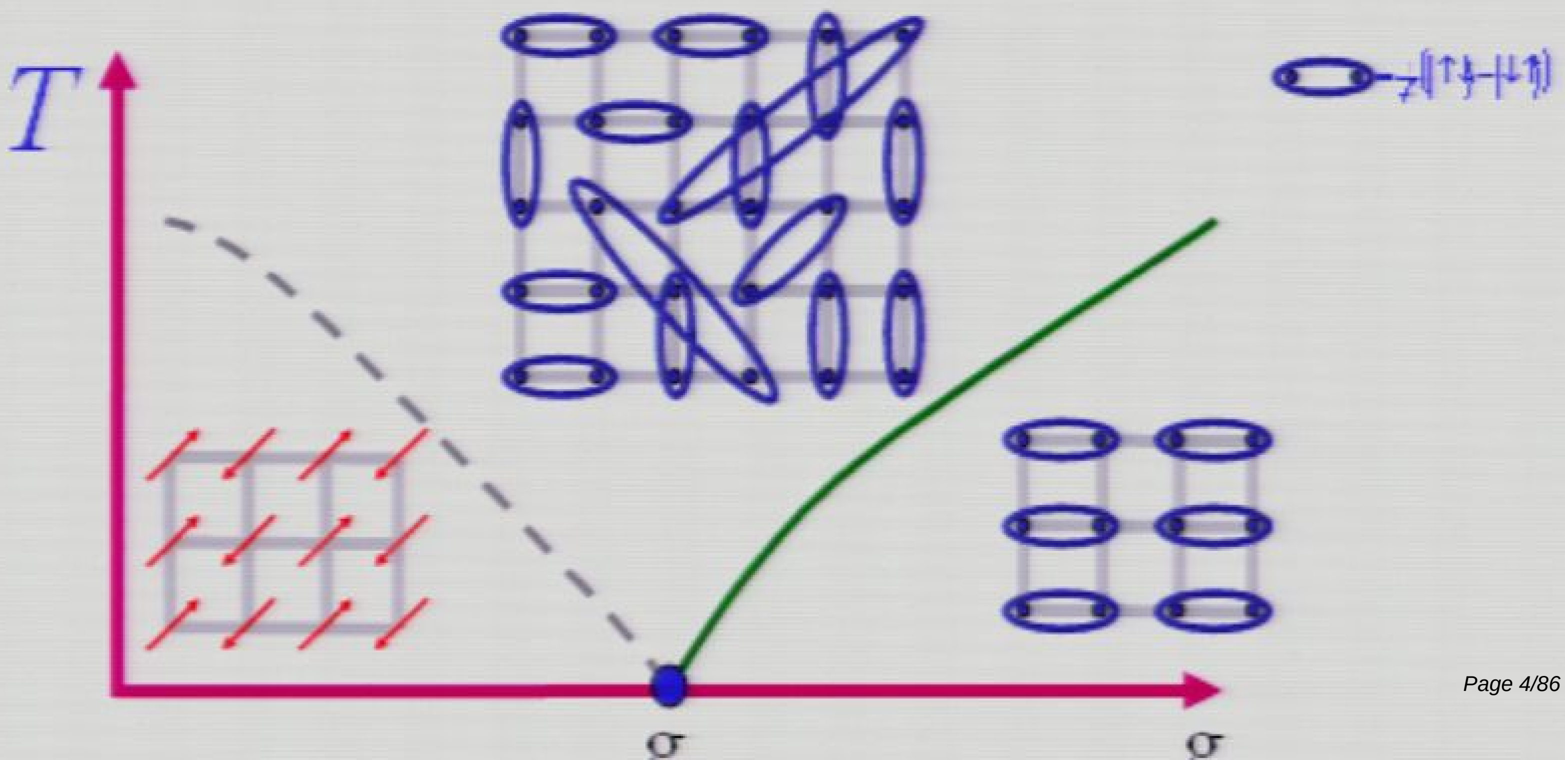
disordered

- spontaneous symmetry breaking
- local order parameter

→ universality class

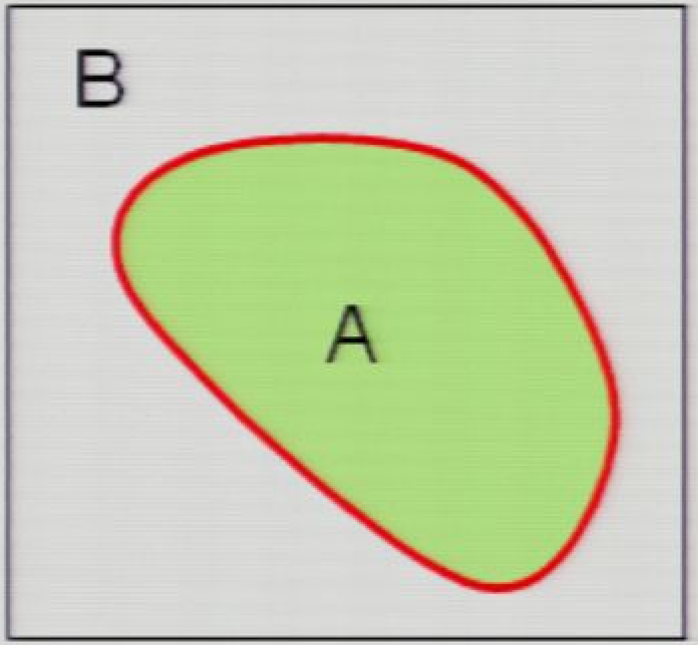
Quantum criticality

- Landau-Ginzburg paradigm describes some, but not all quantum phase transitions
- Quantum critical points realize some of the most non-classical states of matter
  - low energy excitations are neither particles nor waves
  - ground state wave-function is highly entangled



Entanglement entropy

- Goal: to characterize quantum critical points via entanglement properties of the many-body wave-function



$$\rho = |\psi\rangle\langle\psi|$$

$$\rho_A = tr_B \rho$$

$$\rho_A = \sum_i p_i |i\rangle\langle i|$$ - mixed state

- Distribution of p_i - entanglement spectrum

- Entanglement entropy

$$S_A = -tr \rho_A \log \rho_A \qquad S_A = S_B$$

- inherently quantum mechanical, non-local observable

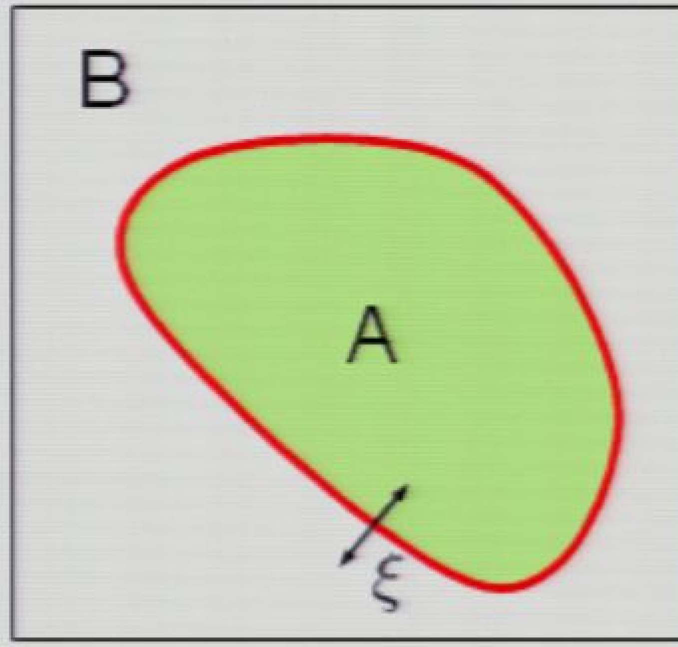
Entanglement entropy away from QCP

- Local Hamiltonian
- finite correlation length ξ

$$S \propto |\partial A|$$

$|\partial A|$ - area (length) of the boundary

- Area law: entanglement is local to the boundary



- For a generic state in the Hilbert space

$$S \sim \text{volume}(A)$$

Ground states of local Hamiltonians are very special!
(form set of measure zero in the Hilbert space)

M. Srednicki (1993)

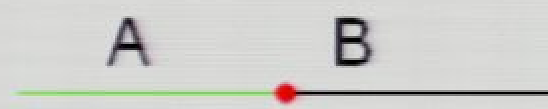
The effect of dimensionality; relation to DMRG

- $$\rho_A = \sum_i p_i |i\rangle\langle i|$$

How many states do we need to keep to "saturate" ρ_A ?

$$N_s \sim e^S \sim e^{C|\partial A|}$$

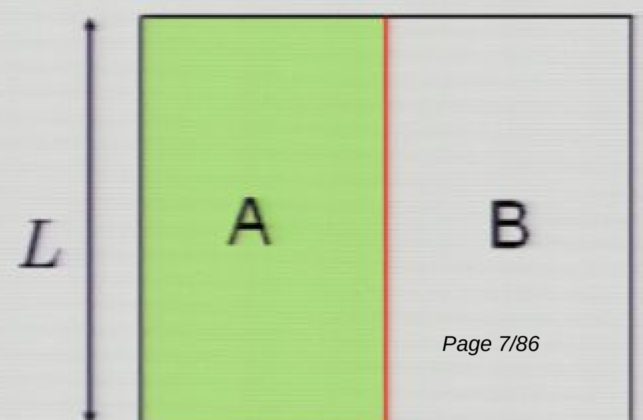
- Dimension $d = 1$



$|\partial A|$ - number of boundary points, N_s - non-extensive

- Higher dimensions

$$N_s \sim e^{CL^{d-1}}$$ ☹



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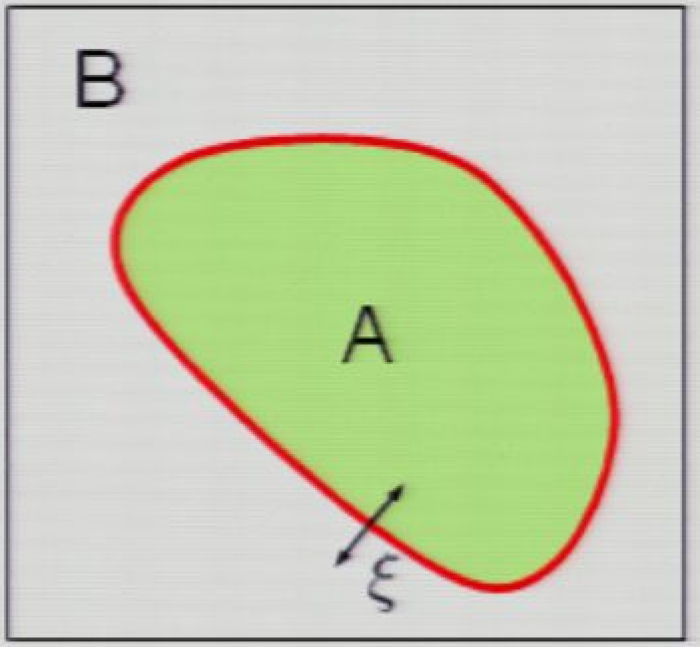

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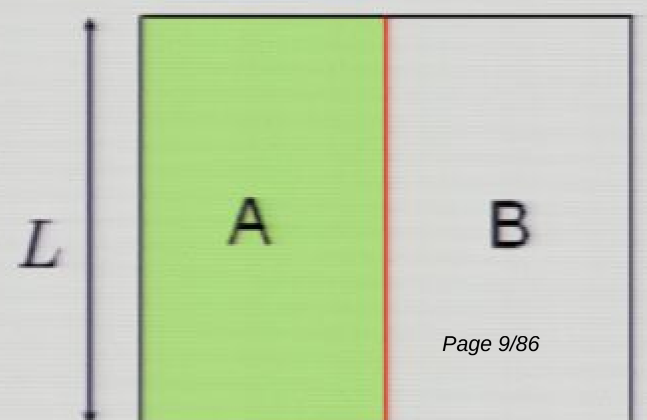
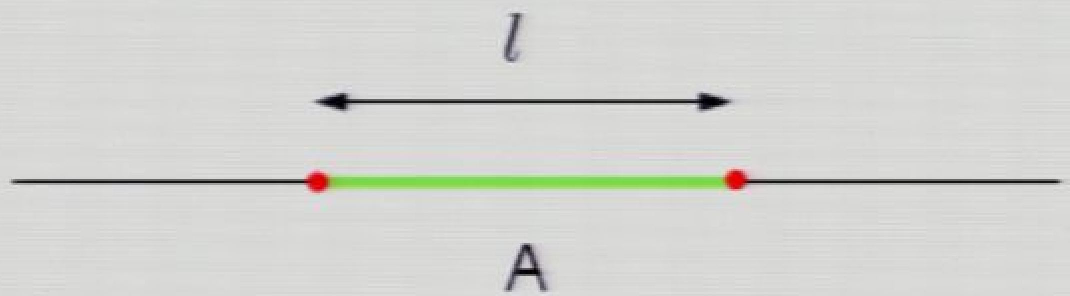


The entanglement entropy at 1d QCPs

- How does the entanglement entropy behave near a QCP?
- Problem completely solved for QCPs in d = 1 with z = 1
  - system described by a conformal field theory (CFT)

$$S = \frac{c}{3} \log l/a$$



c - central charge of the CFT

a - short distance cut-off

S - universal!

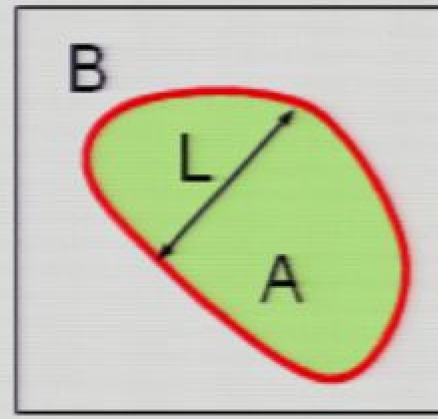
- Perturb the CFT slightly away from criticality

$$S = \frac{c}{3} \log \xi/a$$

C. Holzhey, F. Larsen and F. Wilczek (1994).
P. Calabrese and J. L. Cardy (2004).

Higher dimensional QCPs

- For QCPs described by local field theories



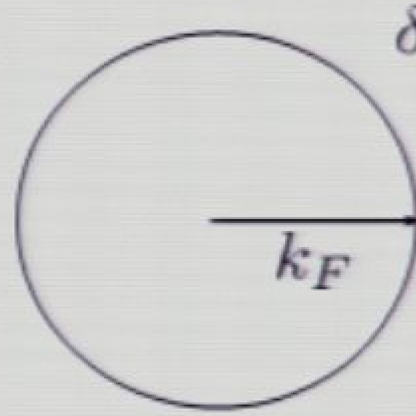
$$S = C\frac{|\partial A|}{a^{d-1}}$$ - area law

- The coefficient C - non-universal

- Violations of the area law: systems with a Fermi surface

$$S \propto (k_F L)^{d-1}\log(k_F L)$$

$$\delta\theta \sim \frac{1}{k_F L}$$



$$N_m \sim (k_F L)^{d-1}$$

One-dimensional modes

$$\omega = v_F|k - k_F|$$

$$S_{1d} = \frac{c}{3}\log l/a, \quad c = \frac{1}{2}$$

$$S = N_m S_{1d}$$

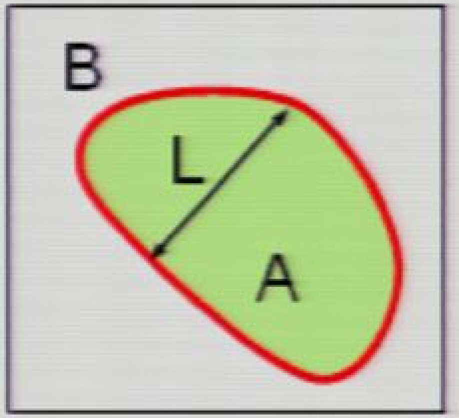
Beyond the leading term

- Any non-universal contributions to S must come from the boundary

$$S = g_{d-1}\frac{L^{d-1}}{a^{d-1}} + g_{d-2}\frac{L^{d-2}}{a^{d-2}} + \ldots + g_0 \log(L/a) + S_0(L/\xi)$$

$g_k L^k$ - integrals of local geometric quantities over the boundary

$$g_{d-1}L^{d-1} = C\int_{\partial A} 1 = C|\partial A|$$



$g_k,\ k > 0$ - non-universal

g_0 - universal

S_0 - universal function, up to additive contributions $\sim g_0[\partial A]$

Beyond the leading term

- Any non-universal contributions to S must come from the boundary

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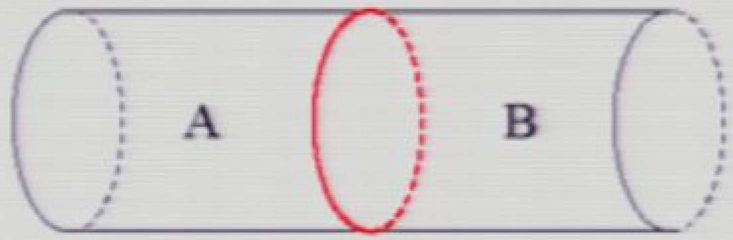
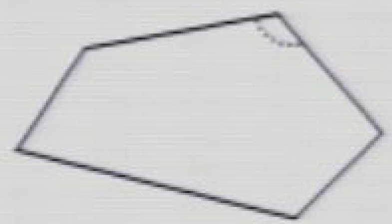
- Straight closed boundary:



$$S = C\frac{|\partial A|}{a^{d-1}} + S_0(L/\xi)$$

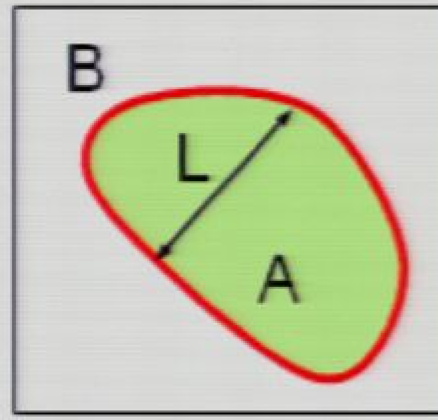
- also believed to be true for general smooth closed boundaries in d = 2

