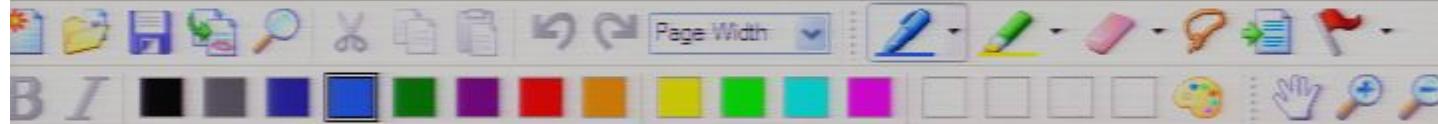


Title: Quantum Field Theory for Cosmology - Lecture 7

Date: Feb 02, 2010 04:00 PM

URL: <http://pirsa.org/10020014>

Abstract: This course begins with a thorough introduction to quantum field theory. Unlike the usual quantum field theory courses which aim at applications to particle physics, this course then focuses on those quantum field theoretic techniques that are important in the presence of gravity. In particular, this course introduces the properties of quantum fluctuations of fields and how they are affected by curvature and by gravitational horizons. We will cover the highly successful inflationary explanation of the fluctuation spectrum of the cosmic microwave background - and therefore the modern understanding of the quantum origin of all inhomogeneities in the universe (see these amazing visualizations from the data of the Sloan Digital Sky Survey. They display the inhomogeneous distribution of galaxies several billion light years into the universe: Sloan Digital Sky Survey).



QFT for Cosmology, Achim Jempf, Winter 2010, Lecture 7

13/01/2006

The driven harmonic oscillator cont'd:

D. Energy eigenstates

* Recall the time dependence of the Hamiltonian:

$$\hat{H}(t) = \omega(a^*(t)a(t) + \frac{1}{2}) - \frac{i}{\sqrt{2\omega}}(a^*(t) + a(t))J(t)$$

$$= \begin{cases} \omega(a_m^*a_m + \frac{1}{2}) & \text{for } t < 0 \\ \text{something} & \text{for } 0 \leq t \leq T \\ \omega(a_{out}^*a_{out} + \frac{1}{2}) & \text{for } T < t \end{cases}$$



QFT for Cosmology, Achim Jempf, Winter 2010, Lecture 7

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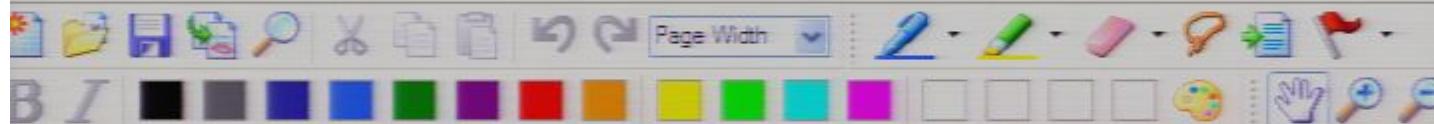
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GL+I for Cosmology, Achim Stumpf, Winter 2010, Lecture +

1/30/2006

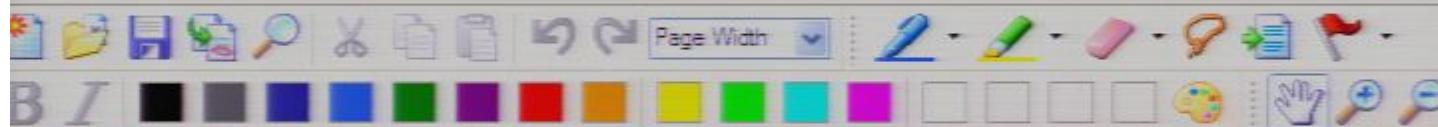
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Here: $a_{out} = a_{in} + J_0$

* For $t < 0$, we diagonalized the Hamiltonian



$$\hat{H}(t) = \omega(a_{in}^* a_{in} + \frac{1}{2}) = \hat{H}_{t=0} = \text{const.}$$



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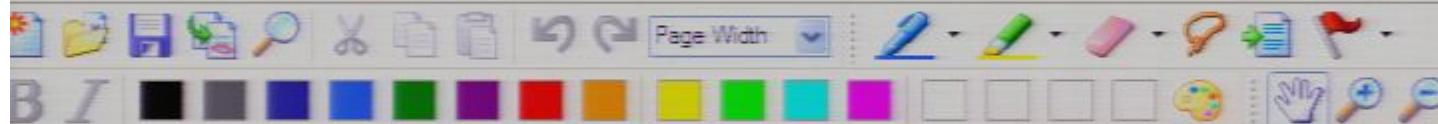
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by using $[a_{in}^*, a_{in}] = 1$ to construct its eigenbasis:

$$\hat{H}_{in} |n_{in}\rangle = E_n^{(in)} |n_{in}\rangle$$



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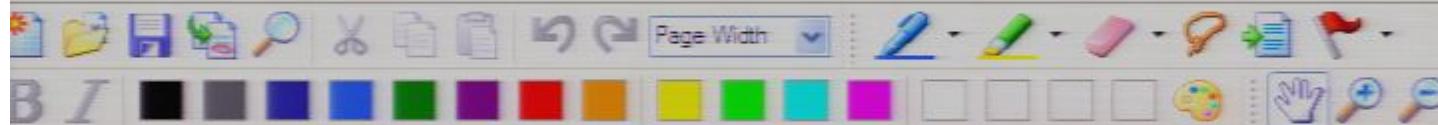
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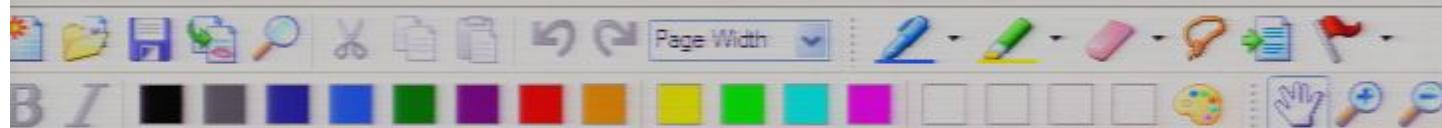
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by using $[a_{in}, a_{in}^+] = 1$ to construct its eigenbasis:

$$\hat{H}_{t=0} |n_{in}\rangle = E_n^{(in)} |n_{in}\rangle$$

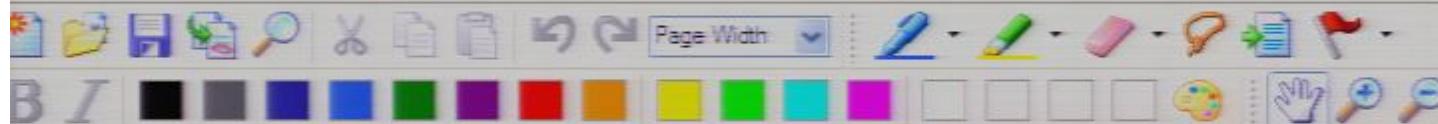
Namely:

$$E_n^{(in)} = \omega(n + \frac{1}{2}) \quad , n = 0, 1, 2, 3 \dots$$

$$|n_{in}\rangle := \frac{1}{\sqrt{n!}} (a_{in}^+)^n |0_{in}\rangle$$

Note: The set $\{|n_{in}\rangle\}$ is a Hilbert basis of the Hilbert space \mathcal{H} .

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* By $t > T$, the Hamiltonian has become a different operator:

$$\hat{H}(t) = \omega(a_{out}^+ a_{out} + \frac{1}{2}) = \hat{H}_{t>T} = \text{const.}$$

What are its eigenvectors $|n_{out}\rangle$ and eigenvalues E_n^{out} ?

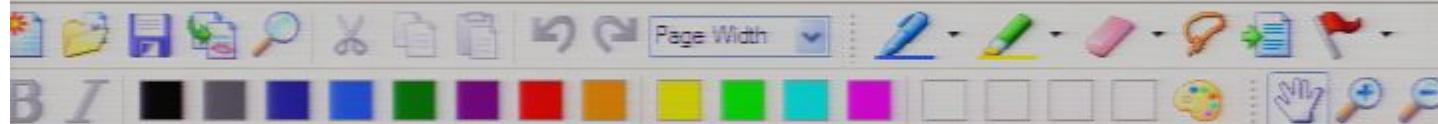
Observation:

We have:

$$[a_{out}, a_{out}^+] = 1$$

\Rightarrow we can construct the eigenbasis of $H_{t>T}$

with the same method as the eigenbasis of H_{in}



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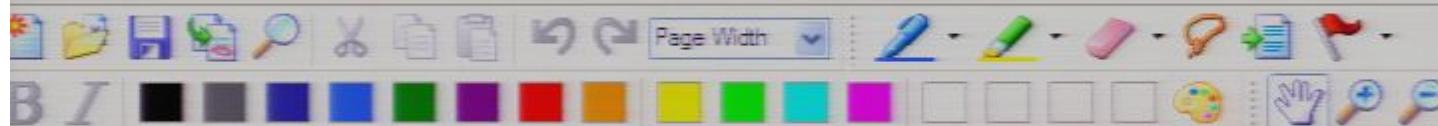
Observation:

We have:

$$[\alpha_{out}, \alpha_{out}^+] = 1$$

\Rightarrow we can construct the eigenbasis of $H_{t>T}$

with the same method as the eigenbasis of $H_{t<0}$:



* There is a unique vector $|0_{out}\rangle \in \mathcal{H}$ obeying:

$$a_{out} |0_{out}\rangle = 0$$

* We define the set of vectors $\{|n_{out}\rangle\}$:

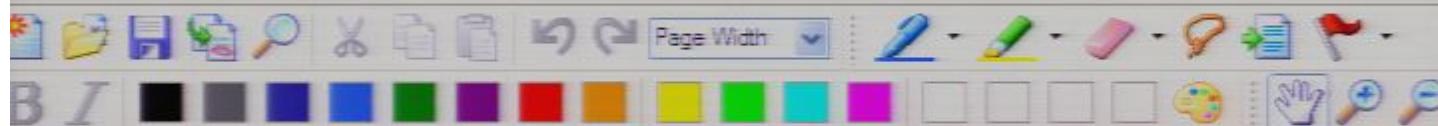
$$|n_{out}\rangle := \frac{1}{\sqrt{n!}} (a_{out}^+)^n |0_{out}\rangle$$

* Proposition:

$$\hat{H}_{ext} |n_{out}\rangle = E_n^{(out)} |n_{out}\rangle \text{ with } E_n = \omega(n + \frac{1}{2}) = E_n^{(in)}$$

* Proposition:

The set $\{|n_{out}\rangle\}$ is a ON Hilbert basis of the Hilbert space \mathcal{H}



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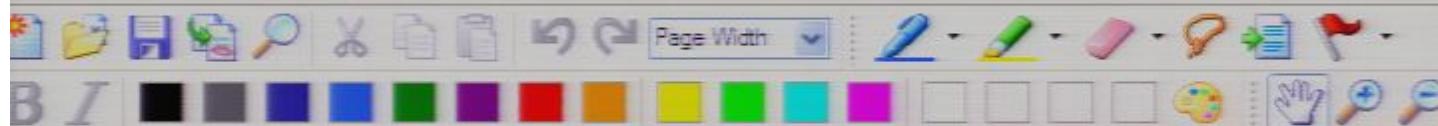
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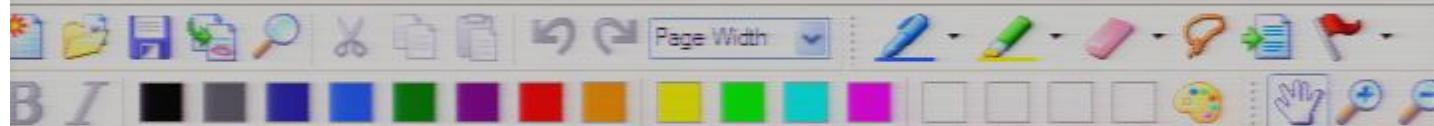
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The operators \hat{H}_{ext} and \hat{H}_{ext}
are different and have different
eigenvectors: $|n_{in}\rangle$ and $|n_{out}\rangle$. Why
are the eigenvalues the same? They
both describe a free oscillator of frequency ω .

* Proposition:

The set $\{|n_{out}\rangle\}$ is a ON Hilbert basis of the Hilbert space \mathcal{H} .

Interpretation:



* Proposition:

$$\hat{H}_{\text{ext}} |n_{\text{out}}\rangle = E_n^{(\text{out})} |n_{\text{out}}\rangle \text{ with } E_n = \omega(n + \frac{1}{2}) = E_n^{(\omega)}$$

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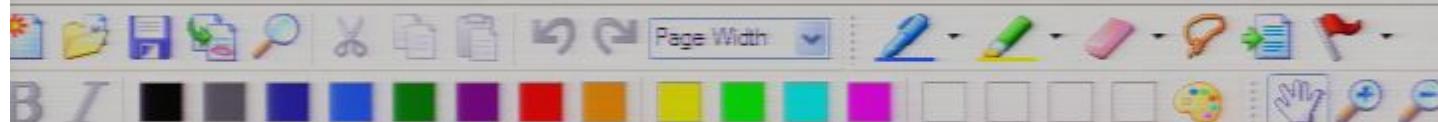
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- * At times $t > T$ the system is in the n^{th} energy eigenstate if it is in the state $|n_{out}\rangle$.
- * However, if the system starts out at $t < 0$ for example in what is then the ground state:

$$|\psi\rangle = |0_{in}\rangle$$

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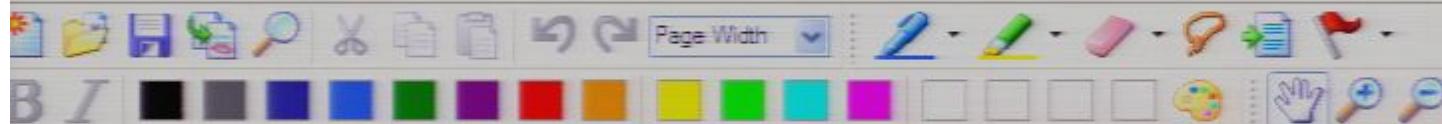
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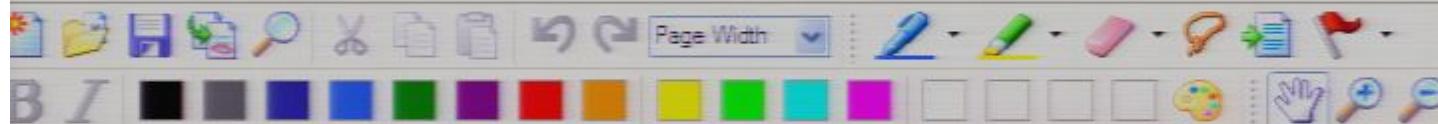
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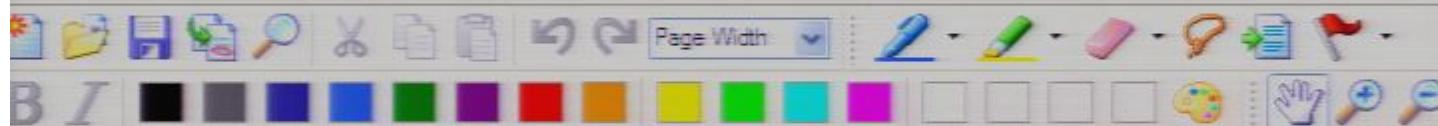
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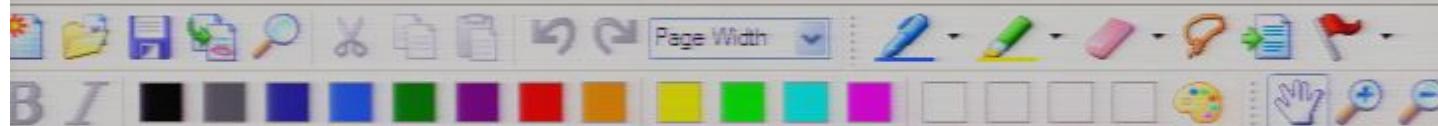
* We expect that $|x\rangle = |0_{in}\rangle \neq |0_{out}\rangle$ because after the force acted, the system's state $|x\rangle = |0_{in}\rangle$ should have acquired the meaning of an excited state.

Which meaning does $|x\rangle = |0_{in}\rangle$ acquire at late times?

* Recall: Both, $\{|n_{in}\rangle\}$ and $\{|n_{out}\rangle\}$ are ON bases of \mathcal{H} .

\Rightarrow Each basis vector $|n_{in}\rangle$ is a linear combination of the basis vectors $\{|n_{out}\rangle\}$ and vice versa.

* Therefore, in particular:



Which meaning does $|0_i\rangle = |0_{in}\rangle$ acquire at late times?

