

Title: Quantum Field Theory for Cosmology - Lecture 10

Date: Feb 11, 2010 05:00 PM

URL: <http://pirsa.org/10020019>

Abstract: <span>This course begins with a thorough introduction to quantum field theory. Unlike the usual quantum field theory courses which aim at applications to particle physics, this course then focuses on those quantum field theoretic techniques that are important in the presence of gravity. In particular, this course introduces the properties of quantum fluctuations of fields and how they are affected by curvature and by gravitational horizons. We will cover the highly successful inflationary explanation of the fluctuation spectrum of the cosmic microwave background - and therefore the modern understanding of the quantum origin of all inhomogeneities in the universe (see these amazing visualizations from the data of the Sloan Digital Sky Survey. They display the inhomogeneous distribution of galaxies several billion light years into the universe: Sloan Digital Sky Survey).</span>

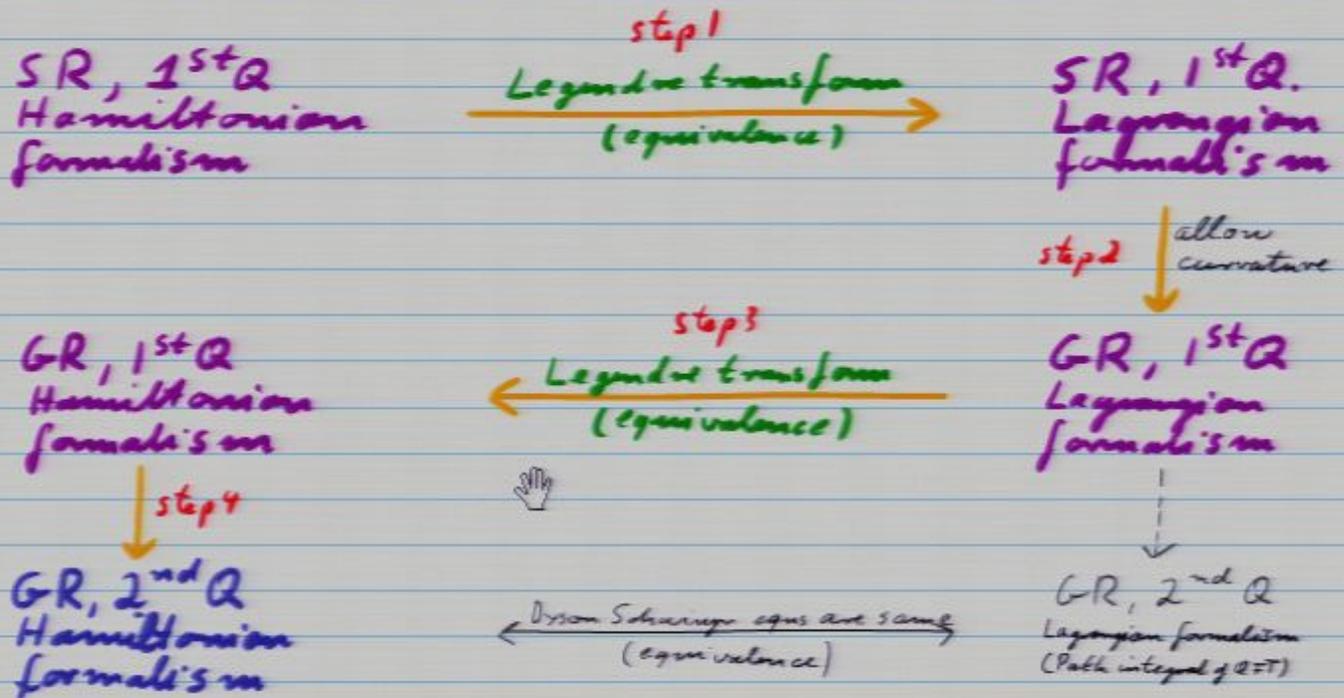


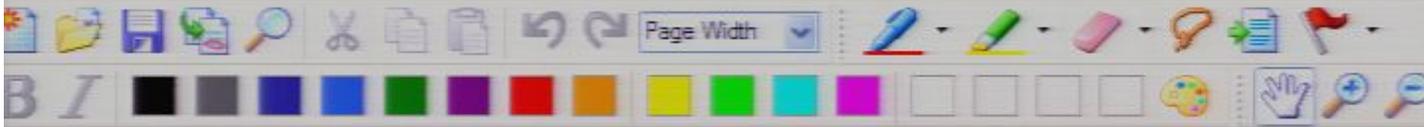
Recall:

(because the Lagrangian framework treats space and time in the same way)

- \* Hamiltonian formulations are suitable for quantization.
- \* Lagrangian formulations are suitable to achieve general relativistic covariance.

→ Strategy:





We already started step 1:

$$\begin{array}{ccc}
 H[\phi, \pi, t] & \begin{array}{c} \xrightarrow{\beta(x,t) := \frac{\delta H}{\delta \pi(x,t)} \quad (T)} \\ \xleftarrow{\pi(x,t) := \frac{\delta L}{\delta \beta(x,t)} \quad (T^{-1})} \end{array} & L[\phi, \beta, t]
 \end{array}$$

Proposition: These equations of motion are equivalent:

Hamiltonian eqns. of motion:

$$\dot{\phi}(x,t) = \frac{\delta H[\phi, \pi, t]}{\delta \pi(x,t)} \quad (H1)$$

$$\dot{\pi}(x,t) = - \frac{\delta H[\phi, \pi, t]}{\delta \phi(x,t)} \quad (H2)$$

Lagrangian eqns. of motion:

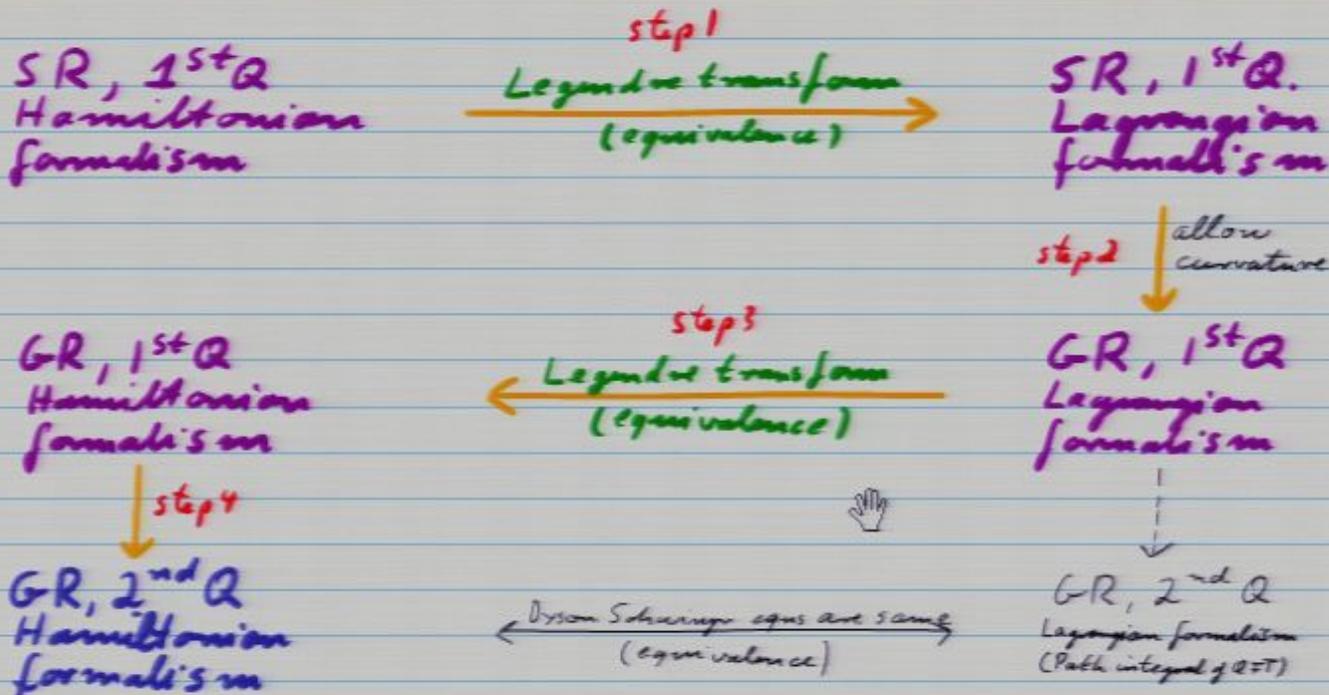
$$\dot{\phi}(x,t) = \beta(x,t) \quad (L1)$$

$$\frac{\delta L}{\delta \phi(x,t)} = \frac{d}{dt} \frac{\delta L}{\delta \beta(x,t)} \quad (L2)$$



\* Lagrangian formulations are suitable to achieve general relativistic covariance.

→ Strategy:



We already started step 1:

$$\beta(x,t) := \frac{\delta H}{\delta \pi(x,t)} (T)$$



GR, 1<sup>st</sup> Q  
Hamiltonian  
formalism

step 4

GR, 2<sup>nd</sup> Q  
Hamiltonian  
formalism

step 3  
Legendre transform  
(equivalence)

Dyson Schwinger eqns are same  
(equivalence)

step 2 allow  
curvature

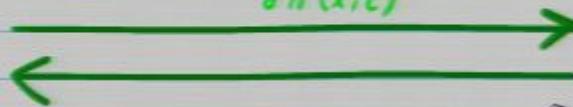
GR, 1<sup>st</sup> Q  
Lagrangian  
formalism

GR, 2<sup>nd</sup> Q  
Lagrangian formalism  
(Path integral of QFT)

We already started step 1:

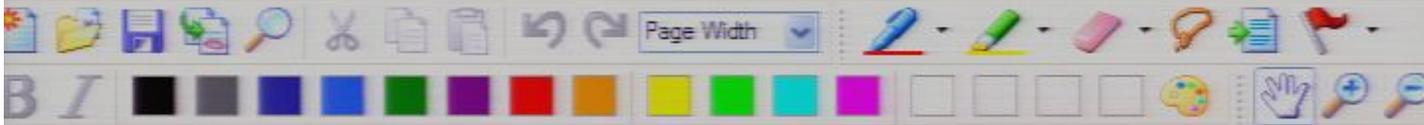
$H[\phi, \pi, t]$

$$\beta(x, t) := \frac{\delta H}{\delta \pi(x, t)} \quad (T)$$



$L[\phi, \beta, t]$

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Proof: We need to show that  $(H1 + H2) \xleftrightarrow{T} (L1, L2)$ .

The case " $\Rightarrow$ "

□ Show L1: Indeed:  $\phi \stackrel{(H1)}{=} \frac{\delta H}{\delta \pi} \stackrel{(T)}{=} \beta \checkmark$

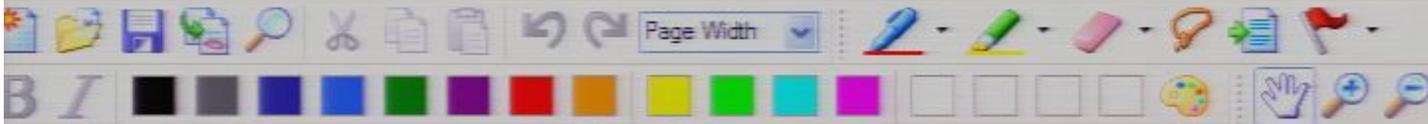
□ Show L2: Indeed:

$$\frac{d}{dt} \frac{\delta L(\phi, \beta, t)}{\delta \beta} \stackrel{(T^{-1})}{=} \frac{d}{dt} \pi$$

$$\stackrel{(H2)}{=} - \frac{\delta H(\phi, \pi, t)}{\delta \phi}$$

$$\stackrel{\text{by def. of } L}{=} - \frac{\delta}{\delta \phi} \left( \int \beta(\phi, \pi) \pi d^3x - L(\phi, \beta(\phi, \pi), t) \right)$$

$$= \frac{\delta L}{\delta t} - \frac{\delta \beta}{\delta t} \pi + \frac{\delta L}{\delta \beta} \frac{\delta \beta}{\delta t}$$



Hamiltonian eqns. of motion:

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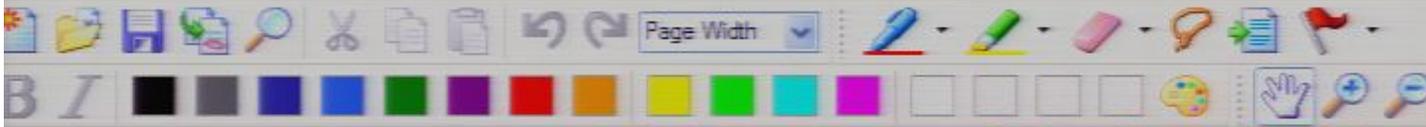
Proof: We need to show that  $(H1 + H2) \xleftrightarrow{T} (L1, L2)$ .

The case " $\Rightarrow$ "

□ Show L1: Indeed:  $\dot{\phi} \stackrel{(H1)}{=} \frac{\delta H}{\delta \pi} \stackrel{(T)}{=} \beta \checkmark$

□ Show L2: Indeed:

$$\frac{d}{dt} \frac{\delta L(\phi, \beta, t)}{\delta \beta} \stackrel{(T^{-1})}{=} \frac{d}{dt} \pi$$



$$\frac{dt}{\delta\beta} \quad \frac{dt}{dt}''$$

$$\frac{(H2)}{\delta\phi} = \frac{\delta H(\phi, \pi, t)}{\delta\phi}$$

$$\frac{\delta L}{\delta\phi} = \frac{\delta}{\delta\phi} \left( \int \beta(\phi, \pi) \pi d^3x - L(\phi, \beta(\phi, \pi), t) \right)$$

$$= \frac{\delta L}{\delta\phi} - \frac{\delta\phi}{\delta\phi} \pi + \frac{\delta L}{\delta\beta} \frac{\delta\beta}{\delta\phi} \quad \checkmark$$

The case " $\Leftarrow$ ": Exercise.

