Title: The princess and the EPR pair

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Abstract: In quantum information, entanglement has often been viewed as a resource. But in this talk, I will look at (pure bipartite) entanglement through the lens of superselection rules. The idea is that it requires quantum communication not only to create entanglement, but also to destroy it in a way that doesn't leak information to the environment. As a result, when communication is scarce, superpositions of different numbers of EPR pairs can be difficult to obtain. This constraint is not a strict superselection rule, but rather an approximate version that gives rigorous bounds on achievable fidelities. After describing the general phenomenon, I will show how it relates to communication complexity, information theory and fairy tales about princesses. This talk is based on 0803.3066, 0909.1557, 0912.5537 and other unpublished work.

The princess and the EPR pair

$\Big($ |  $\rangle$ + |  $\rangle$ $\Big) / \sqrt{2}$

or

Entanglement spread,
communication complexity and
information theory

Aram Harrow
University of Bristol
8 Feb, 2010

entanglement as resource

A familiar story:

Alice and Bob share $|\psi\rangle = \sum_i c_{ij} \, |i\rangle \otimes |j\rangle$.

LO (local operations) + CC (classical communication) are free.

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But what if classical communication isn't free?

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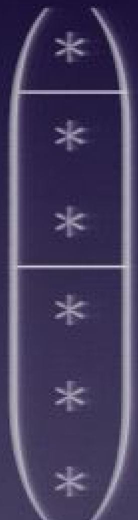
The state space is partitioned according to some observable, such as total particle number.



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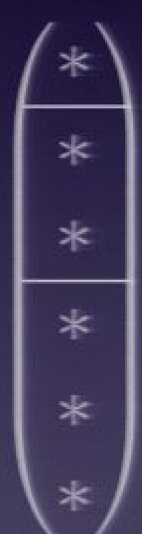

Measurements, Hamiltonians and unitaries are constrained to respect this partition.

entanglement as conserved quantity

Idea:

If Alice and Bob are allowed only local unitaries (LU) then the Schmidt coefficients of their state remain exactly the same.

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Really?

1. $|\Phi_2\rangle^{\otimes k}$ and $|\Phi_2\rangle^{\otimes l}$ are only approximately orthogonal.
2. Technically we can only approximately decompose $|\psi\rangle$ into

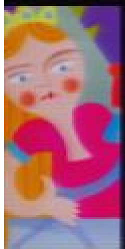
$$\sum_{k\geq 0} \sqrt{p_k}|k\rangle_A|k\rangle_B|\Phi_{\lfloor 2^{(1+\epsilon)k}\rfloor}\rangle_{AB}$$

implications

1. Any transformation using local unitaries and Q qubits of communication has off-diagonal blocks decaying as $\leq 2^{Q-\frac{|k-\ell|}{2}}$

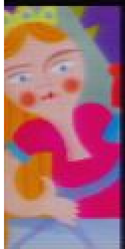
$$\begin{pmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{pmatrix}$$

2. 'Exotic' states, such as $|01\rangle^{\otimes n} \pm |\Phi_2\rangle^{\otimes n} / \sqrt{2}$, should be difficult to create, and are potentially valuable.



A bipartite fairy tale

Traditional version: A mysterious woman appears at the castle claiming to be a princess. That night, a single pea placed under twenty mattresses keeps her from sleeping. The prince realises that she is genuine and immediately asks her to marry him.



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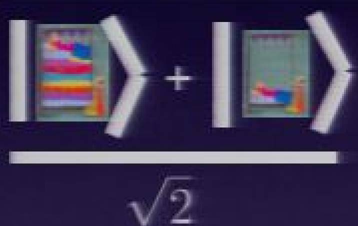
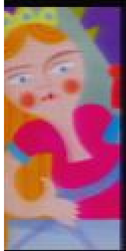


