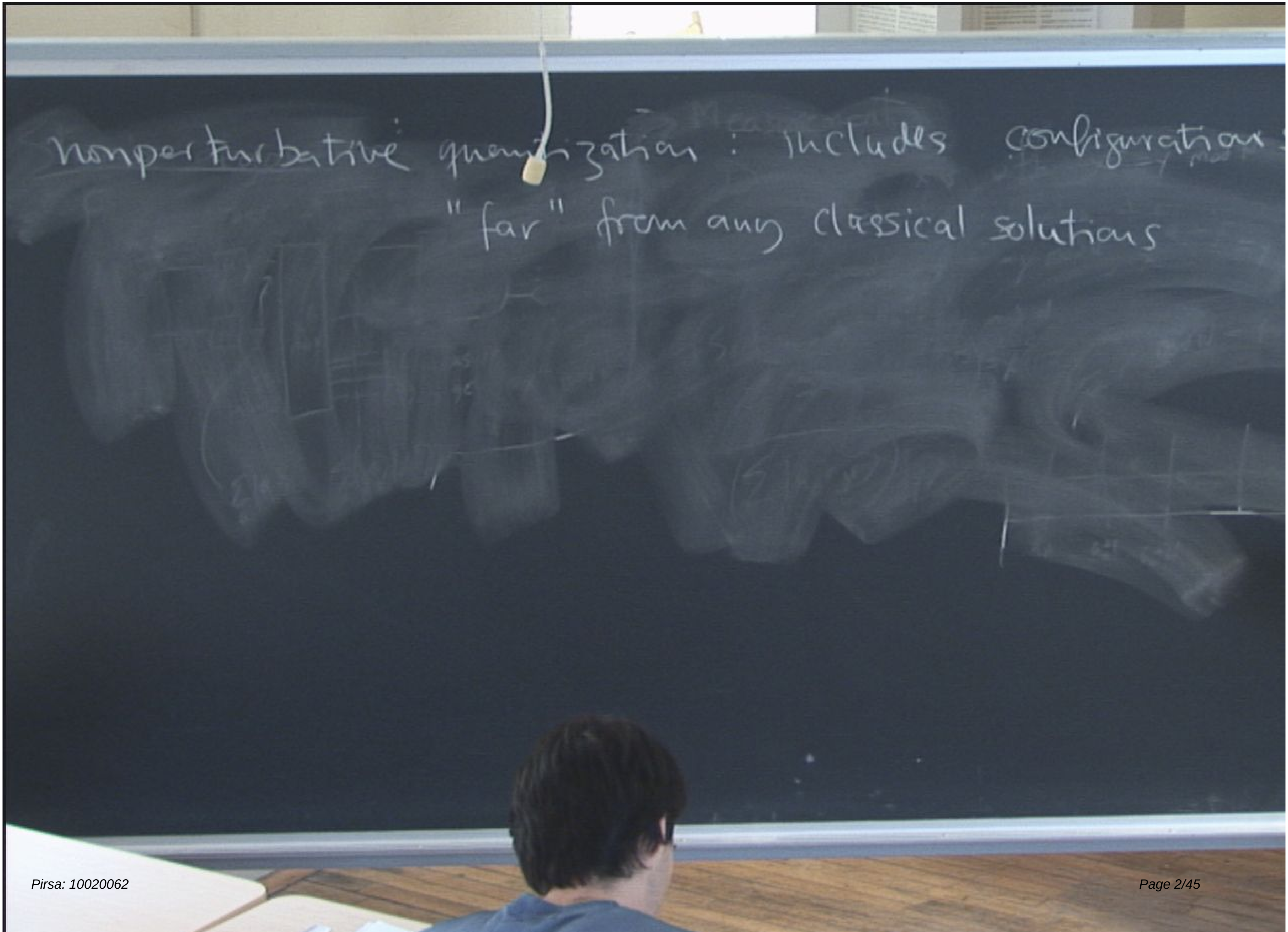
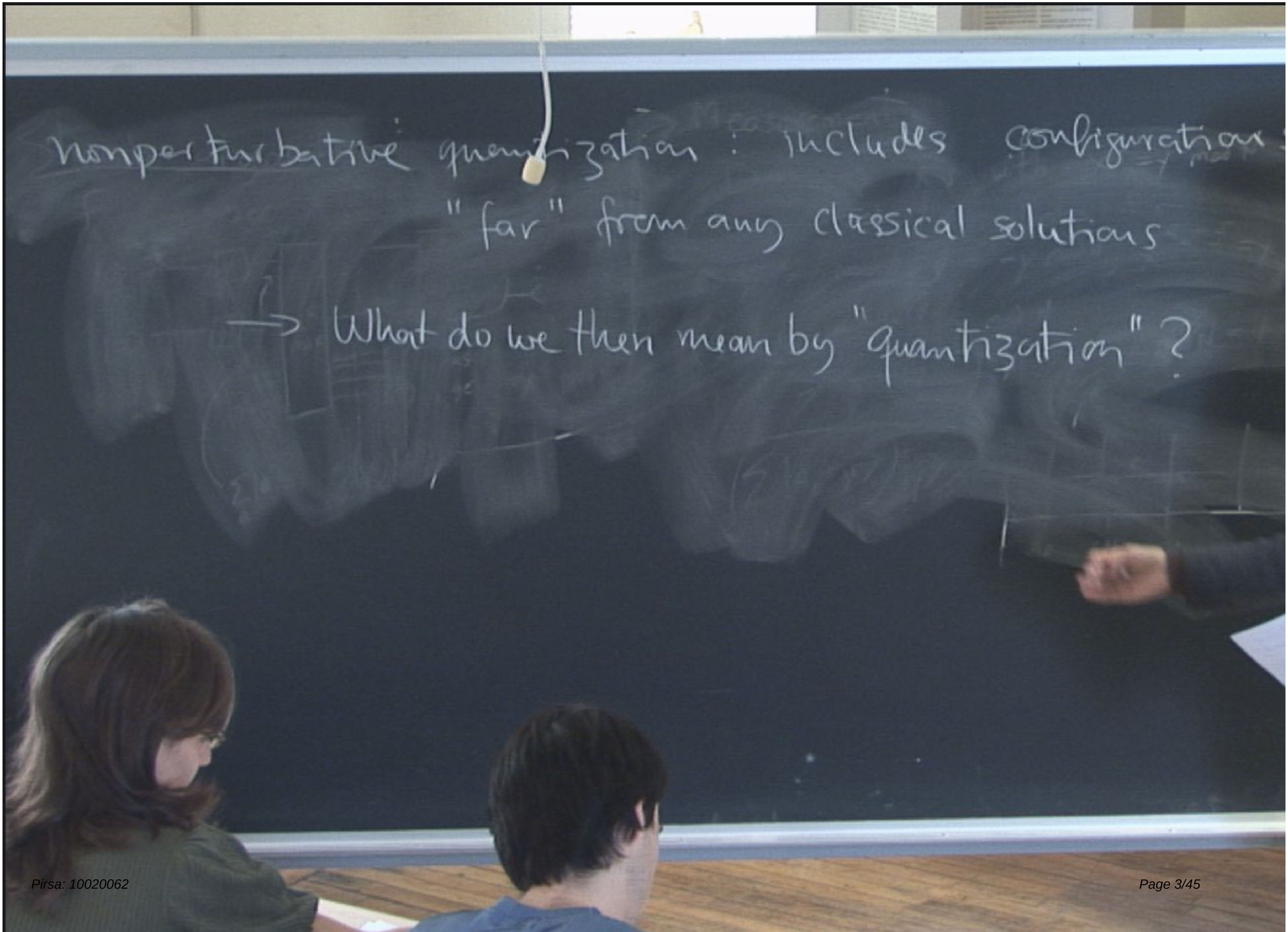
Title: Quantum Gravity - Review (PHYS 638) - Lecture 8

Date: Feb 03, 2010 10:00 AM

URL: http://pirsa.org/10020062

Abstract:


