Title: Diffeomorphism symmetry, triangulation independence and constraints in discrete gravity

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Abstract: Diffeomorphism symmetry is the underlying symmetry of general relativity and deeply intertwined with its dynamics. The notion of diffeomorphism symmetry is however obscured in discrete gravity, which underlies most of the current quantum gravity models. We will propose a notion of diffeomorphism symmetry in discrete models and find that such a symmetry is weakly broken in many models. This is connected to the problem of finding a consistent canonical dynamics for discrete gravity. Finally we will discuss methods to construct models with exact symmetries and elaborate on the connection between diffeomorphism symmetry and triangulation independence.

Motivation

- In the continuum diffeomorphism symmetry is deeply entangled with the dynamics of the theory.
  - canonical theory: dynamics defined by constraints
  - evolution as time reparametrization
  - writing the most general diffeomorphism invariant action

- implementing diffeomorphism symmetry into quantum gravity model could ensure that general relativity (+ more) emerges in semiclassical limit

Questions

- Is there a notion of diffeomorphism symmetry in discrete models?

- Can it help us to adress:
    - ambiguities and anomalies, lattice effects
    - path integral measure (for labels and triangulations)
    - sum over triangulations?

- Relation to triangulation independence?

Overview

A. Criterium for gauge symmetries

B. Do we have gauge symmetries in discrete gravity?

C. Why do we care?

D. Improving the dynamics with renormalization

E. Perturbative Expansion

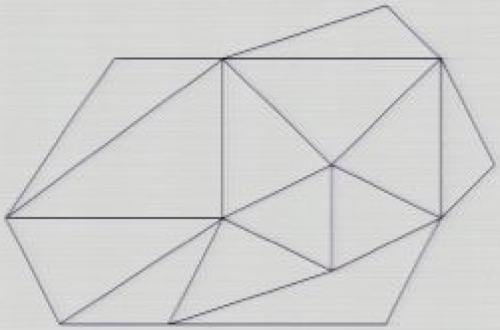
F. Repercussions for canonical formalism

G. Conclusions

Set up: Regge calculus

(classical theory corresponding to spin foam models, lattice loop quantum gravity)

- approximate space time by piecewise flat triangulation
- length variables on edges fix geometry
- discrete action defines dynamics



$$S_{cont} = \int d^D x \sqrt{g}\left(\tfrac{1}{2}R - \Lambda\right)$$

$$\Big\Downarrow$$

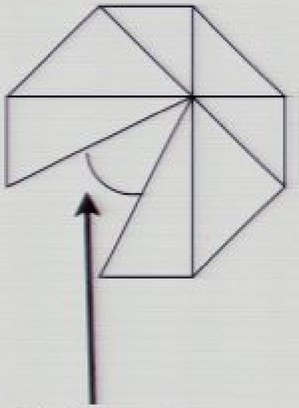
$$S_{discr} = \sum_{\text{hinges}\, h} F_h \epsilon_h - \Lambda \sum_{\text{simplices}\, \sigma} V_\sigma$$

4d:triangles
3d: edges

volume of triangle/edge

deficit angle

volume of 4-simplex/ tetrahedron



deficit angle

A. and B.

Is there a notion of diffeomorphism symmetry in discretized actions?

A. Criterium for gauge symmetries

- criterium: non-uniqueness of solutions for fixed boundary conditions

  - $$\det\left(\frac{\partial^2 S}{\partial x^i \partial x^j}\right)\Big|_{\text{solution}} = 0$$

- existence of symmetries depends on dynamics (that is the action)!
- different solutions might have gauge orbits of different size
- invariance of action not sufficient for gauge symmetry

- criterium relevant for
  - canonical analysis
  - perturbative expansion
  - counting of physical degrees of freedom

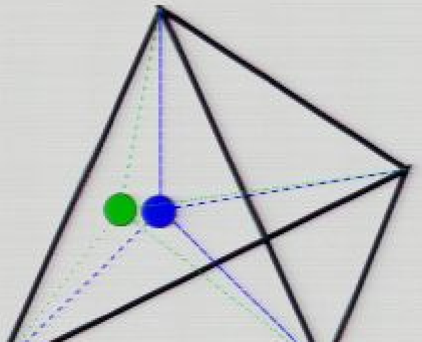
B. Gauge symmetries in Regge calculus?

- for boundary conditions leading to flat solutions: non-uniqueness of solutions!
  ⇒there are gauge symmetries!

- 3d (vanishing cosmological constant): all boundary conditions lead to flat solutions
  ⇒gauge symmetries for all configurations [Freidel, Louapre '02]

- 4d (vanishing cosmological constant): some boundary conditions lead to flat solutions
  ⇒gauge symmetries for these configurations

- gauge modes correspond to changing position of vertices on flat background
  ⇒matched to continuum diffeomorphism symmetry in lineariziation [Rocek and Williams 81]



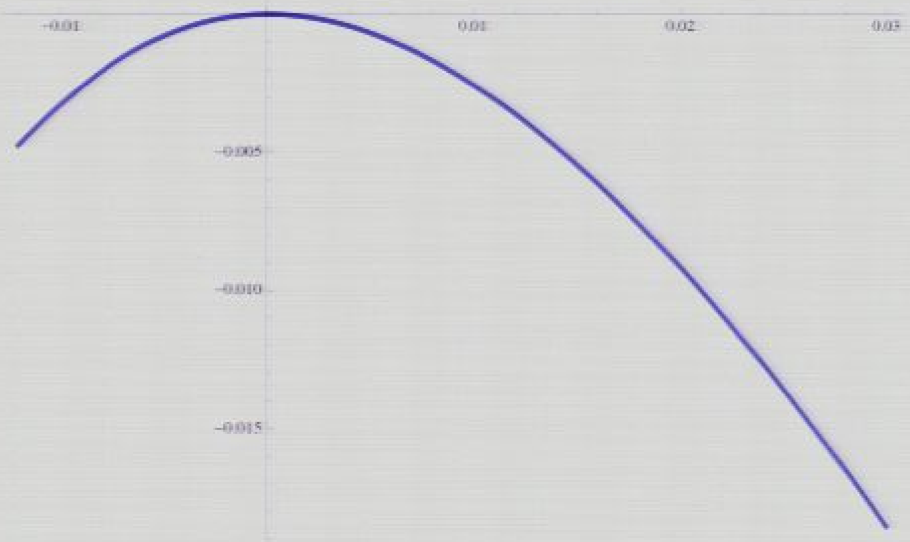
vertex translation acting on
flat solution

Hessian of action evaluated on
flat solutions has null modes

B. Gauge symmetries in Regge calculus?

For (a) curved solution: symmetries are broken.

[Bahr, BD 09]



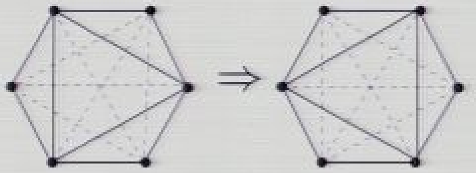
lowest eigenvalues of Hessians as function of deviation parameter from 4d flat solution (curvature)

Symmetry is broken, effect quadratic in curvature.

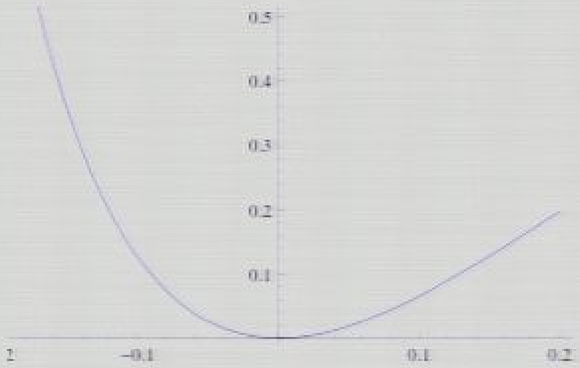
Non-invariance under 3-3 Pachner moves

[Bahr, BD, Hoehn wip]

4d Regge action already invariant under 1-5, 5-1, 2-4 and 4-2 moves.



Triangulation with three 4-simplices and spherica
boundary. There are no inner edges.
3-3 move redefines inner triangle.



Difference of actions evaluated on the two configurations as function of the deficit angle on the inner triangle.

Effect quadratic in curvature.