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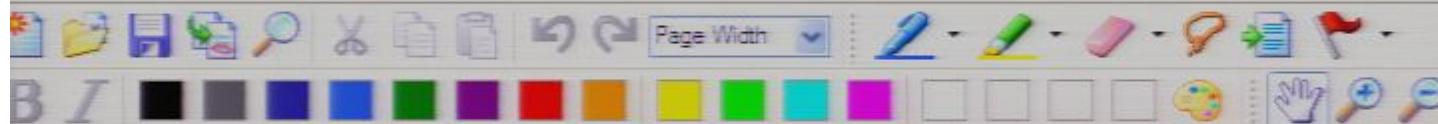
* Therefore, in particular:

There must exist coefficients $A_n \in \mathbb{C}$ so that:

$$|0_i\rangle = \sum_n A_n |n_{out}\rangle$$

↳ "Bogoliubov Transformation"

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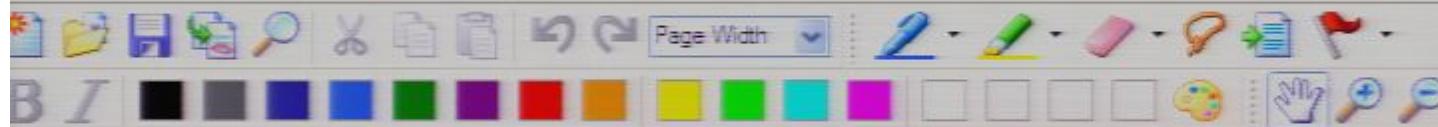
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* Meaning of the λ_n ?



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"Bogoliubov Transformation"

* Meaning of the λ_n ?



□ The system is frozen in state $|x\rangle = |0_n\rangle$.

□ Assume we measure at a time $t > T$ the energy,

i.e., we measure

$$\hat{H}(t) = \omega(a^\dagger a + 1)$$

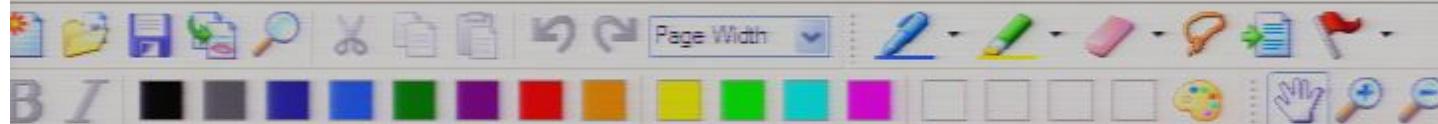


Bogoliubov Transformation

* Meaning of the Λ_n ?

- The system is frozen in state $|y\rangle = |0_m\rangle$.
 - Assume we measure at a time $t > T$ the energy,
i.e., we measure
- $$A(t) = \omega(a_{out}^+ a_{out} + \frac{1}{2})$$
- What is the probability amplitude for finding the energy eigenvalue E_n ?

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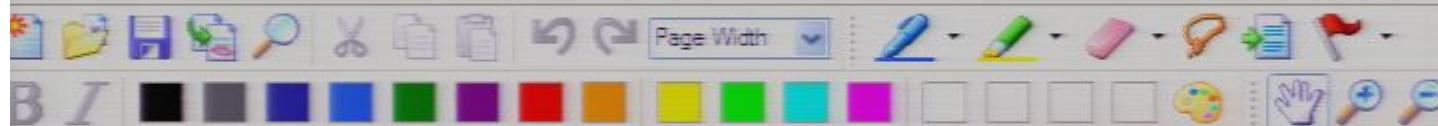
$$\hat{A}(t) = \omega(a_{out}^+ a_{out} + \frac{1}{2})$$

□ What is the probability amplitude for finding the
energy eigenvalue E_n ?

□ Clearly:

$$\text{prob.amp.}(|n_{out}\rangle \text{ at } t > T) = \langle n_{out} | y \rangle$$

$$\text{i.e.:} \quad \text{prob.}(|n_{out}\rangle \text{ at } t > T) = | \langle n_{out} | y \rangle |^2$$



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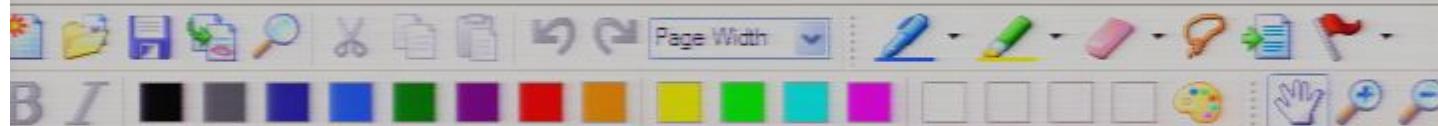
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i.e.: $\text{prob.}(|n_{out}\rangle \text{ at } t > T) = | \langle n_{out} | \psi \rangle |^2$

□ Calculate:

$$\langle n_{out} | \psi \rangle = \langle n_{out} | \phi \rangle \cdot ?$$



$$\text{prob.amp.}(|n_{\text{out}}\rangle \text{ at } t > T) = \langle n_{\text{out}} | \rho \rangle$$

i.e.:

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Calculate:

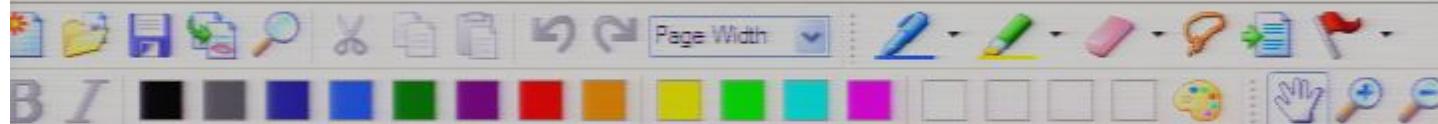
$$\langle n_{\text{out}} | \rho \rangle = \langle n_{\text{out}} | 0_{\text{in}} \rangle$$

$$= \langle n_{\text{out}} | \sum_m A_m | m_{\text{out}} \rangle$$

$$= \Lambda_n$$

\Rightarrow If the oscillator started in its ground state, then at time $t > T$ the probability for finding the oscillator in its n^{th} excited state is given by:

$$\text{prob.}(|n_{\text{out}}\rangle \text{ at } t > T) = |\Lambda_n|^2$$



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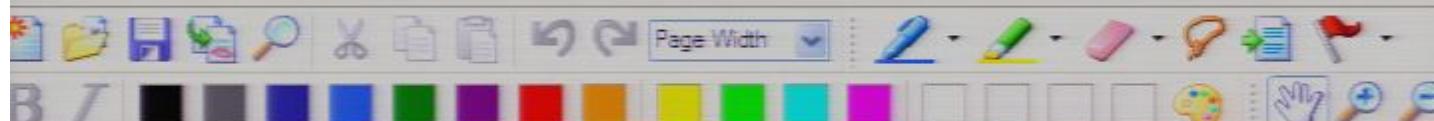
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Remark: In QFT, this will be the prob. for finding n particles



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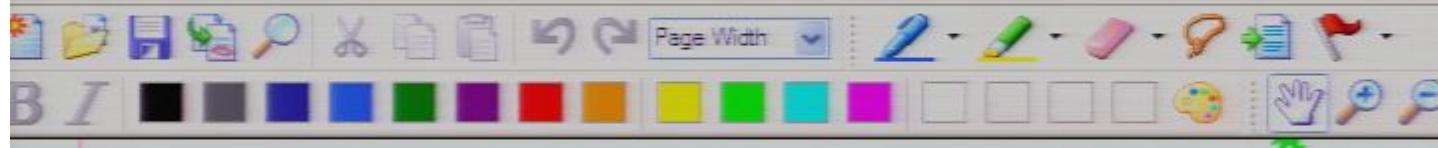
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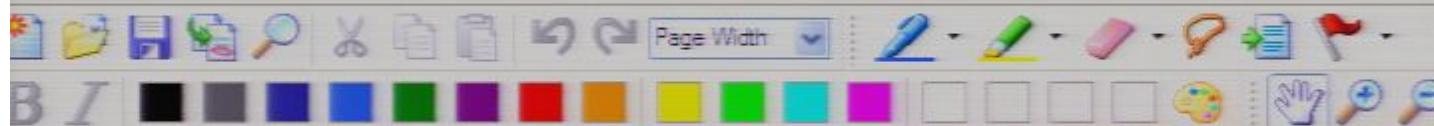
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Calculation of Λ_n :

Proposition: $\Lambda_n = e^{-\frac{1}{2}|\tilde{J}_0|^2} \frac{1}{\sqrt{n!}} \tilde{J}_0^n$



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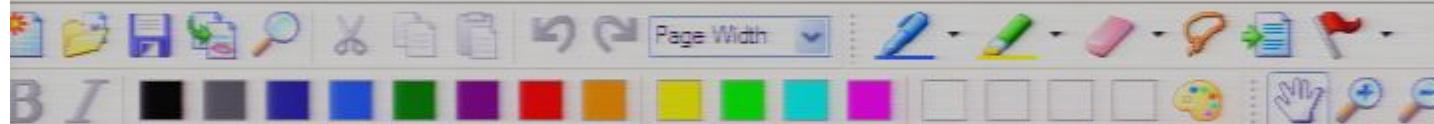
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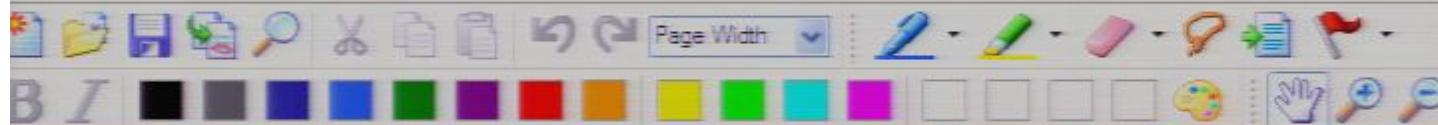
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Calculation of Λ_n :

Proposition: $\Lambda_n = e^{-\frac{1}{2}|\mathcal{J}_0|^2} \frac{1}{\sqrt{n!}} |\mathcal{J}_0^n|$

Proof:

The claim is that $|0_n\rangle = \sum_{n=0}^{\infty} e^{-\frac{1}{2}|\mathcal{J}_0|^2} \frac{1}{\sqrt{n!}} |\mathcal{J}_0^n| |n_{\text{out}}\rangle$.



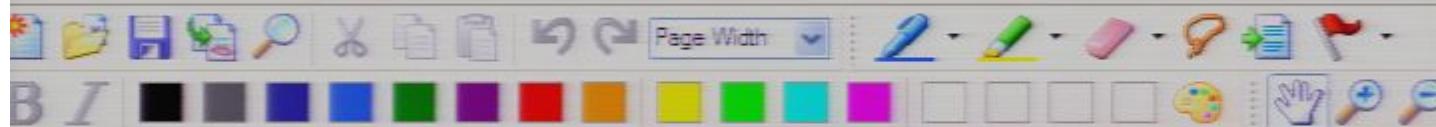
Remark: In QFT, this will be the prob. for finding n particles after the interaction excited the vacuum.

Calculation of Λ_n :

Proposition: $\Lambda_n = e^{-\frac{1}{2}|\mathcal{J}_0|^2} \frac{1}{\sqrt{n!}} |\mathcal{J}_0|^n$

Proof:

The claim is that $|0_n\rangle = \sum_n e^{-\frac{1}{2}|\mathcal{J}_0|^2} \frac{1}{\sqrt{n!}} |\mathcal{J}_0|^n |n_{\text{vac}}\rangle$.



Calculate:

$$\langle n_{out} | \mu \rangle = \langle n_{out} | 0_{in} \rangle$$

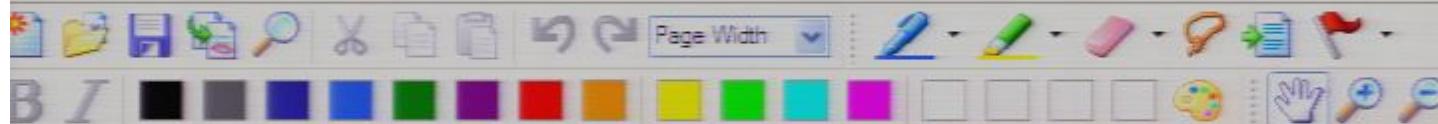
$$= \langle n_{out} | \sum_m A_m | m_{out} \rangle$$

$$= \Lambda_n$$

\Rightarrow If the oscillator started in its ground state, then at time $t > T$ the probability for finding the oscillator in its n 'th excited state is given by:

$$\text{prob.}(|n_{out}\rangle \text{ at } t > T) = |\Lambda_n|^2$$

Remark: In QFT, this will be the prob. for finding n particles after the interaction excited the vacuum.



Bogoliubov Transformation

* Meaning of the Λ_n ?

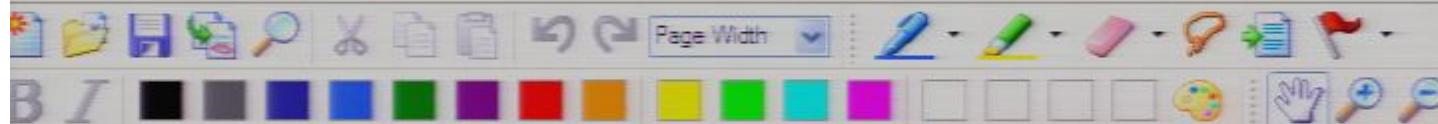
- The system is frozen in state $|y\rangle = |0_m\rangle$.
- Assume we measure at a time $t > T$ the energy,
i.e., we measure

$$\hat{H}(t) = \omega(a_{out}^+ a_{out} + \frac{1}{2})$$

- What is the probability amplitude for finding the energy eigenvalue E_n ?

□ Clearly:

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* Therefore, in particular:

There must exist coefficients $\lambda_n \in \mathbb{C}$ so that:

$$|0_{in}\rangle = \sum_n \lambda_n |n_{out}\rangle$$



Bogoliubov Transformation

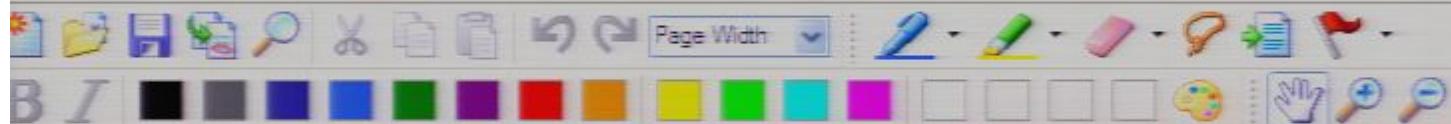
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□ The system is frozen in state $|f\rangle = |0_{in}\rangle$.

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Q What is the probability amplitude for finding the energy eigenvalue E_n ?

Q Clearly:

$$\text{prob.amp.}(|n_{out}\rangle \text{ at } t > T) = \langle n_{out} | \psi \rangle$$

i.e.:

$$\text{prob.}(|n_{out}\rangle \text{ at } t > T) = |\langle n_{out} | \psi \rangle|^2$$

Q Calculate:

$$\langle n_{out} | \psi \rangle = \langle n_{out} | 0_{in} \rangle$$

$$= \langle n_{out} | \sum_m A_m | m_{out} \rangle$$



$$= |\Lambda_n|$$

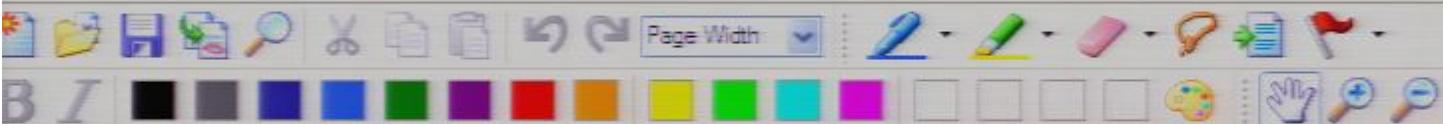
\Rightarrow If the oscillator started in its ground state, then at time $t > T$ the probability for finding the oscillator in its n^{th} excited state is given by:

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Remark: In QFT, this will be the prob. for finding n particles after the interaction excited the vacuum.

Calculation of Λ_n :

Proposition: $\Lambda_n = e^{-\frac{i}{2}|J_0|^2} \frac{1}{n!} J_0^n$



after the interaction excited the vacuum.