- $g_0 \neq 0$ for



- boundaries with corners/endpoints in d = 2,
- boundaries with intrinsic/extrinsic curvature in d = 3

Universal geometric correction



- Smooth, closed boundary 1 < d < 3

$$S = C\frac{|\partial A|}{a^{d-1}} + S_0(L/\xi)$$

- At QCP

$$S = C\frac{|\partial A|}{a^{d-1}} + \gamma$$ - universal geometric constant

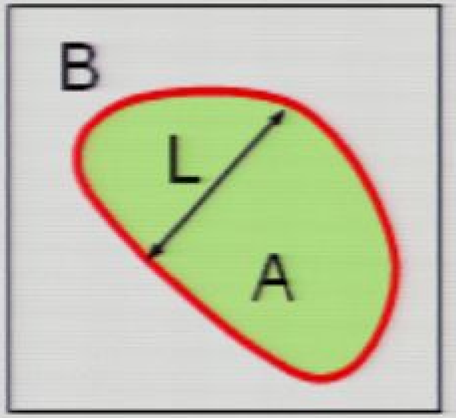
Akin to Privman-Fisher correction to free-energy: $F = V f_\infty + U_0$

Only corrections to scaling, $L^{-\omega}$

- γ captures the non-local entanglement present at the QCP

Relation to massive topological phases

$$S = C\frac{L}{a} + \gamma \qquad d = 2$$



- $\gamma = -\log \mathcal{D}$ - universal constant

$\mathcal{D} = 2$ - Kitaev model

$\mathcal{D} = \sqrt{q}$ - FQHE $\nu = \frac{1}{q}$

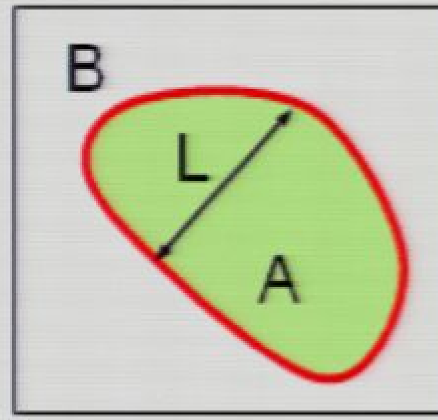
- γ - topological invariant (unlike at QCP)

- $\gamma = 0$ for non-topological gapped phases

A. Kitaev and J. Preskill (2006)

M. Levin and X.-G. Wen (2006)

Away from QCP



- Smooth, closed boundary 1 < d < 3

$$S = C\frac{|\partial A|}{a^{d-1}} + S_0(L/\xi)$$

- Away from QCP $(L \gg \xi)$

$$S = C(t)\frac{|\partial A|}{a^{d-1}} + r\frac{|\partial A|}{\xi^{d-1}}, \quad \xi \sim t^{-\nu} \qquad t = g - g_c$$

$C(t)$: $C + C't$ ξ^{d-1}: $t^{\nu(d-1)}$

$r -$ universal (up to definition of ξ)

No subleading constant term for ordinary (non-topological) gapped phases

Explicit calculations to date

- Free theories (straight boundaries)

$$L = \frac{1}{2}(\partial_\mu \phi)^2 + m^2\phi^2 \qquad\qquad L = \bar{\psi}(i\partial_\mu \gamma_\mu - m)\psi$$

P. Calabrese and J. L. Cardy (2004), Fursaev (2006)
H. Casini and M. Huerta (2005), (2007), H. Casini, M. Huerta and L. Leitao (2009)

- Special multi-critical QCPs in d = 2 with z = 2 (e.g. RK point)

  - correlation functions reduce to those of a classical d = 2 CFT

E. Fradkin and J. E. Moore (2006),
B. Hsu, M. Mulligan, E. Fradkin, and Eun-Ah Kim(2009)

- Holographic calculations (AdS/CFT)

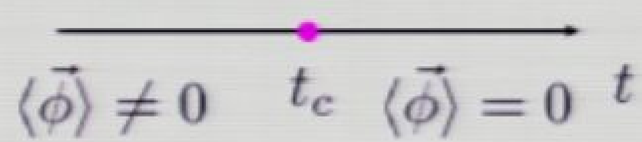
S. Ryu and T. Takayanagi (2006)

The quantum O(N) model

- Simplest interacting QCP in d = 2 (z = 1)

$$S = \int d^d x d\tau \left(\frac{1}{2}(\partial_\tau \vec{\phi})^2 + \frac{c^2}{2}(\nabla \vec{\phi})^2 + \frac{t}{2}\vec{\phi}^2 + \frac{u}{4}(\vec{\phi}^2)^2 \right)$$

$\vec{\phi}$ - N component order parameter



- Microscopic Hamiltonians:

N = 1 – transverse field Ising model
N = 2 – Bose-Hubbard model at integer filling
N = 3 – bilayer S = ½ antiferromagnet on a square lattice

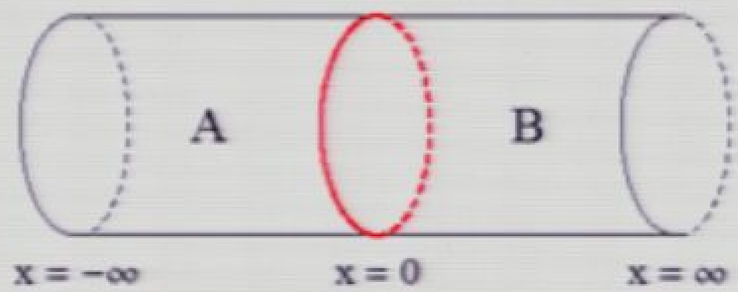
- Critical properties accessible using expansions in

$$\epsilon = 3 - d, \qquad 1/N$$

Entanglement entropy in the O(N) model

- Universal geometric correction at QCP

$$S = C\frac{L^{d-1}}{a^{d-1}} + \gamma$$



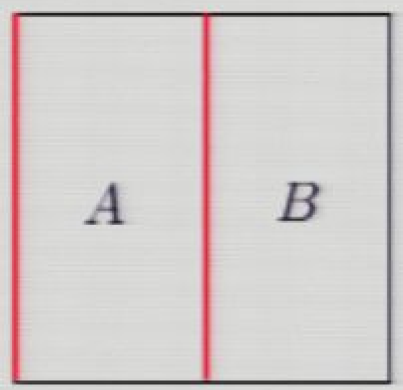
Boundary – d-1 dimensional torus of side L (circle in d = 2)

Twisted boundary conditions: $\phi(x + \hat{n}_i L) = e^{i\varphi}\phi(x)$

- Numerical evidence for finite γ at QCP in the transverse field Ising model (N = 1)

  - Tree tensor networks

  - Half-torus geometry



L. Tagliacozzo, G. Evenbly, G. Vidal (2009)

Entanglement entropy in the O(N) model

- **Universal correlation length correction** $(L \gg \xi)$

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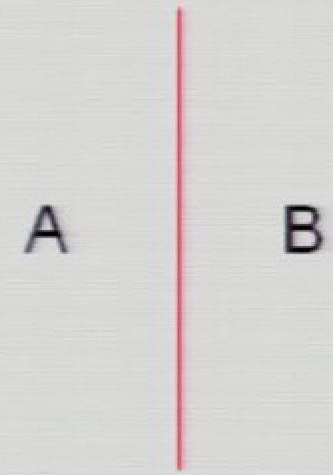
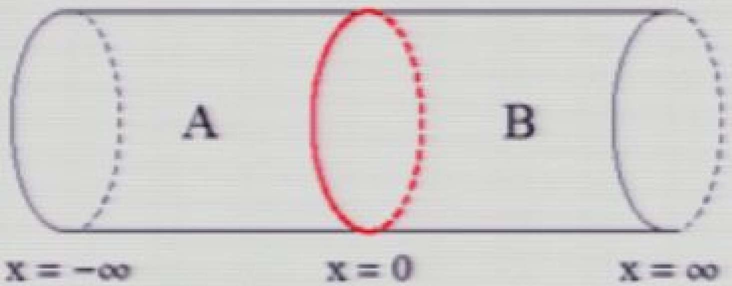


Use $\xi = \frac{c}{m}$, m **- gap to lowest excitation (** $O(N)$ **- multiplet)**

symmetry unbroken phase

Entanglement entropy in the O(N) model

- Universal geometric correction at QCP



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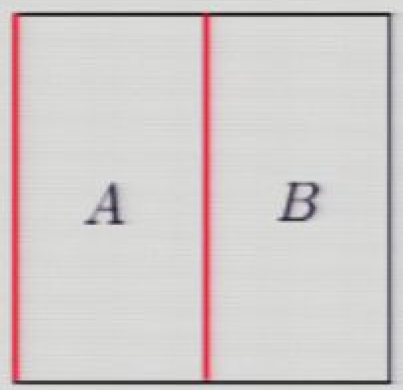

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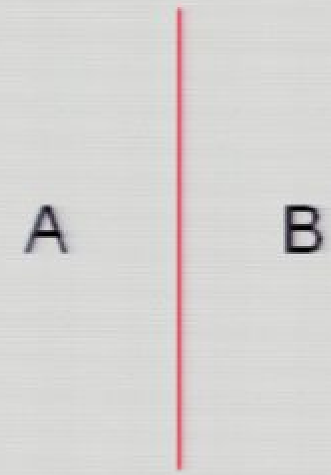

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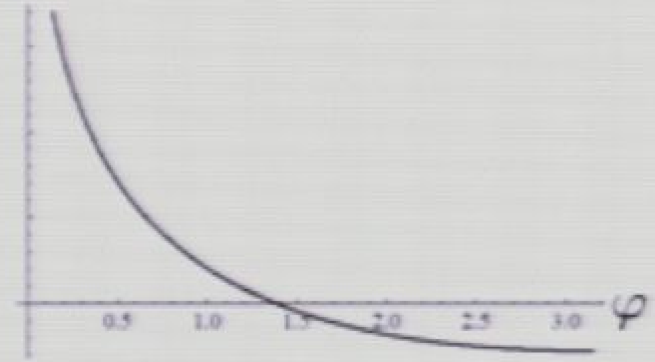
Results: γ



- Result of expansion in $\epsilon = 3 - d$

$$\gamma = -\frac{N\epsilon}{6(N+8)}\left(\log\left|\theta_1(\frac{\varphi(1+i)}{2\pi}, i)\right| - \frac{\varphi^2}{4\pi} - \log\eta(i)\right), \quad \text{Wilson}-\text{Fisher FP}$$

$$\gamma(\varphi = \pi) = -\frac{N\epsilon}{12(N+8)}\log 2$$

$$\gamma(\varphi \to 0) \to -\frac{N\epsilon}{6(N+8)}\log\varphi,$$ - breaks down for $\varphi \lesssim \epsilon^{1/2}$

$$\gamma(\varphi = 0) = -\frac{N\epsilon}{12(N+8)}\log\epsilon$$ - hypothesis

- Compare to Gaussian fixed point in : $d = 3 - \epsilon$

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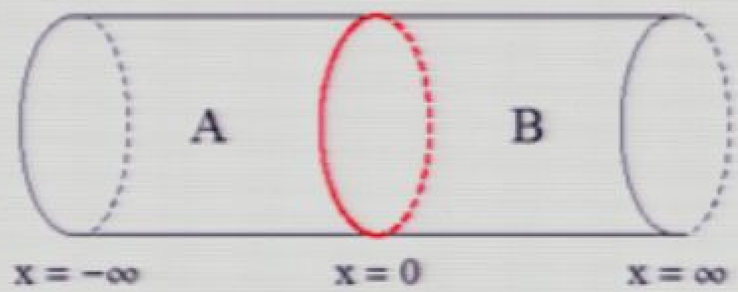

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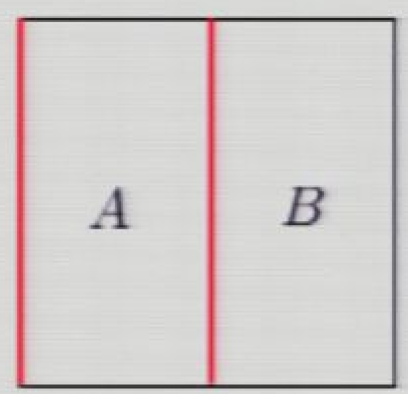

L. Tagliacozzo, G. Evenbly, G. Vidal (2009)

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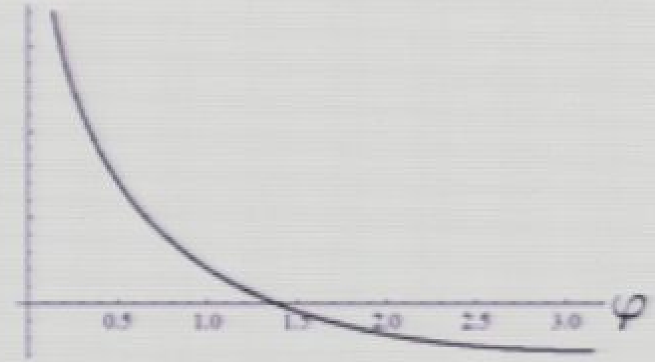

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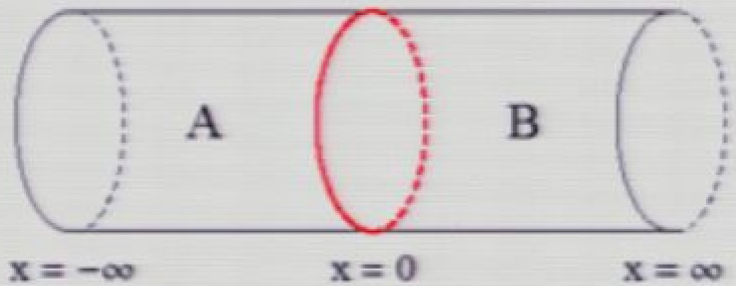

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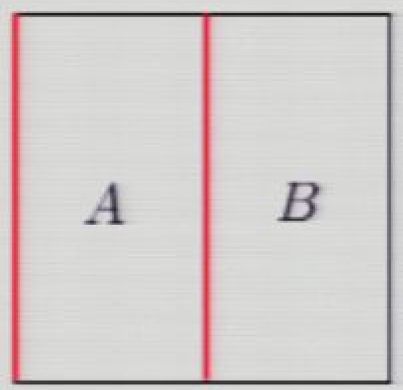

L. Tagliacozzo, G. Evenbly, G. Vidal (2009)

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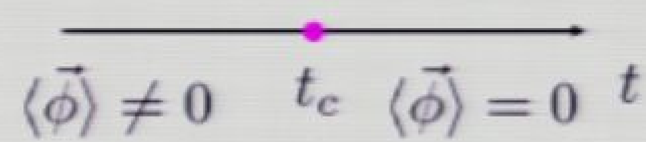