C. Why do we care?

- exact symmetries ⇒ exact (first class) constraints

  [Gambini & Pullin et al 03-05, et al, Bahr & BD 09, BD & Hoehn 09]

- anomalies in quantization (by regularization) vs fixing of ambiguities

  [for instance Perez & Pranzetti 10 in 3d with cosmological constant]

- perturbative expansion around flat geometries is very subtle if symmetries are broken

  [related: Horava-Lifshitz gravity]

- path integral computation: no propagator for pseudo gauge modes

- condition on measure in path integral

- action with exact diffeomorphism symmetry hopefully related to triangulation independent Hamilton-Jacobi functional: control sum over triangulation!

D. Is there a discretization with exact symmetry?

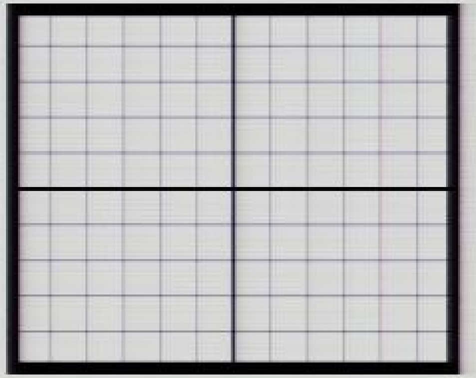
Gauge symmetries are properties of the (discrete) action.
⇒Improve the action.

Construct better actions

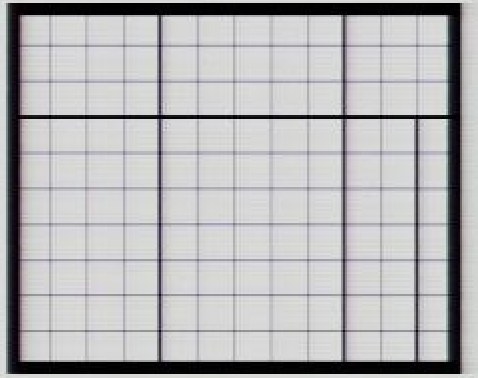
- by renormalization group transformation:
  - fine grain and integrate out fine grained degrees of freedom
  - obtain effective action on coarse grained lattice, capturing dynamics of fine grained lattice

Question: Do we regain local gauge symmetries from continuum?

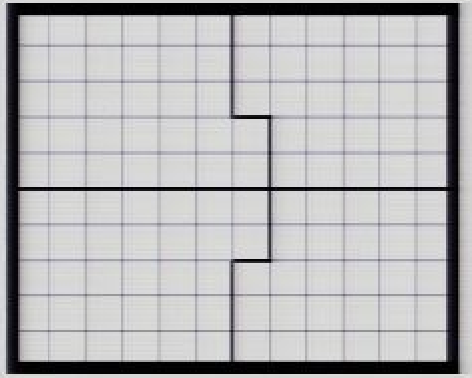
Lattice deformation algebra



coarse graining to
intermediate lattice



alternative
intermediate lattice



vertex
translation

- Dirac's hypersurface deformation algebra (Hamiltonian and diffeomorphism constraints) can be derived as condition that final evolved state does not depend on intermediate hypersurface [Teitelboim '76]
- lattice deformation algebra: independence from intermediate lattice
- subgroup: vertex translation algebra could be similar to hypersurface deformation algebra

Examples

1d discretized systems, perturbatively and non-perturbatively

[quantum: Bahr, Steinhaus & BD wip]

It works!

3d Regge calculus with cosmological constant

[Bahr & BD 09]

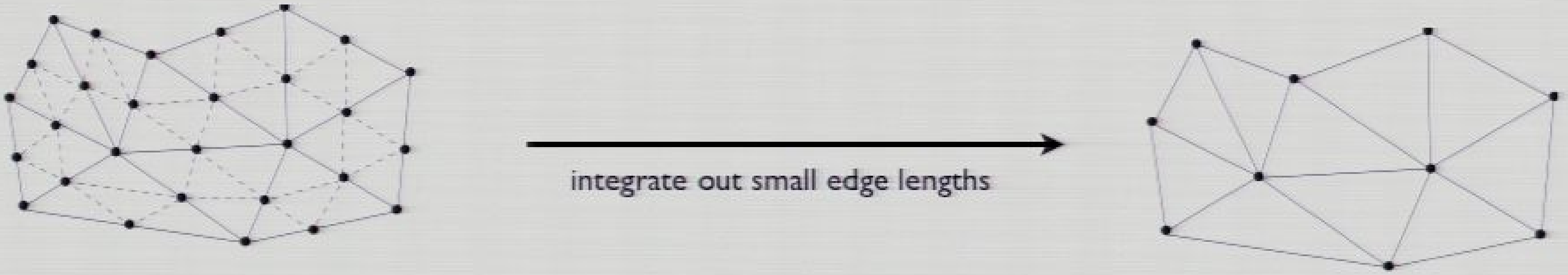
It works!

4d Regge calculus, perturbative expansion

[BD & Hoehn 09]
[Bahr & BD &He wip]

3d Regge calculus with matter

[Banisch & BD wip]



integrate out small edge lengths

3d Regge with cosmological constant

$$S_{\mathcal{T}} = \sum_e l_e \epsilon_e - \Lambda \sum_\sigma V_\sigma$$

action for flat simplices

approximate symmetries, triang. dependent

3d Regge with curved simplices

[B.Bahr, BD 09]

$$S^\kappa_{\mathcal{T}} = \sum_E L_E \epsilon^\kappa_E + 2\kappa \sum_\sigma V^\kappa_\Sigma$$

action for simplices with curvature

$$\kappa = \Lambda$$

exact symmetries, triang. independent

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$$S = \sum_e l_e \epsilon_e - \Lambda \sum_\sigma V_\sigma + \sum_E \alpha_E (L_E - \sum_{e \subset E} l_e)$$

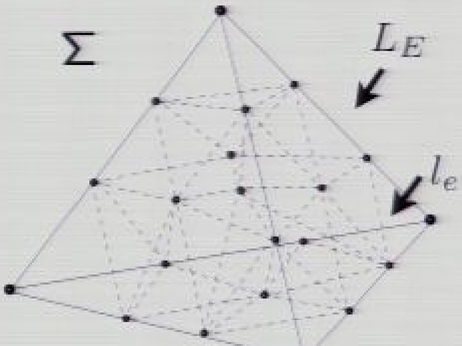
equations of motion

$$0 = L_E - \sum_{e \subset E} l_e$$

$$0 = \epsilon_e - \Lambda \sum_\sigma \frac{\partial V_\sigma}{\partial l_e} - \sum_{E \supset e} \alpha_E$$

resum : $\sum_e l_e$

$$0 = \sum_e l_e \epsilon_e - 3\Lambda \sum_\sigma V_\sigma - \sum_E \alpha_E L_E$$



$\Rightarrow$

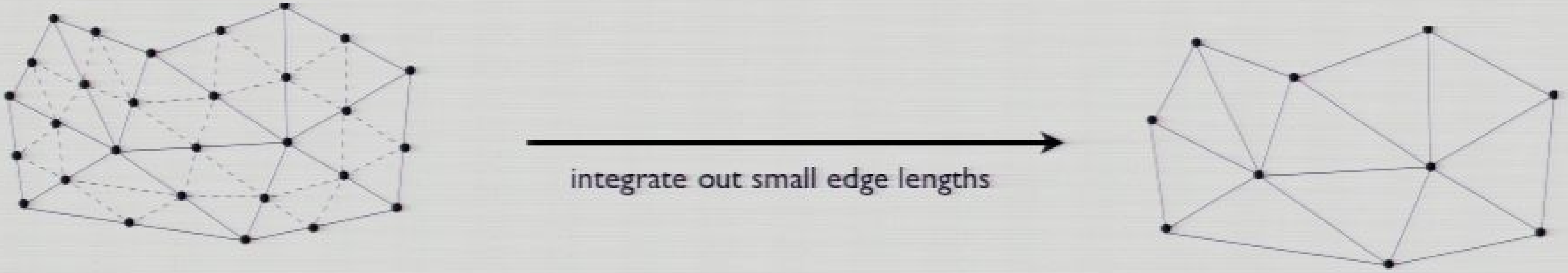
$$S_{|solution} = \sum_E L_E \alpha_E + 2\Lambda \sum_\Sigma V_\Sigma$$

with

$$V_\sigma = \sum_{\sigma \subset \Sigma} V_\sigma$$

$$\alpha_E = \epsilon_e - \Lambda \sum_\sigma \frac{\partial V_\sigma}{\partial l_e}$$

In the infinite refinement limit deficit angles and volume for homogeneously curved tetrahedra.
Obtain fix point action.



