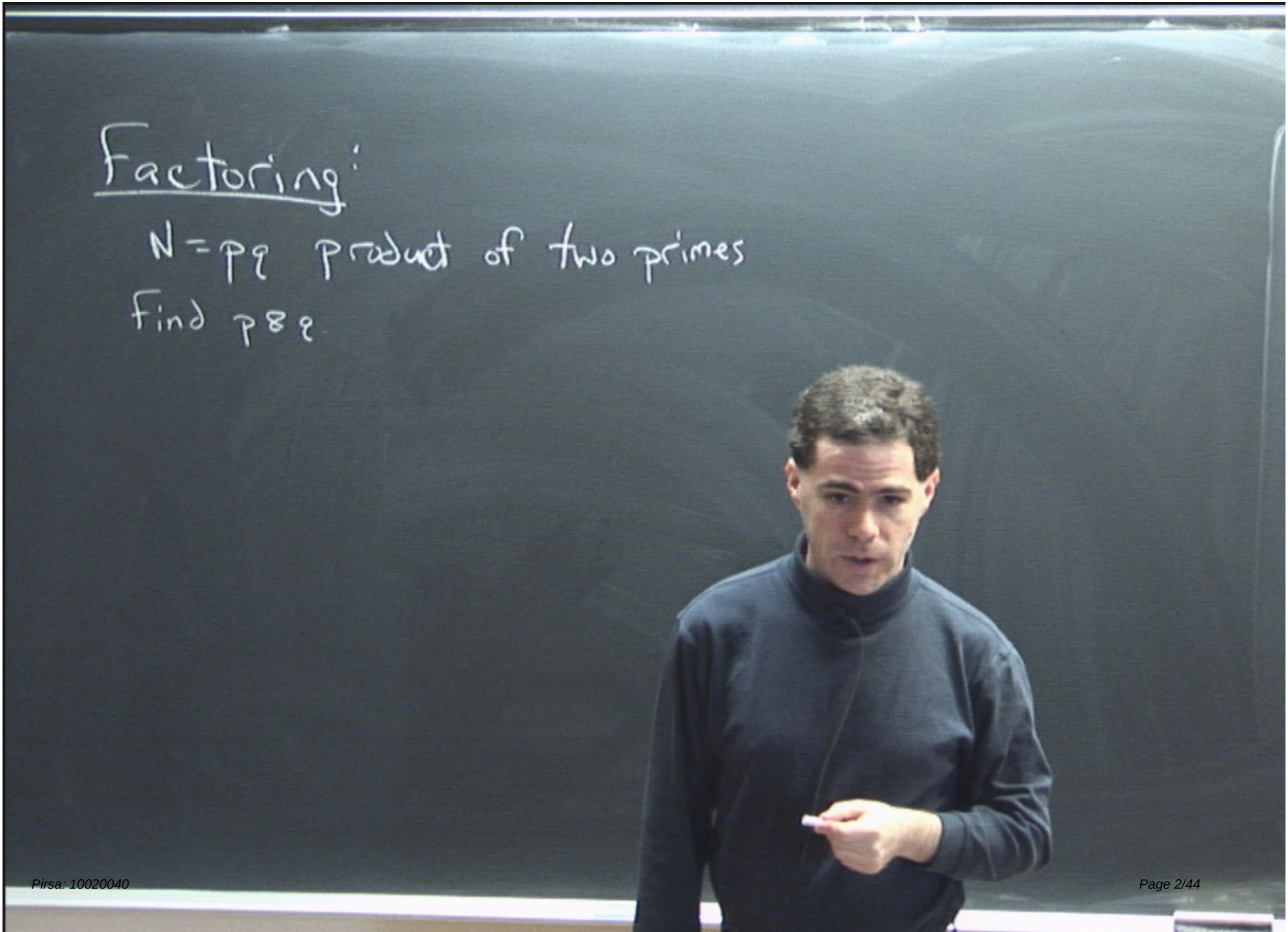
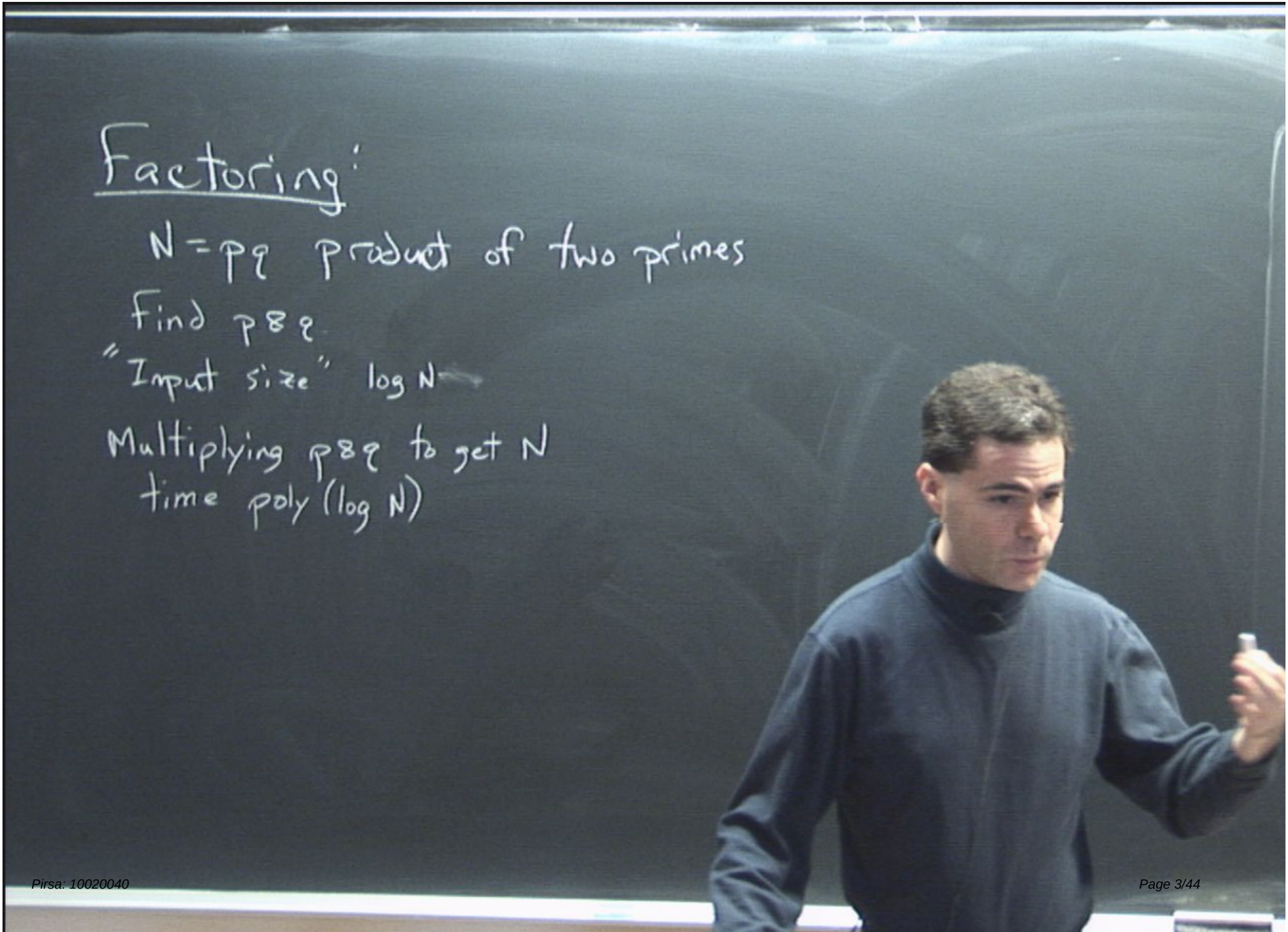
Title: Quantum Information - Review (PHYS 635) - Lecture 8