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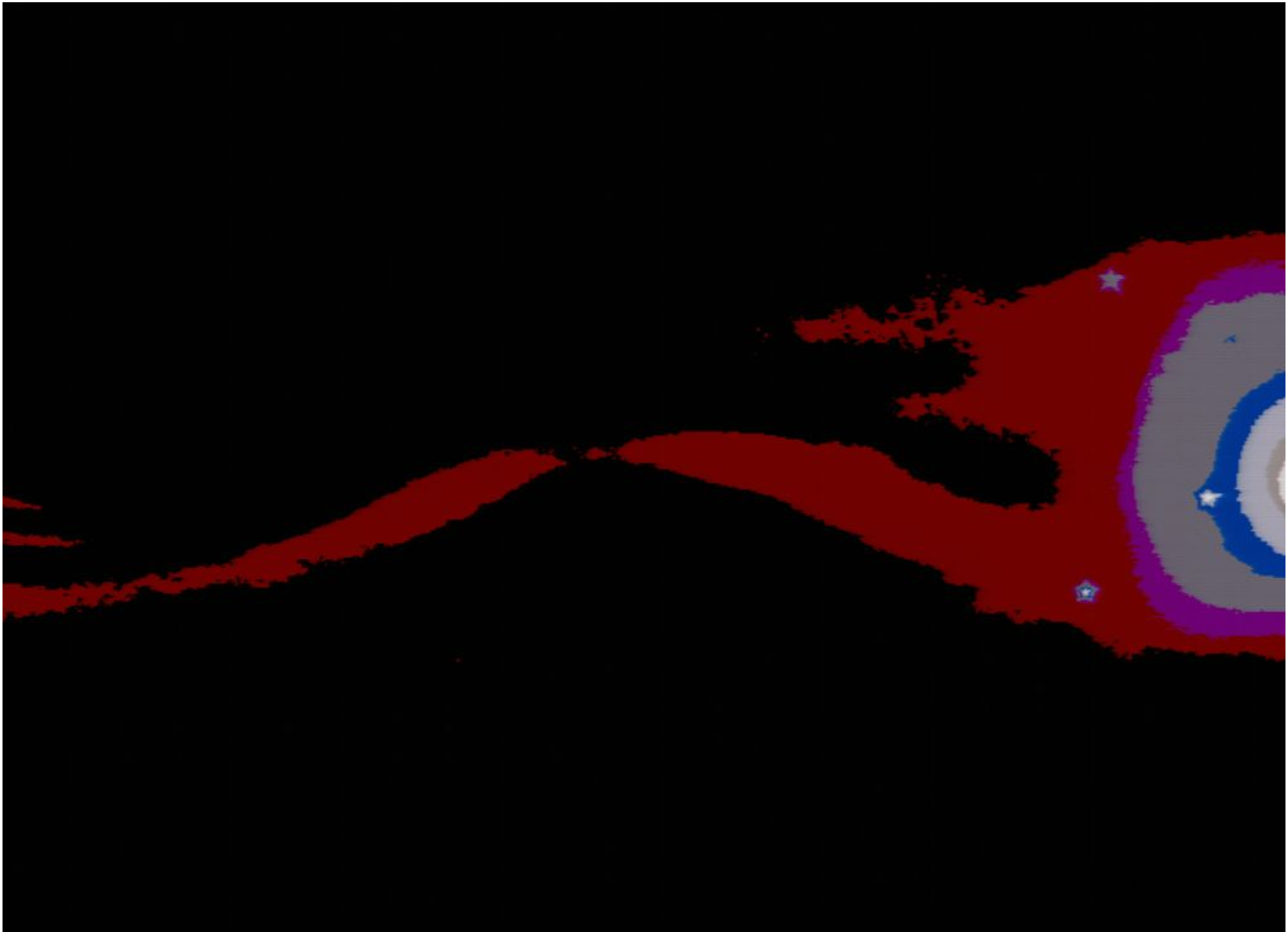
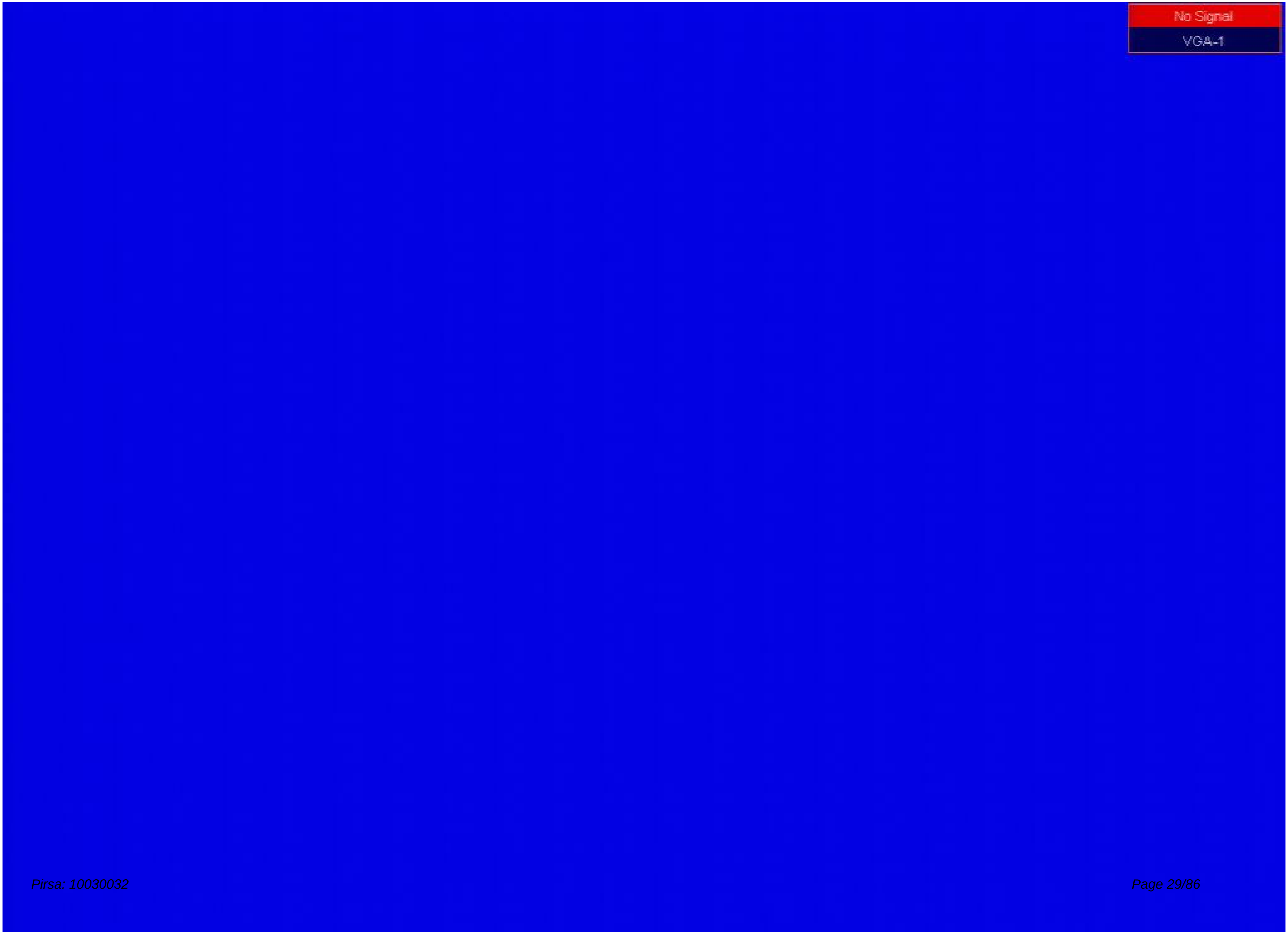
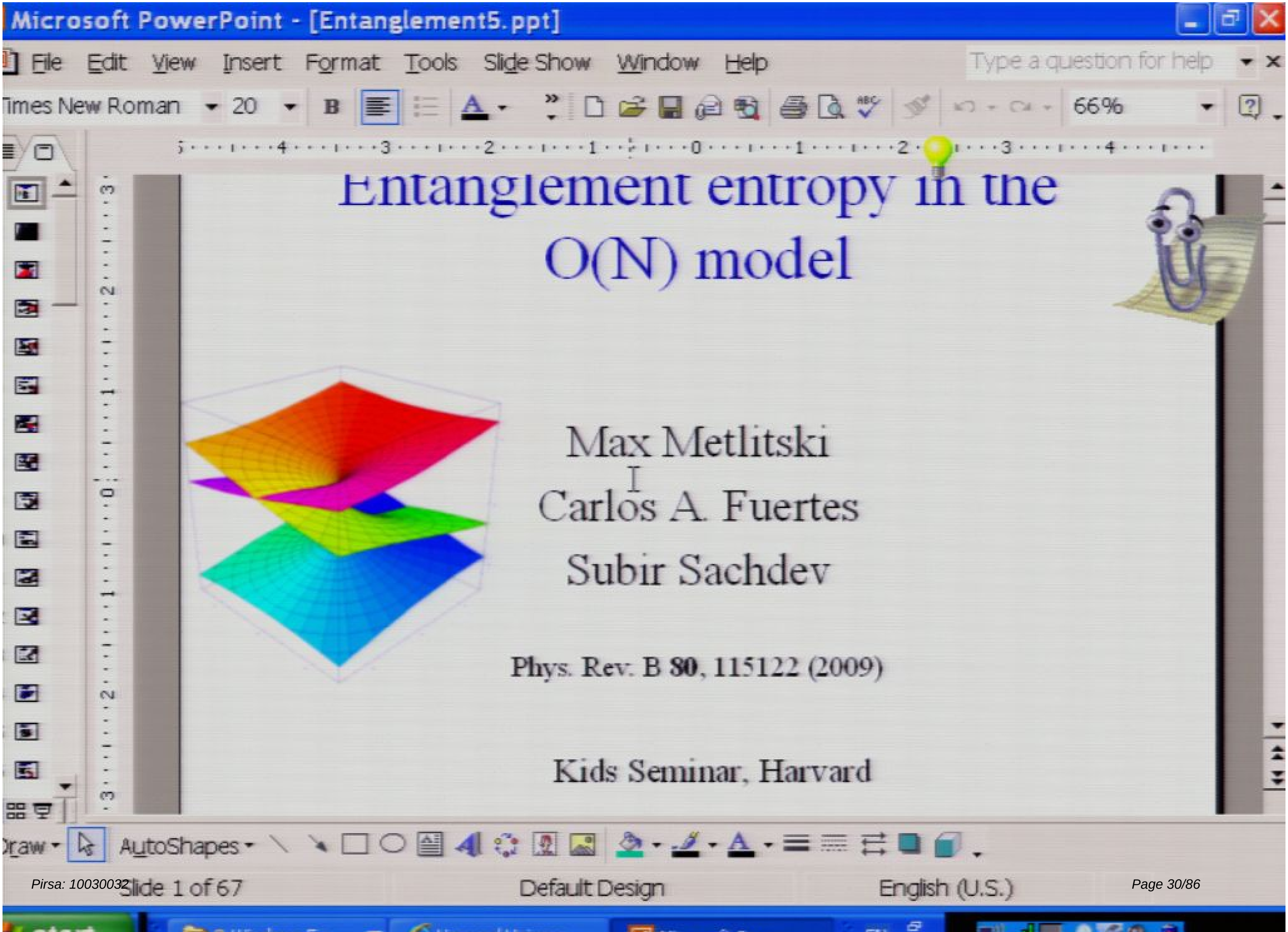



No Signal
VGA-1


Microsoft PowerPoint - [Entanglement5.ppt]
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Type a question for help
Times New Roman 20
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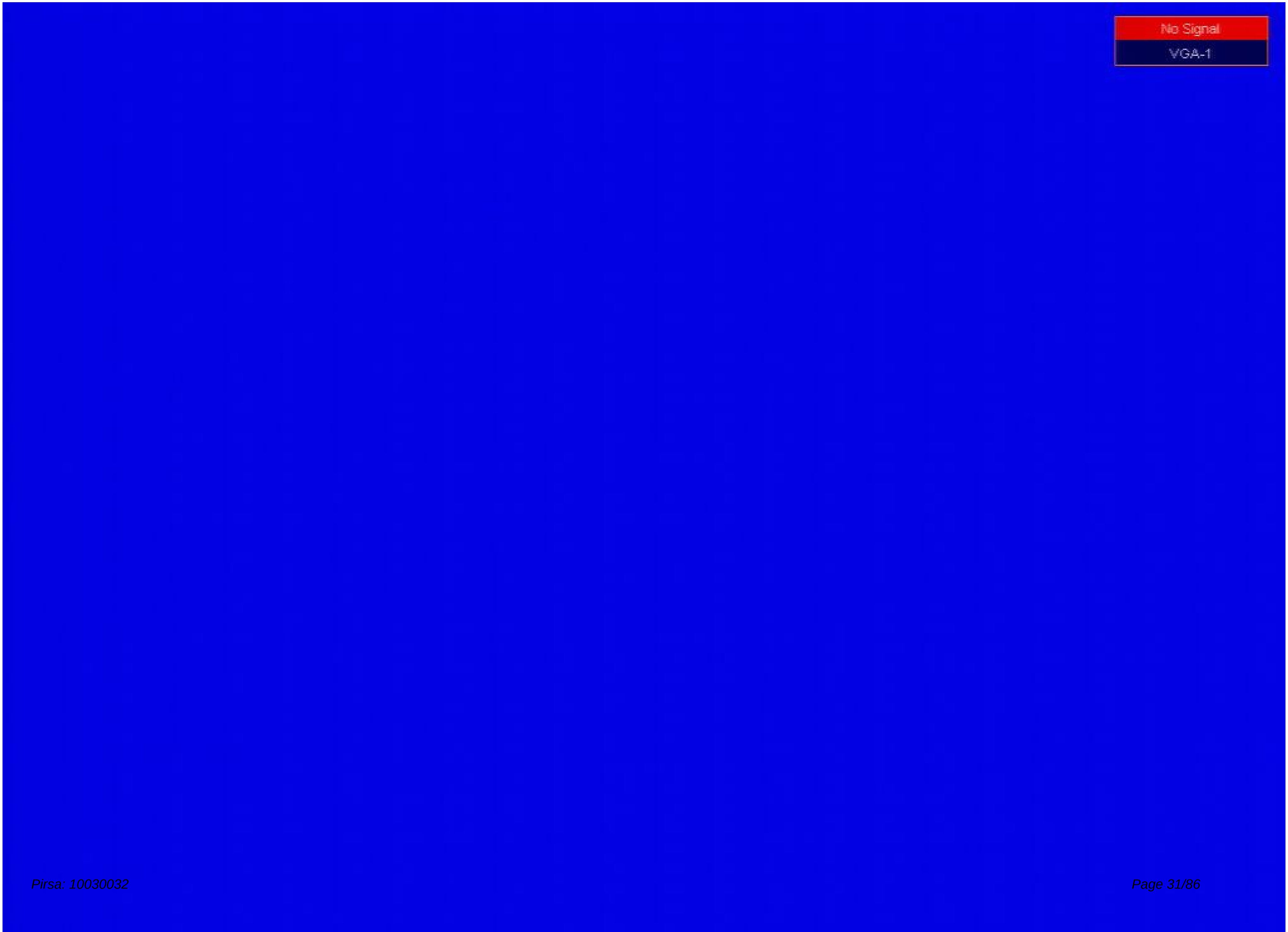
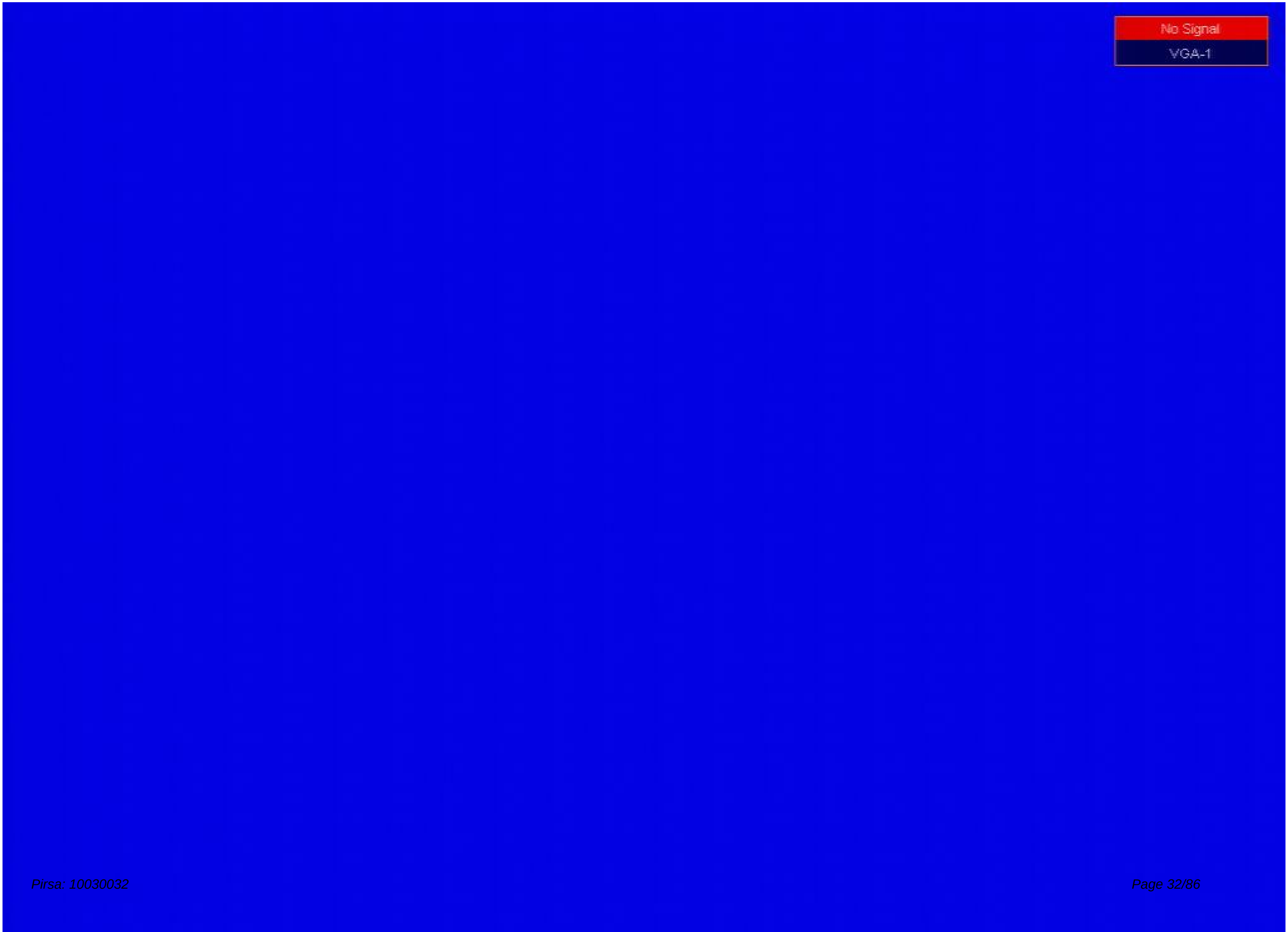
Entanglement entropy in the O(N) model
Max Metlitski
Carlos A. Fuertes
Subir Sachdev
Phys. Rev. B 80, 115122 (2009)
Kids Seminar, Harvard
Draw AutoShapes
Slide 1 of 67
Default Design
English (U.S.)


