Title: Quantum Field Theory for Cosmology - Lecture 19

Date: Mar 23, 2010 04:00 PM

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Abstract:

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# QFT for Cosmology, Achim Kempf, Winter 10, Lecture 19

Note Title 3/20/2006

## Example: QFT in de Sitter spacetime

- The deSitter FRW spacetime can be defined through this scale factor function:

$$\boxed{a(t) := e^{Ht}}$$

Notes:

* $t$ is the time on a comoving observer's wrist watch
* large $H \Leftrightarrow$ large acceleration

where $H>0$ is a constant, the "Hubble constant".

- Exercise: Read Mukhanov's comments on de Sitter space.

- This case is a preparation for inflationary cosmology, where $a(t)$ is assumed close to exponential for a short period in the very early

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## The de Sitter horizon

Proposition: (in particle picture)

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# The de Sitter horizon

**Proposition:** (in particle picture)

Objects (or any observers) who are further apart than a proper distance of $d_H = 2/H$ can never meet, and cannot communicate.

Note: large $H \Leftrightarrow$ small horizon $d_H$

**Proof:**

* Suppose, e.g., an observer in galaxy A at time, say $t=0$, emits a radio signal from his position, say $x=0$, towards an observer in galaxy B.

* The signal travels in a small time $\Delta t$ the the small comoving distance $\Delta x$:

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proper distance

our unit convention here

$$\frac{a(t)\,\Delta x}{\Delta t} = c = 1$$

speed of light

$$\Rightarrow \qquad \frac{dx}{dt} = a^{-1}(t)$$



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$\Rightarrow$ $$x(t) = -\frac{1}{H} e^{-Ht} + C_1$$

Fix the integration constant $C_1$ so that $x(0) = 0$:

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$\Rightarrow$ Any signal (and any massive object) can travel at most the comoving distance $1/H$, no matter how long it travels.

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$\Rightarrow$ Any two observers further apart than a comoving distance of $2/H$ can never meet or communicate!

Interpretation:

In the case where a deSitter exponential expansion



Remark: Notice that the

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Remark: Notice that the de Sitter horizon is

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Interpretation:

In the case where a deSitter exponential expansion lasts forever, between any objects of comoving distance $> 2/H$ space is being created faster the

Remark: Notice that the de Sitter horizon is [illegible] in time.

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Proposition: (in wave picture)

Klein Gordon modes oscillate while their proper

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**Proof:** 1) Let us switch to conformal time: (Thus, need $a(\eta)$!)

- Recall: $\eta(t) := \int^t \frac{1}{a(t')} dt'$

The choices of the integration constant $C_1$ merely mean different fixed shifts in the time coordinate $\eta$ relative to the time coordinate $t$.

here: $\eta(t) = \int^t e^{-Ht'} dt'$

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- **Notice:**

  - As $t \to -\infty$ we have $\eta \to -\infty$.

  - But as $t \to +\infty$ we have $\eta \to C_1$.

- Choose $C_1 = 0$:



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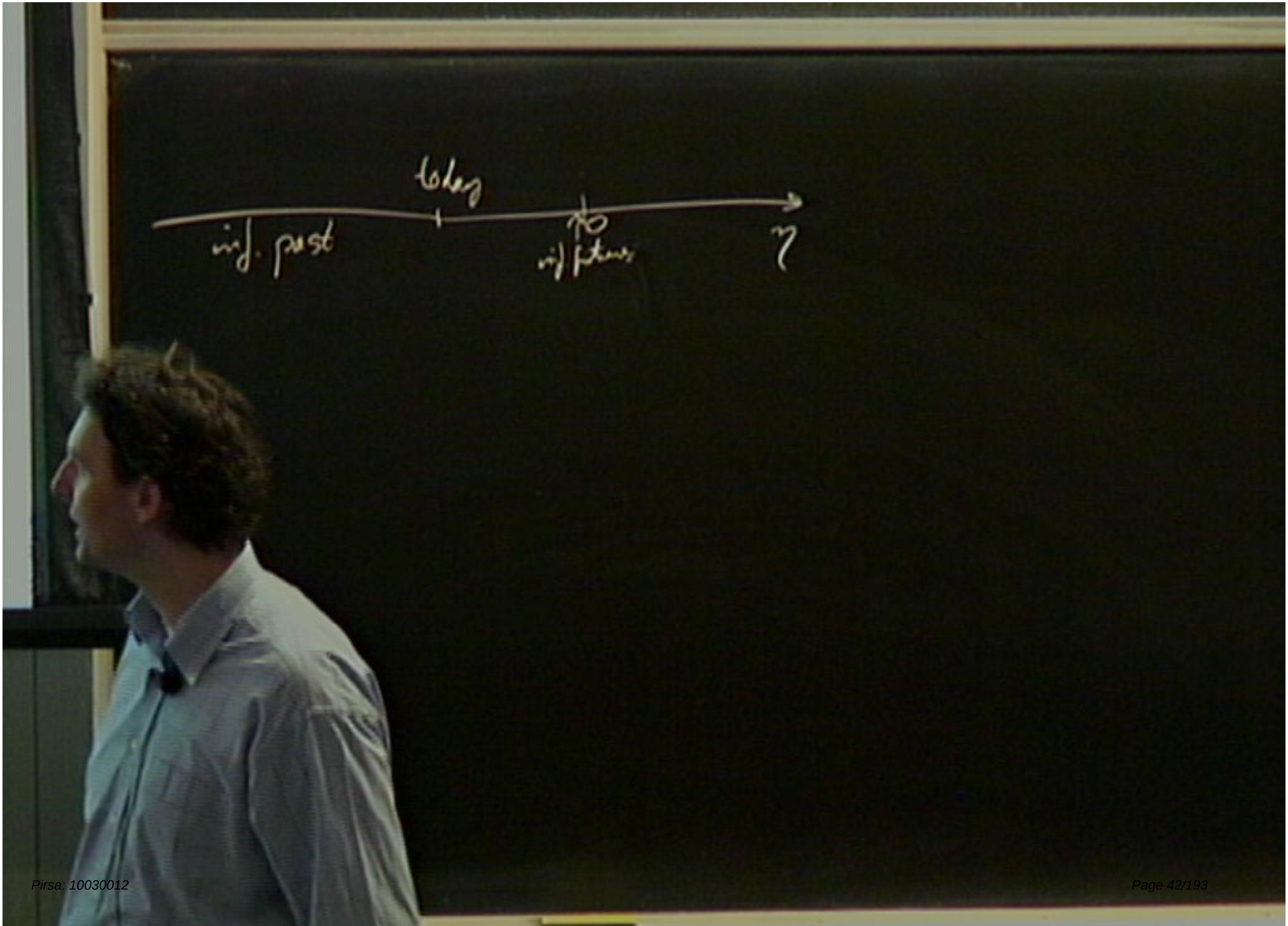
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□ Choose $C' = 0$:

$\Rightarrow$

$$\eta(t) = -\frac{1}{H}\frac{1}{a(t)}$$

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i.e.: $\boxed{a(\eta) = -\frac{1}{H\eta}}$

2) Introduce $\hat{\chi}_k(\eta) := a(\eta)\,\hat{\phi}_k(\eta)$:

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- $\hat{\chi}_k$ obeys this Klein Gordon equation

$$\hat{\chi}_k''(\eta) + \omega_k^2(\eta)\,\hat{\chi}_k(\eta) = 0$$

with:

$$\omega_k^2(\eta) = k^2 + m^2 a^2(\eta) - \frac{a''(\eta)}{a(\eta)}$$

- Exercise: Show that in the de Sitter case this yields:

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## Re-expressed in terms of proper wavelength?

Noting $|\eta| = \frac{1}{Ha}$ and multiplying it with $k = 2\pi/L$ we obtain:

$L$: comoving wavelength

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- A mode oscillates as long as: $|\eta| k \gg 1$

  This now means, if $\frac{2\pi}{H\lambda} \gg 1$, i.e., $\boxed{\text{if: } \lambda \ll \frac{1}{H}}$ ⓐ'

- A mode has imaginary frequency from when:

  $|\eta| k \ll 1$ , i.e., if $\boxed{\lambda \gg \frac{1}{H}}$ ⓑ'

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## The more realistic case of a deSitter expansion of a finite duration

Consider the case that spacetime was exponentially expanding only in a finite time interval:

$$\eta_i < \eta < \eta_f$$

and assume that spacetime was expanding slowly or was even Minkowski before $\eta_i$ and after $\eta_f$.