Date: Feb 03, 2010 09:00 AM

URL: http://pirsa.org/10020040

Abstract: Week 1: Basic topics (Qubits, quantum gates, quantum circuits, density matrices, quantum operations, entropy, entanglement) Week 2: Algorithms and complexity (Languages, complexity classes, oracles, RSA, Deutsch-Jozsa algorithm, Shor's algorithm, Grover's algorithm) Week 3: Information theory and implementations (Overview of implementations, quantum error correction, quantum cryptography, quantum information theory)


Factoring:
N = pq product of two primes
Find p & q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time poly (log N)

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Given N & m, does N have
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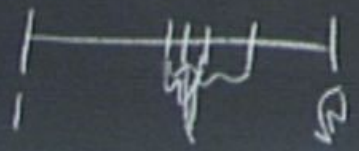

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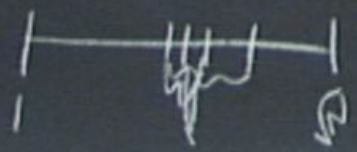

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RSA ("Rivest-Shamir-Adleman")

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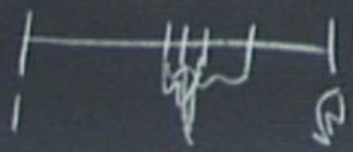

RSA ("Rivest-Shamir-Adleman")

Public-key cryptosystem

A B

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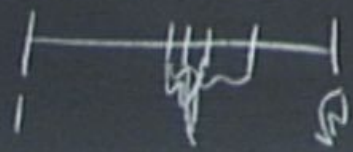

RSA ("Rivest-Shamir-Adleman")

Public-key cryptosystem

A

Private key

B

Public key

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encrypt → Public key

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A ← decrypt — B

A: Private key (d)
q derived from p, q $(N = pq)$

B: encrypt Public key (N, e)

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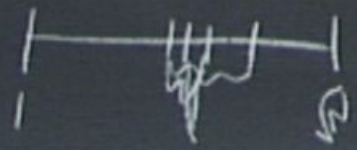

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Encryption:

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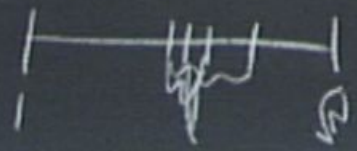

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Encryption: $x \longmapsto x^e \bmod N = y$

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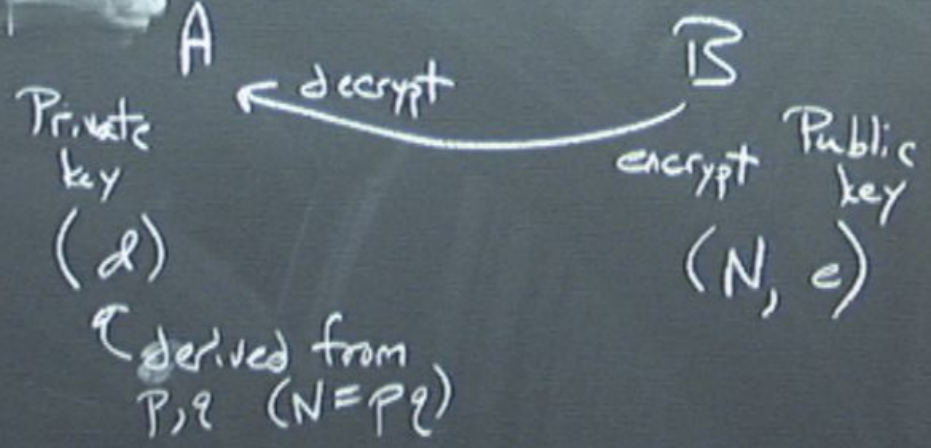

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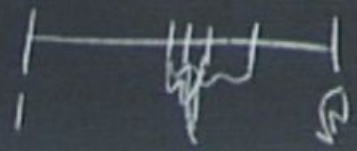

