Title: Foundations and Interpretation of Quantum Theory - Lecture 15

Date: Mar 11, 2010 02:30 PM

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Abstract:

The Born Rule

Given ρ and $\{E_i\}$,

quantum state

POVM measurement

$$\boxed{p(i) = \operatorname{tr} \rho E_i}$$

"The Born Rule"

The Born Rule

Given ρ and $\{E_i\}$,

quantum state

POVM measurement

$$\boxed{p(i) = \mathrm{tr}\,\rho E_i}$$

"The Born Rule"

NOT a law of nature.

RATHER something we should strive for.

Not like

$$\vec{F} = m\vec{a}$$

Not like

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{\varepsilon_0}\rho \qquad \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} \qquad \vec{\nabla}\cdot\vec{B} = 0$$

Not like

$$G_{\mu\nu} = 8\pi T_{\mu\nu}$$

THE TEN COMMANDMENTS

- Thou shalt not kill.
- Thou shalt not steal.
- Thou shalt not covet thy neighbor's wife.

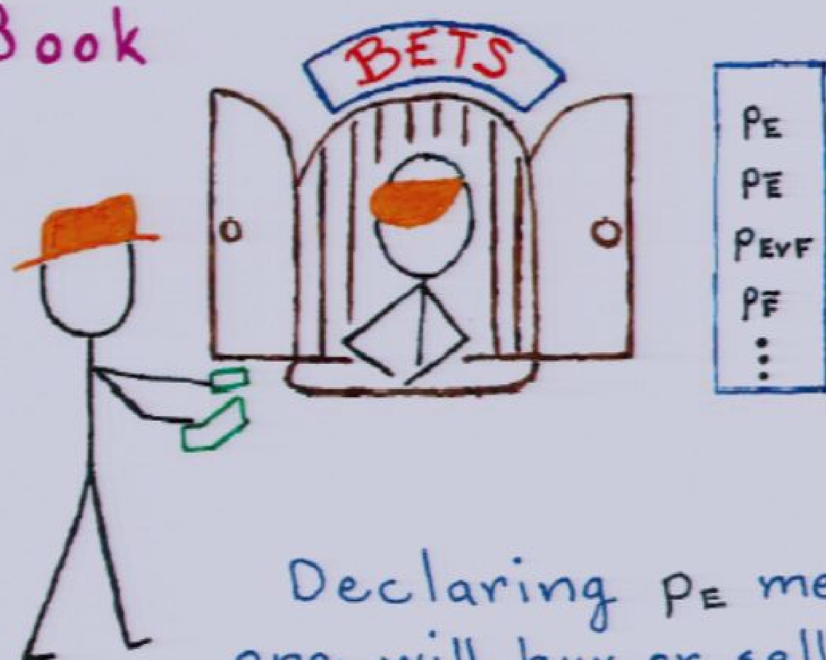
⋮

- The firstling of an ass thou shalt redeem with a lamb.

⋮

Defining Probability

Dutch Book



Declaring p_E means one will buy or sell a lottery ticket

$\boxed{\text{Worth \$1 if } E}$

for $\$p_E$.

Dutch Book

Normative Rule:

Never declare p_E, $p_{\bar{E}}$, $p_{E \vee F}$, etc. that will lead to sure loss.

Example 1:

If $p_E < 0$, bookie will sell ticket for negative money. Sure loss!

Example 2:

If $p_E > 1$, bookie will buy ticket for more than it is worth in best case. Sure loss.

Example 3:

Suppose E and F mutually exclusive.

Worth \$1 if $E \vee F$

Worth \$1 if E

Worth \$1 if F

buying this is equivalent to buying these two

So must have $p_{E \vee F} = p_E + p_F$.

Example 4:

Worth $\$\frac{m}{n}$ if E

Price? $\$\frac{m}{n} p_E$ of course.

Example

One contemplates taking

$p(F) = 0.75$

$p(E|F) = 0.50$

$p(E \wedge F) = 0.70$.

One could gamble that way,
but it wouldn't be too wise:

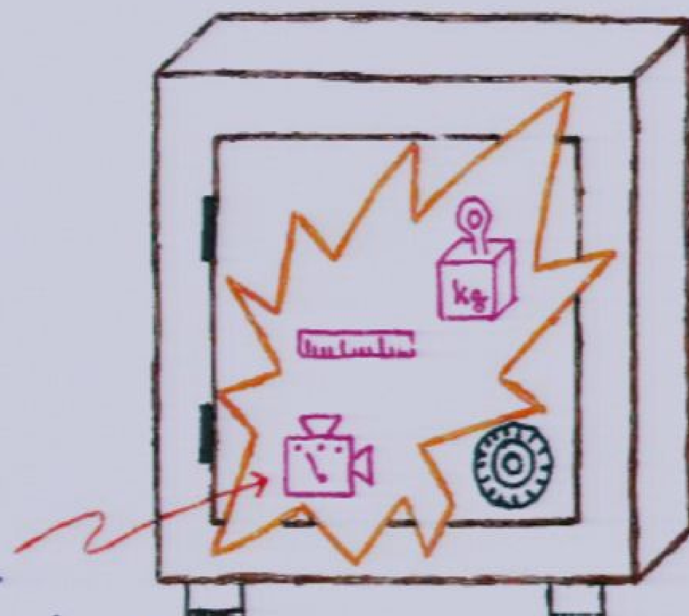
Not coherent.

Normative Rule:

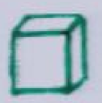
Strive for coherence.