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- **Recall:** The time when a mode, $k$, crosses the horizon is:

$$\eta_{hor}(k) \approx -\frac{\sqrt{2}}{k} \qquad (\text{for } m \ll H)$$




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## ⟹ Three classes of modes:

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⟹ Three classes of modes:

1. "Small" modes:

These are modes which are so small that by the time their proper wavelength would reach the Hubble

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## ⇒ Three classes of modes:

### 1. "Small" modes:

These are modes which are so small that by the time their proper wavelength would reach the Hubble horizon length the de Sitter period is already over:

$$\eta_{hor}(k) >> \eta_f$$

Recall: Both sides are negative

i.e.:

$$\frac{\sqrt{2}}{k} \ll |\eta_f|$$

Recall: $\eta_{hor} \approx \frac{\sqrt{2}}{k}$

$$L \ll |\eta_f|$$

Their quantum fluctuations do not get "amplified", as we will see.

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2. "Medium" size modes:

These are the modes which do cross the horizon because

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The quantum fluctuations of those modes are of cosmological interest.

## 3. "Large" modes:

These are modes which were larger than the horizon already at $\eta_i$. In the inflationary scenario they are today very much larger than the visible universe. They may only contribute, effectively, like a cosmological constant - and may even be the origin of $\Lambda$.

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Quantum fluctuations in de Sitter space.

The usual ansatz

$$\hat{\chi}_k(\eta) = \frac{1}{\sqrt{2}}\left(v_k^*(\eta)\, a_k + v_k(\eta)\, a_k^+\right)$$

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(not complex conjugation, just another symbol)

generalizations of sine and cosine

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generalizations of sine and cosine

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generalizations of sine and cosine

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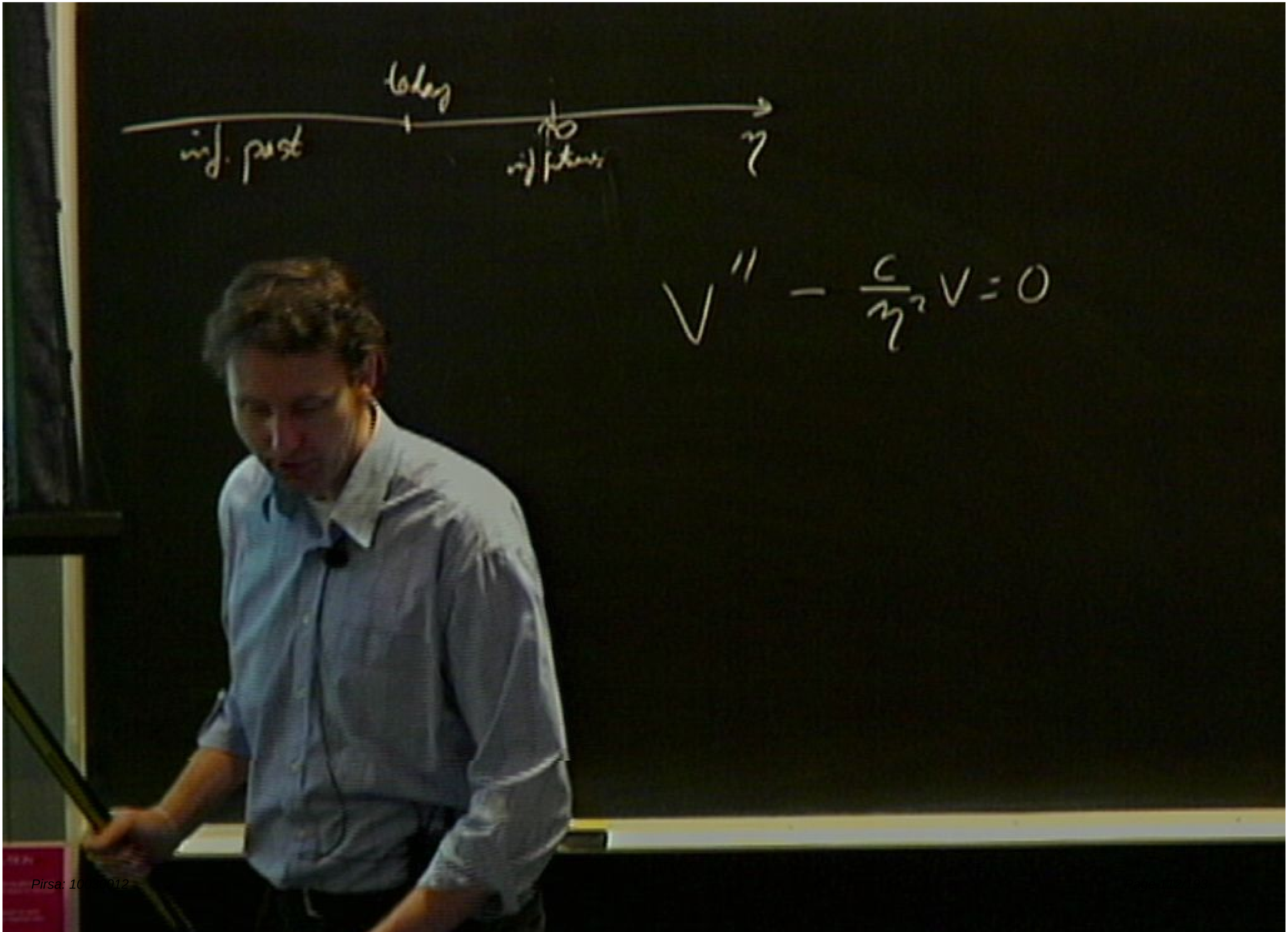