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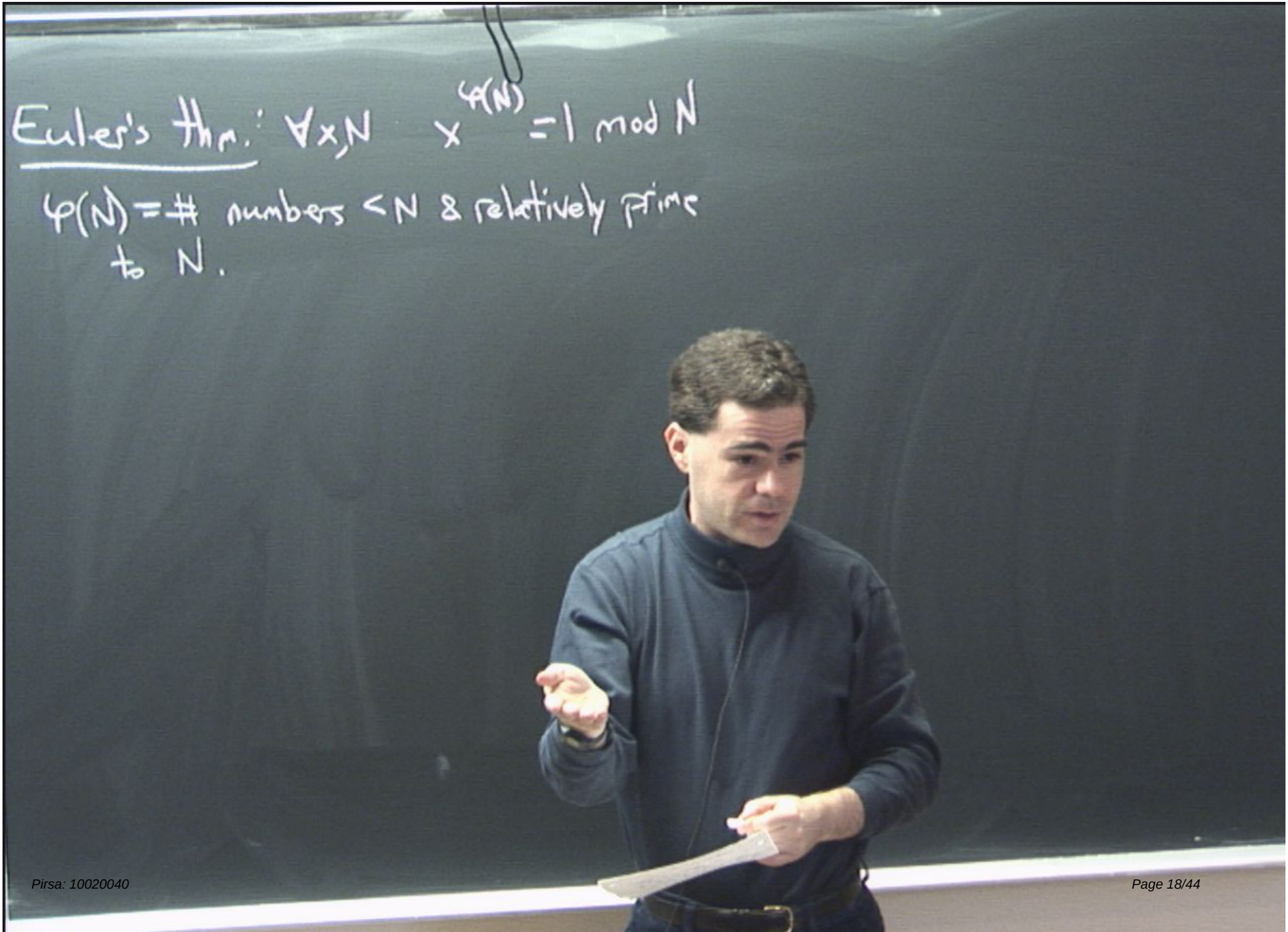
$x^{ed} = x \bmod N$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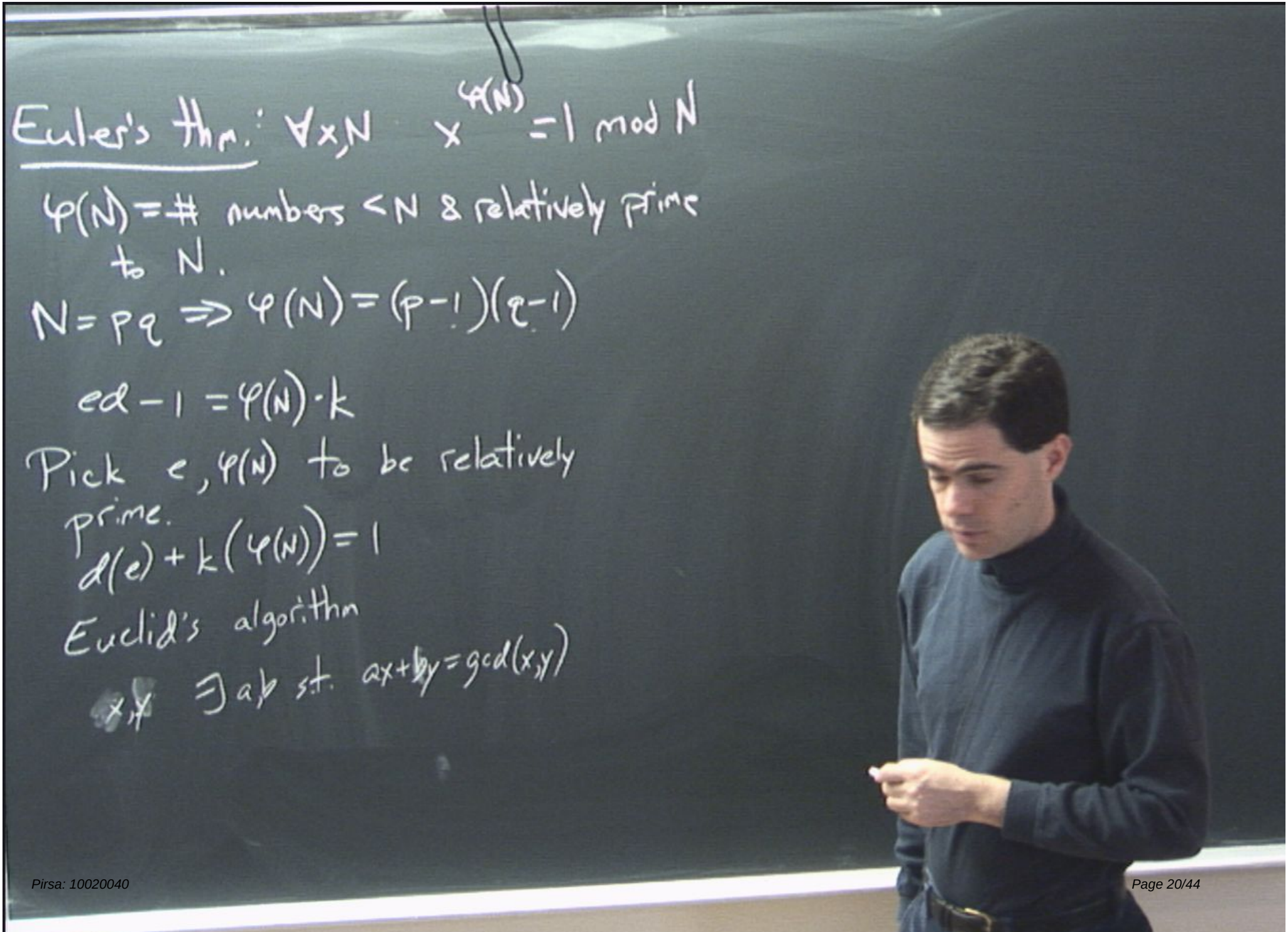
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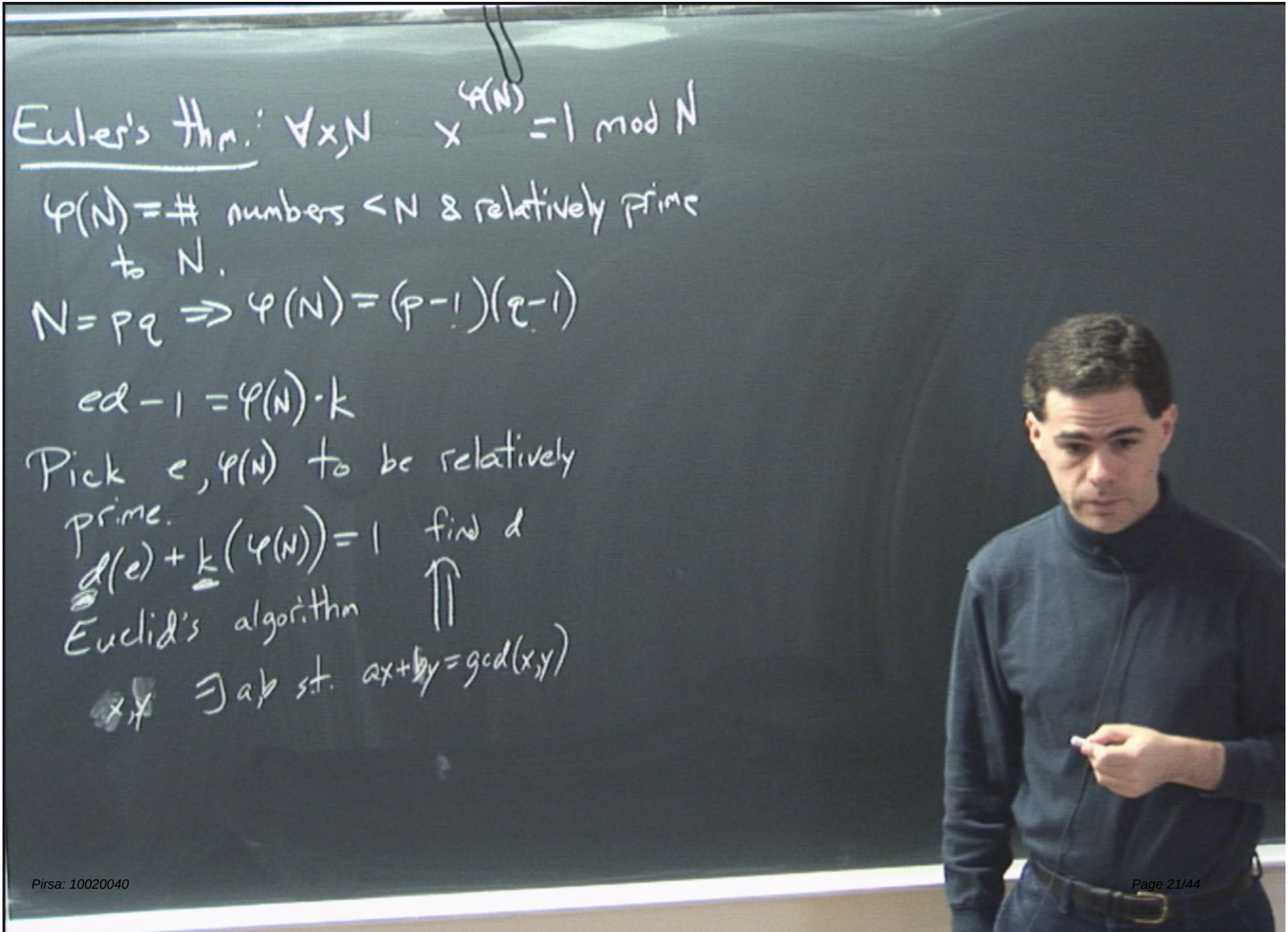
$d(e) + k(\varphi(N)) = 1$
Euclid's algorithm
$x,y \ \exists a,b$ s.t. $ax + by = gcd(x,y)$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Finding r in general ⇒ Factoring N

$=$

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- Find x with even order $r/2$ integer $y = x^{r/2}$ then $y^2 = 1 \bmod N$, $y \neq$

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- Find x with even order $r/2$ integer $y = x^{r/2}$ then $y^2 \equiv 1 \bmod N$, $y \not\equiv 1 \bmod N$

- Choose random x, with constant prob. x has even order r & $x^{r/2} \not\equiv -1 \bmod N$.