nonperturbative quantization : includes configurations
"far" from any classical solutions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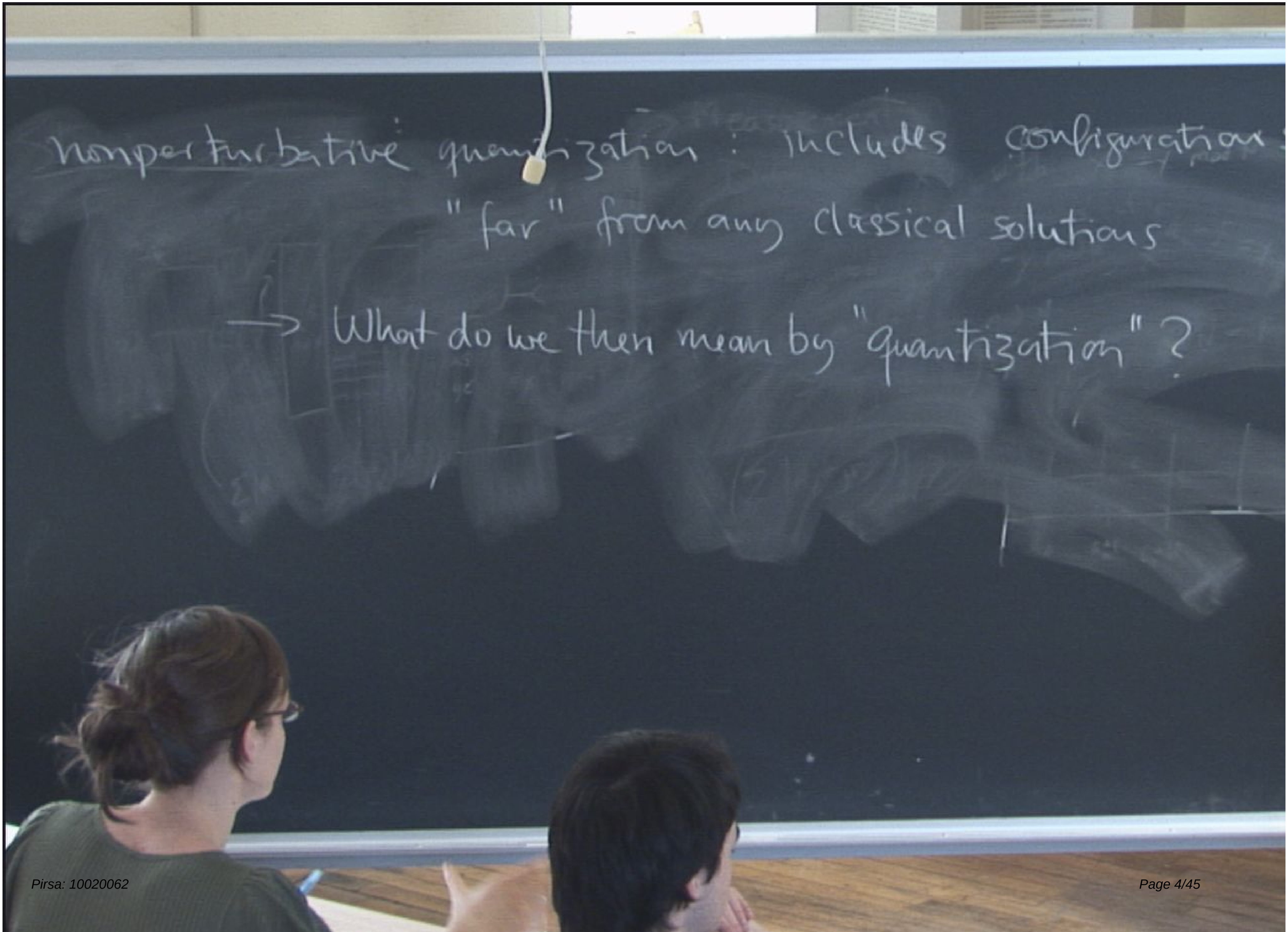
→ What do we then mean by "quantization"?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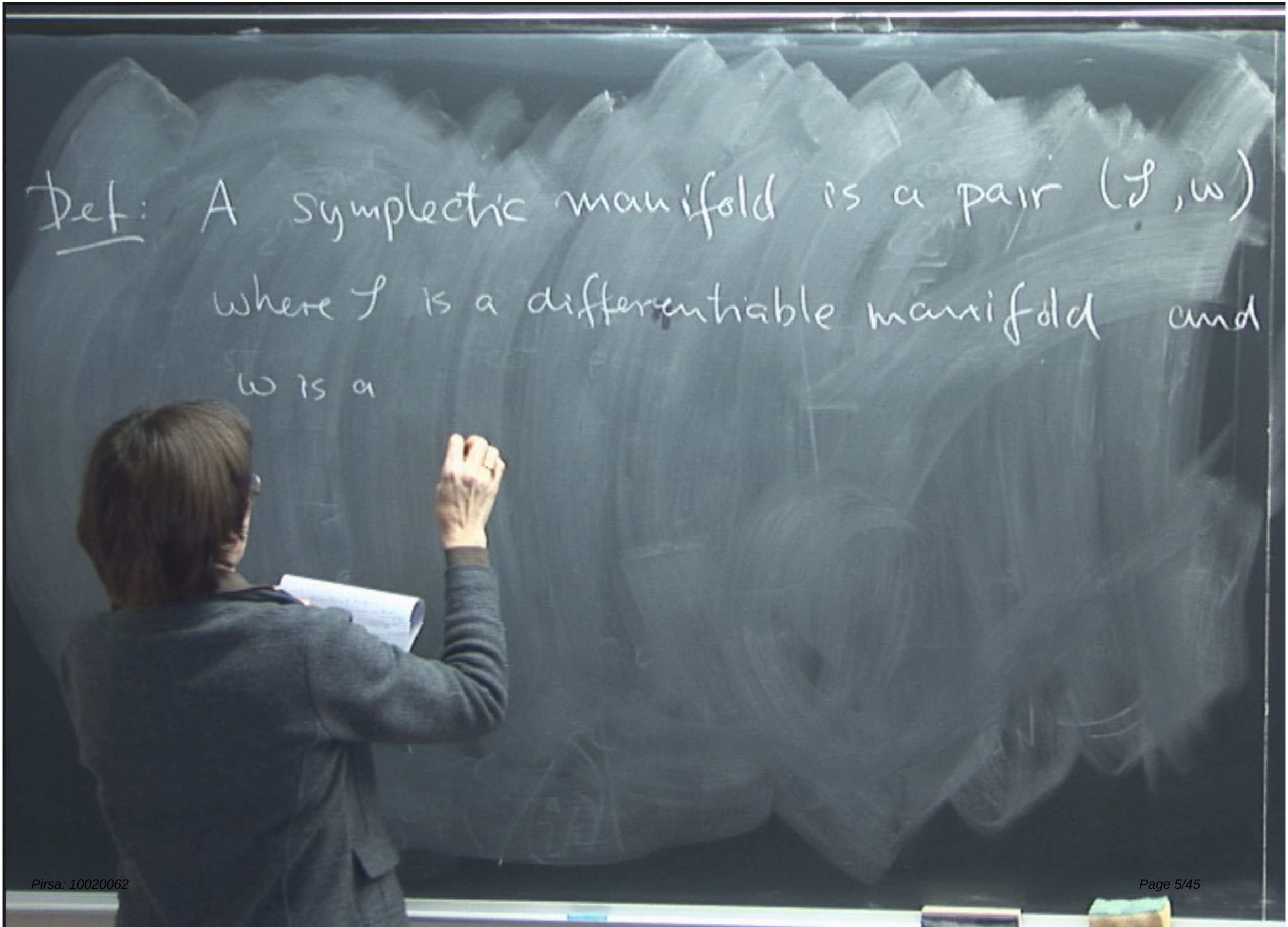
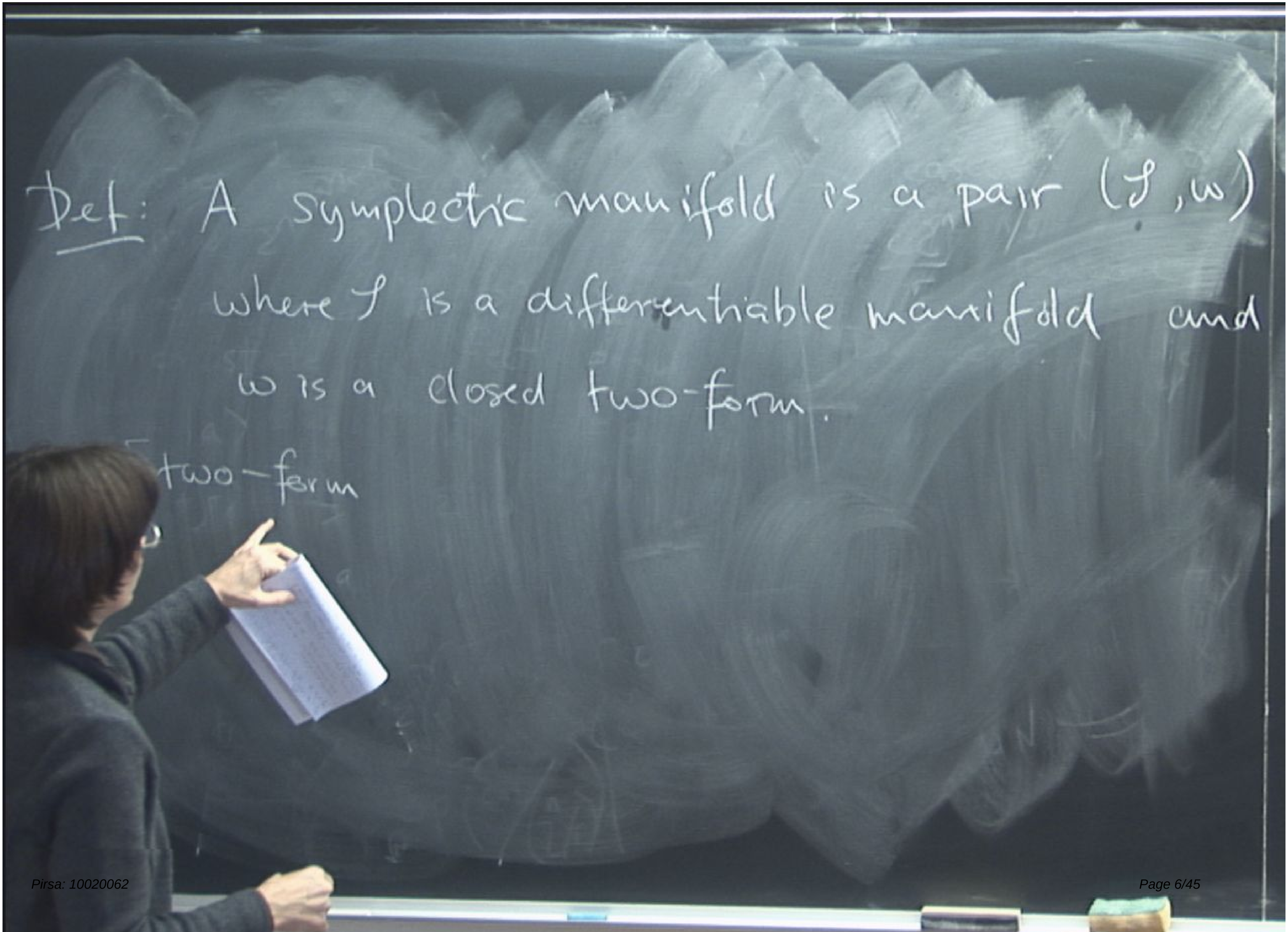

Def: A symplectic manifold is a pair $(\mathcal{S}, \omega)$
where $\mathcal{S}$ is a differentiable manifold and
ω is a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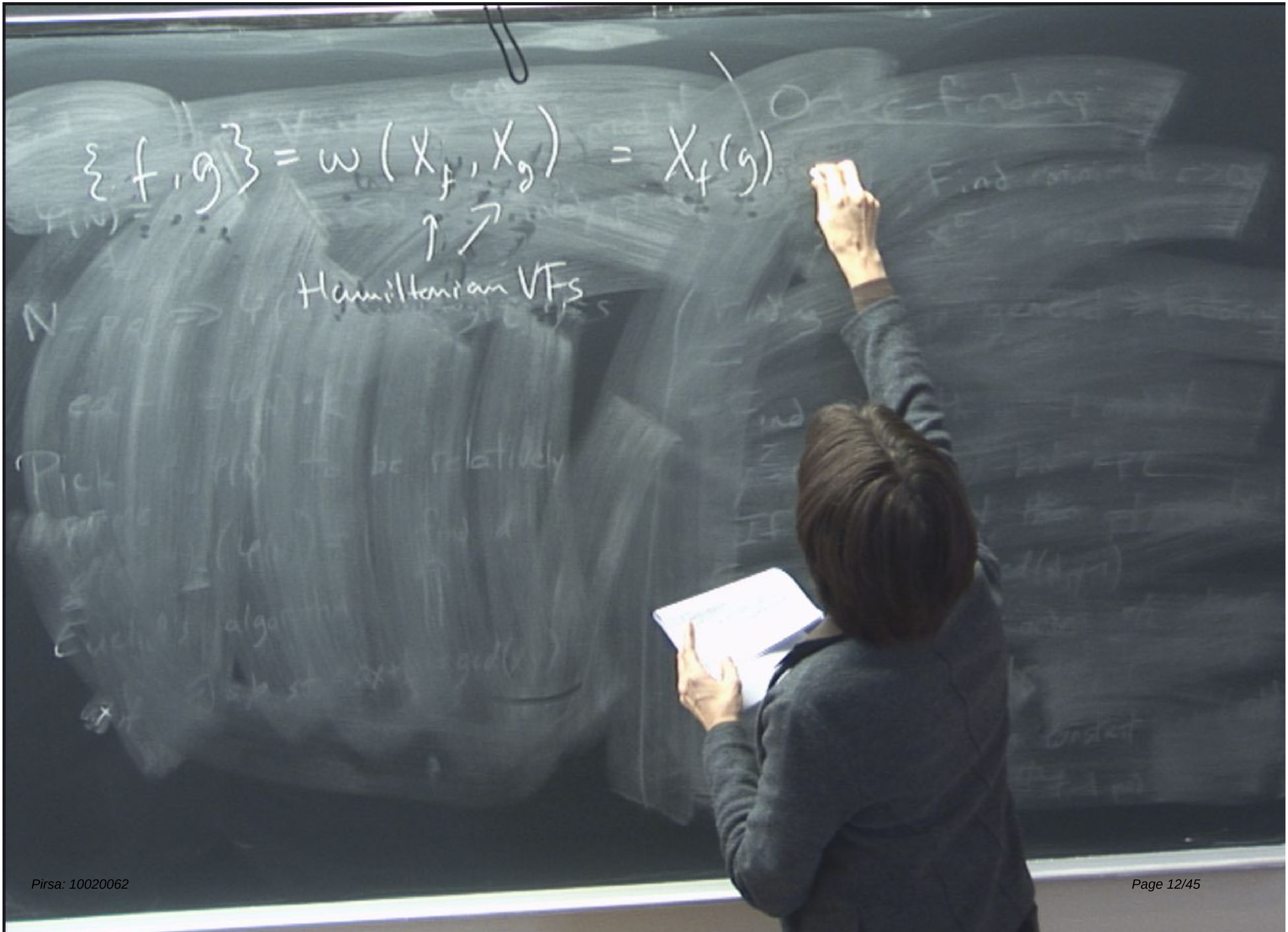
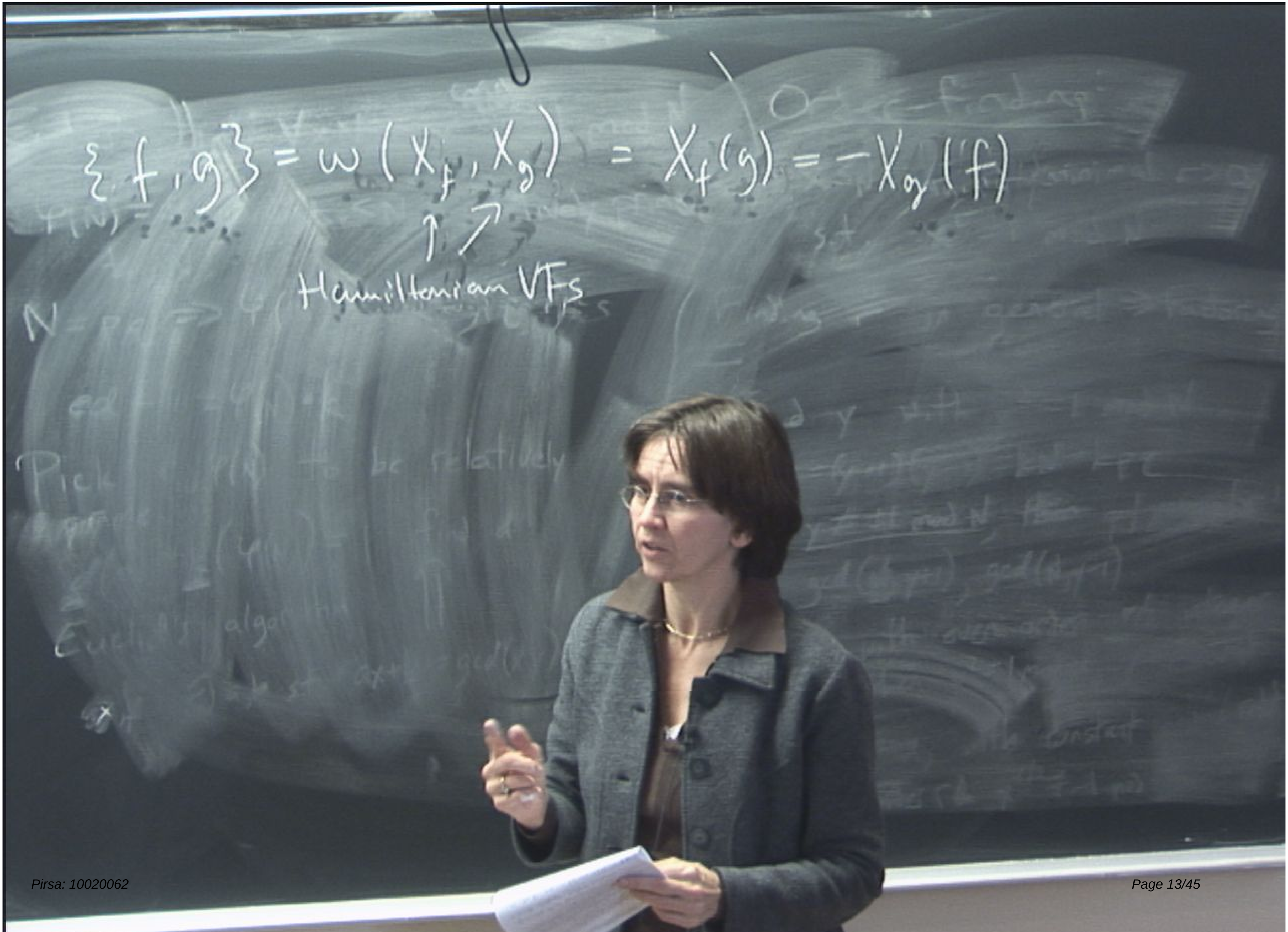

