Title: Quantum Field Theory for Cosmology - Lecture 9

Date: Feb 09, 2010 04:00 PM

URL: http://pirsa.org/10020015

Abstract: This course begins with a thorough introduction to quantum field theory. Unlike the usual quantum field theory courses which aim at applications to particle physics, this course then focuses on those quantum field theoretic techniques that are important in the presence of gravity. In particular, this course introduces the properties of quantum fluctuations of fields and how they are affected by curvature and by gravitational horizons. We will cover the highly successful inflationary explanation of the fluctuation spectrum of the cosmic microwave background - and therefore the modern understanding of the quantum origin of all inhomogeneities in the universe (see these amazing visualizations from the data of the Sloan Digital Sky Survey. They display the inhomogeneous distribution of galaxies several billion light years into the universe: Sloan Digital Sky Survey).

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QFT for Cosmology, Achim Kempf, Winter 2010, Lecture 9

Mathematical preparations for QFT in curved space:

Plan today:

- Functional derivatives $\frac{\delta F[g]}{\delta g(x)} = ?$
- Example use 1: to make the QFT Schrödinger equation well defined.
- Example use 2: to define the Functional Legendre transform.
- Use both to obtain the Lagrangian formulation of QFT

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Recall:

a) Differentiation of functions of one variable $F(x)$:

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a.) Differentiation of functions of one variable, $F(u)$:

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Recall:

a.) Differentiation of functions of one variable, $F(u)$:

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b.) Differentiation of functions of countably many variables, $F(\{u_j\}_{j=1,2,3,\ldots})$:

$$\frac{dF(\{u_j\}_{j=1,2,\ldots})}{du_i} := \lim_{\varepsilon \to 0} \frac{F(\{u_j + \varepsilon\delta_{ij}\}_{j=1\ldots}) - F(\{u_j\}_{j=1,\ldots})}{\varepsilon}$$

$$= \lim_{\varepsilon \to 0} \frac{F(u_1, \ldots, u_i + \varepsilon, \ldots) - F(u_1, \ldots, u_i, \ldots)}{\varepsilon}$$

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Definition:

c.) Differentiation of functions of uncountably many variables, $F(\{u(x)\}_{x\in\mathbb{R}^n})$:

Note: Since the Dirac delta is not a function but a distribution, which is only defined relative to an integral, the full definition is more technical.

$$\frac{\delta F(\{u(x)\}_{x\in\mathbb{R}^n})}{\dots} := \lim \frac{F(\{u(x)+\varepsilon\delta^n(x-y)\}_{x\in\mathbb{R}^n}) - F(\{u(x)\}_{x\in\mathbb{R}^n})}{\dots}$$

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→ Since F is a "functional", i.e. is mapping functions to numbers

$$F:\ \underset{\substack{\uparrow\\ \text{function}}}{u} \to \underset{\substack{\uparrow\\ \text{short for } \{u(x)\}_{x\in\mathbb{R}^n}}}{F[u]} \in \mathbb{C}$$

we call $\frac{\delta F}{\delta u(x)}$ a **functional derivative.**

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we call $\frac{\delta F}{\delta u(x)}$ a functional derivative.

Example:

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Example:

$$F[u] := \int_{\mathbb{R}} \cos(x)\, u(x)^2\, dx$$

Then:

$$\frac{\delta F}{\delta u(y)} = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\left[\int_{\mathbb{R}} \cos(x)\left(u(x)+\varepsilon\delta(x-y)\right)^2 - \int_{\mathbb{R}} \cos(x)\, u(x)^2\, dx\right]$$

Distribution theory would be needed. But it drops out anyway

$$= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_{\mathbb{R}} \cos(x)\left(\cancel{u(x)^2} + \varepsilon 2 u(x)\delta(x-y) + \varepsilon^2 \delta^2(x-y) - \cancel{u(x)^2}\right) dx$$

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Similarly, one obtains: $\frac{\delta}{\delta u(y)} \int_{\mathbb{R}} f(x)\, u(x)^n\, dx = f(y)\, n\, u(y)^{n-1}$

$\Rightarrow$ Functional derivatives act on polynomials (and suitable power series) in u by removing the integral and reducing the power in u by one, as expected from ordinary derivatives.

Remark: * Worked with $u(x)$.

* Would obtain same result if we used any other continuous or discrete basis of L^2.