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[B.Bahr, BD 09]

$$S_{\mathcal{T}}^\kappa = \sum_E L_E \epsilon_E^\kappa + 2\kappa \sum_\sigma V_\Sigma^\kappa$$

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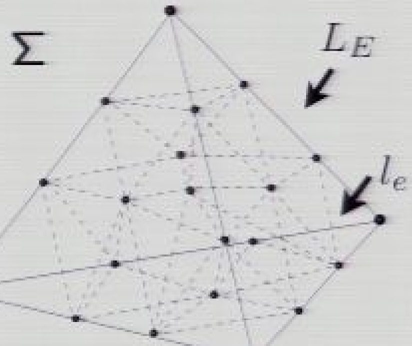

⇒

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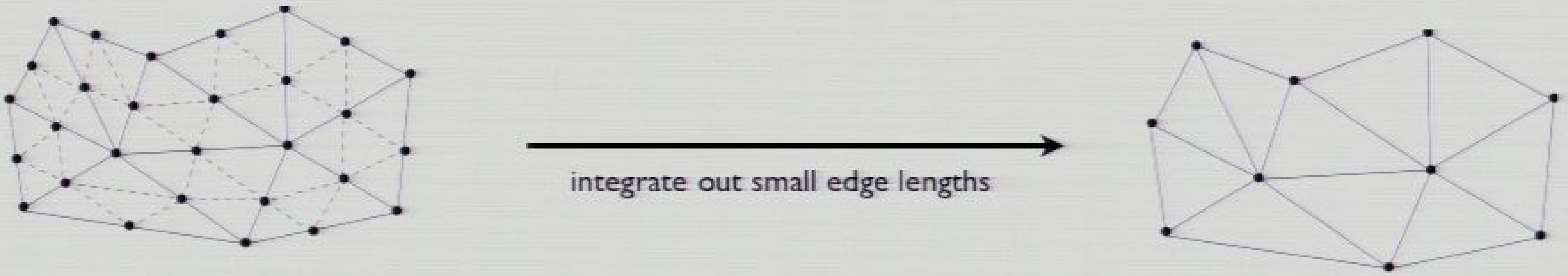


3d Regge with cosmological constant

3d Regge with curved simplices

[B.Bahr, BD 09]

$$S_T = \sum_e l_e \epsilon_e - \Lambda \sum_\sigma V_\sigma$$

$$S_T^\kappa = \sum_E L_E \epsilon_E^\kappa + 2\kappa \sum_\sigma V_\Sigma^\kappa$$

action for flat simplices

action for simplices with curvature

$$\kappa = \Lambda$$

approximate symmetries, triang. dependent

exact symmetries, triang. independent

1d reparametrization invariant systems

continuum:

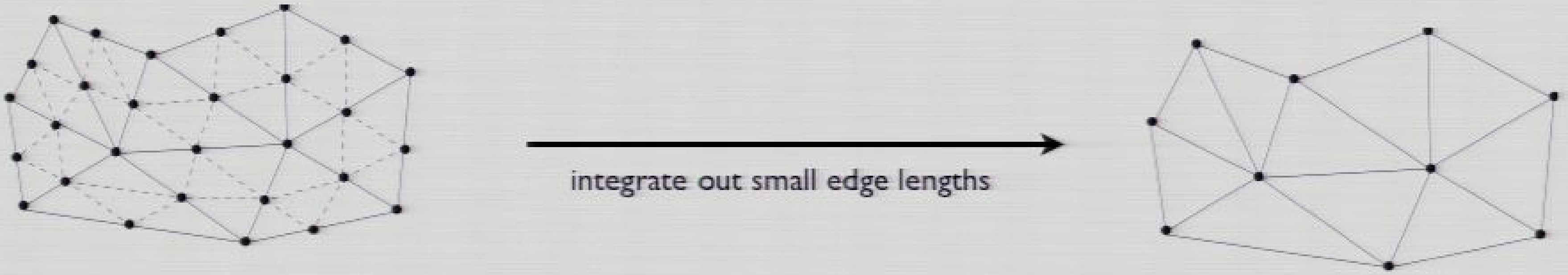
- take q and t as variables
- use auxilary parameter evolution parameter s
- solutions $t(s), q(s)$ invariant under reparametrizations in s

$$L = t' \left(\frac{m}{2} \frac{q'^2}{t'^2} - V(q) \right)$$

discretization $\Big\downarrow$ $s \longrightarrow n$

$$L(n, n+1) = (t_{n+1} - t_n) \left(\frac{m}{2} \frac{(q_{n+1} - q_n)^2}{(t_{n+1} - t_n)^2} - V(\frac{1}{2} q_n + \frac{1}{2} q_{n+1}) \right)$$

- vertex translation symmetry for $V = 0$
- symmetry broken for $V \neq 0$ [Gambini, Pullin '03, Marsden, West '01]



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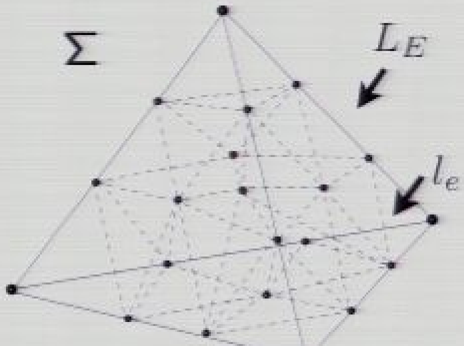
$$\begin{aligned} S &= \sum_e l_e \epsilon_e - \Lambda \sum_\sigma V_\sigma + \sum_E \alpha_E (L_E - \sum_{e \subset E} l_e) \\ & \text{equations of motion} \\ 0 &= L_E - \sum_{e \subset E} l_e \\ 0 &= \epsilon_e - \Lambda \sum_\sigma \frac{\partial V_\sigma}{\partial l_e} - \sum_{E \supset e} \alpha_E \\ & \text{resum : } \sum_e l_e \\ 0 &= \sum_e l_e \epsilon_e - 3\Lambda \sum_\sigma V_\sigma - \sum_E \alpha_E L_E \end{aligned}$$



➡

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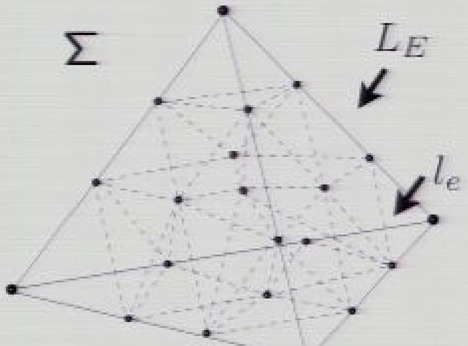

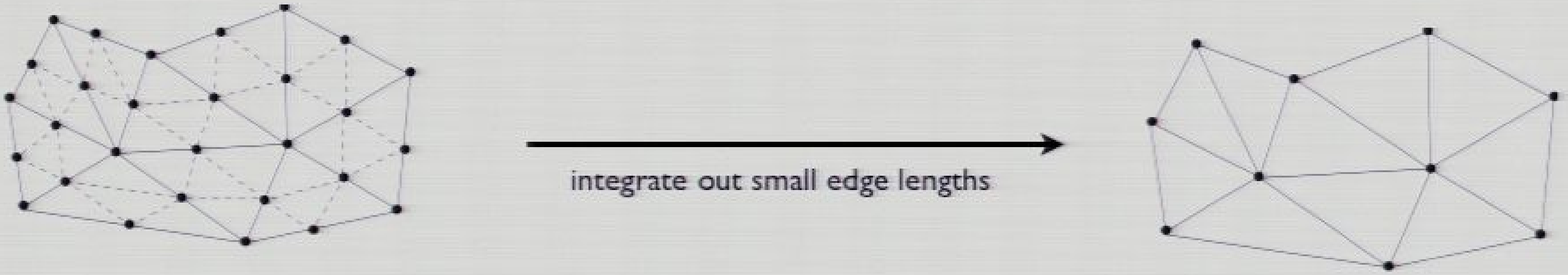

3d Regge with cosmological constant

3d Regge with curved simplices

[B.Bahr, BD 09]

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$$S^{\kappa}_{\mathcal{T}} = \sum_{E} L_E \epsilon^{\kappa}_E + 2\kappa \sum_{\sigma} V^{\kappa}_{\Sigma}$$