No Signal
VGA-1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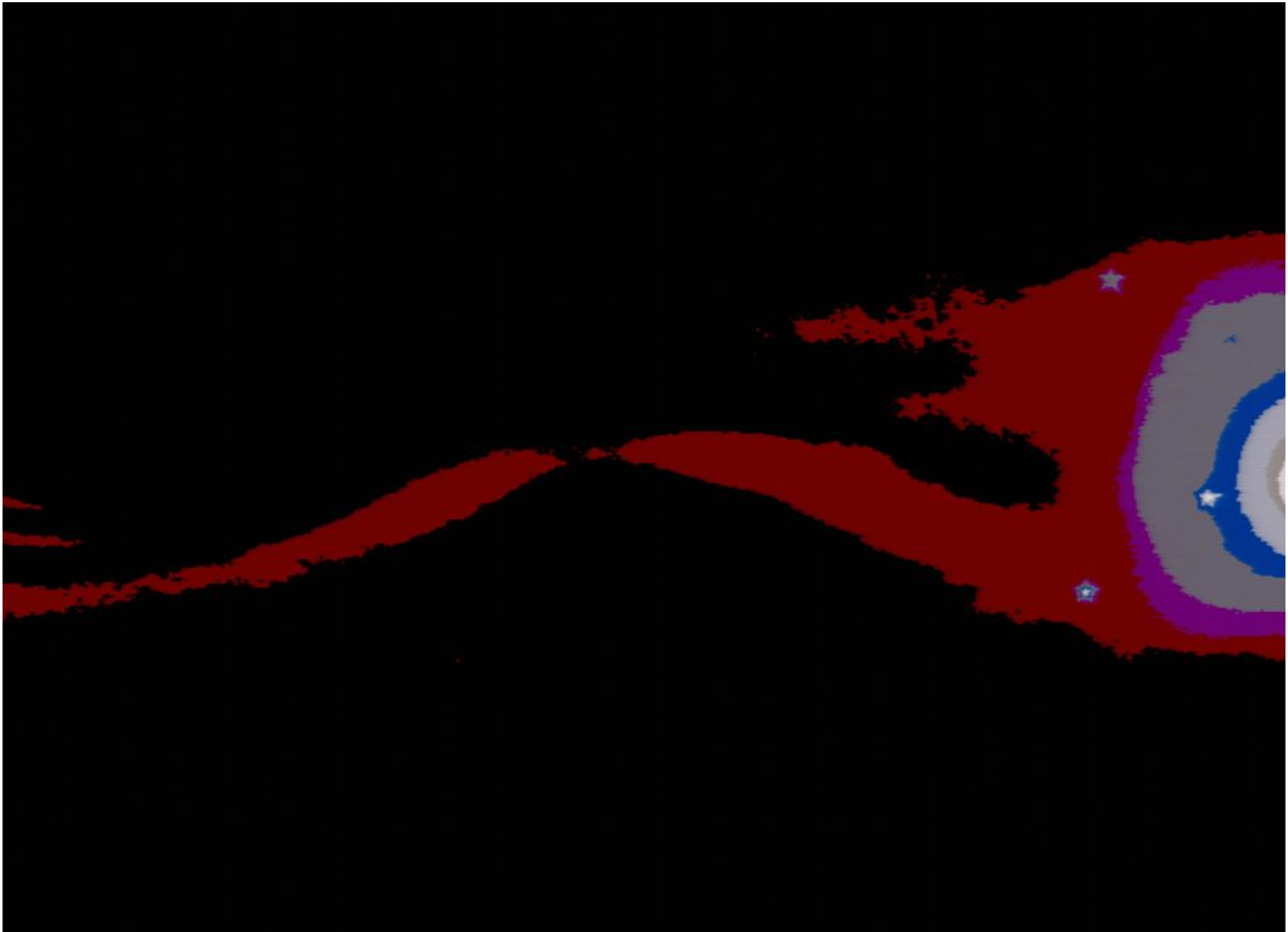
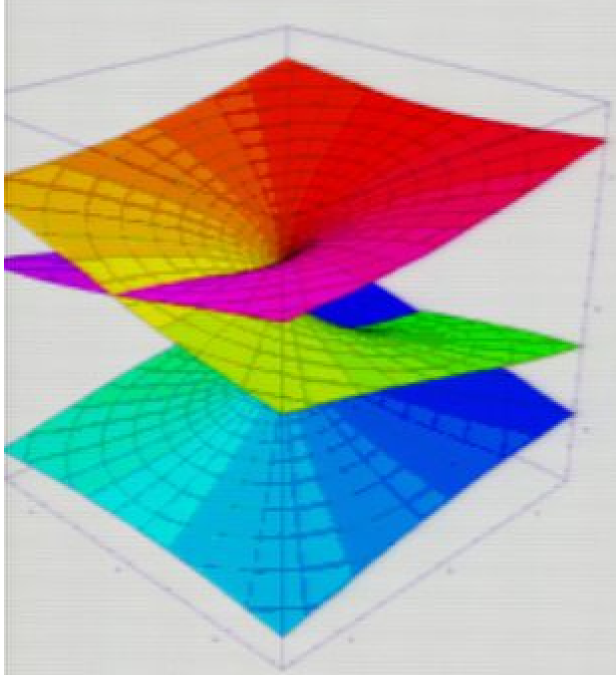


Entanglement entropy in the O(N) model



Max Metlitski

Carlos A. Fuertes

Subir Sachdev

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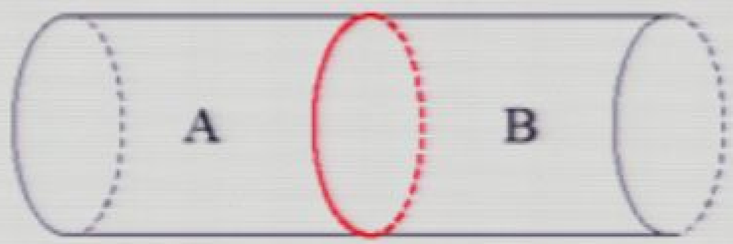
Beyond the leading term

- Any non-universal contributions to S must come from the boundary

$$S = g_{d-1}\frac{L^{d-1}}{a^{d-1}} + g_{d-2}\frac{L^{d-2}}{a^{d-2}} + \ldots + g_0 \log(L/a) + S_0(L/\xi)$$

- Straight closed boundary:



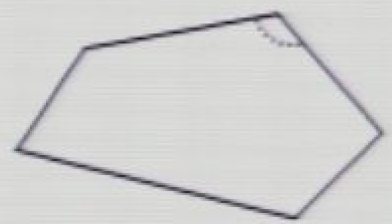
$$S = C\frac{|\partial A|}{a^{d-1}} + S_0(L/\xi)$$

- also believed to be true for general smooth closed boundaries in d = 2

- $g_0 \neq 0$ for

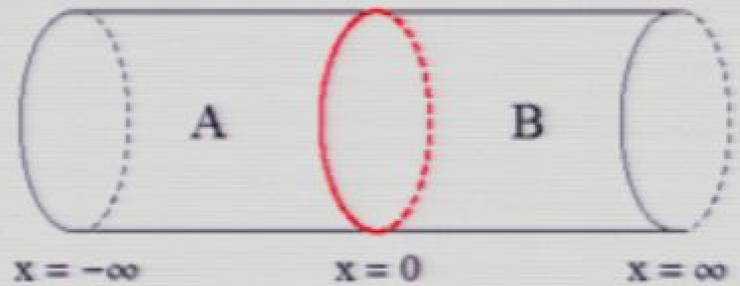
  - boundaries with corners/endpoints in d = 2,

  - boundaries with intrinsic/extrinsic curvature in d = 3



Entanglement entropy in the O(N) model

- Universal geometric correction at QCP



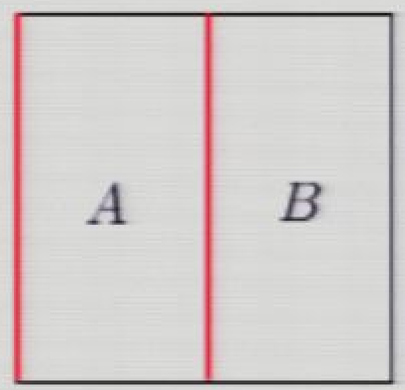
$$S = C\frac{L^{d-1}}{a^{d-1}} + \gamma$$

Boundary – d-1 dimensional torus of side L (circle in d = 2)

Twisted boundary conditions: $\phi(x + \hat{n}_i L) = e^{i\varphi}\phi(x)$

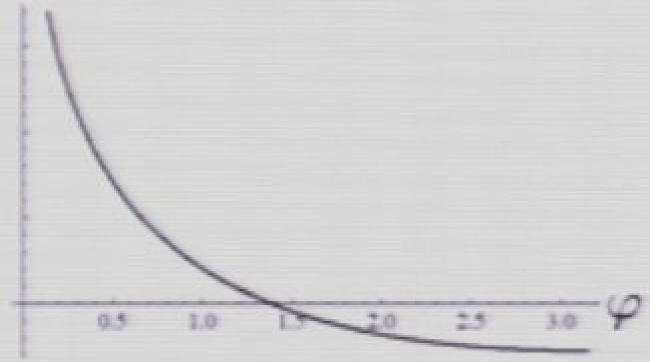
- Numerical evidence for finite γ at QCP in the transverse field Ising model (N = 1)

  - Tree tensor networks
  - Half-torus geometry



L. Tagliacozzo, G. Evenbly, G. Vidal (2009)

Results: γ



- Result of expansion in $\epsilon = 3 - d$

$$\gamma = -\frac{N\epsilon}{6(N+8)}\left(\log\left|\theta_1\left(\frac{\varphi(1+i)}{2\pi}, i\right)\right| - \frac{\varphi^2}{4\pi} - \log\eta(i)\right), \quad \text{Wilson} - \text{Fisher FP}$$

$$\gamma(\varphi = \pi) = -\frac{N\epsilon}{12(N+8)}\log 2$$

$$\gamma(\varphi \to 0) \to -\frac{N\epsilon}{6(N+8)}\log\varphi,$$ - **breaks down for** $\varphi \lesssim \epsilon^{1/2}$

$$\gamma(\varphi = 0) = -\frac{N\epsilon}{12(N+8)}\log\epsilon$$ - **hypothesis**

- Compare to Gaussian fixed point in : $d = 3 - \epsilon$

$$\gamma = -\frac{N}{6}\left(\log\left|\theta_1\left(\frac{\varphi(1+i)}{2\pi}, i\right)\right| - \frac{\varphi^2}{4\pi} - \log\eta(i)\right), \quad \text{Gaussian FP}$$

Results: r

- Result of expansion in $\epsilon = 3 - d$

$$S = C(t)\frac{L^{d-1}}{a^{d-1}} + r\frac{L^{d-1}}{\xi^{d-1}}$$

$$r = -\frac{N}{144\pi}, \text{ Wilson-Fisher FP}$$

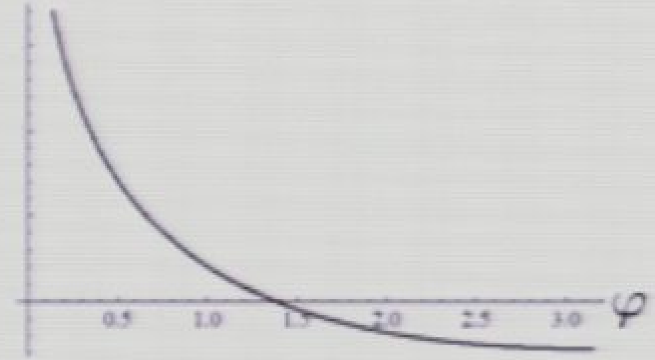
- Analytic correction: $\sim t$

  Singular correction in d = 2: $\sim t^{\nu}$ $\quad \nu < 1$ - wins!

- Compare to Gaussian fixed point in $d = 3 - \epsilon$:

$$r = -\frac{N}{24\pi\epsilon}, \text{ Gaussian FP}$$

Results: γ



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Different $\epsilon \to 0$ limits

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Entanglement largest at the QCP!

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Different $\epsilon \to 0$ limits

Renyi entropy

- $S_n = \frac{1}{1-n} \log tr_A \rho_A^n$ $S = -tr_A \rho_A \log \rho_A = \lim_{n \to 1} S_n$

- In d = 1,

$$S_n = \frac{c}{6}\left(1 + \frac{1}{n}\right) \log(l/a)$$

- In d > 1, area law + subleading universal corrections

$$S_n = C_n \frac{L^{d-1}}{a^{d-1}} + \gamma_n$$

$$S_n = C_n(t) \frac{L^{d-1}}{a^{d-1}} + r_n \frac{L^{d-1}}{\xi^{d-1}}$$

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- Violation of (naïve) large N counting

  Expect: $r_n \sim O(N)$ **Instead:** $r_n \sim O(N^2)$ (but $r \sim O(N)$)

- Computed r_n in large-N expansion in d = 2

n	α_n
2	-0.16515
3	-0.26594
4	-0.32905
5	-0.36743

$$r_n = \frac{3\pi^2 N^2}{128} \frac{n\alpha_n^2}{n-1}$$

$$r_n \sim n - 1, \quad n \to 1$$

- Can compare r_n to lattice Monte-Carlo simulations of O(N) model

P. V. Buividovich and M. I. Polikarpov (2008), M. Caraglio and F. Gliozzi (2008)

Renyi entropy - results

- $$S_n = C_n \frac{L^{d-1}}{a^{d-1}} + \gamma_n$$
  $$S_n = C_n(t) \frac{L^{d-1}}{a^{d-1}} + r_n \frac{L^{d-1}}{\xi^{d-1}}$$

- In dimension $d = 3 - \epsilon$, γ_n, r_n have a discontinuity at

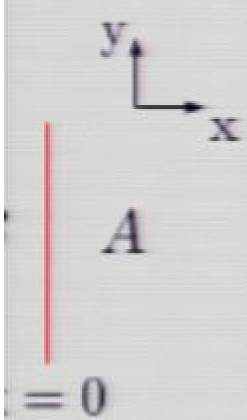
  $$n = n^*, \; 1 < n^* \leq 1 + \frac{3}{4} \frac{N+2}{N+8} \epsilon$$

- New length scale emerges as $n \to n_*$, "phase transition"

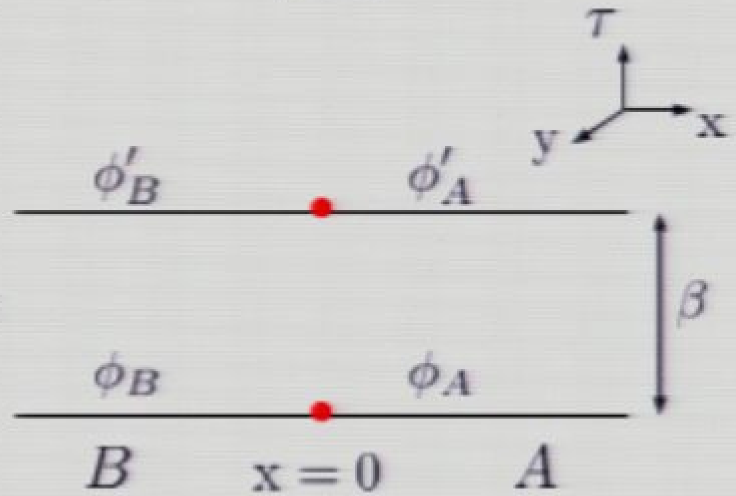
- Large N expansion shows that this phenomena occurs in 2.74 < d < 3 (likely absent in d = 2)

Replica Trick

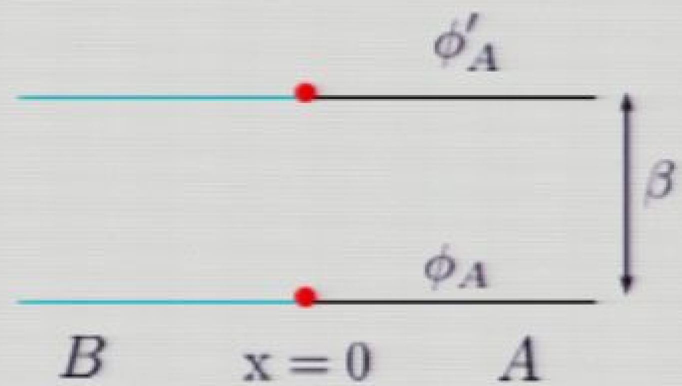
- $S_n = \frac{1}{1-n} \log tr_A \rho_A^n \qquad S = -tr_A \rho_A \log \rho_A = \lim_{n \to 1} S_n$



$\rho = \frac{1}{Z} e^{-\beta H} \qquad \langle \phi'_B, \phi'_A | \rho | \phi_B, \phi_A \rangle = \frac{1}{Z} \times$



$\rho_A = tr_B \rho \qquad \langle \phi'_A | \rho_A | \phi_A \rangle = \frac{1}{Z} \times$



Renyi entropy - results

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- In dimension $d = 3 - \epsilon,\ \gamma_n, r_n$ have a discontinuity at
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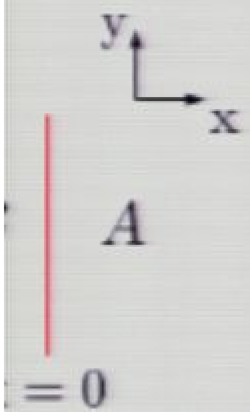

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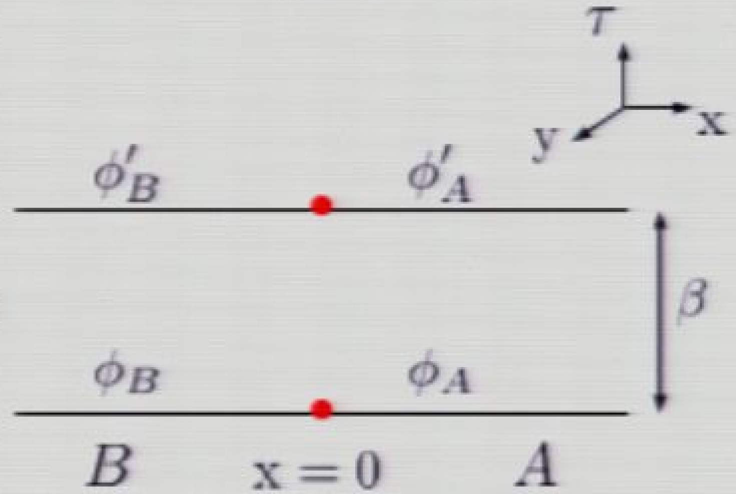