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Note: How can $L^2(\mathbb{R})$ have countable basis? Recall: $L^2(\mathbb{R})$ consists not of functions, but of equivalence classes of functions

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Schrödinger equation of QFT now well defined:

QM: $\quad \hat{q}_i \quad \hat{p}_i \quad i \quad t$

QFT: $\quad \hat{\phi}(x) \quad \hat{\pi}(x) \quad x \quad t$

Similarly, one obtains: $$\frac{\delta}{\delta u(y)} \int_{\mathbb{R}} f(x)\, u(x)^n \, dx = f(y)\, n\, u(y)^{n-1}$$

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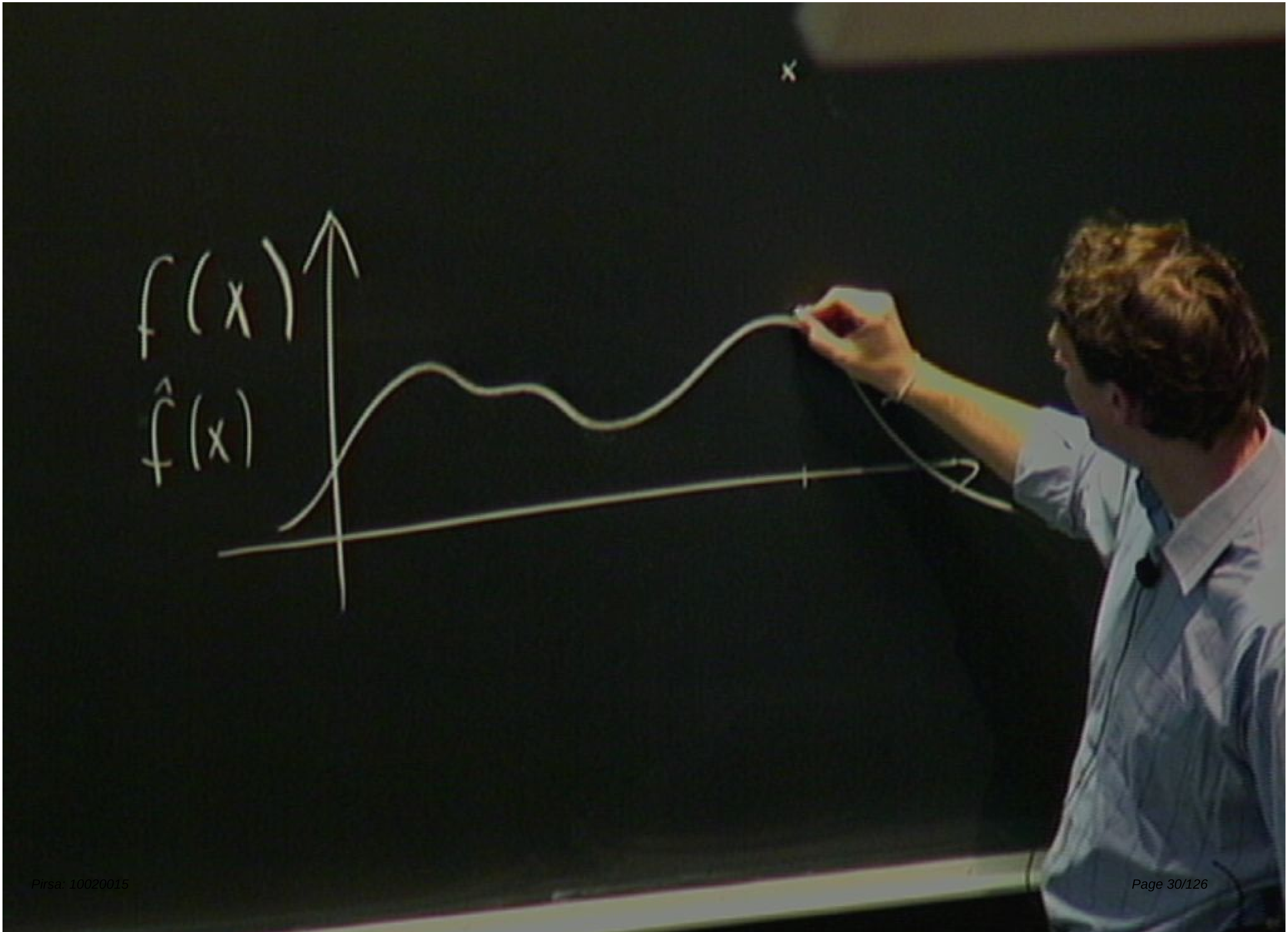
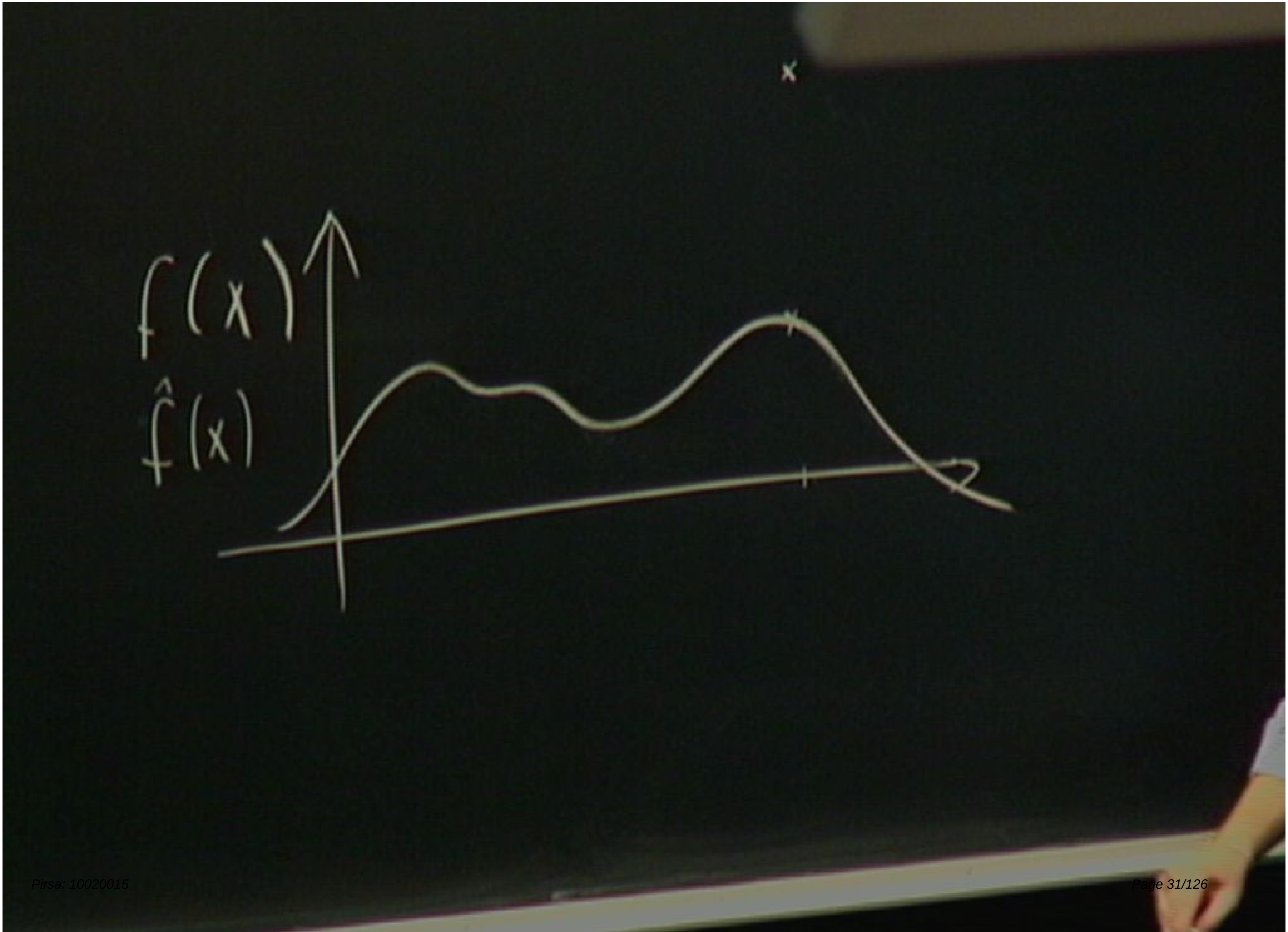