{f,g} = ω(X_f, X_g) = X_f(g)
Hamiltonian VFs


{f, g} = ω(X_f, X_g) = X_f(g) = −X_g(f)
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$$\{f, g\} = \omega(X_f, X_g) = X_f(g) = -X_g(f)$$

$\uparrow \nearrow$

Hamiltonian VFs

ex. $X_{q_i} = \partial/\partial p_i \,, \quad X_{p_i} = -\partial/\partial q_i$

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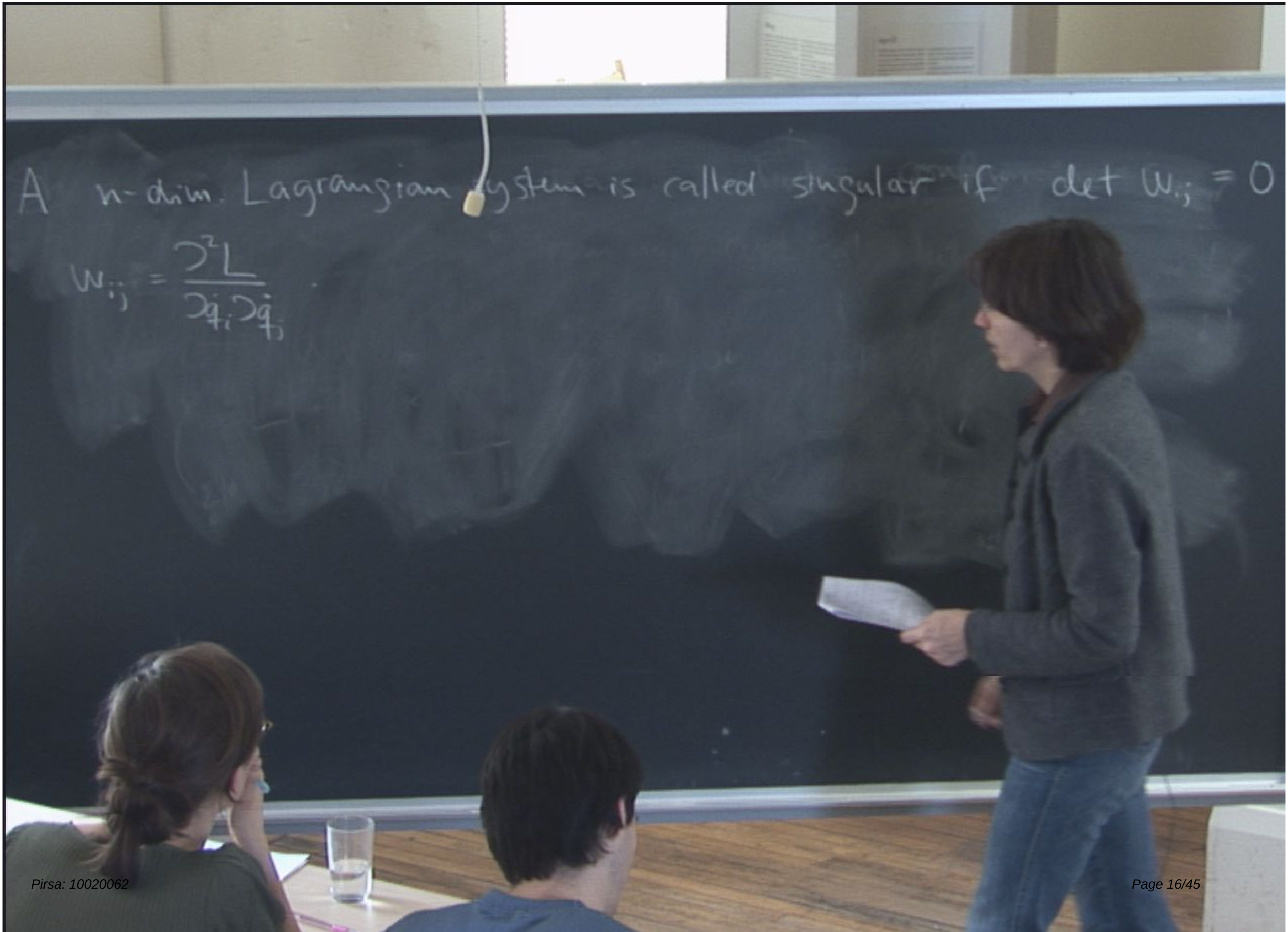
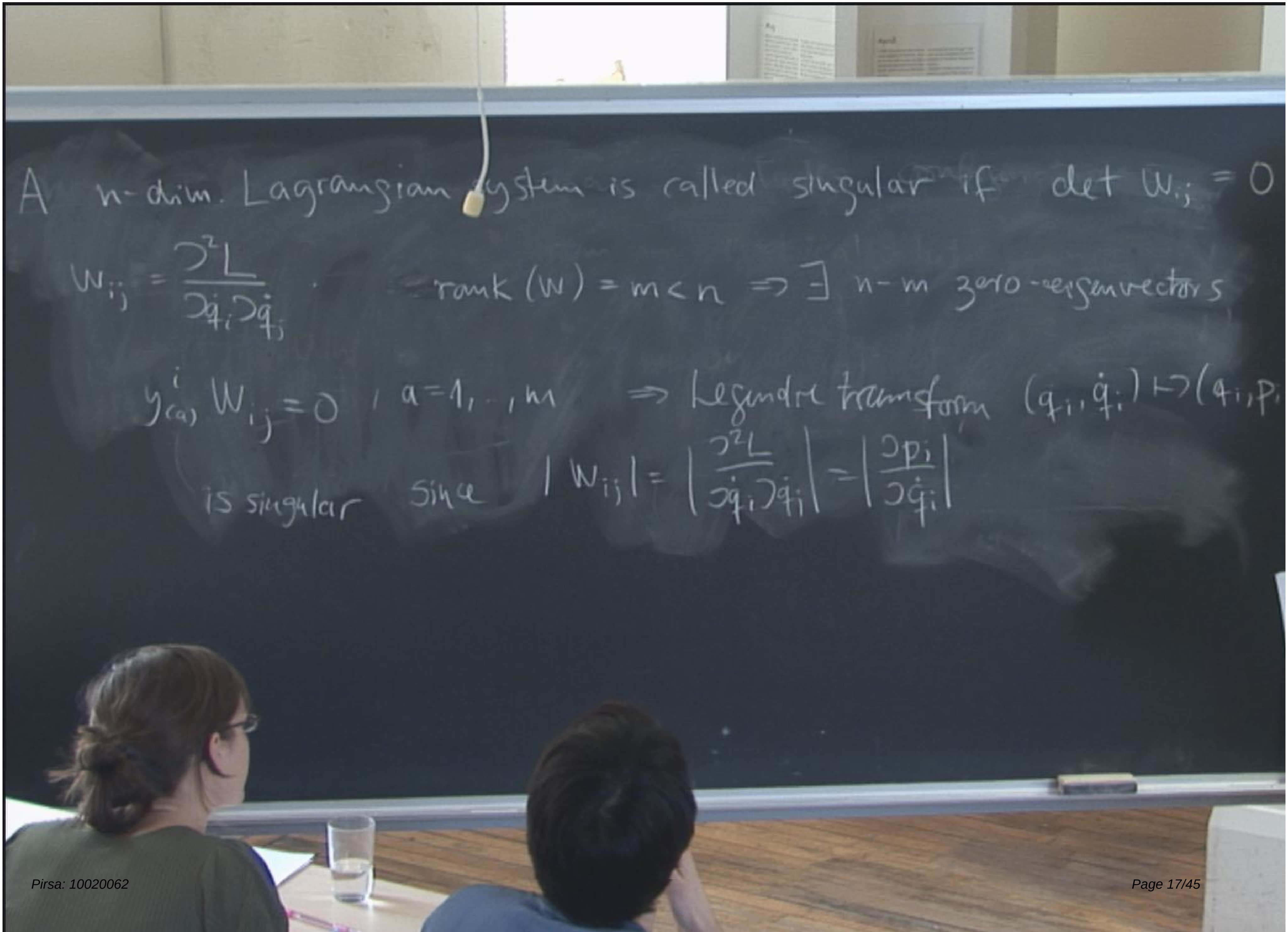
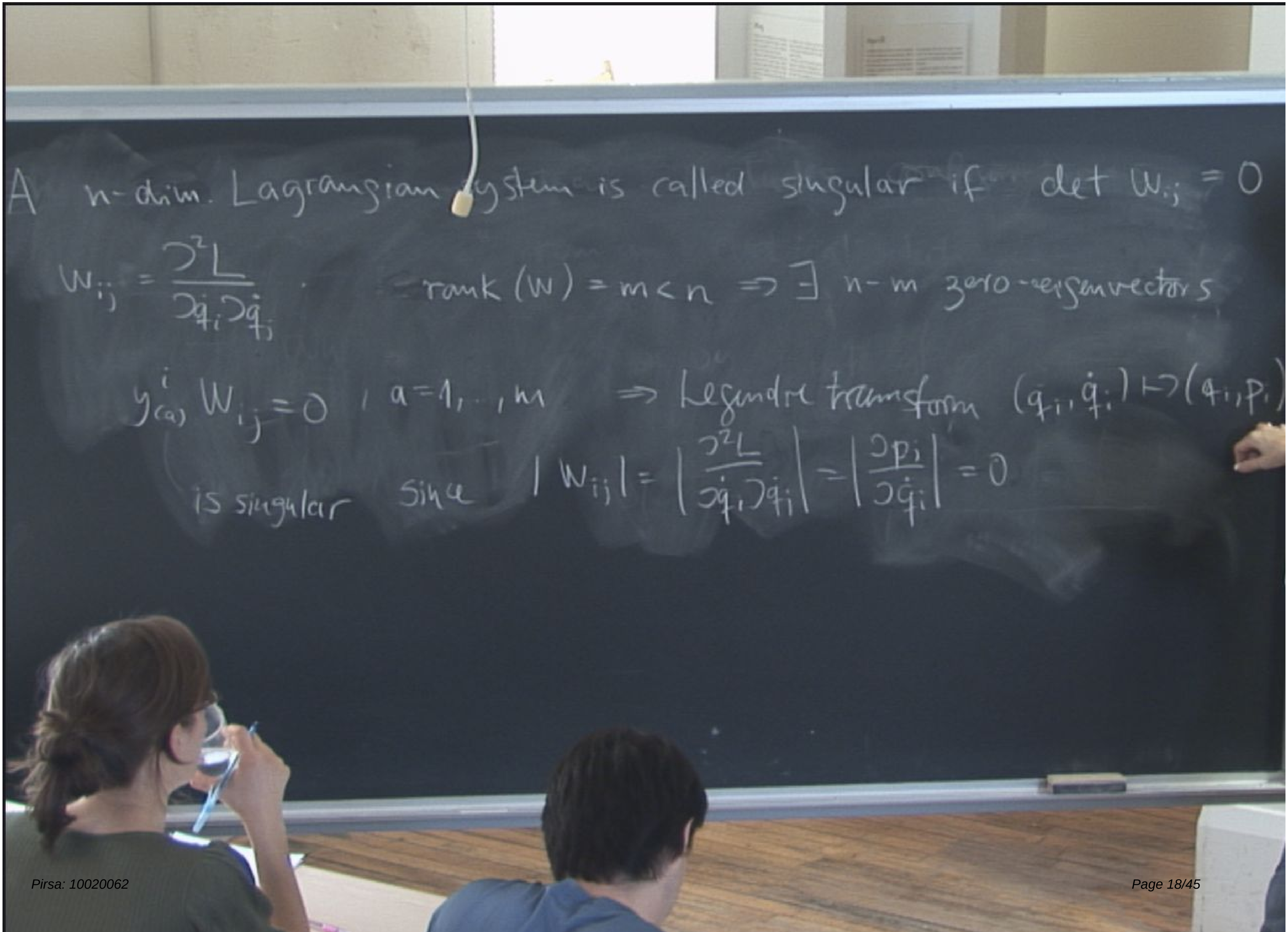
Constrained Hamiltonian systems


