Title: Quantum Field Theory for Cosmology - Lecture 14

Date: Mar 04, 2010 05:00 PM

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Abstract:

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QFT for Cosmology, Achim Kempf, Winter 10, Lecture 14

Quantum field theory on FRW spacetimes.

Friedmann Robertson Walker (FRW) spacetimes:

- While galaxies are causing some curvature locally, the universe is found to be spatially very flat on larger scales. (E.g., for triangles of billions of light years in size, Pythagoras' theorem holds with very good precision, the angles add up to 180°, etc)

- Thus, one often considers the simplifying approximation that spacetime has no spatial curvature at all.

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- This leaves the possibility that spacetime may contract or to expand, and potentially differently in different directions.

- The experimental evidence so far supports the simplifying assumption that the cosmic expansion is isotropic.

Remark: It is known that the Einstein equations allow for highly nontrivial evolutions of non-isotropic spacetimes, see, e.g., the text by Wainwright & Ellis.

- With these assumptions, let us choose these coordinates:

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$$g_{\mu\nu}(t,\vec{x}) = \begin{pmatrix} 1 & & & \\ & -a^2(t) & & \\ & & -a^2(t) & \\ & & & -a^2(t) \end{pmatrix}$$

because our coordinate system's unit of length means over time a larger and larger proper length.

Recall: $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ is "proper 4-distance".

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$$d = \sqrt{|g_{\mu\nu}(t_0)\,\Delta x^\mu \Delta x^\nu|}$$

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Dynamics of $a(t)$:

- The function $a(t)$ is calculated this way:

1. Calculate the energy momentum tensor $T_{\mu\nu}(t,\vec{x})$ contributions of at least the most important fields, say $\varphi_i(x,t)$.

2. Solve, simultaneously:

* The equations of motion for the fields φ_i
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"Semi-classical approximation" → One uses $\bar{T}_{\mu\nu}(x) := \langle \Omega | T_{\mu\nu}(t,\vec{x}) | \Omega \rangle$.

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and therefore $\eta(t) = \int_{t_0}^{t} \frac{1}{a(t')}\, dt'$

t_0 ← yields arbitrary integration constant.

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and therefore $\eta(t) = \int_{t_0}^{t} \frac{1}{a(t')}\, dt'$

t_0 ← yields arbitrary integration constant.

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The Klein Gordon field in FRW spacetimes

- Neglecting a potential $V(\phi)$ for now, we obtain the action of the "free K.G. field on the FRW background":

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- ## Strategy:

We try to change from $\phi(\eta,\vec{x})$ to a new field variable, say $\chi(\eta,\vec{x})$, so that the equation of motion for χ has no "friction"-type term.

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$$\chi'' - \Delta\chi + \left(m^2a^2 - \frac{a''}{a}\right)\chi = 0 \qquad (EoM\,1)$$

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$$0 = \frac{\delta S}{\delta \phi} = \frac{\delta S}{\delta \chi} \frac{\delta \chi}{\delta \phi}$$

if $\delta S/\delta\chi$ vanishes then also $\delta S/\delta\phi$ vanishes.

* Thus, we may calculate the equation of motion directly in terms of χ from $S[\chi]$, to obtain:

Exercise: verify!

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to the field $x(\eta, \vec{x})$, i.e., the Legendre transform of x':

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$$\pi^{(x)}(\eta, \vec{x}) := \frac{\delta L}{\delta x'(\eta, \vec{x})} = x'(\eta, \vec{x}) \quad (\text{EoM } 2)$$

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$$S_{K.G.} = \int \left(\frac{1}{2} a^{-2}(\eta)\, \eta^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} - \frac{1}{2} m^2 \phi^2\right) a^4 \, d\eta \, d^3x$$

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they have no reason to be the same!

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- Should there not be one Hamiltonian for all variables?

- **Answer:** Yes, and it is, of course $\hat{H}^{(\phi)}$.