today
inf. past
inf. future
η
V'' − c/η² V = 0

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Thus: every mode function $v_k$ is a linear combination for example, by these two real-valued Bessel functions

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## How to identify the state of the system?

Strategy:

a. Check if modes start out in an adiabatic

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Strategy:

a. Check if modes start out in an adiabatic regime (the small and medium ones do).

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Indeed, we see from the K.G. eqn.

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Indeed, we see from the K.G. eqn.

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Thus, $v_k$ is the usual Minkowski mode function at early times:

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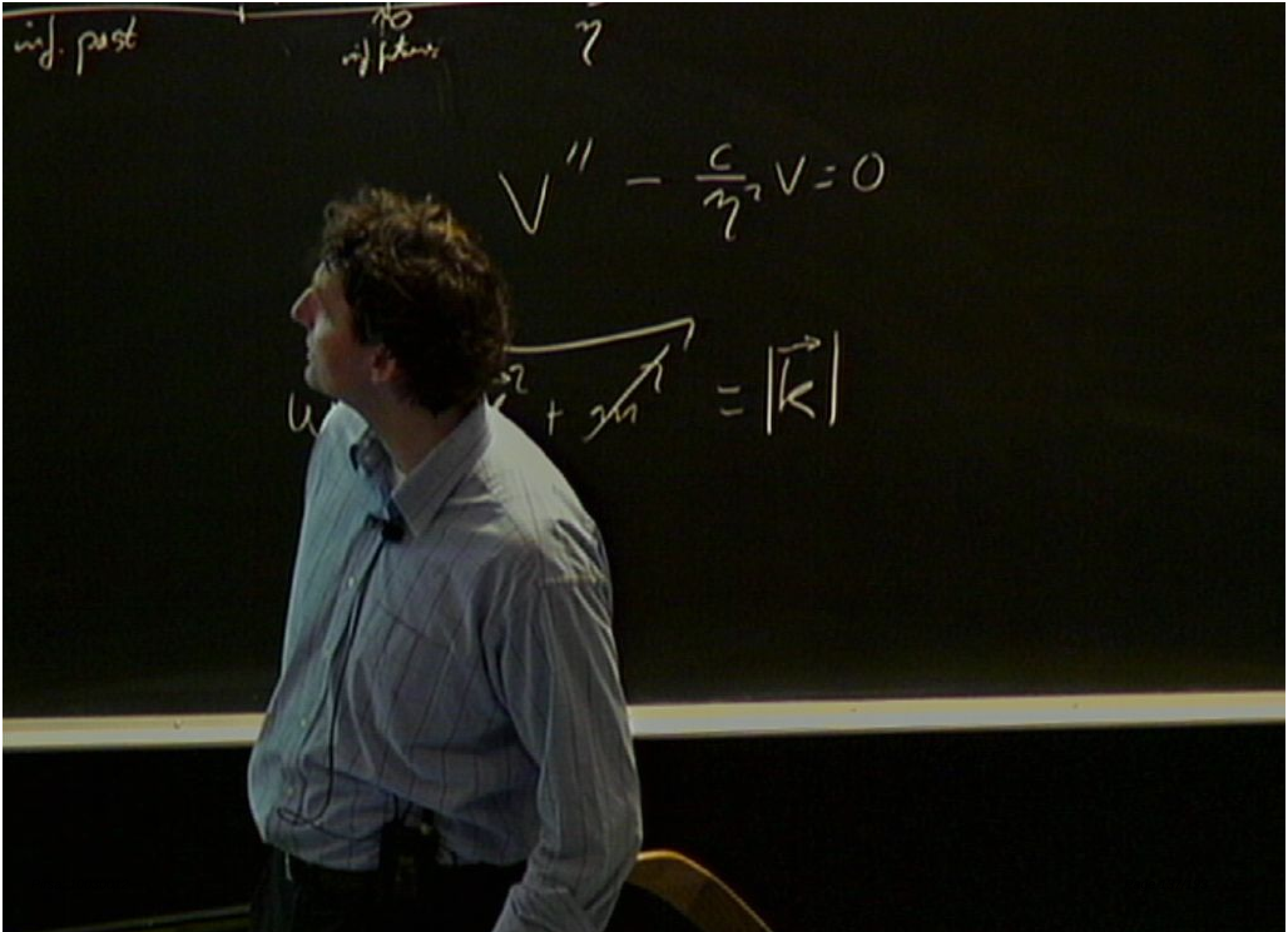
inf. past
inf. future
η
V'' − c/η² V = 0
= |k⃗|

