Title: Quantum Field Theory for Cosmology - Lecture 2

Date: Jan 14, 2010 05:00 PM

URL: http://pirsa.org/10010074

Abstract: This course begins with a thorough introduction to quantum field theory. Unlike the usual quantum field theory courses which aim at applications to particle physics, this course then focuses on those quantum field theoretic techniques that are important in the presence of gravity. In particular, this course introduces the properties of quantum fluctuations of fields and how they are affected by curvature and by gravitational horizons. We will cover the highly successful inflationary explanation of the fluctuation spectrum of the cosmic microwave background - and therefore the modern understanding of the quantum origin of all inhomogeneities in the universe (see these amazing visualizations from the data of the Sloan Digital Sky Survey. They display the inhomogeneous distribution of galaxies several billion light years into the universe: Sloan Digital Sky Survey).

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QFT for Cosmology, Achim Kempf, Winter 2010, Lecture 2

"A taste of quantum fields"

We will deal with wave phenomena,

i.e., with oscillating amplitudes.

$\Rightarrow$ Harmonic oscillators will arise in the formalism!

Plan:

1. Recall harmonic oscillators
2. Relativistic fields
3. 2nd quantization

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Classical:

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1. Harmonic oscillator

Classical:

- Hamiltonian: $H = \frac{p^2}{2\mu} + \mu \frac{\omega^2}{2} q^2, \; \mu = \text{mass}$

- Eqns of motion: $\dot{p} = -\omega^2 q, \quad \dot{q} = \frac{p}{\ldots}$

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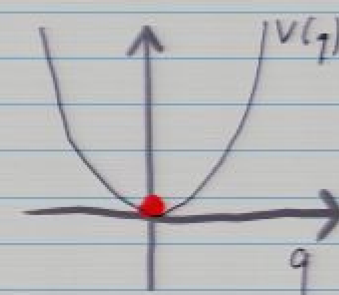
- Lowest energy solution: (later relevant for "vacuum")

$$q(t) = 0, \; p(t) = 0$$

i.e. $H(t) = 0$ for all t



- "Nothing moves, certainly"

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Quantum:

As always when quantizing:

- H and Eqns of motion unchanged.
- But, the canonically conjugate pairs of variables (here, q and p) no longer commute:

▫ Hamiltonian: $\hat{H} = \frac{\hat{p}^2}{2\mu} + \frac{\omega^2}{2}\hat{q}^2$

▫ Eqns of motion: $\dot{\hat{p}} = -\omega^2 \hat{q}, \quad \dot{\hat{q}} = \frac{\hat{p}}{\mu}$

▫ And of course:

$$[\hat{q}(t), \hat{p}(t)] = i\hbar 1$$

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4. Harmonic oscillators in fields ⇒ vacuum fluctuations

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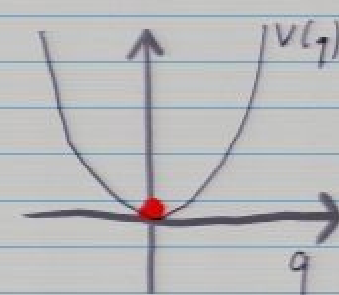

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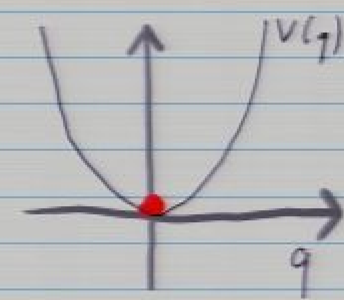

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- $\Rightarrow \hat{q}(t), \hat{p}(t), \hat{H}$ etc are operator-valued.

- Lowest energy solution now?

The lowest energy state, $|\psi_0\rangle$, obeys:

$$\hat{H}|\psi_0\rangle = E_0 |\psi_0\rangle$$

with $E_0 = \frac{1}{2}\hbar\omega$

- We notice:

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Lowest energy state $|\psi_0\rangle$?