Result so far:

Legendre transform to Lagrangian formulation

$\Rightarrow$  Eqs of motion can be cast in the



The case "  $\Rightarrow$  "

□ Show L1: Indeed:  $\dot{\phi} \stackrel{(H1)}{=} \frac{\delta H}{\delta \pi} \stackrel{(T)}{=} \beta \checkmark$

□ Show L2: Indeed:

$$\frac{d}{dt} \frac{\delta L(\phi, \beta, t)}{\delta \beta} \stackrel{(T^{-1})}{=} \frac{d}{dt} \pi$$

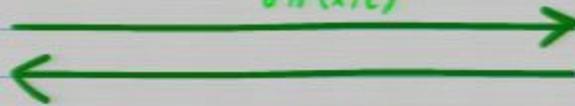
$$\stackrel{(H2)}{=} - \frac{\delta H(\phi, \pi, t)}{\delta \phi}$$

$$\stackrel{\text{by def.}}{\neq L} \frac{\delta}{\delta \phi} \left( \int \beta(\phi, \pi) \pi d^3x - L(\phi, \beta(\phi, \pi), t) \right)$$

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$$H[\phi, \pi, t]$$

$$\beta(x, t) := \frac{\delta H}{\delta \pi(x, t)} \quad (T)$$



$$L[\phi, \beta, t]$$

$$\pi(x, t) := \frac{\delta L}{\delta \beta(x, t)} \quad (T^{-1})$$

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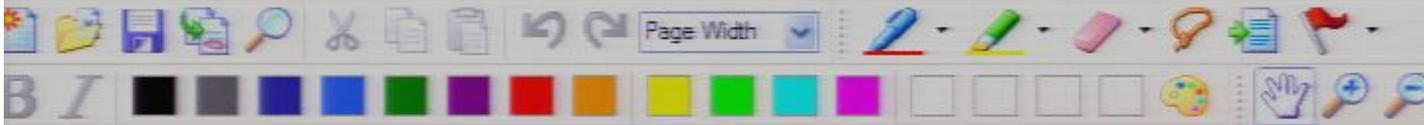
$$\dot{\pi}(x, t) = - \frac{\delta H[\phi, \pi, t]}{\delta \phi(x, t)} \quad (H2)$$

Lagrangian eqns. of motion:

$$\dot{\phi}(x, t) = \beta(x, t) \quad (L1)$$

$$\frac{\delta L}{\delta \phi(x, t)} = \frac{d}{dt} \frac{\delta L}{\delta \beta(x, t)} \quad (L2)$$

Proof: we need to show that  $(H1 + H2) \xrightarrow{T} (L1 + L2)$



We already started step 1:

$$\begin{array}{ccc}
 H[\phi, \pi, t] & \begin{array}{c} \xrightarrow{\beta(x,t) := \frac{\delta H}{\delta \pi(x,t)} \quad (T)} \\ \xleftarrow{\pi(x,t) := \frac{\delta L}{\delta \beta(x,t)} \quad (T^{-1})} \end{array} & L[\phi, \beta, t]
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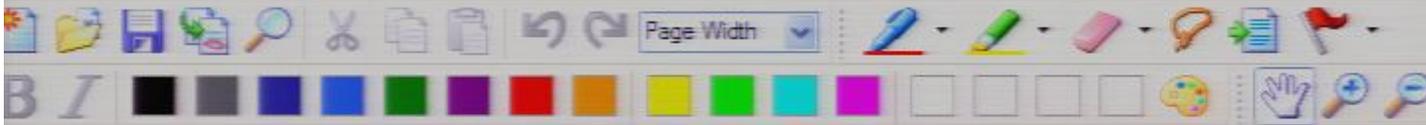
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$$\frac{\delta L}{\delta \phi(x,t)} = \frac{d}{dt} \frac{\delta L}{\delta \beta(x,t)} \quad (L2)$$



Proof: We need to show that  $(H1 + H2) \overset{T}{\longleftrightarrow} (L1, L2)$ .

The case " $\Rightarrow$ "

□ Show L1: Indeed:  $\phi \overset{(H1)}{=} \frac{\delta H}{\delta \pi} \overset{(T)}{=} \beta \checkmark$

□ Show L2: Indeed:

$$\frac{d}{dt} \frac{\delta L(\phi, \beta, \pi, t)}{\delta \beta} \overset{(T^{-1})}{=} \frac{d}{dt} \pi$$

$$\overset{(H2)}{=} - \frac{\delta H(\phi, \pi, t)}{\delta \phi}$$

$$\overset{\text{by def. of } L}{=} - \frac{\delta}{\delta \phi} \left( \int \beta(\phi, \pi) \pi d^3x - L(\phi, \beta(\phi, \pi), t) \right)$$

$$= \frac{\delta L}{\delta \phi} - \frac{\delta \beta}{\delta \phi} \pi + \frac{\delta L}{\delta \beta} \frac{\delta \beta}{\delta \phi}$$



Proof: We need to show that  $(H1 + H2) \xleftrightarrow{T} (L1, L2)$ .

The case " $\Rightarrow$ "

□ Show L1: Indeed:  $\phi \xrightarrow{(H1)} \frac{\delta H}{\delta \pi} \xrightarrow{(T)} \beta \checkmark$

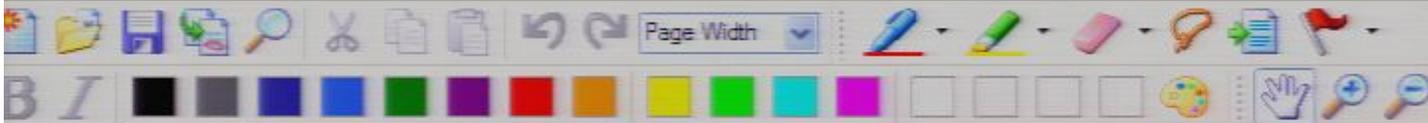
□ Show L2: Indeed:

$$\frac{d}{dt} \frac{\delta L(\phi, \beta, t)}{\delta \beta} \xrightarrow{(T^{-1})} \frac{d}{dt} \pi$$

$$\xrightarrow{(H2)} - \frac{\delta H(\phi, \pi, t)}{\delta \phi}$$

$$\xrightarrow[\text{of } L]{\text{by def.}} - \frac{\delta}{\delta \phi} \left( \int \beta(\phi, \pi) \pi d^3x - L(\phi, \beta(\phi, \pi), t) \right)$$

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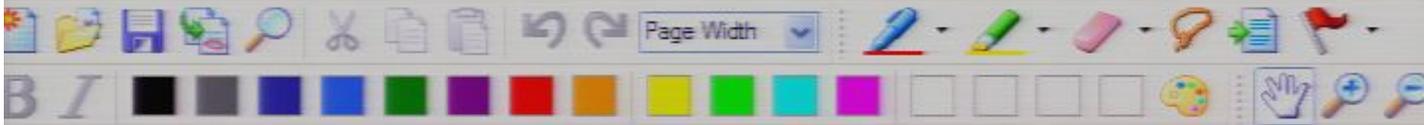
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□ Show L1: Indeed:  $\dot{\phi} \stackrel{(H1)}{=} \frac{\delta H}{\delta \pi} \stackrel{(T)}{=} \beta \quad \checkmark$

□ Show L2: Indeed:

$$\frac{d}{dt} \frac{\delta L(\phi, \beta, t)}{\delta \beta} \stackrel{(T^{-1})}{=} \frac{d}{dt} \pi$$



We already started step 1:

$$\begin{array}{ccc}
 & \beta(x,t) := \frac{\delta H}{\delta \pi(x,t)} \quad (T) & \\
 H[\phi, \pi, t] & \begin{array}{c} \xrightarrow{\hspace{10em}} \\ \xleftarrow{\hspace{10em}} \end{array} & L[\phi, \beta, t] \\
 & \pi(x,t) := \frac{\delta L}{\delta \beta(x,t)} \quad (T^{-1}) &
 \end{array}$$

Proposition: These equations of motion are equivalent:

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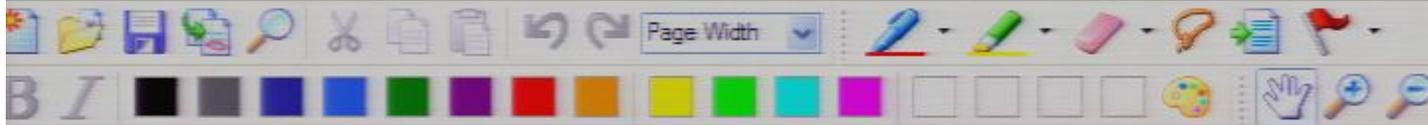
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Proof: We need to show that  $(H1 + H2) \xleftrightarrow{T} (L1, L2)$ .

The case " $\Rightarrow$ "

□ Show L1: Indeed:  $\dot{\phi} \stackrel{(H1)}{=} \frac{\delta H}{\delta \pi} \stackrel{(T)}{=} \beta \checkmark$

□ Show L2: Indeed:

$$\frac{d}{dt} \frac{\delta L(\phi, \beta, t)}{\delta \beta} \stackrel{(T^{-1})}{=} \frac{d}{dt} \pi$$

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Proof: We need to show that  $(H1 + H2) \overset{T}{\iff} (L1, L2)$ .

The case " $\implies$ "

□ Show L1: Indeed:  $\dot{\phi} \stackrel{(H1)}{=} \frac{\delta H}{\delta \pi} \stackrel{(T)}{=} \beta \quad \checkmark$

□ Show L2: Indeed:

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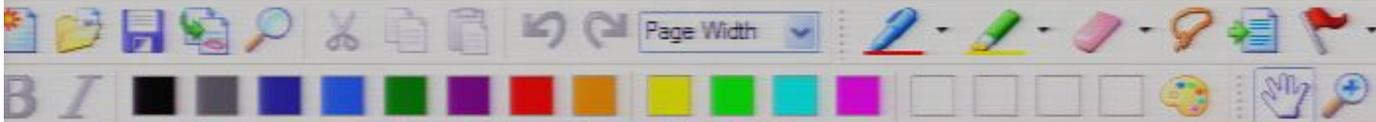
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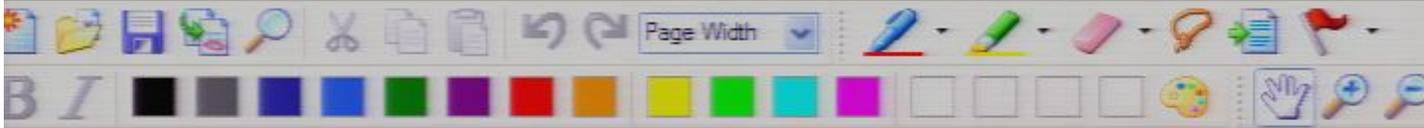
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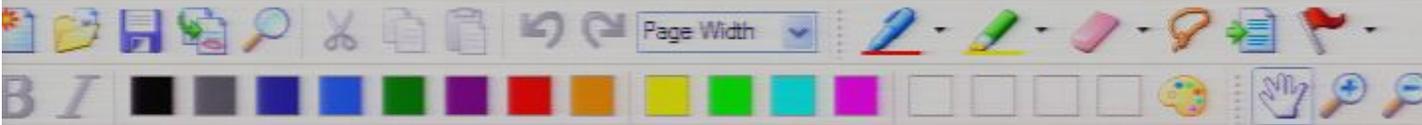
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□ Show L2: Indeed:

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□ Show L2: Indeed:

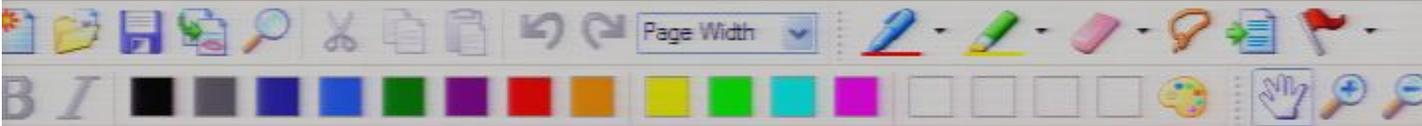
$$\frac{d}{dt} \frac{\delta L(\phi, \beta, t)}{\delta \beta} \stackrel{(T^{-1})}{=} \frac{d}{dt} \pi$$

$$\stackrel{(H2)}{=} - \frac{\delta H(\phi, \pi, t)}{\delta \phi}$$

$$\stackrel{\text{by def.}}{\neq L} - \frac{\delta}{\delta \phi} \left( \int \beta(\phi, \pi) \pi d^3x - L(\phi, \beta(\phi, \pi), t) \right)$$

$$= \frac{\delta L}{\delta \phi} - \cancel{\frac{\delta \beta}{\delta \phi} \pi} + \cancel{\frac{\delta L}{\delta \beta} \frac{\delta \beta}{\delta \phi}} \checkmark$$

The case " $\Leftarrow$ ": See page



The case "  $\Rightarrow$  "

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Result so far:



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Legendre transform to Lagrangian formulation

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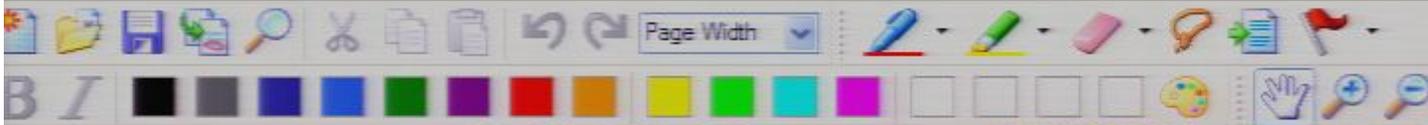
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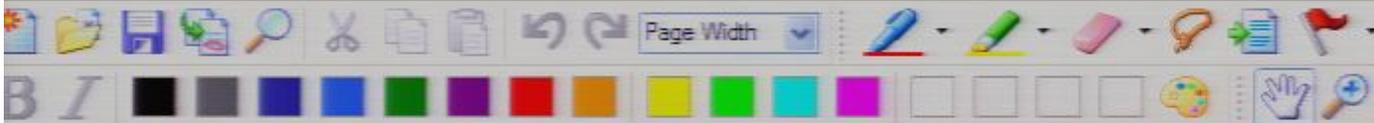
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$\delta\phi$   ~~$\delta\phi$~~   ~~$\delta\beta$~~   $\delta\phi$