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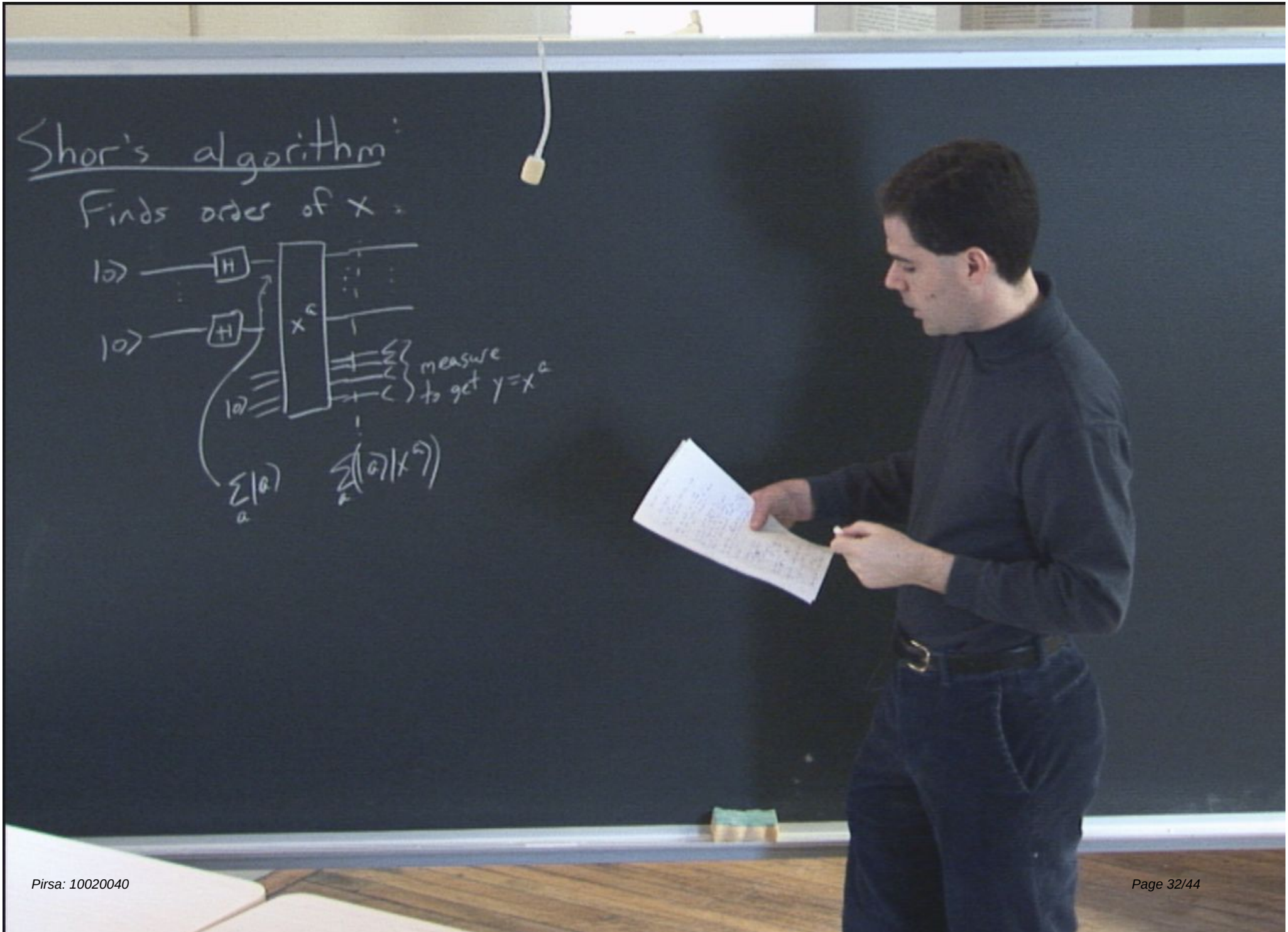
Shor's algorithm:
Finds order of x:
|0⟩
|0⟩
H
H
x^a
measure to get $y = x^a$
|0⟩
$\sum_a |a\rangle$
$\sum_a |a\rangle|x^a\rangle$

Shor's algorithm:

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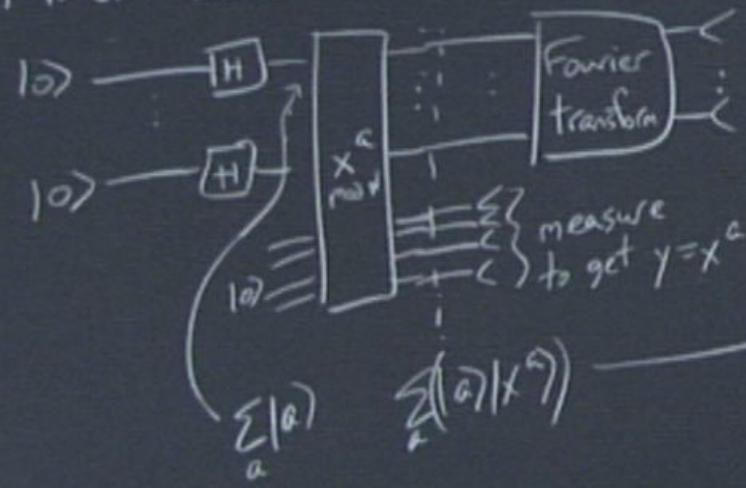


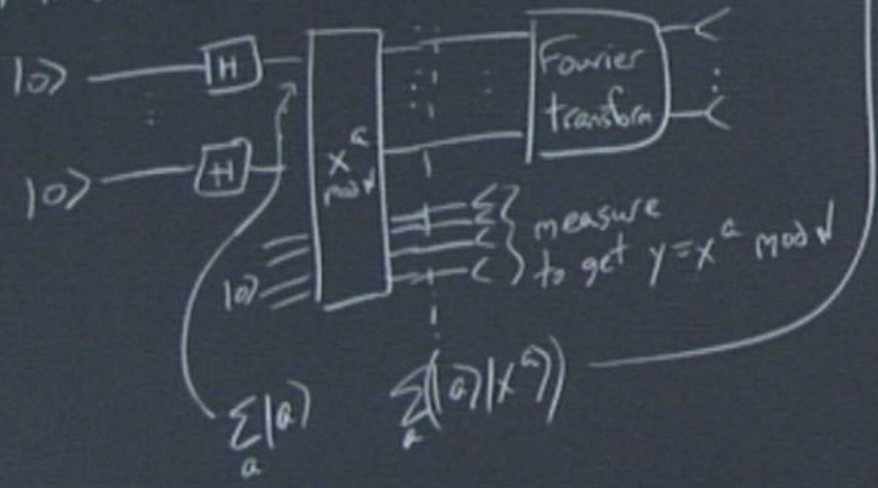
Measurement

Different x's with $x^a = y \bmod N$ are related by multiples of r

$x^{a_0} = y \bmod N$, $x^{a_0+r} = y \bmod N$, x^{a_0+}

Shor's algorithm:

Finds order of x:



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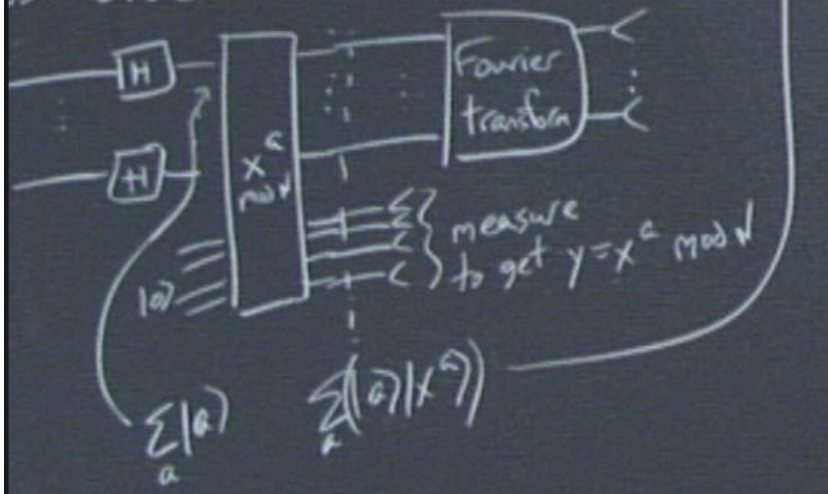
$x^{a_0} = y \bmod N$, $x^{a_0+r} = y \bmod N$, $x^{a_0+jr} = y$

algorithm:

...s order of x.



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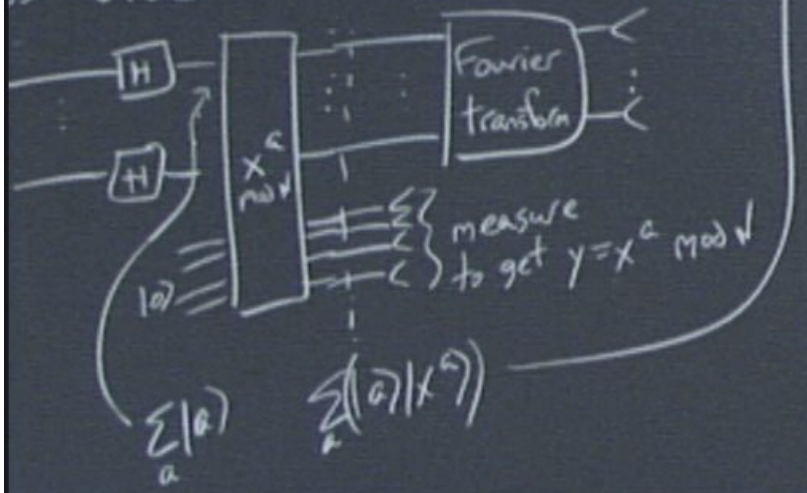
(Suppose $x^{a_0+s} = y \bmod N$

$\Rightarrow x^{-a_0}\left(x^{a_0+s}\right) = x^s = x^{-a_0} y \bmod N = 1 \bmod N$

algorithm:

...s order of x:



Measurement

Different x's with $x^a = y \bmod N$ are related by multiples of r

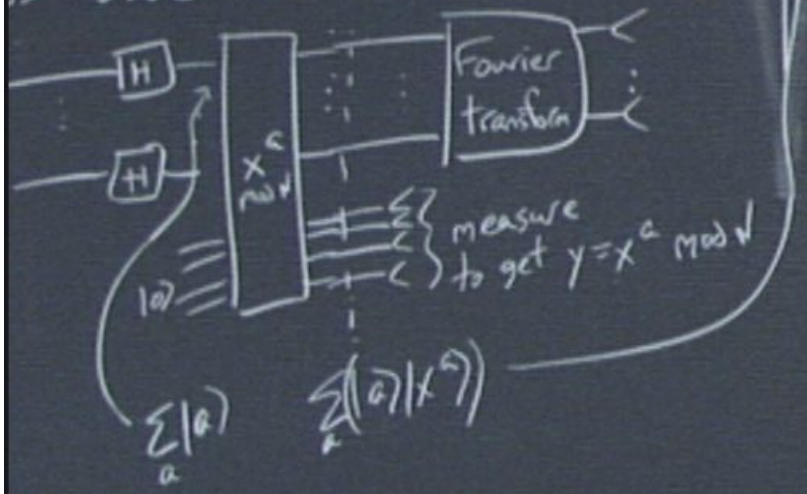
$x^{a_0} = y \bmod N$, $x^{a_0+r} = y \bmod N$, $x^{a_0+jr} = y \bmod N$

(Suppose $x^{a_0+s} = y \bmod N$

$\Rightarrow x^{-a_0}\left(x^{a_0+s}\right) = x^s = \underbrace{x^{-a_0} y}_{1} \bmod N = 1 \bmod N$)

algorithm:

...s order of x:



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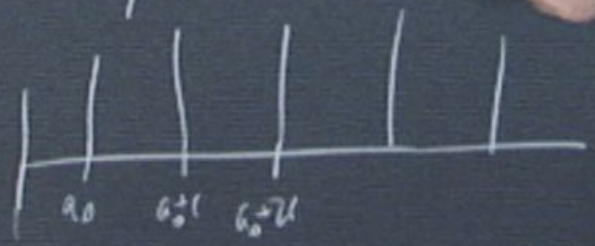
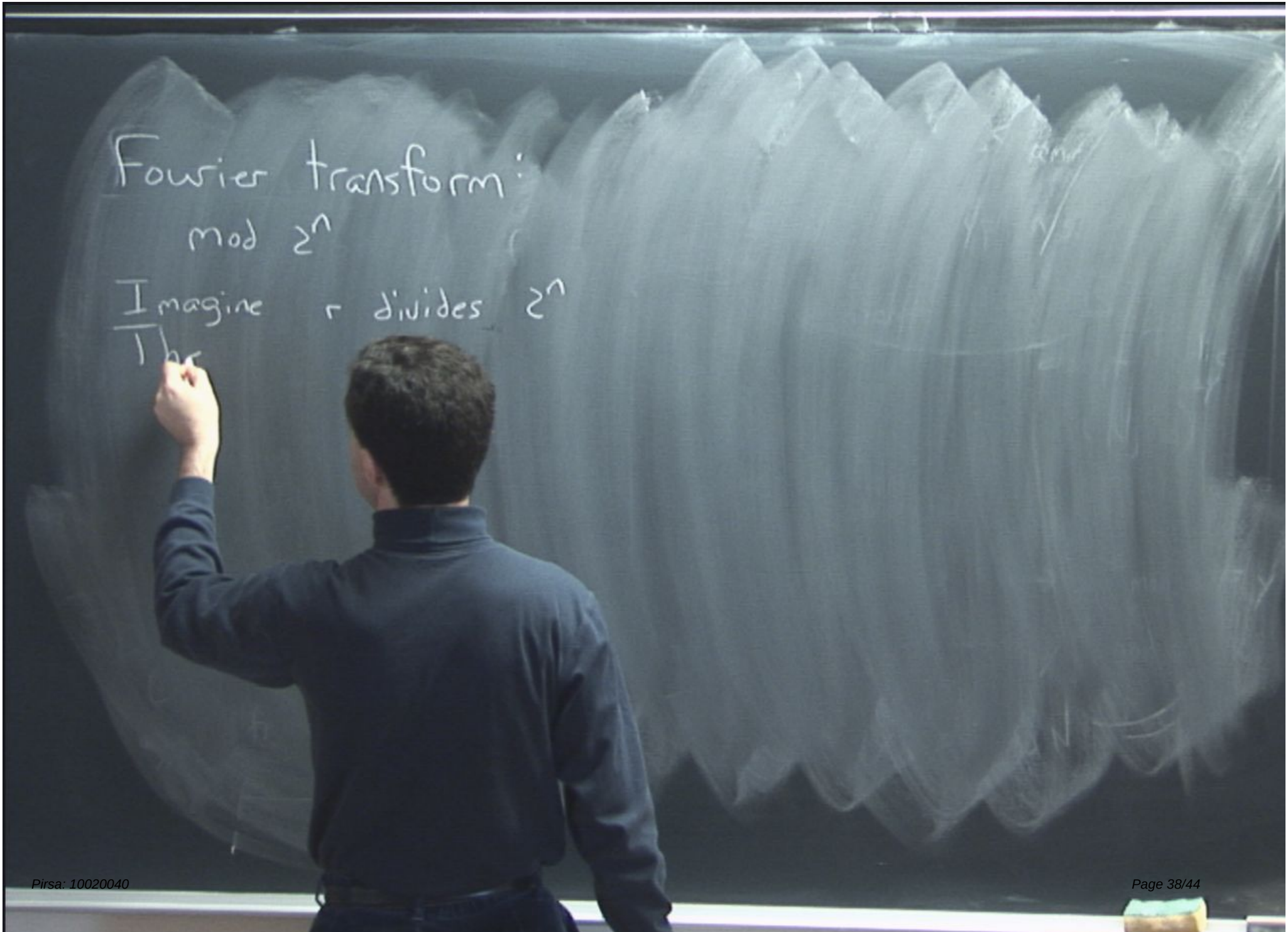
$\left(\sum_j |a_0 + jr\rangle\right)|y\rangle$




