Title: Foundations and Interpretation of Quantum Theory - Lecture 10

Date: Feb 23, 2010 02:30 PM

URL: http://pirsa.org/10020013

Abstract: After a review of the axiomatic formulation of quantum theory, the generalized operational structure of the theory will be introduced (including POVM measurements, sequential measurements, and CP maps). There will be an introduction to the orthodox (sometimes called Copenhagen) interpretation of quantum mechanics and the historical problems/issues/debates regarding that interpretation, in particular, the measurement problem and the EPR paradox, and a discussion of contemporary views on these topics. The majority of the course lectures will consist of guest lectures from international experts covering the various approaches to the interpretation of quantum theory (in particular, many-worlds, de Broglie-Bohm, consistent/decoherent histories, and statistical/epistemic interpretations, as time permits) and fundamental properties and tests of quantum theory (such as entanglement and experimental tests of Bell inequalities, contextuality, macroscopic quantum phenomena, and the problem of quantum gravity, as time permits).

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Generalized Probabilistic or "Operational" Approach

- Whatever else they are, quantum states are compendia of probabilities for the outcomes of all measurements we might make on them. And quantum effects (POVM elements) are compendia of their probabilities in all states they might occur on. Study more general theories in such a states/effects framework.
- Some think a fundamental theory should not be about probabilities of measurement outcomes. We may not know what QM is "really about" or how it "should" be formulated, but studying the structure of these probabilities may help us understand it. Maybe QM is a theory for how small subsystems of the universe look to other, somewhat larger, subsystems...

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Research program: study information processing in general probabilistic theories

What?

Characterize quantum and classical theories within broad framework of "foil theories"...
...in terms of flow and processing of information.

Why?

From pragmatism...

- *Conceptual* understanding of info processing: principles ↔ tasks
  - ...help develop protocols
  - ...understand *limits* to QIP ...
  - ...model info in other complex / concurrent systems?

...to hubris

- Information the essence of quantum physics? ...analogue of Einstein's principle-based accounts of spec/gen relativity?

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Information processing protocols and primitives and the structure of physical theories

Howard Barnum

LANL; soon, Perimeter Institute

July 12, 2009 / 2nd FQXI Conference

Information Sciences Group/CCS-3 (LANL)
Aug. 2009 – Aug. 2010: Perimeter Institute for Theoretical Physics, Waterloo Ontario hnbarnum@aol.com

Collaborators: J. Barrett, DAMTP, University of Cambridge; O. Dahlsten, ETH Zürich; M. Leifer, CQC, University of Waterloo; B. Toner, CWI, Amsterdam; A. Wilce, University of Susquehanna

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Barnum (LANL; soon, Perimeter Institute) Information and 2 / 21

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Birth of quantum mechanics: an informational break with classical physics

Radical principles underlying quantum information processing recognized by QM's founders

- Measurement disturbs state (Bohr, Heisenberg)
- Entanglement (Schrödinger: "The best possible knowledge of a total system does not necessarily include total knowledge of all its parts, not even when these are fully separated from each other and at the moment are not influencing each other at all.")

Same principles now viewed as underlying QIP's power (e.g. QKD)

Possibly, *many* illuminating characterizations/axiomatizations. Existing characterizations or partial characterizations

- Hardy 2001, D'Ariano recently, Alfsén-Shultz, Araki 1980, Quantum logical

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Rough overview of convex operational formalism

- Systems $A, B, C...$
- Convex set $\Omega_A, \Omega_B...$ of states (for each system)
- Convex sets of measurement outcomes $[0, u_A]$.
- Bilinear map: states $\times$ outcomes $\rightarrow$ *probabilities*.
- Convex set of allowable dynamics taking states to states, $\mathscr{D}_A$.
- Perhaps: way(s) of making "composite" systems, or of recognizing compositeness: $C = A \otimes B$

(Looks a bit categorical!)

Main Results

- A set of states is clonable (independent copies) if and only if the states are perfectly distinguishable. (BBLW)
- A set of states is broadcastable (possibly correlated copies) iff it is in the convex hull of a set of clonable states, i.e. in a *classical subset* of states. (BBLW)
- The only information that can be obtained without disturbance is *intrinsically classical* information (information about which "superselection sector" a state is in). (BBLW)
- Exponentially secure bit commitment is possible in any non-classical theory that does not have entanglement. (BDLT)
- Necessary conditions for conclusive teleportation (BBLW)
- Sufficient conditions for deterministic teleportation (BBLW)
- Conditions for "ensemble steering" (generalized Hughston-Jozsa-Wootters Theorem)

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7 of 22

The Convex Operational Framework for (Generalized) Probability Theory

Howard Barnum and (maybe!) Alex Wilce

PI and Susquehanna University

Perimeter Institute and IQC, February 23 & 25 2010

Outline

TALK I: The Framework

(1) Introduction and motivation
(2) Test spaces
(3) Background on convexity
(4) Abstract state spaces
(5) Digression: Symmetric cones and Jordan algebras
(5) Coupled Systems, non-signaling states and entanglement

TALK II: Constraints

(1) Stock-taking
(2) Conditioning and remote evaluation
(3) Teleportation and entanglement swapping
(4) Weak self-duality and Homogeneity
(5) The measurement problem (time permitting)
(6) Hidden variables (time permitting)

Two ways to be puzzled by QM

Altitude

We're going to focus on the second point of view. We begin by surveying formal possibilities from some altitude – i.e., we'll start with a very general, and conceptually innocent, version of probability theory – broad enough to encompass quantum and classical PTs as special cases. Then add constraints ("non-signaling", "weakly self-dual", ...), hoping to single out QM.

Two ways to be puzzled by QM

- QM is linear dynamical theory with a familiar mathematical apparatus, but a mysterious probabilistic interpretation.
- QM is a conservative extension of classical probability theory, with a relatively unproblematic interpretation, but a mysterious mathematical apparatus.

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(Ancient) History

- Roots in von Neumann's work

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- Explicit in Mackey's *Mathematical foundations of QM*, 1963:

  "Ideally, one would like to have a short list of physically plausible assumptions from which one could deduce [the structure of QM]. Short of this one would like a list from which one could deduce a set of possibilities for the structure ... all but one of which could be show to be inconsistent with suitably planned experiments."

- Most of the literature on "quantum logic" revolves around this idea (subject for another talk!)

(Modern) History

Over the past 5-10 years, this "convex operational" framework has been revived (in some cases, rediscovered) by workers in foundational areas of quantum information theory.

Characteristic of recent research:

(Modern) History

Over the past 5-10 years, this "convex operational" framework has been revived (in some cases, rediscovered) by workers in foundational areas of quantum information theory.

Characteristic of recent research:

- Focus on compound systems,
- Focus on *finite dimensional* systems (at least initially)

Some successes

- Extension of signature QIT results to much more general settings – no-cloning, no-broadcasting, teleportation, and more (Barrett, BBLW, others)

Operational Infrastructure (Scaffolding?)

Definition: A **test space** is a collection $\mathfrak{A}$ of non-empty sets, called *tests*, understood as the outcome-sets of various "measurements".

Notation and Jargon:

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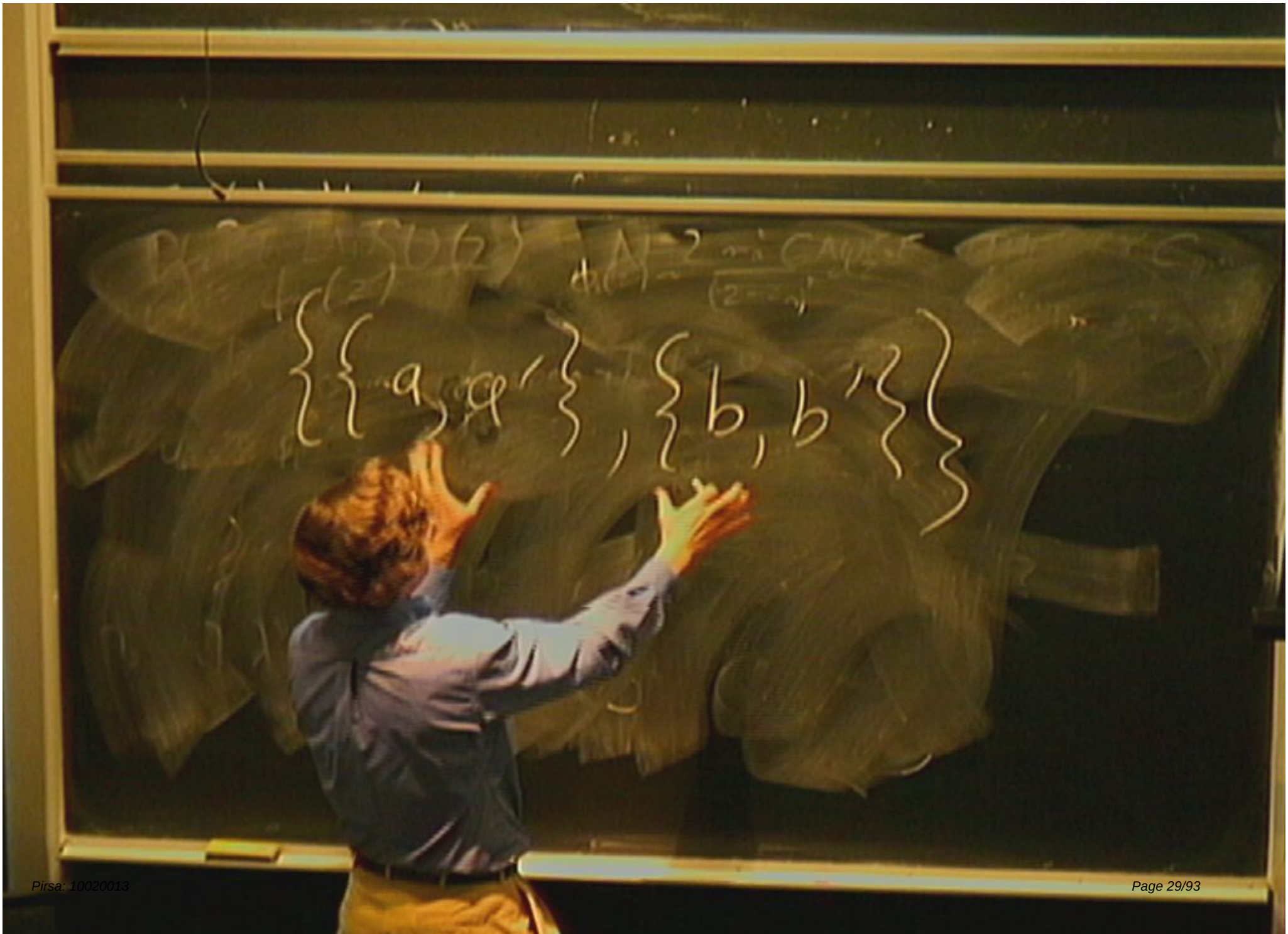
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$$\{\{a, a'\}, \{b, b'\}\}$$