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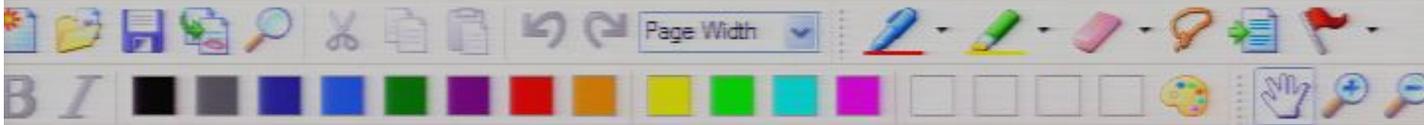
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(Notice: Only a time derivative,   
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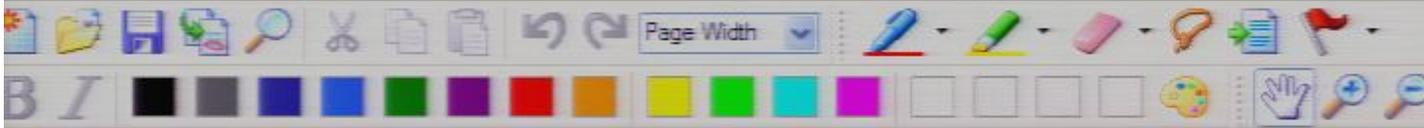
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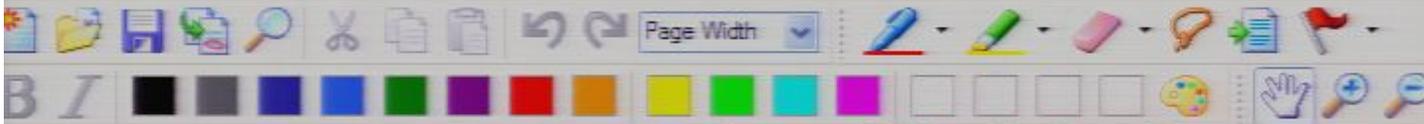
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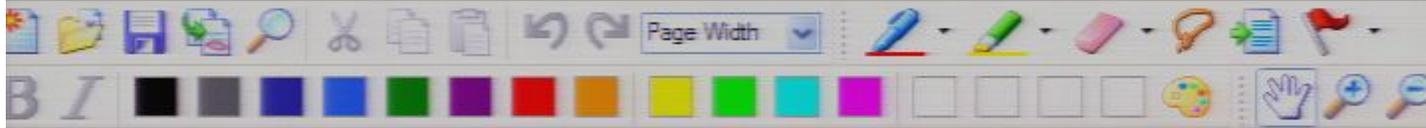
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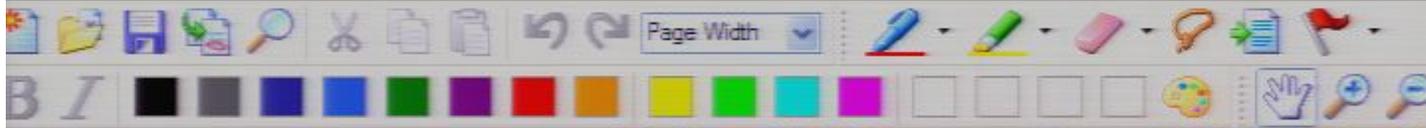
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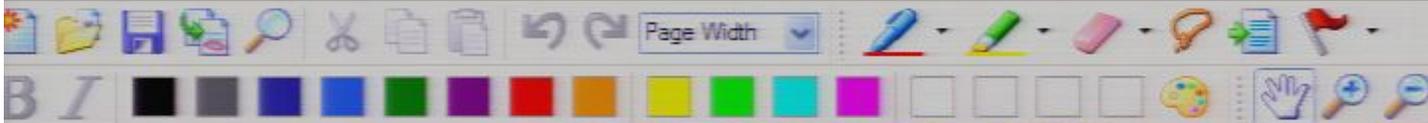
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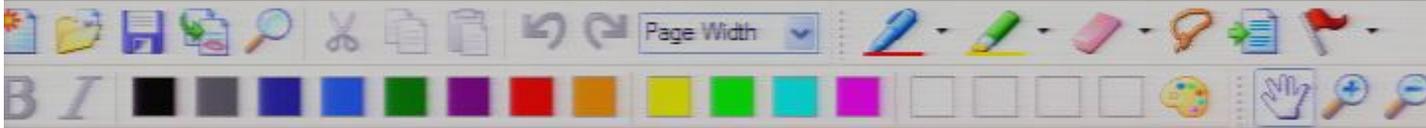
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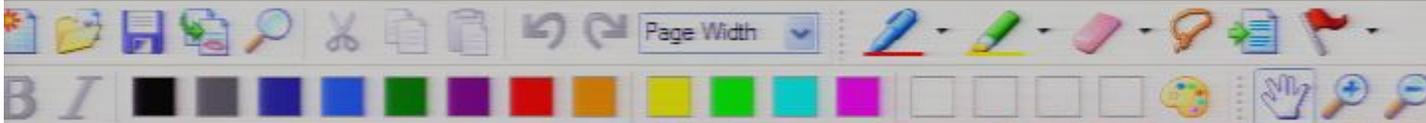
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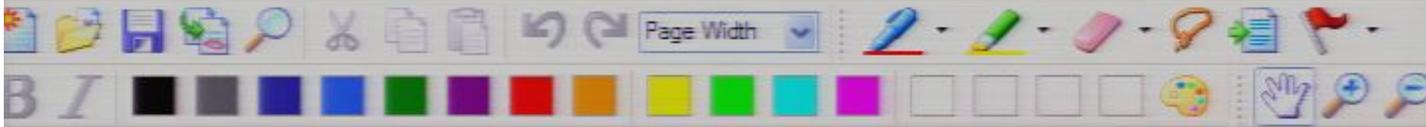
$L_1, L_2$  indeed do contain time and space

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✎ \* Is there a simple way to evaluate the derivatives with respect to  $\frac{\partial \phi}{\partial x_i}$ ?

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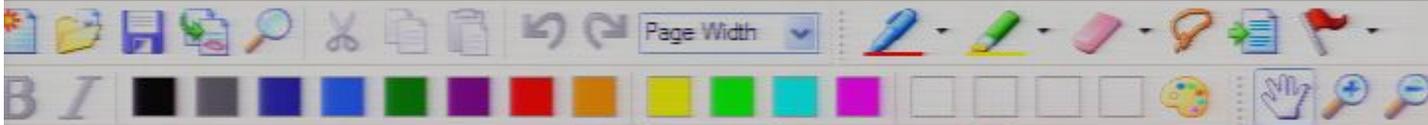
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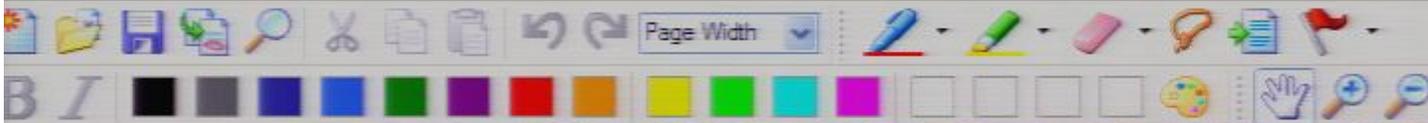


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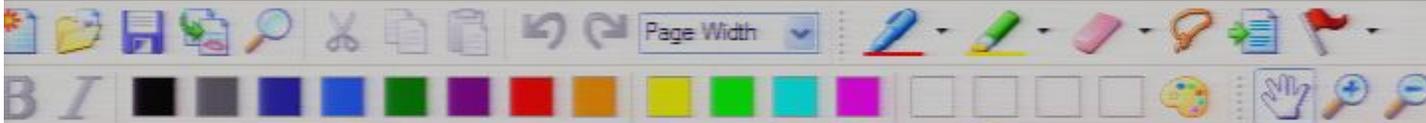
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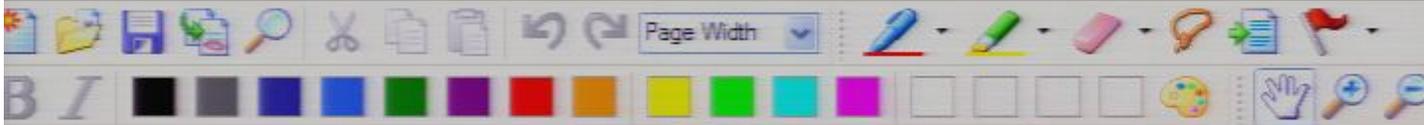
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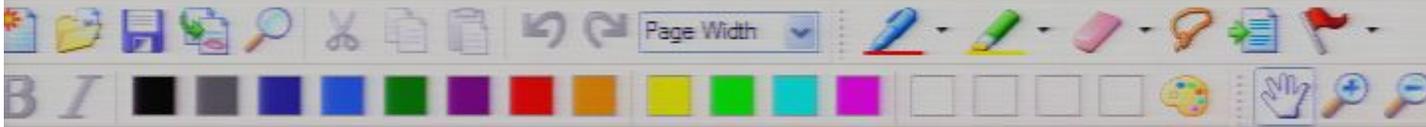
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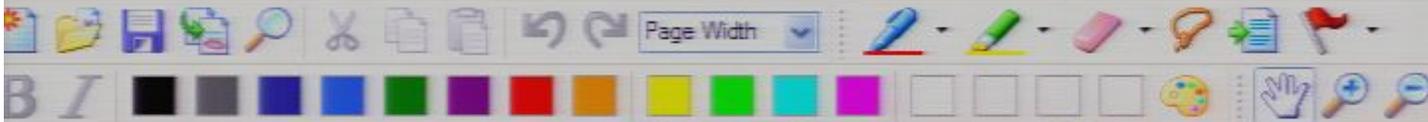
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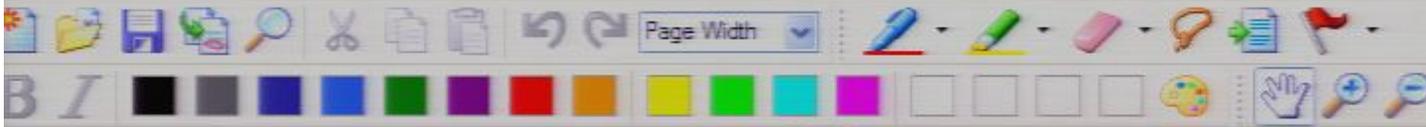
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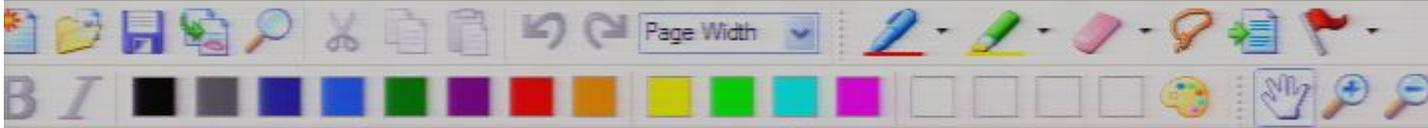
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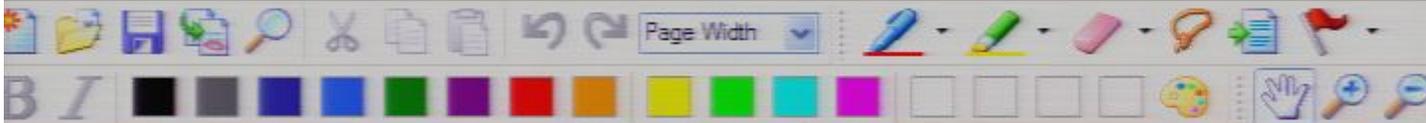
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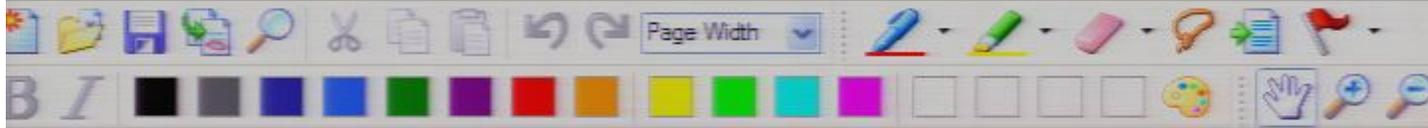
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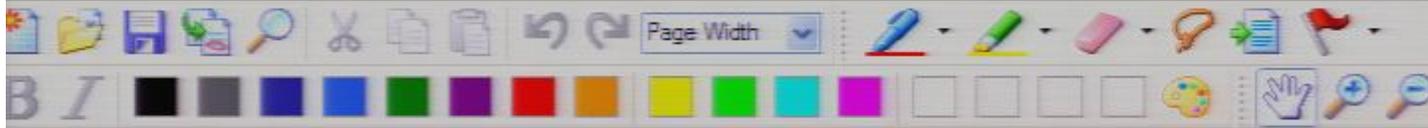
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$$\frac{\delta Z[\partial_x f]}{\delta(\partial_x f(x))} = 2 \partial_x f(x)$$

- Our lemma claims, therefore:

$$\frac{\delta Z[f]}{\delta f(x)} = -\partial_x \frac{\delta Z[\partial_x f]}{\delta(\partial_x f(x))} = -2 \partial_x \partial_x f(x)$$



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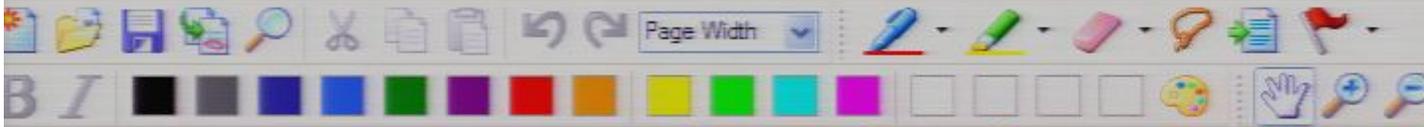
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- Our lemma claims, therefore:

$$\frac{\delta Z[f]}{\delta f(x)} = -\partial_x \frac{\delta Z[\partial_x f]}{\delta(\partial_x f(x))} = -2 \partial_x \partial_x f(x)$$



$$\text{Then: } \frac{\delta z}{\delta f(x)} = - \frac{d}{dx} \frac{\delta z}{\delta \left( \frac{d}{dx} f \right)}$$

On the right hand side  
we view  $\frac{d}{dx} f$  as an  
independent function.

Example:

$$\text{Notation: } \partial_x f(x) = \frac{d}{dx} f(x)$$

$$z[J] := \int_{\mathbb{R}} (\partial_x f(x))^2 dx$$

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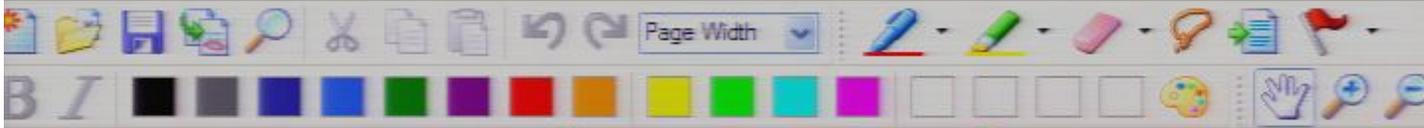
$$Z[f] := \int_{\mathbb{R}} (\partial_x f(x))^2 dx'$$

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□ Let us verify this from first principles!