$$\rho \longleftrightarrow p(h)$$

Bureau of Standards

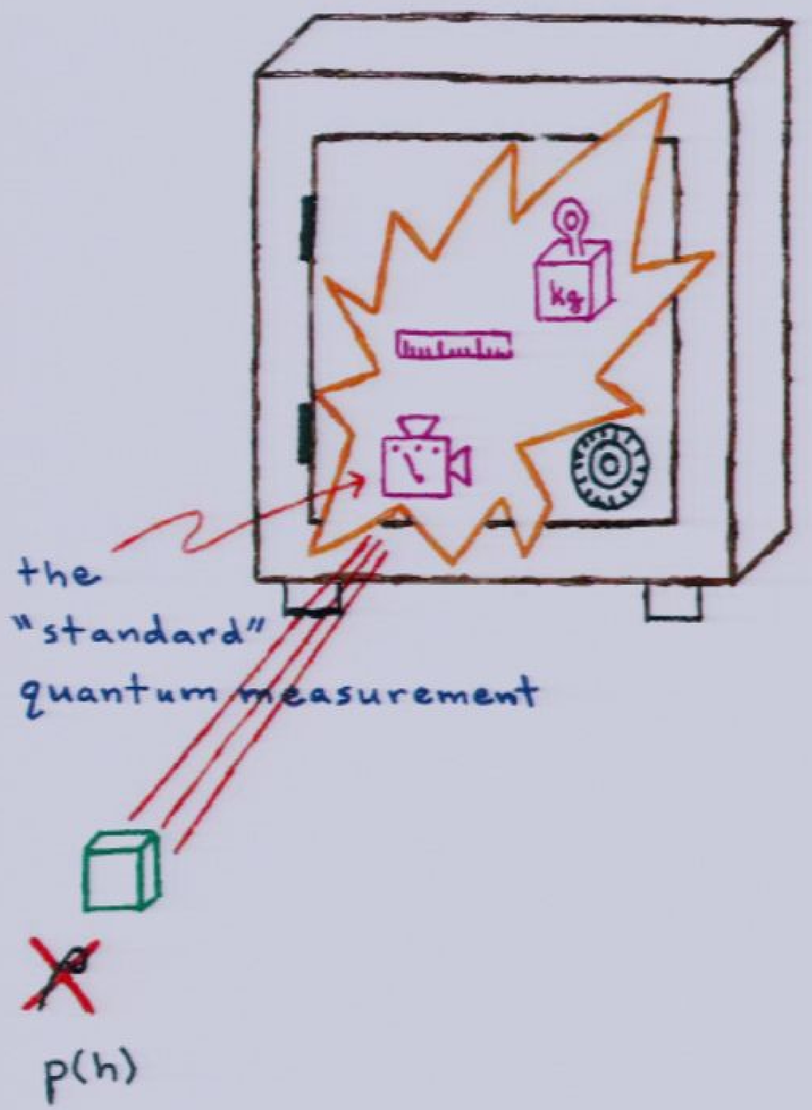


the
"standard"
quantum measurement

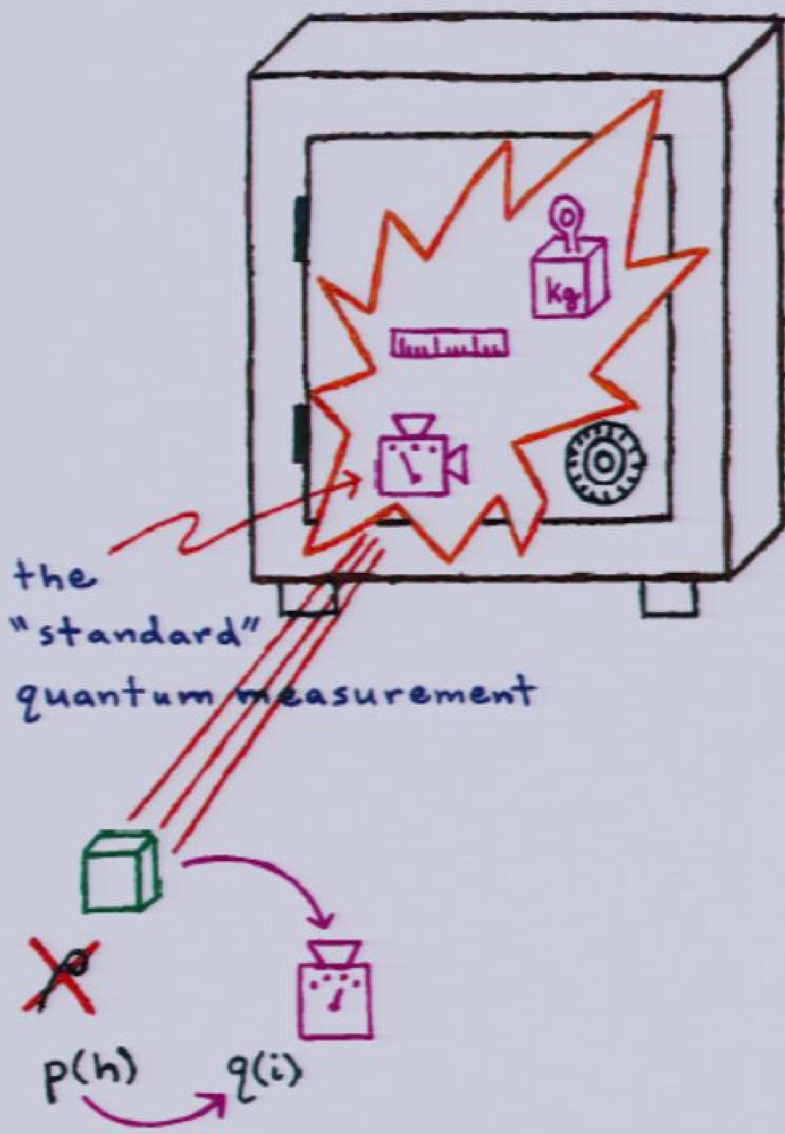


ρ

Bureau of Standards



Bureau of Standards



A Very Fundamental Mmt ?

Suppose d^2 projectors $\Pi_i = |\psi_i\rangle\langle\psi_i|$ satisfying

$$\mathrm{tr}\,\Pi_i \Pi_j = \frac{1}{d+1} \quad , \quad i \neq j$$

exist. ← Called SIC.

Can prove:

1) the Π_i linearly independent

2) $\sum_i \frac{1}{d} \Pi_i = I$

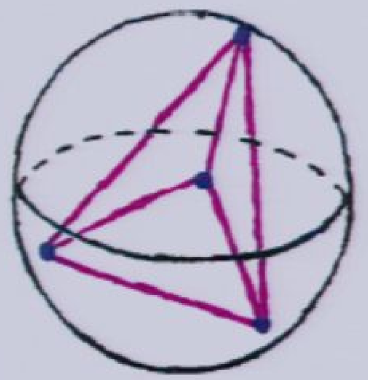
So good for Bureau of Standards.