x
f(x)
f̂(x)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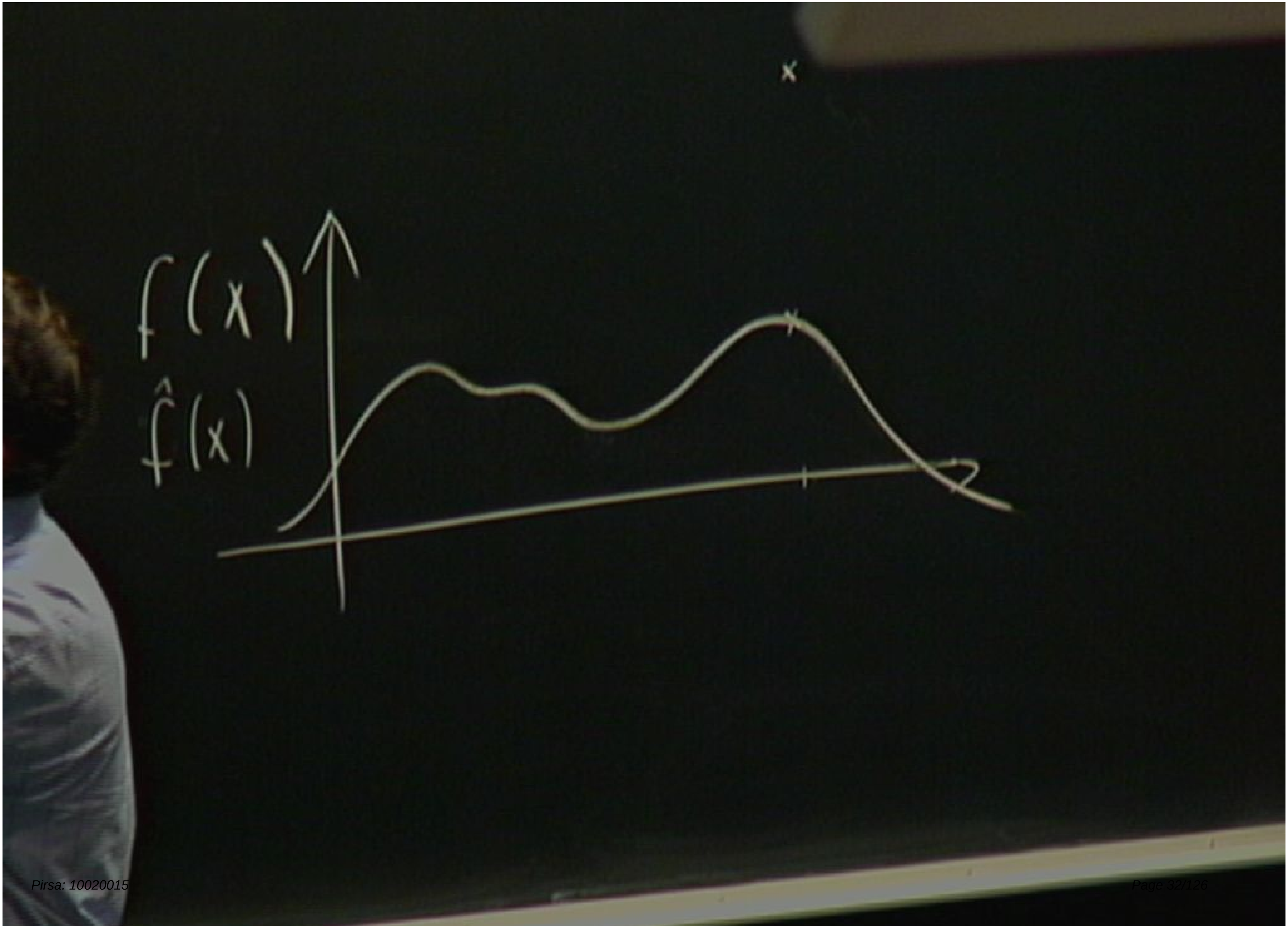