Indeed:



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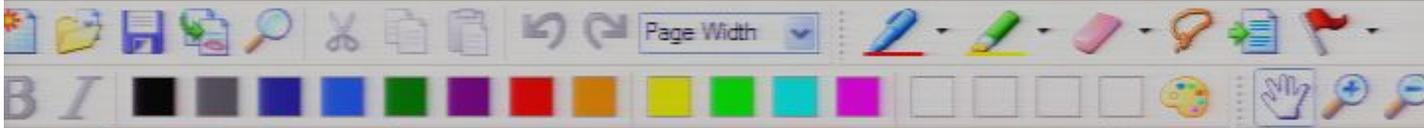
$$\frac{\delta Z[f]}{\delta f(x)} = -2 \partial_x \frac{\delta Z[\partial_x f]}{\delta(\partial_x f(x))} = -2 \partial_x \partial_x f(x)$$

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Indeed:



$$\frac{\delta}{\delta f(x)} Z[f] = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \left[ \int_{\mathbb{R}} \left( \partial_x (f(x') + \varepsilon \delta(x-x')) \right)^2 dx' \right]$$



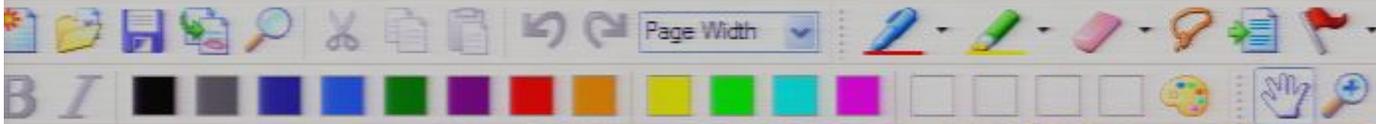
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$$\frac{\delta Z[f]}{\delta f(x)} = -2 \frac{\delta Z[\partial_x f]}{\delta(\partial_x f(x))} = -2 \partial_x \partial_x f(x)$$

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$$\frac{\delta}{\delta f(x)} Z[f] = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left[ \int_{\mathbb{R}} (\partial_x (f(x') + \epsilon \delta(x-x')))^2 dx' - \int_{\mathbb{R}} (\partial_x f(x'))^2 dx' \right]$$

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$$\stackrel{\lim \varepsilon \rightarrow 0}{=} 2 \int_{\mathbb{R}} (\partial_{x'} f(x')) (\partial_{x'} \delta(x-x')) dx'$$

$$\stackrel{\text{int. by parts}}{=} -2 \int_{\mathbb{R}} (\partial_{x'}^2 f(x')) \delta(x-x') dx' + \text{boundary term}$$

$$= -2 \partial_x^2 f(x)$$



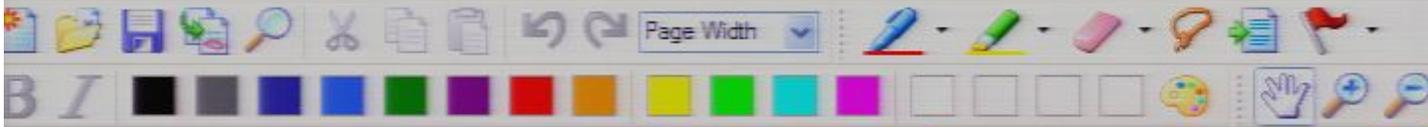
Our lemma claims, therefore:

$$\frac{\delta Z[f]}{\delta f(x)} = -2 \frac{\delta Z[\partial_x f]}{\delta(\partial_x f(x))} = -2 \partial_x \partial_x f(x)$$

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$$\delta f(x) \quad \bar{x} \quad \delta(\partial_x f(x)) \quad \dots$$

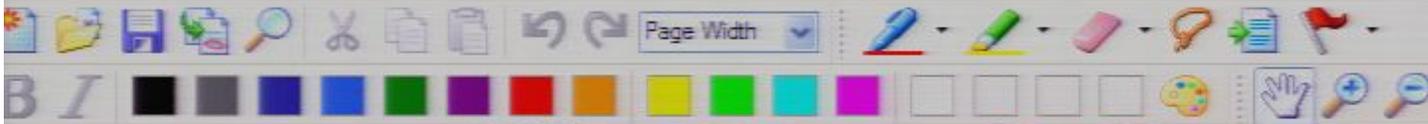
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⊙

$$\stackrel{\lim \epsilon \rightarrow 0}{=} 2 \int_{\mathbb{R}} (\partial_{x'} f(x')) (\partial_{x'} \delta(x-x')) dx'$$



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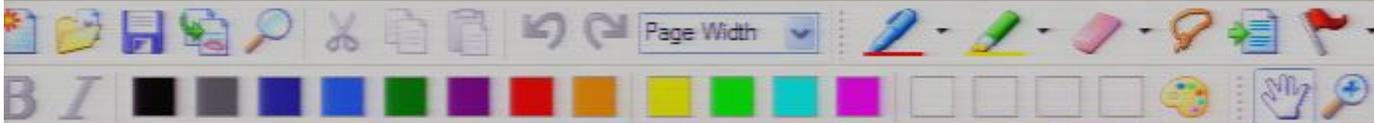
Example:

$$\text{Notation: } \partial_x f(x) = \frac{d}{dx} f(x)$$

$$Z[f] := \int_{\mathbb{R}} (\partial_x f(x))^2 dx$$

□ If we view  $\partial_x f$  as an independent function, then we obtain of course:

$$\frac{\delta Z[\partial_x f]}{\delta(\partial_x f(x))} = 2 \partial_x f(x)$$



Lemma: Consider any functional  $Z$  of the form:

$$Z[f] = \int \text{polynomial} \left( \frac{d}{dx} f \right) dx$$

Then: 
$$\frac{\delta Z}{\delta f(x)} = - \frac{d}{dx} \frac{\delta Z}{\delta \left( \frac{d}{dx} f \right)}$$

On the right hand side we view  $\frac{d}{dx} f$  as an independent function.

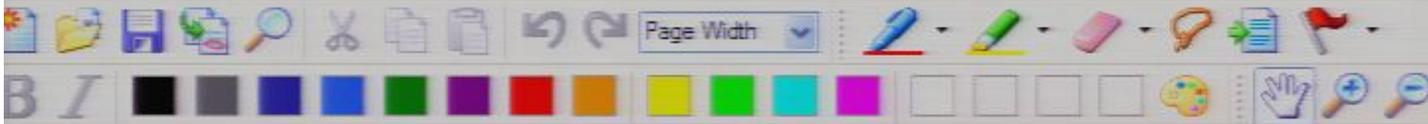
Example:



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↓

$$Z[f] := \int_{\mathbb{R}} (\partial_x f(x))^2 dx$$



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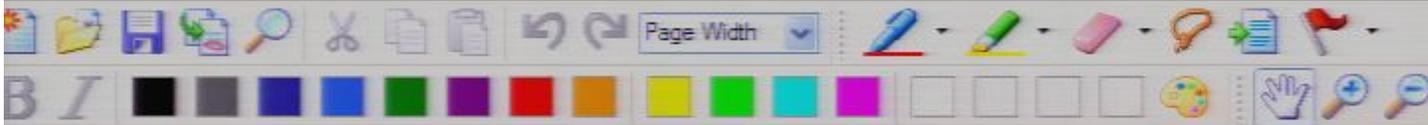
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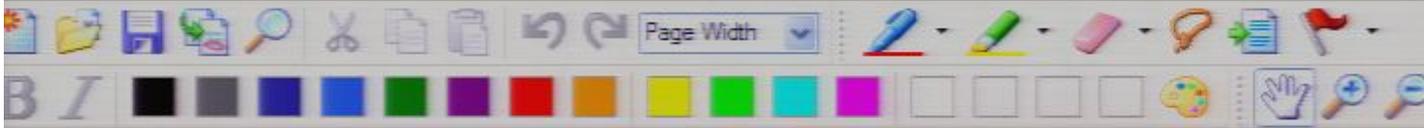
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Recall L2:

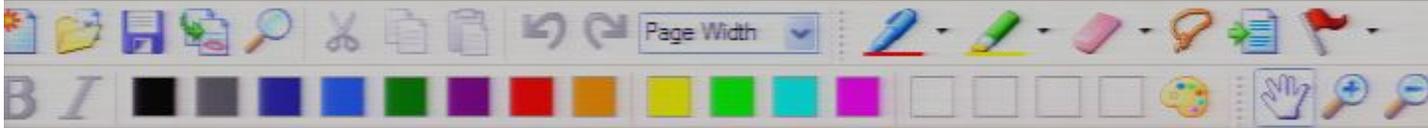
$$\frac{\delta L[\phi, \beta, t]}{\delta \phi(x, t)} = \frac{d}{dt} \frac{\delta L[\phi, \beta, t]}{\delta \dot{\beta}(x, t)}$$

Use lemma:



$$\frac{\delta L[\phi, \beta, t]}{\delta \phi(x, t)} = \frac{\delta L[\phi, \partial_1 \phi, \partial_2 \phi, \partial_3 \phi, \beta, t]}{\delta \phi(x, t)}$$

$$- \sum_{j=1}^3 \frac{\partial}{\partial x^j} \frac{\delta L[\phi, \partial_1 \phi, \partial_2 \phi, \partial_3 \phi, \beta, t]}{\delta (\partial_j \phi(x, t))}$$



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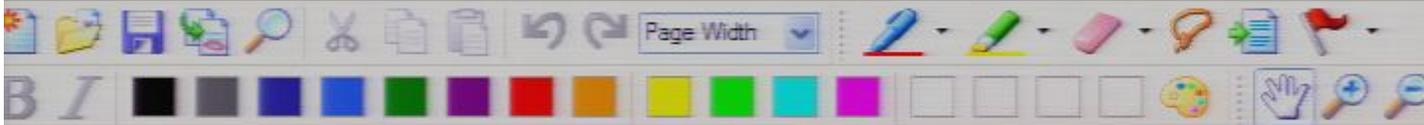
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$\Rightarrow$  L2 takes the form:



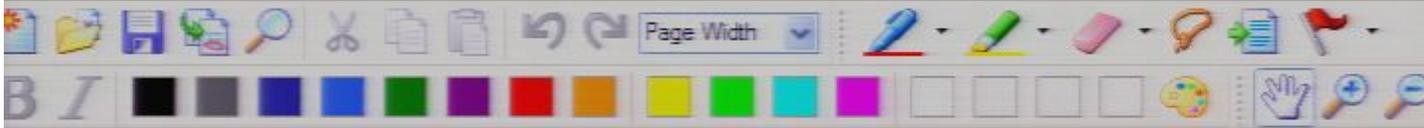
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$\Rightarrow L_2$  takes the form:

$$\frac{\delta L[\phi, \alpha_j \phi, t]}{\delta \phi(x, t)} - \sum_{j=1}^3 \frac{\partial}{\partial x^j} \frac{\delta L[\phi, \alpha_j \phi, t]}{\delta (\alpha_j \phi(x, t))} = \frac{d}{dt} \frac{\delta L[\phi, \alpha_j \phi, t]}{\delta \beta(x, t)}$$



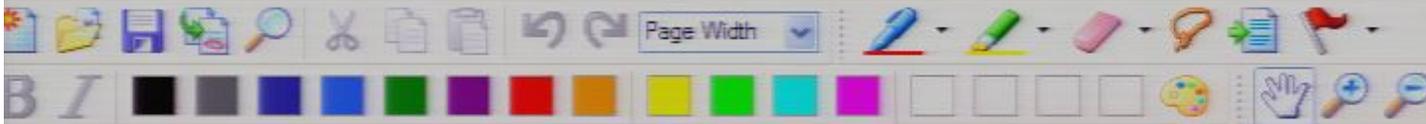
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$$= -2 \partial_x f(x)$$

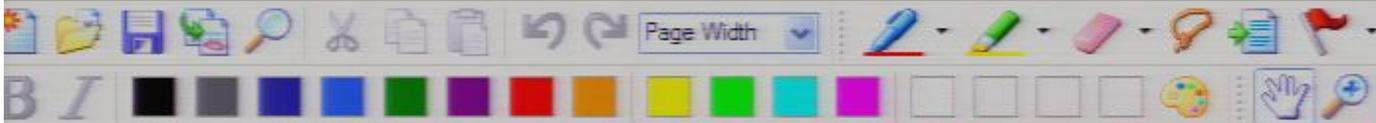
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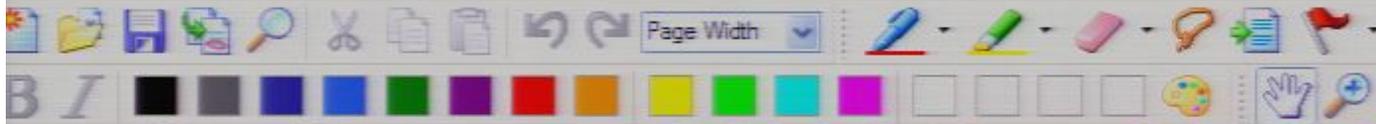
$$\frac{\delta L[\phi, \beta, t]}{\delta \phi(x, t)} = \frac{d}{dt} \frac{\delta L[\phi, \beta, t]}{\delta \dot{\beta}(x, t)}$$

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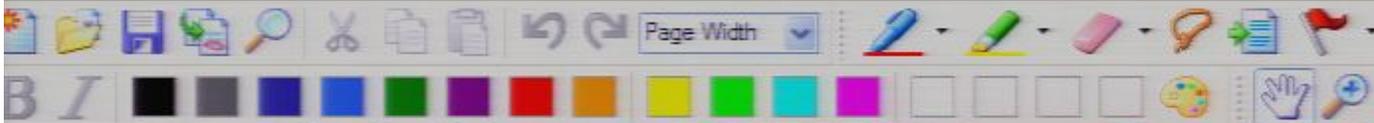
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Recall  $L_2$ :

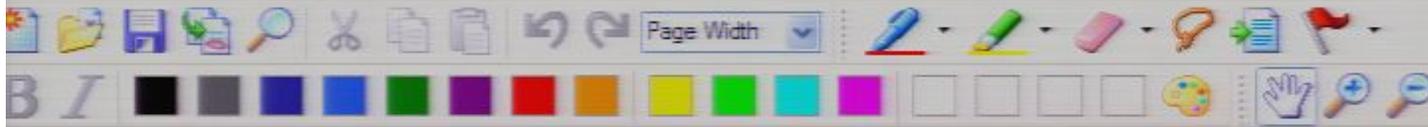
$$\frac{\delta L[\phi, \beta, t]}{\delta \phi(x, t)} = \frac{\partial}{\partial t} \frac{\delta L[\phi, \beta, t]}{\delta \beta(x, t)}$$

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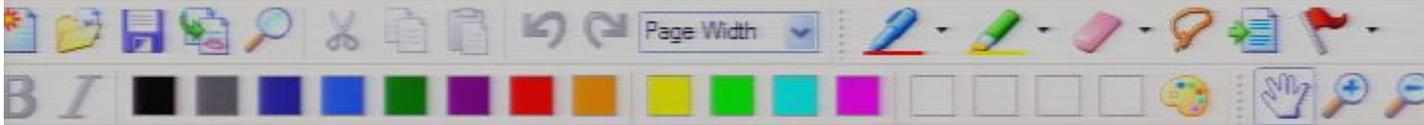
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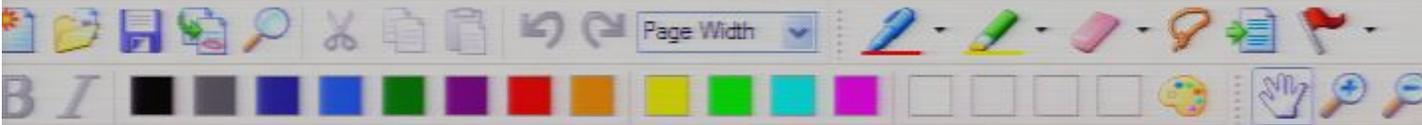
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Recall also  $L_1$ :  $\beta(x, t) = \dot{\phi}(x, t)$