Also
$$p(i) = \frac{1}{d}\,\mathrm{tr}\,\rho\Pi_i$$

$$\boxed{\rho = \sum_i \left[(d+1)p(i) - \frac{1}{d}\right]\Pi_i}$$

SIC Sets

dimension 2



any regular tetrahedron

dimension 3

Let $\omega = e^{\frac{2\pi i}{3}}$.

$$\begin{bmatrix}0\\1\\-1\end{bmatrix} \quad \begin{bmatrix}0\\ \omega\\-\bar{\omega}\end{bmatrix} \quad \begin{bmatrix}0\\ \bar{\omega}\\-\omega\end{bmatrix}$$

$$\begin{bmatrix}-1\\0\\1\end{bmatrix} \quad \begin{bmatrix}-1\\0\\ \bar{\omega}\end{bmatrix} \quad \begin{bmatrix}-1\\0\\ \omega\end{bmatrix}$$

$$\begin{bmatrix}1\\-1\\0\end{bmatrix} \quad \begin{bmatrix}1\\-\omega\\0\end{bmatrix} \quad \begin{bmatrix}1\\-\bar{\omega}\\0\end{bmatrix}$$

Dimension 6

$$|\psi\rangle = \frac{\alpha}{3\sqrt{2}}\begin{pmatrix} f_+ \\ \sigma^5 f_- \\ \sigma^8 f_+ \\ \sigma^{-3} f_- \\ \sigma^8 f_+ \\ \sigma^5 f_- \end{pmatrix} + \frac{\beta_- e^{i\theta_+}}{3\sqrt{2}}\begin{pmatrix} \sigma^8 f_- \\ \sigma^{-7} f_+ \\ f_- \\ \sigma^{-7} f_+ \\ \sigma^8 f_- \\ \sigma^9 f_- \end{pmatrix} + \frac{\beta_+ e^{i\theta_-}}{3\sqrt{2}}\begin{pmatrix} \sigma^8 f_- \\ \sigma^9 f_+ \\ \sigma^8 f_- \\ \sigma^{-7} f_+ \\ f_- \\ \sigma^{-7} f_- \end{pmatrix}$$

where

$$\sigma = e^{i\pi/12}$$

$$f_\pm = \sqrt{3 \pm \sqrt{3}}$$

$$g = \sqrt{6\sqrt{21} - 18}$$

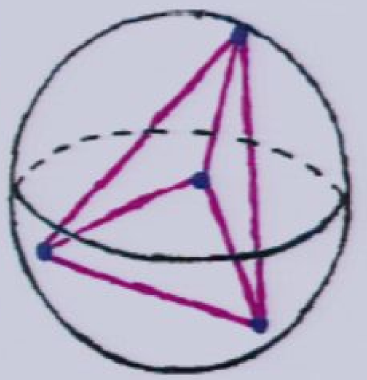
$$\alpha = \sqrt{\frac{7 - \sqrt{21}}{14}}$$

$$\beta_\pm = \sqrt{\frac{7 + \sqrt{21} \pm \sqrt{14\sqrt{21} - 42}}{28}}$$

$$e^{i\theta_\pm} = \frac{1}{2}\left(\sqrt{46 - 6\sqrt{21} \mp 6g} \pm i\sqrt{18 + 6\sqrt{21} \pm 6g}\right)^{\frac{1}{3}}$$

SIC Sets

dimension 2



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$$
\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \quad \begin{bmatrix} 0 \\ \omega \\ -\bar{\omega} \end{bmatrix} \quad \begin{bmatrix} 0 \\ \bar{\omega} \\ -\omega \end{bmatrix}
$$