Fourier transform:
mod 2^
Imagine r divides 2^

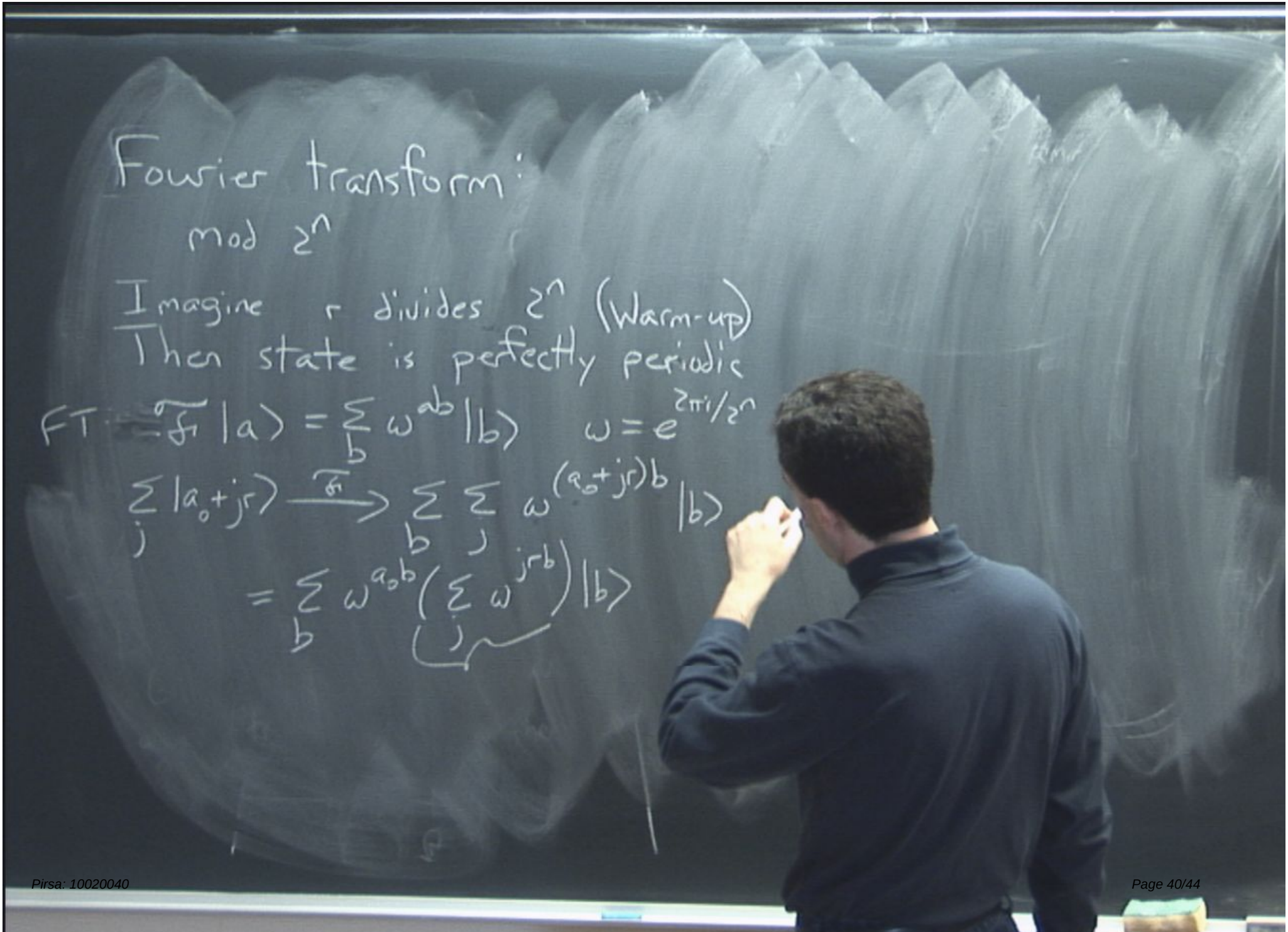
Fourier transform:

mod 2^n

Imagine r divides 2^n (Warm-up)