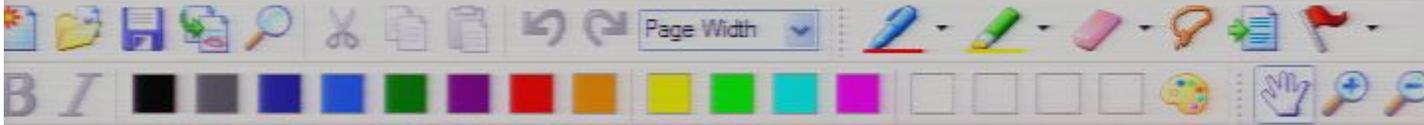
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Recall also  $L_1$ :  $\beta(x, t) = \dot{\phi}(x, t)$



ways to find:

$$\frac{\delta L[\phi, \partial; \phi, t]}{\delta \phi(x, t)} = \sum_{j=1}^3 \frac{\partial}{\partial x^j} \frac{\delta L[\phi, \partial; \phi, t]}{\delta (\partial_j \phi(x, t))} = \frac{d}{dt} \frac{\delta L[\phi, \partial; \phi, t]}{\delta \beta(x, t)}$$

Recall also L1:  $\beta(x, t) = \dot{\phi}(x, t)$

$\Rightarrow$  One often writes, formally:

$$\frac{\delta L[\phi, \partial; \phi, t]}{\delta \phi(x, t)} = \sum_{\mu=0}^3 \frac{\partial}{\partial x^\mu} \frac{\delta L[\phi, \partial; \phi, t]}{\delta (\partial_\mu \phi(x, t))} \quad \left( \partial_0 = \frac{d}{dt} \right)$$

with the understanding  
that  $\frac{\delta L}{\delta \partial_0 \phi} := \frac{\delta L}{\delta \dot{\phi}} \Big|_{\beta = \dot{\phi}}$

However:

Here, we must remember that here the true variational



$$\frac{\delta L[\phi, \partial; \phi, t]}{\delta \phi(x, t)} - \sum_{j=1}^3 \frac{\partial}{\partial x^j} \frac{\delta L[\phi, \partial; \phi, t]}{\delta (\partial_j \phi(x, t))} = \frac{d}{dt} \frac{\delta L[\phi, \partial; \phi, t]}{\delta \beta(x, t)}$$

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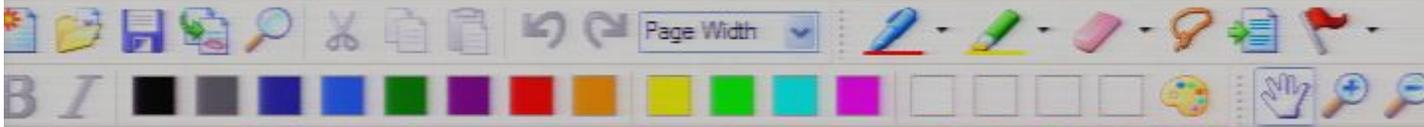
⇒ One often writes, formally:

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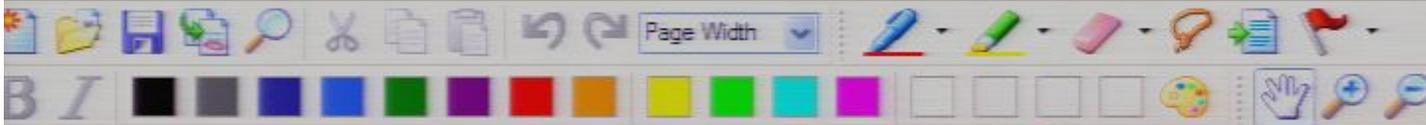
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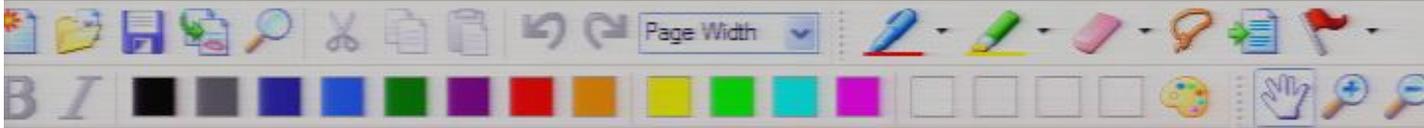
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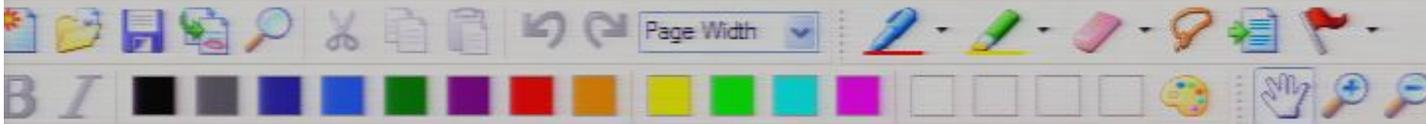
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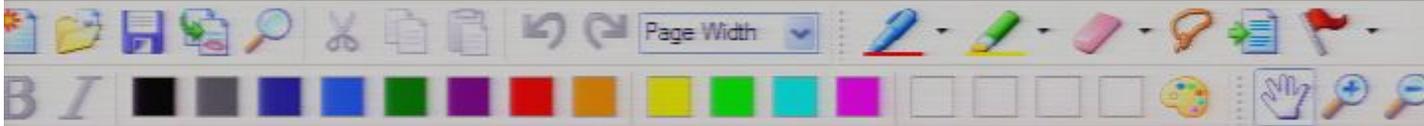
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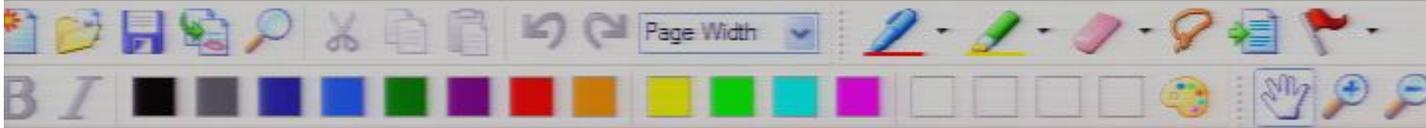
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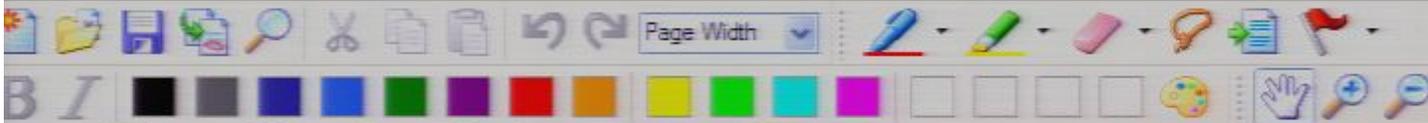
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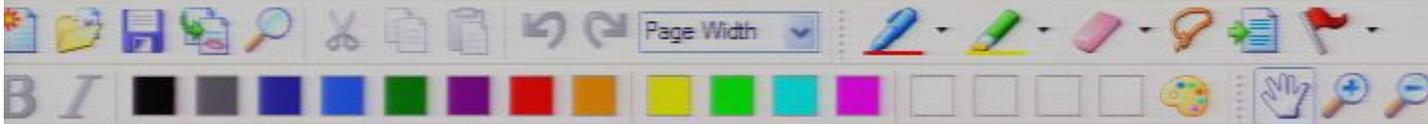
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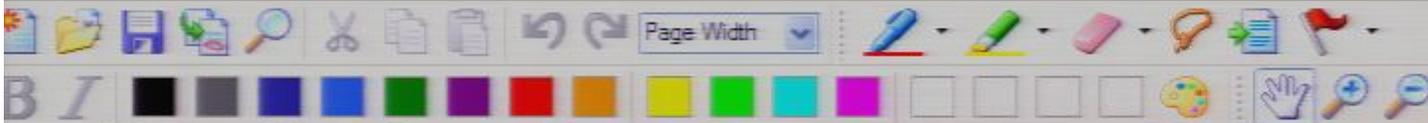


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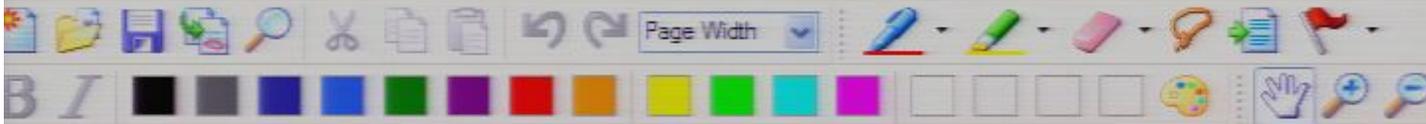
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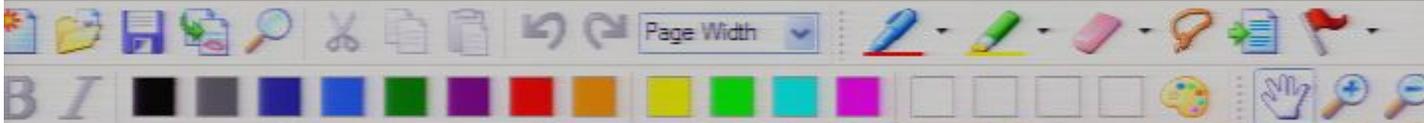
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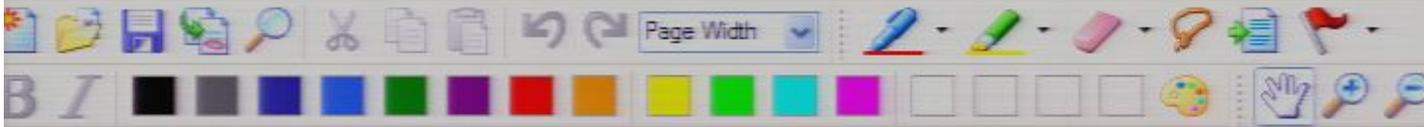
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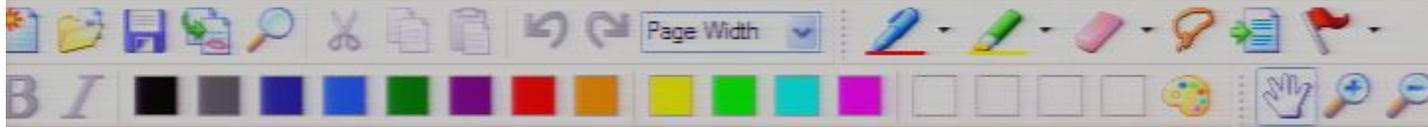
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□ Definition:  $S[\phi] := \int_{\mathbb{R}} L[\phi, \dot{\phi}] dt$

$S[\phi]$  is called the "action of the field evolution  $\phi(x, t)$ "

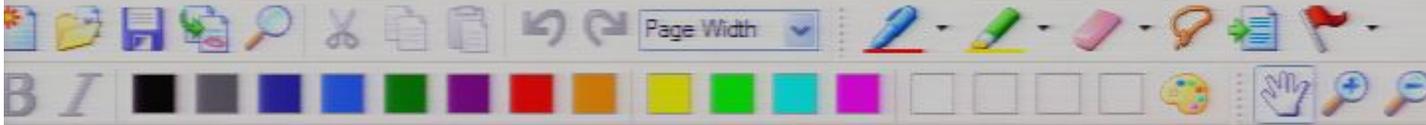
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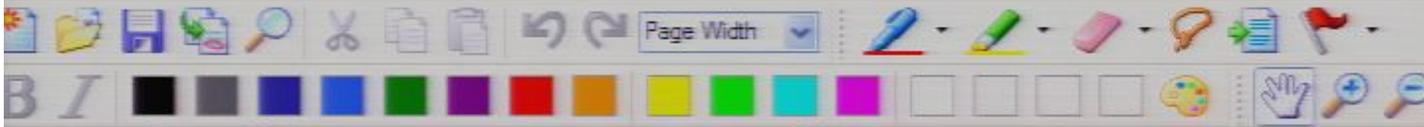
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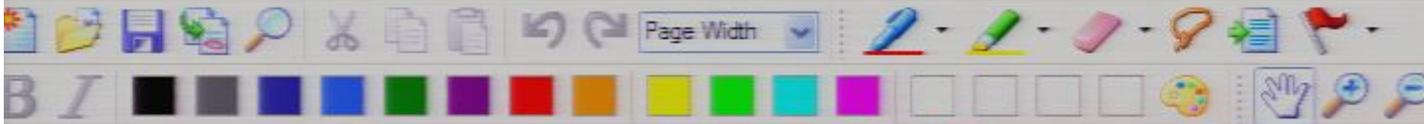
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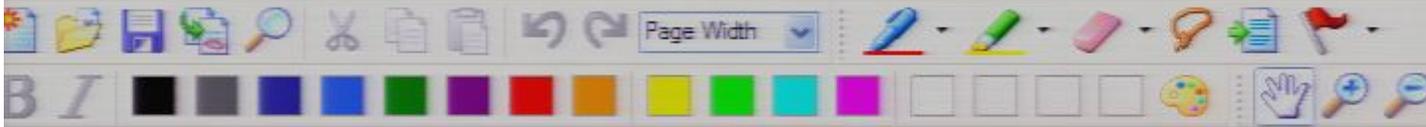
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- \* The action's integrand is called the "Lagrange density"  $\mathcal{L}(x, t)$ :

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\* Notation often used in General Relativity:

a.  $\phi_{, \mu}(x, t) := \frac{\partial}{\partial x^\mu} \phi(x, t)$



$$\begin{array}{ccc} \delta \phi(x, t) & \mu=0 & \delta(\sigma, \mu, \psi) \\ \delta \phi(x, t) & \mu=0 & \delta(\sigma, \mu, \psi) \end{array}$$

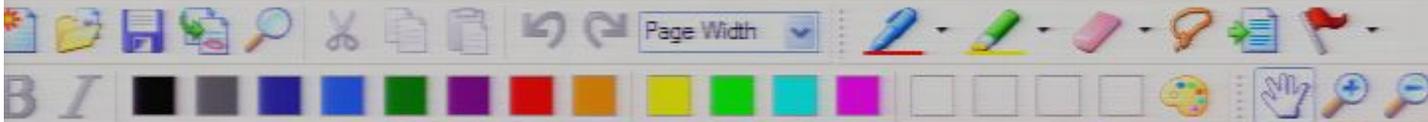
or equivalently:

$$\frac{\delta S[\phi]}{\delta \phi(x, t)} = 0$$

"The action principle"

Notice that the action principle, spelled out, reads:

$$0 = \frac{\delta S[\phi]}{\delta \phi(x, t)} = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left( S[\{\phi(x') + \epsilon \delta^4(x-x')\}_{x' \in \mathbb{R}^4}] - S[\{\phi(x')\}_{x' \in \mathbb{R}^4}] - S[\{\phi(x')\}_{x' \in \mathbb{R}^4}] \right)$$



