Title: Quantum Field Theory for Cosmology - Lecture 4

Date: Jan 21, 2010 05:00 PM

URL: http://pirsa.org/10010075

Abstract: This course begins with a thorough introduction to quantum field theory. Unlike the usual quantum field theory courses which aim at applications to particle physics, this course then focuses on those quantum field theoretic techniques that are important in the presence of gravity. In particular, this course introduces the properties of quantum fluctuations of fields and how they are affected by curvature and by gravitational horizons. We will cover the highly successful inflationary explanation of the fluctuation spectrum of the cosmic microwave background - and therefore the modern understanding of the quantum origin of all inhomogeneities in the universe (see these amazing visualizations from the data of the Sloan Digital Sky Survey. They display the inhomogeneous distribution of galaxies several billion light years into the universe: Sloan Digital Sky Survey).

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QFT for Cosmology, Achim Kempf, Winter 2010, Lecture 4

From the Heisenberg to the Schrödinger picture

Recall Heisenberg picture:

$$\hat{H} = \int_{\mathbb{R}^3} \frac{1}{2}\hat{\pi}^2(x',t) + \frac{1}{2}\hat{\phi}(x',t)(m^2-\Delta)\hat{\phi}(x',t)\, d^3x'$$

The Heisenberg eqns $i\partial_t \hat{Q}(t) = [\hat{Q}(t), \hat{H}]$ yield:

- K.G. eqn: $\dot{\hat{\phi}}(x,t) = \hat{\pi}(x,t)$ and $\dot{\hat{\pi}}(x,t) = (\Delta - m^2)\hat{\phi}(x,t)$

and we have to solve:

- C.C.Rs: $[\hat{\phi}(x,t), \hat{\pi}(x',t)] = i\hbar\,\delta^3(x-x')$

- Hermiticity: $\hat{\phi}^+(x,t) = \hat{\phi}(x,t)$, $\hat{\pi}^+(x,t) = \hat{\pi}(x,t)$

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Our results so far:

in box of size $L \times L \times L$

- We obtained explicit solution $\hat\phi(x,t)$ in the form

(∓) $$\hat\phi(x,t) = L^{-3/2} \sum_k \hat\phi_k(t)\, e^{ikx} \quad \text{for } k = \frac{2\pi}{L}(n_1, n_2, n_3)$$

where $\hat\phi_k(t)$ are uncoupled complex harmonic oscillators:

$$\ddot{\hat\phi}_k(t) = -\omega_k^2 \hat\phi_k(t) \quad \text{with} \quad \omega_k = \sqrt{k^2+m^2}$$

- Then,

$$\hat\phi_k(t) = \frac{1}{\sqrt{2}}\left(\hat q_k(t) + \hat q_{-k}(t)\right) + \frac{i}{\omega_k\sqrt{2}}\left(\hat p_k(t) - \hat p_{-k}\right)$$

is in terms of ordinary quantum harmonic oscillators $\hat q_k, \hat p_k$.

- **Note:** The $\hat\phi(x,t)$ act on the same Hilbert space $\mathcal{H}$ as do the $\hat\phi_k(t)$, and as do the $\hat q_k(t)$ and $\hat p_k(t)$.

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How then to make predictions?

- Assume:
  - $\hat{\phi}(x,t)$, $\hat{\pi}(x,t)$ are known.
  - state $|\Psi\rangle \in \mathcal{H}$ of the K.G. quantum system known.

    ... for example in terms of the $\hat{q}_k(t)$, $\hat{p}_k(t)$ and their action on a Hilbert space.

- Predict, e.g.:
  - expect. value when repeatedly measuring say $\hat{\phi}(x,t)$ at (x,t):

$$\bar{\phi}(x,t) = \langle \Psi | \hat{\phi}(x,t) | \Psi \rangle$$

  - uncertainty in measurement of $\hat{\phi}(x,t)$ at (x,t):

(Recall standard deviation:
$\Delta A = \sqrt{\overline{(A-\bar{A})^2}}$)

$$\Delta\phi(x,t) = \langle \Psi | \left(\hat{\phi}(x,t) - \bar{\phi}(x,t)\right)^2 | \Psi \rangle^{1/2}$$

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$$\hat{\phi}(x,t) \qquad x \in \mathbb{R}$$

we can, equivalently, measure

$$\hat{\phi}_k(t) \qquad k = \frac{2\pi}{L} n, \quad n \in \mathbb{Z}^3$$

because the measurement outcomes are trivially Fourier related.