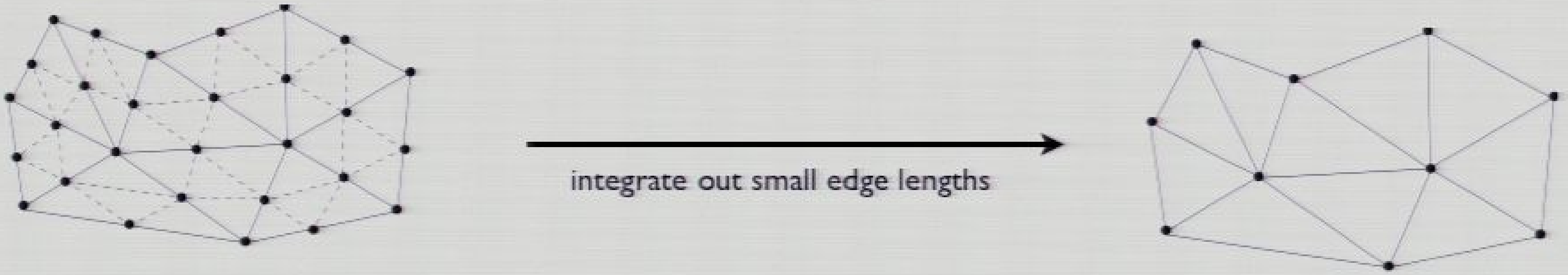
action for flat simplices

action for simplices with curvature

$$\kappa = \Lambda$$

approximate symmetries, triang. dependent

exact symmetries, triang. independent



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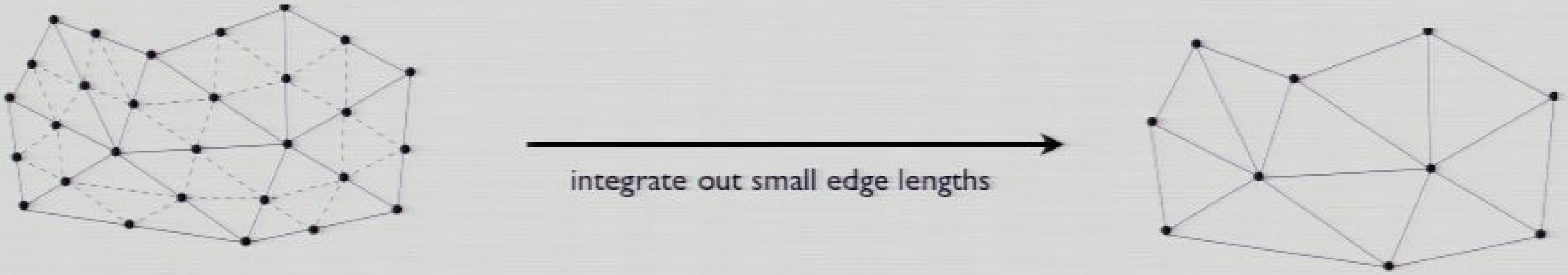

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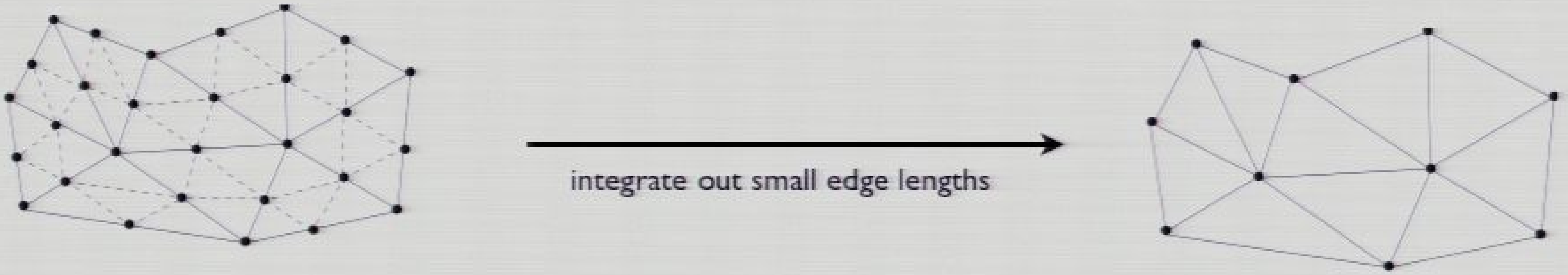

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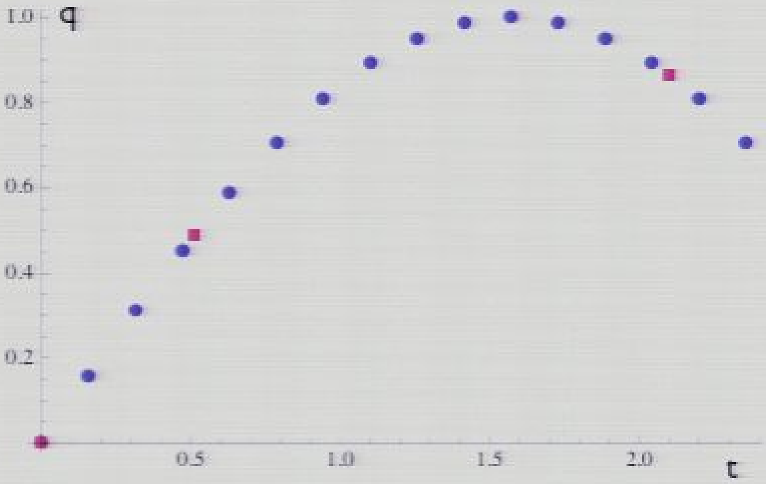
1d reparametrization invariant discrete systems

- There is always a discrete action with exact symmetries!
- trick: use the Hamilton-Jacobi functional of continuum theory as discrete action
- ⇒discrete theory captures exactly continuums dynamic
- can be obtained by integrating out almost all variables ("renormalization group flow") → refinement independent

$$
\begin{aligned}
S_e &= \sum_{n=0}^{N-1} S_{HJ}^{s_n,s_{n+1}}(t_n,q_n,t_{n+1},q_{n+1}) \\
&= \sum_{n=0}^{N-1} \int_{s_n}^{s_{n+1}} ds\; L(t(s),q(s)) \quad .
\end{aligned}
$$

continuums solution



Remark: piecewise linear approximation introduces errors

3d Regge with cosmological constant

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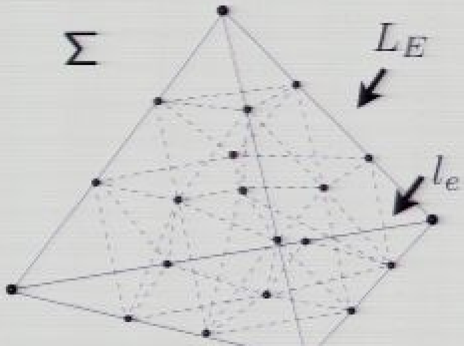

⇒

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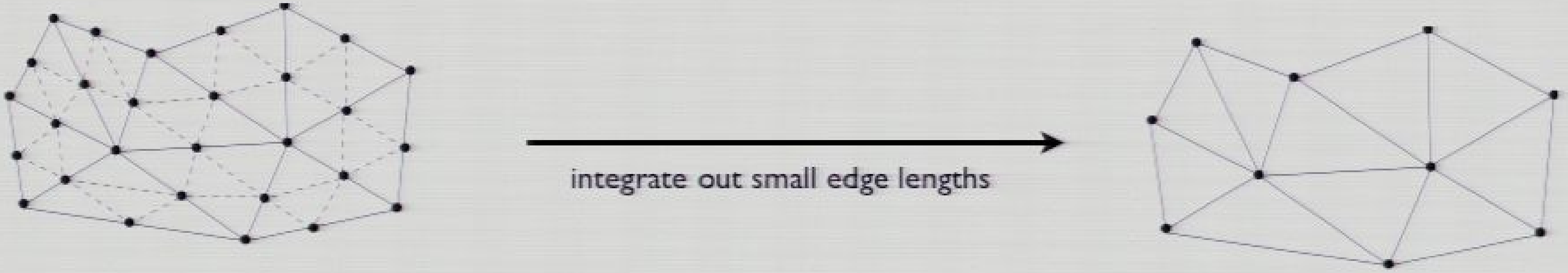