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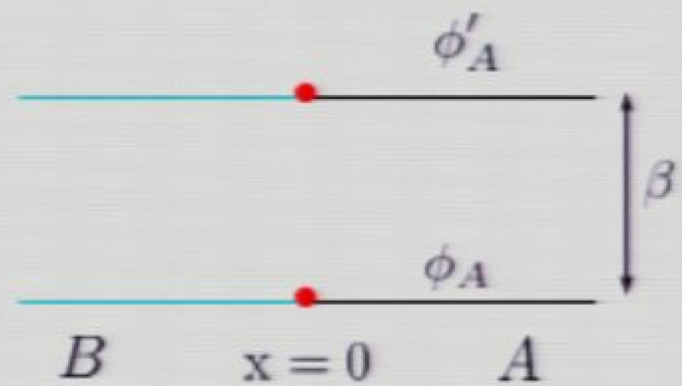

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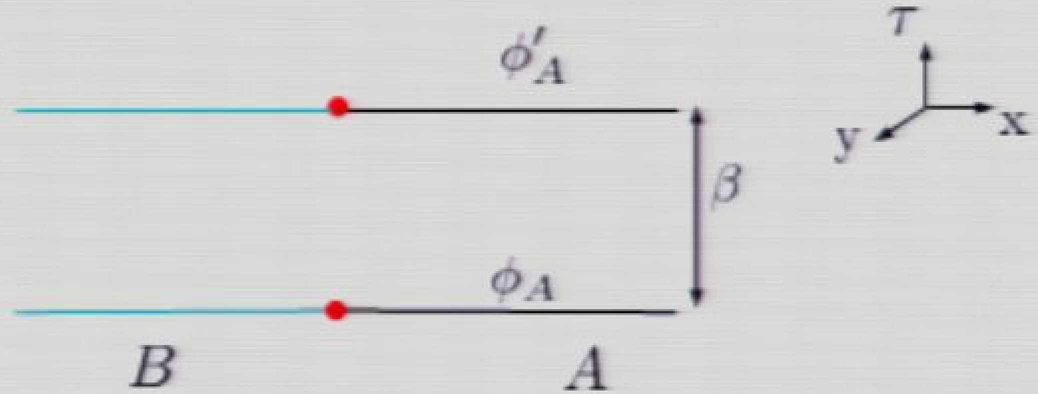
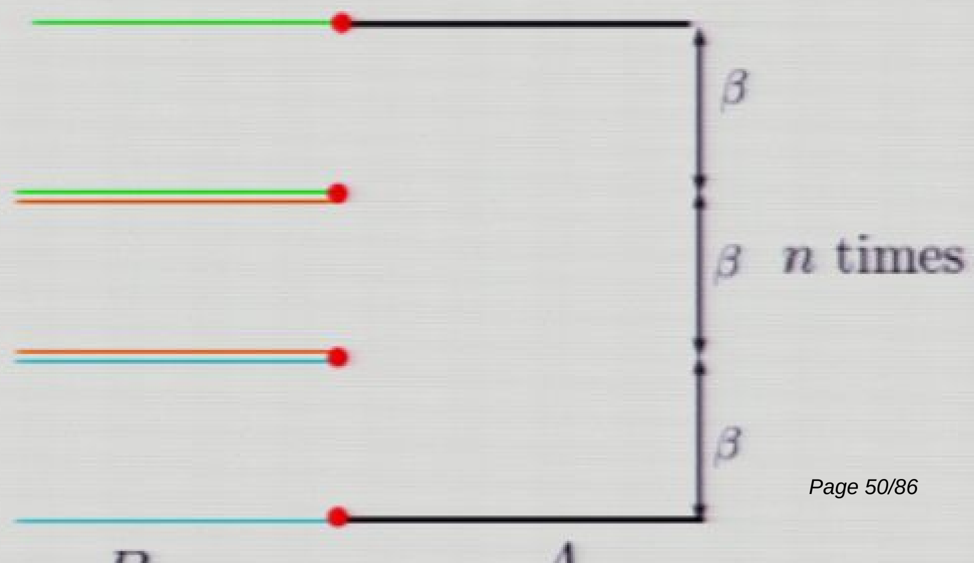
$\rho_A = tr_B \rho$

$$\langle \phi'_A | \rho_A | \phi_A \rangle = \frac{1}{Z} \times$$



$$tr_A \rho_A^n = \int \prod_{i=1}^{n} d\phi_A^i \langle \phi_A^1 | \rho_A | \phi_A^n \rangle \ldots \langle \phi_A^3 | \rho_A | \phi_A^2 \rangle \langle \phi_A^2 | \rho_A | \phi_A^1 \rangle =$$

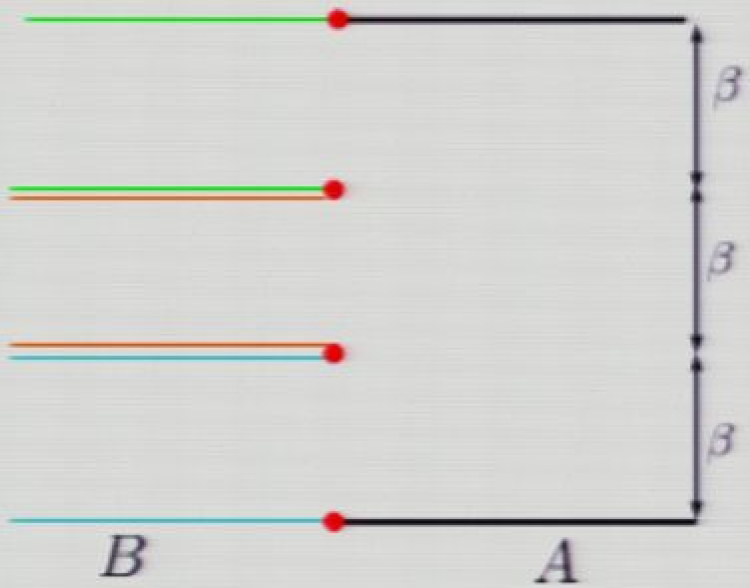
$$\frac{1}{Z^n} \times$$



Riemann surface

- $S_n = \frac{1}{1-n} \log tr_A \rho_A^n \qquad S = -tr_A \rho_A \log \rho_A = \lim_{n\to 1} S_n$

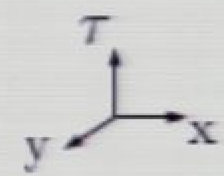
$tr_A \rho_A^n = \frac{1}{Z^n} \times$



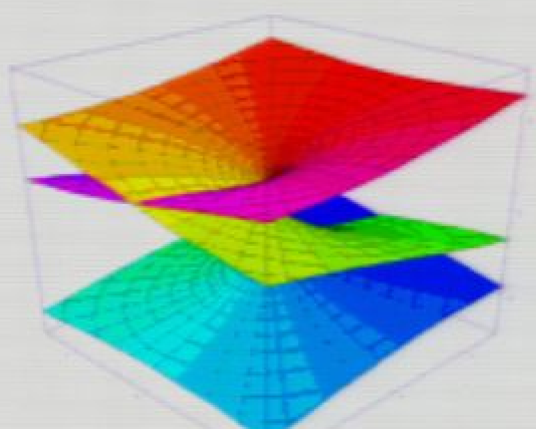
$\beta \to \infty$

n-sheeted Riemann surface C_n

$(\mathrm{x}, \tau) = (0,0)$



$$tr_A \rho_A^n = \frac{Z_n}{Z^n}$$



$(\mathrm{x}, \tau) = r(\cos\theta, \sin\theta)$

$ds^2 = dr^2 + r^2 d\theta^2 + dx_\perp^2$

$\theta \sim \theta + 2\pi n$

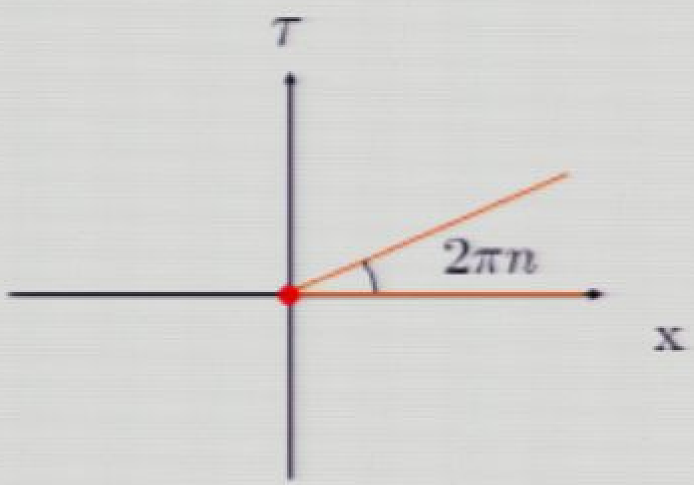
Conical singularity

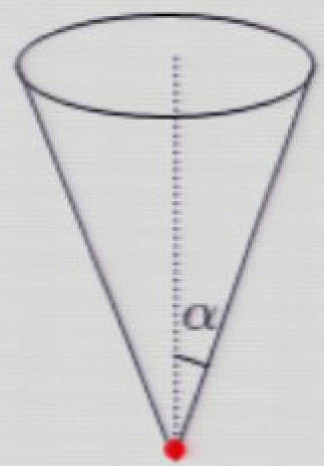
$(\mathrm{x}, \tau) = r(\cos\theta, \sin\theta)$

$ds^2 = dr^2 + r^2 d\theta^2 + dx_\perp^2$

$\theta \sim \theta + 2\pi n$

- Let $n < 1$



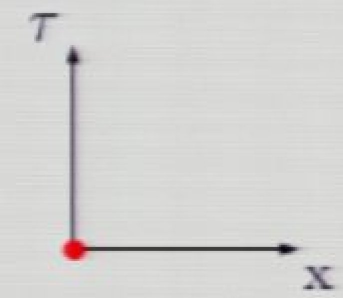


$n = \sin\alpha$

Why is d = 1 easy?

- Confomal group in 1+1d consists of all analytic (or anti-analytic) mappings

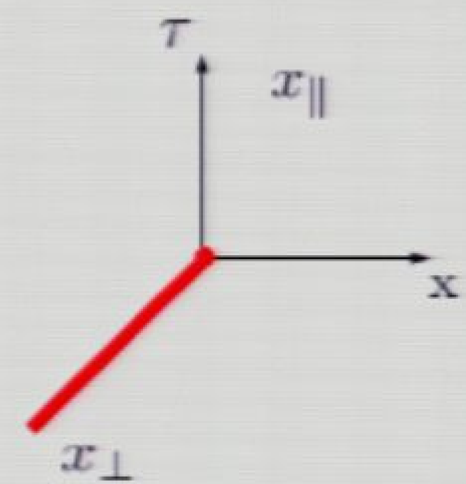
  can map the n-sheeted Riemann surface to ordinary complex plane via



$$f : \mathbb{C}_n \to \mathbb{C}, \quad w = z^{1/n}$$

$$ds^2_{\mathbb{C}} = dw d\bar{w} = \left|\frac{dw}{dz}\right|^2 dz d\bar{z} = \left|\frac{dw}{dz}\right|^2 ds^2_{\mathbb{C}_n}$$

- This is no-longer a confomal symmetry for d > 1



$$ds^2_{\mathbb{C}} = dw d\bar{w} + dx^2_{\perp} = \left|\frac{dw}{dz}\right|^2 dz d\bar{z} + dx^2_{\perp}$$

Have to work harder!

Explicit calculations on an n-sheeted surface

Finite size correction

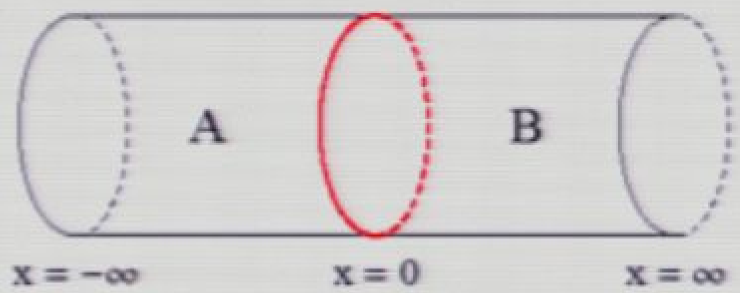
$$S_n = C_n \frac{L^{d-1}}{a^{d-1}} + \gamma_n \qquad S_n = \frac{1}{1-n} \log tr_A \rho_A^n$$

$$tr \rho_A^n = \frac{Z_n}{Z^n}$$

- **Twisted boundary conditions**

$$\phi(x + L\hat{n}_i) = e^{i\varphi_i}\phi(x)$$

$$i = 1 .. d-1$$



- **Momentum along the boundary is quantized**

$$k_\perp^i = \frac{2\pi n_i + \varphi_i}{L}$$

Finite size correction: Gaussian theory

$$L = \frac{1}{2}(\partial_\mu \phi)^2$$

- **Tower of decoupled 1+1 dimensional massive theories:** $m^2 = \vec{k}_\perp^2$

$$\log \frac{Z_n}{Z^n} = -\frac{N}{2}\left(Tr\log(-\partial^2)_n - nTr\log(-\partial^2)_1\right)$$

$$= -\frac{N}{2}\sum_{\vec{k}_\perp}\left[Tr_{\|}\log(-\partial_{\|}^2 + \vec{k}_\perp^2)_n - nTr_{\|}\log(-\partial_{\|}^2 + \vec{k}_\perp^2)_1\right]$$

$$= -\frac{N}{2}\sum_{\vec{k}_\perp}\log\frac{Z_n}{Z^n}\Big|_{d=1}\left(m^2 = \vec{k}_\perp^2\right)$$

- **In a massive d = 1 Gaussian theory**

$$\log\frac{Z_n}{Z^n}\Big|_{d=1} = \frac{1}{24}\left(n - \frac{1}{n}\right)\log(m^2) \qquad m \neq 0 \Rightarrow \varphi \neq 0$$

P. Calabrese and J. L. Cardy (2004).

Finite size correction: Gaussian theory

$$S_n = C_n \frac{L^{d-1}}{a^{d-1}} + \gamma_n$$

- $d = 2$

$$\begin{aligned}\gamma_n &= -\frac{N}{12}\left(1+\frac{1}{n}\right)\log(2|\sin\varphi/2|) \\ \gamma &= -\frac{N}{6}\log(2|\sin\varphi/2|)\end{aligned}$$

- $d = 3$

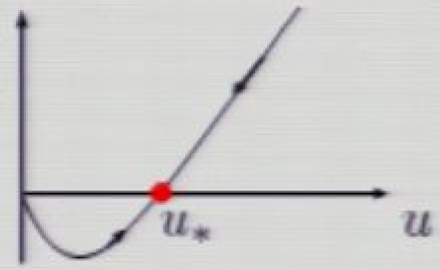
$$\begin{aligned}\gamma_n &= \frac{\pi N}{6}\left(1+\frac{1}{n}\right)g(\vec{\varphi}) \\ \gamma &= \frac{\pi N}{3}g(\vec{\varphi})\end{aligned}$$

$$g(\vec{\varphi}) = -\frac{1}{2\pi}\left(\log\left|\theta_1\left(\frac{\varphi_1+i\varphi_2}{2\pi}, i\right)\right| - \frac{\varphi_2^2}{4\pi} - \log\eta(i)\right)$$

Interacting theory

- $L = \frac{1}{2}(\partial_\tau \vec{\phi})^2 + \frac{1}{2}(\nabla \vec{\phi})^2 + \frac{t}{2}\vec{\phi}^2 + \frac{u}{4}(\vec{\phi}^2)^2$

- **Theory flows to Wilson-Fisher fixed point**



$$\beta(u) = -\frac{du}{dl} = -\epsilon u + \frac{(N+8)}{8\pi^2}u^2$$

$$u_* = \frac{8\pi^2\epsilon}{N+8}$$

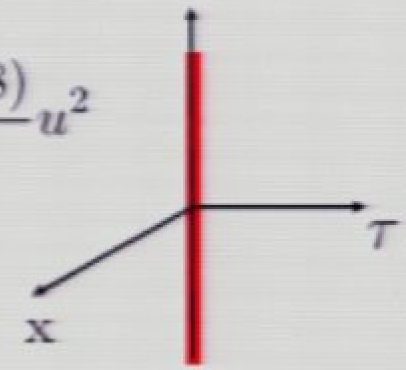
- **Can we do mean-field at leading order in ε?**

$$\gamma_n = \frac{\pi N}{6}\left(1+\frac{1}{n}\right) g(\vec{\varphi})$$
$$\gamma = \frac{\pi N}{3} g(\vec{\varphi})$$

No!

$$g(\vec{\varphi}) = -\frac{1}{2\pi}\left(\log\left|\theta_1\left(\frac{\varphi_1 + i\varphi_2}{2\pi}, i\right)\right| - \frac{\varphi_2^2}{4\pi} - \log\eta(i)\right)$$

"Boundary" perturbations

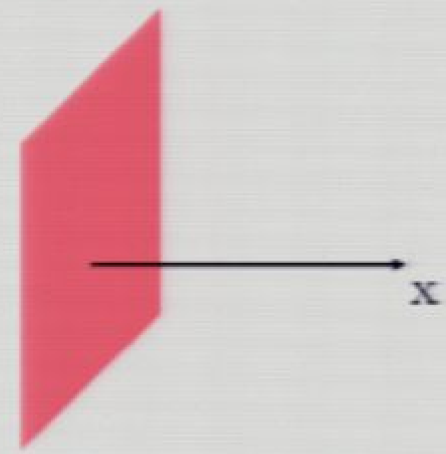


- Boundary cannot renormalize bulk: $\beta(u) = -\epsilon u + \frac{(N+8)}{8\pi^2} u^2$
- New operators can be induced at the boundary!