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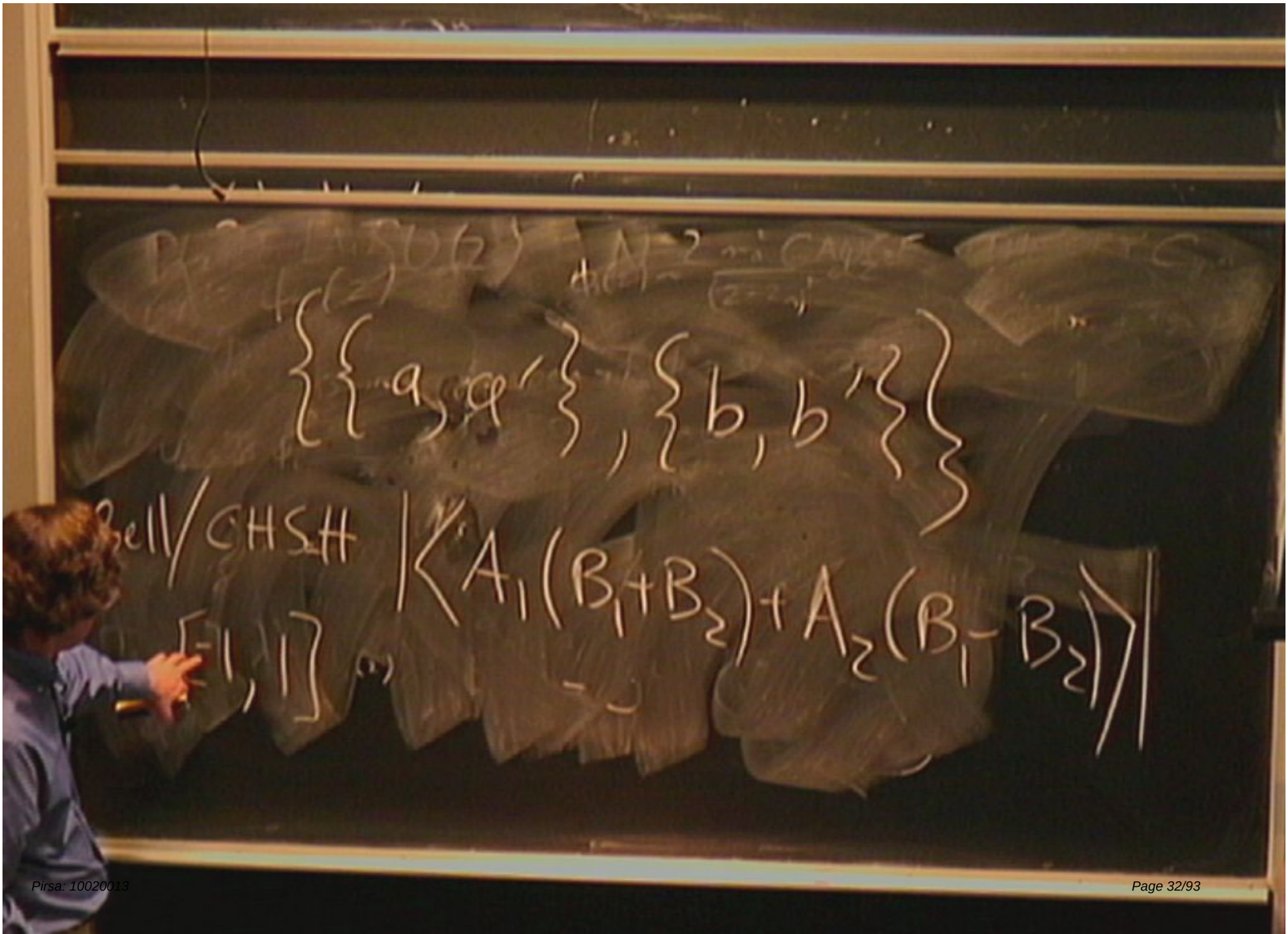
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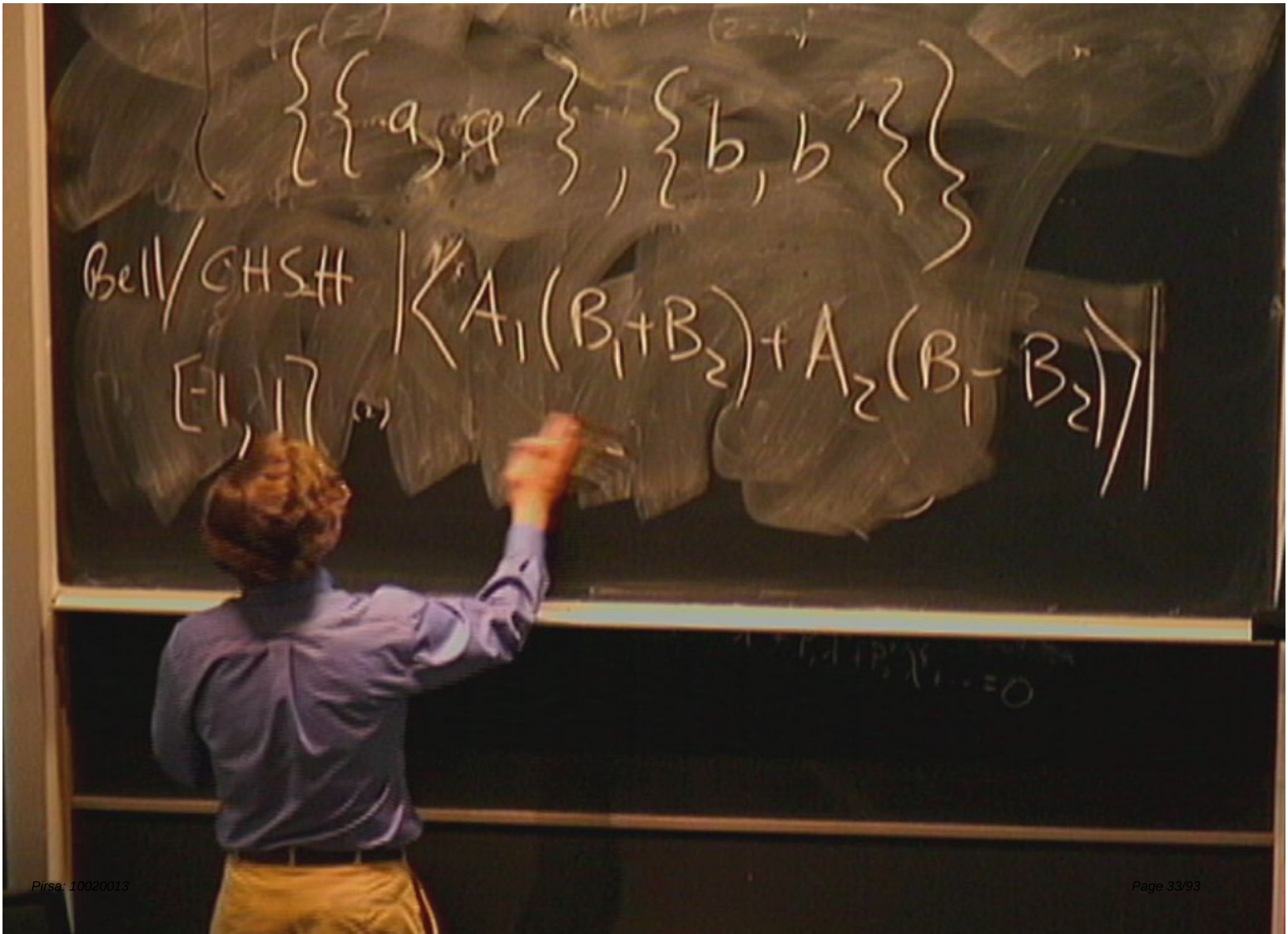
|⟨A1(B1+B2)+A2(B1-B2)⟩|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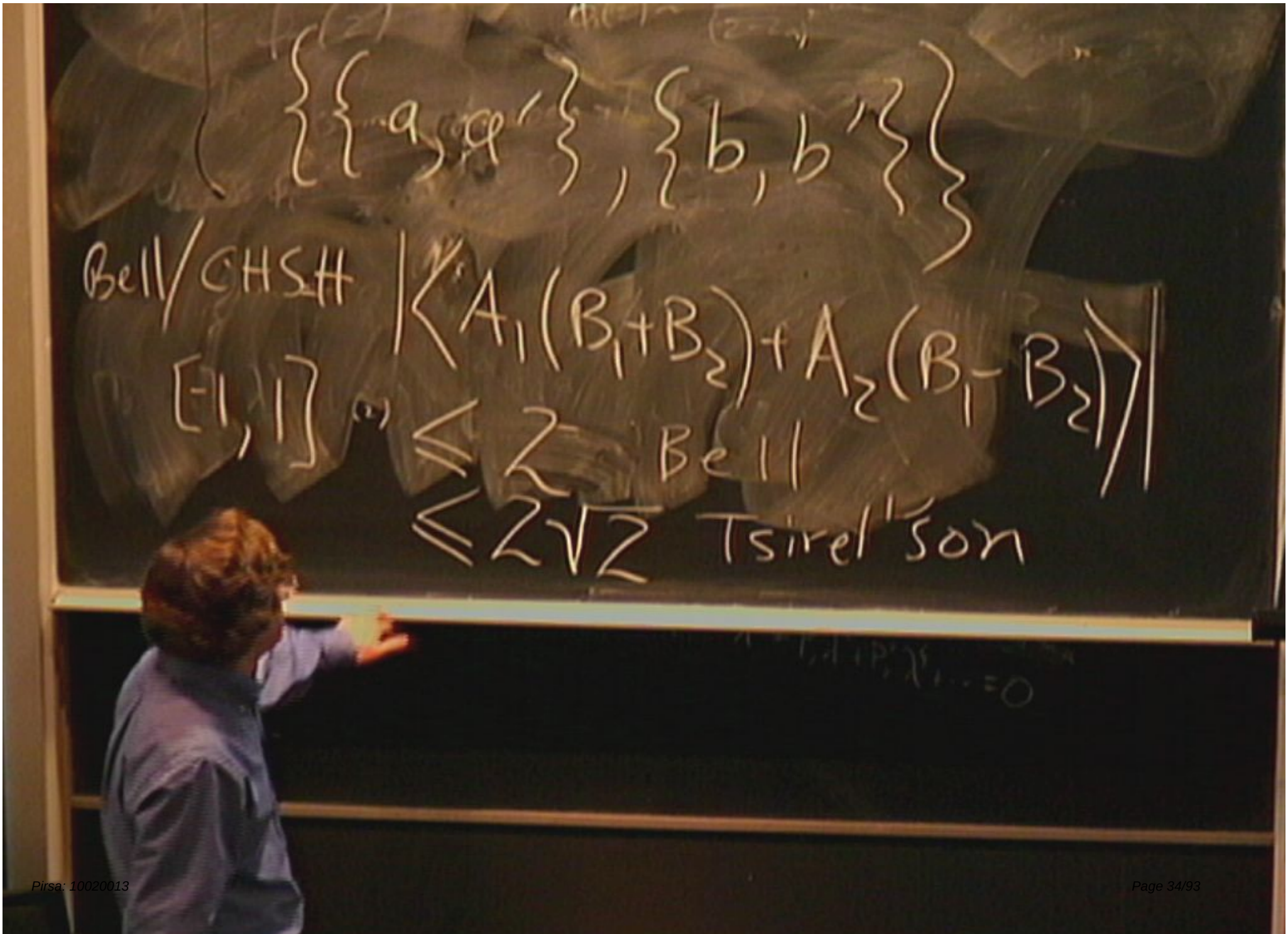
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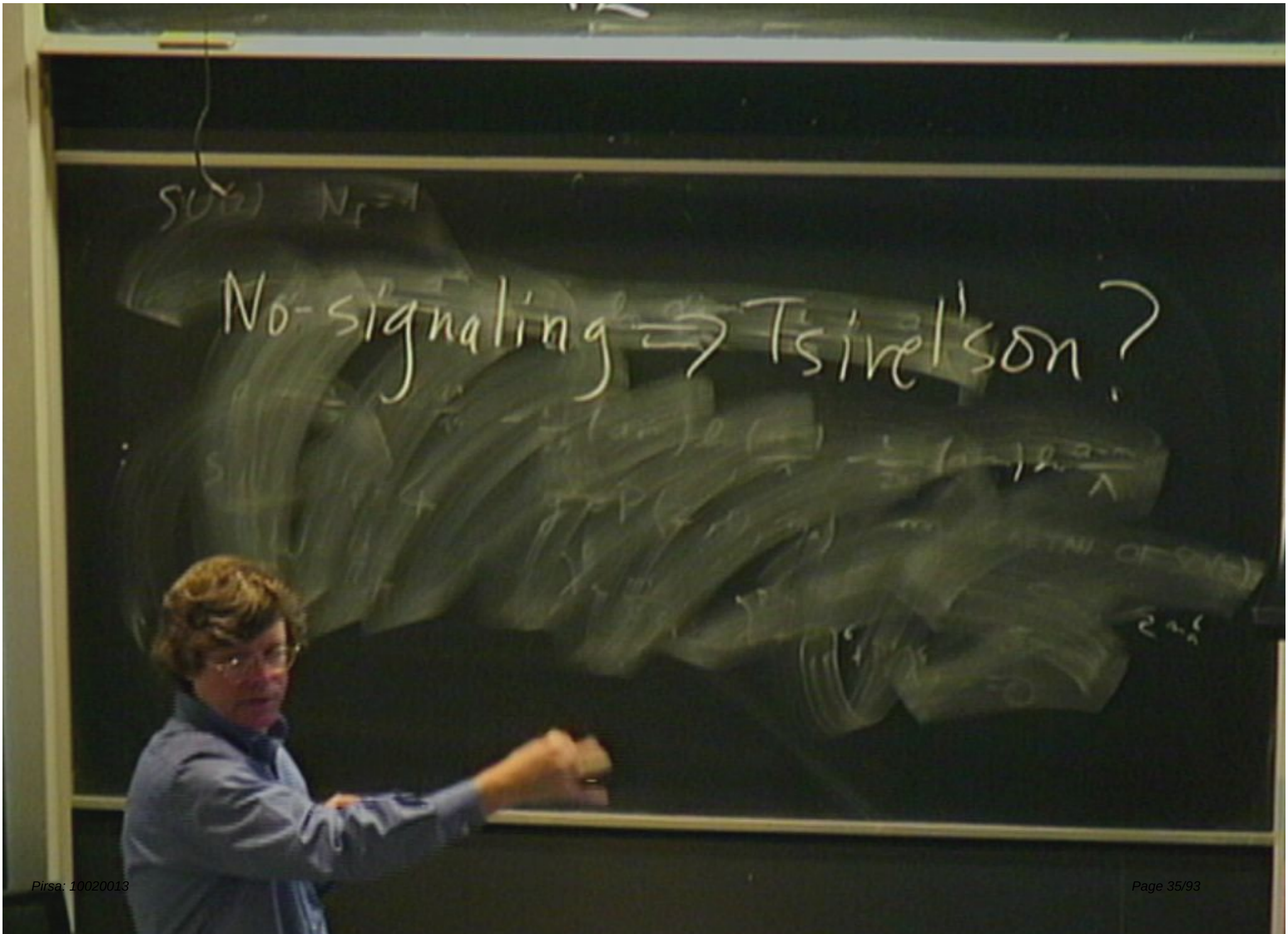
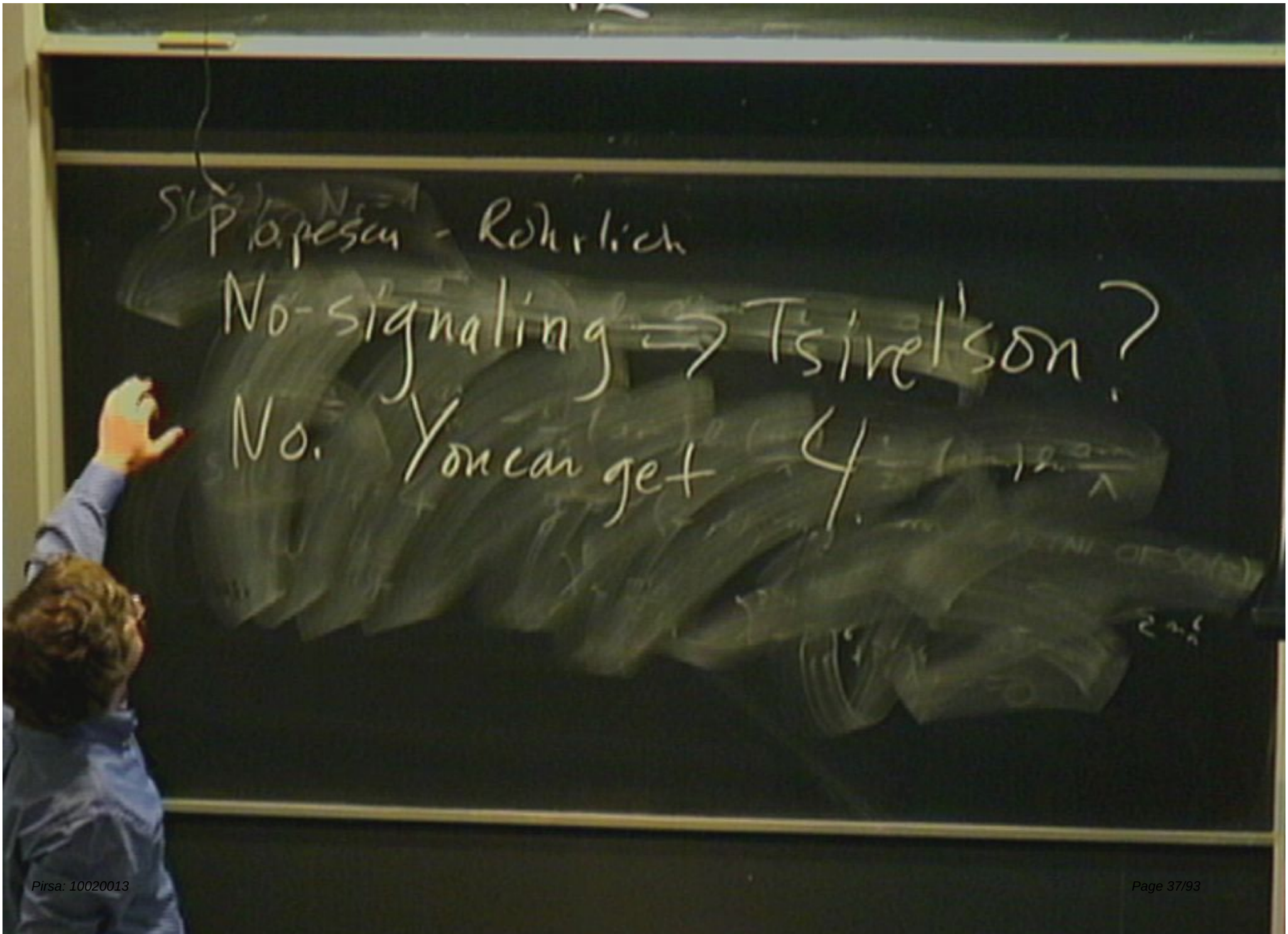

No-signaling ⇒ Tsirel'son?