Calculation of Λ_n :

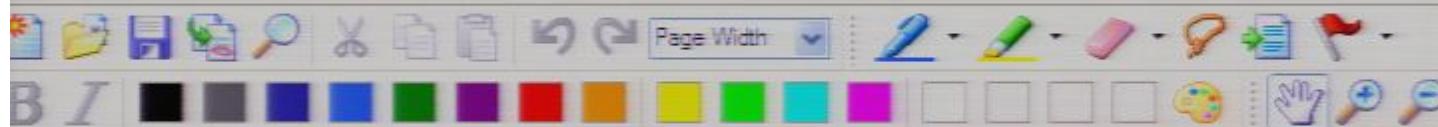
Proposition: $\Lambda_n = e^{-\frac{1}{2}|\mathcal{J}_0|^2} \frac{1}{\sqrt{n!}} |\mathcal{J}_0^n\rangle$

Proof:

The claim is that $|0_n\rangle = \sum_n e^{-\frac{1}{2}|\mathcal{J}_0|^2} \frac{1}{\sqrt{n!}} |\mathcal{J}_0^n\rangle |n_{\text{out}}\rangle$.

We need to check that indeed:

$$a_{in} |0_n\rangle = 0$$



Calculation of Λ_n :

Proposition: $\Lambda_n = e^{-\frac{1}{2}J_0 l^2} \frac{1}{\sqrt{n!}} J_n$

Proof:

The claim is that $|0_n\rangle = \sum_n e^{-\frac{1}{2}J_0 l^2} \frac{1}{\sqrt{n!}} J_n |n_{\text{out}}\rangle$.

We need to check that indeed:

$$a_{in} |0_n\rangle = 0$$

Using $a_{out} = a_{in} + J_0$ it is equivalent to check that:

$$(a_{out} - J_0) |0_n\rangle = 0$$



□ Clearly:

$$\text{prob.amp.}(|n_{\text{out}}\rangle \text{ at } t > T) = \langle n_{\text{out}} | \rho \rangle$$

i.e.:

$$\text{prob.}(|n_{\text{out}}\rangle \text{ at } t > T) = |\langle n_{\text{out}} | \rho \rangle|^2$$

□ Calculate:

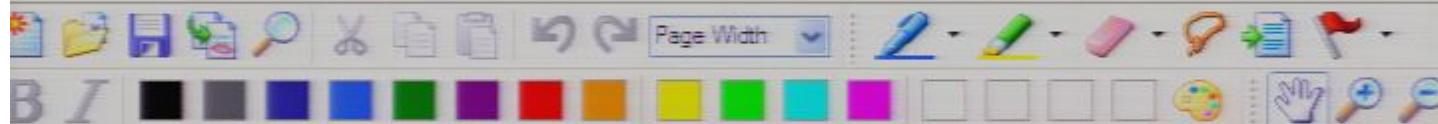
$$\langle n_{\text{out}} | \rho \rangle = \langle n_{\text{out}} | 0_{\text{in}} \rangle$$

$$= \langle n_{\text{out}} | \sum_m A_m | m_{\text{out}} \rangle$$

$$= \Lambda_n$$

\Rightarrow If the oscillator started in its ground state, then

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↳ "Bogoliubov Transformation"

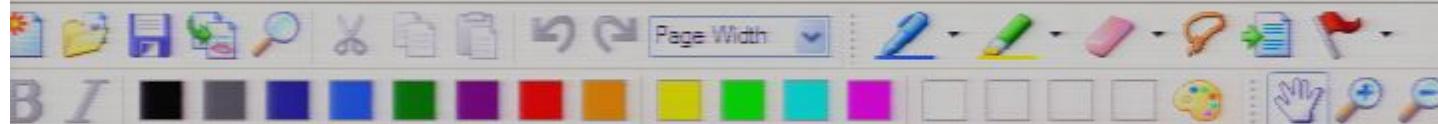
* Meaning of the λ_n ?

□ The system is frozen in state $|x\rangle = |0_{in}\rangle$.

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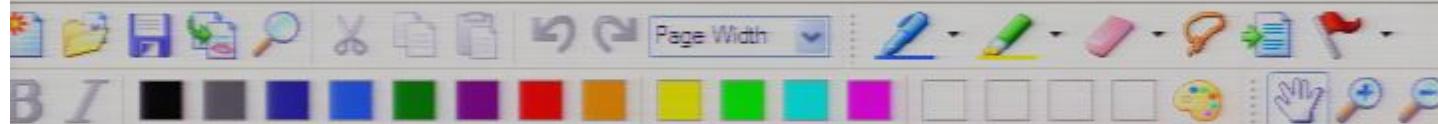
$$|0_{in}\rangle = \sum \lambda_n |n_{out}\rangle$$

└ "Bogoliubov Transformation"

* Meaning of the λ_n ?

□ The system is frozen in state $|y\rangle = |0_{in}\rangle$.

□ Assume we measure at a time $t > T$ the energy,
i.e., we measure



$$= \langle n_{out} | \sum_m |\Lambda_m | m_{out} \rangle$$

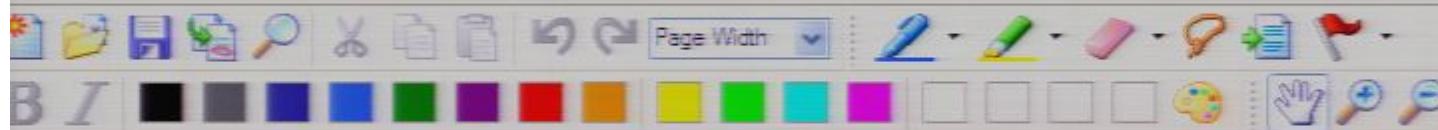
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Remark: In QFT, this will be the prob. for finding n particles after the interaction excited the vacuum.

Calculation of Λ_n :



Calculation of Λ_n :

Proposition: $\Lambda_n = e^{-\frac{1}{2}|\mathbf{J}_0|^2} \frac{1}{\sqrt{n!}} |\mathbf{J}_n\rangle$

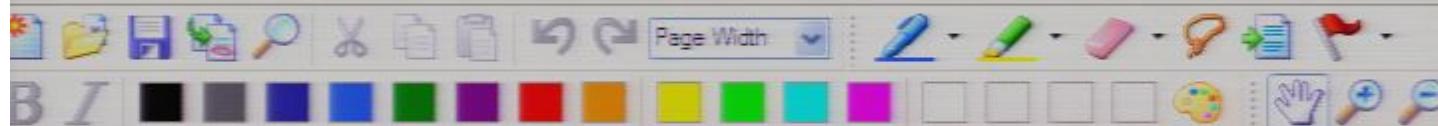
Proof:

The claim is that $|\psi_n\rangle = \sum_{\mathbf{m}} e^{-\frac{1}{2}|\mathbf{J}_{\mathbf{m}}|^2} \frac{1}{\sqrt{\mathbf{m}!}} |\mathbf{J}_{\mathbf{m}}\rangle$.

We need to check that indeed:

$$a_{in} |\psi_n\rangle = 0$$

Using $a_{out} = a_{in} + \mathbf{J}_0$ it is equivalent to check that:



Calculation of Λ_n :

Proposition: $\Lambda_n = e^{-\frac{1}{2} \langle J_s \rangle^2} \frac{1}{\sqrt{n!}} J_s^n$

Proof:

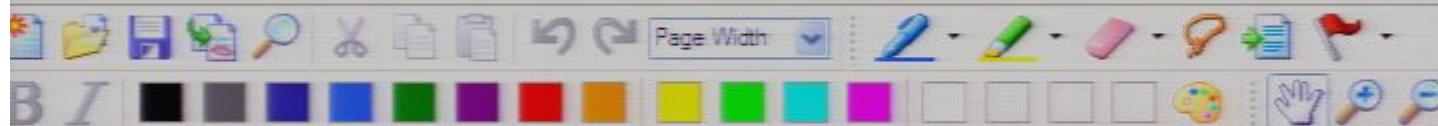
The claim is that $|0_n\rangle = \sum_n e^{-\frac{1}{2} \langle J_s \rangle^2} \frac{1}{\sqrt{n!}} J_s^n |n_{\text{out}}\rangle$.

We need to check that indeed:

$$a_{in} |0_n\rangle = 0$$

Using $a_{out} = a_{in} + J_s$ it is equivalent to check that:

$$(a_{out} - J_s) |0_n\rangle = 0$$



Calculation of Λ_n :

Proposition: $\Lambda_n = e^{-\frac{1}{2}|\mathbf{J}_0|^2} \frac{1}{\sqrt{n!}} |\mathbf{J}_0|^n$

Proof:

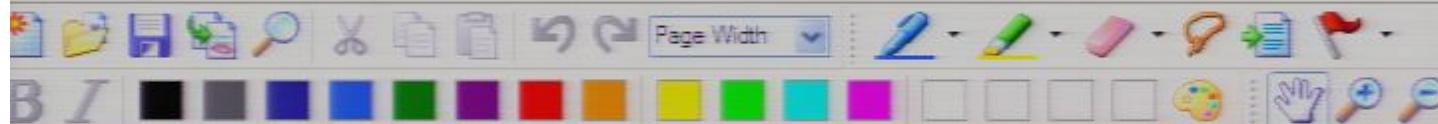
The claim is that $|\psi_n\rangle = \sum_n e^{-\frac{1}{2}|\mathbf{J}_0|^2} \frac{1}{\sqrt{n!}} |\mathbf{J}_0|^n |\psi_{n0}\rangle$.

We need to check that indeed:

$$a_{in} |\psi_{n0}\rangle = 0$$

Using $a_{out} = a_{in} + \mathbf{J}_0$ it is equivalent to check that:

$$(a_{out} - \mathbf{J}_0) |\psi_{n0}\rangle = 0$$



Proof:

The claim is that $|0_m\rangle = \sum_n e^{-\frac{1}{2}|\beta_n|^2} \frac{1}{\sqrt{n!}} \beta_n^\dagger |0_{n+1}\rangle$.

We need to check that indeed :

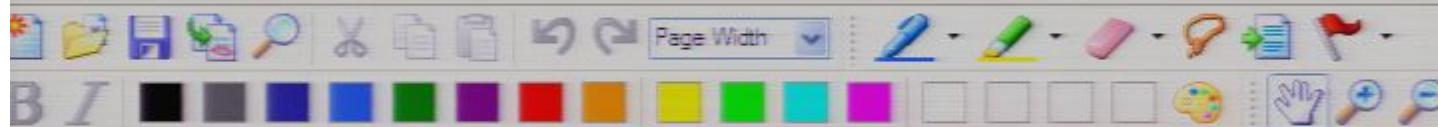
$$\alpha_m |0_m\rangle = 0$$

Using $\alpha_{out} = \alpha_m + \beta_0$ it is equivalent to check that :

$$(\alpha_{out} - \beta_0) |0_m\rangle = 0$$

Indeed :

$$(\alpha_{out} - \beta_0) \sum_n e^{-\frac{1}{2}|\beta_n|^2} \frac{1}{\sqrt{n!}} \beta_n^\dagger |0_{n+1}\rangle$$



$$a_{in}|0_{in}\rangle = 0$$

Using $a_{out} = a_{in} + j_0$ it is equivalent to check that:

$$(a_{out} - j_0)|0_{in}\rangle = 0$$



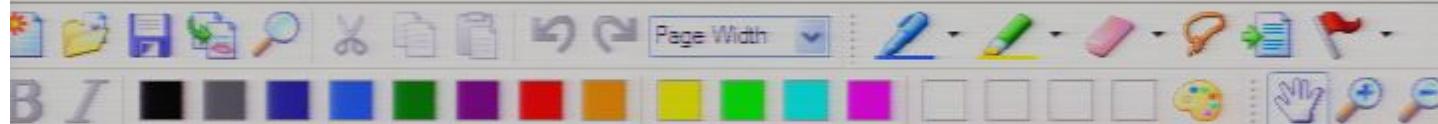
Indeed:

$$(a_{out} - j_0) \sum_n e^{-\frac{1}{2}(j_0)^2} \frac{1}{\sqrt{n!}} j_0^n |n_{out}\rangle$$

$$= (a_{out} - j_0) \sum_n e^{-\frac{1}{2}(j_0)^2} j_0^n \underbrace{\frac{1}{\sqrt{n!}}}_{\frac{1}{n!}} \underbrace{\frac{1}{\sqrt{n!}}}_{\frac{1}{n!}} (a_{out}^+)^n |0_{out}\rangle$$

$$= (a_{out} - j_0) e^{-\frac{1}{2}(j_0)^2} e^{j_0 a_{out}^+} |0_{out}\rangle$$

$$= e^{-\frac{1}{2}(j_0)^2} (-1) e^{j_0 a_{out}^+} |0_{out}\rangle$$



Indeed:

$$(a_{out} - J_0) \sum_n e^{-\frac{1}{2}(J_0)^2} \frac{1}{\sqrt{n!}} J_0^n |0_{out}\rangle \quad \frac{1}{n!}$$

$$= (a_{out} - J_0) \sum_n e^{-\frac{1}{2}(J_0)^2} J_0^n \underbrace{\frac{1}{\sqrt{n!}}}_{\text{!}} \frac{1}{\sqrt{n!}} (a_{out}^+)^n |0_{out}\rangle$$

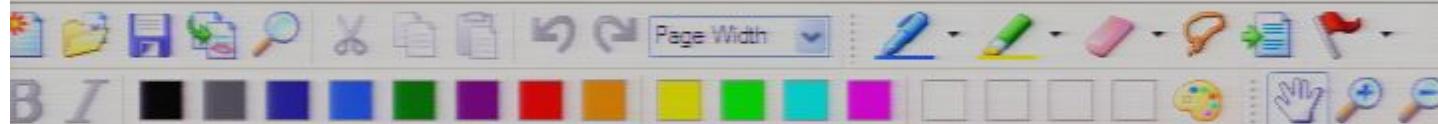
$$= (a_{out} - J_0) e^{-\frac{1}{2}(J_0)^2} e^{J_0 a_{out}^+} |0_{out}\rangle$$

$$= e^{-\frac{1}{2}(J_0)^2} (a_{out} - J_0) e^{J_0 a_{out}^+} |0_{out}\rangle$$

(Exercise:
Check this)

Using $[a_{out}, a_{out}^+] = 1$ yields $[a_{out}, e^{J_0 a_{out}^+}] = J_0 e^{J_0 a_{out}^+}$ \Rightarrow

$$= e^{-\frac{1}{2}(J_0)^2} \left((J_0 - J_0) e^{J_0 a_{out}^+} + e^{J_0 a_{out}^+} a_{out}^+ \right) |0_{out}\rangle$$



Indeed:

$$(a_{out} - J_0) \sum_n e^{-\frac{1}{2}(J_0)^2} \frac{1}{\sqrt{n!}} J_0^n |0_{out}\rangle$$

$$= (a_{out} - J_0) \sum_n e^{-\frac{1}{2}(J_0)^2} J_0^n \underbrace{\frac{1}{\sqrt{n!}}}_{\frac{1}{n!}} \underbrace{\frac{1}{\sqrt{n!}}}_{\frac{1}{n!}} (a_{out}^+)^n |0_{out}\rangle$$

$$= (a_{out} - J_0) e^{-\frac{1}{2}(J_0)^2} e^{J_0 a_{out}^+} |0_{out}\rangle$$

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$$= 0 \quad \checkmark$$

$$aa^+ - a^+ a = 1$$

$$\partial_x x - x \partial_x = 1$$

$$\partial_x x$$

$$aa^+ - a^+ a = I$$

$$\partial_x x - x \partial_x = I$$

$$(\partial_x x - x \partial_x - I) f(x)$$

$$\partial_x x f(x) - x \partial_x f(x) - f(x)$$

$$\alpha \alpha^+ - \alpha^+ \alpha = 1$$

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$$(\partial_x x - x \partial_x - 1) f(x)$$

$$\partial_x x f(x) - x \partial_x f(x) - f(x)$$

$$f(x) + x \partial_x f(x) - y \partial_y f(x) - f(x)$$

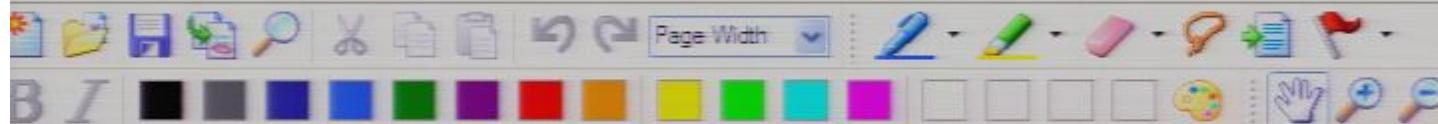
$$a a^+ - a^+ a = I$$

$$\boxed{\partial_x x - x \partial_x = I}$$

$$(\partial_x x - x \partial_x - I) f(x)$$

$$\partial_x x f(x) - x \partial_x f(x) - f(x)$$

$$f(x) + x \partial_x f(x) - y \partial_y f(x) - f(x) = 0$$



Indeed:

$$(a_{out} - J_0) \sum_n e^{-\frac{1}{2}(J_0)^2} \frac{1}{\sqrt{n!}} J_0^n |0_{out}\rangle$$

$$= (a_{out} - J_0) \sum_n e^{-\frac{1}{2}(J_0)^2} J_0^n \underbrace{\frac{1}{\sqrt{n!}}}_{\frac{1}{n!}} \underbrace{\frac{1}{\sqrt{n!}}}_{\binom{n}{2}} (a_{out}^+)^n |0_{out}\rangle$$