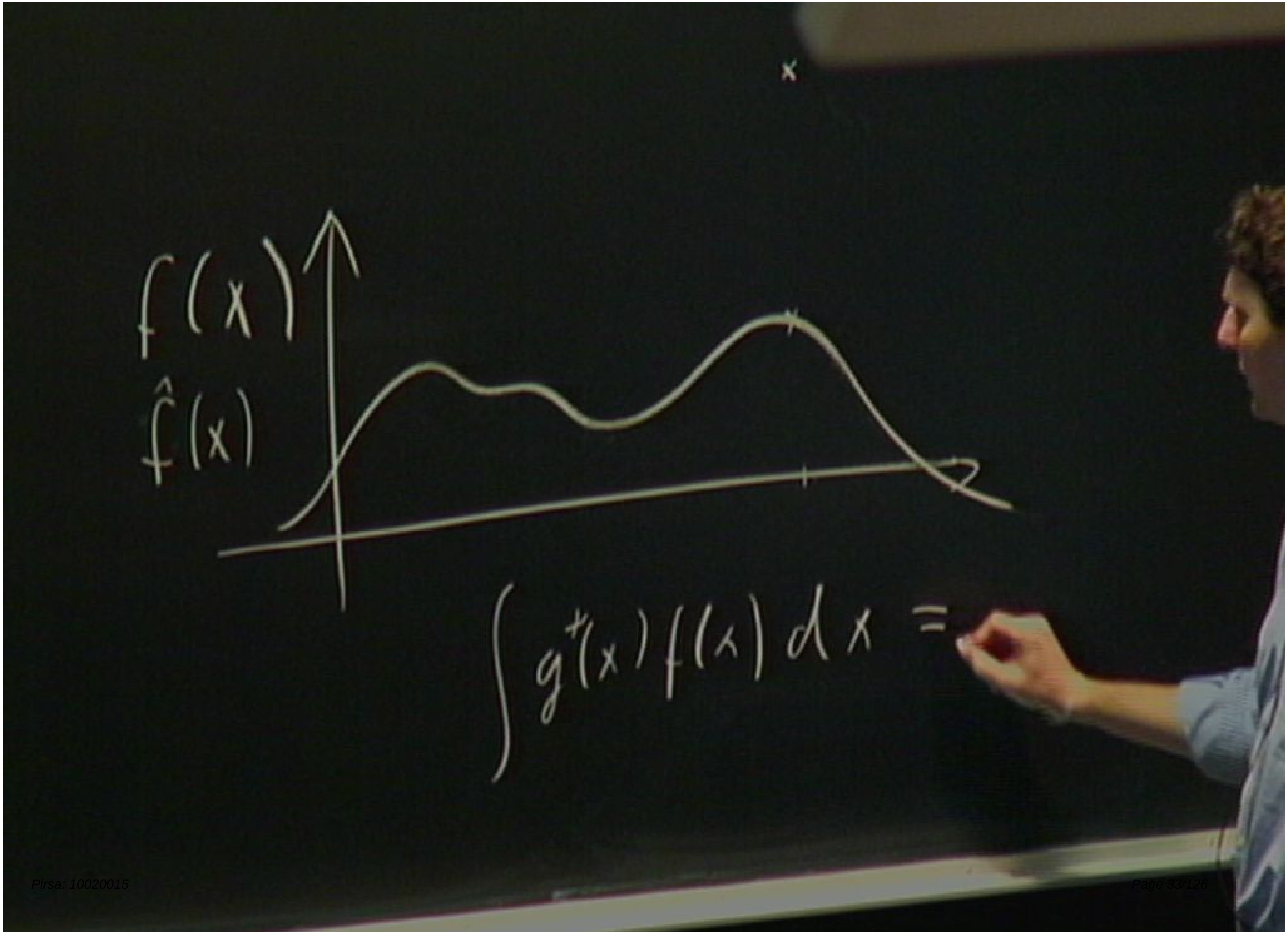
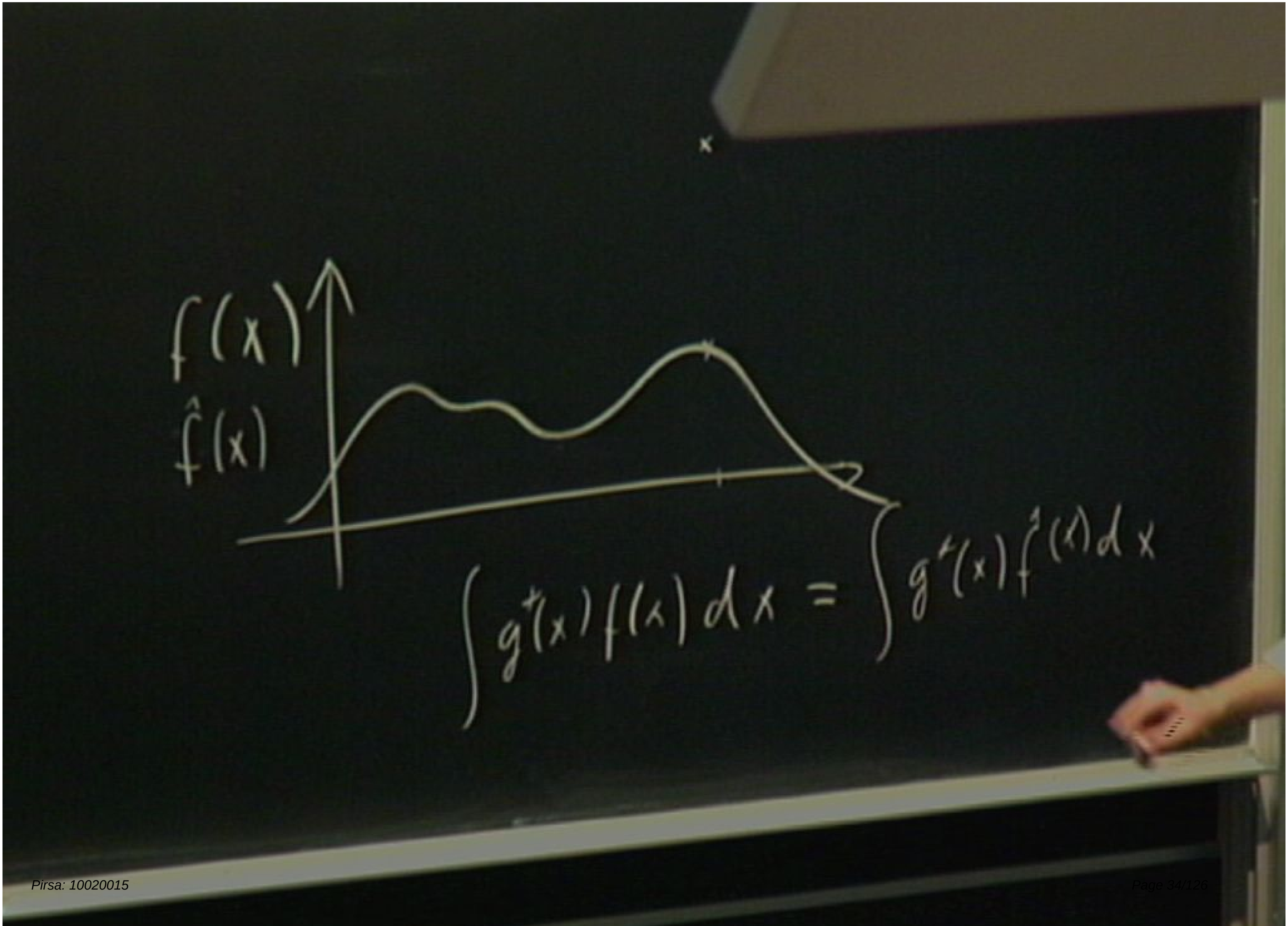