What is prob. amplitude for finding any ϕ_k?

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* From this, one can work out (exercise!):

$$-\omega_k \phi_k \phi_k^* / 2$$

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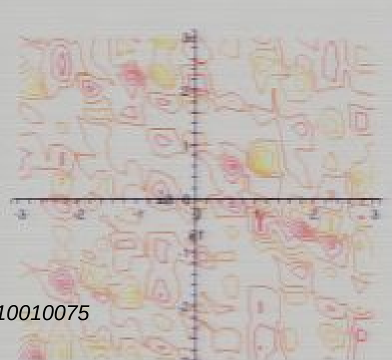
Visualization:



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2.) Fourier transform to obtain a $\phi(x)$.

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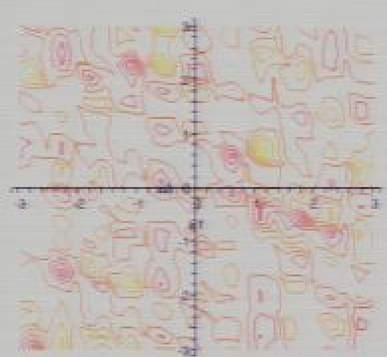

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- Assume that at a time t all the observables $\hat{\phi}(x,t)$ are simultaneously being measured.

(We can because $[\hat{\phi}(x,t), \hat{\phi}(x',t)] = 0$)

- At each x we obtain a real-valued measurement outcome, say $f(x)$.

- Thus, the system collapses into a state

$$|f\rangle \in \mathcal{H}$$

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Definition: If $f: \mathbb{R}^3 \to \mathbb{R}$ is an arbitrary function, we denote by

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Hilbert basis: The set

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of all joint eigenvectors of the $\hat\phi(x,t)$ for all $x \in \mathbb{R}^3$ can be used to form a "complete ON basis" of $\mathcal{H}$. (up to functional analytic subtleties).

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$p = \{p_1, p_2, \ldots, p_N\}$

or, e.g., also the $\{\hat{\pi}(x)\}$.

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A: The QFT Schrödinger equation:

- For <u>every</u> quantum theory, we have in the Schrödinger picture of the time evolution:

$$i\hbar \frac{d}{dt}|\Psi(t)\rangle = \hat{H}|\Psi(t)\rangle$$

- Which form does it take for $\Psi[f,t]$?

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now independent of time!

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Exercise: check it. Can you solve for excited states?

Ground state solution in QFT reads, similarly:

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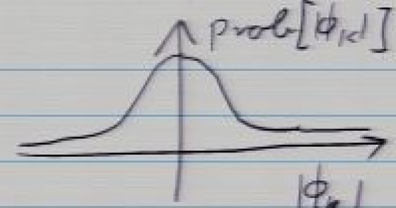
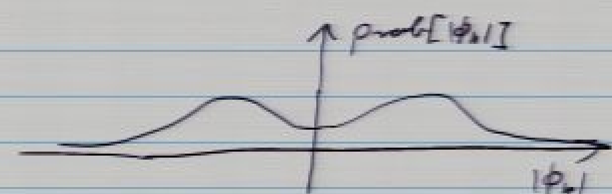
- But if an oscillator is excited, then its wave function spreads out – classically its amplitude of oscillation would increase.

⇒ If a mode k is excited then the prob. distribution of the ϕ_k spreads:



ground state



example of excited state

- The more a mode k is excited, the more likely is a

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$$\Psi(x,t) = N e^{-\frac{1}{2}\omega x^2 - i\omega_k t}$$

Exercise: check it. Can you solve for excited states?