A n-dim. Lagrangian system is called singular if det W_ij = 0
W_ij = ∂²L / ∂q̇_i ∂q̇_j


A n-dim. Lagrangian system is called singular if det $W_{ij} = 0$
$W_{ij} = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}$. rank $(W) = m < n \Rightarrow \exists$ $n-m$ zero-eigenvectors
$y^i_{(a)} W_{ij} = 0$, $a = 1, \dots, m$ $\Rightarrow$ Legendre transform $(q_i, \dot{q}_i) \mapsto (q_i, p_i)$
is singular since $|W_{ij}| = \left|\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}\right| = \left|\frac{\partial p_i}{\partial \dot{q}_i}\right|$



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$\Rightarrow$ can't solve $\forall\ \dot{q}_i = \dot{q}_i(q,p)$

$\Rightarrow$ not all momenta are independent, but we have

m relations $\phi_a(q,p) = 0$, $a = 1, \dots, m$,

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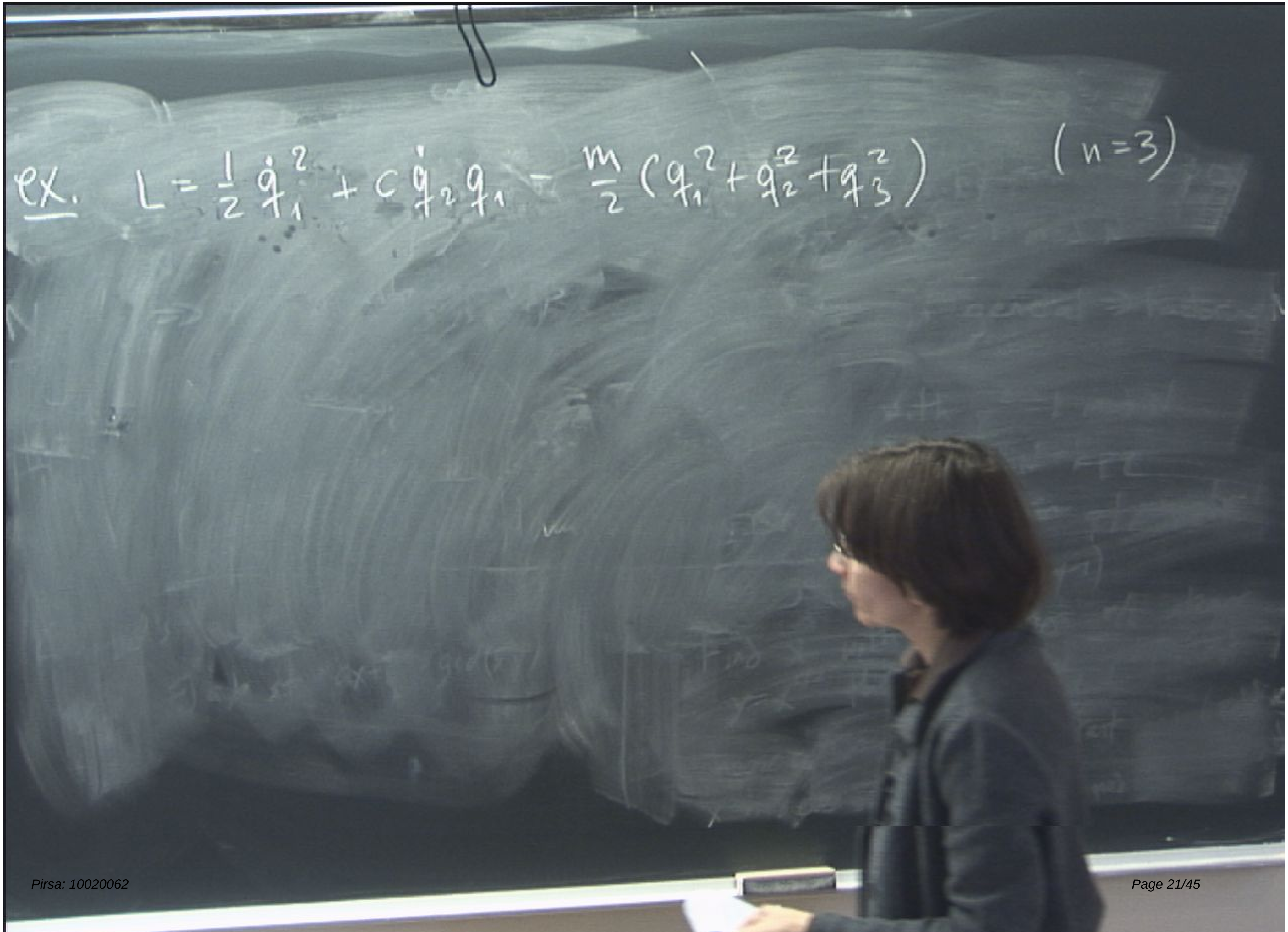
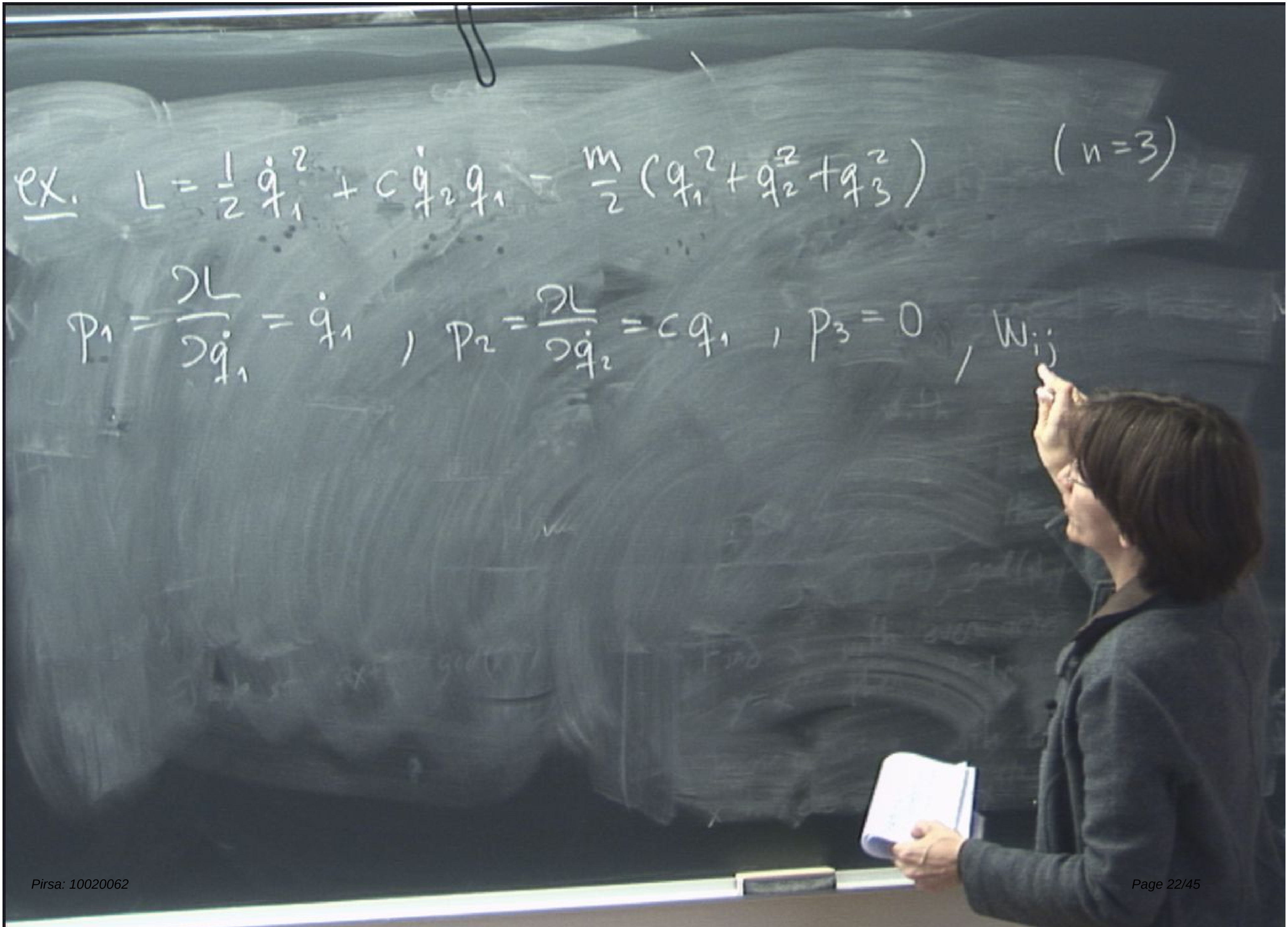
m relations $\phi_a(q,p) = 0$, $a = 1, \dots, m$,

so-called "constraints"


ex. $L = \frac{1}{2}\dot{q}_1^2 + c\,\dot{q}_2 q_1 - \frac{m}{2}(q_1^2 + q_2^2 + q_3^2)$
$(n=3)$


ex. $L = \frac{1}{2}\dot{q}_1^2 + c\dot{q}_2 q_1 - \frac{m}{2}(q_1^2 + q_2^2 + q_3^2)$ $(n=3)$
$p_1 = \frac{\partial L}{\partial \dot{q}_1} = \dot{q}_1$, $p_2 = \frac{\partial L}{\partial \dot{q}_2} = c q_1$, $p_3 = 0$, W_{ij}

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$$\phi_1(q,p) = p_2 - c\,q_1 = 0 \,, \quad \phi_2(q,p) = p_3 = 0$$