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Evidence for Existence

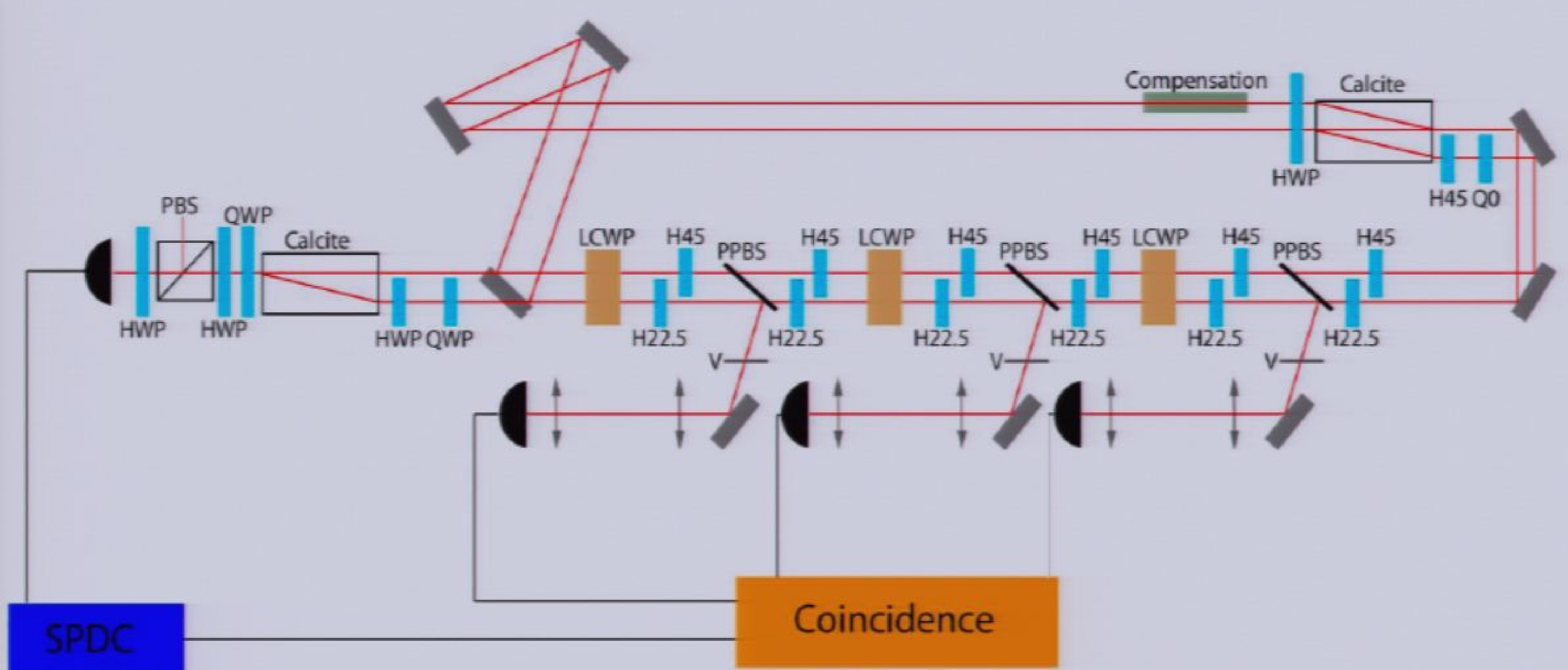
Analytical Constructions

$d = 2 - 13,\ 14,\ 15,\ 19$

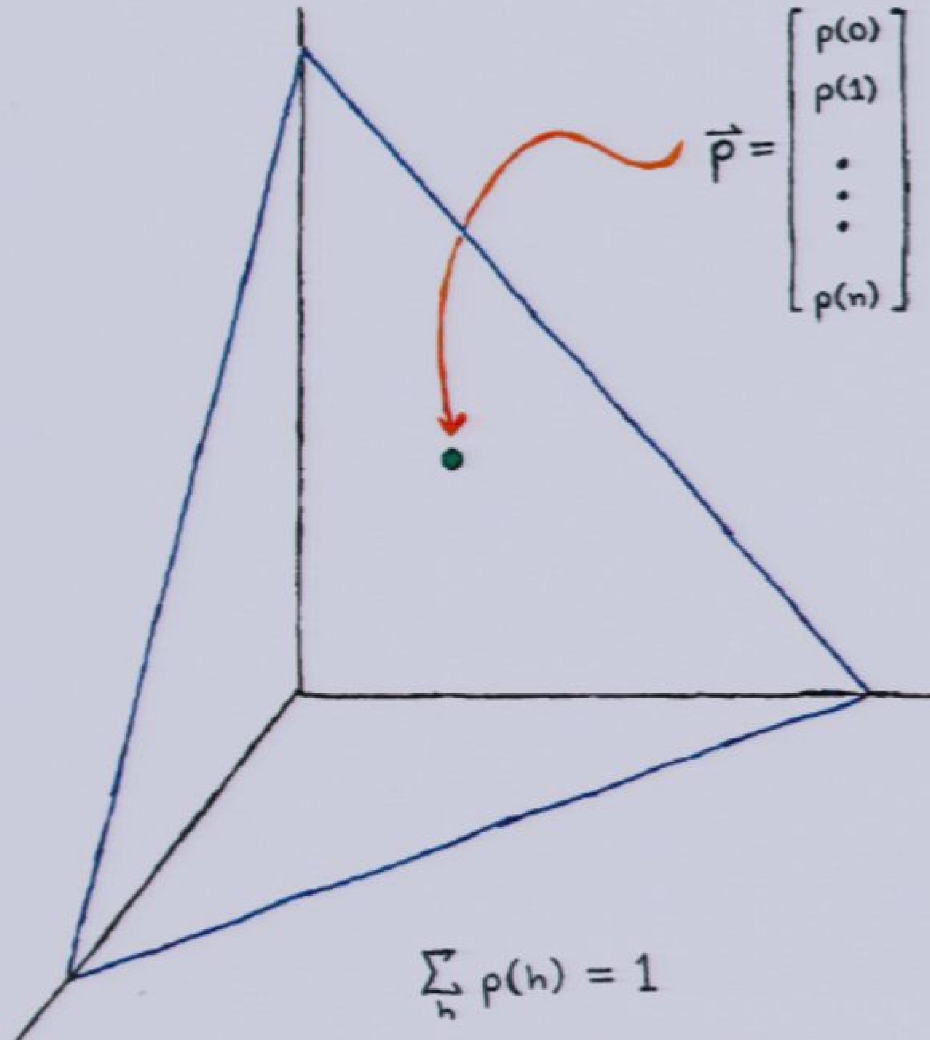
Numerical ($\epsilon \leq \cancel{10^{-11}}$) 10^{-38}!

$d = 2 - \cancel{47}\ 67$

Medendorp, Torres-Ruiz, Shalm, CAF, Steinberg, at QELS 2010

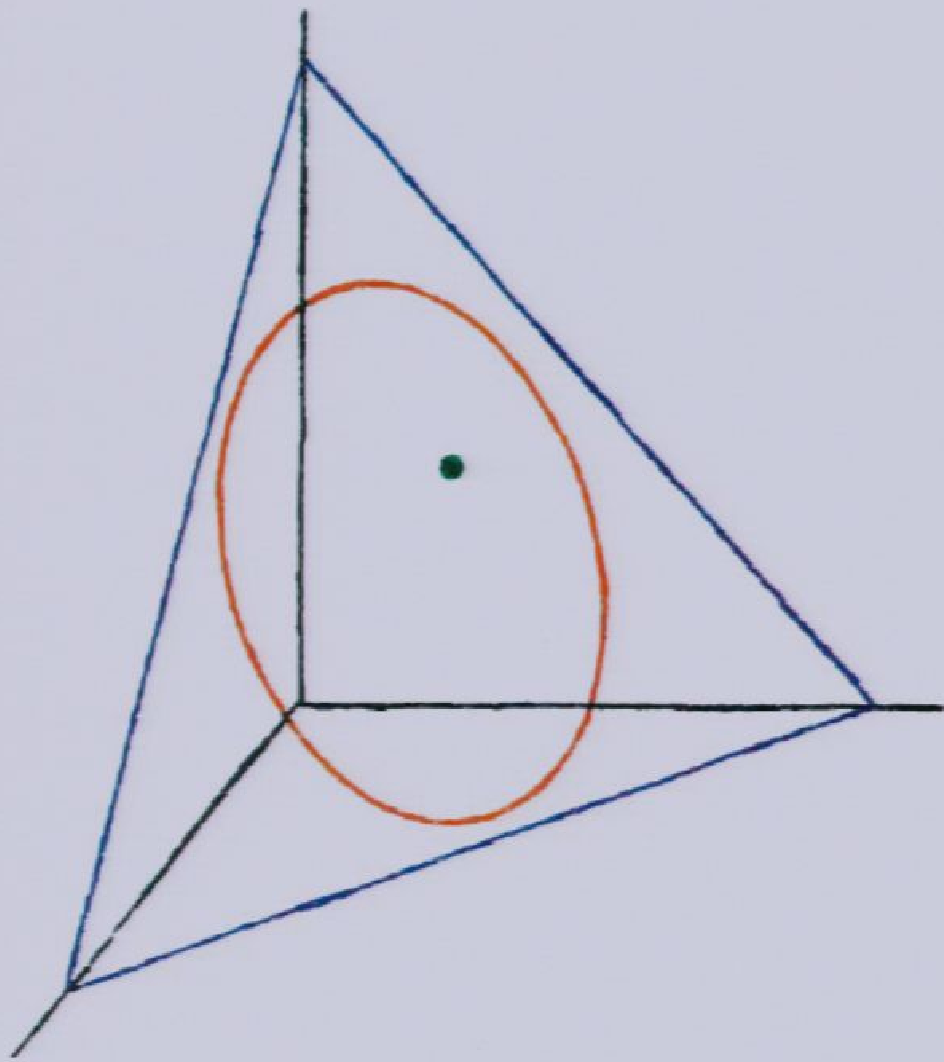


Probability Simplex



$$\sum_h p(h) = 1$$

$$p(h) \geq 0 \quad \forall h$$



Pure States in SIC Language

Conditions

$$\rho^\dagger = \rho \quad , \quad \mathrm{tr}\,\rho^2 = \mathrm{tr}\,\rho^3 = 1$$