$$S_b = \frac{c}{2} \int d^{d-1} x_\perp \, \phi^2(r=0, x_\perp)$$

- Analogy: classical boundary critical phenomena in O(N) model

$$S_b = \frac{c}{2} \int d^{D-1} x_\perp \phi^2(\mathrm{x}=0, \vec{x}_\perp)$$



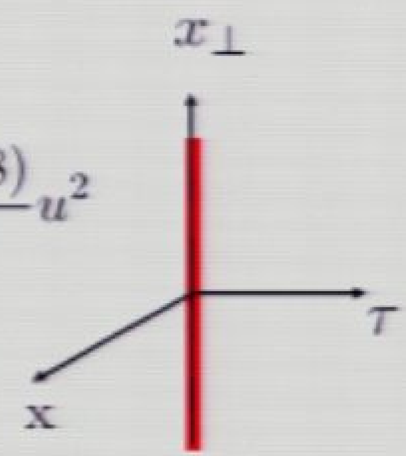
- Surface universality classes

$c > c_{sp}$ - ordinary

$c < c_{sp}$ - extra-ordinary

$c = c_{sp}$ - special (multi-critical)

"Boundary" perturbations



- Boundary cannot renormalize bulk: $\beta(u) = -\epsilon u + \frac{(N+8)}{8\pi^2} u^2$
- New operators can be induced at the boundary!

$$S_b = \frac{c}{2} \int d^{d-1} x_\perp \, \phi^2(r=0, x_\perp)$$

- marginal at the Gaussian fixed point

- In the interacting theory in the absence of conical singularity

$$[\vec{\phi}^2] = D - \frac{1}{\nu}, \qquad [c] = \frac{1}{\nu} - 2 < 0 \qquad \beta(c) = -\frac{dc}{dl} = -\left(\frac{1}{\nu} - 2\right) c$$

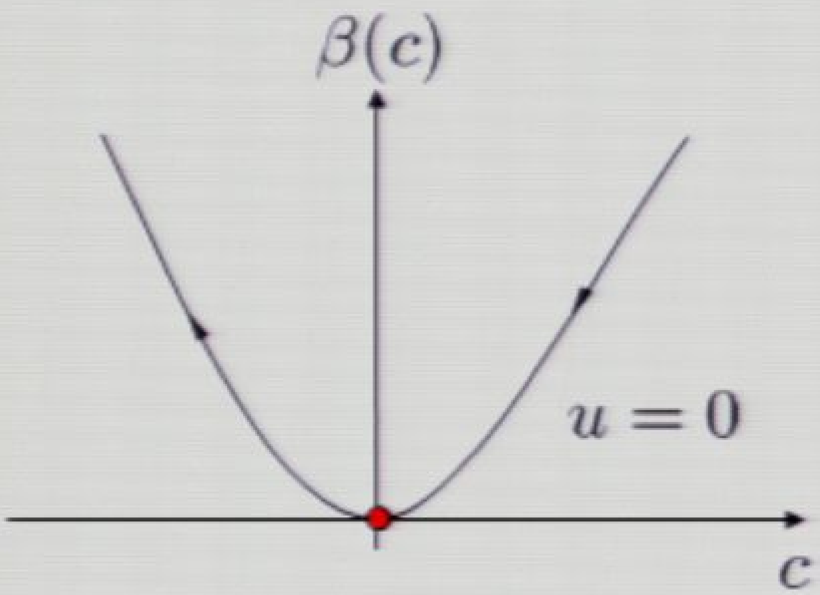
$c-$ irrelevant

- Conical singularity can change the flow!

Boundary flow

$$\beta(c) = \frac{(N+2)u}{24\pi}\left(n - \frac{1}{n}\right) + \frac{(N+2)uc}{8\pi^2} + \frac{c^2}{2\pi n}$$

- **Non-interacting theory:** $\beta(c) = \dfrac{c^2}{2\pi n}$ $\qquad c(p) = \dfrac{c}{1 + \frac{c}{2\pi n}\log \Lambda/p}$



$$L = \frac{1}{2}(\partial_\mu \vec{\phi})^2 + \frac{1}{2}V(x)\vec{\phi}^2 \qquad V(x) = c\delta^2(x_\parallel)$$

$$-\nabla^2\phi + V(x)\phi = E\phi$$

- Repulsive $c > 0$

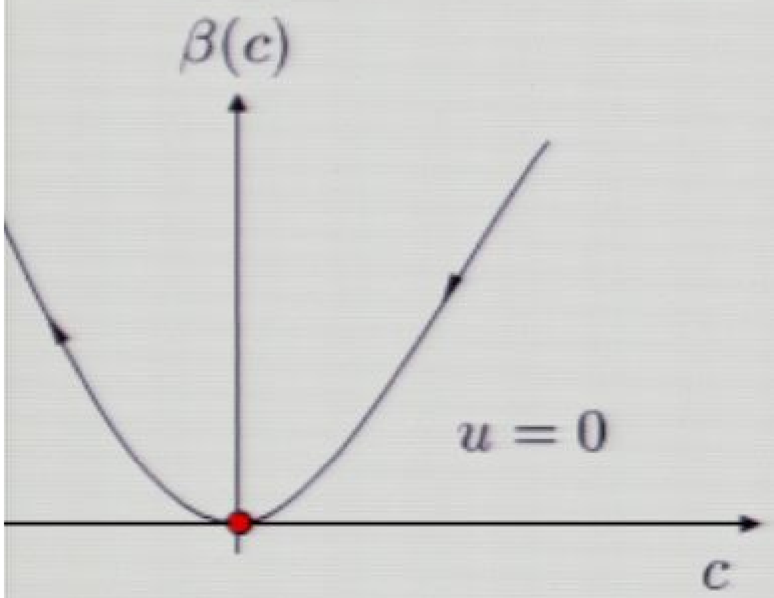
  - logarithmic flow to weak coupling

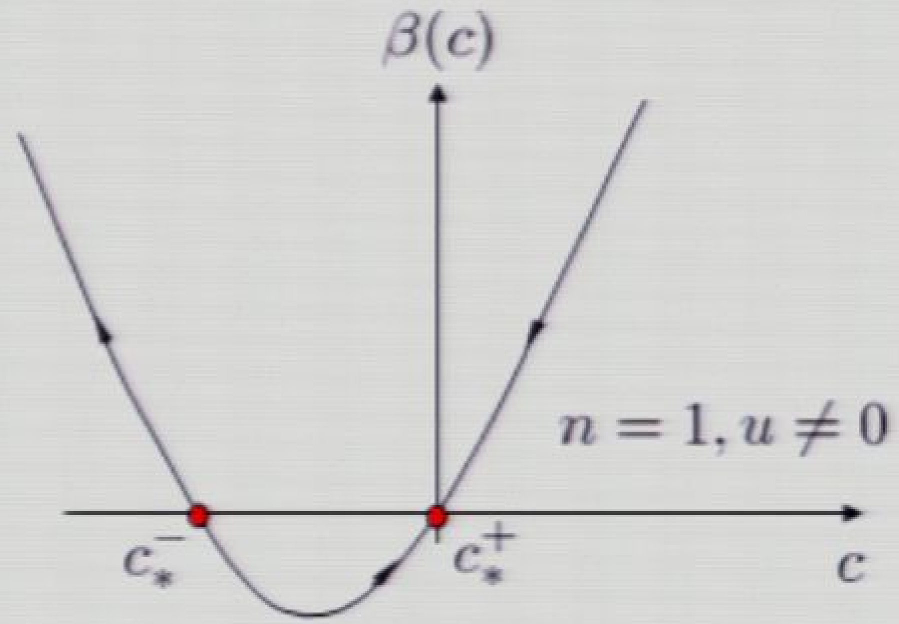
- Attractive $c < 0$

  bound state forms $E \propto -\Lambda^2 e^{-\frac{4\pi}{c}}$

Boundary flow

- **Interacting theory,** $n = 1$ $\qquad \beta(c) = \frac{(N+2)uc}{8\pi^2} + \frac{c^2}{2\pi}$



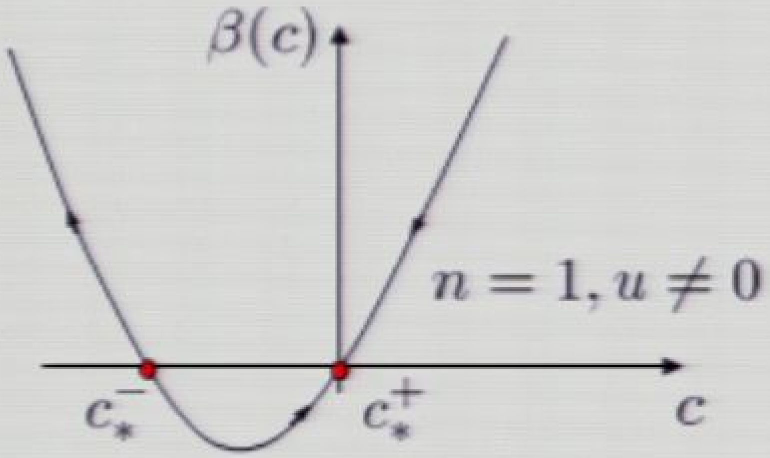


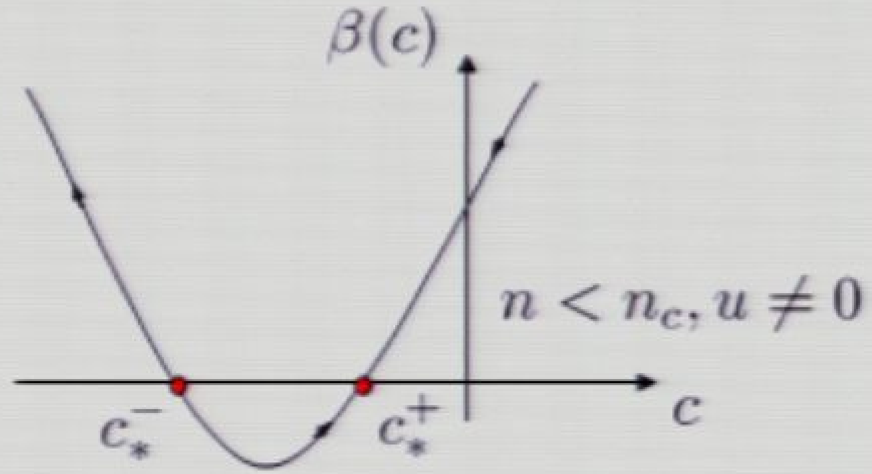
- **Stable fixed point** $c_*^+ = 0 \qquad [\vec{\phi}^2] = D - \frac{1}{\nu}, \qquad [c] = \frac{1}{\nu} - 2 < 0$
- **Unstable fixed point** $c_*^- = -2\pi \frac{N+2}{N+8}\epsilon$

- **Flow** $c \to -\infty$ **is towards another fixed point (large N)**

Boundary flow: full theory

$$n < n_c = 1 + \frac{3}{4}\frac{N+2}{N+8}\epsilon \qquad \beta(c) = \frac{(N+2)u}{24\pi}\left(n - \frac{1}{n}\right) + \frac{(N+2)uc}{8\pi^2} + \frac{c^2}{2\pi n}$$



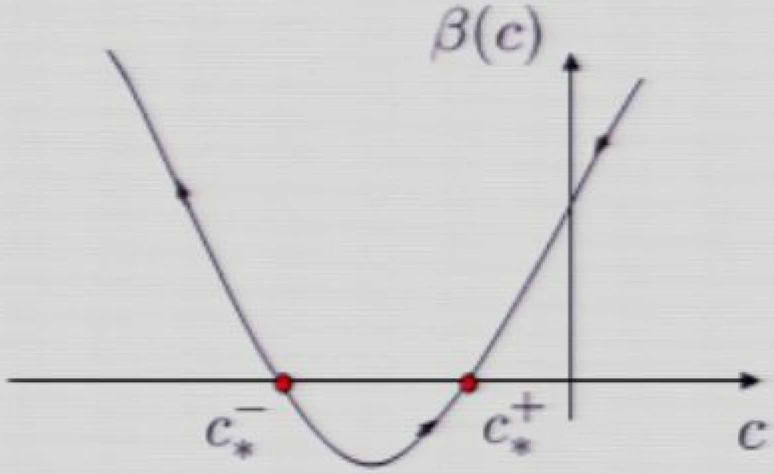


$$c_*^{\pm} = \pi\left(-\frac{N+2}{N+8}n\epsilon \pm \sqrt{\left(\frac{N+2}{N+8}\right)^2 n^2\epsilon^2 - \frac{2}{3}\frac{N+2}{N+8}(n^2-1)\epsilon}\right)$$

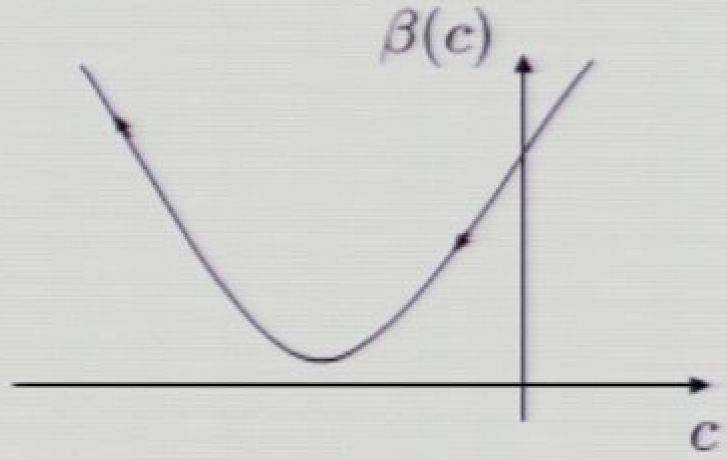
- Main effect: the stable critical point is shifted to $c_+^* \neq 0$
- c_+^* controls the $n \to 1$ limit and hence, the entanglement entropy!

Boundary flow: full theory

$n < n_c$



$n > n_c$



Must have a "phase transition" at $n = n_*, \quad 1 < n_* \le n_c = 1 + \frac{3}{4}\frac{N+2}{N+8}$

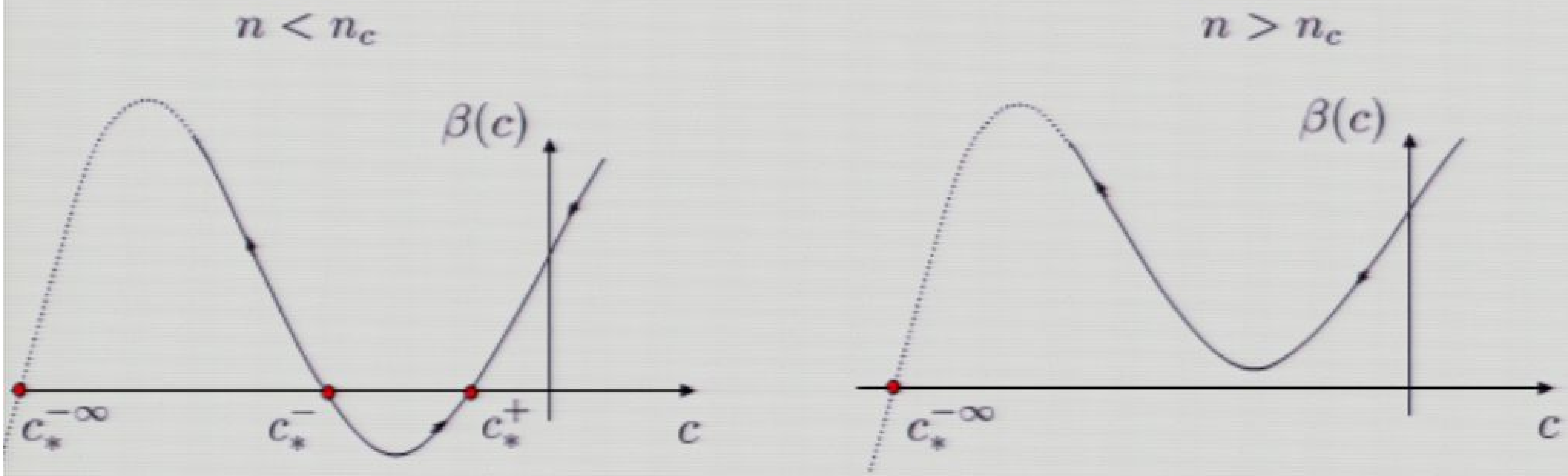
$$S_n = C_n \frac{L^{d-1}}{a^{d-1}} + \gamma_n$$

$$S_n = C_n(t) \frac{L^{d-1}}{a^{d-1}} + r_n \frac{L^{d-1}}{\xi^{d-1}}$$

r_n, γ_n **discontinuous at** $n = n_*$

$L \to \infty$ and $n \to n_*$ **don't commute**

Boundary flow: full theory + large N



Must have a "phase transition" at $n = n_*, \quad 1 < n_* \leq n_c = 1 + \frac{3}{4}\frac{N+2}{N+8}$

- Conical singularity universality classes:

$n < n_* : \quad c_*^+$

$n > n_* : \quad c_*^{-\infty}$

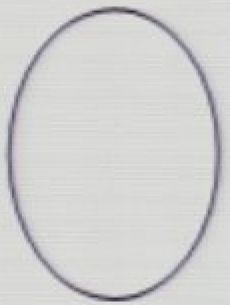
$n = n_* : \quad c_*^-$

Finite size correction: ε-expansion

- Cannot use Gaussian result at leading order! (boundary correction)

$$S_b = \frac{c}{2}\int d^{d-1}x_\perp\, \phi^2(r=0, x_\perp)$$

$\log \frac{Z_n}{Z^n} =$



free theory

$+$

boundary correction

$$= N\left[\frac{\pi}{6}\left(\frac{1}{n} - n\right) - \frac{c}{2n}\right] g(\varphi, \varphi)$$

- Set c to the fixed point value $c = c_*^+$

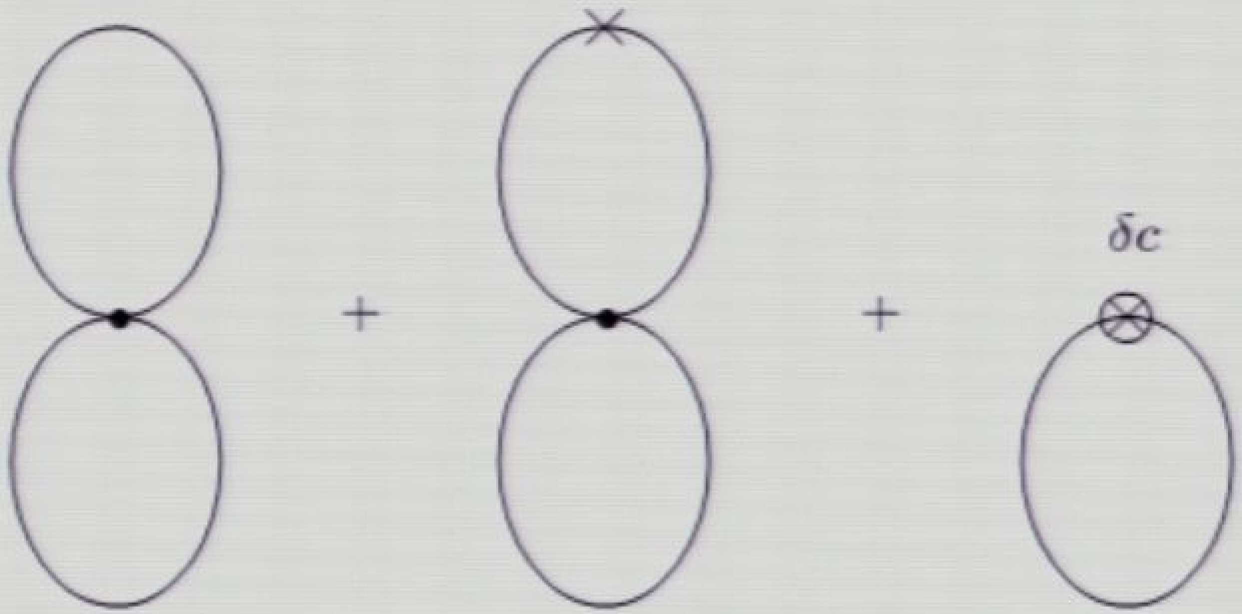
$1 - n \gg \epsilon: \quad c_*^+ \sim \sqrt{(1-n)\epsilon}$ - boundary correction is subleading

$|1 - n| \ll \epsilon: \quad c_*^+ \approx -\frac{2\pi}{3}(n-1)$ - boundary correction cancels with free theory!