inf. past $\qquad \eta_0$ inf. future $\qquad \eta$

$$V'' - \frac{c}{\eta^2} V = 0$$

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**Technical observation:** At early times, $\eta << 0$:

$$u_k(\eta) \approx \sqrt{\frac{2}{\pi}} \cos(k|\eta| + \text{const})$$

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Fluctuations with large prop...

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- Let us evaluate the fluctuation spectrum

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$A_k$

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for massless fields in Minkowski space:

Fluctuations with large proper spatial extent $\lambda$ are suppressed.

# Case 2: Medium size modes.

They are those with $k$ so that in

$$v_k''(\eta) + \left(k^2 + \frac{m^2}{H^2\eta^2} - \frac{2}{\eta^2}\right) v_k(\eta) = 0$$

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## Case 1: Very small modes

- They are those with $k$ large enough, so that in

$$v_k''(\eta) + \left(k^2 + \frac{m^2}{H^2\eta^2} - \frac{2}{\eta^2}\right) v_k(\eta) = 0$$

the $k^2$ term dominates all through the expansion.

- These modes never cross the horizon and we have, approximately:



I.e., the Bessel functions in

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In terms of $u_k$, $\bar{u}_k$ the mode function $v_k$ reads:

$$v_k(\eta) = \underbrace{\sqrt{\frac{\pi}{2k}}}_{= A_k} u_k(\eta) - i \underbrace{\sqrt{\frac{\pi}{2k}}}_{= B_k} \bar{u}_k(\eta)$$

i.e.: $$\boxed{v_k(\eta) = \sqrt{\frac{\pi|\eta|}{2}} \left( J_m(k|\eta|) - i Y_m(k|\eta|) \right)}$$

**Proof:** Exercise.

d. Now we can calculate $\delta\phi_k$ at the end of the exponential expansion, $\eta_f$, namely:

$$\delta\phi_k(\eta_f)^2 = a^{-2}(\eta_f) k^3 |v_k(\eta_f)|^2$$

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Recall: This is the usual fluctuation spectrum for massless fields in Minkowski space:

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$$\Rightarrow \delta\phi_L(\eta_f) \approx H\left(\frac{|\eta|}{L}\right)^{\overbrace{\frac{3}{2}-n}^{\simeq 0}} \cdot \Gamma(n) \frac{2^n}{\pi}$$

independent of $\eta_f$! ⇒ May as well evaluate right after horizon crossing.