P/R

No-signaling $\Rightarrow$ Tsirel'son?

No. You can get 4.


Popescu - Rohrlich
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Classical: Let $\mathfrak{A} = \{E\}$ where E is a finite set, $\Omega = \Delta(E)$, the set of all probability weights on E.

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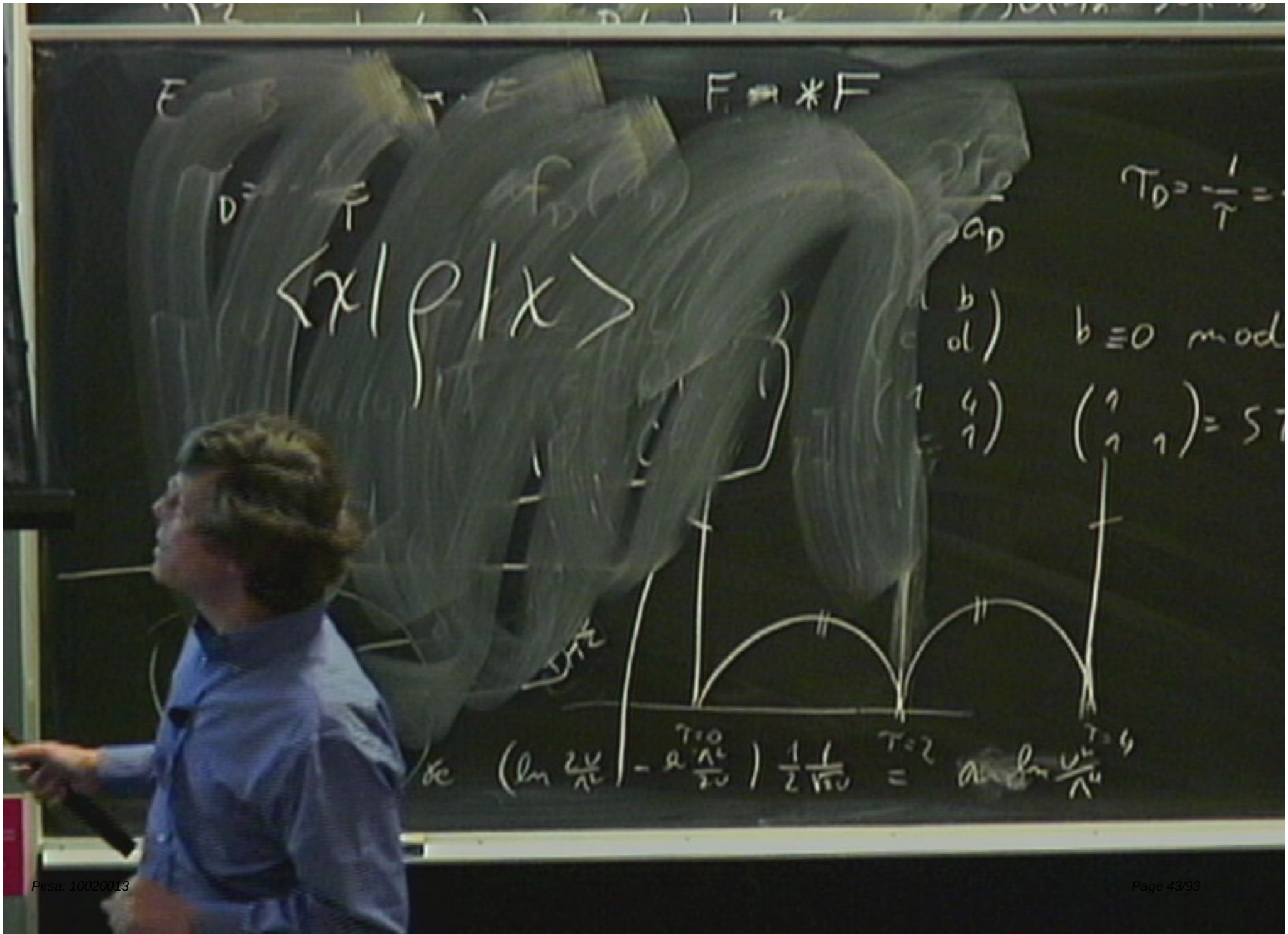
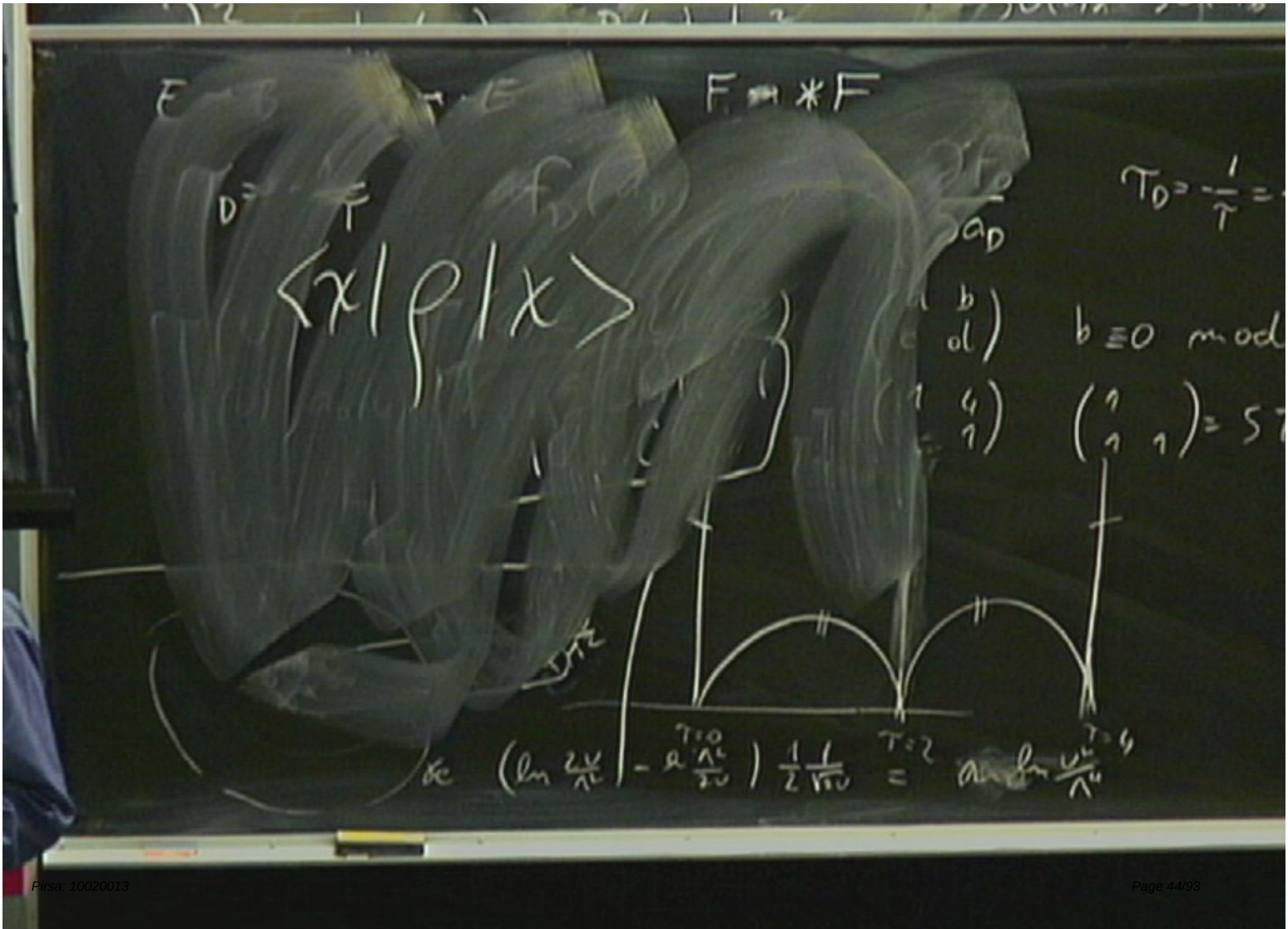
Quantum: Let $\mathfrak{A} = \mathfrak{F}(\mathbf{H})$, the set of orthonormal bases for a Hilbert space $\mathbf{H}$, $\Omega = \Omega(\mathbf{H})$ states of the form $\alpha(x) = \langle \rho x, x \rangle$, ρ a density operator on $\mathbf{H}$. (Gleason's Theorem tells us that all states have this form, if $\dim \mathbf{H} > 2$.)


F = *F
$\langle x | \rho | x \rangle$
a_D
$b \equiv 0$ mod
T=0
T=2


F = *F
$\langle x | \rho | x \rangle$
$\tau_D = -\frac{1}{T} =$
a_D
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$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = 5$
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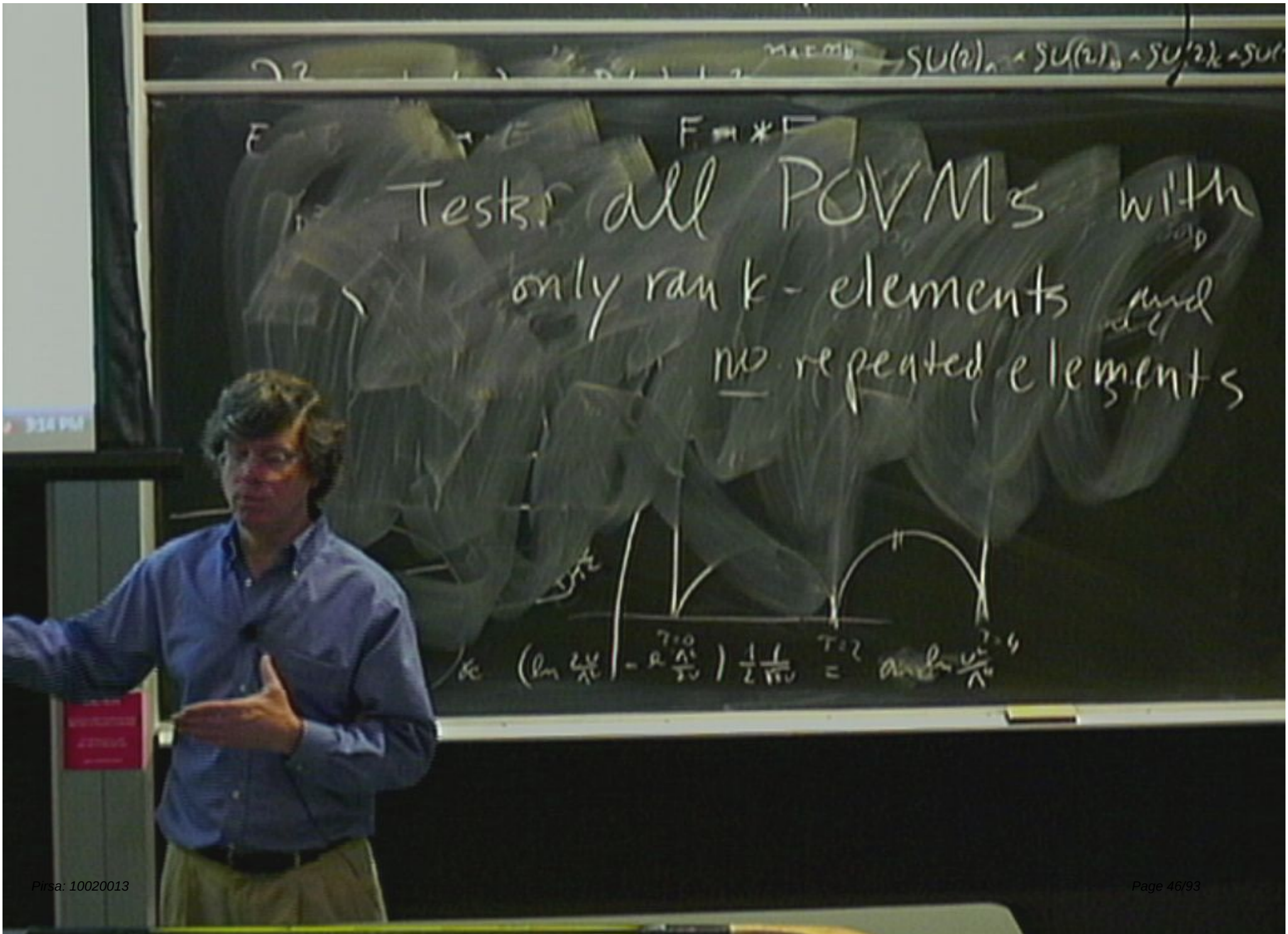
Tests: all POVMs with
only rank-1 elements
no repeated elements

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Common Features

Classical and Quantum examples actually have a lot in common:

- For every outcome x, there is a *unique* state ε_x with $\varepsilon_x(x) = 1$.

Entomological Example

Firefly box goes here....

Note this lacks both features.