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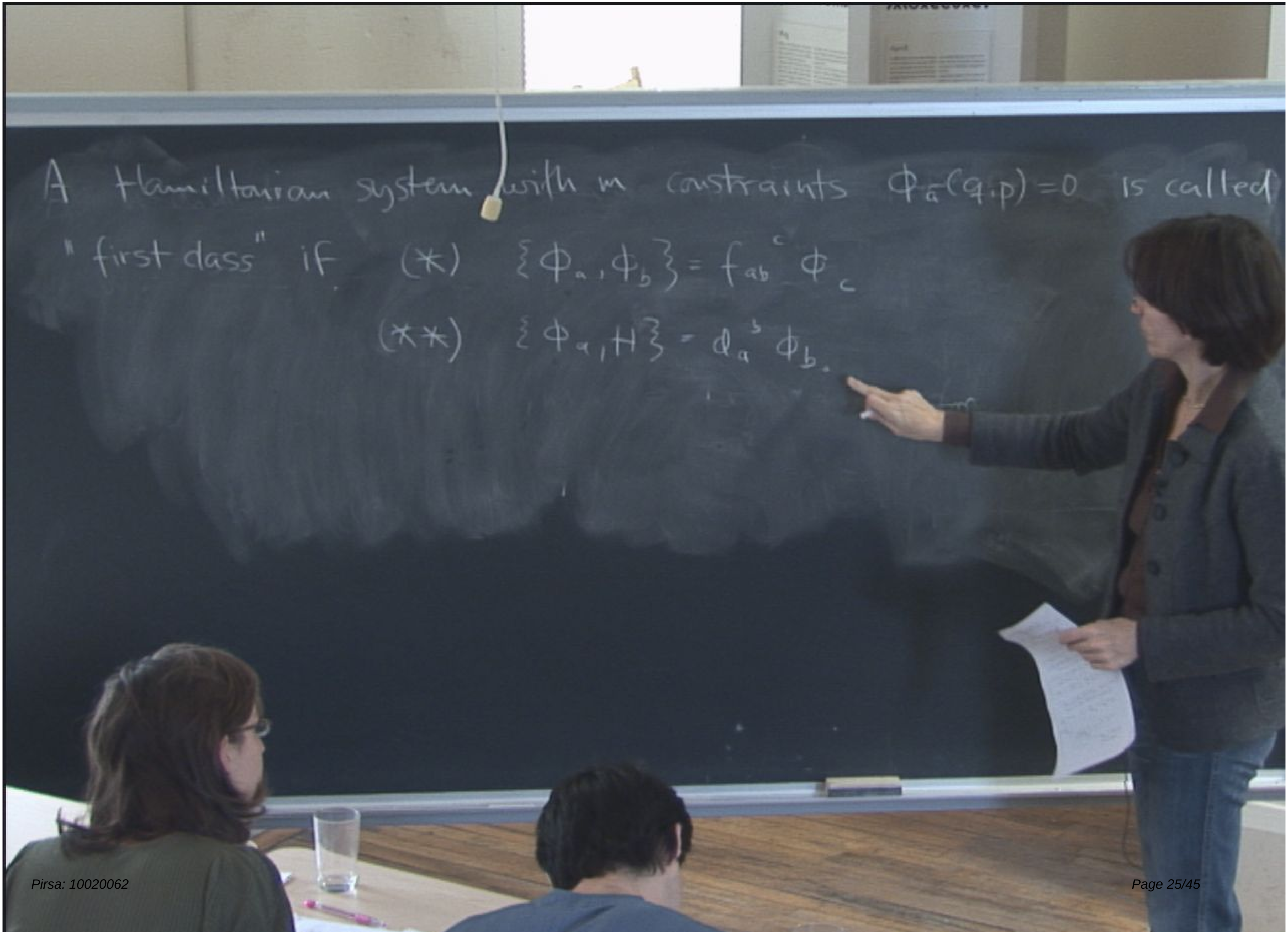
$$\phi_1(q,p) = \boxed{p_2 - c q_1 = 0} \,,\quad \phi_2(q,p) = \boxed{p_3 = 0}$$

$\Rightarrow \mathcal{P}_c$ is 4-dimensional.



A Hamiltonian system with m constraints $\phi_a(q,p)=0$ is called
"first class" if (*) $\{\phi_a,\phi_b\} = f_{ab}{}^c \phi_c$
(**) $\{\phi_a, H\} = d_a{}^b \phi_b$.

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$$(*) \quad \{\phi_a, \phi_b\} = f_{ab}{}^{c}\,\phi_c$$

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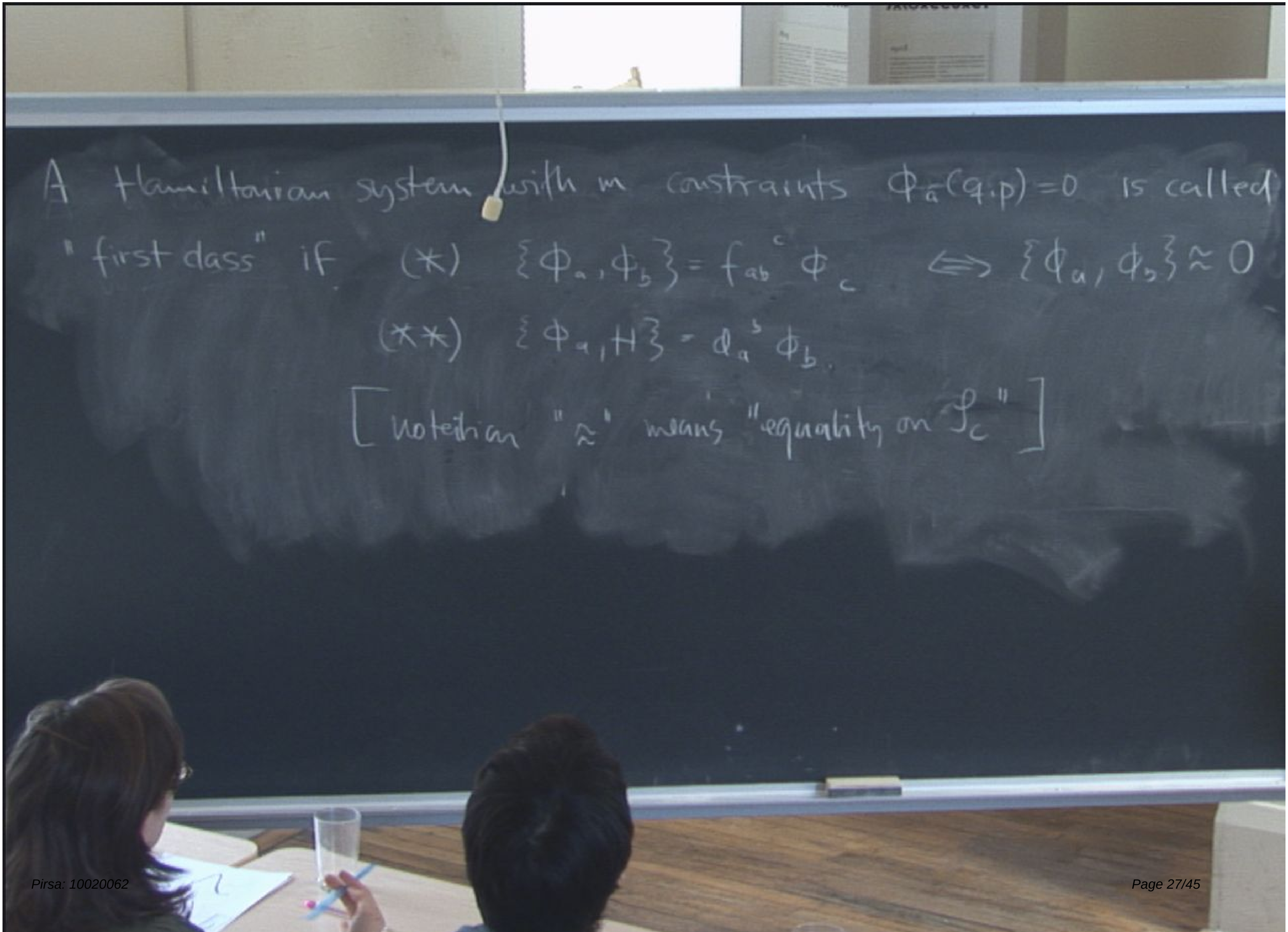
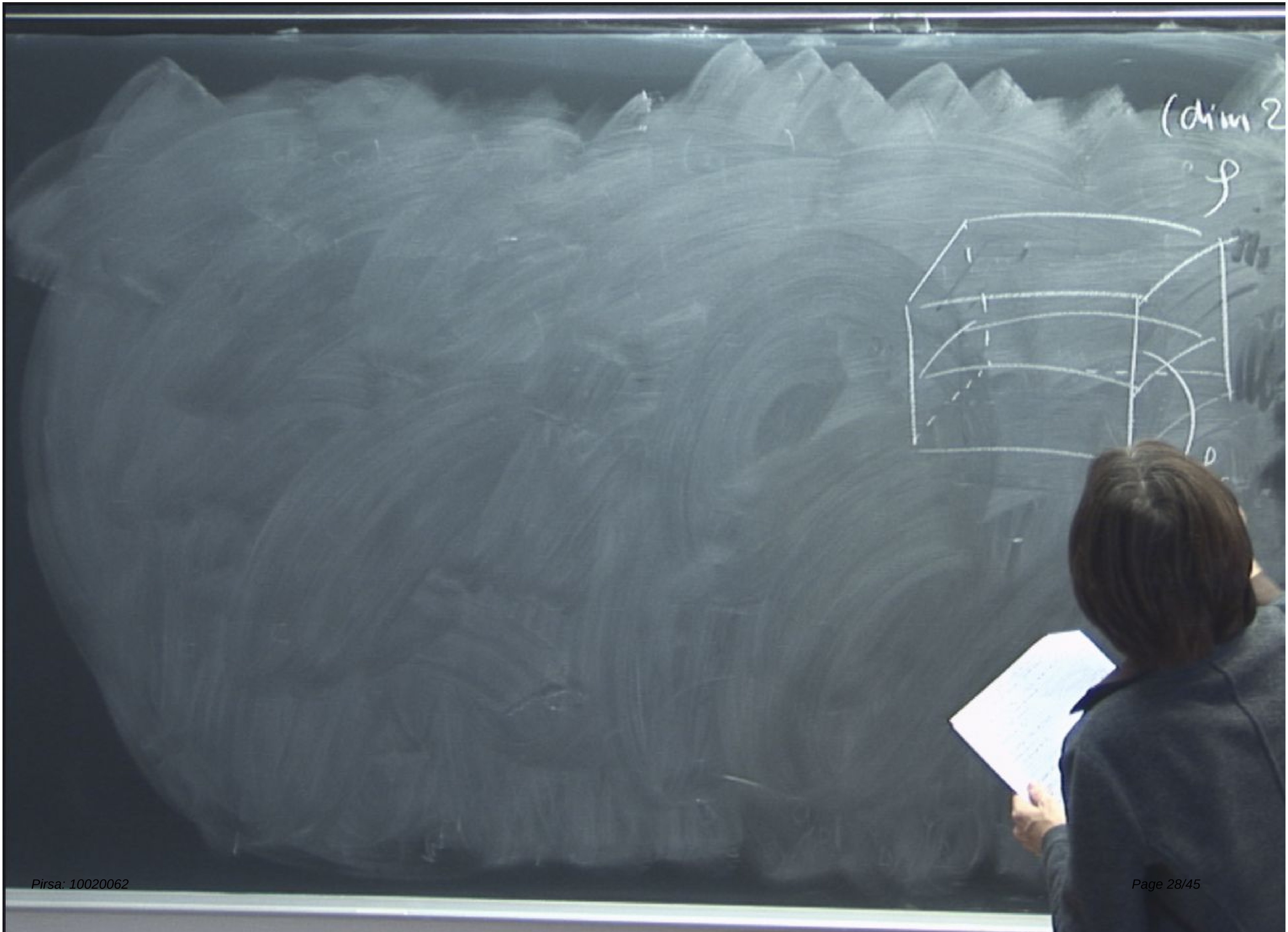
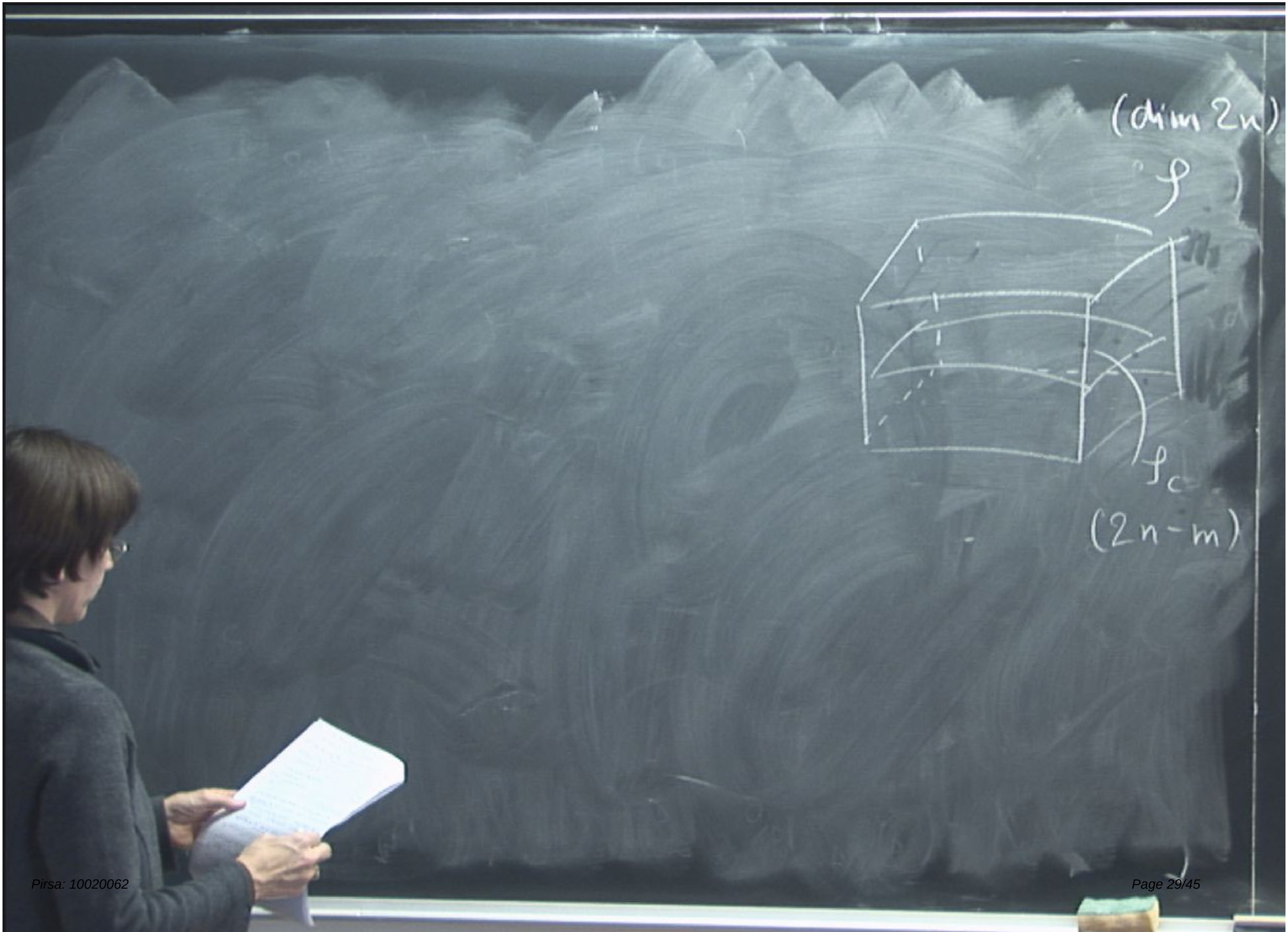
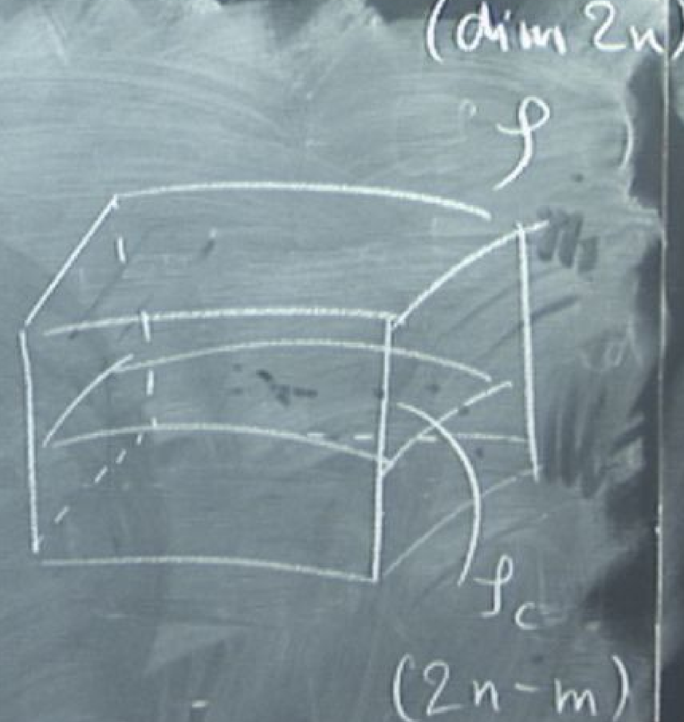
[notation "$\approx$" means "equality on Γ_c"