translate to

$$\sum_i p(i)^2 = \frac{2}{d(d+1)}$$

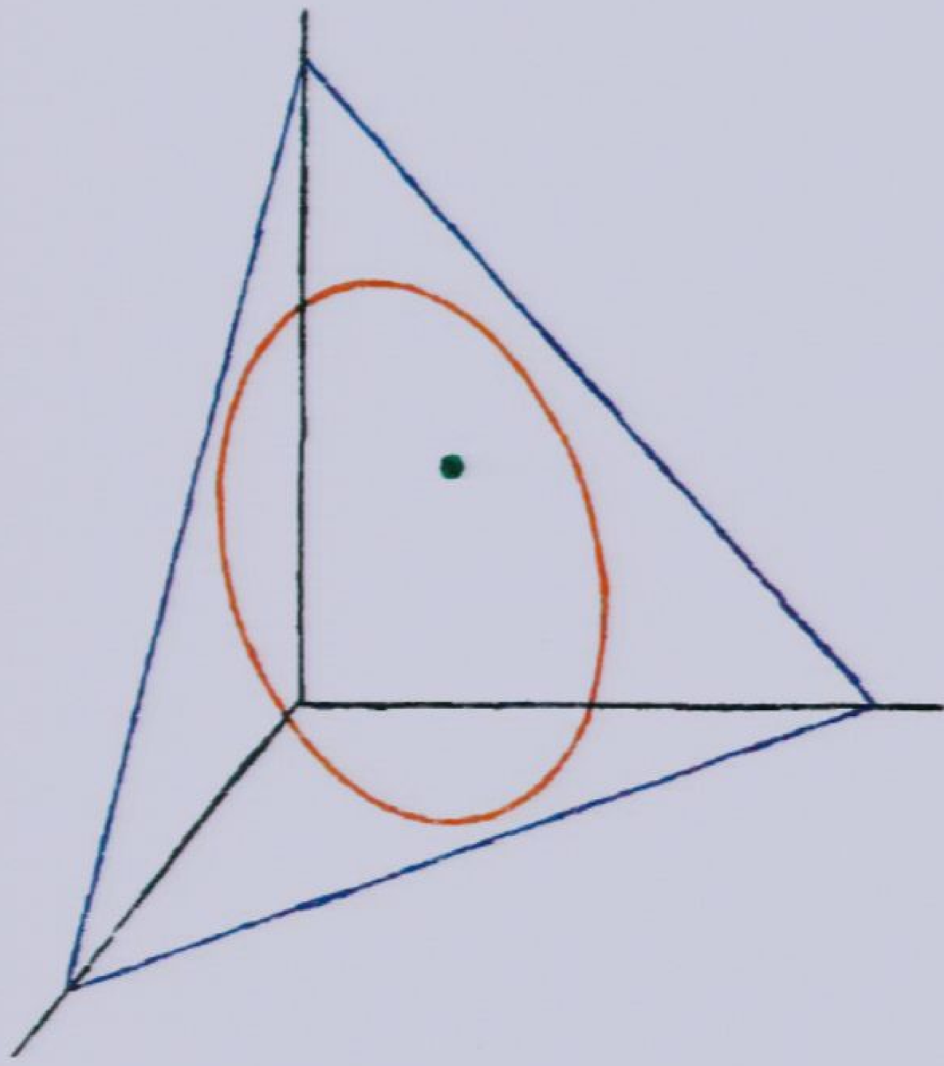
and

$$\sum_{jkl} c_{jkl}\, p(j)\, p(k)\, p(l) = \frac{d+7}{(d+1)^3}$$

where

$$c_{jkl} = \mathrm{Re}\ \mathrm{tr}\, \Pi_j \Pi_k \Pi_l$$

Could these be independently motivatable physical constants?



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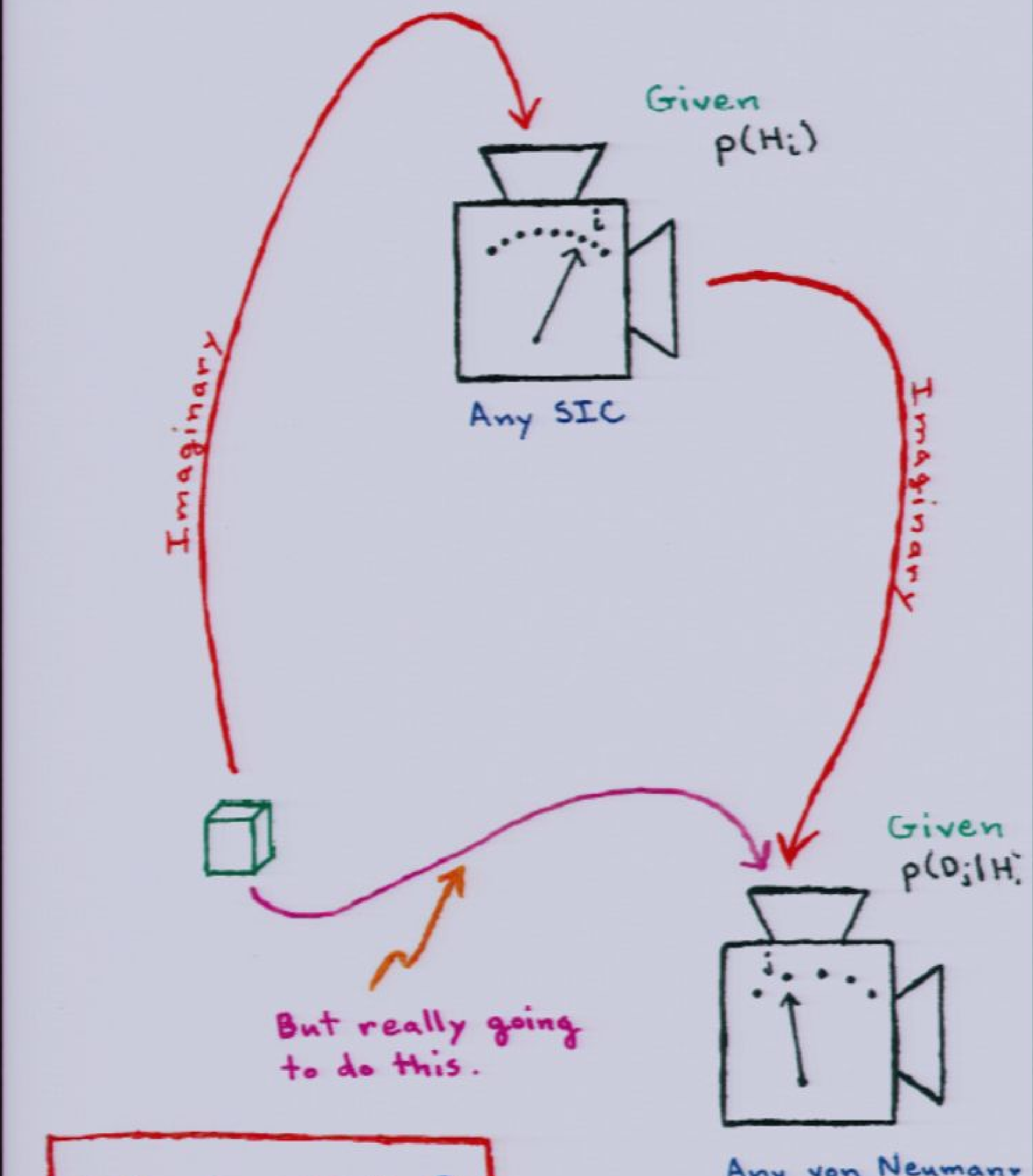