Beyond leading order in ε

- **Need partition function in finite geometry to** $O(u^1)$

$$\log \frac{Z_n}{Z^n} =$$



- **Need** $\beta(c)$ **to** $O(u^2)$

$$\beta(c_r) = (N+2)u_r\left(1 - \frac{7}{2}\frac{u_r}{8\pi^2}\right)\frac{n-1}{12\pi} + (N+2)\frac{u_r}{8\pi^2}\left(1 - \frac{5}{2}\frac{u_r}{8\pi^2}\right)c_r$$

Calculating $\beta(c)$ to $O(u^2)$

$$S_b = \frac{c}{2}\int d^{d-1}x_\perp \, \phi^2(r=0, x_\perp)$$

- Calculations simplify in the limit, $n \to 1$ relevant for entanglement entropy

  $c_*^+ \sim n-1$ (keep only linear terms in c , $n-1$)

- Can use dimensional regularization (and minimal subtraction)!

$$\beta(c_r) = A(u_r) + \eta_2(u_r)c_r$$

effect of conical singularity

$$A \sim n-1$$

bulk renormalization of ϕ^2 operator

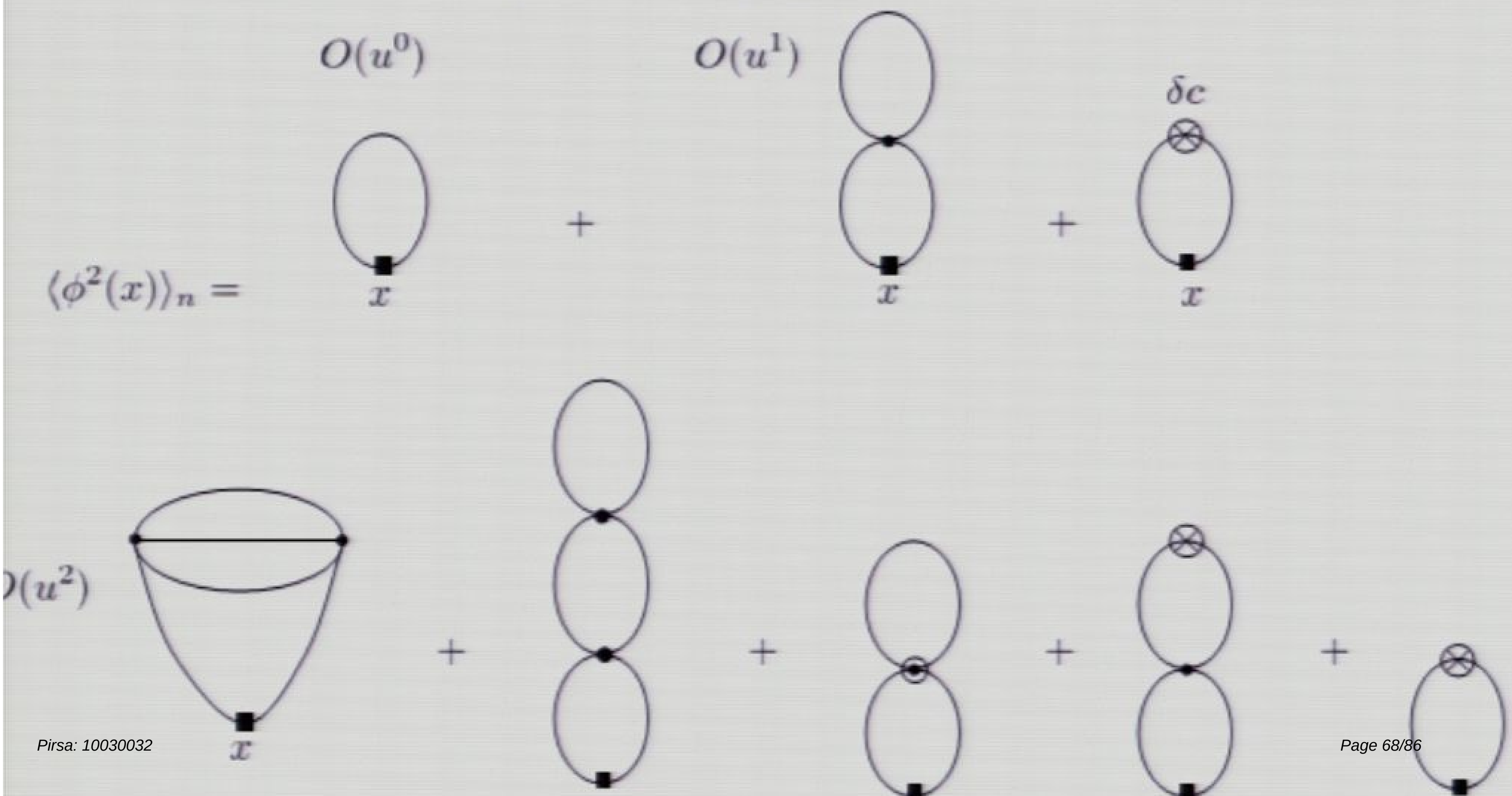
$$\frac{1}{\nu} = 2 + \eta_2(u_{r*})$$

$$\eta_2(u_r) = -(N+2)\frac{u_r}{8\pi^2}\left(1 - \frac{5}{2}\frac{u_r}{8\pi^2}\right)$$

Calculating β(c) to O(u^2)

$$\beta(c_r) = A(u_r) + \eta_2(u_r)c_r$$

- Can determine $A(u_r)$ from $\langle \phi^2(x)_r \rangle_n$ at the critical point



Calculating β(c) to O(u^2)

- Need propagators in $D=4-\epsilon$ on n-sheeted Riemann surface

$D=4:$

$$G_n(r,r',\theta,\vec{x}_\perp)=\frac{\sinh(\eta/n)}{8\pi^2 nrr'\sinh\eta(\cosh(\eta/n)-\cos(\theta/n))},\quad \cosh\eta=\frac{r^2+r'^2+\vec{x}_\perp^2}{2rr'}$$

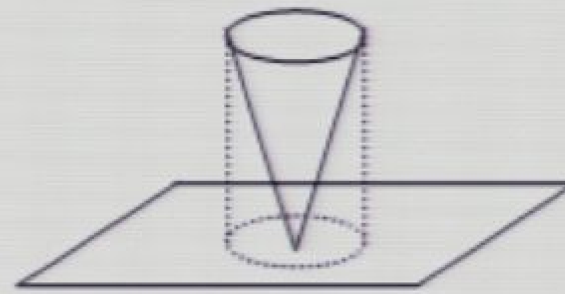
$$\langle\phi^2(x)\rangle_n-\langle\phi^2(x)\rangle_1=\frac{N}{48\pi^2r^2}\left(\frac{1}{n^2}-1\right)$$ - only input of Riemann surface to $O(u^1)$

- Insufficient for present $O(u^2)$

Expansion in n-1

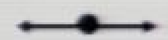
$$S = \int d^D x \sqrt{\det g} \left(g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{t}{2}\phi^2 + \frac{u}{4}\phi^4 \right)$$

$$(\mathrm{x}, \tau) = r(\cos\theta, \sin\theta)$$



$$ds^2 = dr^2 + r^2 d\theta^2 + dx_\perp^2 \qquad \theta \sim \theta + 2\pi n$$

- Rescale coordinates



$\times \frac{1}{\sqrt{n}}$

$$\tilde{r} = \sqrt{n} r, \quad \varphi = \theta / n \qquad \varphi \sim \varphi + 2\pi$$

$$\tilde{\tau} = \tilde{r}\cos\varphi, \quad \tilde{\mathrm{x}} = \tilde{r}\sin\varphi$$

- normal one-sheeted plane

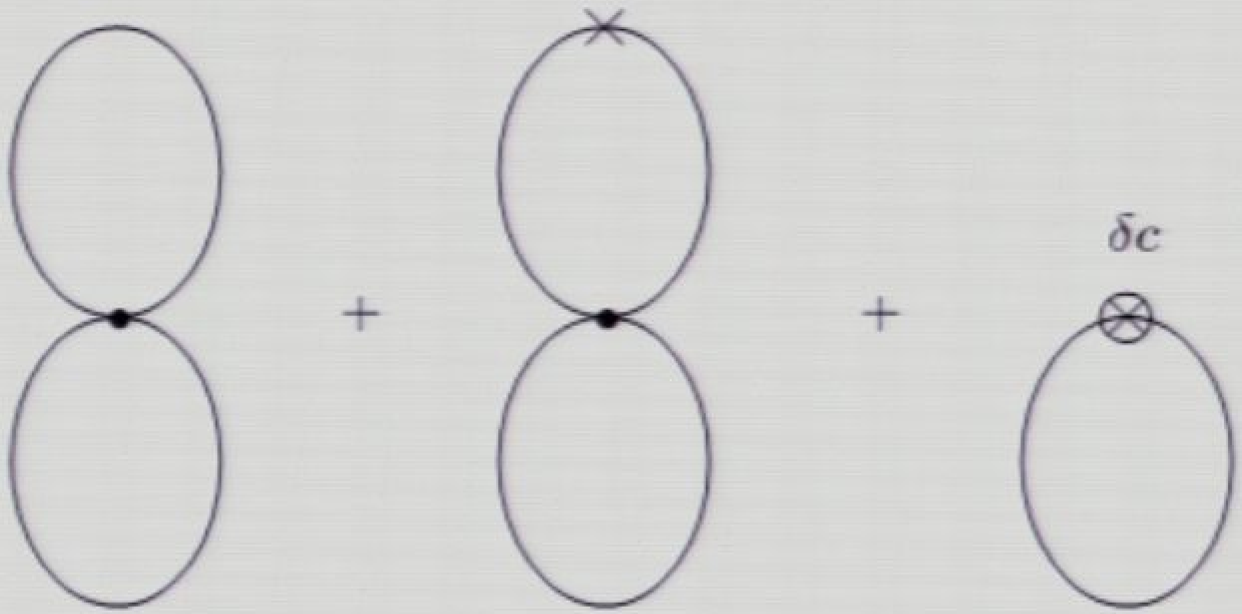
$$g_{\alpha\beta} = n\delta_{\alpha\beta} + \left(\frac{1}{n} - n\right)\frac{\tilde{x}_\alpha \tilde{x}_\beta}{\tilde{x}^2} \qquad \det g = 1$$

$$\delta G_n(x, x') = (n-1)\int d^D y \left(\delta_{\alpha\beta} - \frac{2y_\alpha y_\beta}{y_\parallel^2}\right)\partial_\alpha G_1(x-y)\partial_\beta G_1(x'-y)$$

Finite size correction: beyond leading order in ε

- **Contributions to partition function to** $O(u^1)$

$$\log \frac{Z_n}{Z^n} =$$



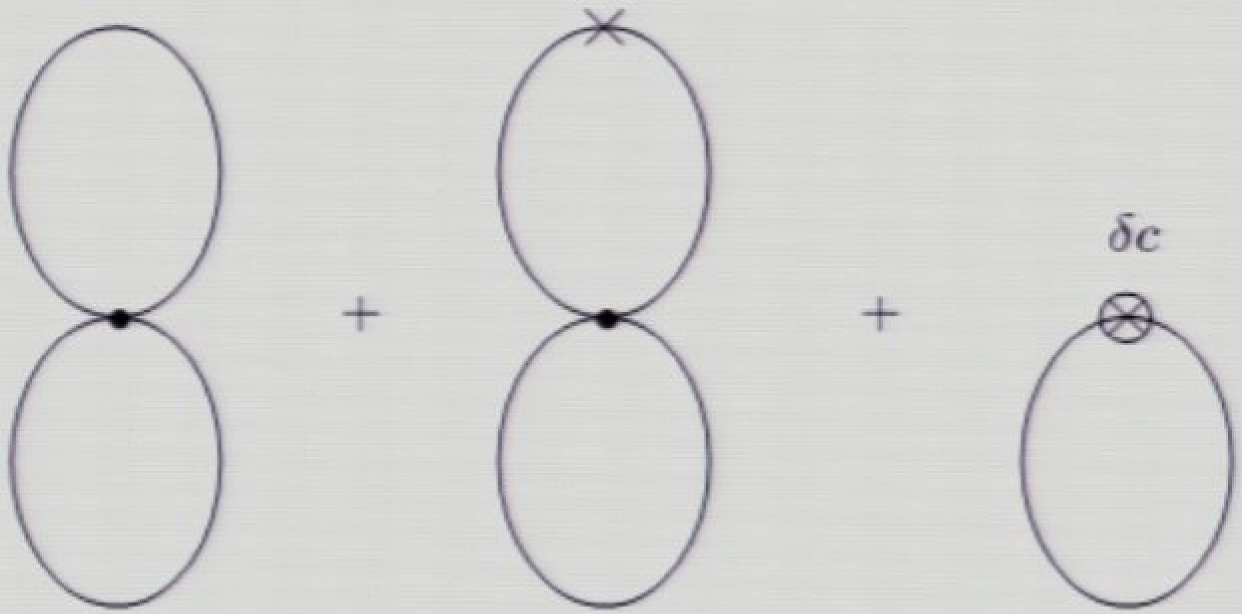
$$\gamma = \frac{N\pi\epsilon}{3(N+8)} g(\varphi,\varphi)$$ - **Wilson-Fisher FP**

$$\gamma = \frac{N\pi}{3} g(\varphi,\varphi)$$ - **Gaussian FP** $d = 3$

Finite size correction: beyond leading order in ε

- Contributions to partition function to $O(u^1)$

$$\log \frac{Z_n}{Z^n} =$$



$$\gamma = \frac{N\pi\epsilon}{3(N+8)} g(\varphi,\varphi)$$

- Wilson-Fisher FP

$$\gamma = \frac{N\pi}{3} g(\varphi,\varphi)$$

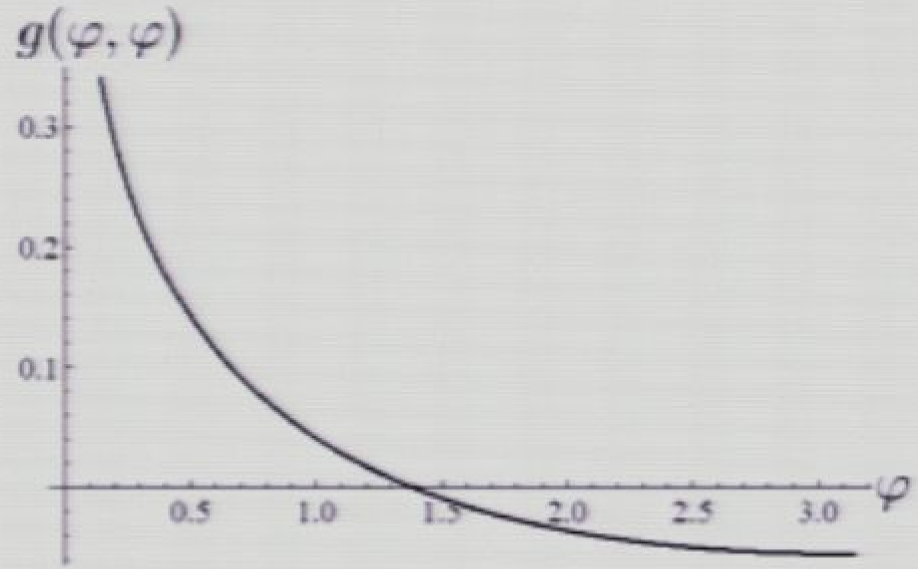
- Gaussian FP $d = 3$

Twist dependence

$$\gamma = \frac{N\pi\epsilon}{3(N+8)} g(\varphi,\varphi)$$

$$g(\pi,\pi) = -\frac{1}{4\pi}\log 2$$

$$g(\varphi,\varphi) \to -\frac{1}{2\pi}\log\varphi$$



Breakdown of perturbation theory for $\varphi \to 0$

Zero twist

- Breakdown of perturbation theory for $\varphi \to 0$

$$\phi(x) = \frac{1}{L^{(D-2)/2}} \phi_0(x_\parallel) e^{i\vec{\varphi}\cdot\vec{x}_\perp/L} + \tilde{\phi}(x)$$

"quasi-zero mode" all other modes

- Effective action for the zero mode

$$S_0 = \int d^2x_\parallel \left(\frac{1}{2}(\partial_\parallel \vec{\phi}_0)^2 + \frac{1}{2} m_{D=2}^2 \vec{\phi}_0^2 + \frac{u_{D=2}}{4} (\vec{\phi}_0^{\,2})^2 \right)$$

$m_{D=2}^2 \sim \frac{\vec{\varphi}^2}{L^2}$ perturbative if $\frac{u_{D=2}}{m_{D=2}^2} \ll 1$ $\varphi^2 \ll u \sim \epsilon$

$u_{D=2} \sim \frac{u}{L^2}$

Guess: $\gamma \approx -\frac{N\epsilon}{12(N+8)} \log \epsilon$

Correlation length correction

- $$S_n = C_n(t)\frac{L^{d-1}}{a^{d-1}} + r_n \frac{L^{d-1}}{\xi^{d-1}} \qquad r_n \sim O(N^2)$$

  $$C_n + C_n' t \qquad t^{\nu(d-1)} \qquad \nu(d-1) = 1 - \nu_1$$

  $$\nu_1 = \frac{3\epsilon}{N+8}, \qquad d = 3 - \epsilon, \qquad \nu_1(d=2) = \frac{32}{3\pi^2 N}, \quad N \to \infty$$

- -Use RG to separate singular and analytic contributions

  $\log \frac{Z_n}{Z_1^n} \sim N$ - at each order in 1/N

  $\log \frac{Z_n}{Z_1^n} \sim N^2$ - from RG

limits $N \to \infty$ and $\xi \to \infty$ do not commute!