For comparison, recall case 1, small modes, whose fluctuation ...

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$\delta\phi_\lambda = 1/\lambda$

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$\Rightarrow$ The medium sized modes get amplified just enough so that the usual suppression of fluctuations of large spatial extent is compensated.

$\Rightarrow$ The quantum fluctuations of a comoving mode

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even though their physical wavelength grows!

Indeed: $\delta\phi_L(\eta_f)$ does not depend on $\eta_f$: Fluctuations stay of same amplitude during de Sitter expansion.

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As we'll see, in a suitable

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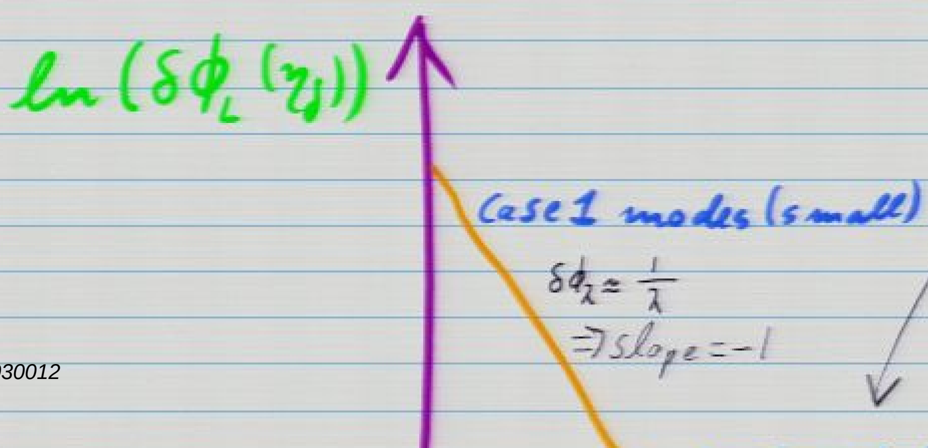
$\Rightarrow$ After exponential expansion:

As we'll see, in a suitable model of very early universe cosmology, "medium size" can be arranged to mean modes that are today at cosmological scales.




unknown significance (depends on assumptions

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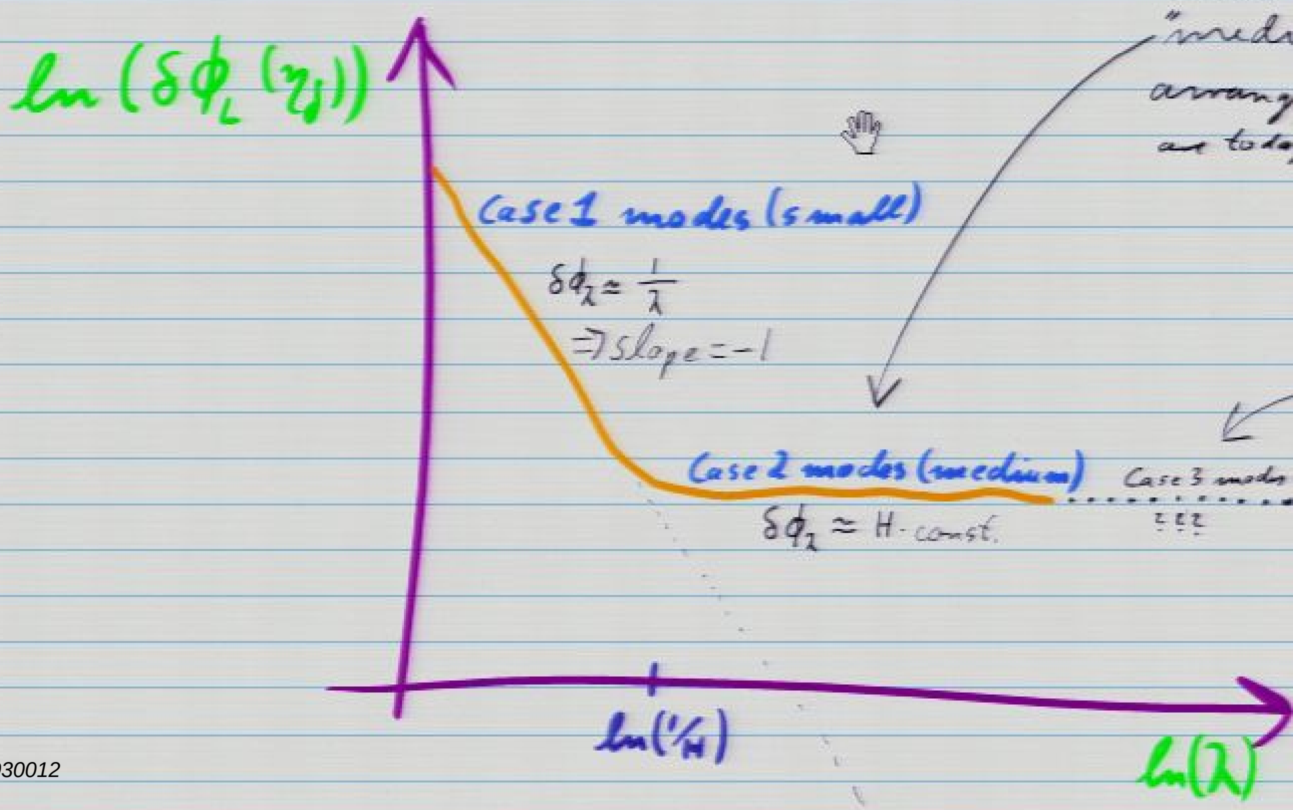
## ⇒ After exponential expansion:

As we'll see, in a suitable model of very early universe cosmology, "medium size" can be arranged to mean modes that are today at cosmological scales.

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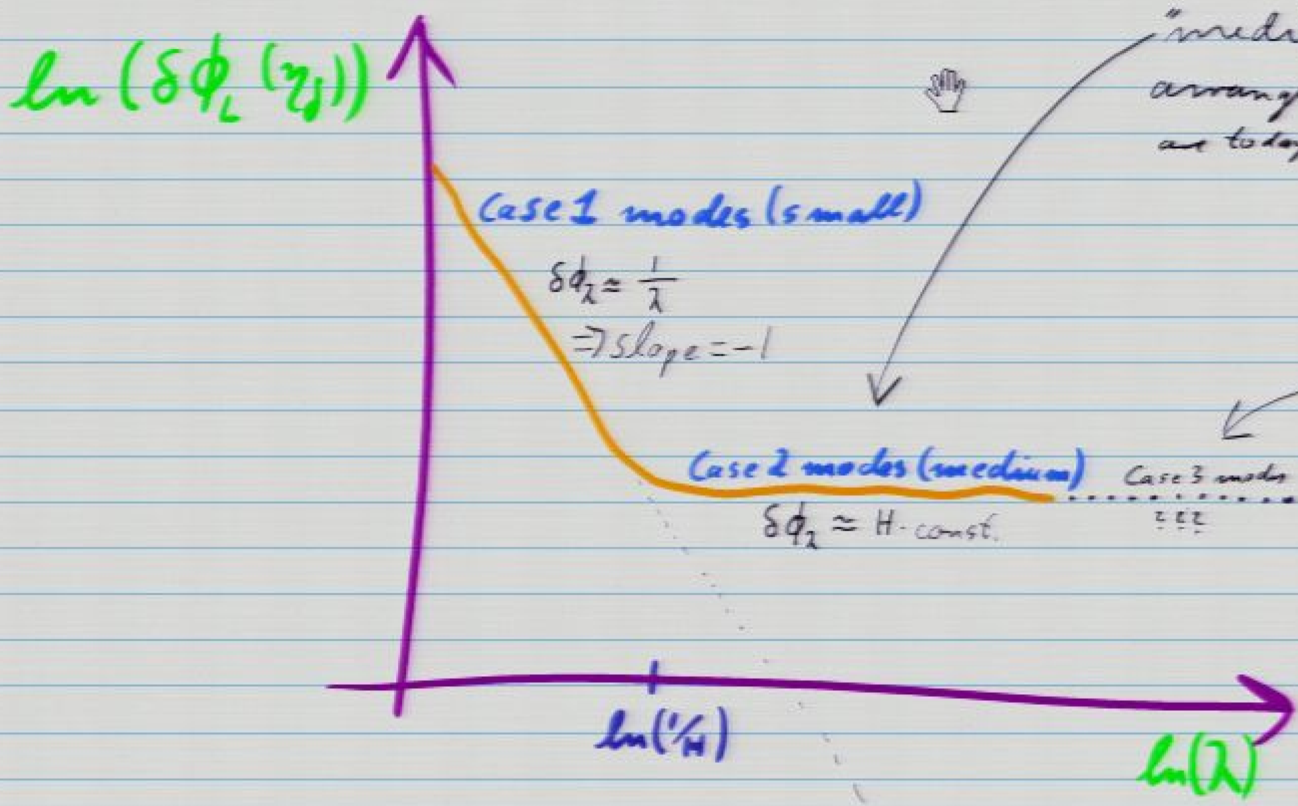