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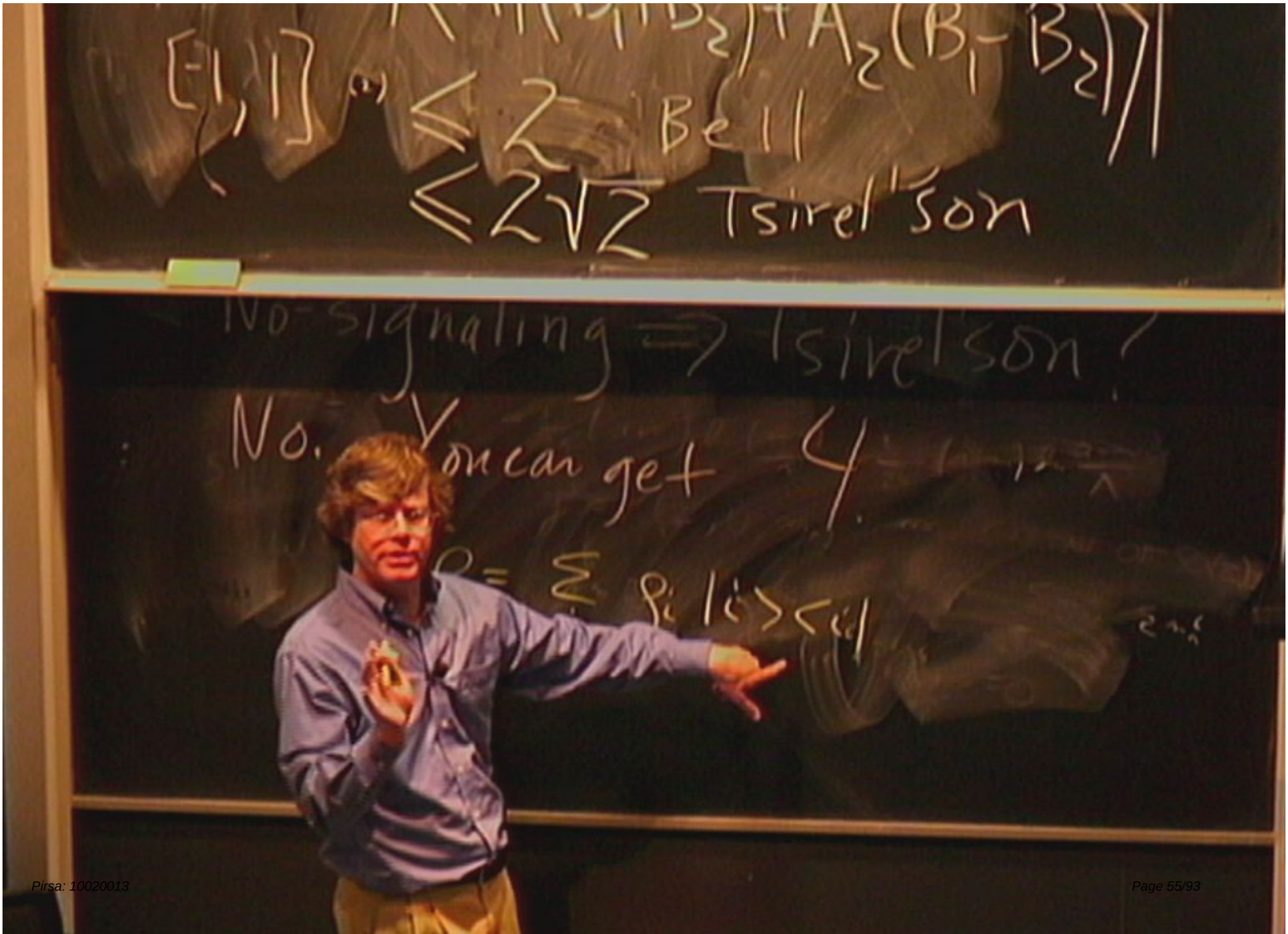
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No. You can get 4
ρ = Σ pi |i><i|

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$\leq 2\sqrt{2}$ Tsirel'son

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No. You can get 4.

$\rho = \sum_i \rho_i |i\rangle\langle i|$

From Operational to Convex

The state space $\Omega(\mathfrak{A})$ is a *convex* subset of $\mathbb{R}^X$:

$$\alpha, \beta \in \Omega \Rightarrow t\alpha + (1-t)\beta \in \Omega \; \forall t \in [0,1].$$

Observation: If all tests $E \in \mathfrak{A}$ are finite, $\Omega(\mathfrak{A})$ is closed, *hence, compact*, in $[0,1]^X$.

Fact [Shultz, 1967]: Any compact convex set can be represented as $\Omega(\mathfrak{A})$ for a suitable locally finite test space $\mathfrak{A}$.

Much (most?) of the probabilistic machinery associated with $\mathfrak{A}$ "lives" in the convex geometry of $\Omega(\mathfrak{A})$. For many purposes, we can discard $\mathfrak{A}$ altogether.

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$$\omega = \sum_i \lambda_i \omega_i$$

extremal

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extremal

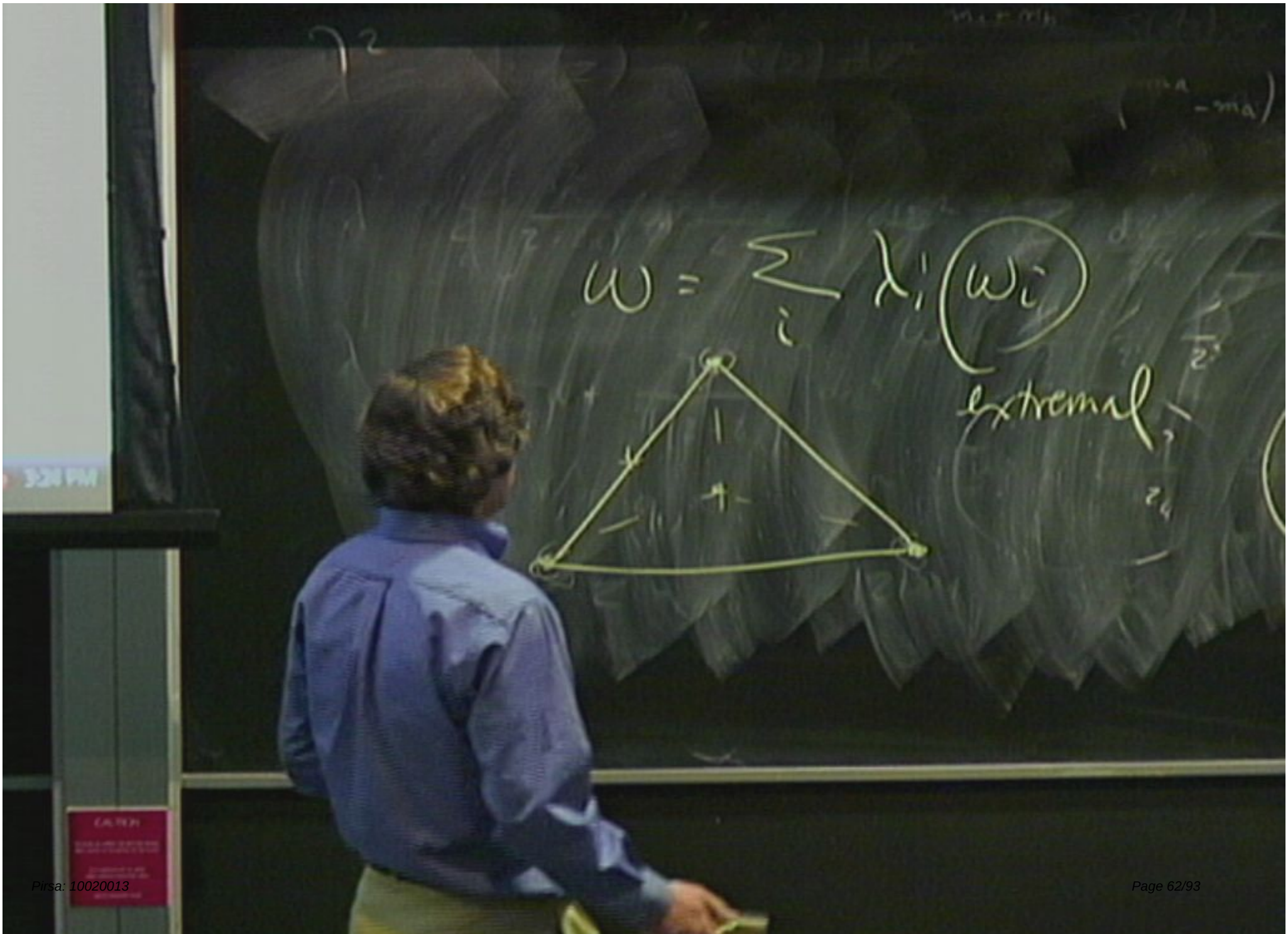
$[-1,1] \quad \leq 2$ Bell

$\leq 2\sqrt{2}$ Tsirel'son

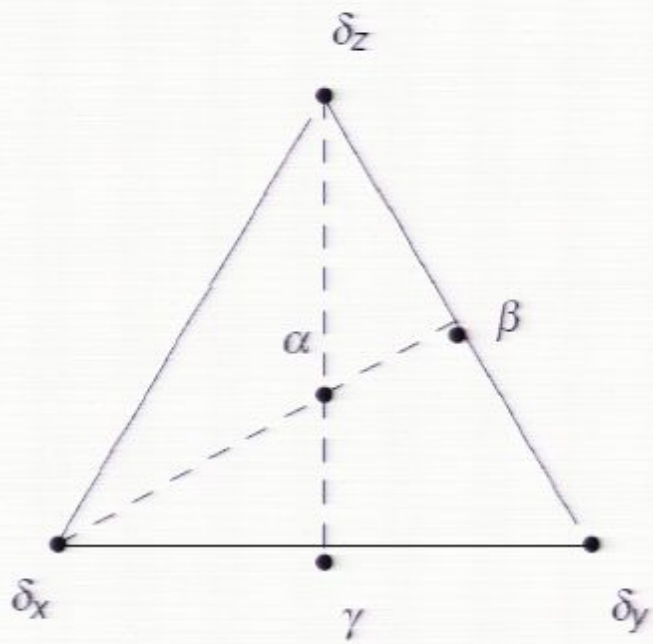
Simplex in $\mathbb{R}^d$

convex hull of $d+1$ points

(or fewer)


$\omega = \sum_i \lambda_i \omega_i$
extremal

Geometrically, the state space $\Delta(E)$ of a classical test space $\mathfrak{A} = \{E\}$ is a *simplex*. For $E = \{x, y, z\}$, this is a triangle:



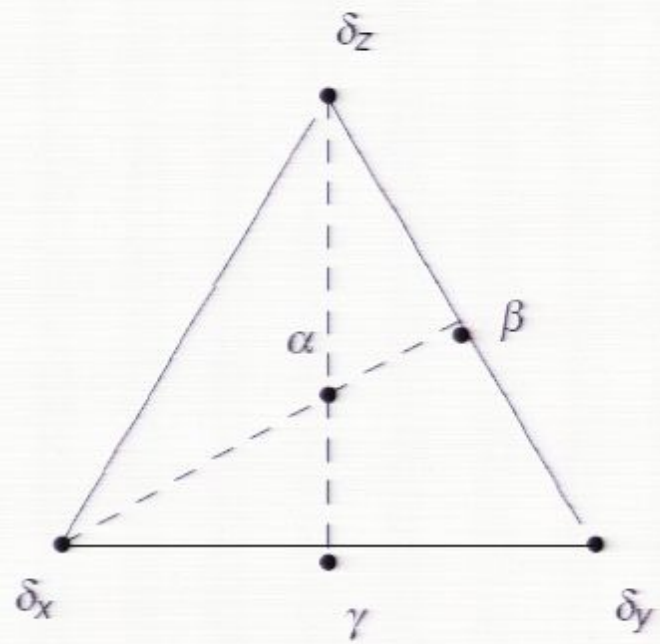
Here

$$\beta = \frac{1}{2}\delta_z + \frac{1}{2}\delta_y, \gamma = \frac{1}{2}\delta_x + \frac{1}{2}\delta_y$$

and

$$\alpha = \frac{1}{3}\delta_x + \frac{2}{3}\beta = \frac{1}{3}\delta_x + \frac{1}{3}\delta_y + \frac{1}{3}\delta_z = \frac{1}{3}\delta_z + \gamma.$$

Geometrically, the state space $\Delta(E)$ of a classical test space $\mathfrak{A} = \{E\}$ is a *simplex*. For $E = \{x, y, z\}$, this is a triangle:



Here

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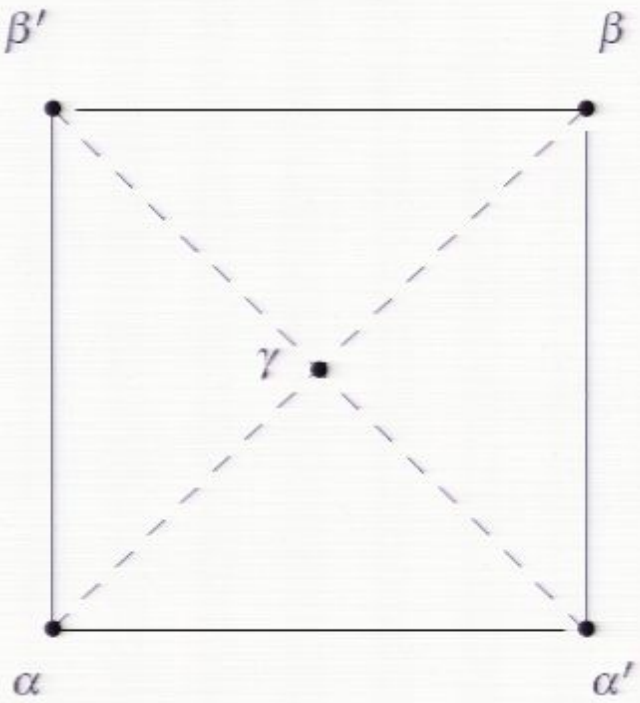
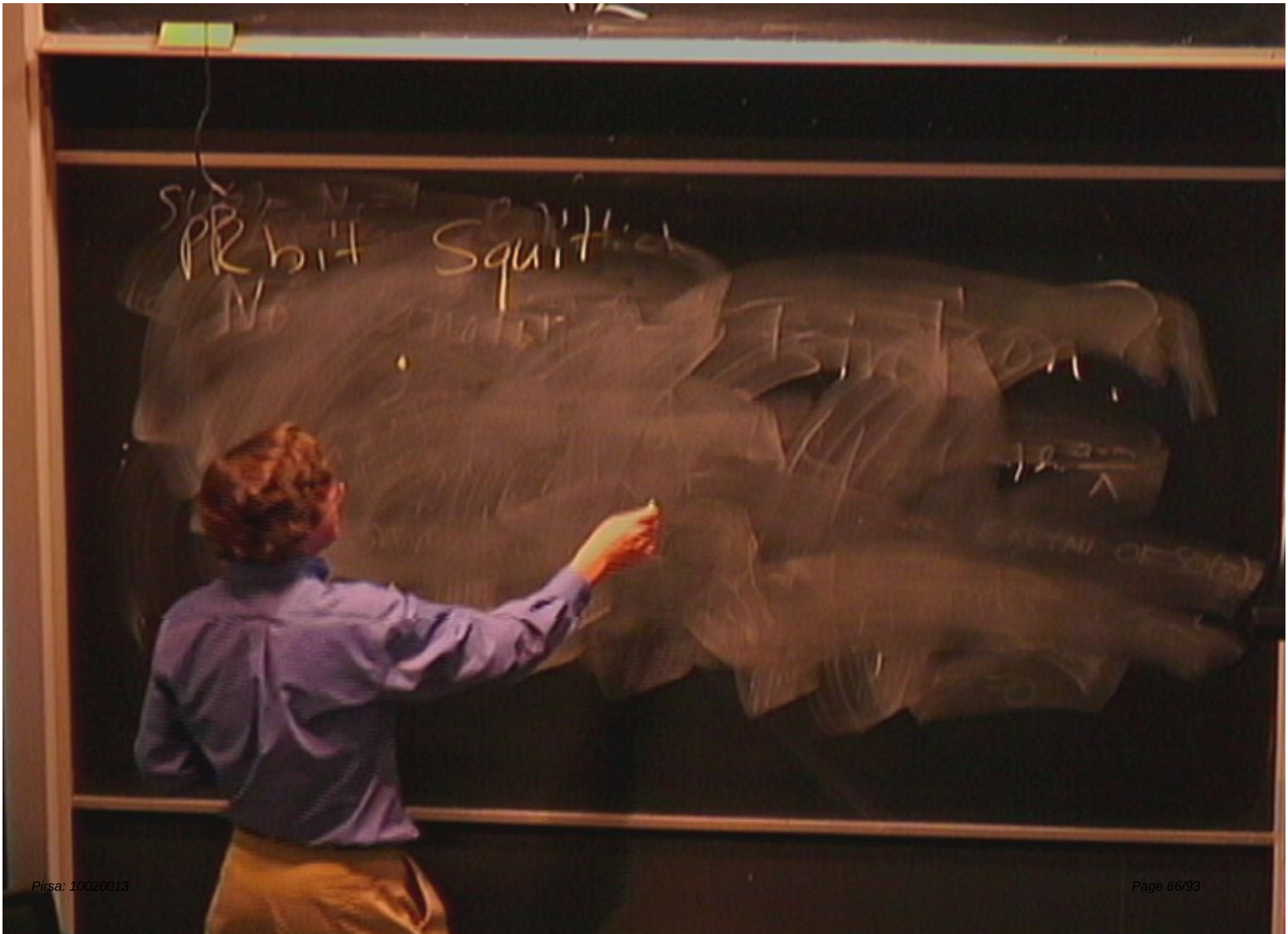
For contrast, let $\mathfrak{A} = \{\{a, a'\}, \{b, b'\}\}$. The state space is a square, with vertices $\alpha(a, b) = 1$, $\alpha'(a', b') = 1$, $\beta(a, b') = 1$ and $\beta'(a', b) = 1$. We can represent the center as an equally-weighted average of *either* pair of opposite vertices.