Given
p(H_i)
Any SIC
Imaginary
Imaginary
Given
p(D_j|H
But really going
to do this.
Any von Neumann

Laws of Probability

H_i – various hypotheses one might have

D_j – data values one might gather

Given: $p(D_j | H_i)$ ← expectations for data given hypothesis

$p(H_i)$ ← expectations for hypotheses themselves

Question: What expectations should one have for the D_j ?

Answer: $P(D_j) = \sum_i p(H_i) p(D_j | H_i)$

In this case,

$$p(D_j) \neq \sum_i p(H_i)\, p(D_j|H_i) .$$

As Ballentine (1986) points out, there are hidden conditionals

$p(D_j)$ really $p(D_j|C_1)$

$p(H_i)$ really $p(H_i|C_2)$

$p(D_j|H_i)$ really $p(D_j|H_i, C_2)$

$$\underbrace{p(D_j)}_{\text{Quantum}} = (d+1)\underbrace{\sum_i p(H_i)\,p(D_j|H_i)}_{\text{(Usual) Bayesian}} - 1$$

Magic!

Law of Total Probability:

$$p(D_j) = \sum_i p(H_i)\, p(D_j|H_i)$$

The Born Rule:

$$q(D_j) = \mathrm{tr}\, \hat{\rho}\, \hat{D}_j$$

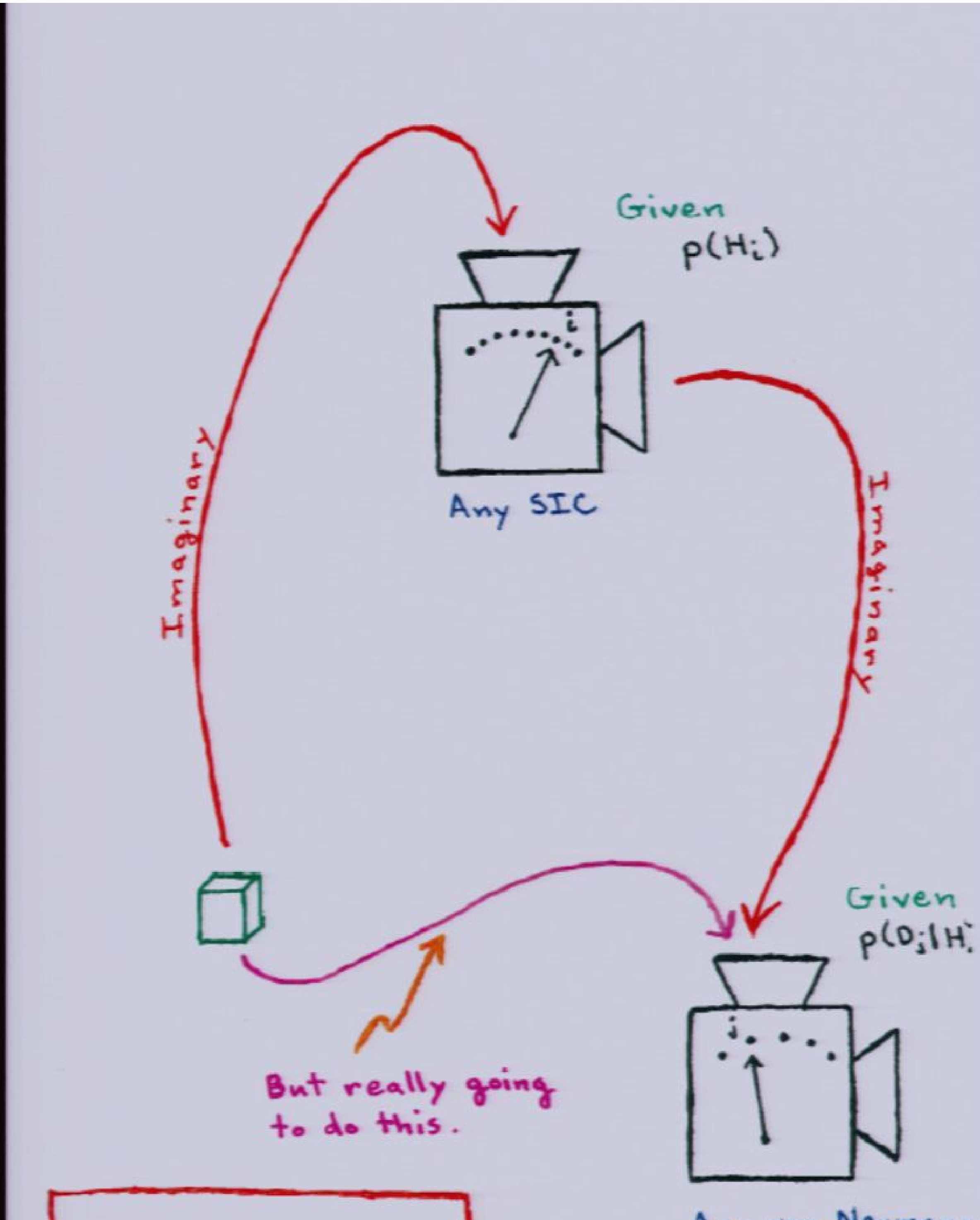
$$= (d+1)\, p(D_j) - 1$$

dimensionality of the system

Could we take diagram and modified Law of Total Probability

$$p(D_j) = (d+1)\sum_i p(H_i)\,p(D_j|H_i) - 1$$

as a fundamental postulate of quantum mechanics?