Ground state solution in QFT reads, similarly:

$= (\vec{k}^2 + m^2)^{1/2}$

$$\Psi[\tilde{f},t] = N e^{-\sum_k \left(\frac{1}{2}\omega_k \tilde{f}_k \tilde{f}_{-k} - i\omega_k t\right)}$$

Exercise: verify

... which we had already claimed before.

Generic wave functionals

- Assume the system is in a state, $|\alpha\rangle$, other than $|\Psi_0\rangle$.

$\Rightarrow$ Not for all modes' oscillators is $|\alpha\rangle$

the ground state.

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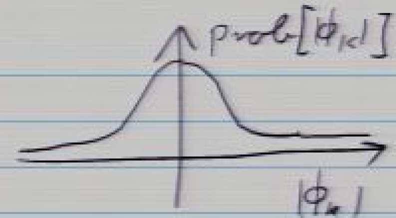

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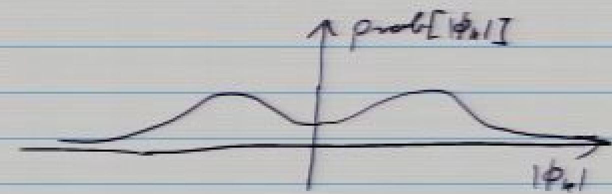

example of excited state

The more a mode k is excited, the more likely is a measurement of $\hat{\phi}_k$ to yield a $f_k = \phi_k$ with a large modulus $|\phi_k|$.

$\Rightarrow$ If, e.g., a mode k is very highly excited then $|\phi_k|$ is likely very large, i.e., a measurement

Can you produce a

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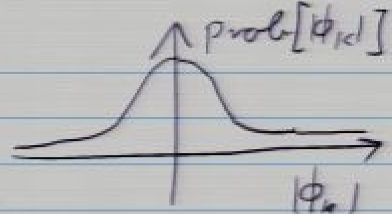

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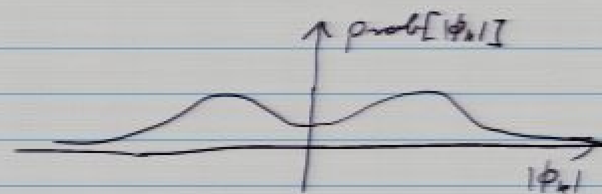

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Can you produce a plot?

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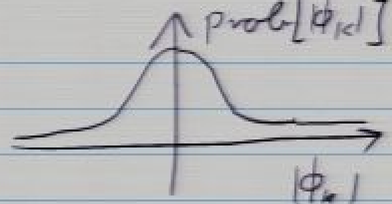

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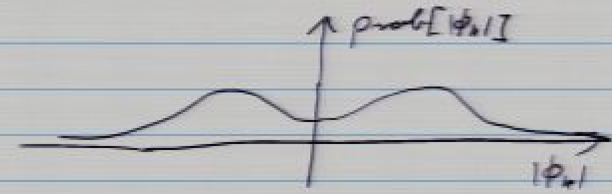

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The particle interpretation

- General states, i.e., states $|\alpha\rangle$ other than the vacuum state $|\Psi_0\rangle$ are states "with particles". Why?

- Recall:

$$\hat{H} = \sum_k \left(\frac{1}{2} \hat{\pi}_k^+(t) \hat{\pi}_k(t) + \frac{1}{2} \hat{\phi}_k^+(t) \left(k^2 + m^2\right) \hat{\phi}_k(t) \right)$$

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* If we increase the n_k of a mode k by 1

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* But recall from special relativity: $E^2 - p^2 = m^2$.

$\Rightarrow$ $E_{particle} = \sqrt{k^2_{particle} + m^2_{particle}} = \omega_k$

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→ Interpretation (which works at least in Minkowski space:)

$$\boxed{\text{Mode excitation = particle creation}}$$

- **Example:**

If the QFT is, e.g., in the above state $|E\rangle$ then we have 3 and 7 particles of momentum k_a and k_b respectively.

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  ⇒ This interpretation above is not always applicable!

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