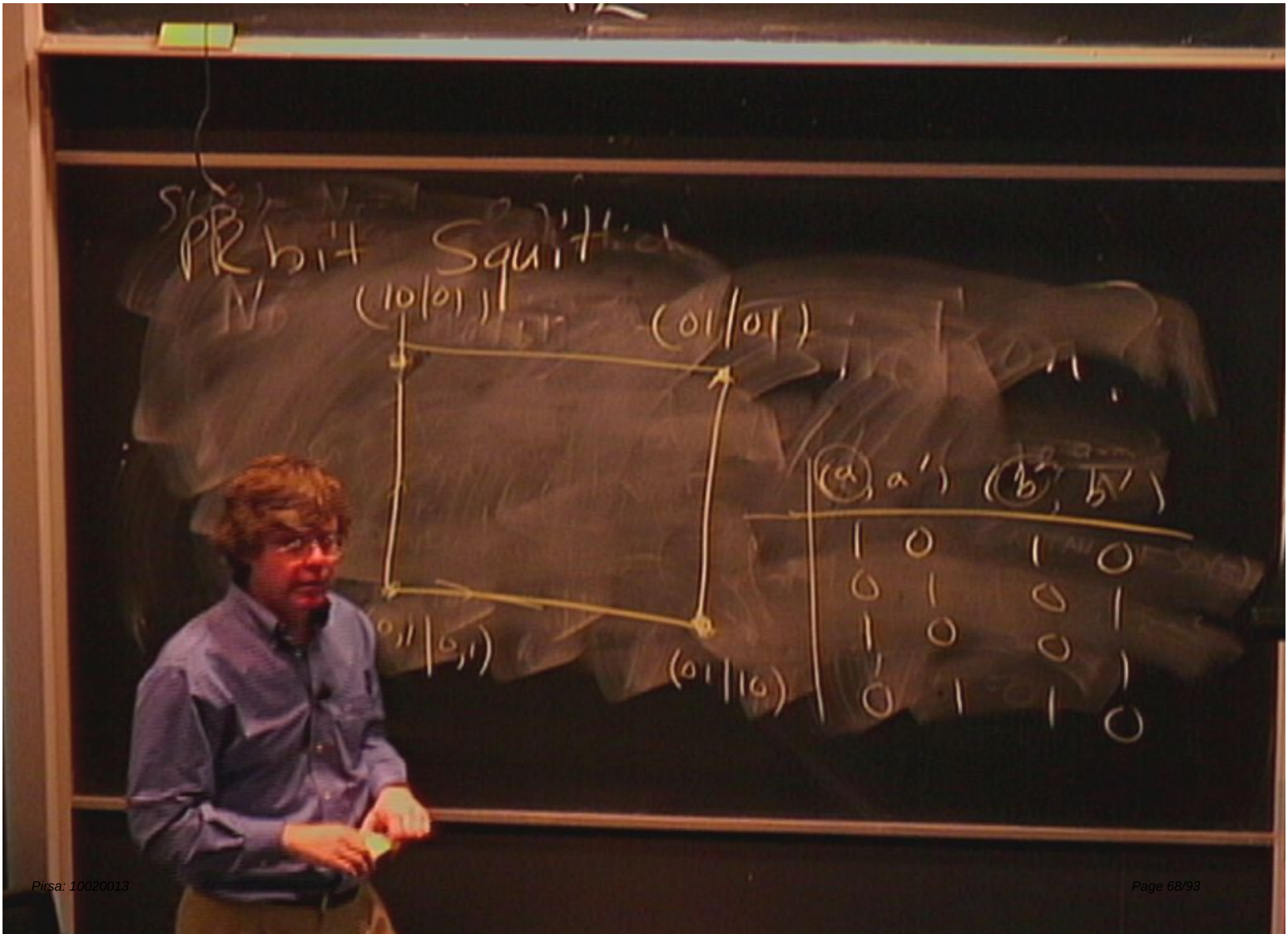


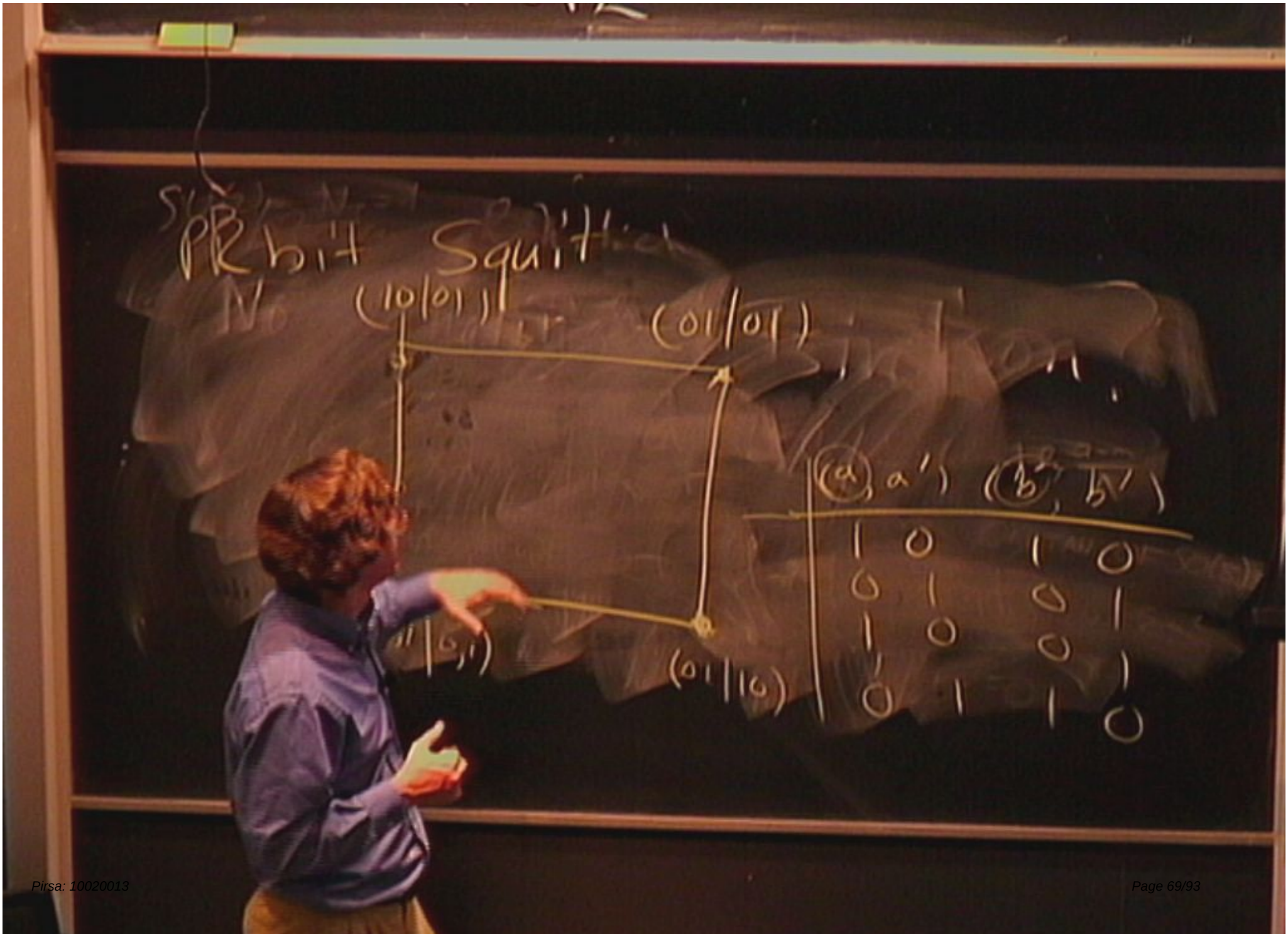
$$\gamma = \frac{1}{2}\alpha + \frac{1}{2}\beta = \frac{1}{2}\alpha' + \frac{1}{2}\beta'$$


PRbit Squitt

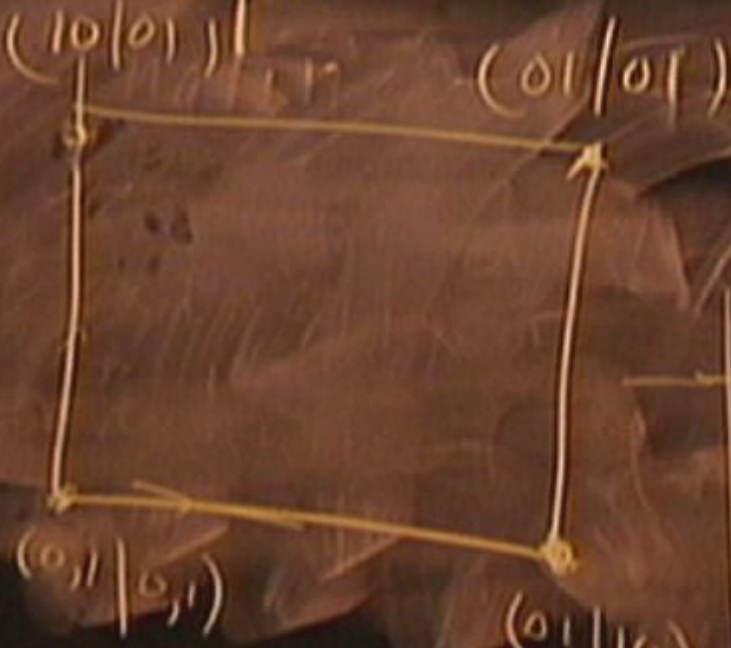
PRbit Squitch

(a, a')	(b, b')
1 0	1 0
0 1	0 1
1 0	0 1
0 1	1 0


PRbit Squitch
(10|01)
(01|01)
(01|10)
(a, a')
(b, b')


PRbit Squitch
(10|01)
(01|01)
(01|10)
(a, a')
(b, b')