$f(x)$
$\hat{f}(x)$
x


f(x)
f̂(x)
∫ g*(x) f(x) dx =


$f(x)$
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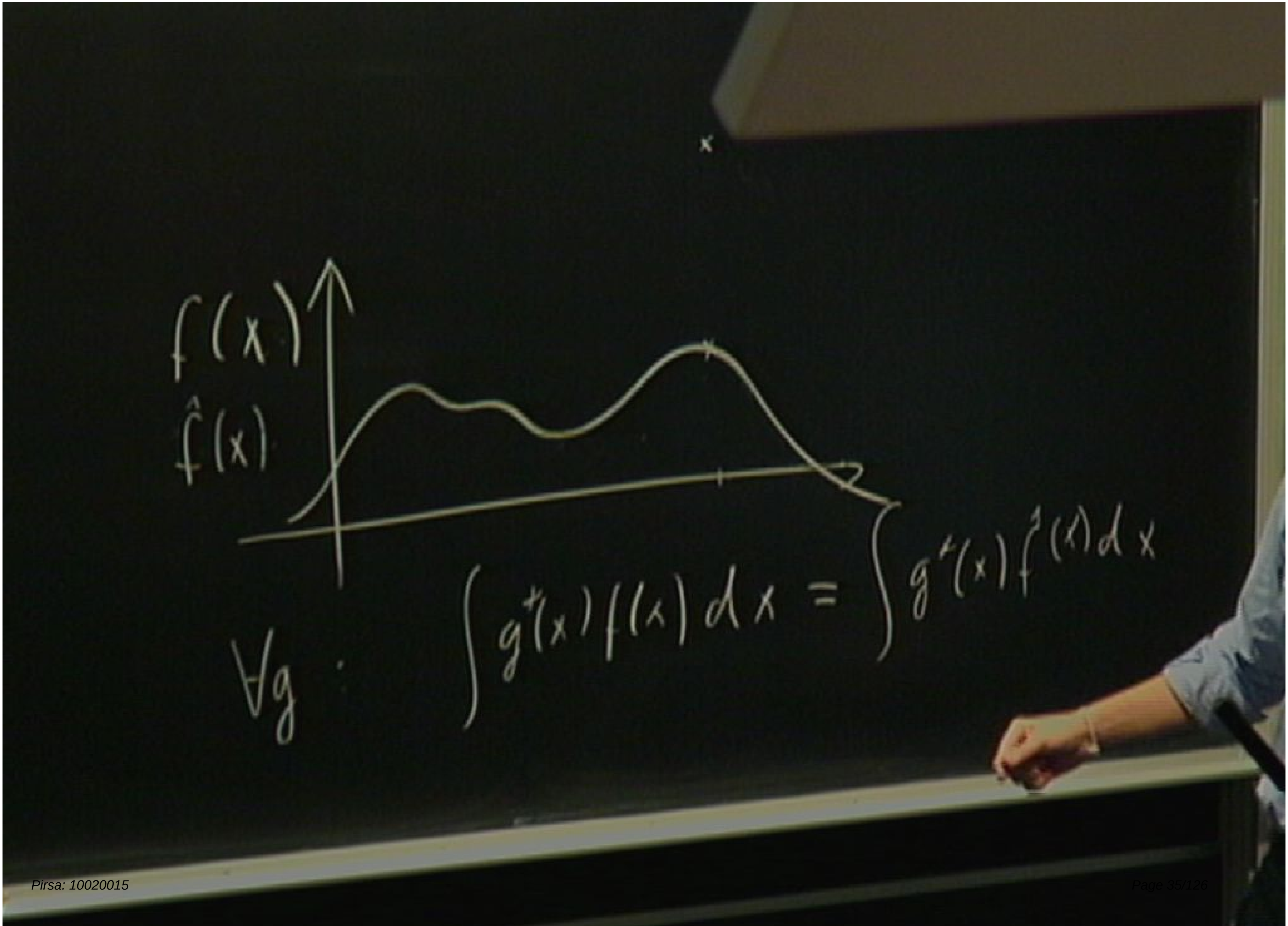
$\int g^*(x) f(x)\, dx = \int g^*(x) \hat{f}(x)\, dx$