from any orthogonal state. In particular, she can distinguish it from



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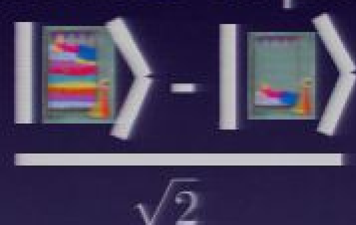
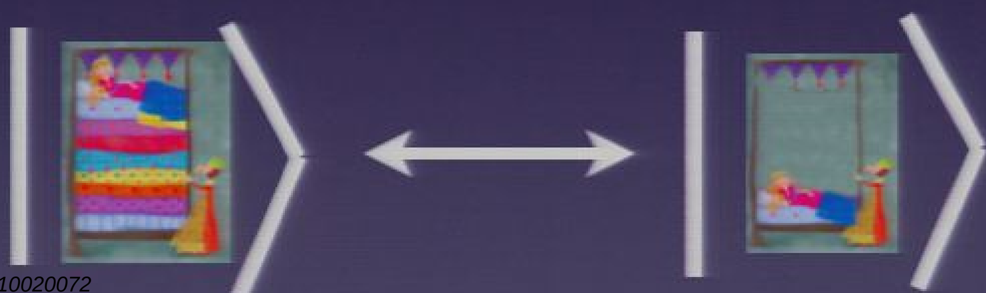
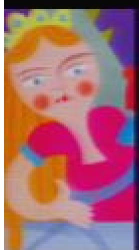


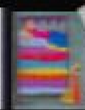
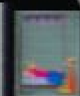
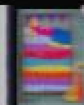
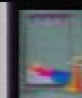
However! Adding or removing lots of mattresses is difficult.



requires

Should he marry her?



Distinguishing (|  ⟩ + |  ⟩) / $\sqrt{2}$ from (|  ⟩ - |  ⟩) / $\sqrt{2}$ with a reversible quantum circuit allows us to apply a phase (-1) to one of the states.

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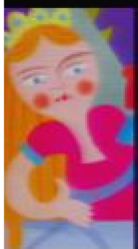
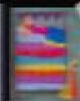
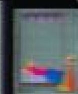
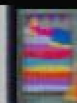
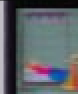
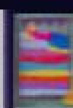
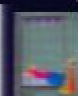
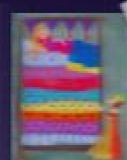
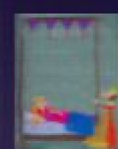
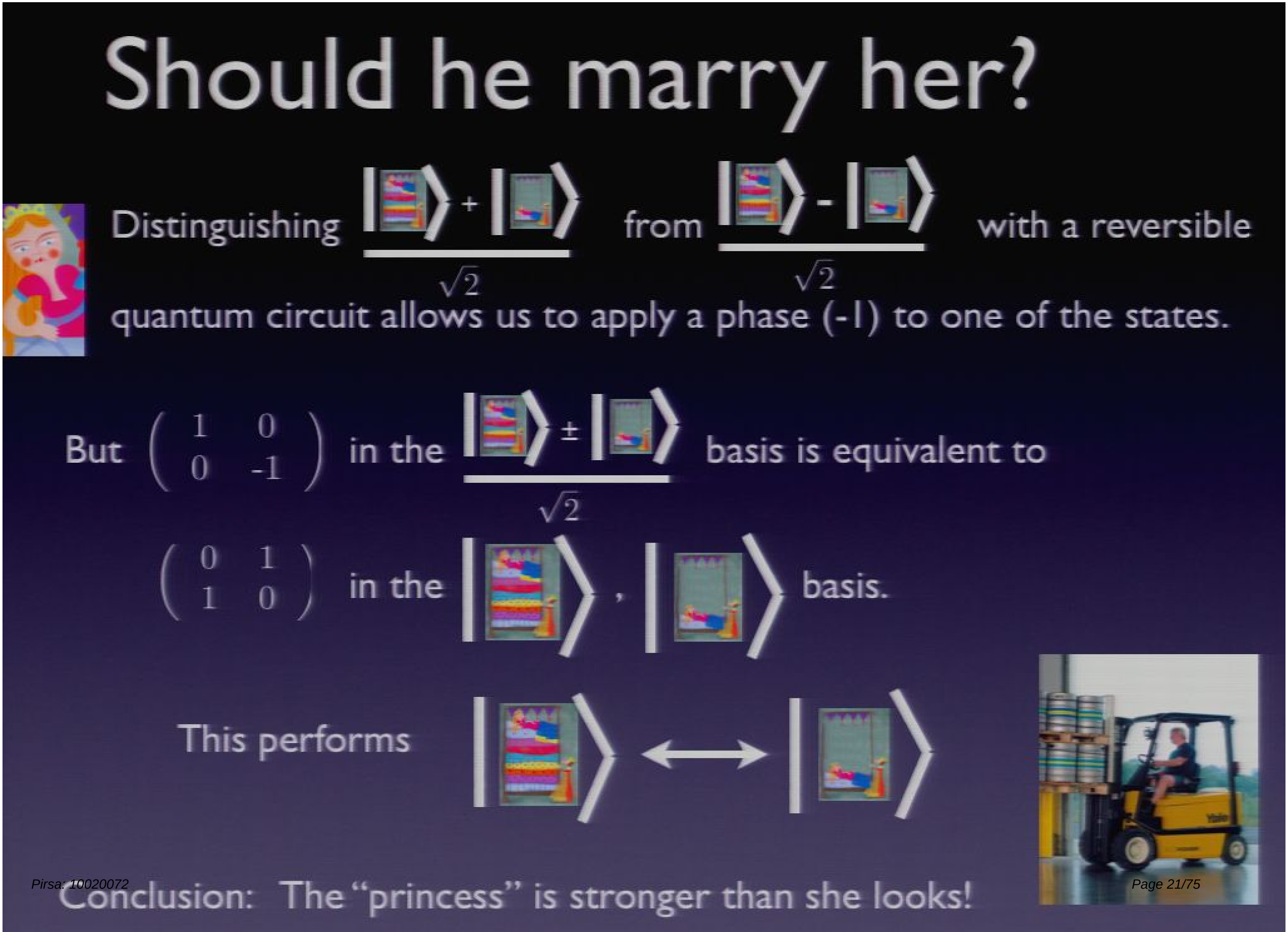