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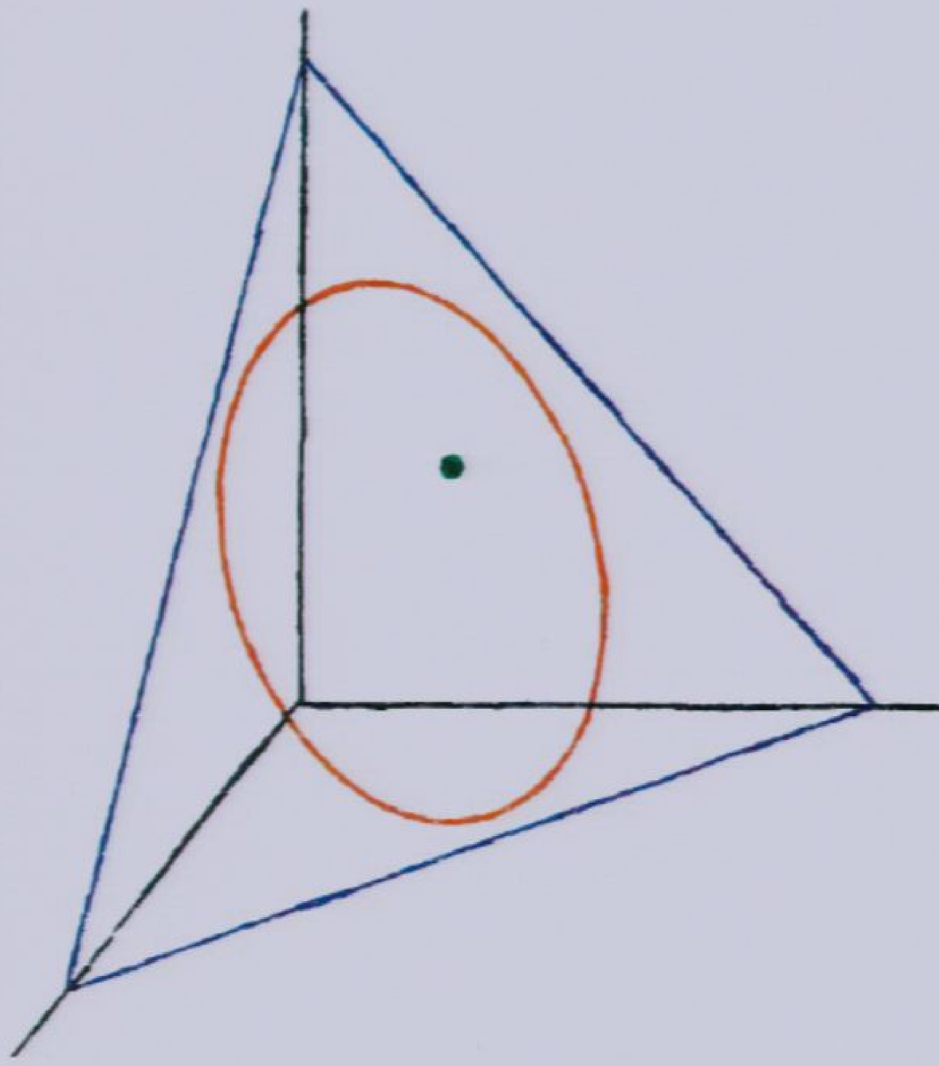

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For instance, with extra assumption that there exist measurements for which

$$p(D_j) = \delta_{jk}$$

then must have, for any two valid $\vec{p}$ and $\vec{q}$

$$\frac{1}{d(d+1)} \leq \sum_i p(H_i) q(H_i) \leq \frac{2}{d(d+1)} .$$

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Homework

containing the $\vec{e}_x$

Call a set $\mathcal{S} \subseteq \Delta_{d^2}$ within the probability simplex

a) consistent if for any $\vec{p}, \vec{q} \in \mathcal{S}$

$$\frac{1}{d(d+1)} \leq \vec{p} \cdot \vec{q} \leq \frac{2}{d(d+1)} ,$$

b) maximal if adding any further $\vec{p} \in \Delta_{d^2}$ makes it inconsistent

Example: If $\mathcal{S}$ is set of quantum states, it is consistent & maximal.

Problem: Characterize all such $\mathcal{S}$; compare to quantum.

Examples

1) Take $\vec{q} = \vec{p}$. Consequently must have

$$\vec{p} \cdot \vec{p} \leq \frac{2}{d(d+1)}$$

Same as quantum.

2) Consider a subset $\{\vec{p}_k\} \subseteq \mathcal{S}$ with $k = 1, \dots, m$ such that

$$\vec{p}_k \cdot \vec{p}_k = \frac{2}{d(d+1)}$$

$$\vec{p}_k \cdot \vec{p}_\ell = \frac{1}{d(d+1)} \qquad k \neq \ell.$$

How large can m be?

Answer: d, same as quantum

Challenge

What further postulates must be made to recover precisely quantum state space?