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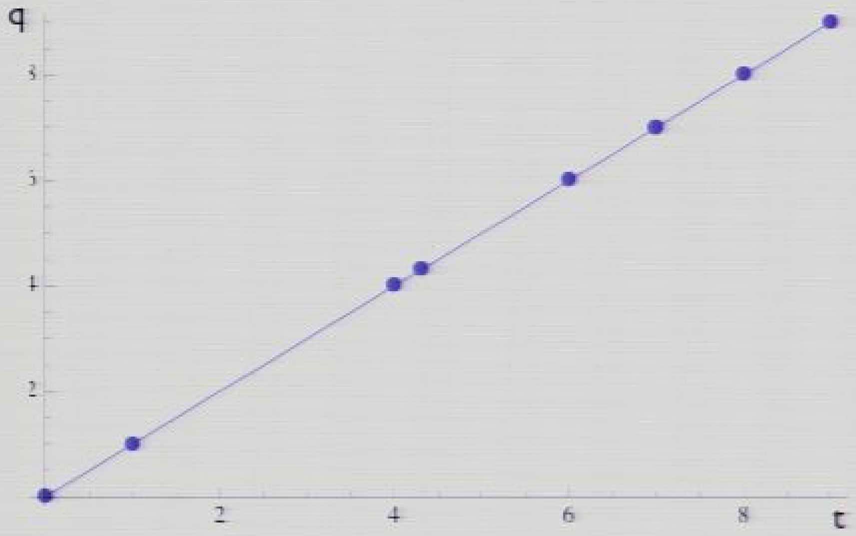
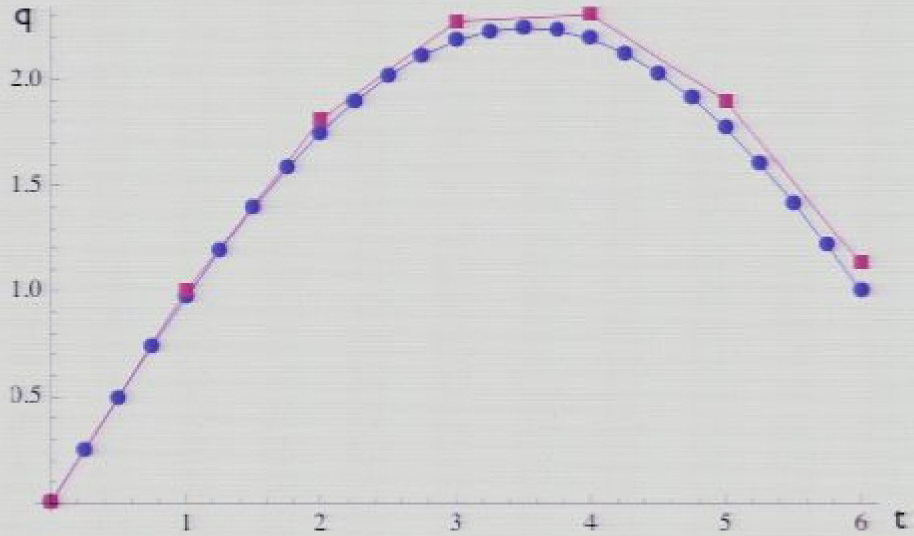
Examples





- vanishing potential
- position of vertices arbitrary
- one gauge mode
- refinement independent

- quadratic potential
- position of vertices fixed
- one pseudo gauge mode
- refinement dependent

Remark: piecewise linear approximation added by hand!

- linearization around solution: kinetic term of pseudo gauge mode vanishing: no propagator
- but gauge breaking in potential

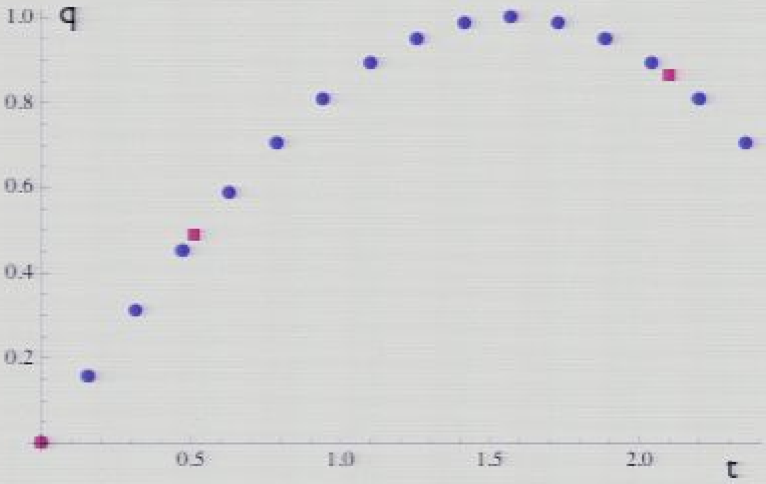
1d reparametrization invariant discrete systems

- There is always a discrete action with exact symmetries!
- trick: use the Hamilton-Jacobi functional of continuum theory as discrete action
- ⇒discrete theory captures exactly continuums dynamic
- can be obtained by integrating out almost all variables ("renormalization group flow") → refinement independent

$$S_e = \sum_{n=0}^{N-1} S_{HJ}^{s_n, s_{n+1}}(t_n, q_n, t_{n+1}, q_{n+1})$$

$$= \sum_{n=0}^{N-1} \int_{s_n}^{s_{n+1}} ds\ L(t(s), q(s)) \quad .$$

continuums solution



Remark: piecewise linear approximation introduces errors

Existence of symmetries depends on the dynamics.

This dynamics can be improved by constructing actions that approximate continuum dynamics very well/perfectly.

Interpretation of discrete building blocks depends on dynamics.

Do not see them literally as (flat) blocks but as representing coarse grained quantities.

4d?

- action will be non-local, but might be triangulation independent [Bahr, BD, He wip]

- impossible to solve equation of motion non-perturbatively:
  ⇒expansion around flat space

- What are the properties of this expansion?
  To which order are the gauge symmetries/ triangulation independence realized?

Regge calculus	Parametrized (an-)harmonic oscillator
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E. Perturbative expansion

[BD, Höhn 09]

$$x^i = x_0^i + \varepsilon\, x_1^i + \varepsilon^2\, x_2^i + \ldots$$

$$\longrightarrow \qquad S = \varepsilon^2 \frac{1}{2} S_{ij} x_1^i x_1^j + \varepsilon^3 S_{ij} x_2^i x_1^j + \varepsilon^3 \frac{1}{3!} S_{ijk} x_1^i x_1^j x_1^k + \ldots$$

↓ solutions not unique

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We will see:

- Typically: consistent expansion only possibly for specific choices of background gauge.
- For other choices: $x_1 \sim \varepsilon^{-1}$
- Precise relation with invariance properties of (truncated) Hamilton-Jacobi functional.

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linear order:

$$S_{ij}(x_0)\ y_g^i(x_0) = 0, \quad y_g^i(x_0) \text{ null vectors with index } g$$

$$x_O^i = x_O^g y_g^i + x_O^p y_p^i \quad \text{coordinate transformation to gauge and physical modes}$$

→ x_0^g and x_1^g remain free

→ x_1^p determined

first non-linear order:

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But there are still the EOM for the gauge modes!

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in particular:

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Interpretation: background parameters get fixed such that dependence of Hamilton-Jacobi functional on these parameters is minimal.

Hamilton-Jacobi functional for linearized Regge

Does the Hamilton-Jacobi functional for linearized Regge calculus depend on background gauge?

Yes! (for a specific example) [BD, Hoehn 09]

-also the case for the parametrized (an-)harmonic oscillator

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now background gauge arbitrary
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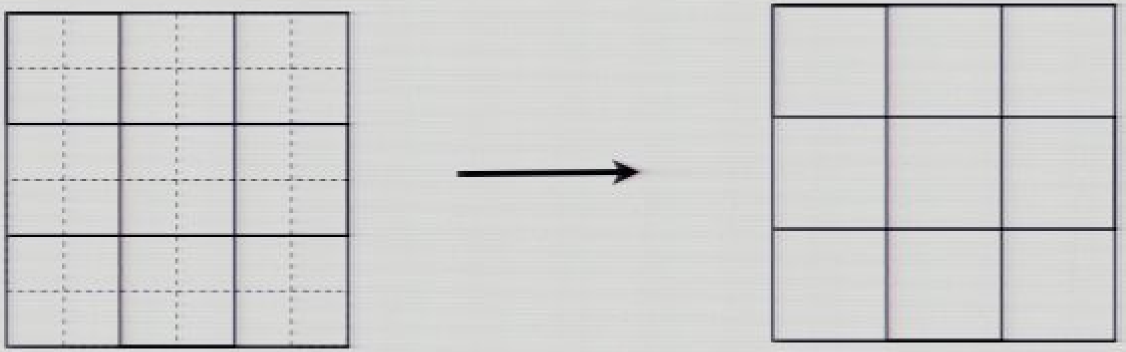
It works not only for the harmonic oscillator but also for the anharmonic one!