Distinguishing (|⟩ + |⟩) / $\sqrt{2}$ from (|⟩ - |⟩) / $\sqrt{2}$ with a reversible quantum circuit allows us to apply a phase (-1) to one of the states.

But $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ in the (|⟩ ± |⟩) / $\sqrt{2}$ basis is equivalent to

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ in the |⟩, |⟩ basis.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This performs
Conclusion: The "princess" is stronger than she looks!

The relevance to entanglement

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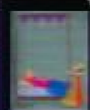
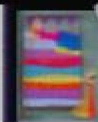
$|01\rangle^{\otimes n} =$ |  ⟩

$|\Phi_2\rangle^{\otimes n} =$ |  ⟩

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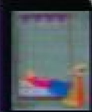
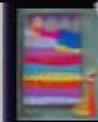
  $|01\rangle^{\otimes n} =$ |  ⟩ $|\Phi_2\rangle^{\otimes n} =$ |  ⟩

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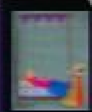
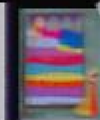
  $|01\rangle^{\otimes n} =$ |  $\rangle$ $|\Phi_2\rangle^{\otimes n} =$ |  $\rangle$

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- This bound holds even given unlimited EPR pairs.

  Why? Because for any m, the same argument applies to the states

  $$|\Phi_2\rangle^{\otimes m} \otimes (|01\rangle^{\otimes n} \pm |\Phi_2\rangle^{\otimes n} / \sqrt{2})$$

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For EPR pairs $r\lambda_1 = 1$.

[P. Hayden, A. Winter. quant-ph/0204092]

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- Approximate versions also exist.

- If $|\psi\rangle = \sum_k \sqrt{p_k}\, |k\rangle|k\rangle|\Phi_2\rangle^{\otimes k}$, then

  $\log(r) \approx \max\{k : p_k > 0\}$ and $\log(\lambda_1) \approx -\min\{k : p_k > 0\}$.

  So the spread of $|\psi\rangle \approx$ the diameter of the support of p.

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- Example: If $|\psi\rangle$ is an entangled state, then $|\psi\rangle^{\otimes n}$ is very close to a state with spread $O(\sqrt{n})$.
  Therefore, $O(\sqrt{n})$ bits of communication are necessary and sufficent to prepare $|\psi\rangle^{\otimes n}$ from EPR pairs. (a.k.a. entanglement dilution.) [Harrow and Lo; quant-ph/0204096]

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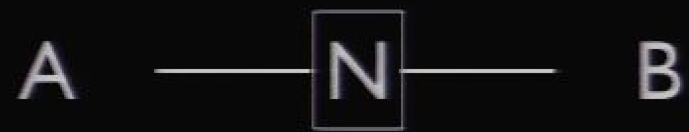

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Any noisy channel N using input distribution p^A can code at rate $C_{N,p} = H(A)_p + H(B)_p - H(AB)_p$.