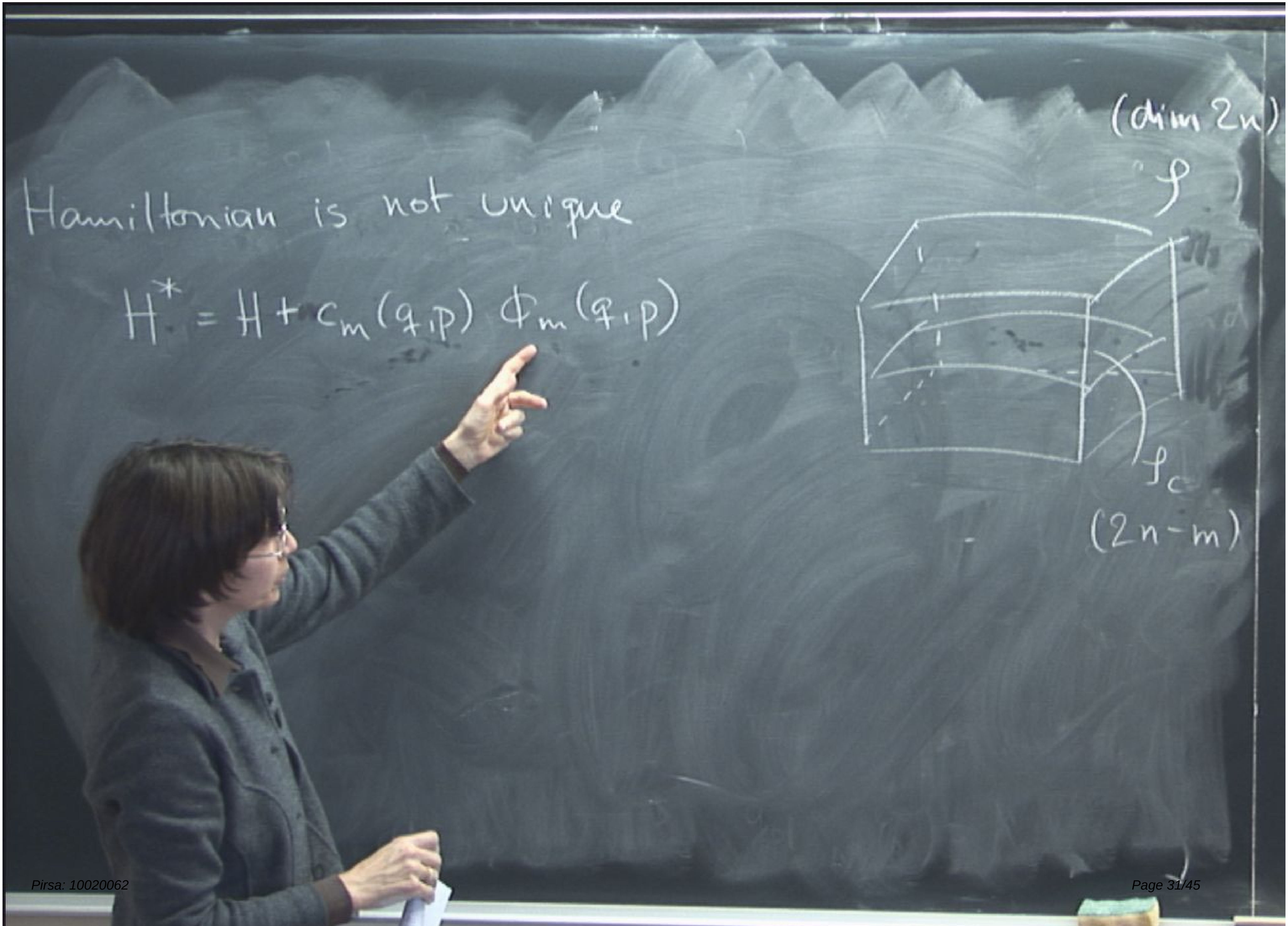

A Hamiltonian system with m constraints $\phi_{\bar{a}}(q,p)=0$ is called
"first class" if (*) $\{\phi_a, \phi_b\} = f_{ab}{}^{c}\,\phi_c \iff \{\phi_a, \phi_b\} \approx 0$
(**) $\{\phi_a, H\} = d_a{}^{b}\,\phi_b$.
[notation "$\approx$" means "equality on Σ_c"]


(dim 2


(dim 2n)
ρ
ρ_c
(2n - m)

Hamiltonian is not unique




Hamiltonian is not unique
$H^* = H + c_m(q,p)\,\phi_m(q,p)$
(dim 2n)
ρ
m
ρ_c
(2n − m)

Hamiltonian is not unique

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... evolution of some $g(q,p) \in C^\infty(\Gamma)$

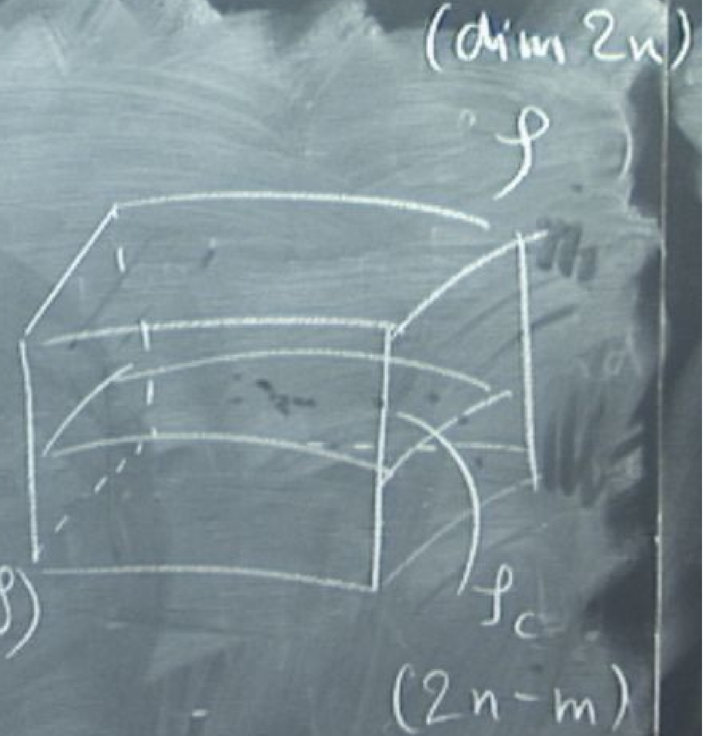
$$= \{g, H^*\}$$

Hamiltonian is not unique

$$H^* = H + c_m(q,p)\,\phi_m(q,p)$$

time evolution of some $g(q,p) \in C^\infty(\mathcal{S})$

$$\dot{g} = \{g, H^*\} = \{g, H\}$$



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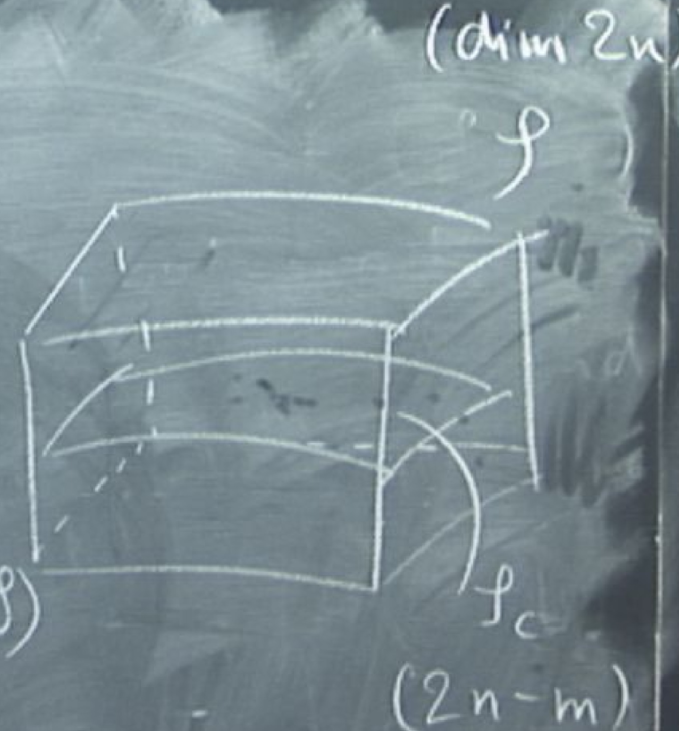

$$H^* = H + c_m(q,p)\,\phi_m(q,p)$$

time evolution of some $g(q,p) \in C^\infty(\Gamma)$

$$\dot{g} = \{g, H^*\} = \{g, H\} + c_m\{g, \phi_m\} + \underbrace{\phi_m\{g, c_m\}}_{\approx 0}$$