Then state is perfectly periodic

FT: $\frac{1}{\sqrt{r}} \sum_b \omega^{ab} |b\rangle \quad \omega = e^{2\pi i / 2^n}$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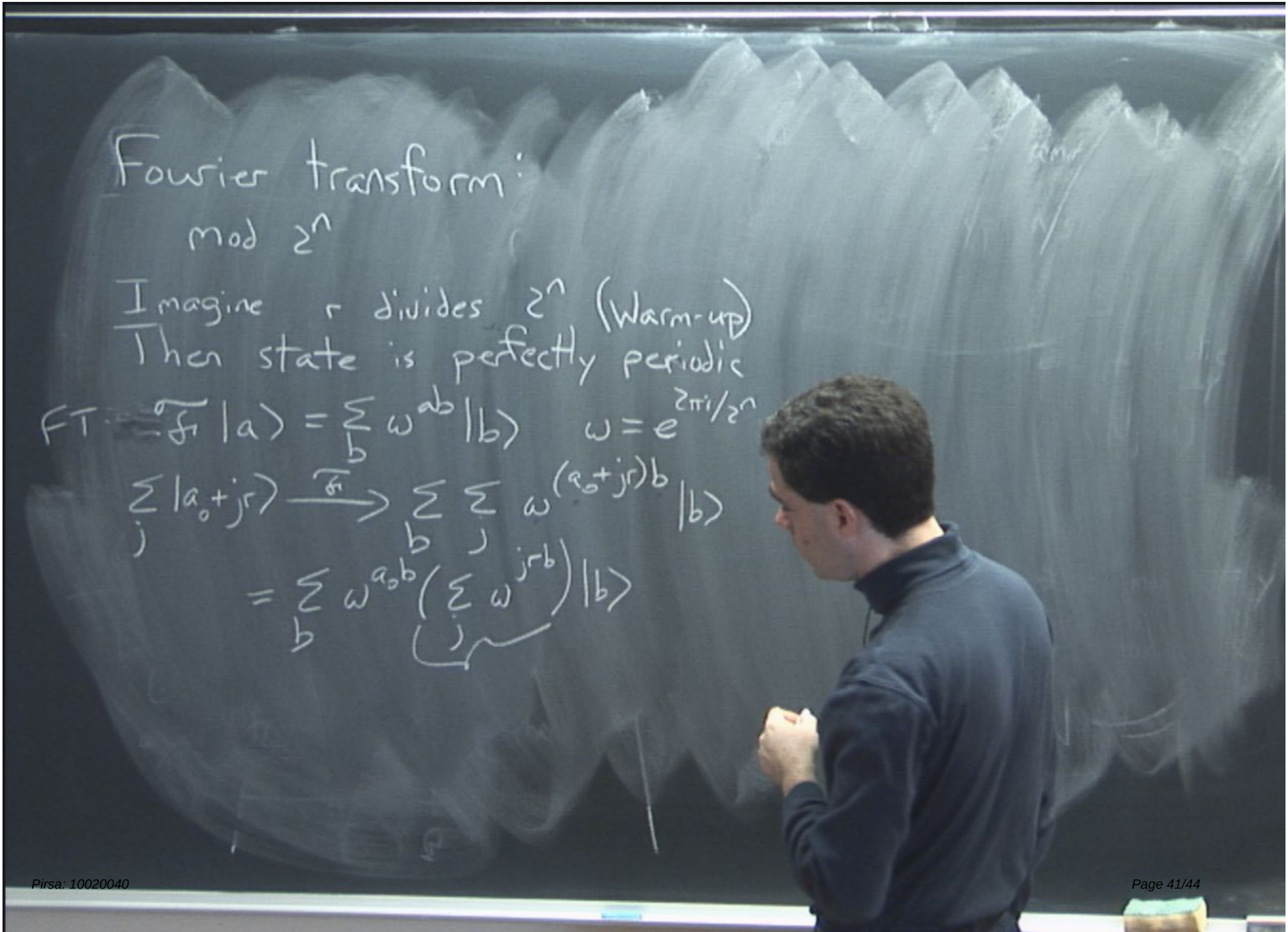
FT $|a\rangle = \sum_b \omega^{ab} |b\rangle$ $\omega = e^{2\pi i/2^n}$
$\sum_j |a_0 + jr\rangle \xrightarrow{FT} \sum_b \sum_j \omega^{(a_0+jr)b} |b\rangle$
$= \sum_b \omega^{a_0 b} \left(\sum_j \omega^{jrb} \right) |b\rangle$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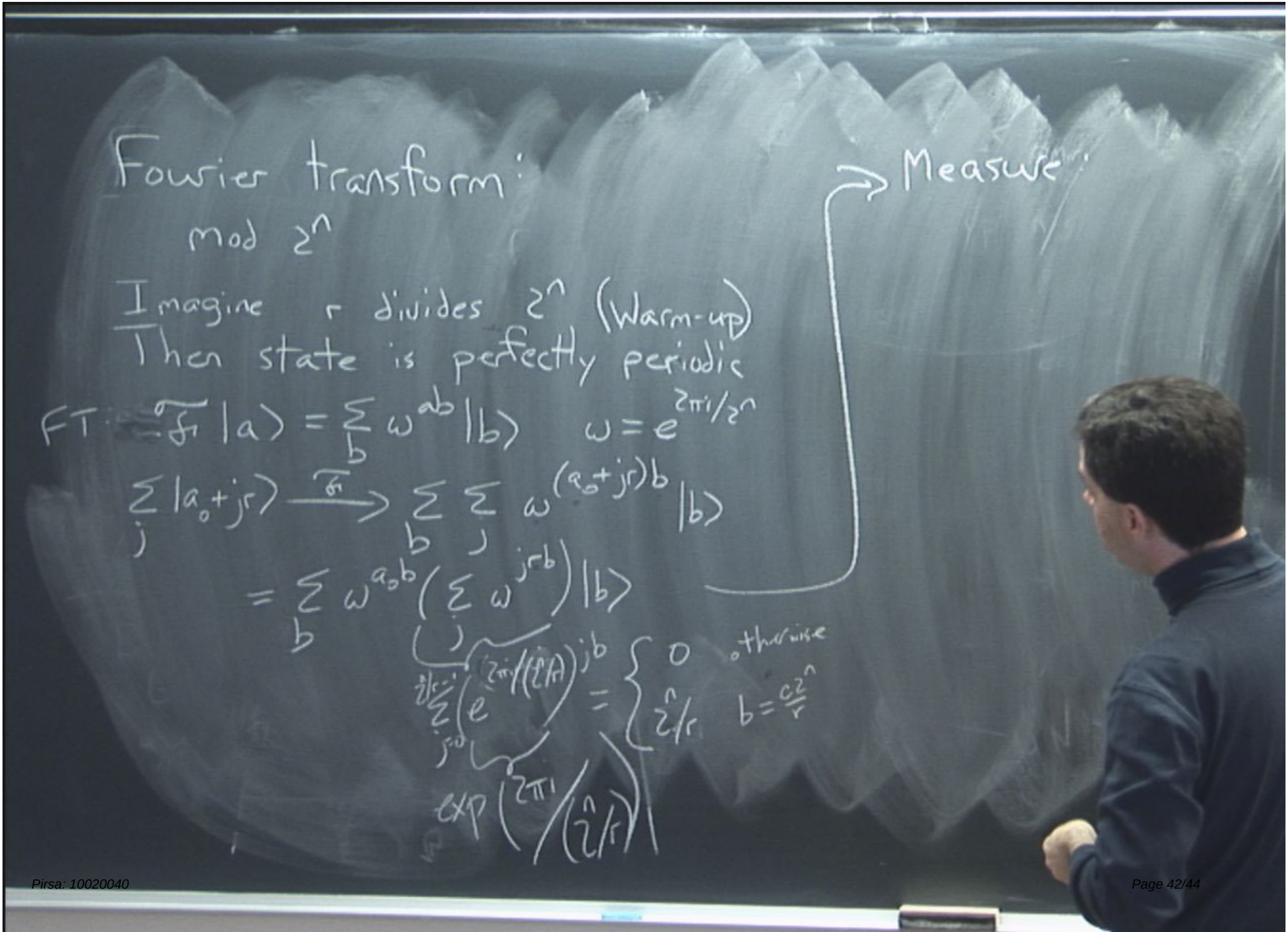