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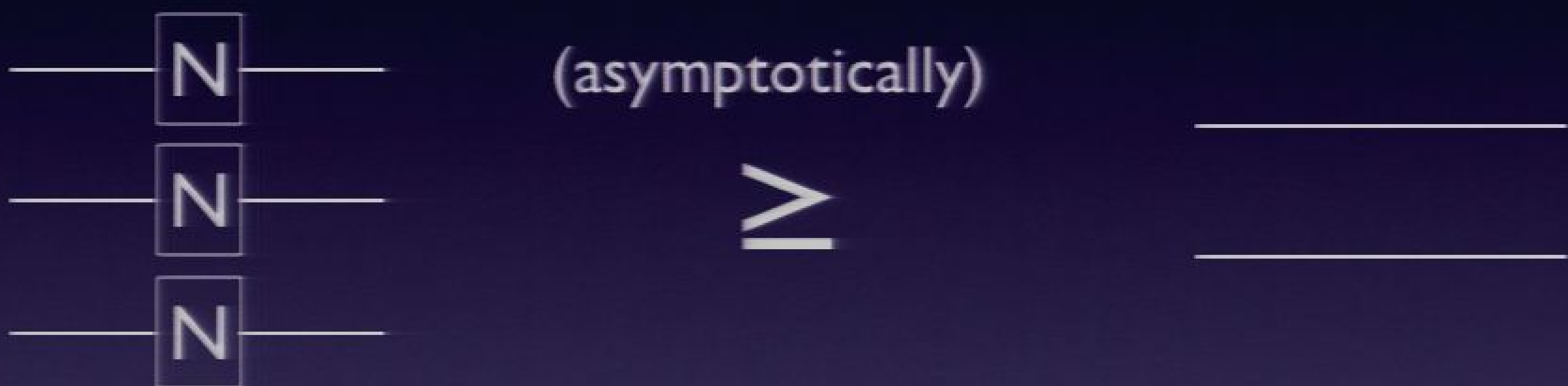

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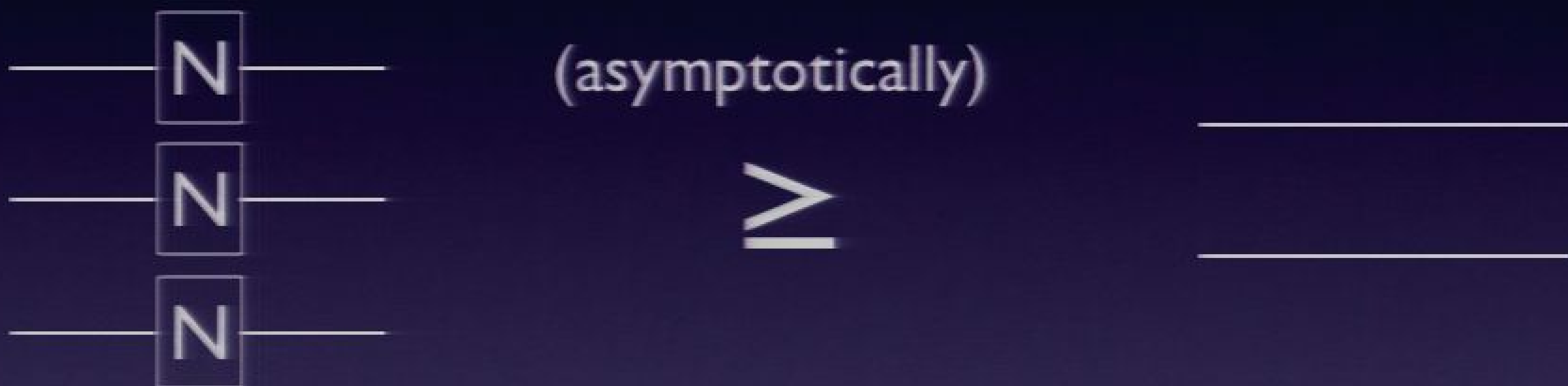