PRbit Squit



(a, a')		(b, b')	
1	0	1	0
0	1	0	1
1	0	0	1
0	1	1	0

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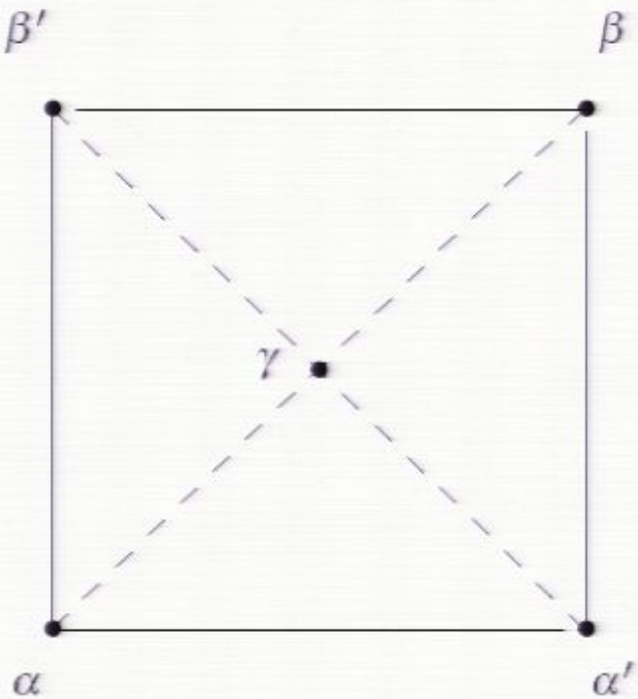
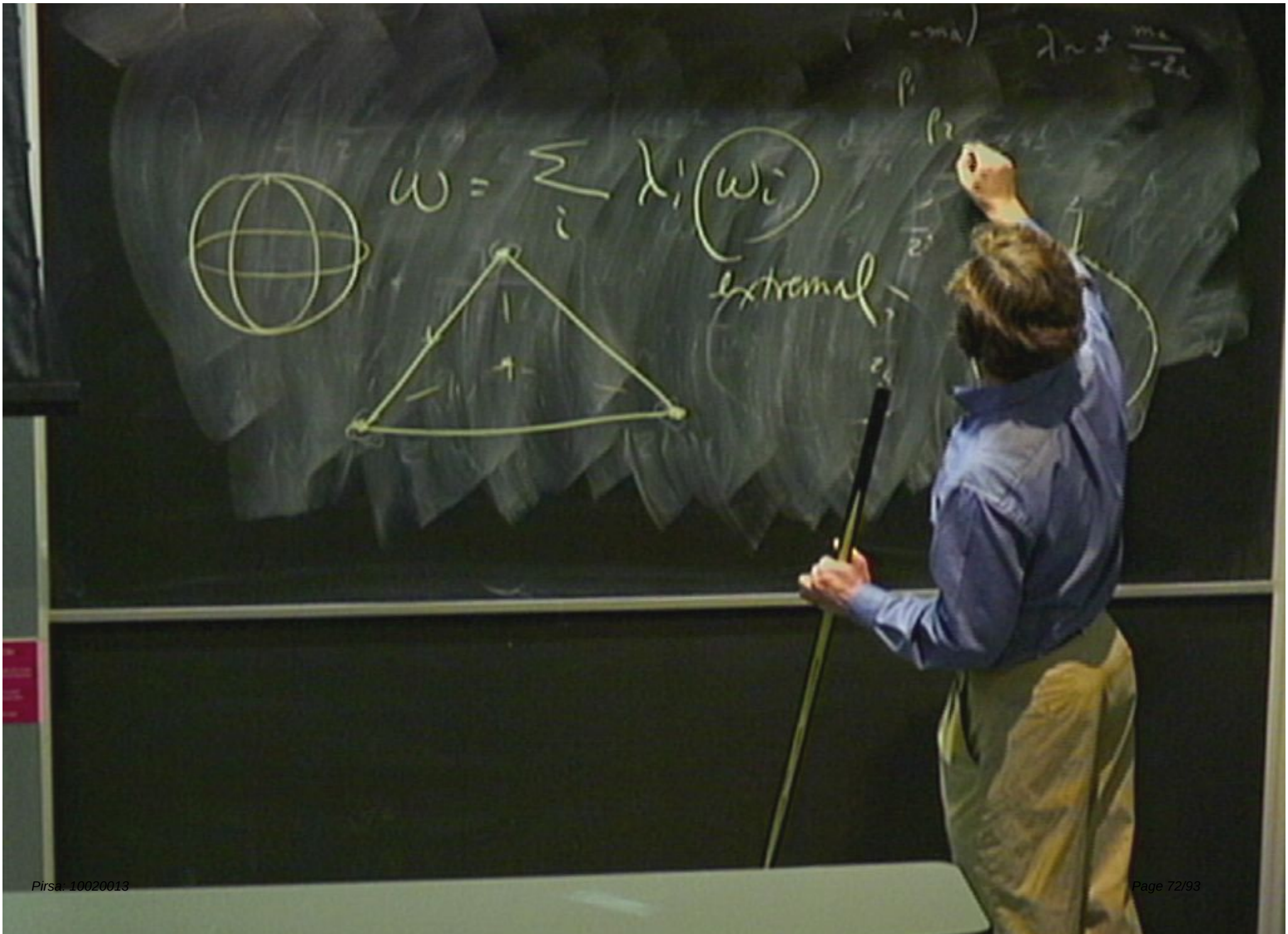
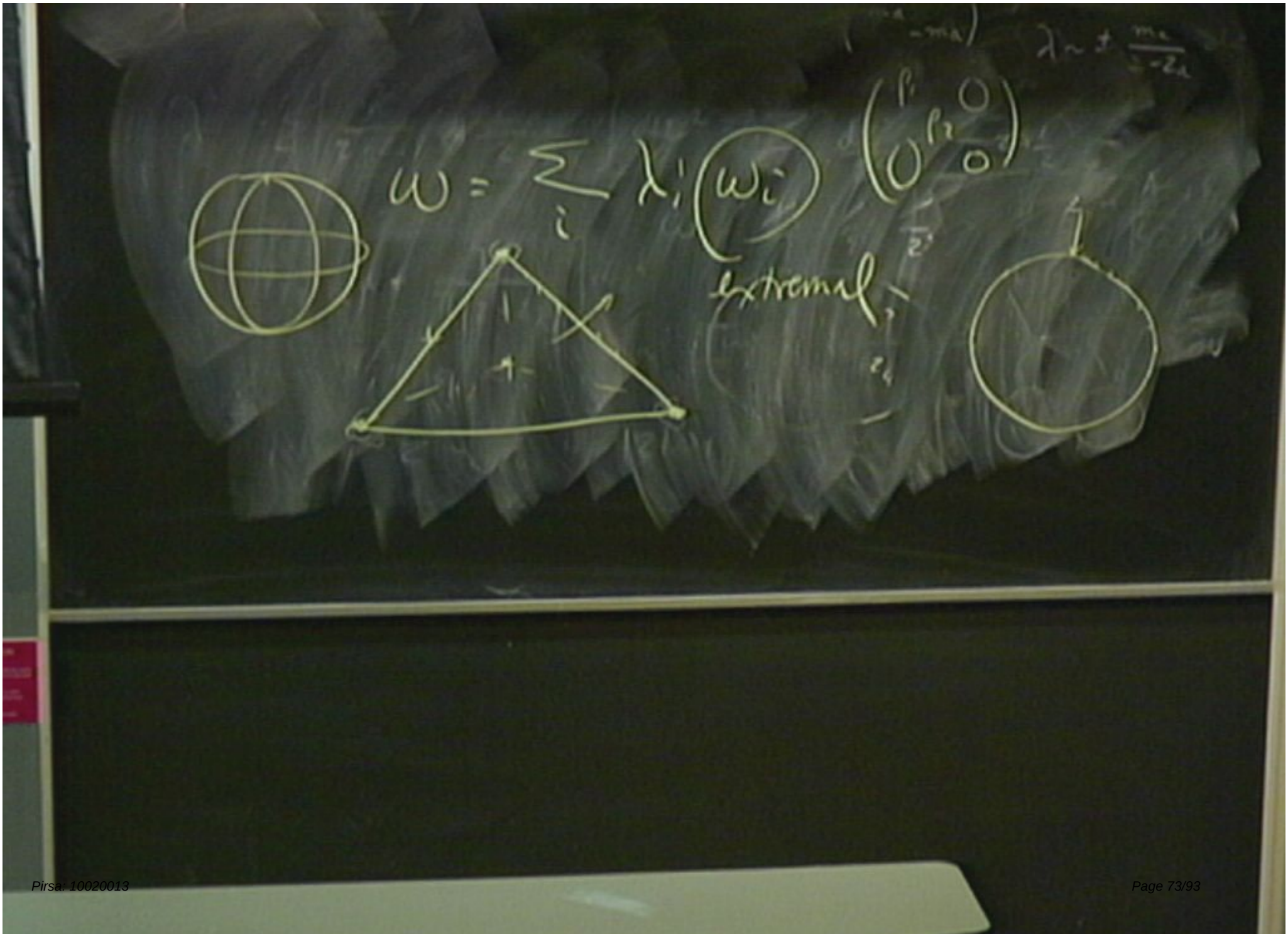


$$\gamma = \frac{1}{2}\alpha + \frac{1}{2}\beta = \frac{1}{2}\alpha' + \frac{1}{2}\beta'$$


$\omega = \sum_i \lambda_i \omega_i$
extremal


$\omega = \sum_i \lambda_i \omega_i$
extremal

Cones

Definition

A (closed, convex) **cone** in a real vector space A is a closed set $K \subseteq A$ such that

(a) $a \in K, \lambda \geq 0 \Rightarrow \lambda a \in K$

(b) $a, b \in K \Rightarrow a + b \in K$

(c) $K \cap -K = \{0\}$.

Any cone K induces a partial order on A, namely $a \leq b$ iff $b - a \in K$.

K is **generating** iff

$$A = K - K = \{a - b | a, b \in K\}.$$

An **ordered linear space** is a real vector space A with a specified *generating* cone $K =: A_+$.

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Positive Linear Mappings

Definition

Let A and B be ordered linear spaces. A linear mapping $\phi : A \to B$ is **positive** iff $\phi(a) \geq 0$ for all $a \geq 0$ in A.

We write $\mathscr{L}_+(A, B)$ for the cone of positive mappings $A \to B$. This is a cone in the space $\mathscr{L}(A, B)$ of all linear mappings $A \to B$. In finite dimensions, it is generating – so $\mathscr{L}(A, B)$ is again an ordered linear space.

Standing Assumption: Henceforth, *all vector spaces are finite-dimensional!*

Cones

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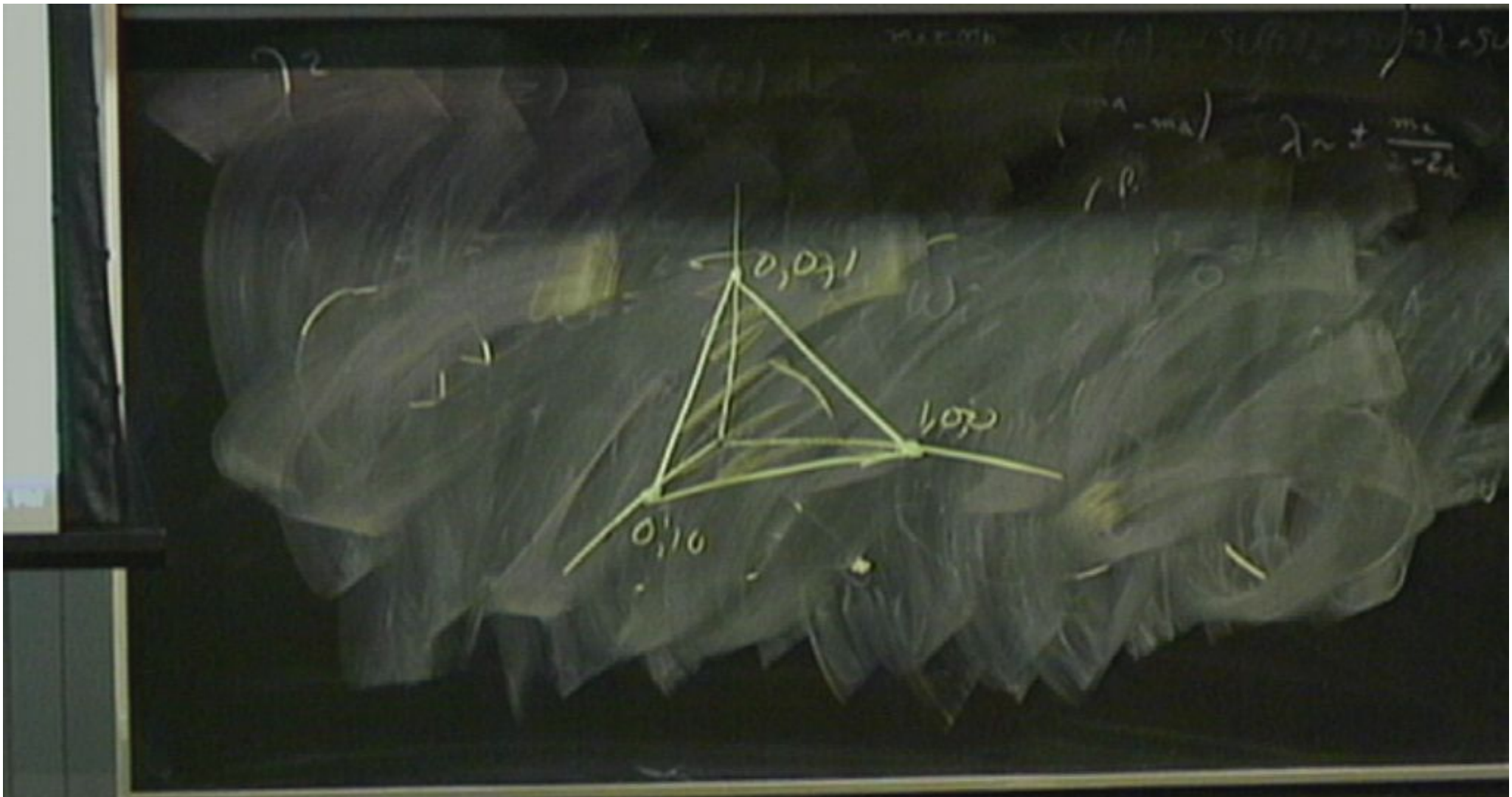
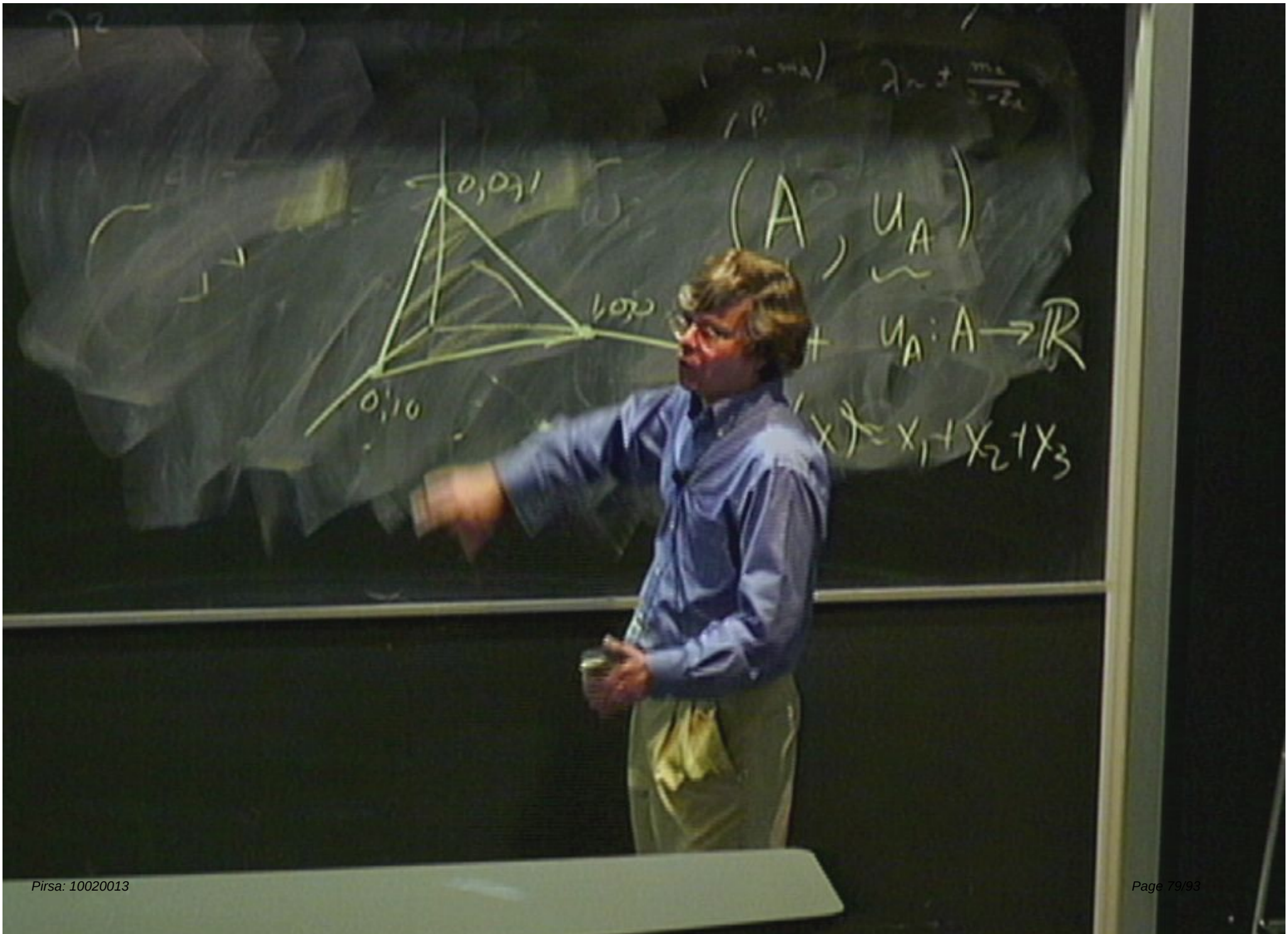
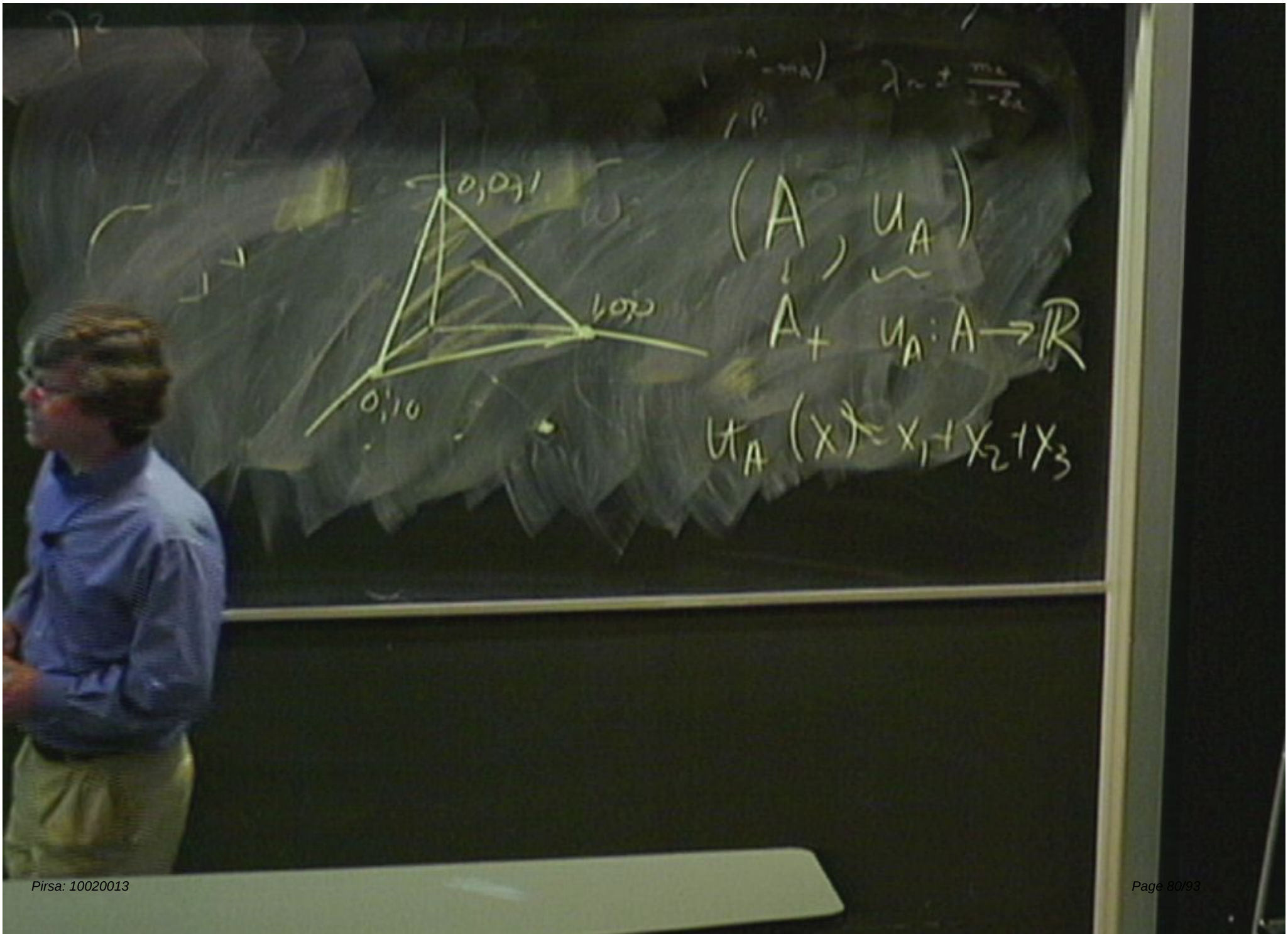
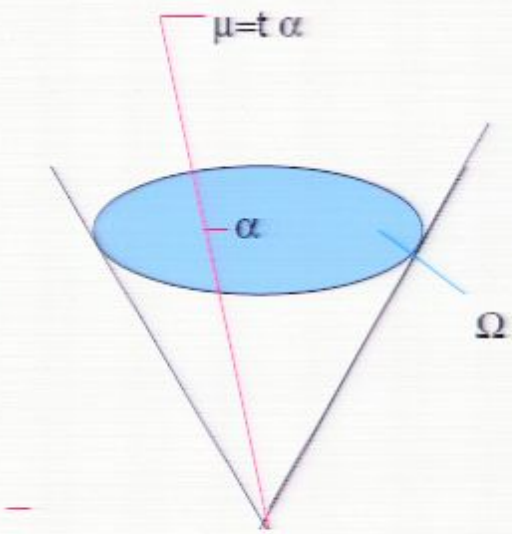