x
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∀g : ∫ g*(x) f(x) dx = ∫ g*(x) f̂(x) dx

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QFT:	$\hat{\phi}(x)$	$\hat{\pi}(x)$	x	t

Plays role of $V(\hat{q},t)$ although the first term is usually not considered to be part of the QFT's potential.

QM: $$\hat{H}(t) = \sum_{j=1}^{n} \frac{\hat{p}_j^{\,2}}{2} + V(\hat{q},t)$$

↑ all $\hat{q}_j$

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continuum or discrete basis of L^2.

* E.g. other basis (continuum): e^{ixp}, i.e. use $\tilde{u}(p)$
* E.g. other basis (countable): $H_n(x)e^{-x^2}$, i.e. use $\check{u}_n$

Hermite polynomials

$\Rightarrow$ Functional differentiation is, up to basis change, usual differentiation

Note: How can $L^2[\mathbb{R}]$ have countable basis? Recall: $L^2[\mathbb{R}]$ consists not of functions, but of equivalence classes of functions.

Example application 1:

Schrödinger equation of QFT now well defined:

QM: $\hat{q}_i$ $\hat{p}_i$ i t

QFT: $\hat{\phi}(x)$ $\hat{\pi}(x)$ x t

Plays role of $V(\hat{q},t)$ although the first term is usually not considered to be part of the QFT's potential.

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Example: $W(\hat{\phi}) = \frac{1}{4!}\hat{\phi}^4(x,t)$

In general: $W(\hat{\phi})$ also contains other fields

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$$\Psi[\{\phi(x)\}_{x\in\mathbb{R}^3}, t] = \langle\{\phi(x)\}_{x\in\mathbb{R}^3}|\Psi(t)\rangle$$

Probability amplitude for finding function $\phi(x)$ when measuring $\hat{\phi}(x)$ at t.

Simplified notation: QM: $\psi(q,t) = \langle q|\psi(t)\rangle$

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Exercise: Verify that $\hat{\phi}(x), \hat{\pi}(x)$ obey the CCRs.

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QM: Schrödinger equation:

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$$i\frac{d}{dt}\Psi(q,t) = \sum_{j=1}^{n} -\frac{1}{2}\frac{\partial^2}{\partial q_j^2}\Psi(q,t) + V(q,t)\Psi(q,t)$$

Recall: It is to be solved for all q

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Remark: With W it can be solved only perturbatively.

Exercise: Set $W=0$. Fourier transform to k variables in box

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Exercise: Set $W=0$. Fourier transform to k variables in box regularization. Verify that the wave functional Ψ_0 of the vacuum state obtained before does obey the Schr. eqn

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2. Move from special to general relativity (GR)

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3. Transform GR result back to Hamilton formalism.
4. Apply 2nd quantization.

SR, 1^{st}Q Hamiltonian formalism — "Legendre transform" equivalence → SR, 1^{st}Q. Lagrangian formalism

SR, 1^{st}Q Lagrangian formalism ↓ allow curvature ↓ GR, 1^{st}Q Lagrangian formalism

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GR, 1^{st}Q Hamiltonian formalism ↓ as outlined already ↓ GR, 2^{nd}Q Hamiltonian formalism

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The Legendre transform (LT):

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The Legendre transform (LT):

- Assume given a function, $F(u)$.