$S[\phi]$  is called the action of the field evolution  $\phi(x, t)$

Then, the "Euler Lagrange field equations" are

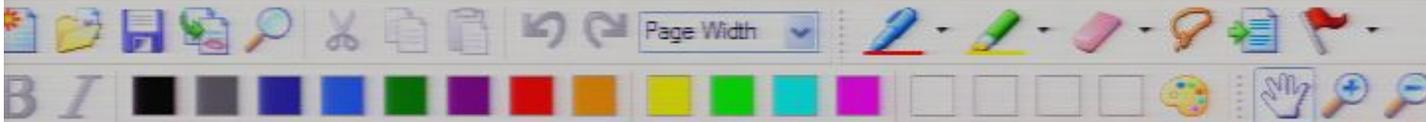
$$\frac{\delta S[\phi, \partial_\mu \phi]}{\delta \phi(x, t)} - \sum_{\mu=0}^3 \frac{\partial}{\partial x^\mu} \frac{\delta S[\phi, \partial_\mu \phi]}{\delta (\partial_\mu \phi)} = 0$$

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Notice that the action principle, spelled out, reads:



$$-S[\phi(x)]_{x \in \mathbb{R}^4}$$

Example:

□ The Klein Gordon action:

$$S[\phi] := \frac{1}{2} \int_{\mathbb{R}^4} (\partial_0 \phi)^2 - \sum_{i=1}^3 (\partial_i \phi)^2 - m^2 \phi^2 d^4 x$$

□ Using either the action principle or directly the Euler Lagrange field equations, one obtains indeed the Klein Gordon equation (Exercise: verify):

## ▣ Definitions:

- \* The action's integrand is called the "Lagrange density"  $\mathcal{L}(x, \phi)$ :

$$S[\phi] = \int_{\mathbb{R}^4} \mathcal{L}(x, \phi) d^4x$$

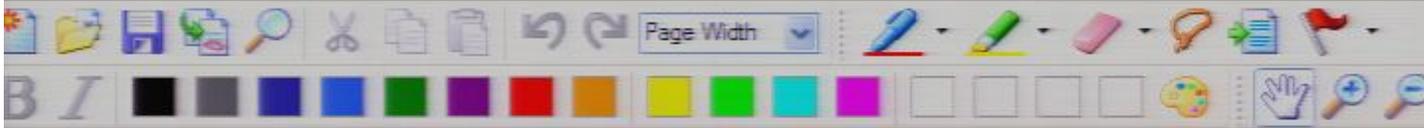
- \* One often formally writes:

$$\frac{\partial \mathcal{L}}{\partial \phi} - \sum_{\mu=0}^3 \frac{\partial}{\partial x^\mu} \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} = 0 \quad (\mathcal{L})$$

- \* Notation often used in General Relativity:

a.  $\phi_{, \mu}(x, t) := \frac{\partial}{\partial x^\mu} \phi(x, t)$

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E.g., equation (2) can be written as:

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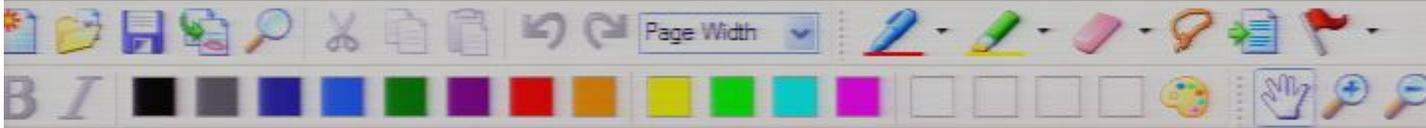
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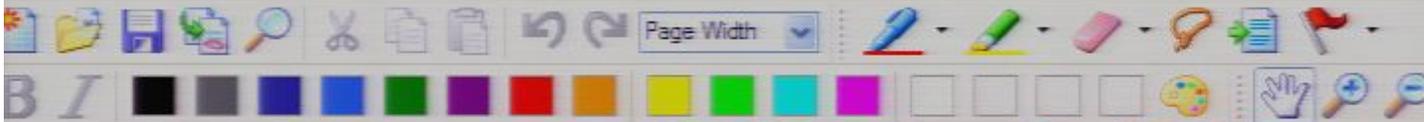
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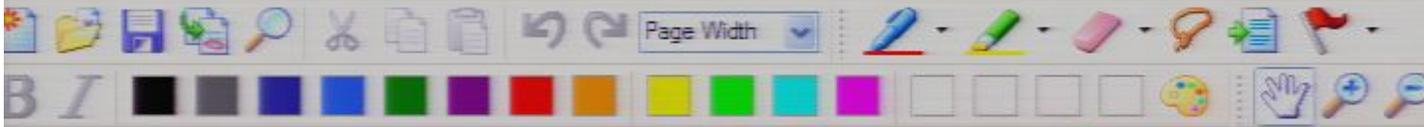
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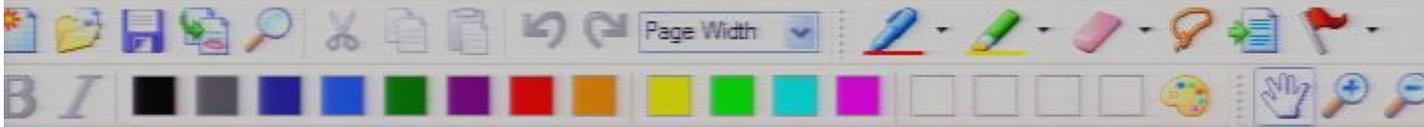
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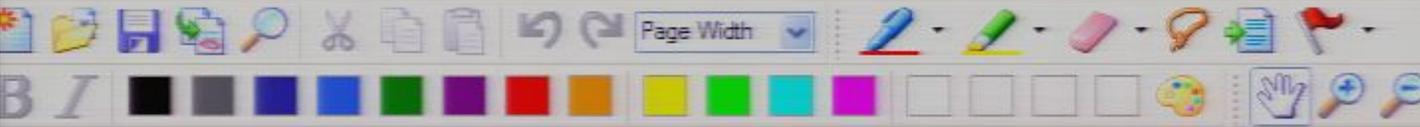
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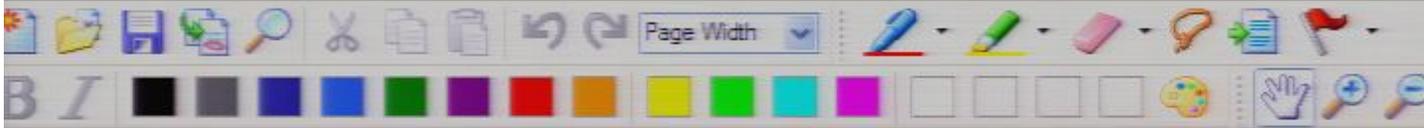
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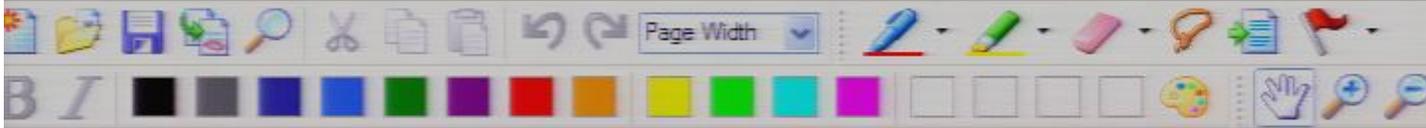
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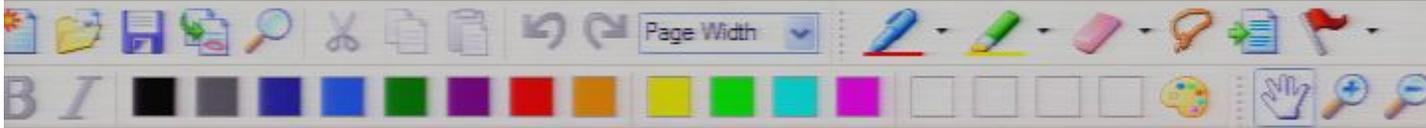
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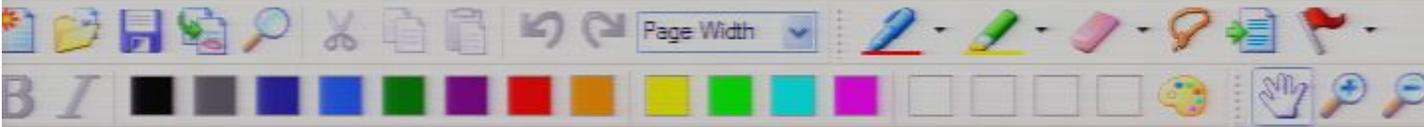
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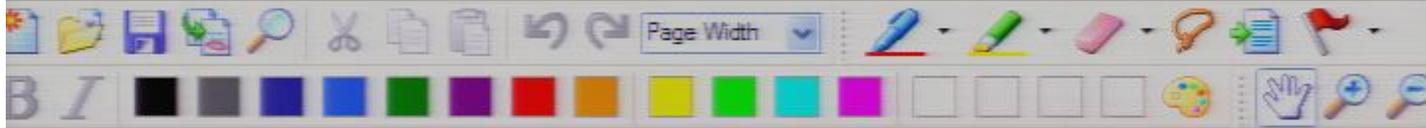
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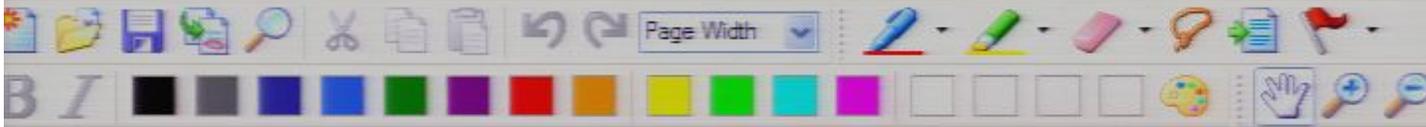