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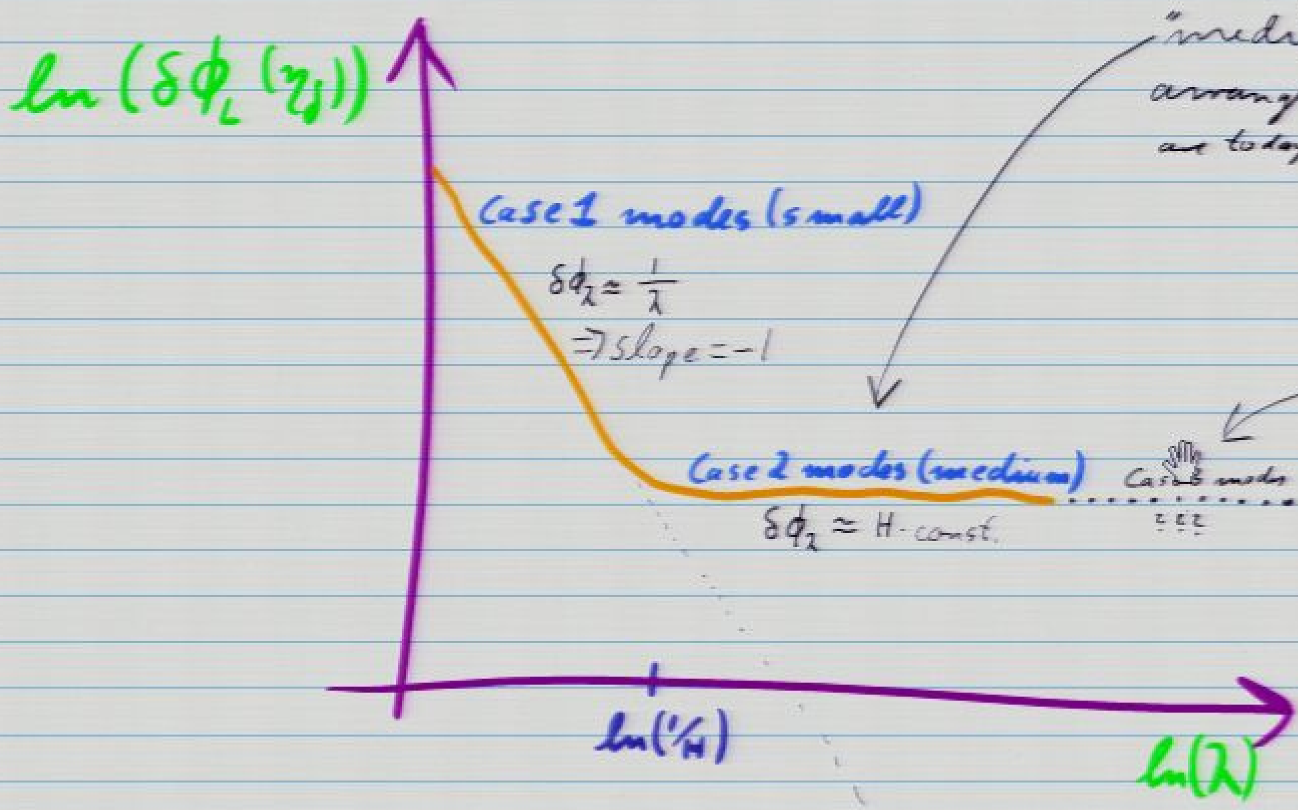


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