Quadratic order



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need to project on orthogonal subspace, then invert

wip: evaluation for the Regge action, geometric interpretation?

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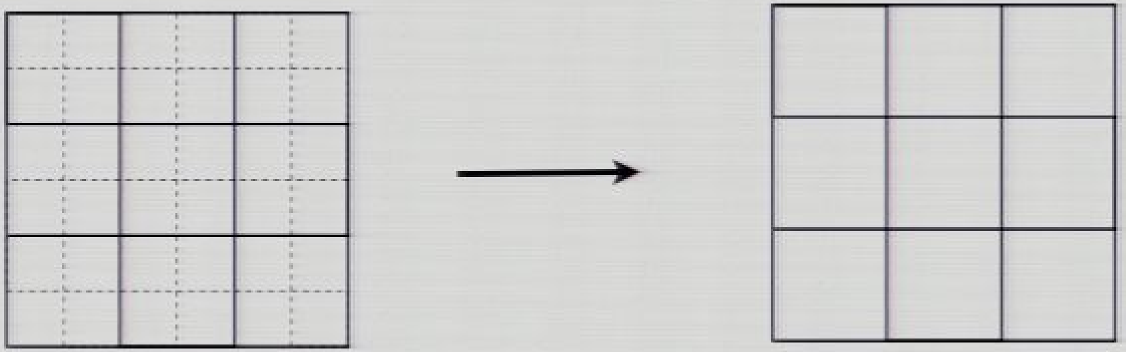

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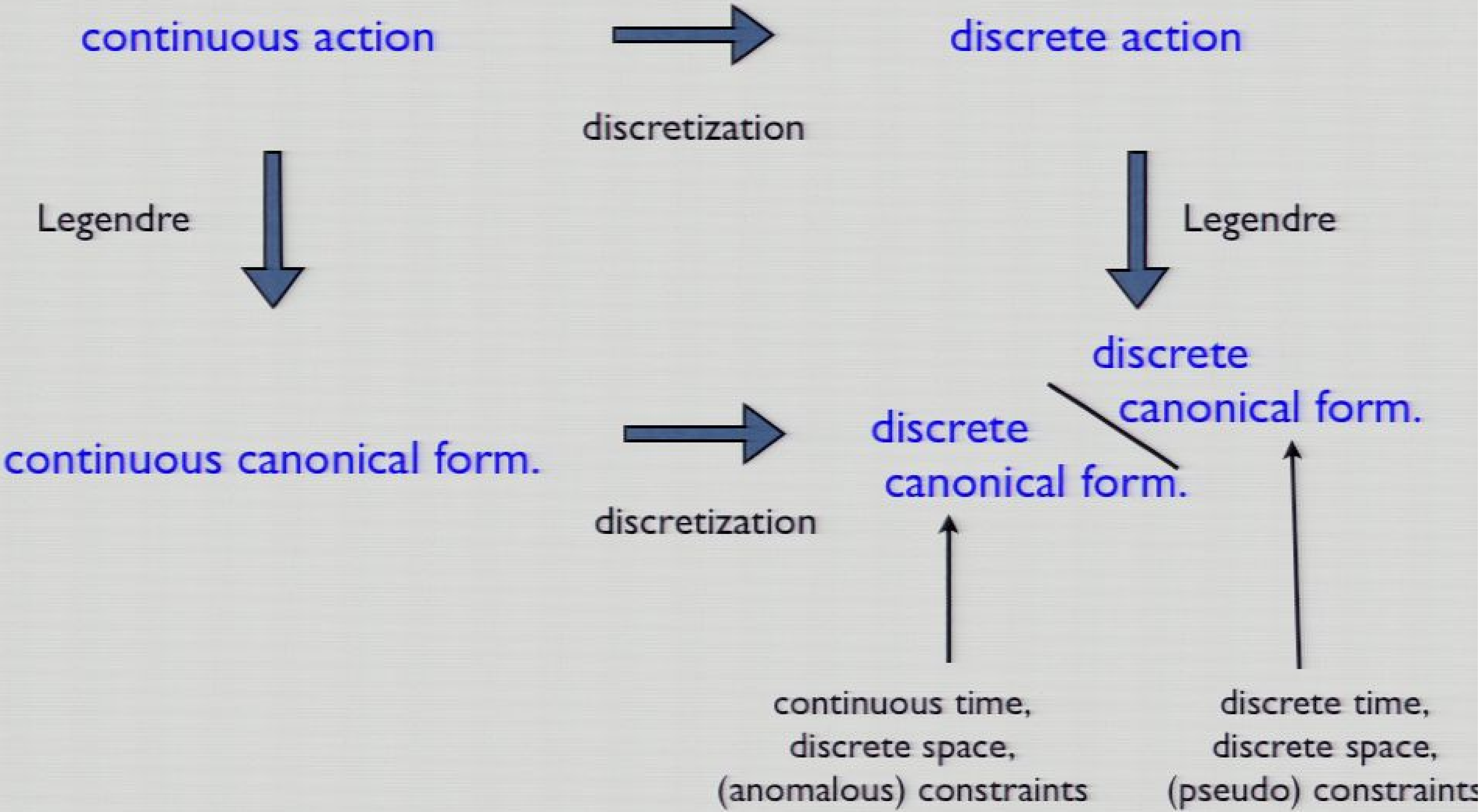
Repercussions for canonical framework and quantization?

typical problem of lattice approaches:
anomalous constraint algebra, inconsistent dynamics

a) Canonical formalism reproducing exactly solutions and (broken) symmetries of discretized action?

b) Constraints? Constraint algebra? Anomalies?

Canonical Framework



continuous action

discretization

discrete action

Legendre

Legendre

continuous canonical form.

discretization

discrete canonical form.

discrete canonical form.

continuous time,
discrete space,
(anomalous) constraints

discrete time,
discrete space,
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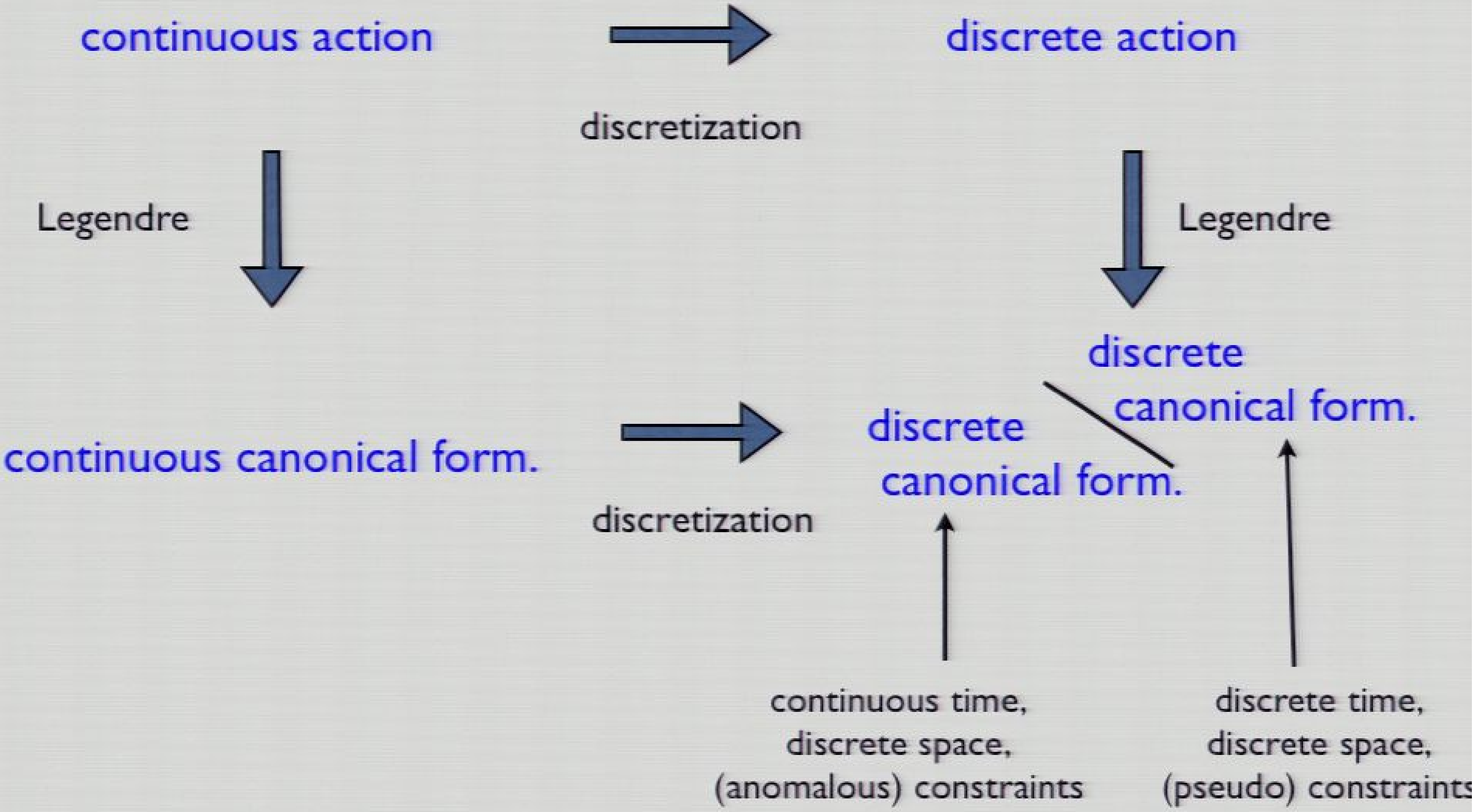
Canonical Framework