Consider eigenbasis $|q\rangle$ of $\hat{q}$:

$$\hat{q}|q\rangle = q|q\rangle \quad \text{for } q \in \mathbb{R}$$

$$\langle q|q'\rangle = \delta(q-q')$$

Then, recall:

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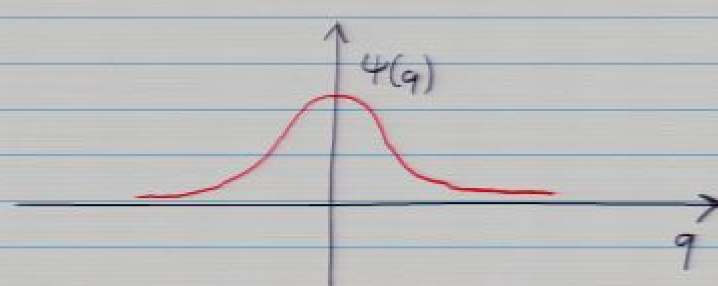

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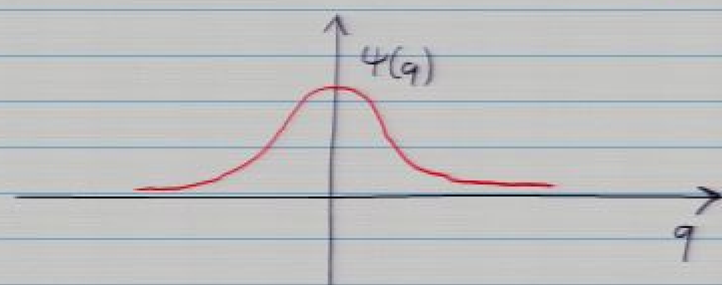

□ Is oscillator at resting position $q=0$?

In lowest energy state, $|\psi_0\rangle$, we have:

$$\bar{q} = \langle \psi_0 | \hat{q} | \psi_0 \rangle = \int_{-\infty}^{+\infty} \psi_0^*(q)\, q\, \psi_0(q)\, dq = 0$$

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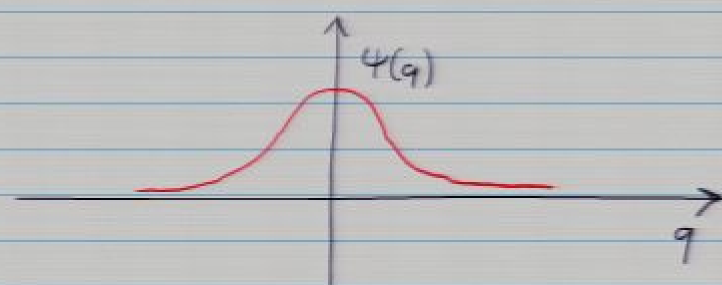

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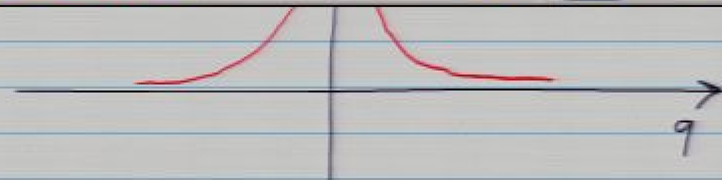

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But, there are quantum fluctuations!

$$\Delta q = \langle\psi_0|(\hat{q} - \bar{q})^2|\psi_0\rangle^{1/2} = \sqrt{\hbar \ldots}$$

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choose simple case without a potential

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$$i\hbar \frac{\partial}{\partial t} \psi(x,t) = -\frac{\hbar^2}{2m} \Delta \, \psi(x,t) \qquad (S)$$

Laplacian: $\Delta = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}$

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But special relativity demands:

$$\frac{E^2}{c^2} - \vec{p}^{\,2} = m^2c^2 \qquad \left(\text{Namely: } p_\mu p^\mu = m^2c^4\right)$$

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(units chosen so that $c=1$, $\hbar=1$)

Or, also $(\Box + m^2)\psi = 0$ with d'Alembertian $\Box = \partial_t^2 - \Delta$

- Nonrelativistic limit ok?