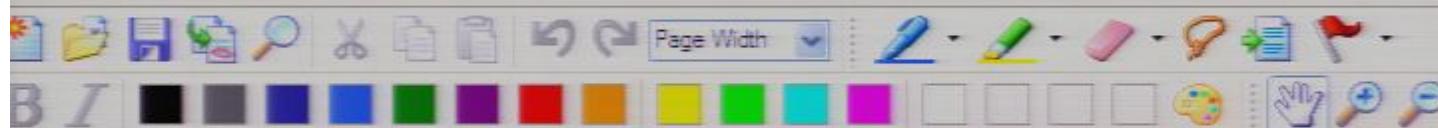
$$= (a_{out} - J_0) e^{-\frac{1}{2}(J_0)^2} e^{J_0 a_{out}^+} |0_{out}\rangle$$

$$= e^{-\frac{1}{2}(J_0)^2} (a_{out} - J_0) e^{J_0 a_{out}^+} |0_{out}\rangle$$

(Exercise: check this) \rightarrow (Using $[a_{out}, a_{out}^+] = 1$ yields $[a_{out}, e^{J_0 a_{out}^+}] = J_0 e^{J_0 a_{out}^+}$) \Rightarrow

$$= e^{-\frac{1}{2}(J_0)^2} \left((J_0 + J_0) e^{J_0 a_{out}^+} + e^{J_0 a_{out}^+} a_{out} \right) |0_{out}\rangle$$

$$= 0 \quad \checkmark$$



(Exercise:
check this) \rightarrow (Using $[a_{\text{out}}, a_{\text{out}}^\dagger] = 1$ yields $[a_{\text{out}}, e^{\lambda_0 a_{\text{out}}^\dagger}] = \lambda_0 e^{\lambda_0 a_{\text{out}}^\dagger}$) \Rightarrow

$$= e^{-\frac{1}{2}|\lambda_0|^2} \left((\lambda_0 - \lambda_0) e^{\lambda_0 a_{\text{out}}^\dagger} + e^{\lambda_0 a_{\text{out}}^\dagger} \cancel{a_{\text{out}}} \right) |0_{\text{out}}\rangle$$

$$= 0 \quad \checkmark$$

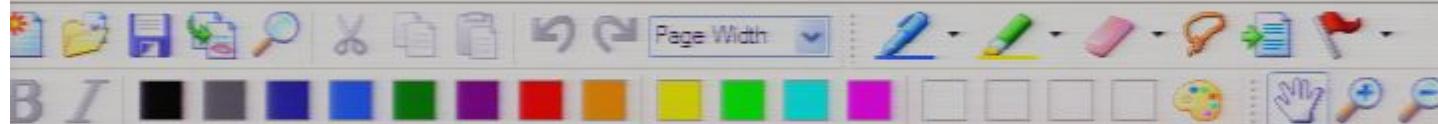
* Exercise: Verify that $|0_m\rangle = \sum_n e^{-\frac{1}{2}|\lambda_n|^2} \frac{1}{\sqrt{n!}} \lambda_n^n |n_{\text{out}}\rangle$

is indeed normalized: $\langle 0_m | 0_m \rangle = 1$

* Definition:

We have shown that if a harmonic oscillator is prepared in the state $|x\rangle = |0_m\rangle$, then after the driving its state always:

$$a_{\text{out}} |x\rangle = \lambda_0 |x\rangle \quad (a_m |x\rangle = 0)$$



* Exercise: Verify that $|0_m\rangle = \sum_n e^{-\frac{1}{2}J_n^2} \frac{1}{\sqrt{n!}} J_n |n\rangle$

is indeed normalized: $\langle 0_m | 0_m \rangle = 1$

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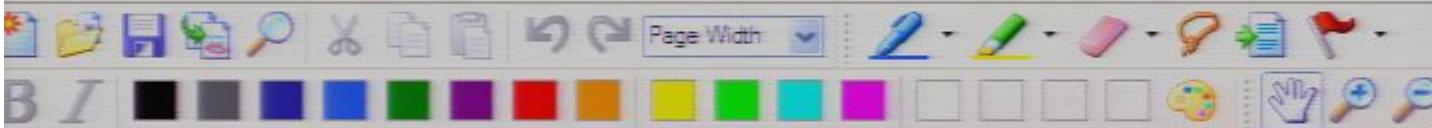
$$\text{and } |x\rangle = J_0 |x\rangle \quad (\text{as } |x\rangle = 0)$$

Eigenstates of annihilation operators are called "Coherent states".

* Recall:

$$\text{For coherent states: } \Delta q \Delta p = \frac{\hbar}{2}$$

i.e. they have least possible fluctuations in both q and p .



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Antennas produce coherent electromagnetic states.

Driven mode oscillators in QFT



we need to prove that $a_{in}|0_{in}\rangle = 0$
we need to prove that $a_{out}|0_{in}\rangle = 0$

$$a_{in}|0_{in}\rangle = 0$$

Using $a_{out} = a_{in} + j_0$ it is equivalent to check that:

$$(a_{out} - j_0)|0_{in}\rangle = 0$$

Indeed:

$$(a_{out} - j_0) \sum_n e^{-\frac{1}{2}(j_0)^2} \frac{1}{\sqrt{n!}} j_0^n |n_{out}\rangle$$

$$= (a_{out} - j_0) \sum_n e^{-\frac{1}{2}(j_0)^2} j_0^n \underbrace{\frac{1}{\sqrt{n!}}}_{\frac{1}{n!}} \underbrace{\frac{1}{\sqrt{n!}}}_{\frac{1}{(n+1)!}} (a_{out}^+)^n |0_{out}\rangle$$

$$= (a_{out} - j_0) e^{-\frac{1}{2}(j_0)^2} e^{a_{out}^+} |0_{out}\rangle$$



We need to check that indeed:

$$a_{in}|0_{in}\rangle = 0$$

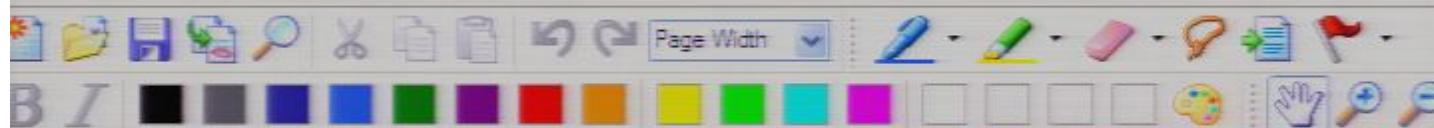
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* Exercise: Verify that $|10_m\rangle = \frac{1}{\sqrt{n!}} |0\rangle \otimes |n\rangle$

is indeed normalized: $\langle 0, |10_m\rangle = 1$

* Definition:

We have shown that if a harmonic oscillator is prepared in the state $|x\rangle = |10_m\rangle$, then after the driving its state always:

$$a_{\text{out}} |x\rangle = j_0 |x\rangle \quad (\text{as } |x\rangle = 0)$$

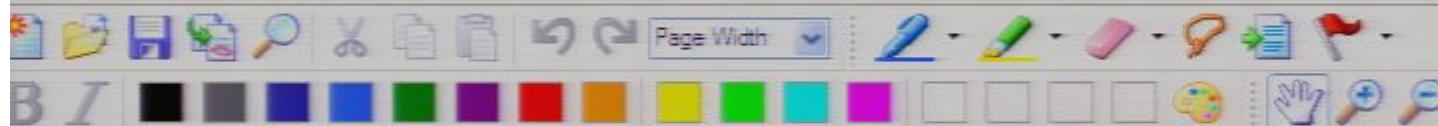
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Antennas produce coherent electromagnetic states.



* Exercise: Verify that $|0_m\rangle = \sum_n e^{-\frac{1}{2}J_0 J_n^2} \frac{1}{\sqrt{n!}} J_0^n |n_m\rangle$

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$$\text{and } |x\rangle = J_0 |x\rangle \quad (\text{as } |x\rangle = 0)$$

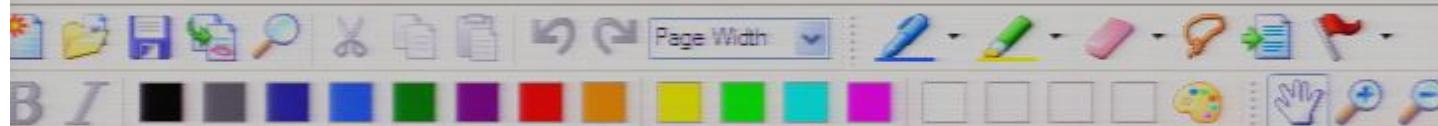
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Driven mode oscillators in QFT

Recall:

$$\hat{H}(t) = \frac{1}{2} \int_{\mathbb{R}^3} \hat{\pi}^2(x, t) - \hat{\phi}(x, t) (\Delta - m^2) \hat{\phi}(x, t) + j(x, t) \hat{\phi}(x, t) d^3x$$

Example interpretation:

* $\hat{\phi}(x, t)$ may be viewed as a slightly simplified version



* Recall:

For coherent states: $\Delta q \Delta p = \frac{\hbar}{2}$.

i.e. they have least possible fluctuations in both q and p .

Antennas produce coherent electromagnetic states.

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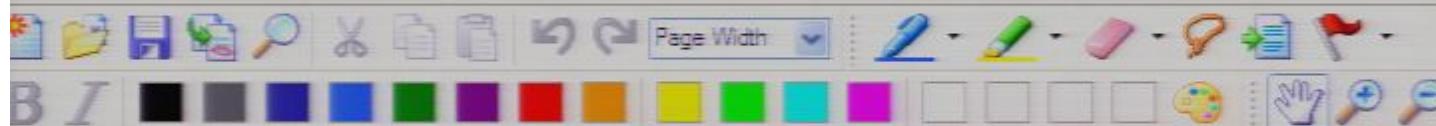
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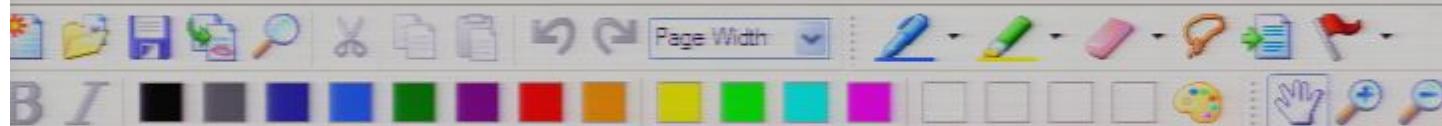
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* $\phi(x,t)$ may be viewed as a slightly simplified version of the quantum electromagnetic field.

* $J(x,t)$ may be viewed as a simplified version of a given classical electric current.



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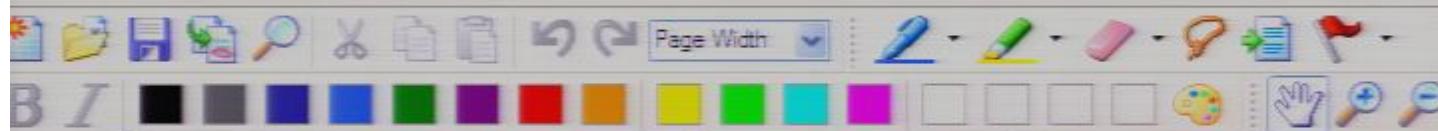
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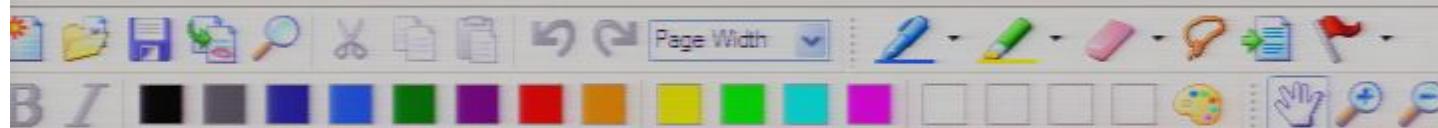
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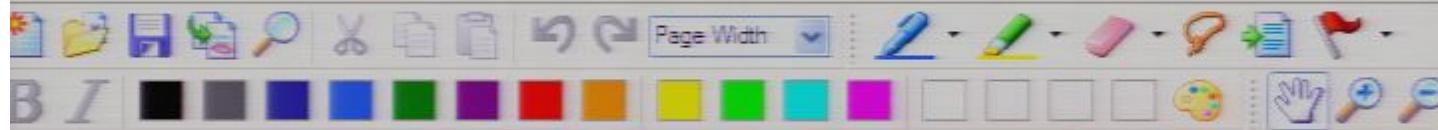
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In- and out periods

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$$\frac{J}{R^3}$$

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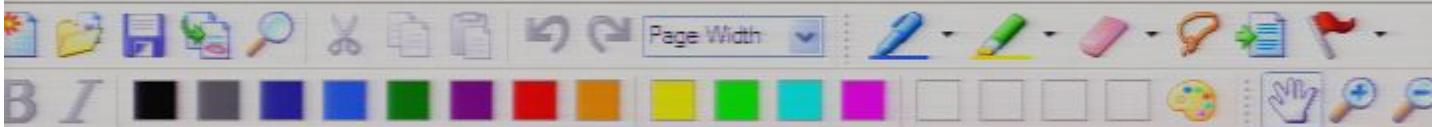
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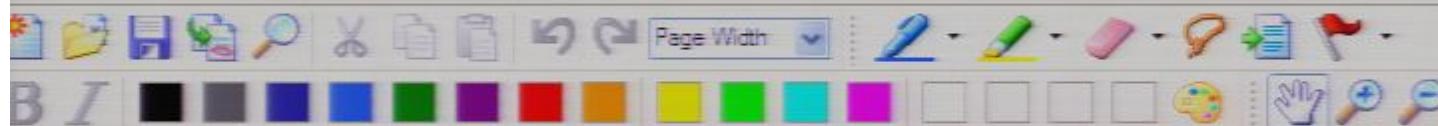
In- and out periods

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$$J(x,t) = 0 \text{ for all } t \notin [0, T]$$

\Rightarrow It suffices to consider the periods $t < 0$ and $t > T$ in both of which $J(x,t) = 0$ (and then to relate the bases).