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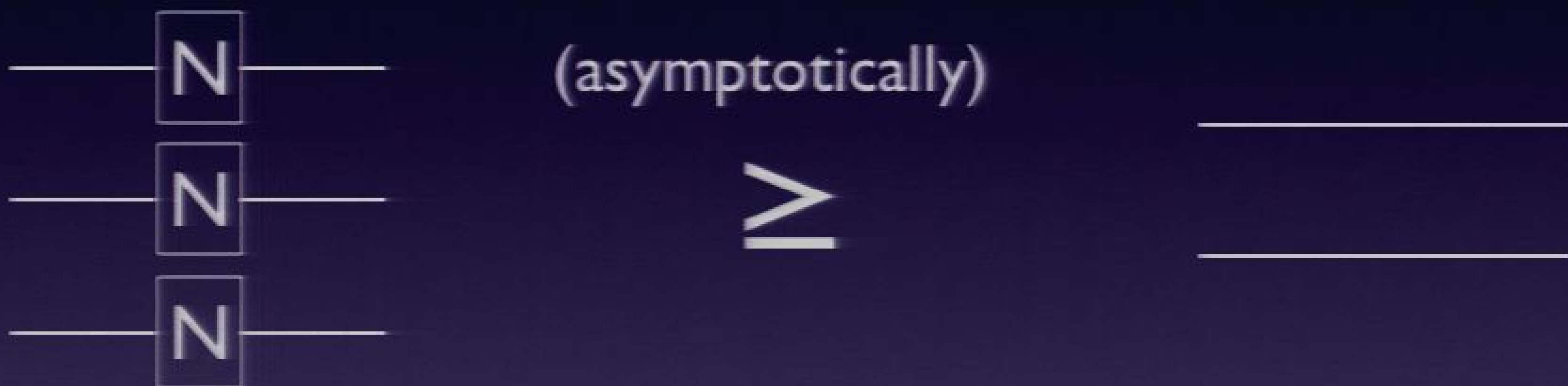

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<u>On general inputs</u>:

The capacity and simulation cost are replaced by $C(N) = \max_p C_{N,p}$.

Randomness cost for simulation is $\max_p H(B)_p - C(N)$.

Simulating quantum channels

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- Coding with quantum channels: Using shared EPR pairs, a quantum channel $\mathcal{N}$ can send noiseless qubits at rate $\max_\rho Q_{\mathcal{N},\rho} = \max_\rho (H(A)_\rho + H(B)_\rho - H(AB)_\rho) / 2$.

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[Bennett, Devetak, Harrow, Shor, Winter; arXiv:0912.5537]
- However, it does not follow that $\mathcal{N}^{\otimes n}$ can be simulated on arbitrary inputs using $\max_\rho(Q_{\mathcal{N},\rho})$ qubits of communication and $\max_\rho(E_{\mathcal{N},\rho})$ shared EPR pairs!

Simulating quantum channels

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- Quantum Reverse Shannon Theorem: For a quantum channel $\mathcal{N}$ and an input distribution ρ, $\mathcal{N}^{\otimes n}$ can be simulated on $\rho^{\otimes n}$ using $Q_{\mathcal{N},\rho}$ qubits of communication and $E_{\mathcal{N},\rho} = H(B)_\rho - Q_{\mathcal{N},\rho}$ shared EPR pairs. [Bennett, Devetak, Harrow, Shor, Winter; arXiv:0912.5537]
- However, it does not follow that $\mathcal{N}^{\otimes n}$ can be simulated on arbitrary inputs using $\max_\rho(Q_{\mathcal{N},\rho})$ qubits of communication and $\max_\rho(E_{\mathcal{N},\rho})$ shared EPR pairs!
- Problem: suppose that the input to $\mathcal{N}^{\otimes n}$ is $(\rho^{\otimes n} + \sigma^{\otimes n})/2$ with $Q_{\mathcal{N},\rho} = Q_{\mathcal{N},\sigma}$ but $E_{\mathcal{N},\rho} > E_{\mathcal{N},\sigma}$. Then the naive method of combining the two simulations will require creating $n(E_{\mathcal{N},\rho} - E_{\mathcal{N},\sigma})$ entanglement spread.

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- This requires either extra communication (forward or back) or embezzling states.