This extra term is there if the variable $\hat{a}$ has also explicit time-dependence, e.g., $\hat{a} = \cos(at + bt^2)\hat{q} + c\hat{p}$, or here: $\hat{x} = \frac{1}{a}\hat{\phi}$

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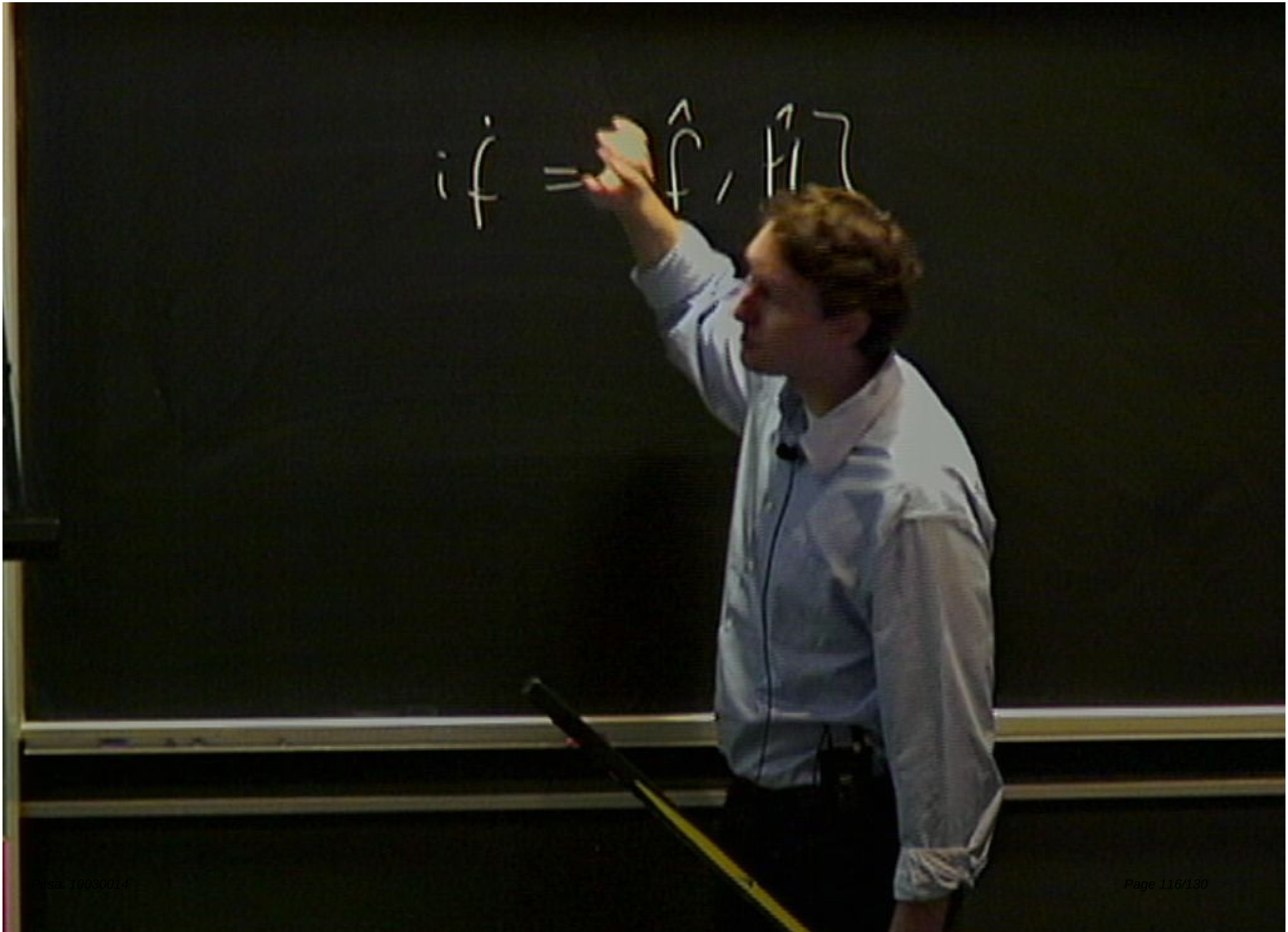
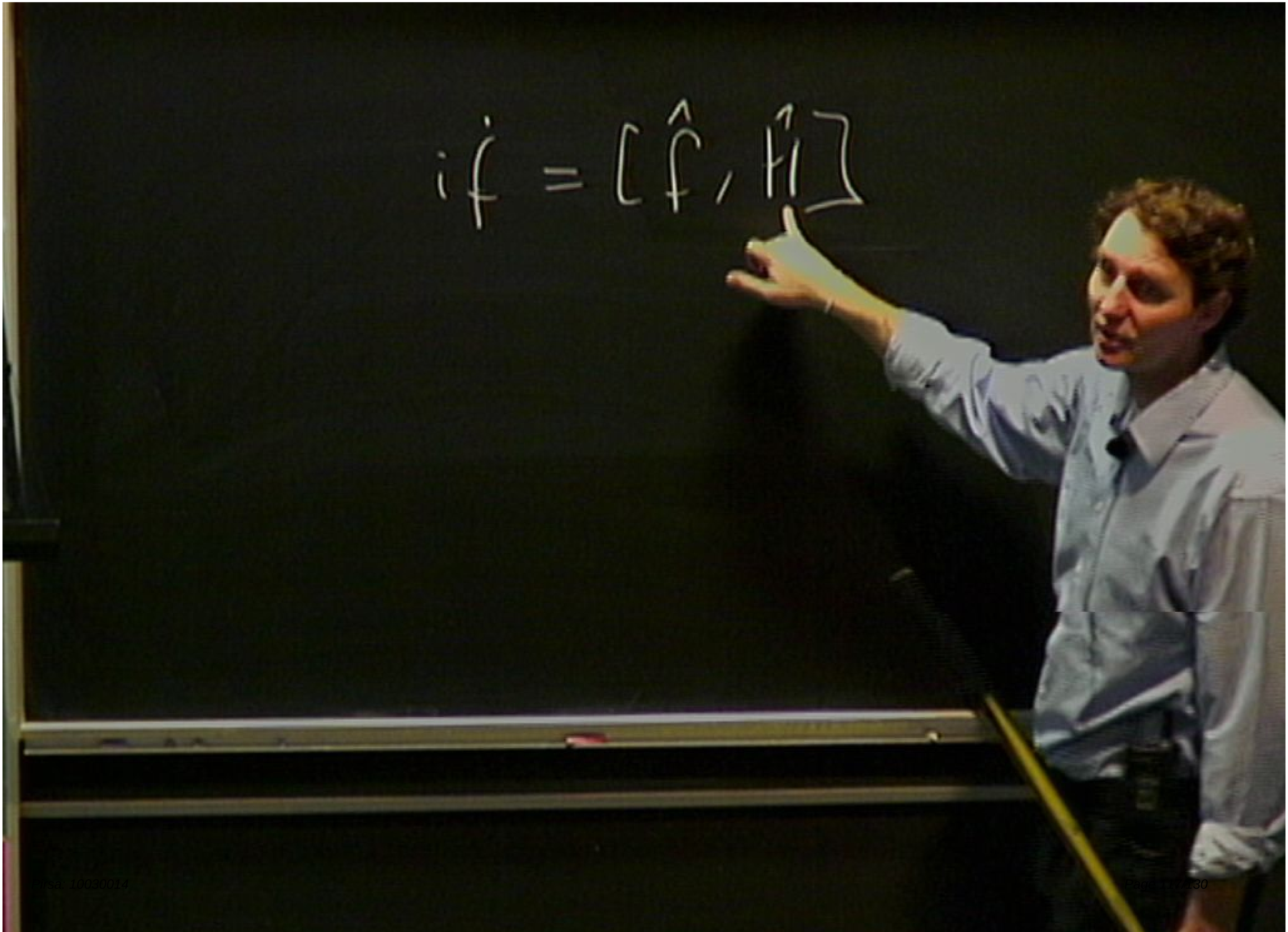
Recall that in QM: $i\hat{Q}' = [\hat{Q}, \hat{H}] + i\frac{\partial}{\partial t}\hat{Q}$