0,0,1
1,0,0
0,1,0


0,0,1
1,0,0
0,1,0
(A, u_A)
u_A : A → ℝ
x = x_1 + x_2 + x_3


0,0,1
1,0,0
0,1,0
(A, u_A)
A_+ u_A : A → ℝ
u_A(x) = x_1 + x_2 + x_3

Cone-bases



Definition

A **base** for a cone K is a convex set $\Omega \subseteq K$ such that, for every $\mu \in K \setminus \{0\}$, there exists a unique $\alpha \in \Omega$ and scalar $t \geq 0$ with $\mu = t\alpha$.

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Howard Barnum

Los Alamos National Laboratory

February 14, 2008 / SQUINT Tutorial

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Cones in $\mathbf{R}^n$

Cones in $\mathbf{R}^n$

A *(closed, convex, full, pointed) cone* in $\mathbf{R}^n$ is a subset

1. Closed under nonnegative scalar multiplication
2. Closed under addition (with the above, makes it convex)
3. Topologically closed
4. Full (nonempty interior, equivalently (since finite dimension), linearly generates (spans) R^n)).
5. Containing no (linear) subspace ("pointed")

A cone with all these properties is called *regular*.
A *base* for a cone V_+ is a set ("slice") $B := H \cap V_+$, where H is an affine hyperplane, such that V_+ is the set of nonnegative multiples of B. Equivalently, H is the plane $u(x) = 1$ for some u in the interior of V_+^*, equivalently some u such that the subspace $\{x \in V : u(x) = 0\}$ intersects V_+ only in 0.

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The dual cone

Definition (Dual vector space)

W^* := the vector space of *linear* functionals from W to R.

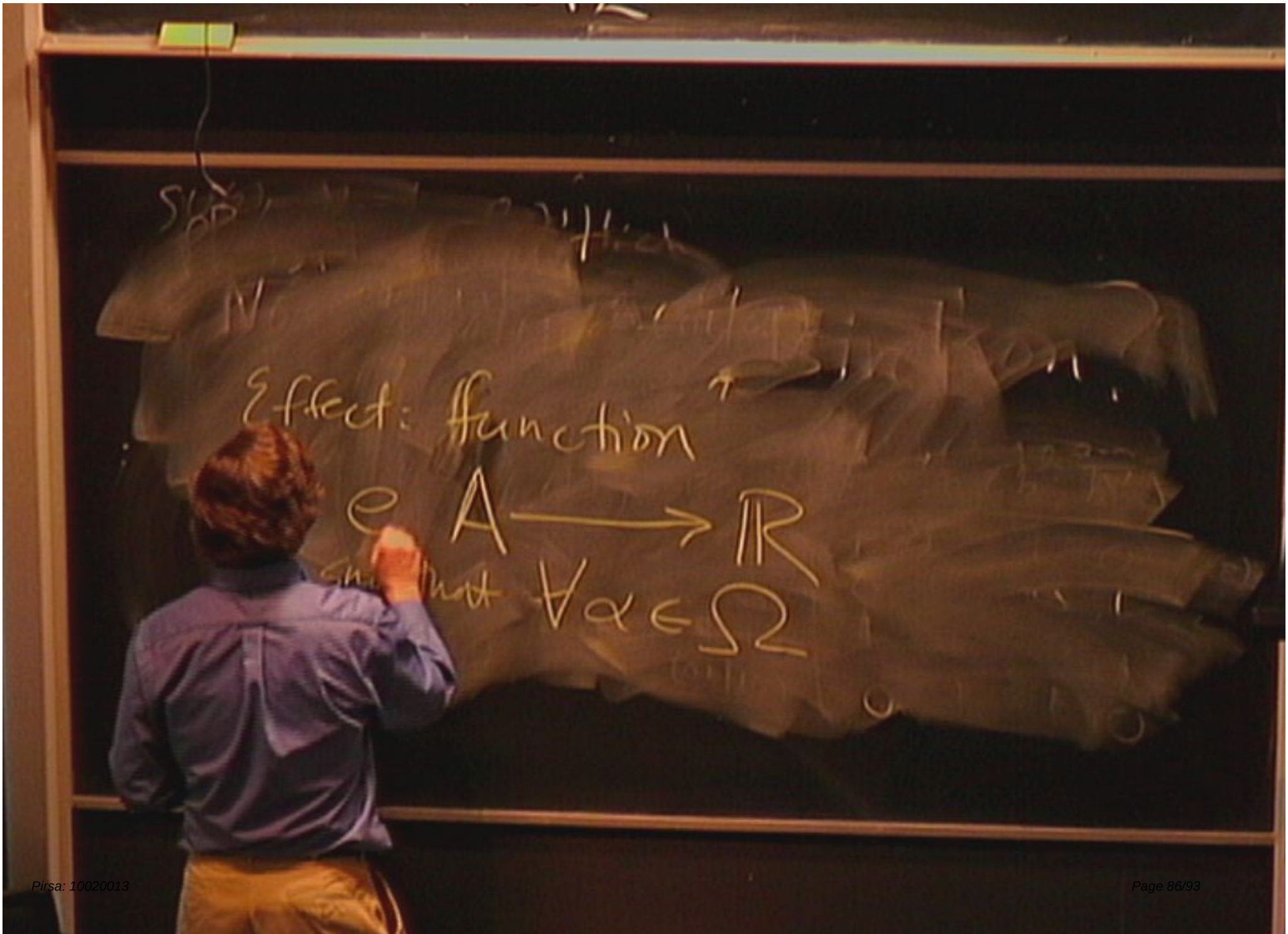
In finite dimensions, $(V^*)^* =: V^{**}$ is canonically isomorphic to V, via the linear map that identifies $x \in V$ with the linear functional $\hat{x}$ on V^* defined by the condition: $\hat{x}[f] = f[x]$ for all $f \in V^*$.

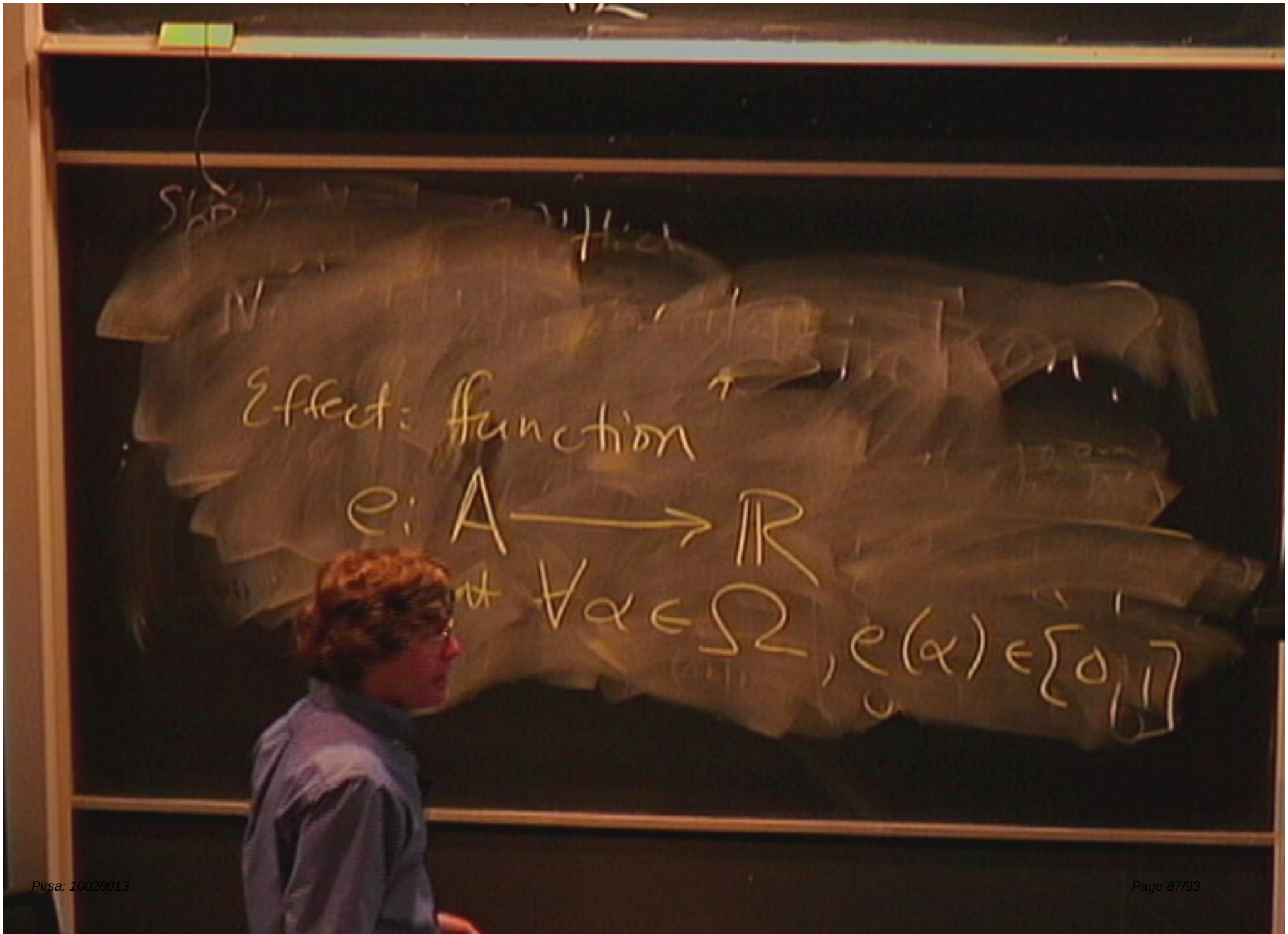
Definition (Dual cone $K^* \subseteq V^*$)