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- Question: When do other forms of entanglement help more than EPR pairs? Simulating noisy quantum channels. More examples to follow.
- Communication complexity: Special case in which Alice holds $x \in \{0,1\}^n$, Bob holds $y \in \{0,1\}^n$ and they want to compute the bit $f(x,y)$.

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- When communication is not free, EPR pairs are one of the weakest forms of entanglement.
- On the other hand, there is a family of $k \times k$-qubit "embezzling states" [van Dam and Hayden. quant-ph/0201041]

$$|\zeta_k\rangle \propto \sum_{i=1}^{2^k} \frac{1}{\sqrt{i}} |i\rangle \otimes |i\rangle$$

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- The proper definition of "free entanglement" is thus closer to "an embezzling state of arbitrary finite size" than "unlimited EPR pairs."
In particular, the entangled state in LOSE operations can be taken to be an embezzling state w.l.o.g.

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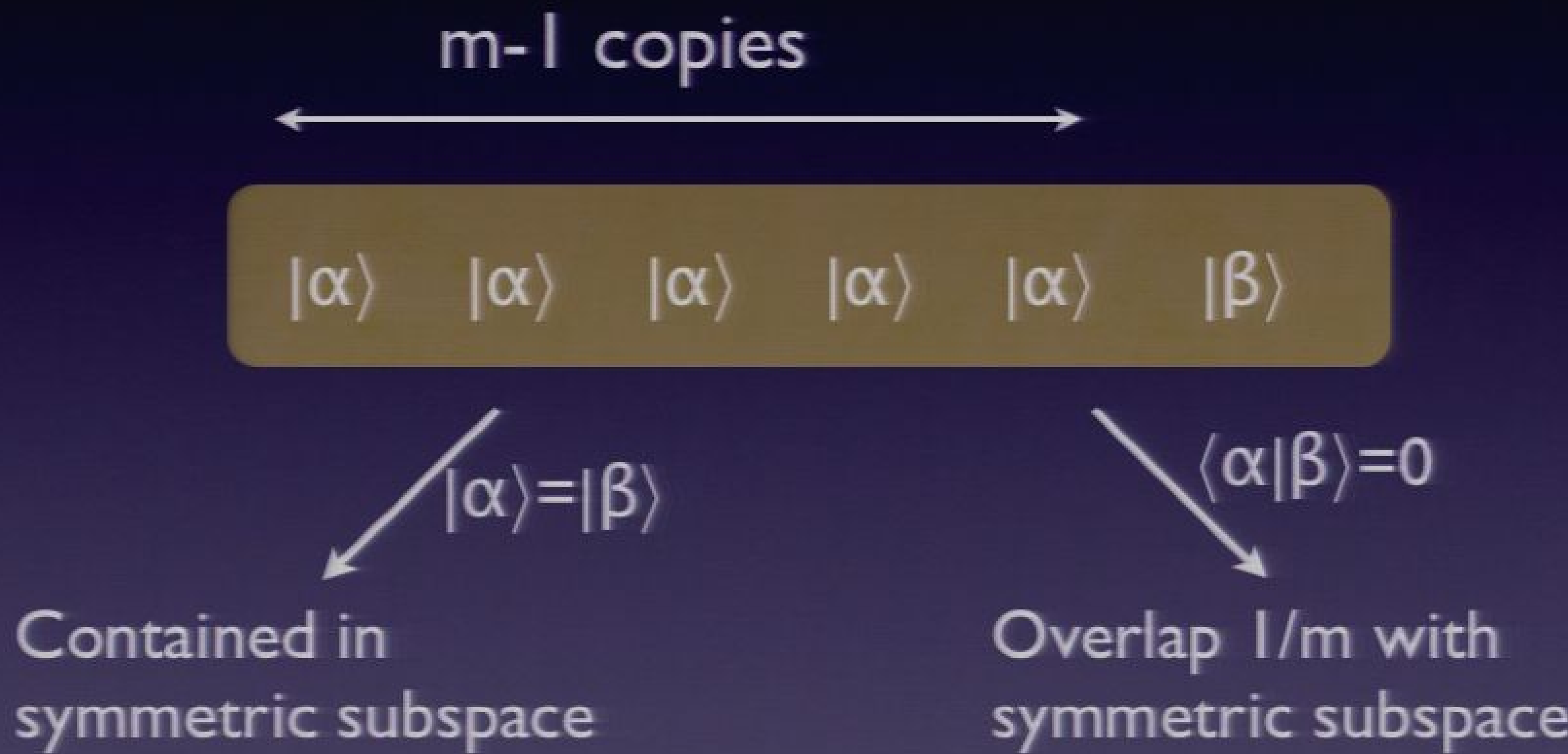