The free (i.e., undriven) QFT : $(t < 0 \text{ or } t > T)$



In- and out periods



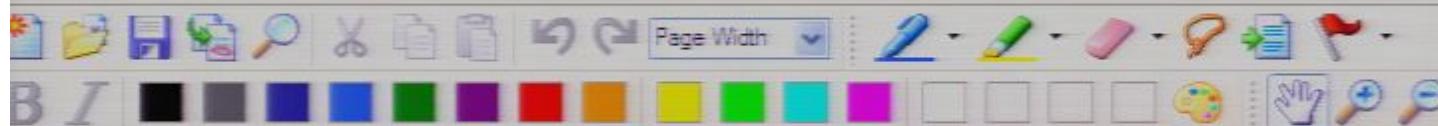
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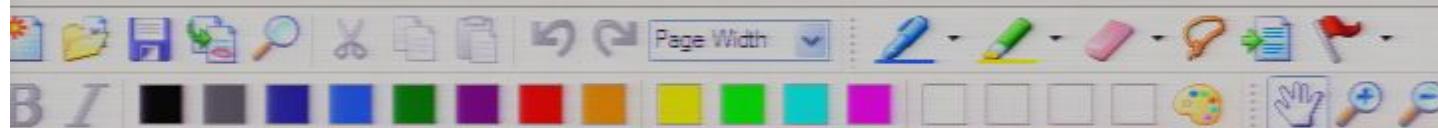
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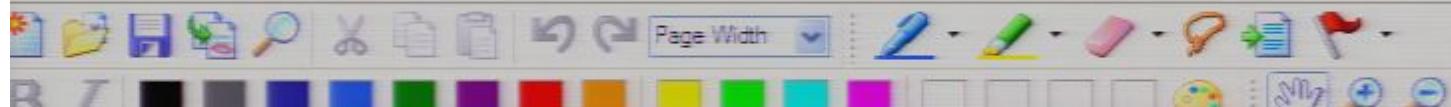
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$\int_{\mathbb{R}^3}$

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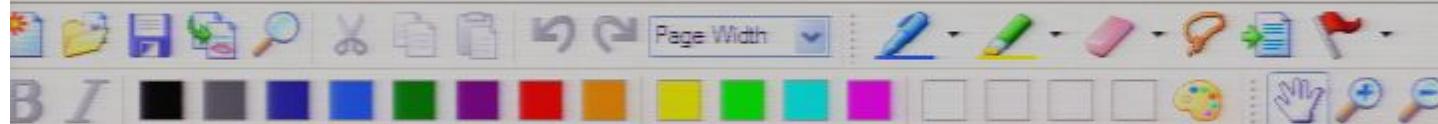
* Fourier transformed,

$$\hat{\phi}_k(t) := \frac{1}{(2\pi)^3 \sqrt{}} \int_{\mathbb{R}^3} \hat{\phi}(x, t) e^{-ikx} d^3x$$

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$$\ddot{\hat{\phi}}_k(t) + \underbrace{(k^2 + m^2)}_{= \sum_{i=1}^3 k_i^2} \hat{\phi}_k(t) = 0$$

(EoM)



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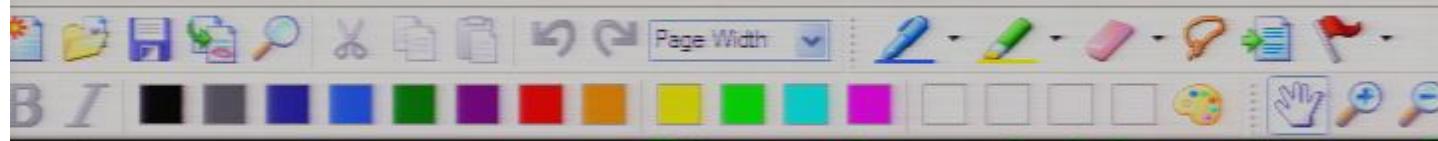
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Solution strategy due to Fock:



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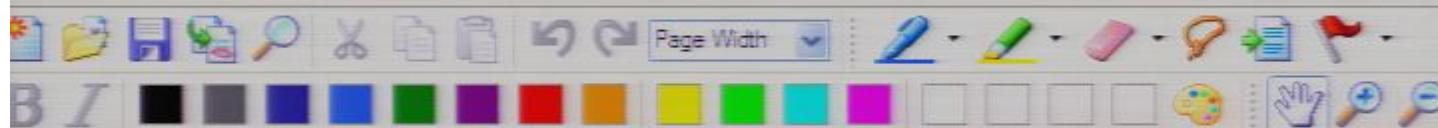
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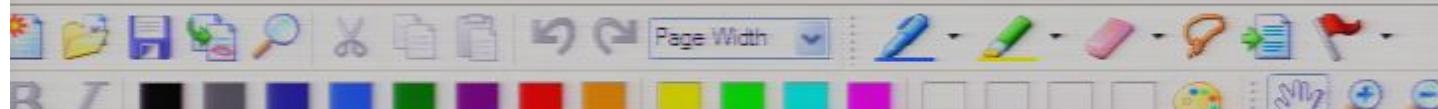
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$$(2\pi)^{3/2} J_{R^3}$$

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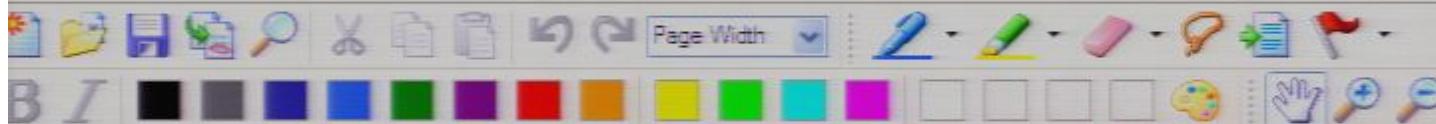
Solution strategy due to Fock:

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□ Introduce new variables:

QM:

$$a(t) := \sqrt{\omega} \hat{a}(t) + i \frac{1}{\sqrt{\omega}} \hat{p}(t)$$



Definition: ω_n^2

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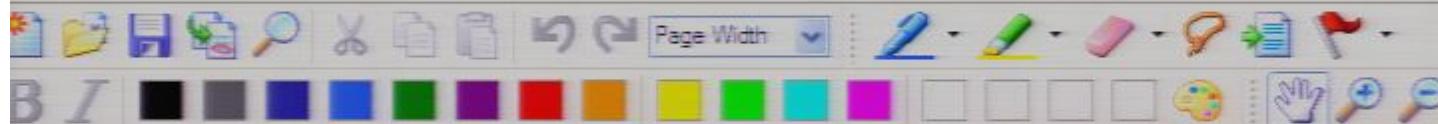
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QFT:

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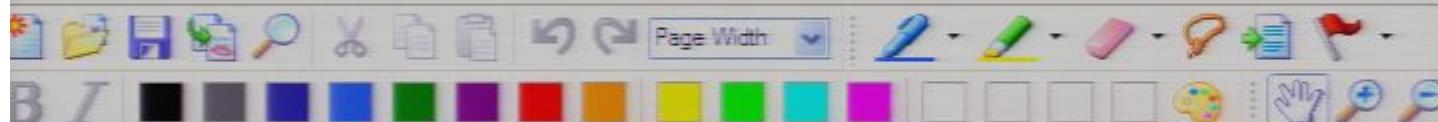
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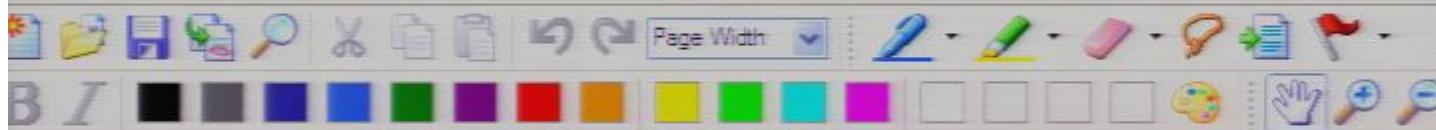
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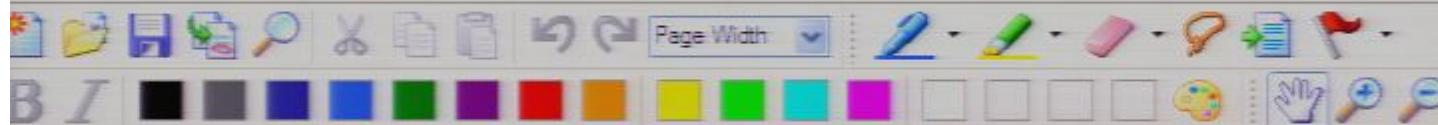


QM:

$$\alpha(t) := \sqrt{\frac{\omega}{2}} \hat{q}(t) + i \frac{1}{\sqrt{2\omega}} \hat{p}(t)$$

PIT:

$$\alpha(t) := \sqrt{\omega_0} \hat{\phi}_0(t) + i \frac{1}{\sqrt{2\omega_0}} \hat{\phi}_0^\dagger(t)$$



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□ Equation of motion and CCRs:

$$\text{QM: } \dot{a}(t) = -i\omega a(t)$$

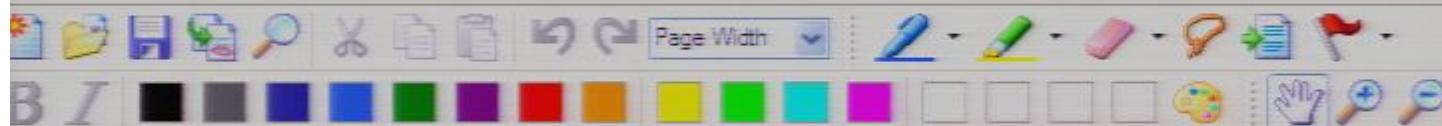
$$[a(t), a^*(t)] = 1$$

Exercise: verify

$$\text{QFT: } \dot{a}_k(t) = -i\omega_k a_k(t)$$

$$[a_k(t), a_{k'}^*(t)] = \delta'(k-k')$$

□ Remark: Valid only while no force and while ω is constant.



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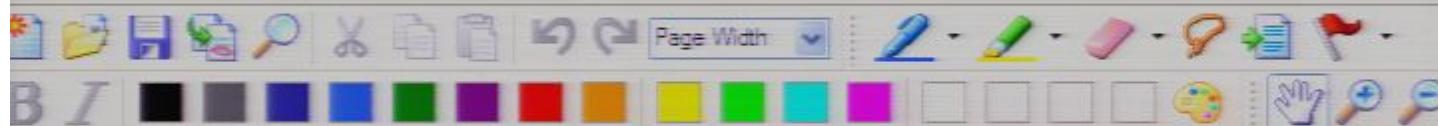
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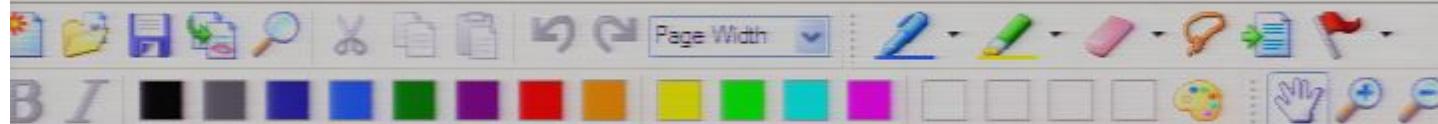
□ Remark: Valid only while no force and while ω is constant.

□ Solution, using an initial condition:

QM:

$$a(t) = e^{-i\omega t} a_0$$

$$[a_0, a_0^*] = 1$$



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Value

□ Equation of motion and CCRs:

QM:

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$$[a(t), a^+(t)] = 1$$

Exercise: verify

QFT:

$$\dot{a}_k(t) = -i\omega_k a_k(t)$$

$$[a_k(t), a_{k'}^+(t)] = \delta^3(k-k')$$

□ Remark: Valid only while no force and while ω is constant.

□ Solution, using an initial condition:

QM:

$$a(t) = e^{-i\omega t} a_{in},$$

$$[a_{in}, a_{in}^+] = 1$$

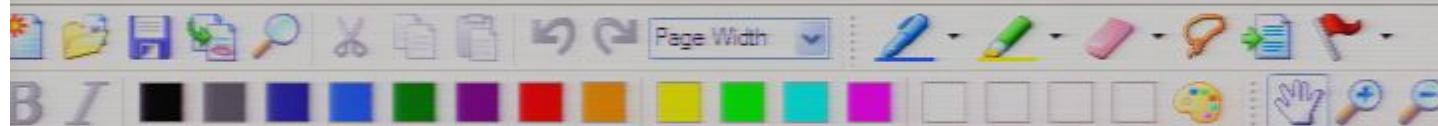
QFT:

$$a_k(t) = e^{-i\omega_k t} a_{in_k},$$

$$[a_{in_k}, a_{in_k}^+] = \delta^3(k-k')$$

□ Explicitly \Rightarrow

$$QM: \quad \hat{a}(t) = \frac{1}{\sqrt{2}} \left(e^{-i\omega t} a_{in} + e^{i\omega t} a_{in}^+ \right)$$



□ Solution, using an initial condition:

QM: $a(t) = e^{-i\omega t} a_{in}$, $[a_{in}, a_{in}^+] = 1$

QFT: $\hat{a}_k(t) = e^{-i\omega_k t} a_{in_k}$, $[a_{in_k}, a_{in_k}^+] = \delta^3(k-k')$

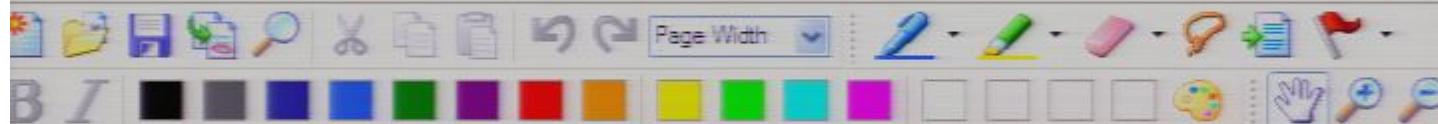
□ Explicitly \Rightarrow

QM: $\hat{q}(t) = \frac{1}{\sqrt{2}} \left(\frac{e^{-i\omega t}}{\sqrt{\omega}} a_{in} + \frac{e^{i\omega t}}{\sqrt{\omega}} a_{in}^+ \right)$

QFT: $\hat{\phi}_k(t) = \frac{1}{\sqrt{2}} \left(\frac{e^{-i\omega_k t}}{\sqrt{\omega_k}} a_{in_k} + \frac{e^{i\omega_k t}}{\sqrt{\omega_k}} a_{in_k}^+ \right)$ (S)

Exercise:
verify

(i.e.: $\hat{\phi}(x, t) = \int \frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_k}} \left(a_{in_k} e^{-i\omega_k t + ikx} + a_{in_k}^+ e^{i\omega_k t - ikx} \right) d\vec{k}$)



□ Solution, using an initial condition:

QM:

$$a(t) = e^{-i\omega t} a_{in}, \quad [a_{in}, a_{in}^+] = 1$$

QFT:

$$a_k(t) = e^{-i\omega_k t} a_{in_k}, \quad [a_{in_k}, a_{in_k}^+] = \delta^3(k-k')$$

□ Explicitly \Rightarrow

QM:

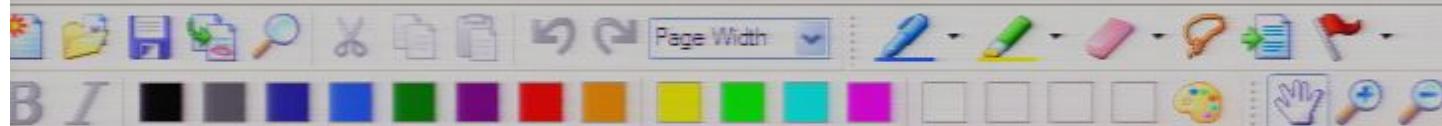
$$q(t) = \frac{1}{\sqrt{2}} \left(\frac{e^{-i\omega t}}{\sqrt{\omega}} a_{in} + \frac{e^{i\omega t}}{\sqrt{\omega}} a_{in}^+ \right)$$

QFT:

$$\hat{\phi}_k(t) = \frac{1}{\sqrt{2}} \left(\frac{e^{-i\omega_k t}}{\sqrt{\omega_k}} a_{in_k} + \frac{e^{i\omega_k t}}{\sqrt{\omega_k}} a_{in_k}^+ \right) \quad (S)$$

Exercise:
verify

i.e.: $\hat{\phi}(x,t) = \int \frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_k}} \left(a_{in_k} e^{-i\omega_k t + ikx} + a_{in_k}^+ e^{i\omega_k t - ikx} \right) d^3 k$



QM:

$$a(t) = e^{-i\omega t} a_{in}, \quad [a_{in}, a_{in}^+] = 1$$

QFT:

$$a_k(t) = e^{-i\omega_k t} a_{in}, \quad [a_{in}, a_{in}^+] = \delta^3(k-k')$$

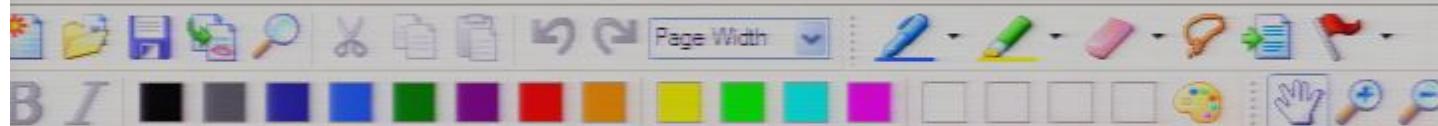
□ Explicitly \Rightarrow

QM: $\hat{q}(t) = \frac{1}{\sqrt{2}} \left(\frac{e^{-i\omega t}}{\sqrt{\omega}} a_{in} + \frac{e^{i\omega t}}{\sqrt{\omega}} a_{in}^+ \right)$

QFT: $\hat{\phi}_k(t) = \frac{1}{\sqrt{2}} \left(\frac{e^{-i\omega_k t}}{\sqrt{\omega_k}} a_{in} + \frac{e^{i\omega_k t}}{\sqrt{\omega_k}} a_{in}^+ \right) \quad (S)$

Exercise:
verify

(i.e.: $\hat{\phi}(x,t) = \int \frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_k}} \left(a_{in} e^{-i\omega_k t + ikx} + a_{in}^+ e^{i\omega_k t - ikx} \right) d\vec{k}$)



QFT:

$$a_k(t) := \sqrt{\frac{\omega_k}{2}} \hat{\phi}_k(t) + i \frac{1}{\sqrt{2\omega_k}} \hat{\pi}_k(t)$$

□ Equation of motion and CCRs:

QM:

$$\dot{a}(t) = -i\omega a(t)$$

$$[a(t), a^\dagger(t)] = 1$$

Exercise: verify

QFT:

$$\dot{a}_k(t) = -i\omega_k a_k(t)$$

$$[a_k(t), a_{k'}^\dagger(t)] = \delta^3(k-k')$$

□ Remark: Valid only while no force and while ω is constant.