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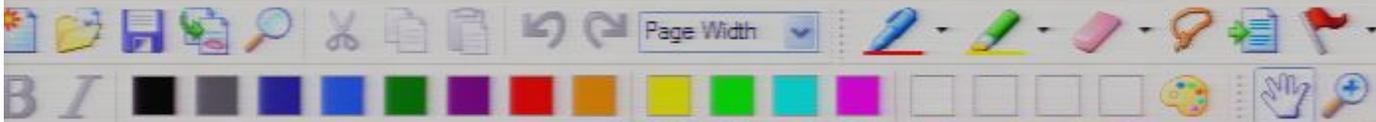
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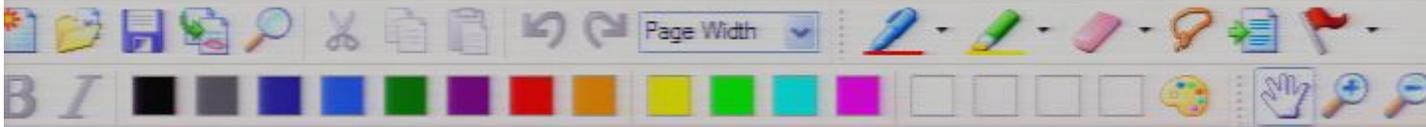
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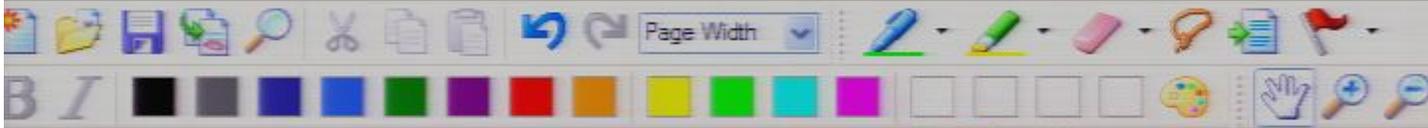
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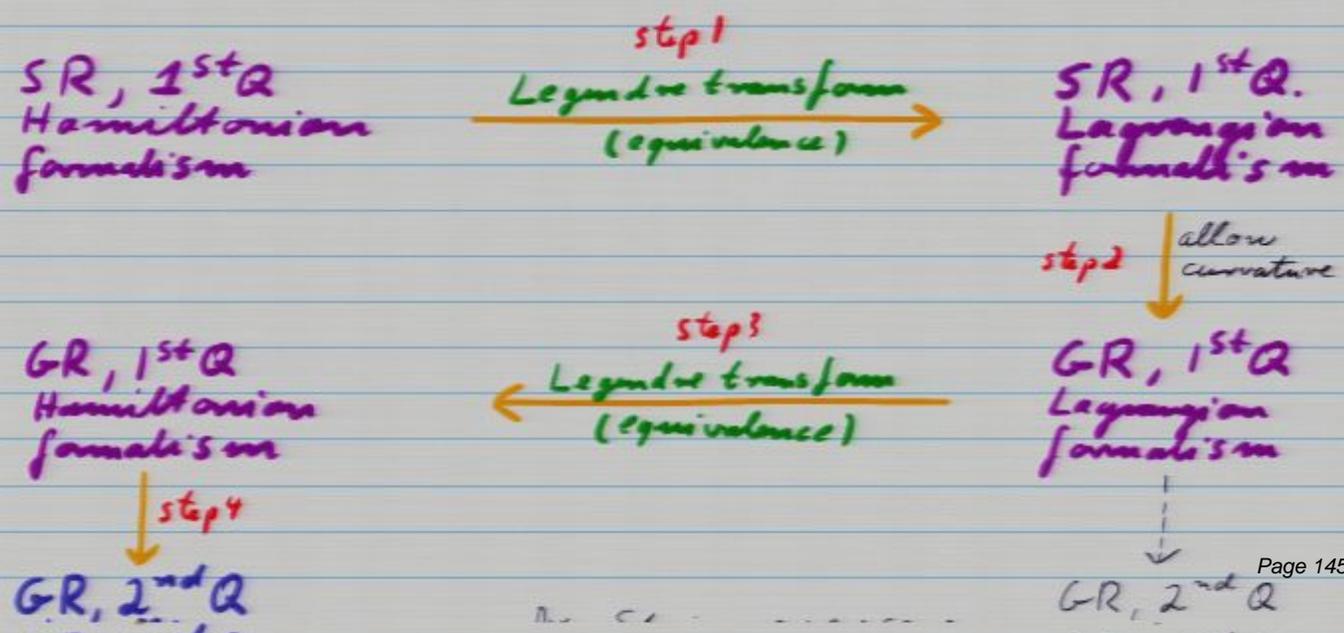
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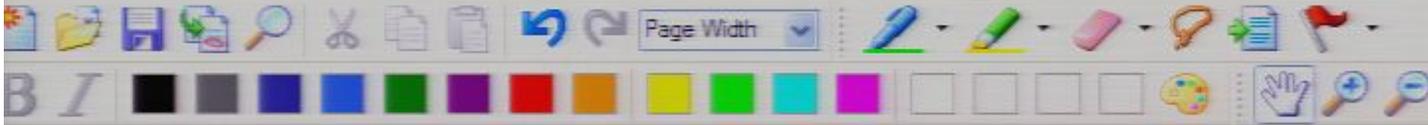
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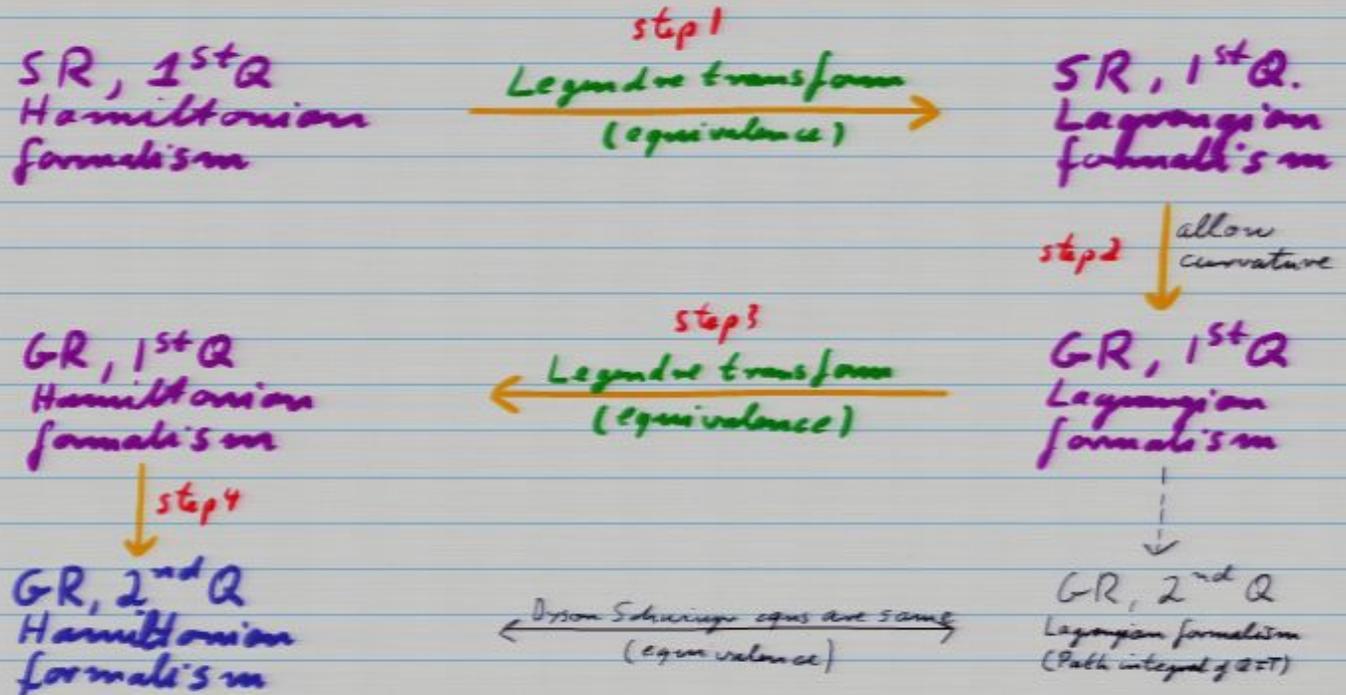
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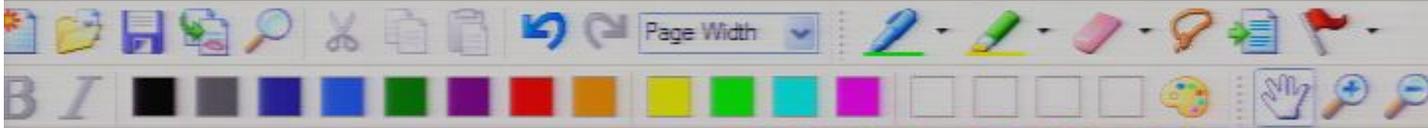
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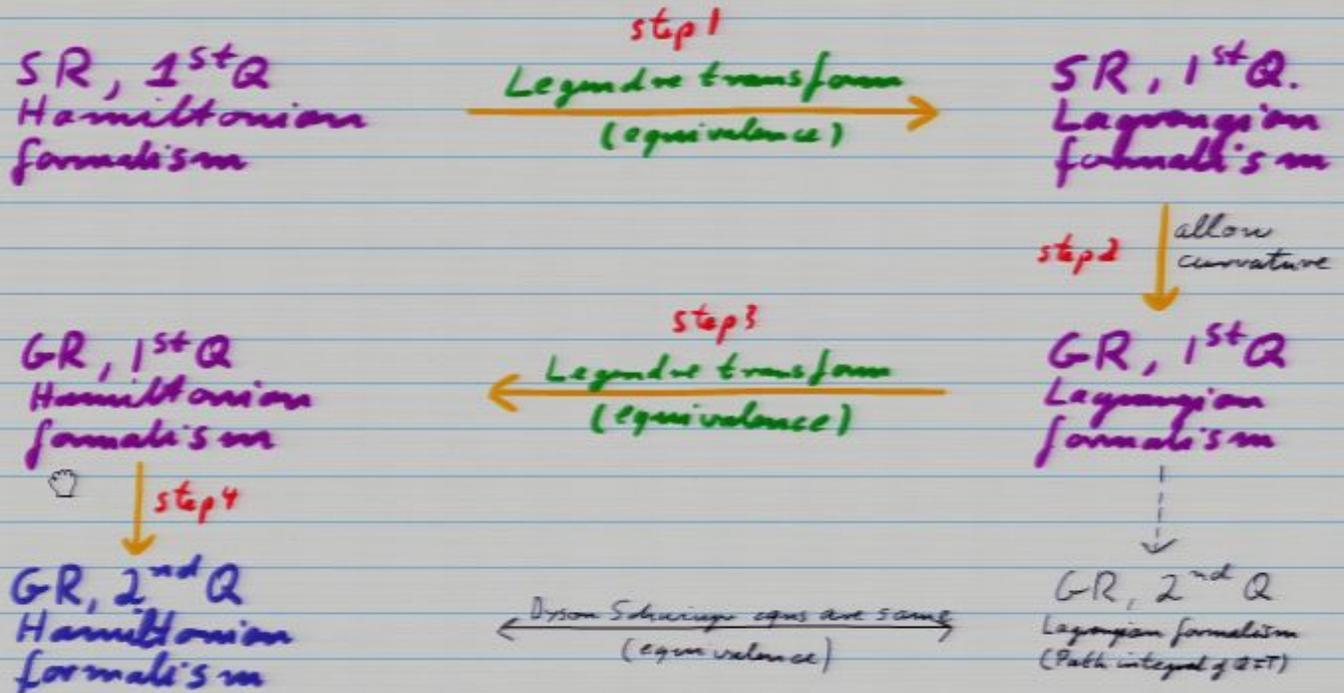
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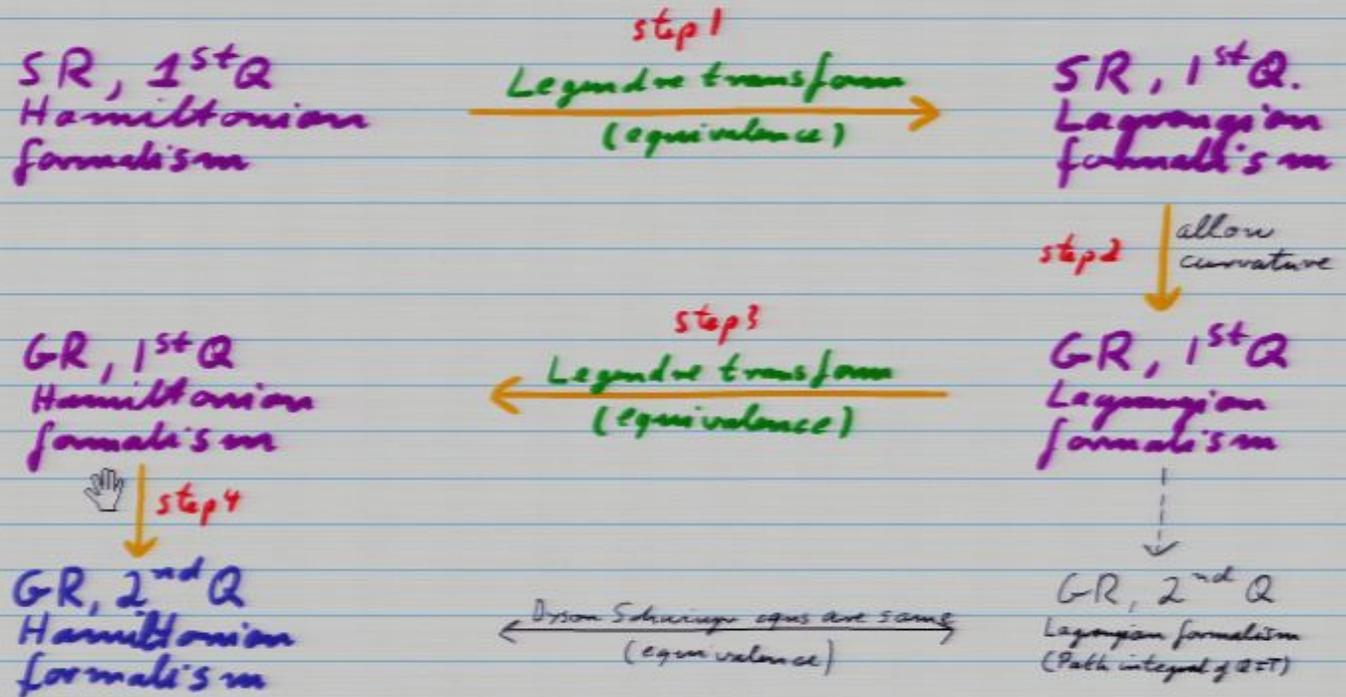


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Now that we have a beautifully covariant Lagrangian formulation:

Step 2: How to allow for curvature of space-time?

Strategy:

- A. Within special relativity, allow not just inertial rectangular coordinate systems but allow arbitrary coordinate systems.
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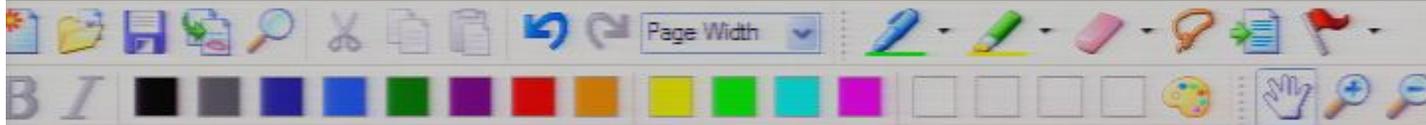
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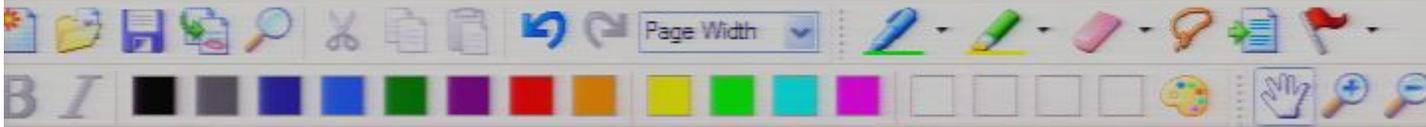


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↑ the inverse matrix to  $g_{\mu\nu}$ . In special relativity, both are the same:  $\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$

□ The E.L. eqns read

$$\frac{\delta S[\phi, \{\phi_{,\mu}\}]}{\delta \phi(x,t)} = \partial_\mu \frac{\delta S[\phi, \{\phi_{,\mu}\}]}{\delta (\phi_{,\mu}(x,t))}$$

and yield

$$-m^2 \phi = \partial_\mu g^{\mu\nu} \phi_{,\nu}$$

i.e., of course:  $(\square + m^2)\phi = 0$

We have now completed Step 1:

Using these definitions, the K.G. action now reads:

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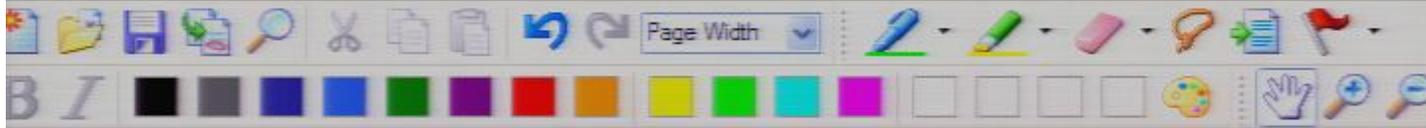


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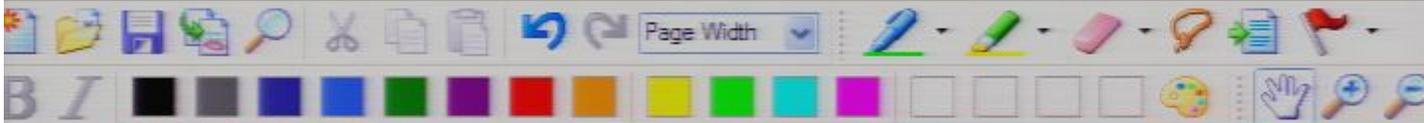


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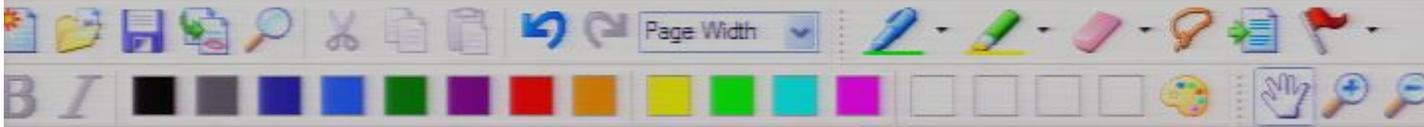
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A. Arbitrary coordinate systems



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## A. Arbitrary coordinate systems

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$$S[\phi] = \frac{1}{2} \int_{\mathbb{R}^4} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - m^2 \phi^2 d^4x$$

□ If we change to arbitrary coordinates

$$x^\mu \rightarrow \tilde{x}^\mu = \tilde{x}^\mu(x)$$

then:  $\phi(x) \rightarrow \tilde{\phi}(\tilde{x}) = \phi(x(\tilde{x}))$

(recall that  $\sum_{\nu=0}^3$  is implied)