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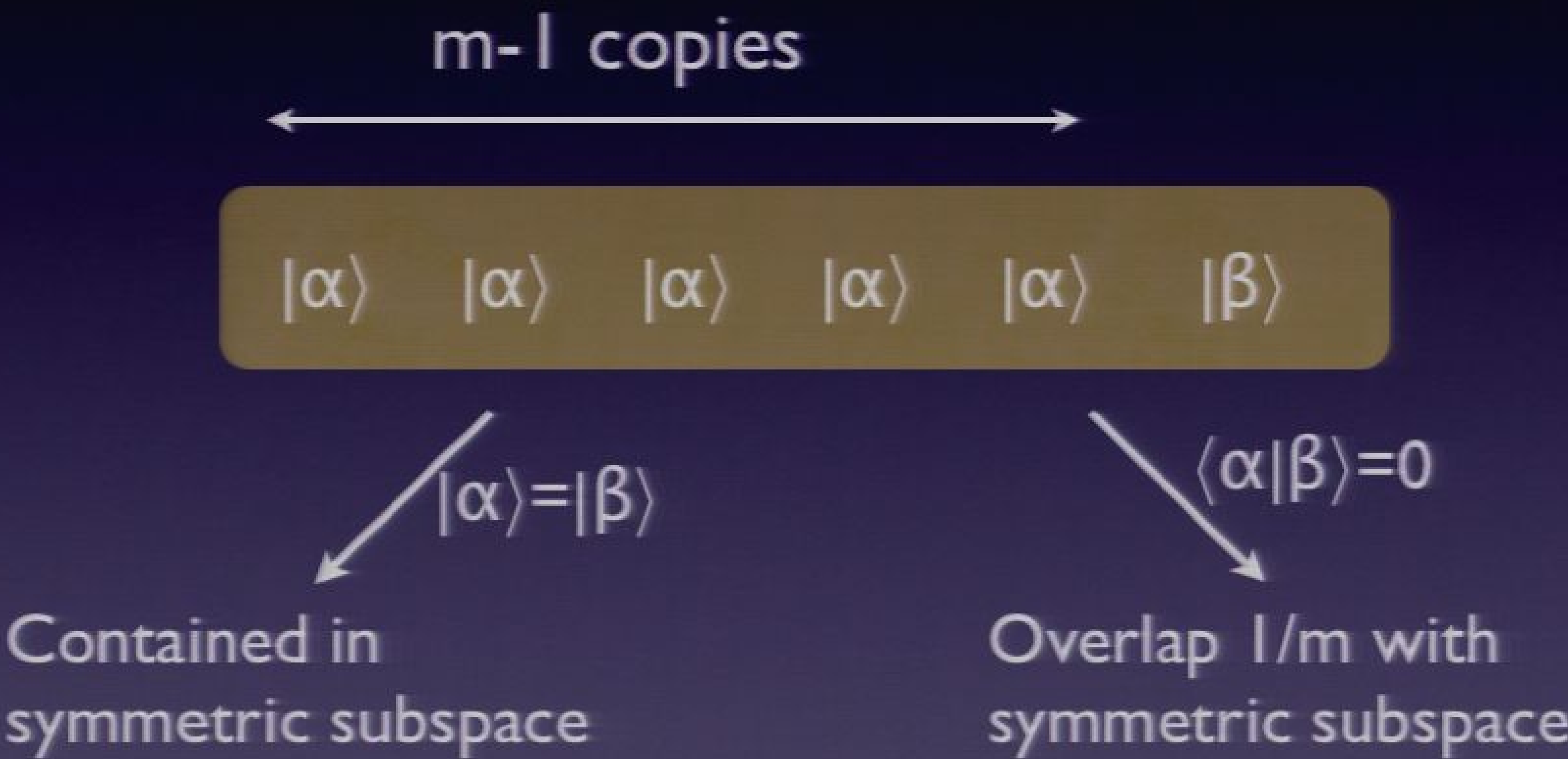

Problem reduces to projecting onto symmetric subspace.