□ Solution, using an initial condition:

QM:

$$a(t) = e^{-i\omega t} a_{in},$$

$$[a_{in}, a_{in}^\dagger] = 1$$

QFT:

$$a_k(t) = e^{-i\omega_k t} a_{in_k},$$

$$[a_{in_k}, a_{in_{k'}}^\dagger] = \delta^3(k-k')$$

□ Explicitly \Rightarrow



□ Equation of motion and CCRs:

QM:

$$\dot{a}(t) = -i\omega a(t)$$

$$[a(t), a^+(t)] = 1$$

Exercise: verify

QFT:

$$\dot{a}_k(t) = -i\omega_k a_k(t)$$

$$[a_k(t), a_{k'}^+(t)] = \delta^3(k-k')$$

□ Remark: Valid only while no force and while ω is constant.

□ Solution, using an initial condition:

QM:

$$a(t) = e^{-i\omega t} a_i ,$$

$$[a_i, a_i^+] = 1$$

QFT:

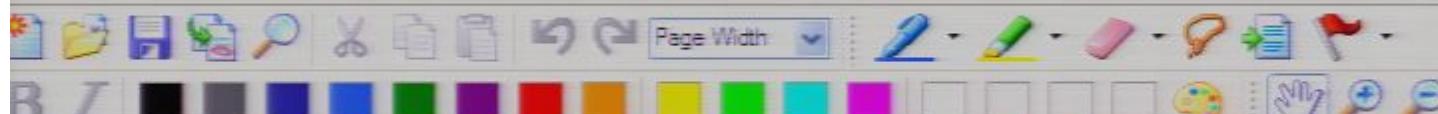
$$a_k(t) = e^{-i\omega t} a_{i,k} ,$$

$$[a_{i,k}, a_{i,k'}^+] = \delta^3(k-k')$$

□ Explicitly \Rightarrow

$$\text{QM: } a(t) = \frac{1}{2} \left(e^{-i\omega t} a_i + e^{i\omega t} a_i^+ \right)$$

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 $\sqrt{2}\omega_0$

□ Equation of motion and CCRs:

QM:

$$\dot{a}(t) = -i\omega a(t)$$

$$[a(t), a^+(t)] = 1$$

QFT:

$$\dot{a}_k(t) = -i\omega_k a_k(t)$$

$$[a_k(t), a_{k'}^+(t)] = \delta^3(k-k')$$

Exercise: verify

□ Remark: Valid only while no force and while ω is constant.

□ Solution, using an initial condition:

QM:

$$a(t) = e^{-i\omega t} a_i ,$$

$$[a_i, a_i^+] = 1$$

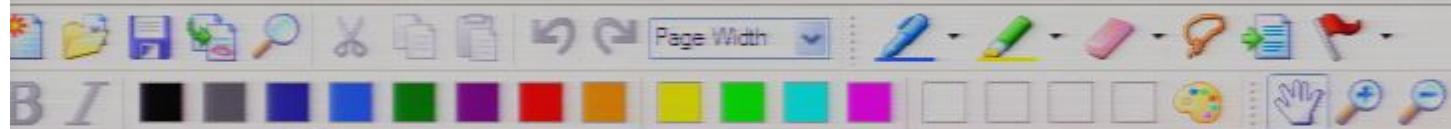
QFT:

$$a_k(t) = e^{-i\omega t} a_{i_k} ,$$

$$[a_{i_k}, a_{i_k}^+] = \delta^3(k-k')$$

□ Explicitly \Rightarrow

$$a(t) = e^{-i\omega t} a_i + \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega' e^{-i\omega t} a_{i_k}$$



□ Remark: Valid only while no force and while ω is constant.

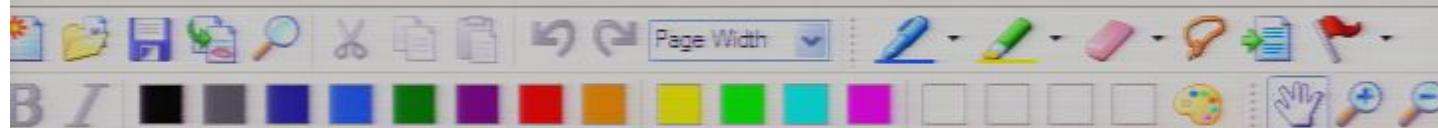
□ Solution, using an initial condition:

QM: $a(t) = e^{-i\omega t} a_{in}$, $[a_{in}, a_{in}^+] = 1$

QFT: $a_k(t) = e^{-i\omega_k t} a_{in_k}$, $[a_{in_k}, a_{in_k}^+] = \delta^3(k-k')$

□ Explicitly \Rightarrow

QM: $q(t) = \frac{1}{\sqrt{2}} \left(\frac{e^{-i\omega t}}{\sqrt{\omega}} a_{in} + \frac{e^{i\omega t}}{\sqrt{\omega}} a_{in}^+ \right)$



QFT:

$$\dot{a}_k(t) = -i\omega_k a_k(t)$$

Exercise: verify

$$[a_k(t), a_{k'}^+(t)] = \delta^3(k-k')$$

□ Remark: Valid only while no force and while ω is constant.

□ Solution, using an initial condition:

QM:

$$a(t) = e^{-i\omega t} a_i , \quad [a_i, a_i^+] = 1$$

QFT:

$$a_k(t) = e^{-i\omega_k t} a_{i,k} , \quad [a_{i,k}, a_{i,k}^+] = \delta^3(k-k')$$

□ Explicitly \Rightarrow

QM:

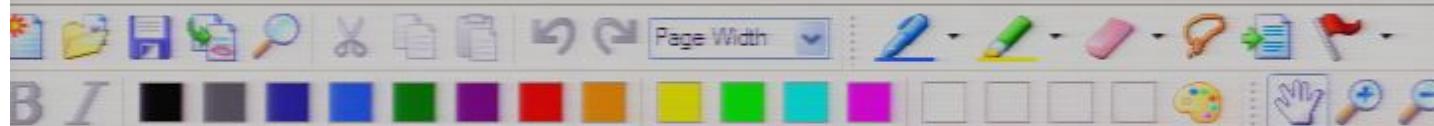
$$q(t) = \frac{1}{\sqrt{2}} \left(\frac{e^{-i\omega t}}{\sqrt{\omega}} a_i + \frac{e^{i\omega t}}{\sqrt{\omega}} a_i^+ \right)$$

QFT:

$$\hat{q}_k(t) = \frac{1}{\sqrt{2}} \left(\frac{e^{-i\omega_k t}}{\sqrt{\omega_k}} a_{i,k} + \frac{e^{i\omega_k t}}{\sqrt{\omega_k}} a_{i,k}^+ \right)$$

(5)

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QFT:

$$\dot{a}_k(t) = -i\omega_k a_k(t)$$

$$[a_k(t), a_{k'}^+(t)] = \delta^3(k-k')$$

Remark: Valid only while no force and while ω is constant.

Solution, using an initial condition:

QM:

$$a_i(t) = e^{-i\omega_i t} a_{i0} ,$$

$$[a_{i0}, a_{i0}^+] = 1$$

QFT:

$$a_k(t) = e^{-i\omega_k t} a_{k0} ,$$

$$[a_{k0}, a_{k'0}^+] = \delta^3(k-k')$$

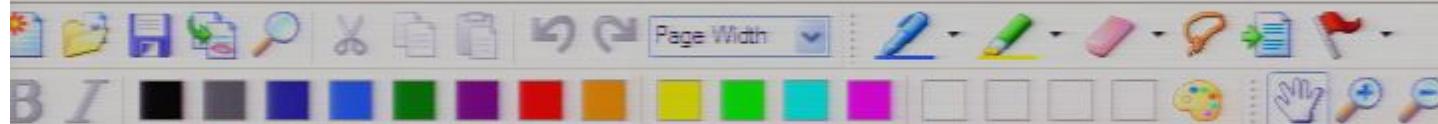
Explicitly \Rightarrow

QM: $q(t) = \frac{1}{\sqrt{2}} \left(\frac{e^{-i\omega_i t}}{\sqrt{\omega_i}} a_{i0} + \frac{e^{i\omega_i t}}{\sqrt{\omega_i}} a_{i0}^+ \right)$

QFT: $\hat{\phi}_k(t) = \frac{1}{\sqrt{2}} \left(\frac{e^{-i\omega_k t}}{\sqrt{\omega_k}} a_{k0} + \frac{e^{i\omega_k t}}{\sqrt{\omega_k}} a_{k0}^+ \right)$

(S)

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QM: $\hat{q}(t) = \frac{1}{\sqrt{2}} \left(\frac{e^{-i\omega t}}{\sqrt{\omega}} a_{in} + \frac{e^{i\omega t}}{\sqrt{\omega}} a_{in}^+ \right)$

QFT: $\hat{\phi}_k(t) = \frac{1}{\sqrt{2}} \left(\frac{e^{-i\omega_k t}}{\sqrt{\omega_k}} a_{in,k} + \frac{e^{i\omega_k t}}{\sqrt{\omega_k}} a_{in,k}^+ \right) \quad (S)$

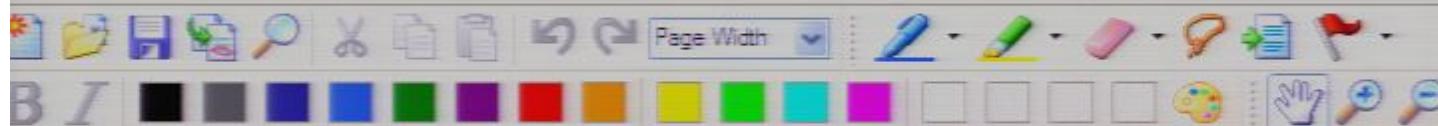
Exercise:
verify

i.e.: $\hat{\phi}(x,t) = \int \frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_k}} \left(a_{in,k} e^{-i\omega_k t + ikx} + a_{in,k}^+ e^{i\omega_k t - ikx} \right) d\vec{k}$

The Hilbert space of states:

- * Analogous to the case of QM, there is a vector, $|0_{in}\rangle \in \mathcal{H}$, which obeys:

a. $|0_{in}\rangle = \theta$. now for all vectors $|$



QFT:

$$a_k(t) = e^{-i\omega_k t} a_{k_0},$$

$$[a_{k_0}, a_{k_0}^+] = \delta^3(k-k')$$

QM: Explicitly \Rightarrow

$$q(t) = \frac{1}{\sqrt{2}} \left(\frac{e^{-i\omega t}}{\sqrt{\omega}} a_{k_0} + \frac{e^{i\omega t}}{\sqrt{\omega}} a_{k_0}^+ \right)$$

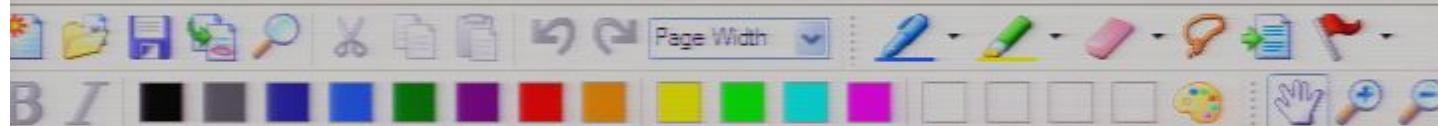
$$QFT: \quad \hat{\phi}_k(t) = \frac{1}{\sqrt{2}} \left(\frac{e^{-i\omega_k t}}{\sqrt{\omega_k}} a_{k_0} + \frac{e^{i\omega_k t}}{\sqrt{\omega_k}} a_{k_0}^+ \right) \quad (S)$$

Exercise:

verify

(i.e.: $\hat{\phi}(x,t) = \int \frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_k}} \left(a_{k_0} e^{-i\omega_k t + ikx} + a_{k_0}^+ e^{i\omega_k t - ikx} \right) dk$)

The Hilbert space of states:



$$\text{QFT: } a_k(t) = e^{-i\omega_k t} a_{in}, \quad [a_{in}, a_{in'}^\dagger] = \delta^3(\mathbf{k}-\mathbf{k}')$$

□ Explicitly \Rightarrow

$$\text{QM: } \hat{q}(t) = \frac{1}{\sqrt{2}} \left(\frac{e^{-i\omega t}}{\sqrt{\omega}} a_{in} + \frac{e^{i\omega t}}{\sqrt{\omega}} a_{in}^\dagger \right)$$

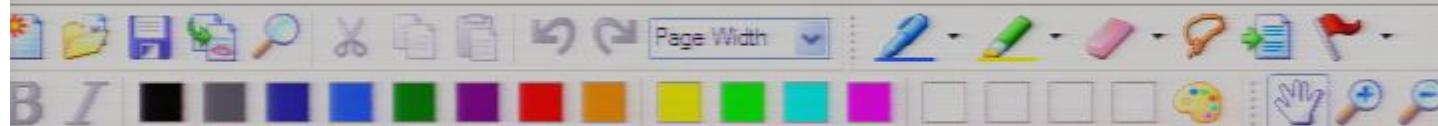
$$\text{QFT: } \hat{\phi}_k(t) = \frac{1}{\sqrt{2}} \left(\frac{e^{-i\omega_k t}}{\sqrt{\omega_k}} a_{in} + \frac{e^{i\omega_k t}}{\sqrt{\omega_k}} a_{in}^\dagger \right) \quad (\text{S})$$

Exercise:

verify

$$\left(\text{i.e.: } \hat{\phi}(x, t) = \int \frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_k}} \left(a_{in} e^{-i\omega_k t + ikx} + a_{in}^\dagger e^{i\omega_k t - ikx} \right) d^3 k \right)$$

The Hilbert space of states:



\square Explicitly \Rightarrow

$$\text{QM: } q(t) = \frac{1}{\sqrt{2}} \left(\frac{e^{-i\omega t}}{\sqrt{\omega}} a_{in} + \frac{e^{i\omega t}}{\sqrt{\omega}} a_{in}^+ \right)$$

$$\text{QFT: } \hat{\phi}_k(t) = \frac{1}{\sqrt{2}} \left(\frac{e^{-i\omega_k t}}{\sqrt{\omega_k}} a_{in_k} + \frac{e^{i\omega_k t}}{\sqrt{\omega_k}} a_{in_k}^+ \right) \quad (S)$$

Exercise:

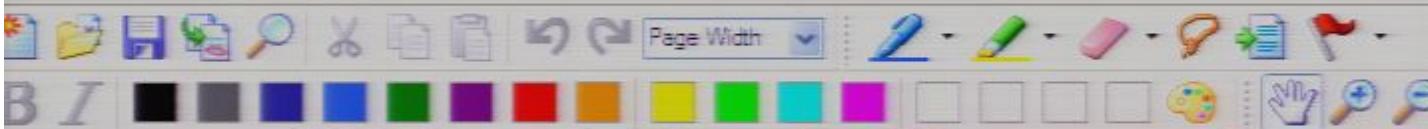
verify

$$\left(\text{i.e.: } \hat{\phi}(x, t) = \int \frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_k}} \left(a_{in_k} e^{-i\omega_k t + ikx} + a_{in_k}^+ e^{i\omega_k t - ikx} \right) dk \right)$$



The Hilbert space of states:

- * Analogous to the case of QM, there is a vector, $|0_m\rangle \in \mathcal{H}$, which obeys:



$$\text{QFT: } \phi_k(t) = \frac{i}{\sqrt{2}} \left(\frac{e^{-i\omega_k t}}{\sqrt{\omega_k}} a_{in,k} + \frac{e^{+i\omega_k t}}{\sqrt{\omega_k}} a_{out,k}^* \right) \quad (5)$$

Exercise:
verify

$$\left(\text{i.e.: } \hat{\phi}(x,t) = \int \frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_k}} \left(a_{in,k} e^{-i\omega_k t + ikx} + a_{out,k}^* e^{i\omega_k t - ikx} \right) d^3k \right)$$

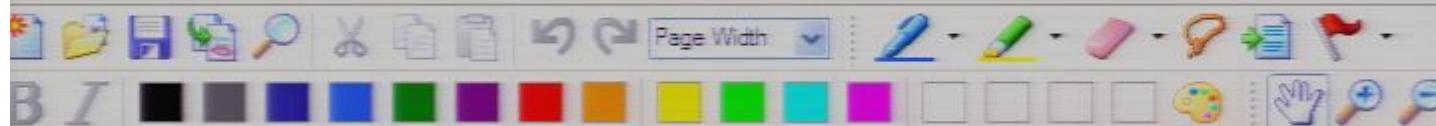
The Hilbert space of states:

- * Analogous to the case of QM, there is a vector, $|0_{in}\rangle \in \mathcal{H}$, which obeys:

$$a_{in,k} |0_{in}\rangle = 0, \text{ now for all vectors } k.$$

- * The Hamiltonian reads (for $t < 0$):

$$\hat{H} = \frac{1}{2} \left(\hat{\pi}_k^* \hat{\pi}_k + \omega^2 \hat{\phi}_k^* \hat{\phi}_k \right) d^3k$$



The Hilbert space of states:



- * Analogous to the case of QM, there is a vector, $|0_m\rangle \in \mathcal{H}$, which obeys:

$$a_{m_k}^{\dagger} |0_m\rangle = 0, \text{ now for all vectors } k.$$

- * The Hamiltonian reads (for $t < 0$):

Exercise: verify →

$$\hat{H} = \frac{1}{2} \int_{\mathbb{R}^3} \pi_{\mathbf{k}}^{\dagger} \pi_{\mathbf{k}} + \omega^2 \phi_{\mathbf{k}}^{\dagger} \phi_{\mathbf{k}} d^3k$$

This \propto is called a "Infrared divergence"

$$= \int_{\mathbb{R}^3} \omega_{\mathbf{k}} (a_{m_{\mathbf{k}}}^{\dagger} a_{m_{\mathbf{k}}} + \frac{1}{2} \delta^3(0)) d^3k$$

This becomes finite, namely $\delta_{\mathbf{k}=0} = 1$ if we use infrared (i.e. box) regularization.