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Question:

How can both be valid generators of time evolution,

i.e., how can we have:

$$i\hat{\phi}' = [\hat{\phi}, \hat{H}^{(\phi)}] \quad \text{and} \quad i\hat{x}' = [\hat{x}, \hat{H}^{(x)}]$$

and yet $\hat{H}^{(\phi)} \neq \hat{H}^{(x)}$?

- Should there not be one Hamiltonian for all variables?

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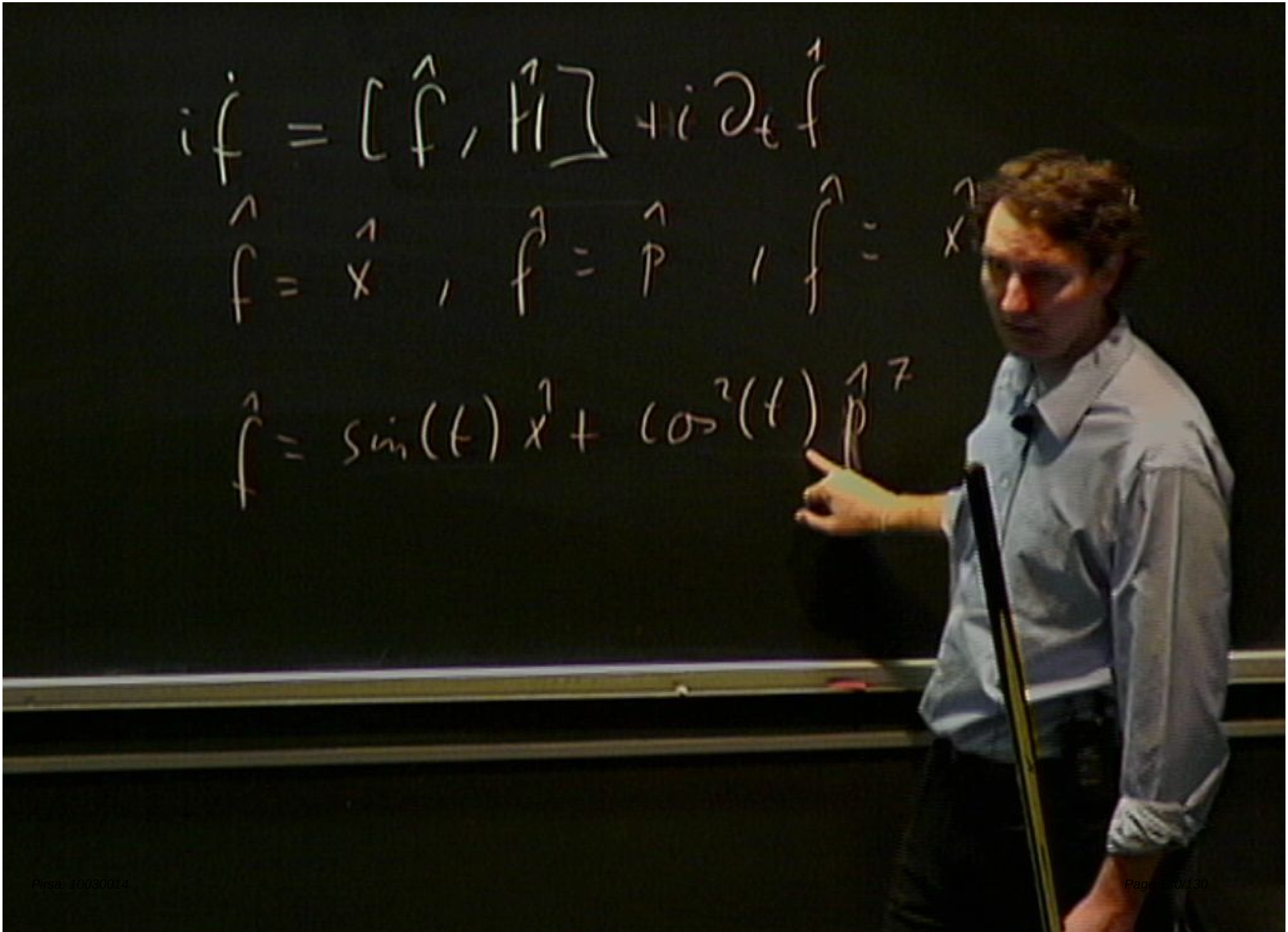
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$$\underbrace{R_{\mu\nu}(x) - \tfrac{1}{2} g_{\mu\nu}(x) R(x) + \Lambda g_{\mu\nu}(x)}_{\text{curvature}} = 8\pi G \underbrace{T_{\mu\nu}(x)}_{\text{"energy momentum"}}$$

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$$T_{00}(\eta,\vec{x}) = a^{-4}\frac{1}{2}\pi^{(\phi)2} + \frac{1}{2}\sum_{i=1}^{3}\phi_{,i}^{\,2} + \frac{a^2}{2} m^2 \phi^2 \qquad (T)$$

- **Exercises:**

a) Verify (T).

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Exercises:

a) Verify (T).

b) Calculate $H^{(\phi)}$.

Notice that $H^{(\phi)}$ is not a scalar.

c) Show that $H^{(\phi)}(\eta) = \int_{\mathbb{R}^3} T^{0}_{\ 0}(\eta,\vec{x})\sqrt{g}\; d^3x$.

d) Calculate $H^{(\phi)}(\eta)$.