Must show that KG eqn reduces

to Schrödinger eqn for small momenta:

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Or, also $(\Box + m^2)\psi = 0$ with d'Alembertian $\Box = \partial_t^2 - \Delta$

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- appear to lead to incurable "divergencies" in QFT.

Note:

- "Graviton" should be a spin 2 particle.

Plan:

1. Recall harmonic oscillators ✓
2. Relativistic fields ✓
3. 2nd quantization
4. Harmonic oscillators in fields ⇒ vacuum fluctuations

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- We will 2nd quantize only the Klein Gordon equation because:
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we will do the general definition later

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- Klein Gordon equation can now be written in the form:

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Quantization conditions:

analogous to:

$$\left[\hat{\phi}(x,t), \hat{\pi}(x',t)\right] = i\hbar\,\delta^3(x-x') \qquad \left[\hat{q}_a(t), \hat{p}_{a'}(t)\right] = i\hbar\,\delta_{aa'}$$

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and we keep the equations of motion from above:

(E1) $$\dot{\hat{\phi}}(x,t) = \hat{\pi}(x,t) \qquad \dot{\hat{q}}_a(t) = \hat{p}_a(t)$$

(E2) $$\dot{\hat{\pi}}(x,t) = -\left(-\Delta + m^2\right)\hat{\phi}(x,t) \qquad \dot{\hat{p}}_a(t) = -K\hat{q}_a(t)$$

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Is there a Hamiltonian for 2nd quantization? Yes!

analogous to:

$$\hat{H} = \int_{\mathbb{R}^3} \frac{1}{2}\hat{\pi}^2(x,t) + \frac{1}{2}\hat{\phi}(x,t)(m^2 - \Delta)\hat{\phi}(x,t)\, d^3x \qquad \hat{H} = \sum_a \frac{\hat{p}_a^2}{2} + \frac{\omega_a^2}{2}\hat{q}_a^2$$

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Exercise: check this

- Proposition:

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<u>Exercise</u>: check this

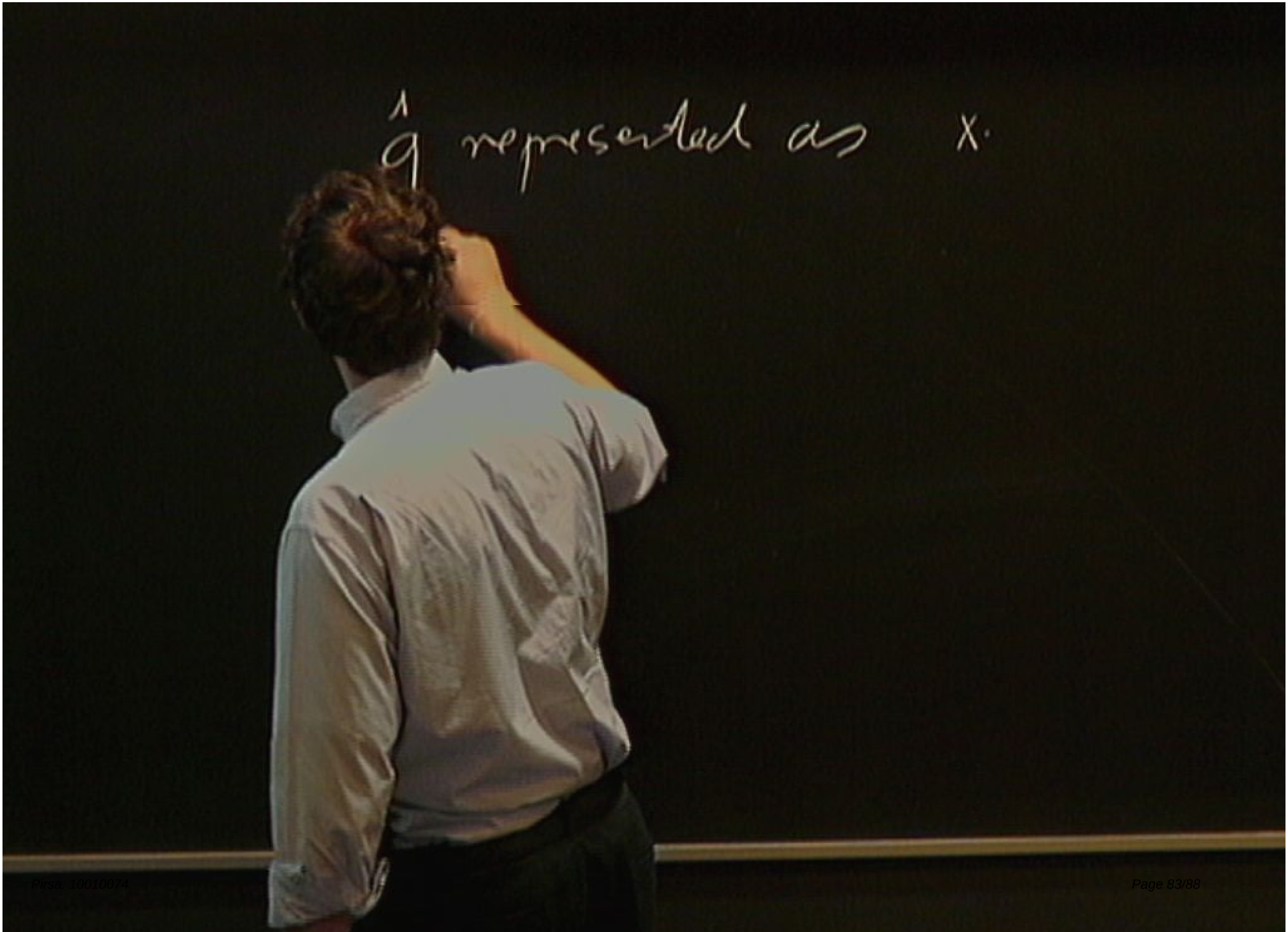
- Proposition:

With this definition of $\hat{H}$, the Heisenberg equations $i\hbar\dot{\hat{f}} = [\hat{f}, \hat{H}]$

$$i\hbar\,\dot{\hat{\phi}}(x,t) = [\hat{\phi}(x,t), \hat{H}] \qquad i\hbar\,\dot{\hat{q}}_a(t) = [\hat{q}_a(t), \hat{H}]$$

$$i\hbar\,\dot{\hat{\pi}}(x,t) = [\hat{\pi}(x,t), \hat{H}] \qquad i\hbar\,\dot{\hat{p}}_a(t) = [\hat{p}_a(t), \hat{H}]$$

yield the proper eqns of motion: E1, E2.


$\hat{q}$ represented as x.

$$\left.\begin{array}{l} \hat{q}_a \text{ represented as } x_a \\ \hat{p}_a \text{ represented as } -i\hbar\partial_{x_a} \end{array}\right\} \text{acting on } \psi(x)$$

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acting on $\psi(x)$

$[\hat{q}_a, \hat{p}_b] = i\hbar\delta_{ab}$ repr. by $x_a(-i\partial_{x_b})\psi(x) - (-i\partial_{x_b})x_a\psi(x) = 0$

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$\hat{q}_a$ represented as x_a
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$i\delta_{ab}\Psi(x)$

$= \delta_{ab}\, i\Psi(x)$

$[\hat{\phi}(x), \hat{\pi}(x')] = i\hbar\delta^3(x-x')$