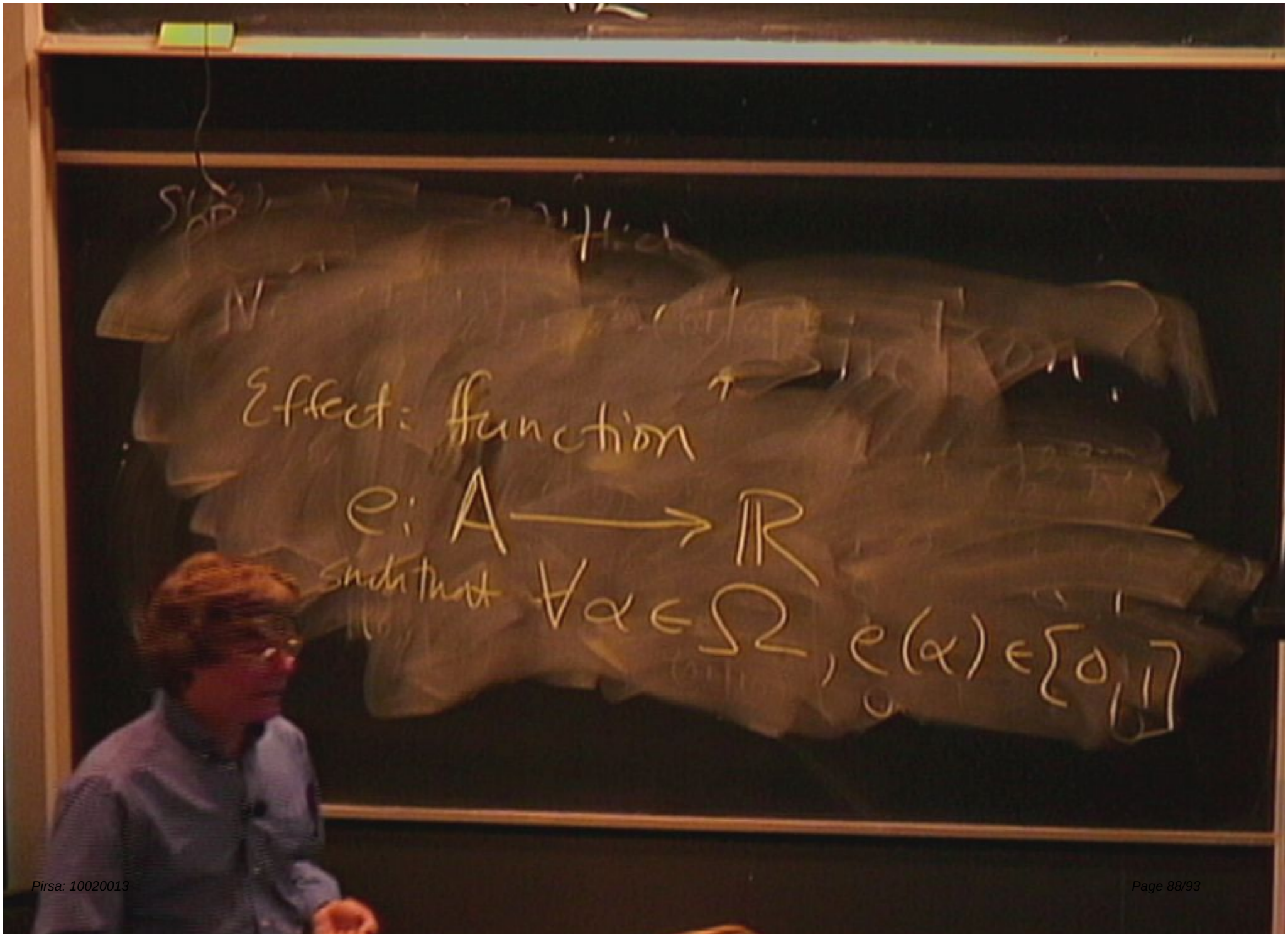
Let K be a closed convex cone in $V \simeq \mathbf{R}^n$.

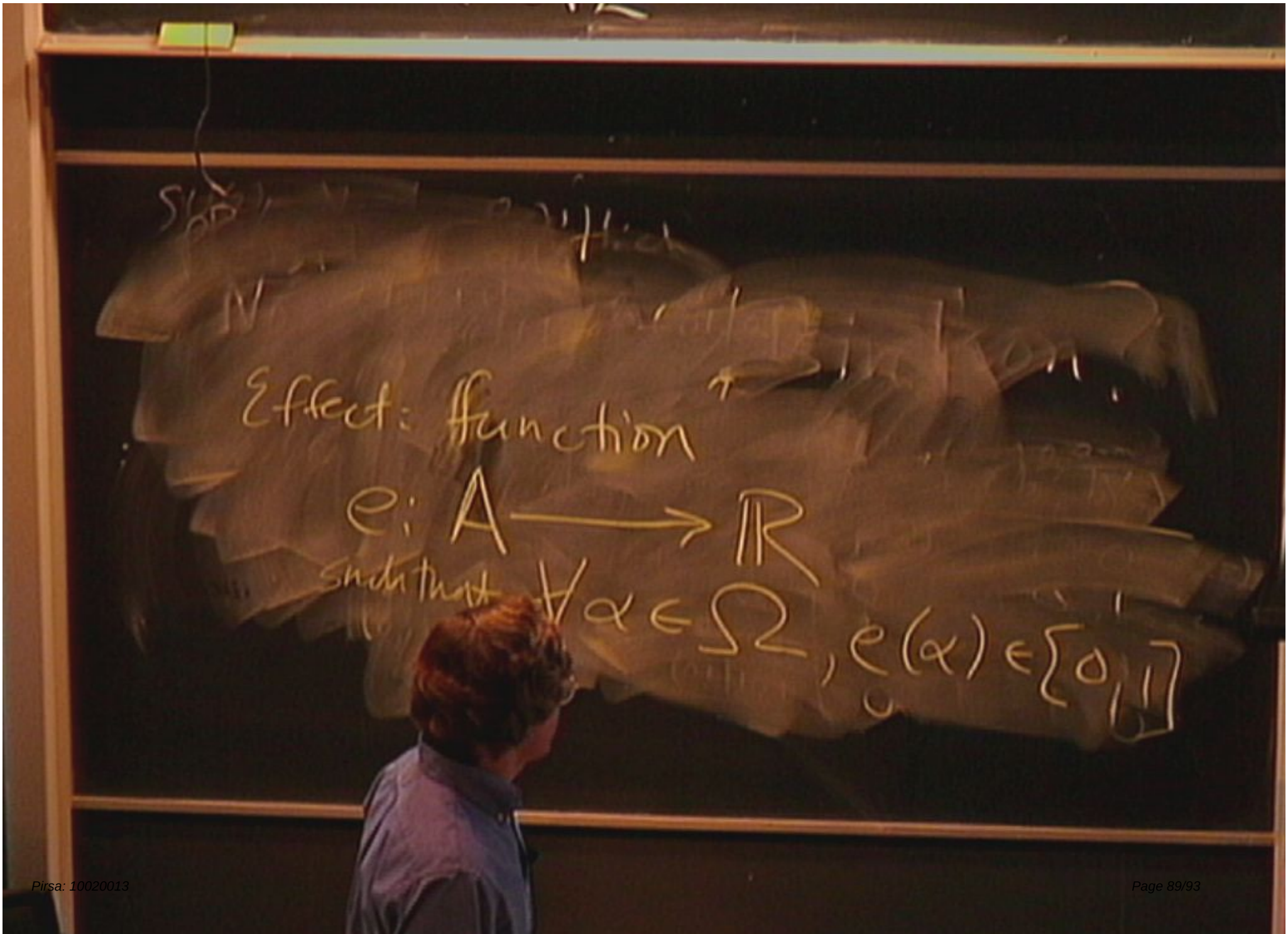
$$K^* := \{f \in V^* : \forall \omega \in K \; f(\omega) \geq 0\} \tag{5}$$

NB: if K is not full (pointed), K^* is not pointed (full).
Since K is closed and convex, $(K^*)^*$ is canonically isomorphic to V, i.e. its image under the canonical isomorphism from $(V^*)^*$ is V.


Effect: function
$e : A \longrightarrow \mathbb{R}$
$\forall \alpha \in \Omega$


Effect: function
$e: A \longrightarrow \mathbb{R}$
$\forall \alpha \in \Omega, e(\alpha) \in [0,1]$


Effect: function
e: A → ℝ
such that ∀α ∈ Ω, e(α) ∈ [0,1]


Effect: "function"
$e: A \longrightarrow \mathbb{R}$
such that $\forall \alpha \in \Omega, e(\alpha) \in [0,1]$

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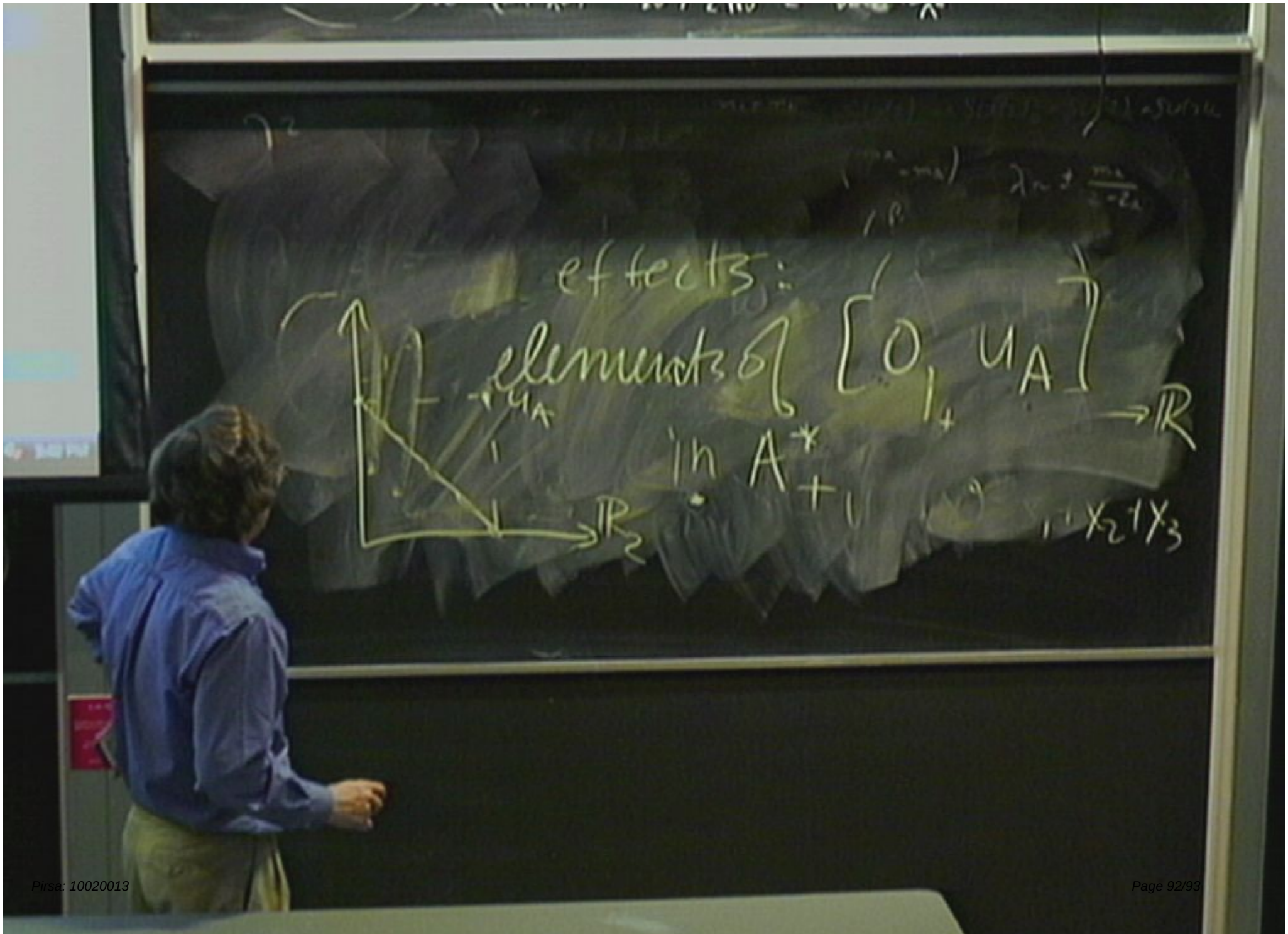
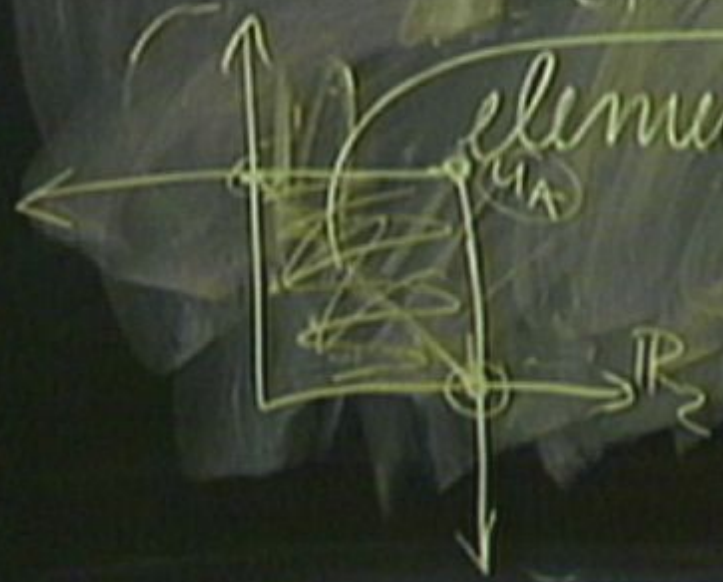

effects:
elements of [0, u_A]
in A*_+
→ ℝ
ℝ_2

effects:

elements of $[0, u_A]$ in A^*_+



$\to \mathbb{R}$