I.e. the convex hull of

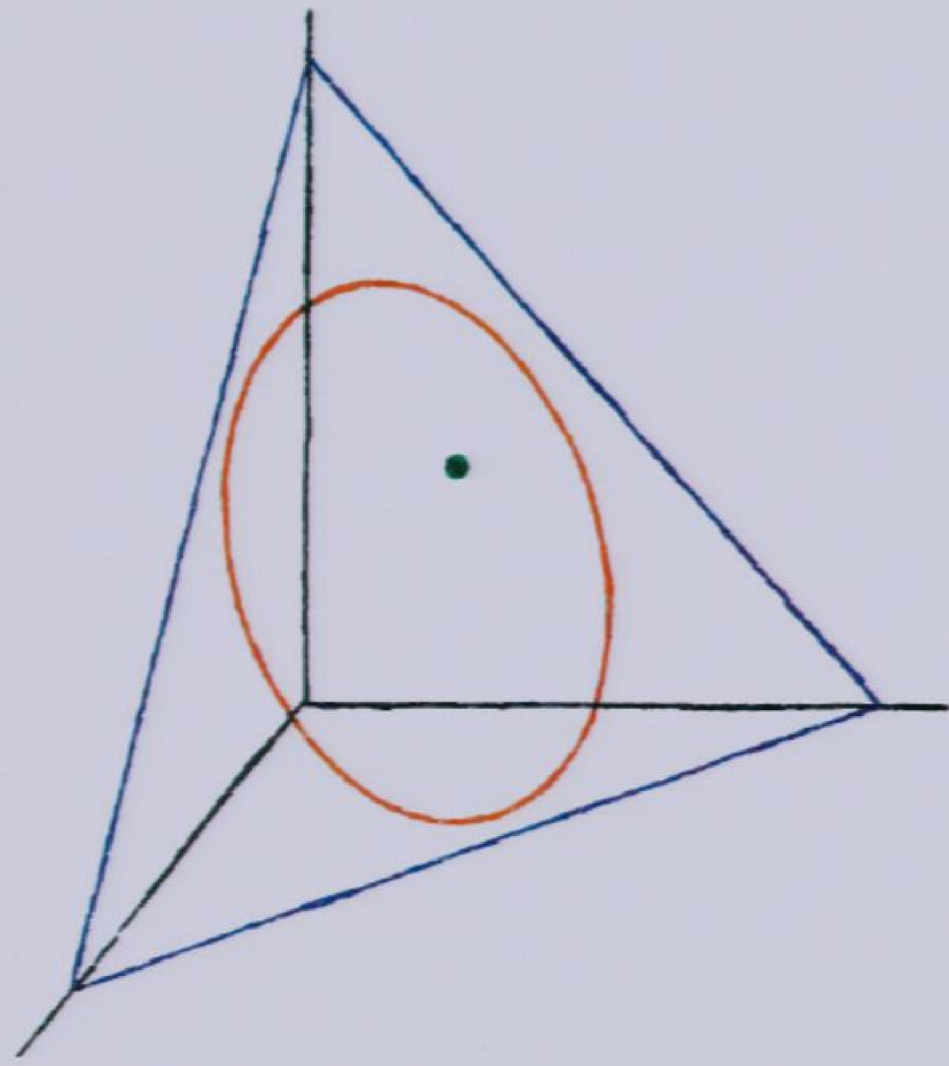
1) $\sum_i p(i)^2 = \frac{2}{d(d+1)}$

2) $\sum_{ijk} c_{ijk}\, p(i)\, p(j)\, p(k) = \frac{d+7}{(d+1)^3}$

$$p(D_j) = (d+1) \sum_i p(H_i)\, p(D_j \mid H_i) - 1$$

$$p(D_j) = (d+1)\sum_i p(H_i)\,p(D_j \mid H_i) - 1$$

$$p(D_j) = \sum_{ik} M_{jk}\, p(H_i)\, p(D_j \mid H_i)$$



$$1)\ \sum_i p(H_i)\,p(D_j|H_i) - 1$$

$$\sum_{ik} M_{jk}\,p(H_i)\,p(D_k|H_i)$$

$$\begin{pmatrix} a & b & - \\ & a & - & - \\ & & a & \\ b & & & \end{pmatrix}$$

$$\Pi_i = |\psi_i\rangle\langle\psi_i|$$

$$\Pi_i = |\psi_i\rangle\langle\psi_i|$$

$$\sum_i \Pi_i \otimes \Pi_i \propto P_{sym}$$

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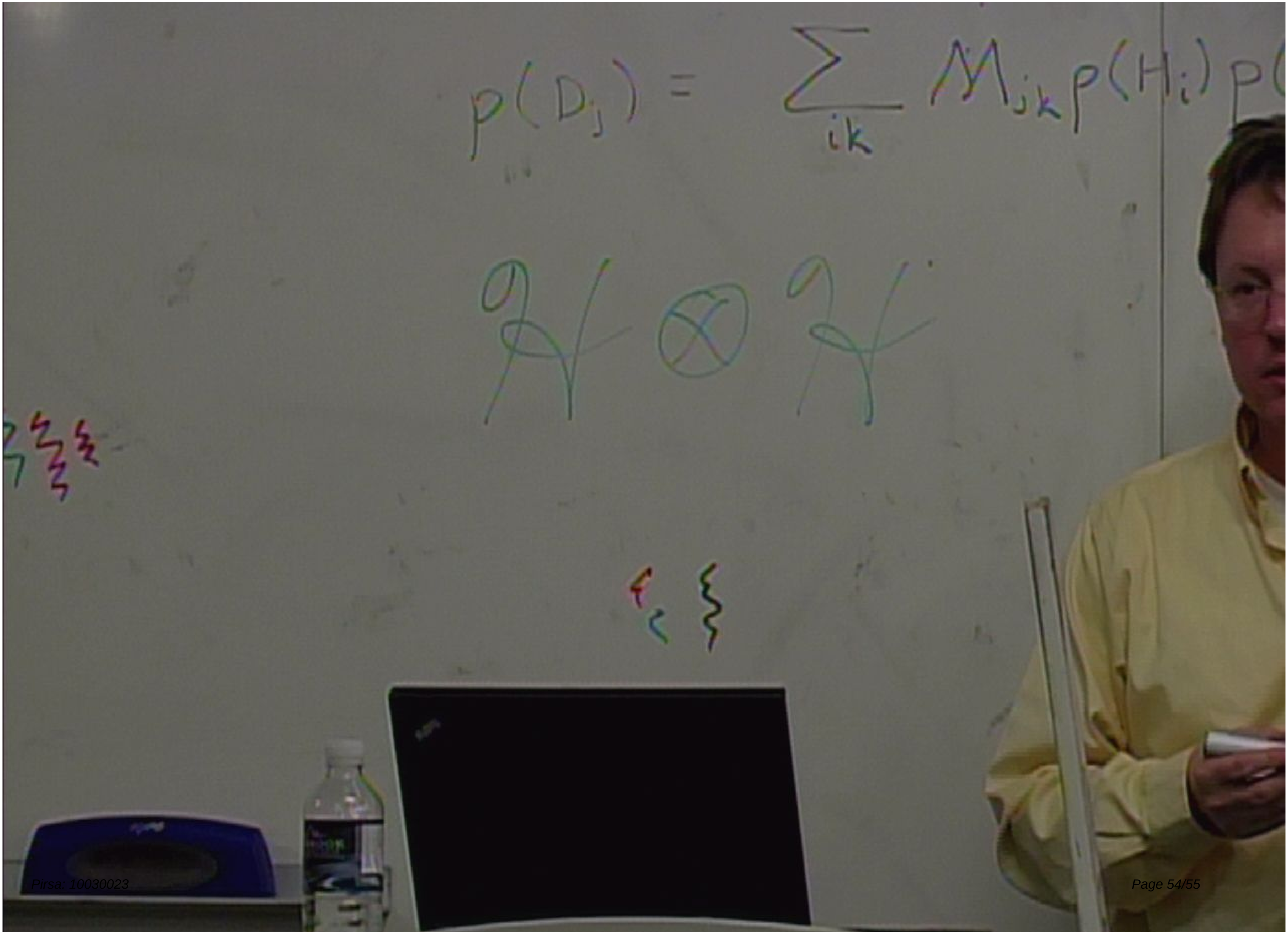
Magic!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$\mathcal{H} \otimes \mathcal{H}$

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$$\underbrace{\mathcal{H} \otimes \mathcal{H}}_{|EPR\rangle}$$