Note: In box, $\hat{H} = \frac{1}{L^{3/2}} \sum_{\mathbf{k}} \omega_{\mathbf{k}} (a_{m_{\mathbf{k}}}^{\dagger} a_{m_{\mathbf{k}}} + \frac{1}{2})$ has still $\sum_{\mathbf{k}} \omega_{\mathbf{k}} \frac{1}{2} = \infty$: "UV divergency".

⇒ We have a harmonic oscillator for each $\mathbf{k} \in \mathbb{R}^3$.

$$|0_{in}\rangle = |0_{in_1}\rangle \otimes |0_{in_2}\rangle \otimes \dots$$

$$|0_{in}\rangle = |0_{in}\rangle_{k_1} \otimes |0_{in}\rangle_{k_2} \otimes \dots$$

$$|\psi\rangle, |\phi\rangle \quad \langle \psi | \phi \rangle = 0.3$$

$$\begin{aligned} |\psi\rangle &= |\alpha\rangle \otimes |\alpha\rangle \otimes |\alpha\rangle \dots \\ &= |\alpha\rangle \otimes |\beta\rangle \dots \end{aligned} \quad \left. \right\} \Rightarrow \langle \psi | \psi \rangle = 0$$

$$|0_m\rangle = |0_{m_1}\rangle \otimes |0_{m_2}\rangle \otimes \dots$$

$$|\alpha\rangle, |\beta\rangle \quad \langle\alpha|\beta\rangle = 0.3$$

$$\begin{aligned} |\phi\rangle &= |\alpha\rangle \oplus |\beta\rangle \quad \dots \\ |\psi\rangle &= |\alpha\rangle \oplus |\beta\rangle \quad \dots \end{aligned} \quad \left. \right\} \Rightarrow \langle\psi|\phi\rangle = 0$$

$$\hat{\phi}_k^+(t) = \hat{\psi}_k \xrightarrow{?} \hat{\phi}(t, x) = \hat{\psi}(x)$$

$$|0_{in}\rangle = |0_{in}\rangle_{\alpha} \otimes |0_{in}\rangle_{\beta} \dots$$

$$|\alpha\rangle, |\beta\rangle \quad \langle\alpha|\beta\rangle = 0.3$$

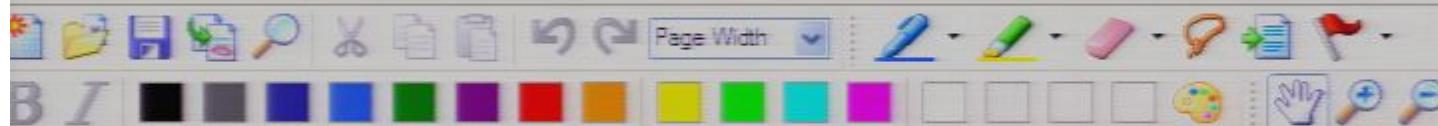
$$\begin{aligned} |\psi\rangle &= |\alpha\rangle \theta_1 |\beta\rangle \theta_2 |\alpha\rangle \dots \\ |\Psi\rangle &= |\beta\rangle \theta_1 |\alpha\rangle \beta \dots \end{aligned} \quad \left. \right\} \Rightarrow \langle\Psi|\psi\rangle$$

$$\hat{\phi}_k^+(t) = \hat{\phi}_k \xrightarrow{?} \hat{\phi}(t, x) = \hat{\phi}(x)$$

$$|0_{in}\rangle = |0_{in}\rangle_k \otimes |0_{in}\rangle_{n_1} \otimes \dots$$

$$|\alpha\rangle, |\beta\rangle \quad \langle\alpha|\beta\rangle = 0.3$$

$$\left. \begin{array}{l} |\psi\rangle = |\alpha\rangle \theta |\alpha\rangle + |\beta\rangle \theta |\beta\rangle \dots \\ |\Psi\rangle = |\alpha\rangle \theta |\beta\rangle \dots \end{array} \right\} \Rightarrow \langle \Psi | \psi \rangle = 0$$



Preview: The case with time-dependent ω :

(will be more important in gravity)

- * Assume that the equation of motion is more general, e.g.:

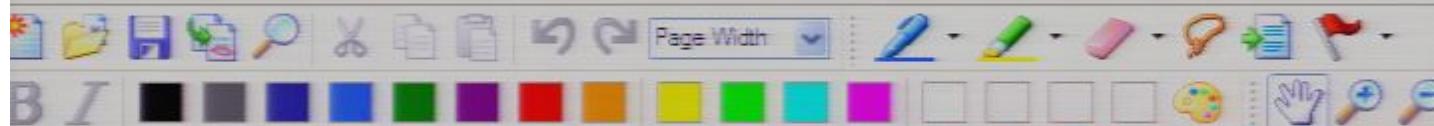
$$\ddot{\phi}_k(t) + \omega_k^2(t) \dot{\phi}_k(t) = 0 \quad (\text{EoM})$$

- * Ansatz:
$$\dot{\phi}_k(t) = \frac{1}{\sqrt{2}} (v_k^+(t) a_{+k} + v_{-k}(t) a_{-k}^*) \quad (\text{A})$$

1 The ansatz guarantees: $\dot{\phi}_k^+(t) = \dot{\phi}_{-k}^-(t)$

□ We use operators $a_{\pm k}$ which obey: $[a_{\pm k}, a_{\pm k'}] = \delta^3(k-k')$

□ Now what properties does $v_k(t)$ need so that $\dot{\phi}_k(t)$ obeys the CCRs between $\dot{\phi}_k(t)$ and $\dot{\pi}_k(t)$ as well as the equation of motion?



$$\phi_k(t) + \omega_k(t) \dot{\phi}_k(t) = 0$$

(EoM)

* Ansatz:

$$\hat{\phi}_k(t) = \frac{1}{\sqrt{2}} (v_k^+(t) a_{+k} + v_{-k}^-(t) a_{-k}^\dagger) \quad (A)$$

(A)

1 The ansatz guarantees: $\hat{\phi}_{k'}(t) = \hat{\phi}_{-k'}(t)$

□ We use operators $a_{\pm k}$, which obey: $[a_{\pm k}, a_{\pm k'}^\dagger] = \delta^3(k-k')$

□ Now what properties does $v_k^-(t)$ need so that $\hat{\phi}_k(t)$ obeys the CCRs between $\hat{\phi}_k(t)$ and $\hat{\pi}_k(t)$ as well as the equation of motion?

* Proposition:

The ansatz (A) solves the equation of motion and



* Ansatz:

$$\hat{\phi}_k(t) = \frac{1}{\sqrt{2}} (v_k^+(t) a_{+k} + v_{-k}(t) a_{-k}^*)$$

(A)

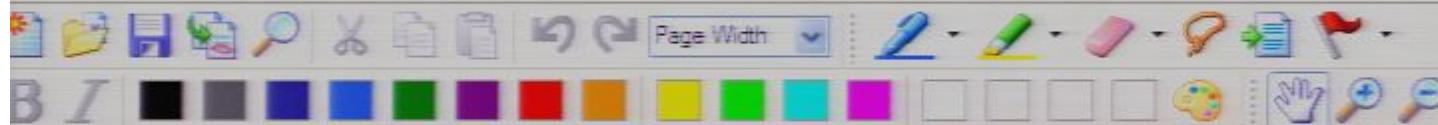
1 The ansatz guarantees: $\hat{\phi}_{n+}^+(t) = \hat{\phi}_{-n}^-(t)$

□ We use operators $a_{\pm k}$, which obey: $[a_{\pm k}, a_{\pm k'}] = \delta^3(k-k')$

□ Now what properties does $v_k(t)$ need so that $\hat{\phi}_k(t)$ obeys the CCRs between $\hat{\phi}_k(t)$ and $\hat{\pi}_k(t)$ as well as the equation of motion?

* Proposition:

The ansatz (A) solves the equation of motion and the commutation relation, if:



* Proposition:

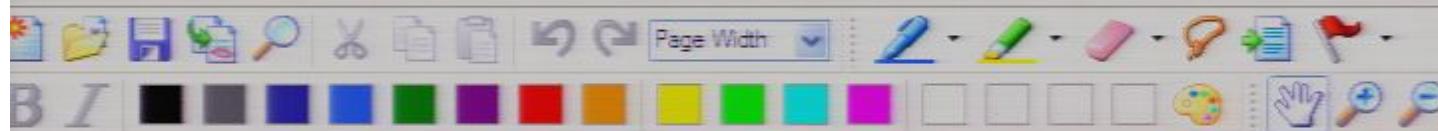
The ansatz (A) solves the equation of motion and the commutation relation, if:

1. The mode function v_n solves the EoM for ϕ_n :

$$\ddot{v}_n(t) + \omega_n^2(t) v_n(t) = 0$$

2. The mode function v_n obeys the so-called "Wronskian condition":

$$v_n^{*}(t) v_n(t) - v_n(t) v_n^{*} = 2i$$



* Ansatz:

$$\hat{\phi}_k(t) = \frac{1}{\sqrt{2}} (v_k^+(t) a_{-k} + v_{-k}(t) a_{-k}^+)$$

(A)

1 The ansatz guarantees: $\hat{\phi}_k^+(t) = \hat{\phi}_{-k}^-(t)$

□ We use operators a_{-k} , which obey: $[a_{-k}, a_{-k'}] = \delta^3(k-k')$

□ Now what properties does $v_k(t)$ need so that $\hat{\phi}_k(t)$ obeys the CCRs between $\hat{\phi}_k(t)$ and $\hat{\pi}_k(t)$ as well as the equation of motion?

* Proposition:

The ansatz (A) solves the equation of motion and the commutation relation, if:



$$\phi_k(t) + \omega_k^*(t) \phi_k(t) = 0 \quad (\text{EoM})$$

* Ansatz:

$$\hat{\phi}_k(t) = \frac{1}{\sqrt{2}} (v_k^*(t) a_{in} + v_{-k}(t) a_{in}^*) \quad (A)$$

1 The ansatz guarantees: $\hat{\phi}_n^+(t) = \hat{\phi}_{-n}(t)$

□ We use operators a_{in} , which obey: $[a_{in}, a_{in}^*] = \delta^3(k-k')$

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* Proposition:

The ansatz (A) solves the equation of motion and

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* Proposition:

The ansatz (A) solves the equation of motion and the commutation relation, if:

1. The mode function v_n solves the EoM for ϕ_n :

$$\ddot{v}_n(t) + \omega_n^2(t) v_n(t) = 0$$

2. The mode function v_n obeys the so-called "Wronskian condition":

$$v_n^*(t) \dot{v}_k(t) - v_k^*(t) \dot{v}_n(t) = 2i$$



* Proposition:

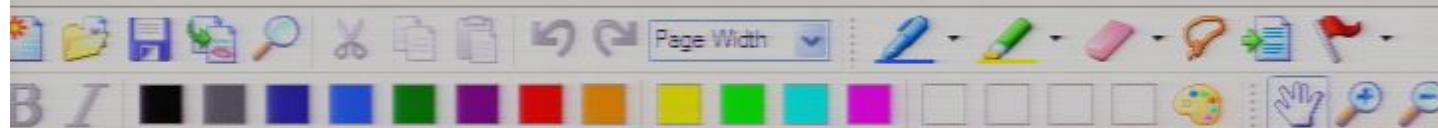
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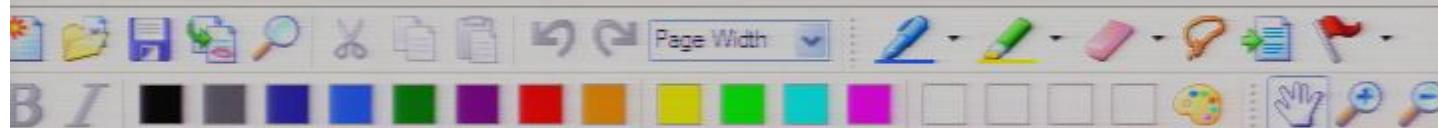
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□ We use operators $a_{\pm k}$, which obey: $[a_{\pm k}, a_{\pm k'}^\dagger] = \delta^3(k-k')$

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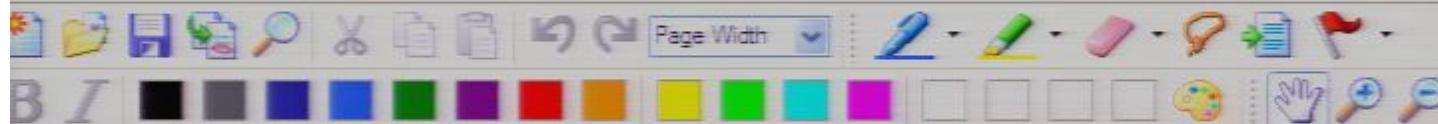
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* Proposition:

The ansatz (A) solves the canonical motion eq.

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* Proposition:

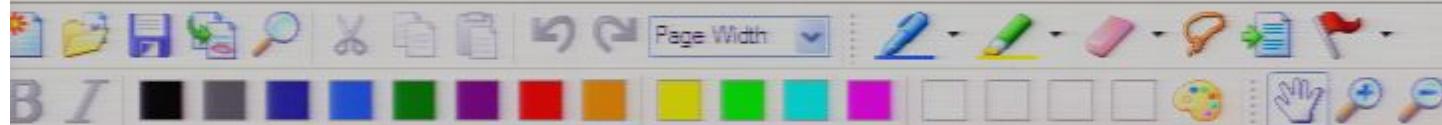
The ansatz (A) solves the equation of motion and the commutation relation, if:

1. The mode function v_k solves the EoM for $\dot{\phi}_k$:

$$\ddot{v}_k(t) + \omega_k^2(t) v_k(t) = 0$$

2. The mode function v_k obeys the so-called "Wronskian condition":

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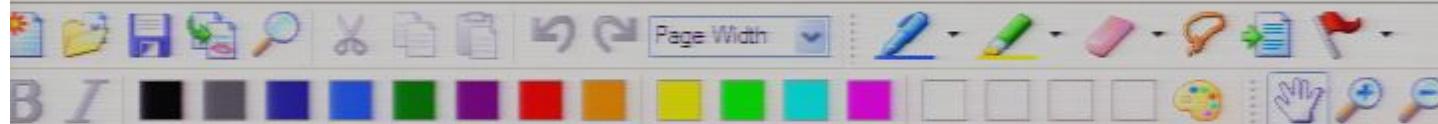
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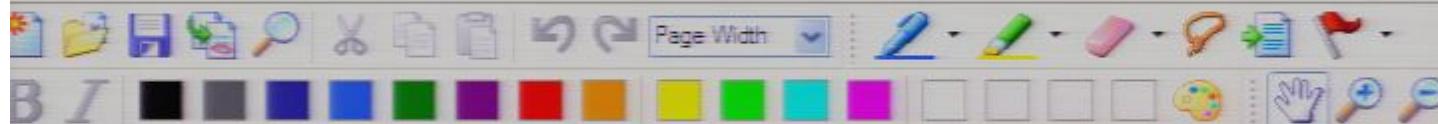
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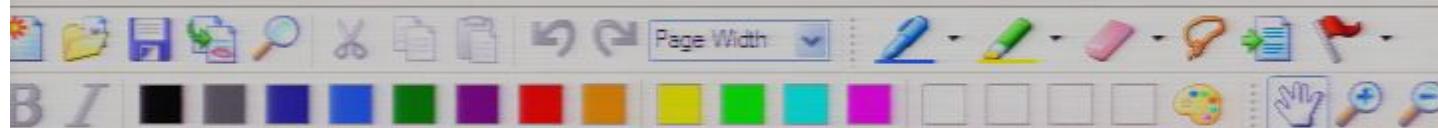
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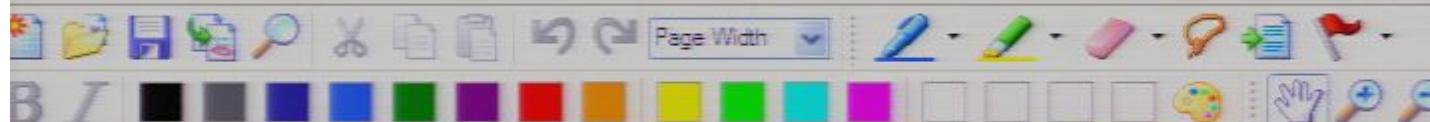
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"Wronskian condition":