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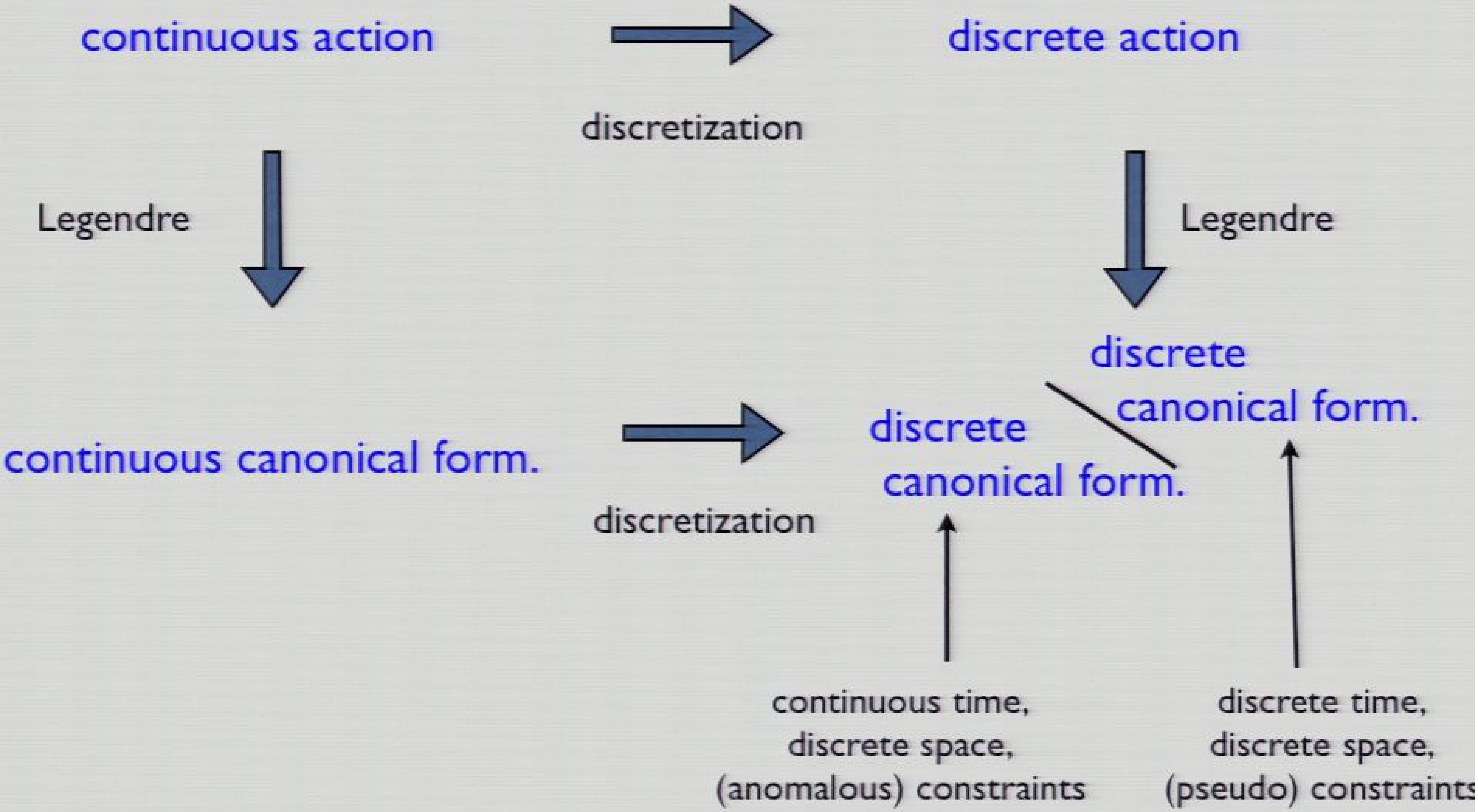

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Canonical Framework

[Bahr, BD '09; BD, Höhn 09]

- evolve spatial triangulation locally by tent moves [Sorkin 75, Barrett et al 97]
- finite time steps
- use action as generating function for time evolution map

[consistent discretizations, Gambini & Pullin et al 03-05]

- reproduces (broken) symmetries exactly [Bahr, BD 09] :

symmetries exact ⇒ eom not independent ⇒constraints (first class)

broken⇒ eom almost not independ. ⇒pseudo-constraints

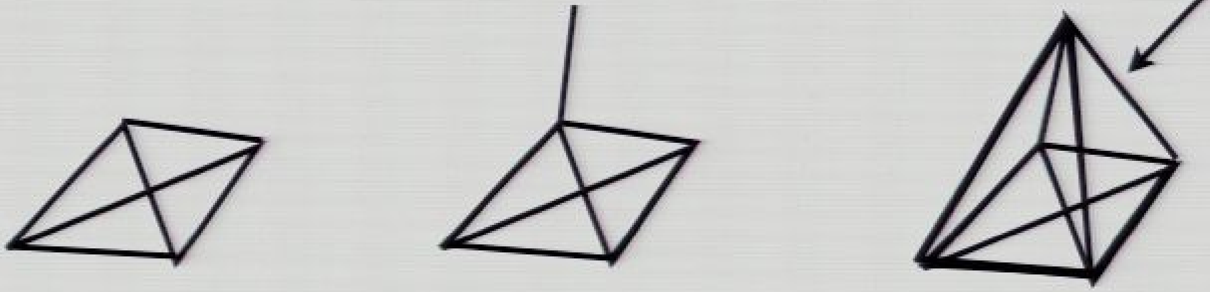
Obtaining anomaly free constraints is equivalent to constructing an action with exact symmetries.

Evolving spatial triangulations with tent moves

[Sorkin 75, Barrett et al 97]

time evolution moves:

- do not change spatial triangulation/ number of variables
- act local, involving only star of a vertex
- can obtain local (pseudo-) constraints based at vertices