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  - Simulating U requires O(n) qubits of communication, even using free EPR pairs.
  - With general entanglement, U can be simulated to accuracy ε using $O(\log 1/\varepsilon)$ qubits of communication.

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- Shared randomness is known to help, but by Newman's theorem, $O(\log n)$ bits of shared randomness always suffice.
- Free EPR pairs are known to help, although all known examples simply use them to turn classical communication into quantum communication.

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Proof: Let $|\psi\rangle = \sum_k \sqrt{p_k}\, |k\rangle|k\rangle|\Phi_2\rangle^{\otimes k}$ be our starting state, and $\mathcal{P}$ a protocol that uses Q qubits of communication. Then

$$\Pr[\text{accept}] - \Pr[\text{reject}] = \langle\psi|\, \mathcal{P}^\dagger\, (\sigma_z \otimes I^{\otimes \text{many}})\, \mathcal{P}\, |\psi\rangle$$

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Thus we can replace $|\psi\rangle$ with a mixture of states with spread $O(Q/\varepsilon)$ and incur error $\leq \varepsilon$.

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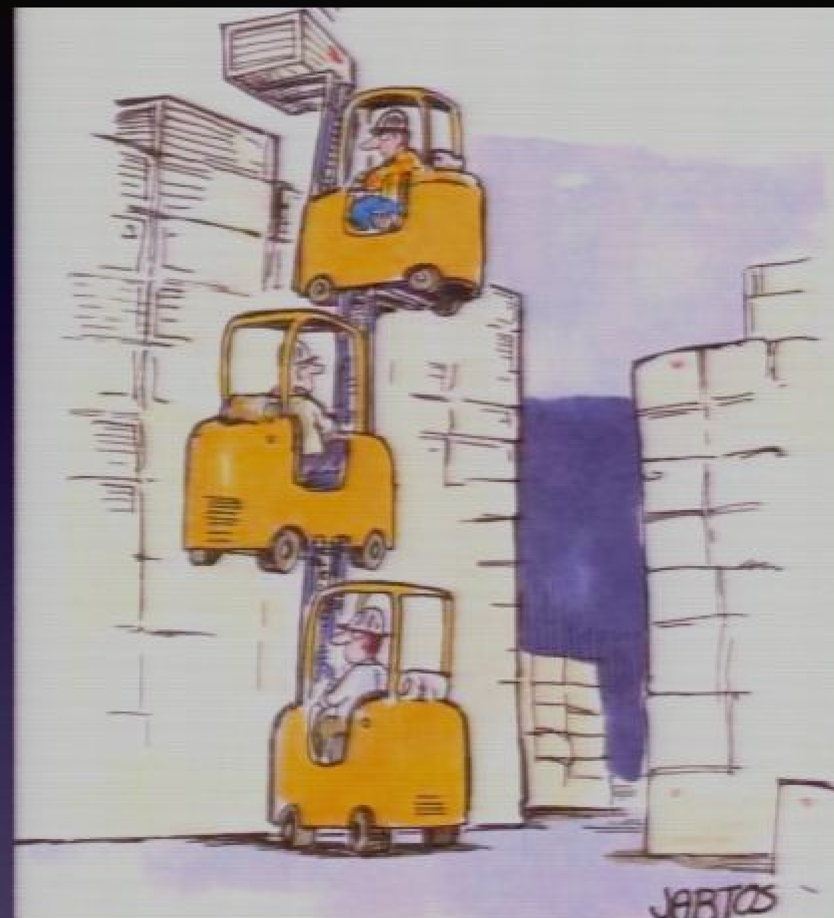
- In communication complexity, how useful even are EPR pairs? Can spread be used to argue that n EPR pairs are not useful for a Q-qubit protocol when $n \gg Q$?

  Black-box reductions *a la* Newman's theorem are no longer possible (consider a protocol that verifies a random sample of EPR pairs). On the other hand, such protocols can be simulated using exp(O(Q)) bits of classical communication [Shi and Zhu, quant-ph/0511071].

And they all lived happily ever after.



The end.