- ◻ we use operators a_{ω_k} ~~and~~ ^{in our way:} $L_{a_{\omega_k}, \pi_{\omega_k}, J} = 0 \text{ (EOM)}$
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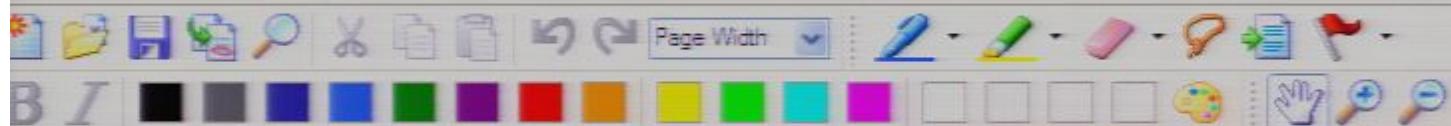
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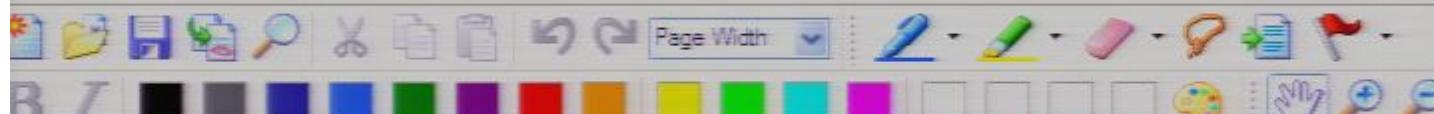


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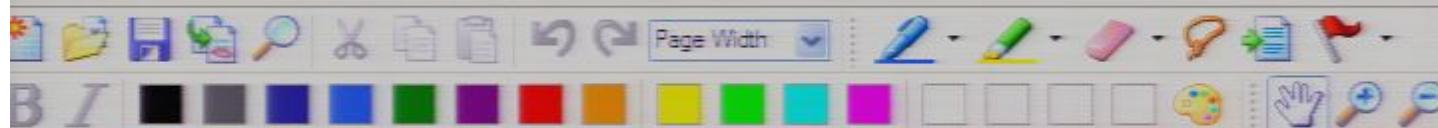
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□ We use operators $a_{\pm k}$, which obey: $[a_{\pm k}, a_{\pm k'}] = \delta^3(k-k')$

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* Proposition:

The ansatz (A) solves the equation of motion and the commutation relation, if:



(will be more important in gravity)

* Assume that the equation of motion is more general, e.g.:

$$\ddot{\phi}_k(t) + \omega_k^2(t) \dot{\phi}_k(t) = 0 \quad (\text{EoM})$$

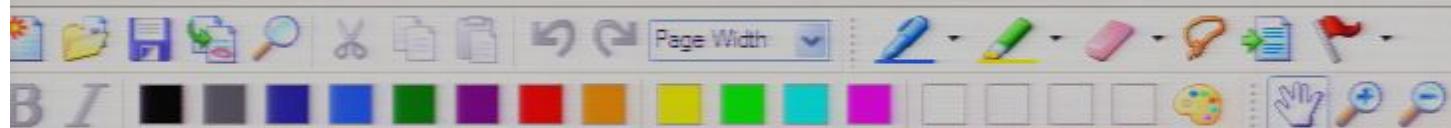
* Ansatz:

$$\dot{\phi}_k(t) = \frac{1}{\sqrt{2}} (v_k^+(t) a_{+k} + v_{-k}(t) a_{-k}^+) \quad (\text{A})$$

1 The ansatz guarantees: $\dot{\phi}_k^+(t) = \dot{\phi}_{-k}(t)$

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□ We use operators a_{in} , which obey: $[a_{in}, a_{in}^d] = \delta'(k-k')$

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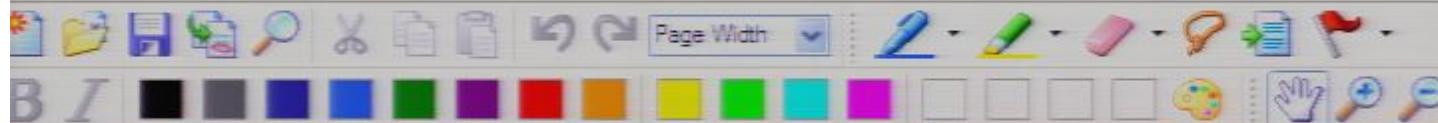


The ansatz (A) solves the equation of motion and the commutation relation, if:

1. The mode function v_k solves the EoM for $\dot{\phi}_k$:

$$\ddot{v}_k(t) + \omega_k^2(t) v_k(t) = 0$$

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the commutation relation, if:

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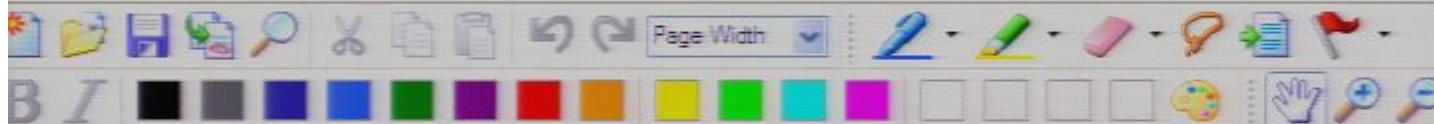
$$\ddot{v}_k(t) + \omega_k^2(t) v_k(t) = 0$$

2. The mode function v_k obeys the so-called
"Wronskian condition":

$$v_k^*(t) \dot{v}_k(t) - v_k(t) \dot{v}_k^* = 2i$$

Proof:

"(1) $\dot{\phi}_k = i \omega_k v_k$ and (2) $\dot{v}_k = -i \omega_k v_k$ "



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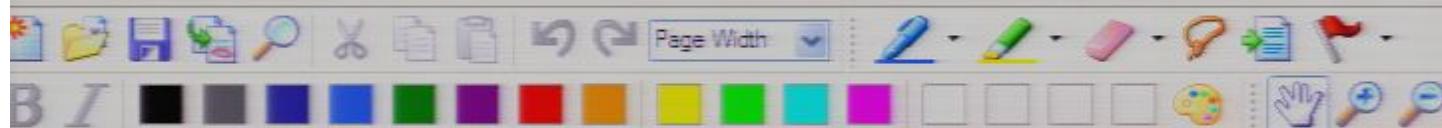
$$\ddot{v}_k(t) + \omega_k^2(t) v_k(t) = 0$$



2. The mode function v_k obeys the so-called
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$$v_k'(t) \dot{v}_k(t) - v_k(t) \dot{v}_k' = 2i$$

* Proof:



* Proof:

□ Check of the equation of motion (EoM):

The EoM is linear in $\dot{\phi}_k(t)$ and the $a_{i_{\text{aux}}}$ are constant.

\Rightarrow If the v_n obey the EoM, so does $\dot{\phi}_n(t)$. ✓

Note: The EoM would not be linear with a $J \neq 0$.

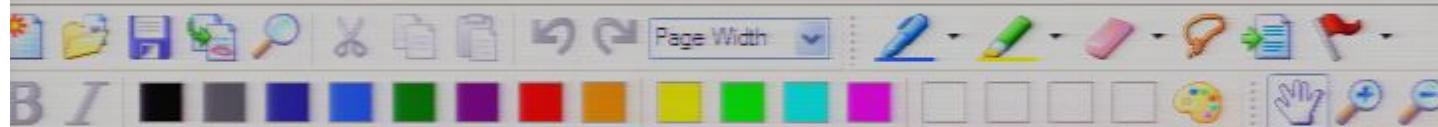
□ Check of the commutation relation:

$$[\dot{\phi}_k(t), \dot{\pi}_{k'}(t)] = [\dot{\phi}_k(t), \dot{\phi}_{k'}(t)]$$

$$= \frac{1}{2} [v_{-k}(t) a_{-k'}^+ + v_k^+(t) a_{-k}, v_{-k'}(t) a_{-k'}^+ + v_{k'}(t) a_{-k'}]$$

$$= \frac{1}{2} (v_k^+(t) v_{k'}(t) - v_{-k}(t) v_{-k'}(t)) \delta^3(k+k') = i \delta^3(k+k')$$

= 2i by commutation condition (note: $k = -k'$)



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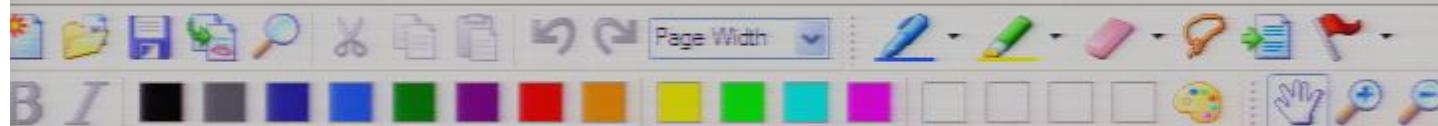
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 &= \frac{i}{2} \left(v_{+k}(t) v_{-k'}(t) - v_{-k}(t) v_{+k'}(t) \right) \delta^3(k+k') = i \delta^3(k+k')
 \end{aligned}$$

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Exercise:

- Show that our solution for $\omega = \text{const}$ is a special case.
- Generally, how large is the set of solutions $v_k(t)$ of both the equation of motion and of the Wronskian condition, i.e., what is the dimension of the



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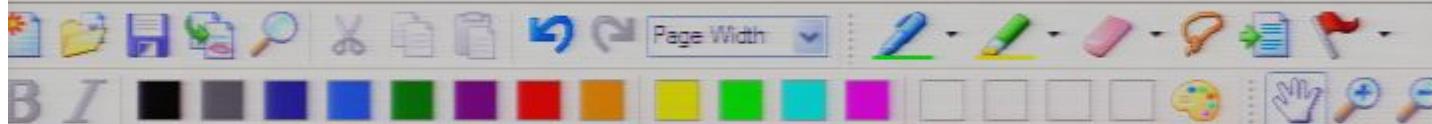
$$= \frac{1}{2} [v_{-k}(t) a_{-k}^\dagger + v_k^\dagger(t) a_{-k}, v_{-k'}(t) a_{-k'}^\dagger + v_{k'}(t) a_{-k'}]$$

$$= \frac{1}{2} (v_k^\dagger(t) v_{k'}(t) - v_{-k}(t) v_{k'}(t)) \delta^3(k+k') = i \delta^3(k+k')$$

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- Show that our solution for $\omega = \text{const}$ is a special case.
- Generally, how large is the set of solutions $v_k(t)$ of both the equation of motion and of the Wronskian condition, i.e., what is the dimension of the solution space?

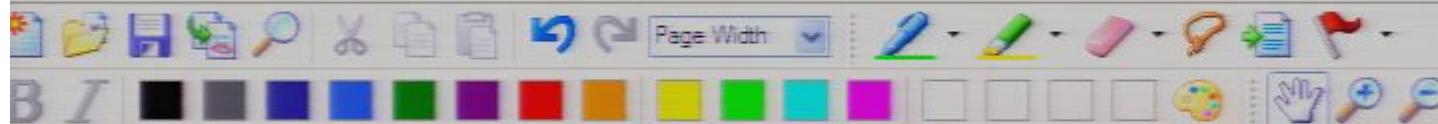


$$= \frac{1}{2} \left(v_k^+(t) \dot{v}_{k'}^-(t) - v_{k'}^-(t) \dot{v}_k^+(t) \right) \delta^3(k+k') = i \delta^3(k+k')$$

$= 2i$ by Wronskian condition (note: $k = -k'$)

Exercise:

- Show that our solution for $w = \text{const}$ is a special case.
- Generally, how large is the set of solutions $v_k^+(t)$ of both the equation of motion and of the Wronskian condition, i.e., what is the dimension of the solution space?



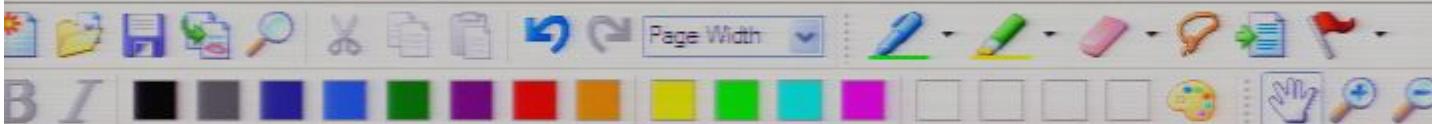
□ Check of the commutation relation:

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 &= \frac{i}{2} [v_{-k}(t) a_{-k}^+ + v_k^*(t) a_{-k}, i v_{k'}(t) a_{-k'}^+ + i v_{k'}^*(t) a_{-k'}] \\
 &= \frac{i}{2} (v_k^*(t) v_{-k'}(t) - v_{-k}(t) v_{k'}^*(t)) \delta^3(k+k') = i \delta^3(k+k')
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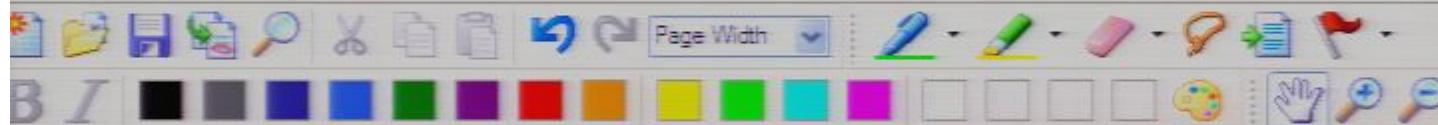
$$= \frac{1}{2} (v_k^+(t) v_{-k'}(t) - v_{-k}(t) v_{k'}^+(t)) \delta^3(k+k') = i \delta^3(k+k')$$

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□ Check of the equation of motion (EoM):

The EoM is linear in $\dot{\phi}_k(t)$ and the a_{ik} are constant.

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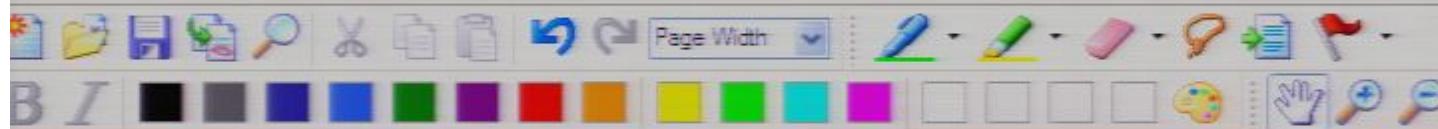
Note: The EoM would not be linear with a $J \neq 0$.

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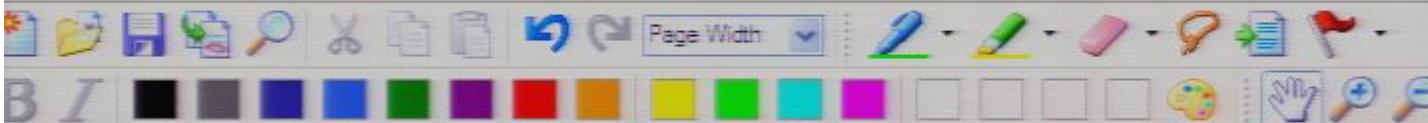
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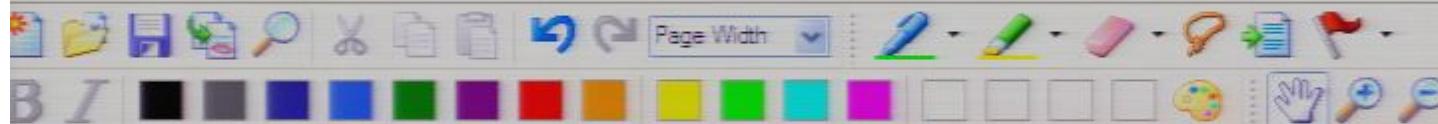


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* Proposition:

The ansatz (A) solves the equation of motion and the commutation relation, if:

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$$\ddot{v}_k(t) + \omega_k^2(t) v_k(t) = 0$$

2. The mode function v_k obeys the so-called "Wronskian condition":

$$v_k^*(t) \dot{v}_k(t) - v_k(t) \dot{v}_k^* = 2i$$