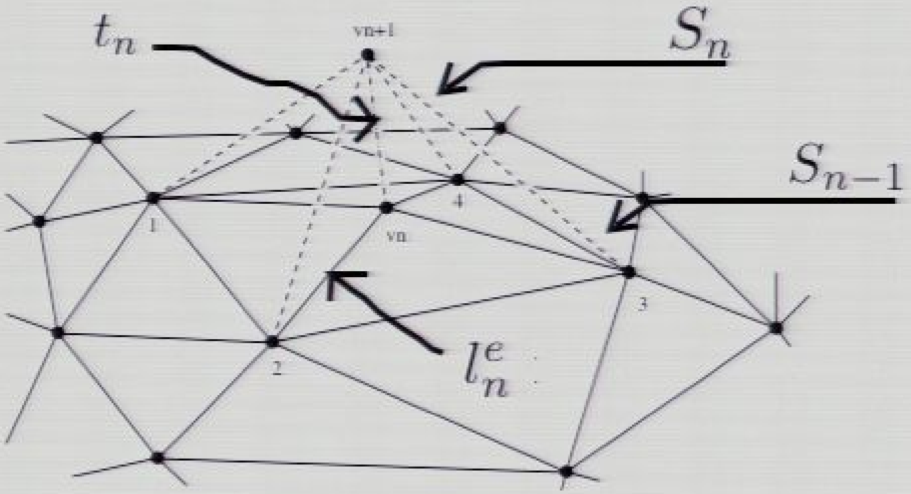


add tent pole on vertex

connect vertices, obtain tent

rtex with star in bigger triangulation

Canonical Framework



equations of motion:

$$0 = \frac{\partial S_n}{\partial t_n} \longrightarrow -p_t^n$$

$$0 = \underset{\substack{\big\downarrow \\ -p_e^n}}{\frac{\partial S_{n-1}}{\partial l_n^e}} + \underset{\substack{\big\downarrow \\ p_e^n}}{\frac{\partial S_n}{\partial l_n^e}}$$

canonical (tent move) transformation:

$$p_t^n := -\frac{\partial S_n}{\partial t_n} \qquad p_e^n := -\frac{\partial S_n}{\partial l_n^e}$$

$$p_t^{n+1} := \frac{\partial S_n}{\partial t_{n+1}} \qquad p_e^{n+1} := \frac{\partial S_n}{\partial l_{n+1}^e}$$

use S_n as generating function for canonical transformation

4-valent vertex: flat dynamics

equation for the tent pole

$$0 = p_t^n = -\frac{\partial S_n}{\partial t_n} = -\sum_{\Delta \supset t} \frac{\partial A_\Delta}{\partial t_n} \epsilon_\Delta$$

solution

$$\epsilon_\Delta = 0$$

momenta associated to edges

$$p_e^n = -\frac{\partial S_n}{\partial l_n^e} = -\sum_{\Delta \supset e} \frac{\partial A_\Delta}{\partial l_n^e} \psi_\Delta - \sum_{\Delta \supset e} \frac{\partial A_\Delta}{\partial l_n^e} \epsilon_\Delta$$

constraints

$$C_e = p_e^n + \sum_{\Delta \supset e} \frac{\partial A_\Delta}{\partial l_n^e}(l_{e'}^n)\ \psi_\Delta(l_{e'}^n)$$

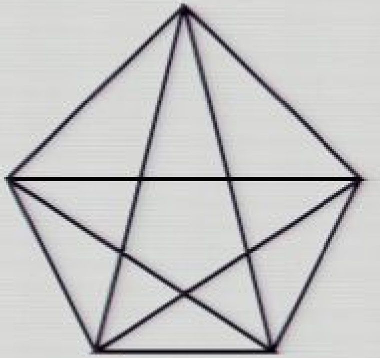
Momenta do not depend on variables at next time step ⇒ constraints.

For higher valent vertices $\epsilon_\Delta \neq 0$, momenta depend (weakly) on variables at next time step

⇒ pseudo constraints.

'Dynamics' for a 4-simplex [BD, Ryan 08, BD, Hoehn 09]

- 3d surface of a 4-simplex: five 4-valent vertices
- apply constraints to every vertex



dihedral angles

$$C_e \;\; = \;\; p_e + \sum_{\Delta \supset e} \frac{\partial A^{\Delta}}{\partial l^e} \xleftarrow{} \psi_{\Delta}(l)$$

geometric meaning?

- symplectic coordinate transformation:

$$A_{\Delta} = A_{\Delta}(l)\,,\; p_{\Delta} = \frac{\partial l^e}{\partial A^{\Delta}} p_e \qquad \longrightarrow \qquad C_{\Delta} \;\; = \;\; p_{\Delta} + \psi_{\Delta}(l)$$

‘Dynamics’ for a 4-simplex [BD, Ryan 08, BD, Hoehn 09]

$$C_\Delta \;=\; p_\Delta + \psi_\Delta(l)$$

- constraints fix the momenta to agree with the dihedral angles as defined by lengths
- are first class! (despite very complicated form of dihedral angles)
- generate deformation of hypersurface (via vertex translations): Hamiltonian and diffeomorphism constraints
- 3d surface of a 4-simplex: zero physical degrees of freedom: no 4d curvature

Higher-valent vertex: (linearized) dynamics [BD, Hoehn 09]

For higher valent vertices $\epsilon_\Delta \neq 0$, momenta depend (weakly) on variables at next time step

$\Rightarrow$ pseudo constraints.

But for the linearized dynamics $\Rightarrow$ constraints.

$$l = {}^0 l + y, \quad p = {}^0 p + \pi, \qquad\qquad S = y^e \frac{\partial S}{\partial l^e \partial l^{e'}} y^{e'}$$

Hessian on flat space has null eigenvectors Y_I^e . → gauge symmetries

→ linearized constraints

$$C_I = Y_I^{e'} \pi_e^n + Y_I^{e'} \left[\frac{\partial}{\partial l_n^{e'}} \sum_{\Delta \supset e} \frac{\partial A_\Delta}{\partial l_n^e} \psi_\Delta \right]_{|l={}^0 l} \quad y_n^e$$

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- constraints give relation between intrinsic and extrinsic geometry
- are first class! (despite very complicated form of dihedral angles)
- generate linearized deformation of hypersurface (via vertex translations): Hamiltonian and diffeomorphism constraints
- preserved by linearized tent move dynamics (analogous to quadratic Hamiltonian)
- split into gauge and physical variables (relation to linearized curvature on inner triangles)

Options

- higher order: obtain pseudo constraints with Regge action - allows inly for discrete time evolution
- alternatively to tent moves:
  - Pachner moves
  - quantization would lead to spin foam picture

- with perfect action: regain continuous time evolution, exact constraints, however non-local constraints and larger phase space (‘higher derivatives’)

Repercussions:

1) action with exact symmetries:

-proper first class constraints, gauge freedom

2) action with broken symmetries:

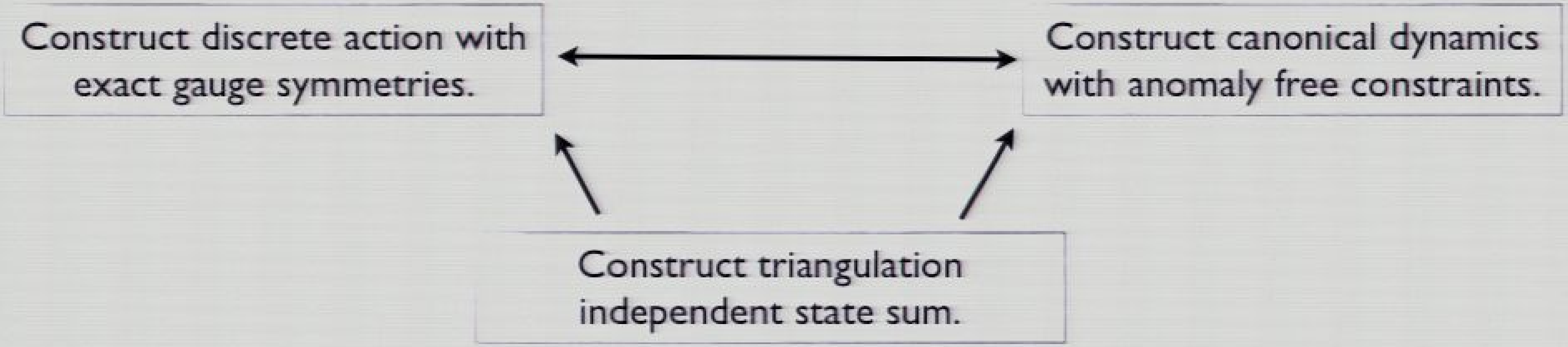
-pseudo constraints with weak dependence on lapse/shift

3) linearized theory inherits symmetries of solution

-exact constraints in linearized theory

-background gauge gets fixed at lowest non-linear order

Connections between problems.



Construct discrete action with exact gauge symmetries.

Construct canonical dynamics with anomaly free constraints.

Construct triangulation independent state sum.

Conclusions

- discrete actions generally break diffeomorphism symmetries
- regaining symmetries by coarse graining, renormalization
- canonical framework exactly mimics covariant symmetries: constraints and pseudo-constraints
- perturbative expansion subtle: background gauge fixed if symmetries are broken

Prospects

- understand triangulation (in-)dependence and investigate non-locality properties of improved actions
- develop lattice deformation algebra:
- improved quantum action/ renormalization in spin foams
- canonical quantization: improve constraints

- Explore general mechanisms and conditions for regaining gauge symmetries.