Violation of large N counting

- Violations of naïve large N counting are quite rare
  - Witten's resolution of U(1) problem in QCD:

E. Witten (1979)

$$0 = \int d^4x \langle F\tilde{F}(x) F\tilde{F}(0) \rangle = -\sum_{\text{glueballs}} \frac{N^2 a_n^2}{M_n^2} - \sum_{\text{mesons}} \frac{N c_n^2}{m_n^2}$$

$$m_{\eta'}^2 \sim \frac{1}{N}$$

  - Theories involving fermi surfaces interacting with bosonic modes

S. S. Lee (2009)

To do list

- Symmetry broken phase

$$S = C(t)\frac{L^{d-1}}{a^{d-1}} + r\frac{L^{d-1}}{\xi^{d-1}} \qquad \xi^{-1} \propto t^{\nu}$$

$$S_+ = C(t)\frac{L^{d-1}}{a^{d-1}} + B_+ t^{\nu(d-1)} L^{d-1}, \;\; t > 0$$

$$S_- = C(t)\frac{L^{d-1}}{a^{d-1}} + B_- |t|^{\nu(d-1)} L^{d-1}, \;\; t < 0$$

$\frac{B_-}{B_+}$ - universal

To do list

- **Upper critical dimension** $(d = 3)$ $\quad \nu = \frac{1}{2}$

$$S = C(t)\frac{L^{d-1}}{a^{d-1}} + r\frac{L^{d-1}}{\xi^{d-1}}$$

$$C + C't \qquad\qquad t^{\nu(d-1)} \to t$$

- **Additional log is possible**

$$S = C(t)\frac{L^2}{a^2} + b\frac{L^2}{\xi^2}\log \xi/a$$

$b-$ **universal**

Gaussian theory: $b = -\frac{N}{24\pi}$

Interacting theory?

To do list

- **Symmetry broken phase**

$$S = C(t)\frac{L^{d-1}}{a^{d-1}} + r\frac{L^{d-1}}{\xi^{d-1}} \qquad \xi^{-1} \propto t^{\nu}$$

$$S_+ = C(t)\frac{L^{d-1}}{a^{d-1}} + B_+ t^{\nu(d-1)} L^{d-1}, \;\; t > 0$$

$$\frac{B_-}{B_+} \quad \text{- universal}$$

$$S_- = C(t)\frac{L^{d-1}}{a^{d-1}} + B_- |t|^{\nu(d-1)} L^{d-1}, \;\; t < 0$$

To do list

- Upper critical dimension $(d = 3)$ $\quad \nu = \frac{1}{2}$

$$S = C(t)\frac{L^{d-1}}{a^{d-1}} + r\frac{L^{d-1}}{\xi^{d-1}}$$

$$C + C't \qquad\qquad t^{\nu(d-1)} \to t$$

- Additional log is possible

$$S = C(t)\frac{L^2}{a^2} + b\frac{L^2}{\xi^2}\log \xi/a$$

$b-$ universal

Gaussian theory: $b = -\frac{N}{24\pi}$

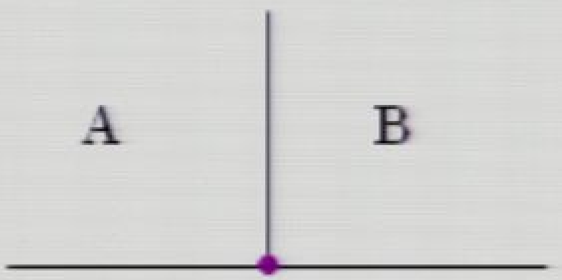
Interacting theory?

To do list

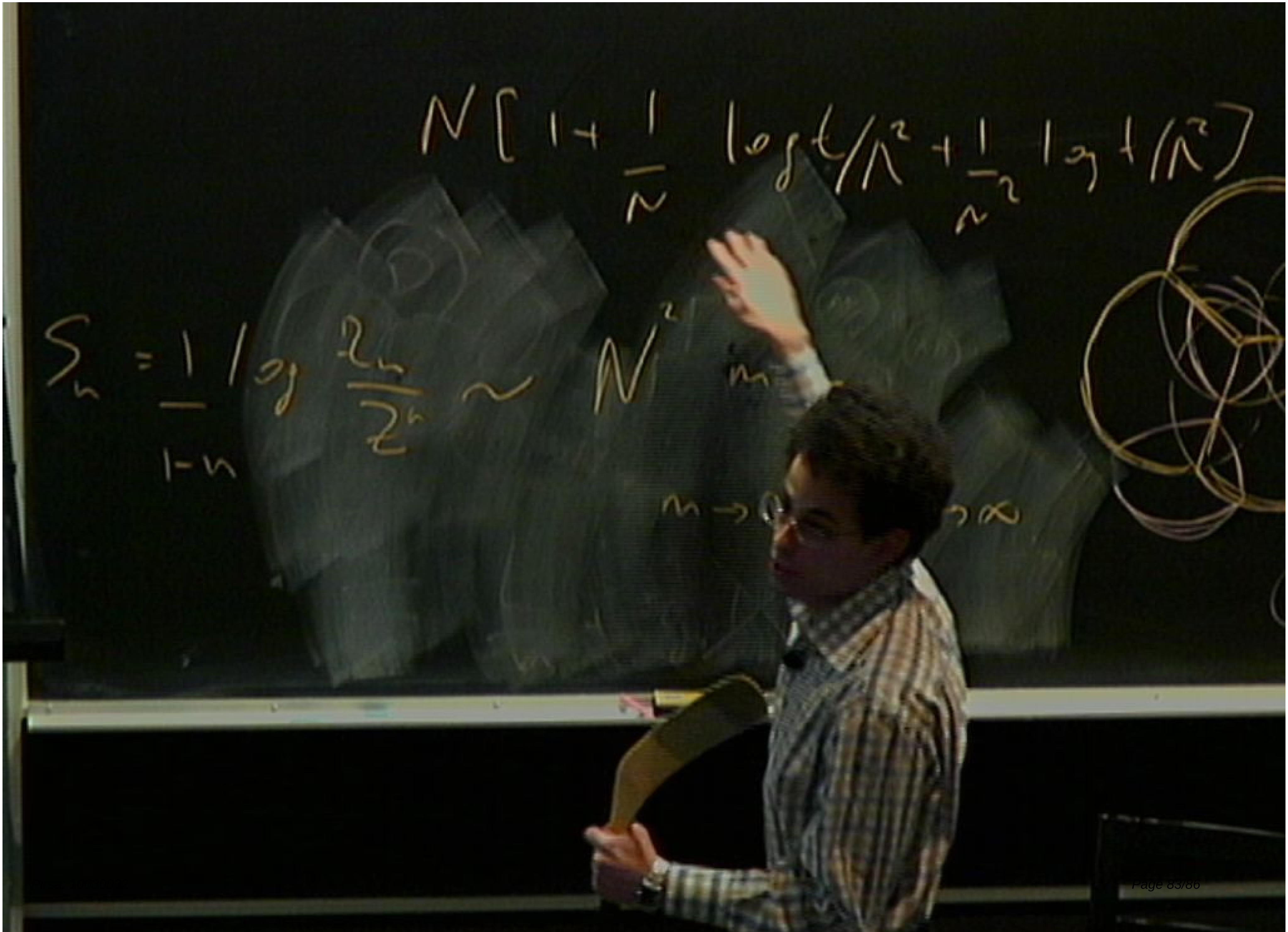
- γ in large-N expansion (solve gap equation numerically)
- Corners/endpoints

$$S = C\frac{L^{d-1}}{a^{d-1}} + c\log L/a$$

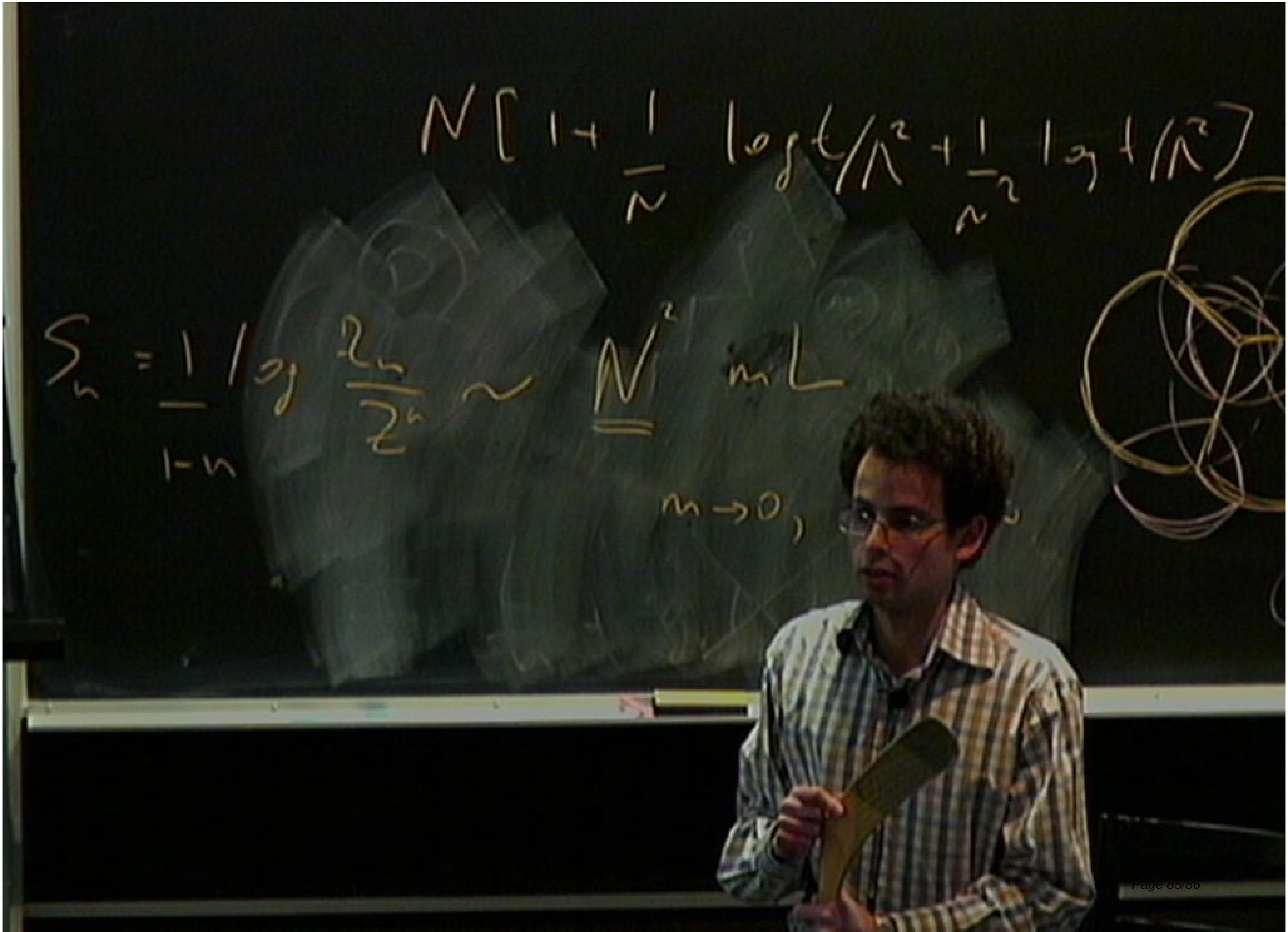
c - universal

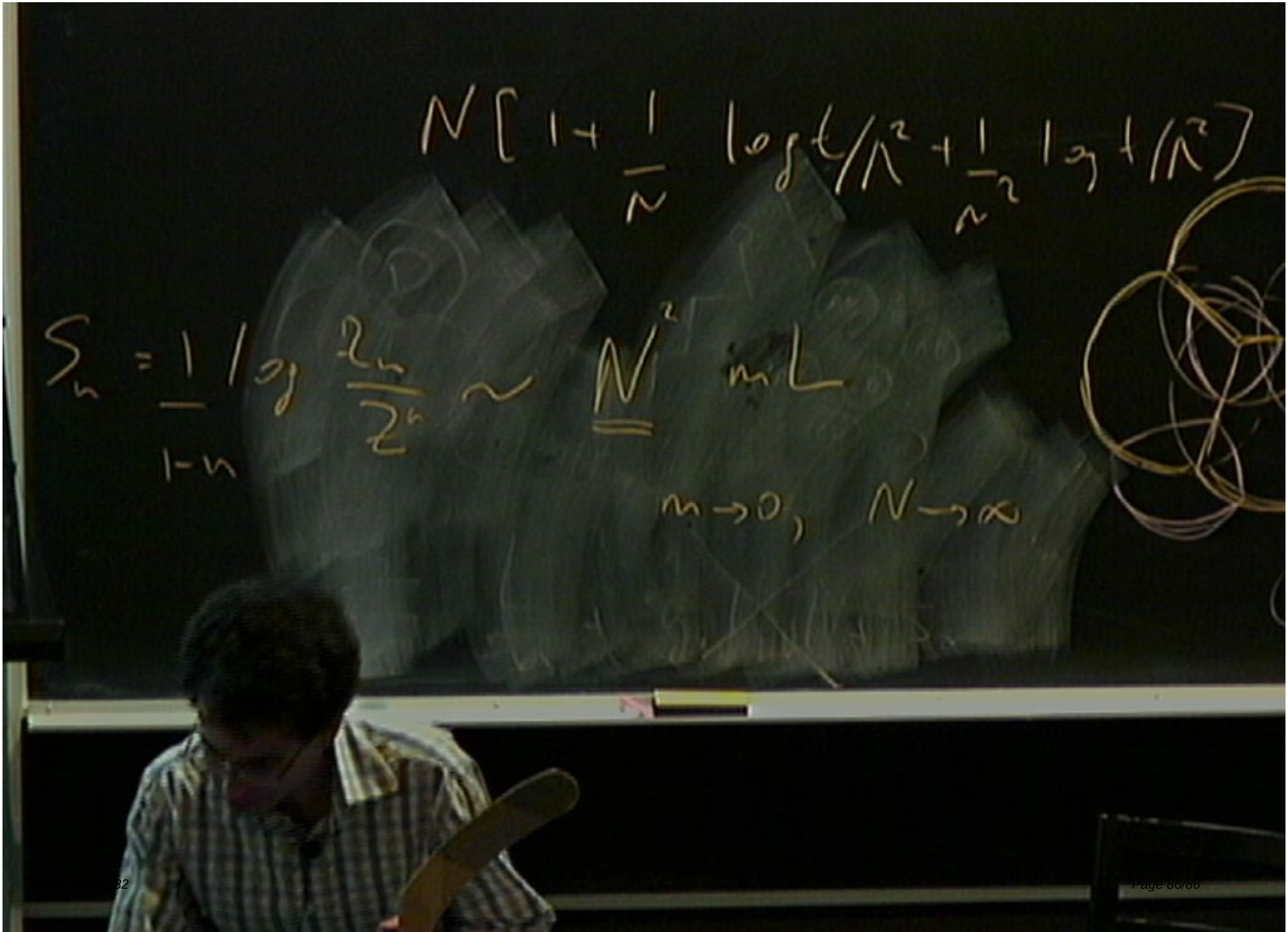


Thank you!


N [1 + 1/N log t/Λ² + 1/N² log t/Λ²]
S_n = 1/(1-n) log Z_n/Z^n ~ N²
n → 0
→ ∞

Thank you!


N[1 + 1/N log t/Λ^2 + 1/N^2 log t/Λ^2]
S_n = 1/(1-n) log 2_n/2^n ~ N^2 mL
m → 0,


N [1 + 1/N log t/Λ² + 1/N² log t/Λ²]
S_n = 1/(1-n) log Z_n/Z^n ~ N² mL
m→0, N→∞