- Define a new variable $w(u)$:

$$w(u) := \frac{dF}{du} \qquad \text{(I)}$$

- Assume that (I) can be solved to obtain:

$$u(w)$$

(that's ok if F is convex, say $F''(u) > 0$ for all u)

- The Legendre transform of F is a new function, G, of w:

$$F(u) \xrightarrow{LT} G(w)$$

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Example:

* Consider $f(a,b,c) := a\,e^{bc}$

* Find LT with respect to b (i.e. while treating a, c as "spectator variables"):

$$f(a,b,c) \xrightarrow[b \to \beta]{LT} g(a,\beta,c)$$

* Define $\beta(a,b,c) := \frac{\partial f}{\partial b} = a\,c\,e^{bc}$

* Invert: $b(a,\beta,c) = \frac{1}{c} \ln \frac{\beta}{ac}$

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Case of countably many variables:

- How to define

- Define: $w_j := \frac{\partial F}{\partial u_j}$

- Assume we can invert to obtain:

$$u_j(\{w_i\})$$

- Define:

$$G(\{w_i\}) := \sum_i w_i u_i(\{w_i\}) - F(\{u_j(\{w_i\})\})$$

(we may also allow for spectator variables)

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← classical mechanics

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$$G[\{w(x)\}_{x\in\mathbb{R}}] := \int_{\mathbb{R}^n} w(x)\, u(x, \{w(x')\}_{x'\in\mathbb{R}^n})\, d^nx - F[\{u(x, \{w(x')\})\}]$$

Note: We still have that $LT \circ LT = id$.

← classical mechanics

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Case of uncountably many variables:

- How to define

$$F[\{u(x)\}_{x\in\mathbb{R}^n}] \xrightarrow{LT} G[\{w(x)\}_{x\in\mathbb{R}^n}] \; ?$$

- Define:

$$w(x) := \frac{\delta F}{\delta u(x)}$$

- Assume we can solve to obtain:

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classical mechanics

Application to CM:

* Assume the Hamiltonian $H(q,p)$ is given.

Recall: Poisson bracket

$\{q, p\} = 1$

* Hamilton equations for arbitrary $f(q,p)$:

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See my notes to AMATH 673:

Dirac showed: Quantization consists in keeping the Poisson bracket definition and the Hamilton equations <u>unchanged</u> while allowing q,p noncommutativity in such a way that the Poisson algebra structure stays. This fixes noncommutativity to be $\hat{q}\hat{p}-\hat{p}\hat{q}=i\hbar$ and $\{\hat{f},\hat{g}\}=\frac{1}{i\hbar}[\hat{f},\hat{g}]$

* From this, one can prove the eqns of motion for q,p:

$$\dot{q} = \frac{\partial H(q,p)}{\partial p}, \qquad \dot{p} = -\frac{\partial H(q,p)}{\partial q} \qquad (EoM)$$

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* Concretely:

QM: $\{q, p\} = i\hbar \ldots$

CM: $\{f(q), g(p)\} = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial g}{\partial q}\frac{\partial f}{\partial p}$

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$$\Rightarrow \quad L(q,b) = L(q,\dot{q}) = \dot{q}\,p(q,\dot{q}) - H(q, p(q,\dot{q}))$$

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The equations of motion (EoM) now take the form:

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Proof: Exercise

Example: $H = \frac{p^2}{2m} + \frac{\omega^2}{2} q^2 \xleftrightarrow{LT} L = \frac{1}{2}\dot{q}^2 - \frac{\omega^2}{2} q^2$

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↙ classical (not conformal) field theory

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classical (not conformal) field theory

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- Assume Hamiltonian $H(\phi, \pi)$ is given.

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- This yields the eqns of motion:

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$$\varsigma(x,t) := \frac{\delta H}{\delta \pi(x,t)}$$

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Concretely:

$$\rho(x,t) := \frac{\delta H}{\delta \pi(x,t)}$$

$$= \dot{\phi}(x,t)$$

Thus:

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$$\xi(x,t) := \frac{\delta \ldots}{\delta \pi(x,t)}$$

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Thus:

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Exercise: Check

Euler Lagrange eqn.

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$$H(\phi,\pi) = \int_{\mathbb{R}^3} \frac{\pi^2(x,t)}{2} + \frac{1}{2}\phi(x,t)\left(m^2-\Delta\right)\phi(x,t)\,d^3x$$

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QFT for Cosmology, Achim Kempf, Winter 2010, Lecture 9

Mathematical preparations for QFT in curved space:

Plan today:

- Functional derivatives $\frac{\delta F[g]}{\delta g(x)} = ?$
- Example use 1: to make the QFT Schrödinger equation well defined.
- Example use 2: to define the Functional Legendre transform.
- Use both to obtain the Lagrangian formulation of QFT – which will be starting point for QFT on curved space.