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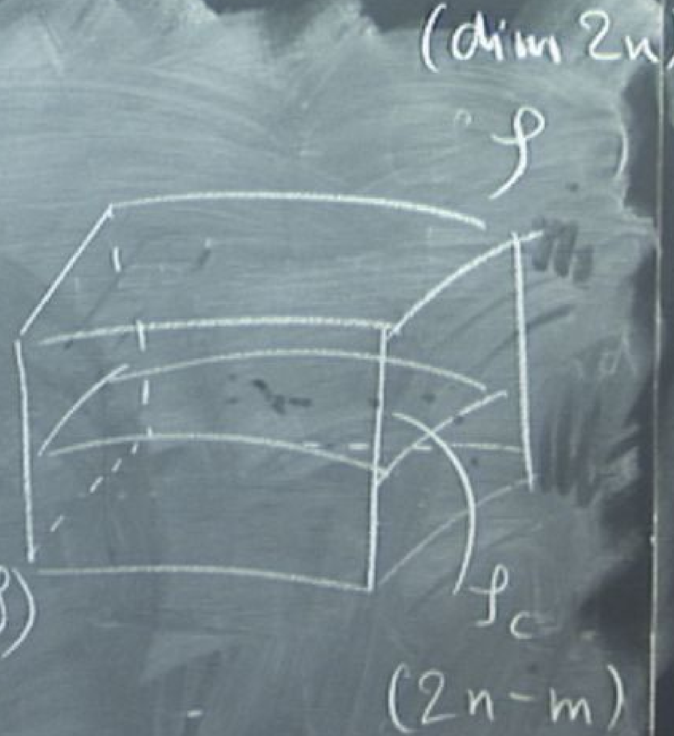
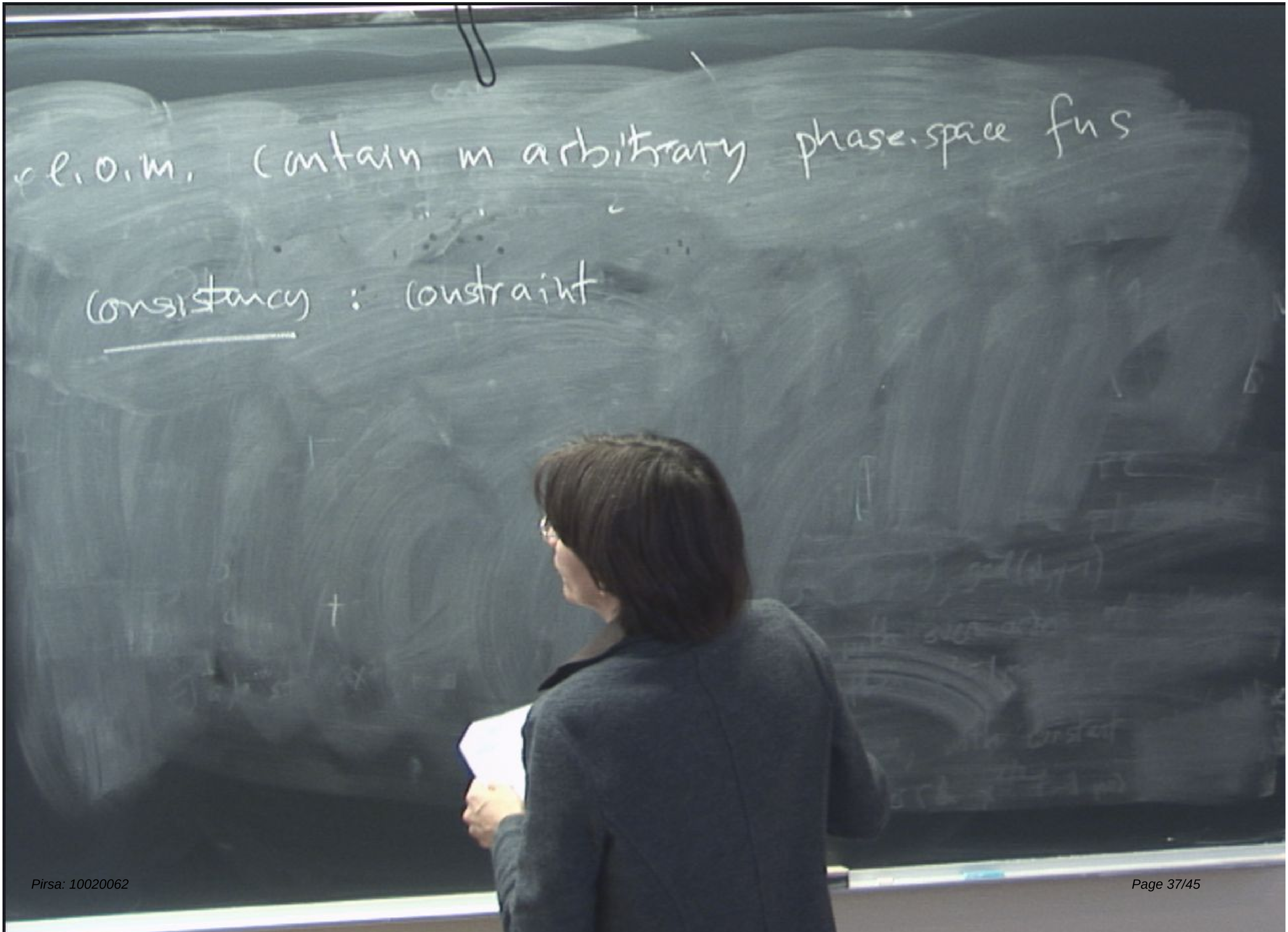
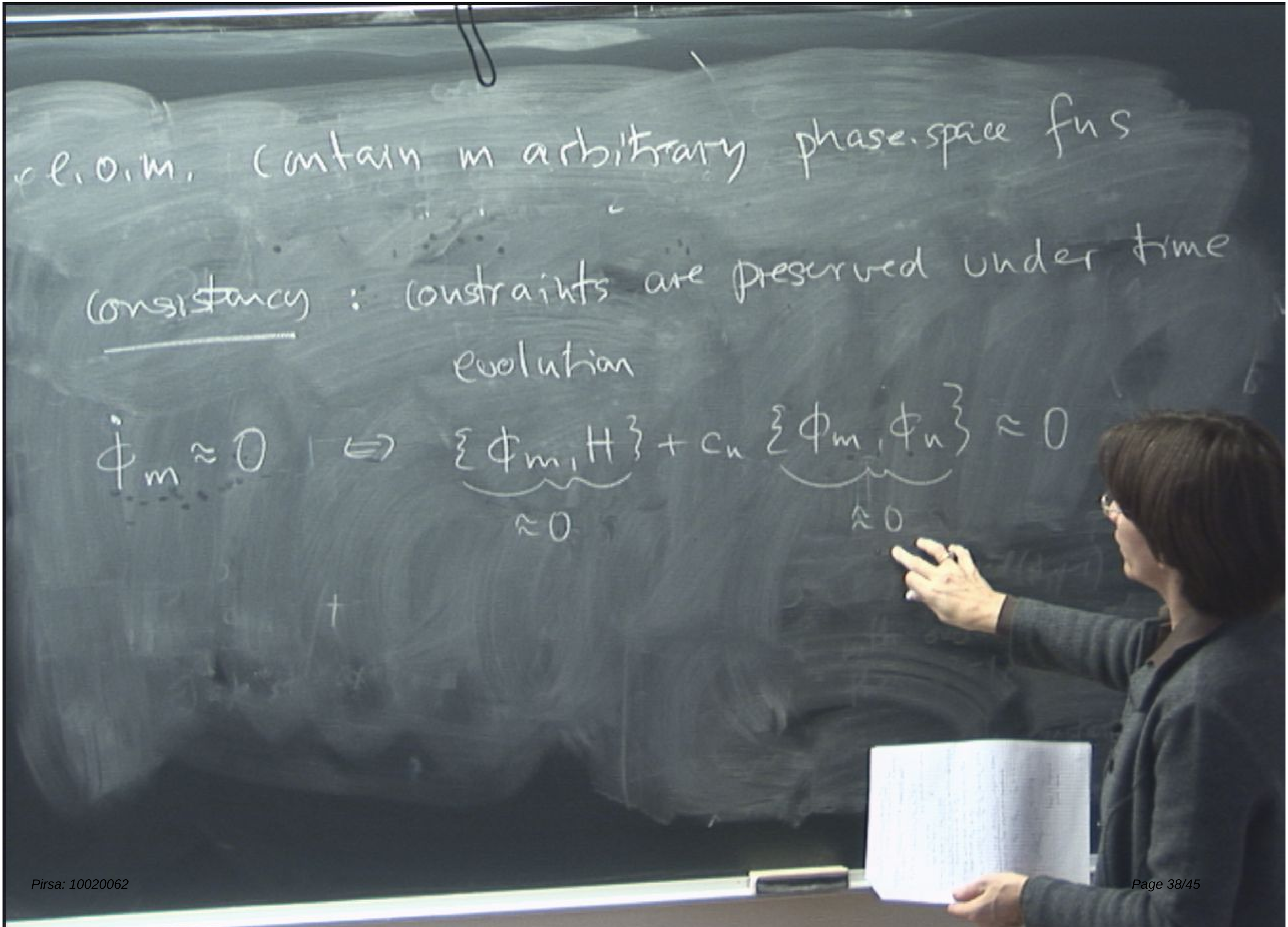
in particular
$$\dot{q}_i \approx \frac{\partial H}{\partial p_i} + c_m \frac{\partial \phi_m}{\partial p_i}$$

$$\dot{p}_i \approx -\frac{\partial H}{\partial q_i} - c_m \frac{\partial \phi_m}{\partial q_i}$$




e.o.m. contain m arbitrary phase.space fns
consistency : constraint



e.o.m. contain m arbitrary phase.space fns
consistency : constraints are preserved under time evolution
$\dot{\phi}_m \approx 0 \Leftrightarrow \underbrace{\{\phi_m, H\}}_{\approx 0} + c_n \underbrace{\{\phi_m, \phi_n\}}_{\approx 0} \approx 0$