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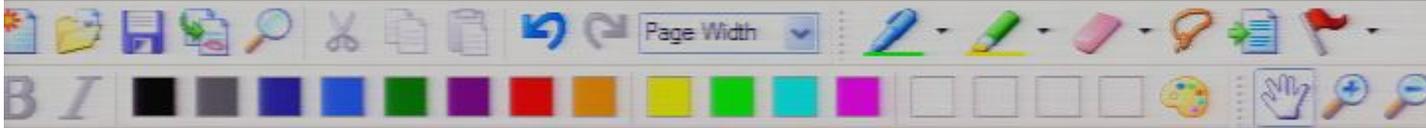
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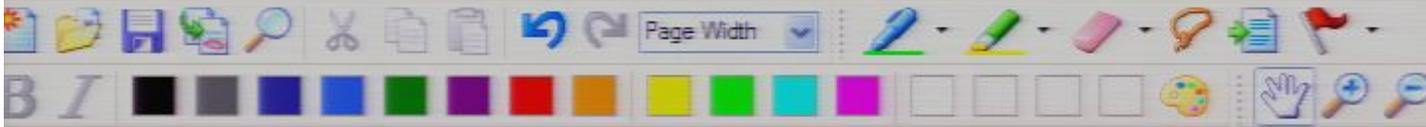
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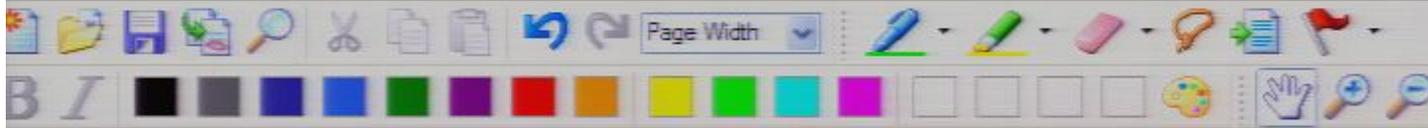
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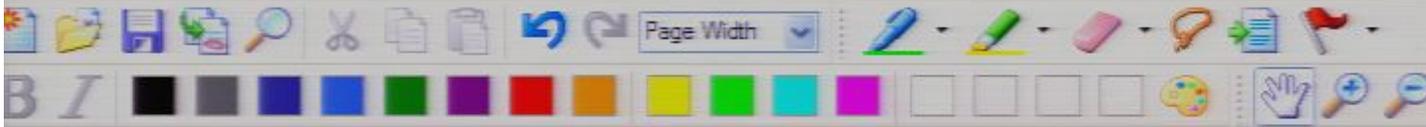
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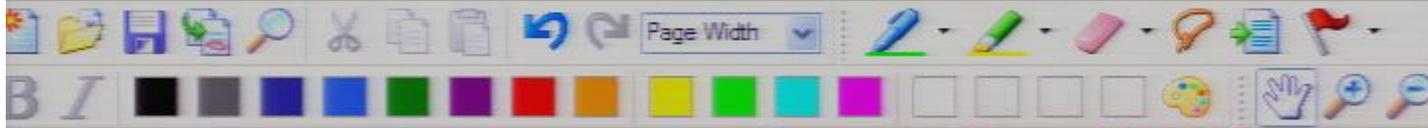
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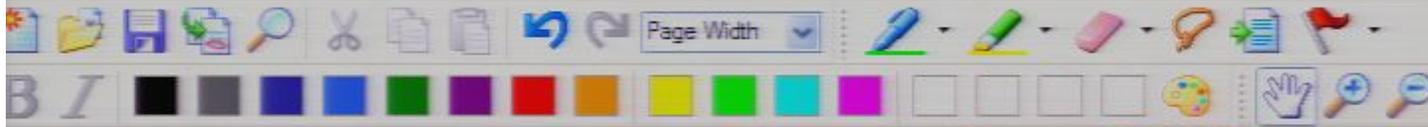
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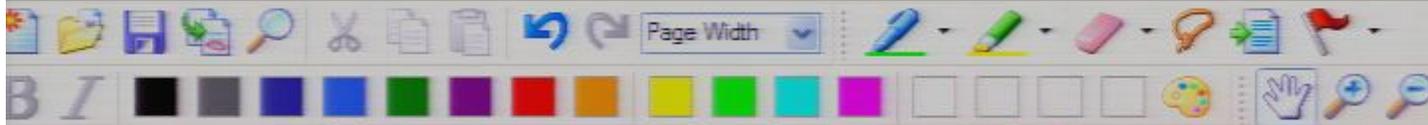
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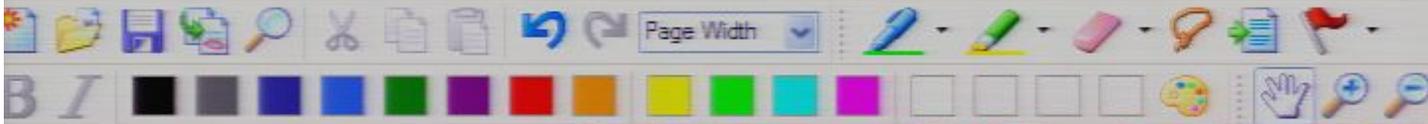
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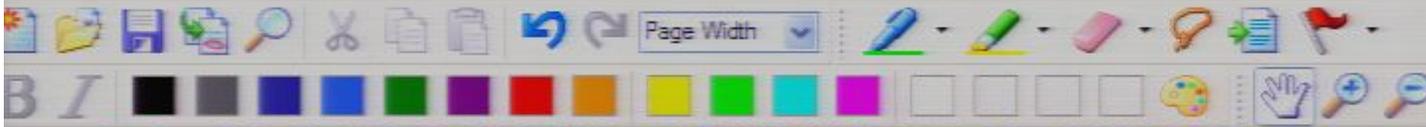
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$$\begin{aligned} g^{\mu\nu}(x) \left( \frac{\partial}{\partial x^\mu} \phi(x) \right) \left( \frac{\partial}{\partial x^\nu} \phi(x) \right) &\rightarrow \tilde{g}^{\mu\nu}(\tilde{x}) \left( \frac{\partial}{\partial \tilde{x}^\mu} \phi(\tilde{x}) \right) \left( \frac{\partial}{\partial \tilde{x}^\nu} \phi(\tilde{x}) \right) \\ &= g^{\alpha\beta}(x) \frac{\partial \tilde{x}^\mu}{\partial x^\alpha} \frac{\partial \tilde{x}^\nu}{\partial x^\beta} \frac{\partial x^\alpha}{\partial \tilde{x}^\mu} \frac{\partial x^\beta}{\partial \tilde{x}^\nu} \left( \frac{\partial}{\partial x^\alpha} \phi(x) \right) \left( \frac{\partial}{\partial x^\beta} \phi(x) \right) \end{aligned}$$



## 1.1 Arbitrary coordinate systems

□ Reconsider the K.G. action:

$$S[\phi] = \frac{1}{2} \int_{\mathbb{R}^4} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - m^2 \phi^2 d^4x$$

□ If we change to arbitrary coordinates

$$x^\mu \rightarrow \tilde{x}^\mu = \tilde{x}^\mu(x)$$

then:  $\phi(x) \rightarrow \tilde{\phi}(\tilde{x}) = \phi(x(\tilde{x}))$

(recall that  $\sum_{\nu=0}^3$  is implied)

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$$\frac{\partial x^a}{\partial \tilde{x}^b} \frac{\partial \tilde{x}^b}{\partial x^c} = \frac{\partial x^a}{\partial x^c} = \delta_{ac}$$

$$\begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix}$$



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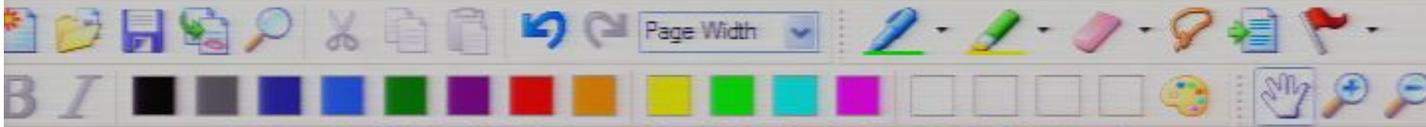
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Therefore, if we transform

$$g^{\mu\nu}(x) \rightarrow \tilde{g}^{\mu\nu}(\tilde{x}) = \frac{\partial \tilde{x}^\mu}{\partial x^\alpha} \frac{\partial \tilde{x}^\nu}{\partial x^\beta} g^{\alpha\beta}(x(\tilde{x}))$$

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## Terminology:

\* We say that we let  $g^{\mu\nu}(x)$  transform as a contravariant tensor of rank 2.

$\uparrow$  because upper indices                       $\uparrow$  because 2 indices

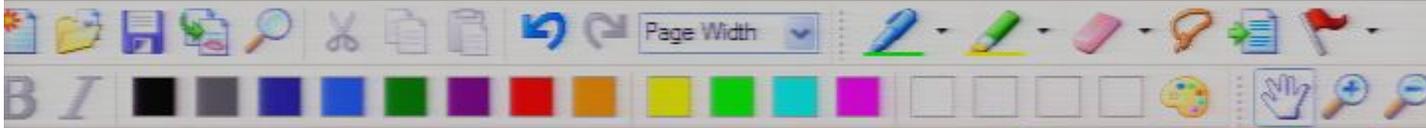
\* Since  $g^{\mu\nu}(x) g_{\nu\sigma}(x) = \delta^{\mu}_{\sigma}$  we have

$$g_{\mu\nu}(x) \rightarrow \tilde{g}_{\mu\nu}(\tilde{x}) = \frac{\partial x^{\alpha}}{\partial \tilde{x}^{\mu}} \frac{\partial x^{\beta}}{\partial \tilde{x}^{\nu}} g_{\alpha\beta}(x(\tilde{x}))$$

which is called a covariant rank 2 tensor.

Is  $S[\phi]$  now coordinate system independent?

No, not yet!



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How can we modify the action  $S'[\phi]$  so that:

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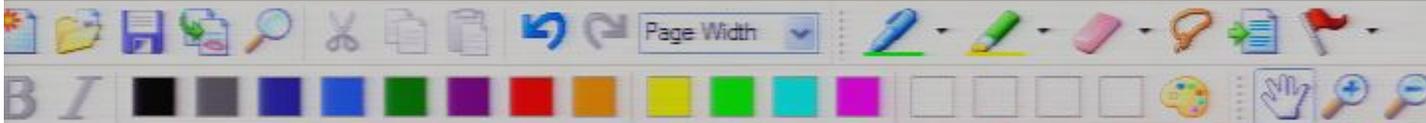
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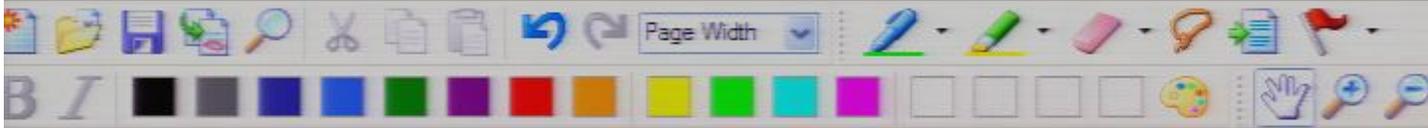
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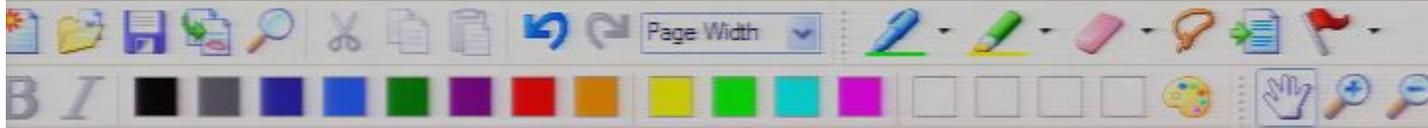
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Modify the action to include a "Volume factor":

$$S[\phi] := \frac{1}{2} \int_{\mathbb{R}^4} \left( g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - m^2 \phi^2 \right) \sqrt{-\det(g_{\mu\nu})} d^4x$$

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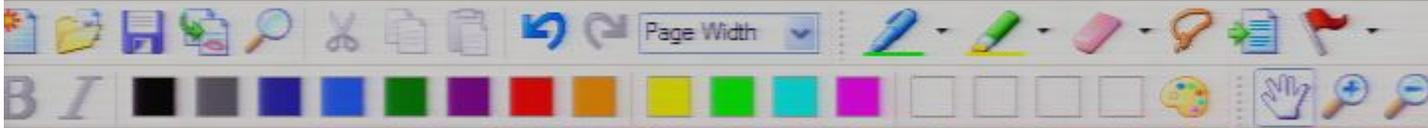
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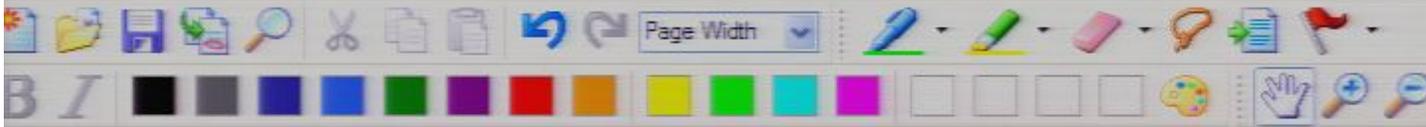
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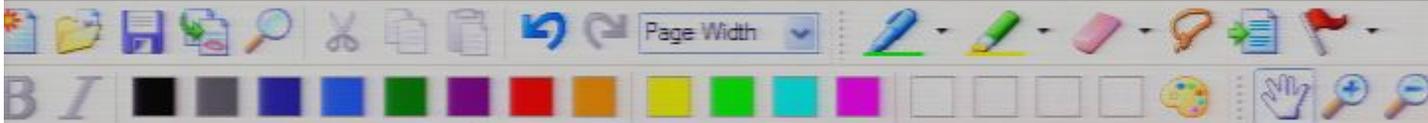
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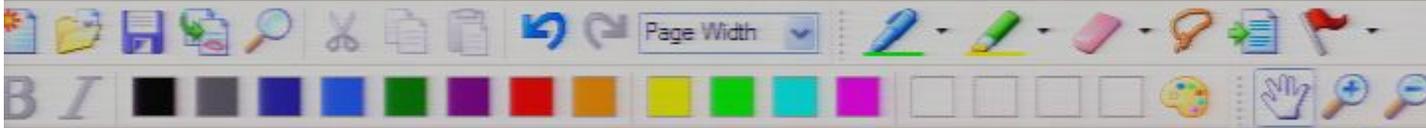
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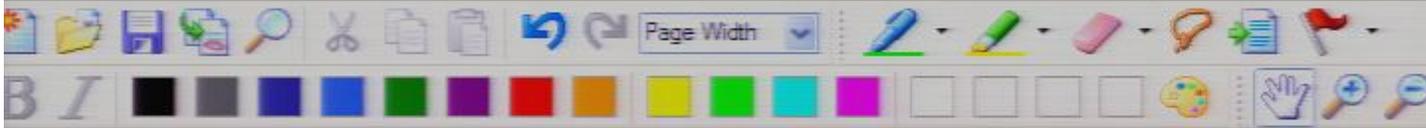
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\* The trivial metric  $g_{\mu\nu}(x) = \eta_{\mu\nu} = \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix}$

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