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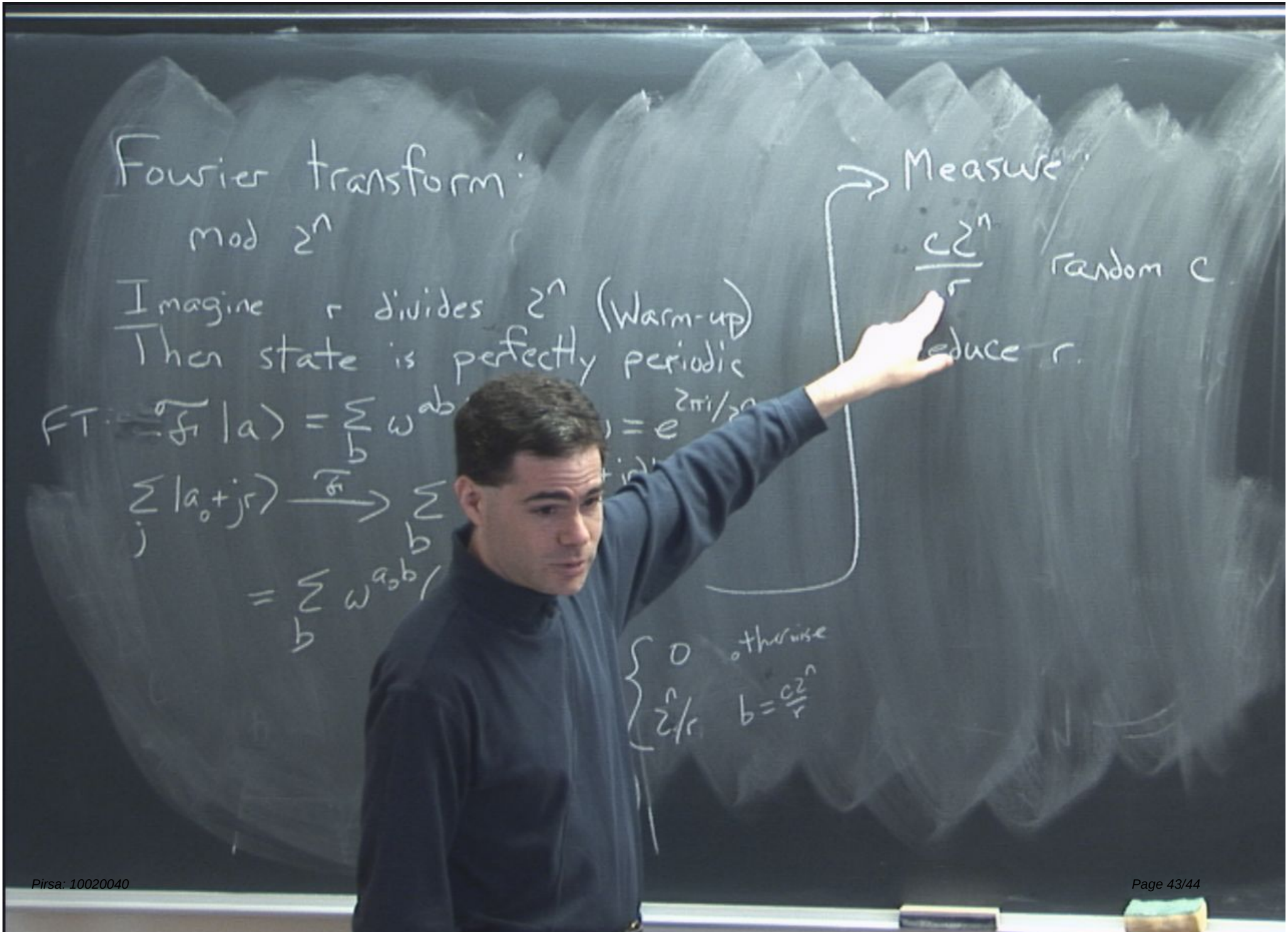
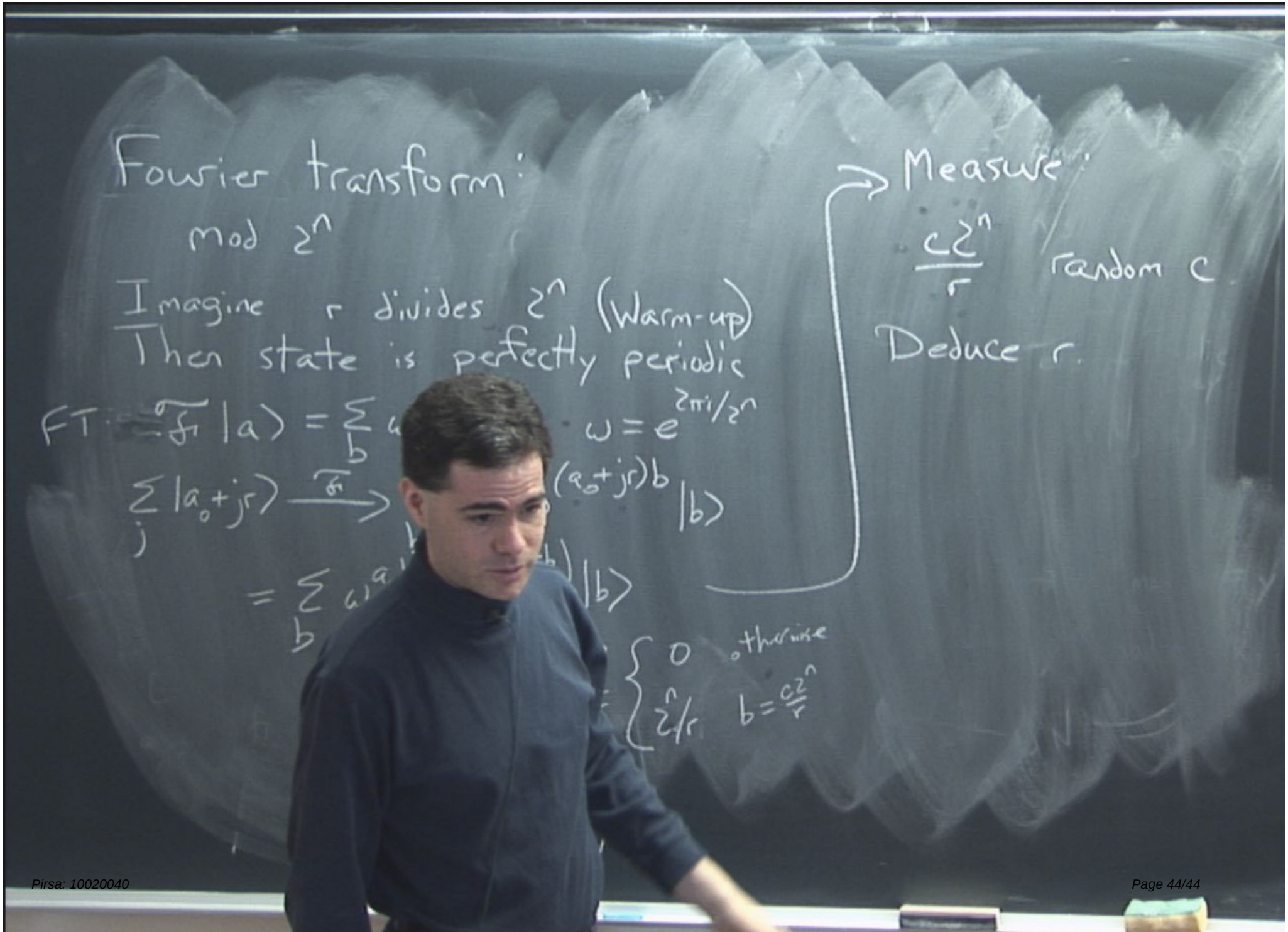
$\sum_{j=0}^{2^n/r} e^{2\pi i/(2^n/r)\, jb} = \begin{cases} 0 & \text{otherwise} \\ 2^n/r & b = c\frac{2^n}{r} \end{cases}$
$\exp\left(2\pi i/(2^n/r)\right)$


Fourier transform:
mod 2^n
Imagine r divides 2^n (Warm-up)
Then state is perfectly periodic
Measure
$\frac{c2^n}{r}$
random c
educe r.
otherwise
$b = \frac{c2^n}{r}$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Deduce r.