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(i) $X_H \phi_m \approx 0$

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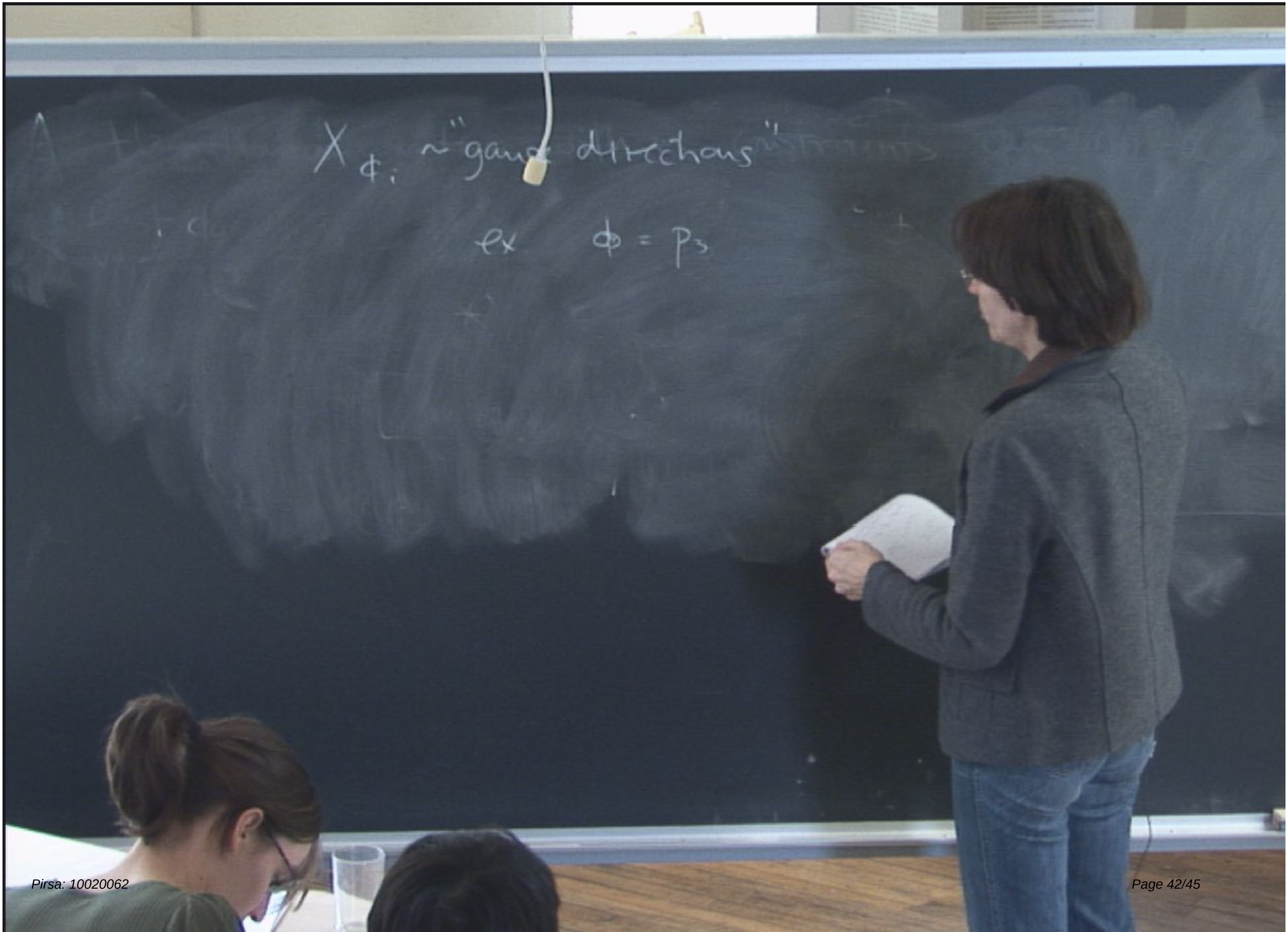
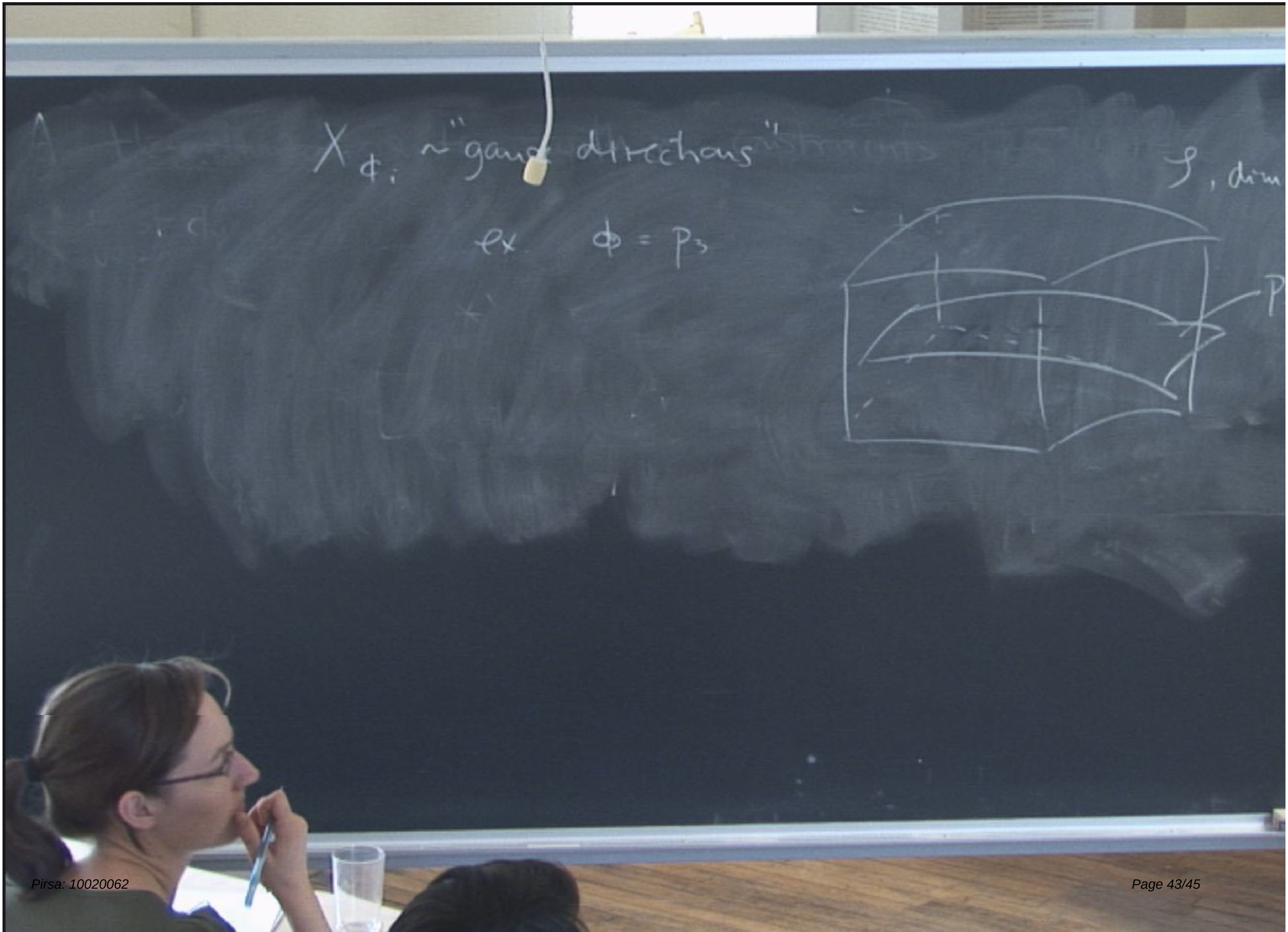
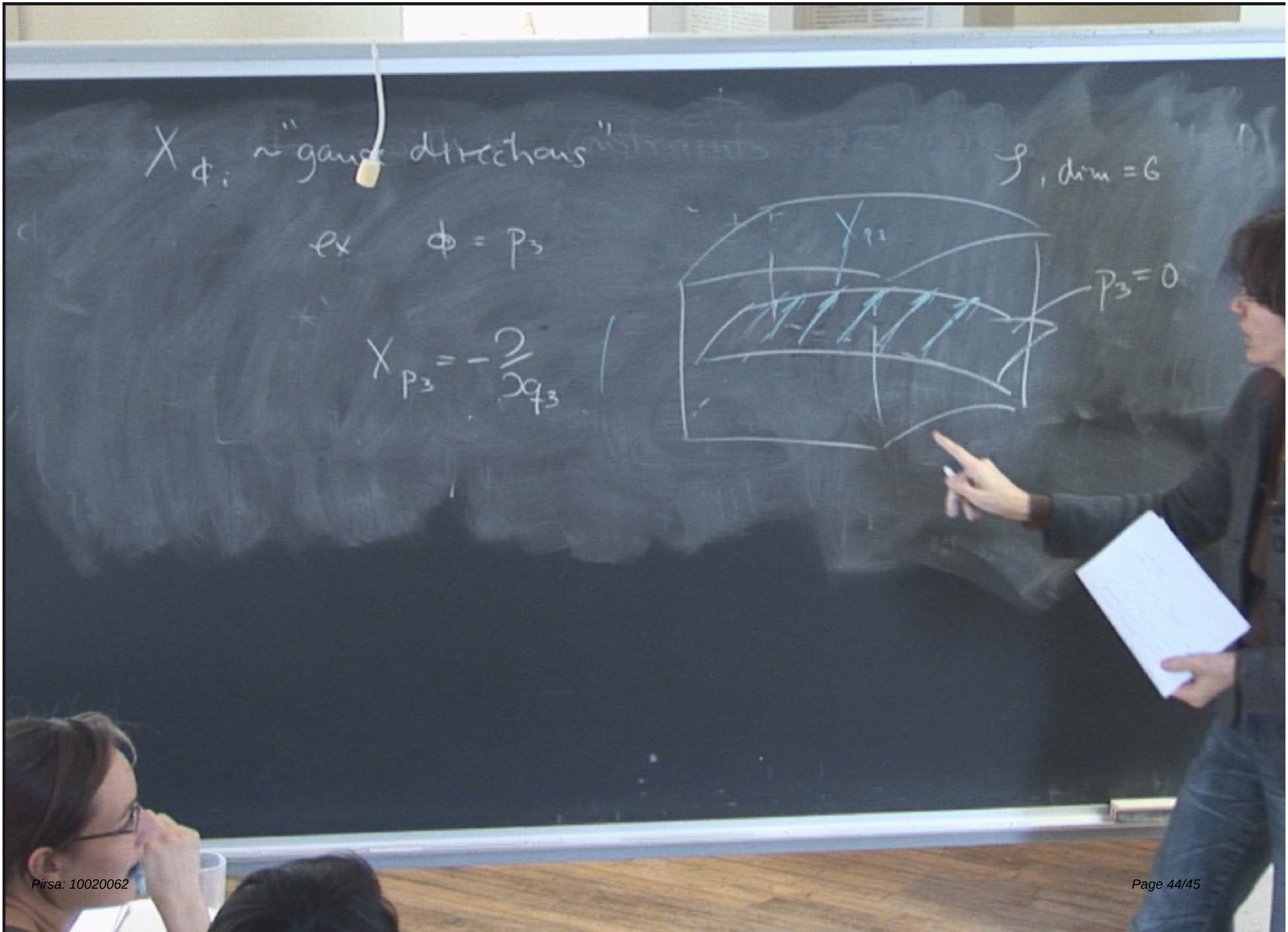
$$\{\phi_m, H\} \approx 0$$

(i) $X_H \phi_m \approx 0$ (flow of Hamiltonian stays on $\mathcal{S}_c$)

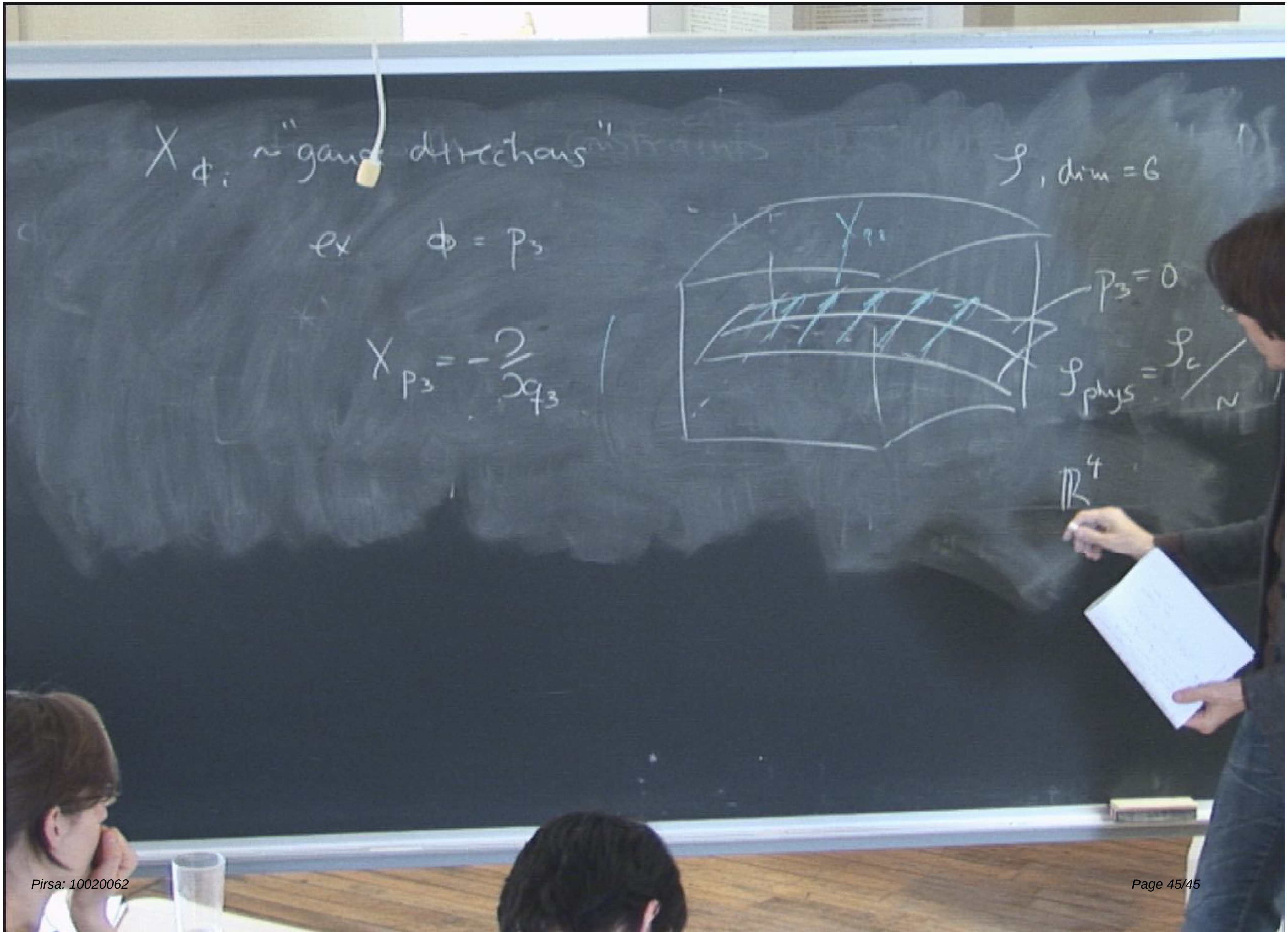
(ii) H is unchanged when moving in the direction of the X_{ϕ}


X_{φ_i} ~ "gauge directions"
ex φ = p_3


X_φi ~ "gauge directions"
ex. Φ = p3
ρ, dim



X_{φ_i} ~ "gauge directions"
ex φ = p_3
X_{p_3} = −∂/∂q_3
X_{q_3}
p_3 = 0
S, dim = 6



X_φi ~ "gauge directions"
ex φ = p3
X_p3 = -∂/∂q3
X_q3
S, dim = 6
p3 = 0
S_phys = S_c / N
R^4