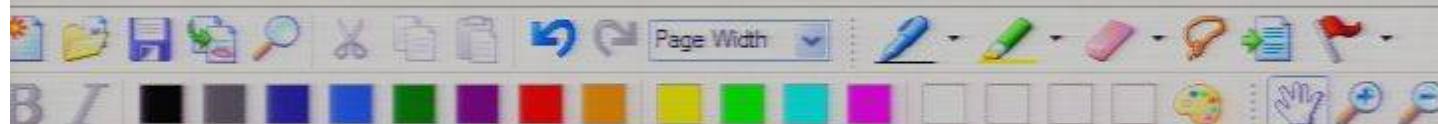


Title: Quantum Field Theory for Cosmology - Lecture 5

Date: Jan 26, 2010 04:00 PM

URL: <http://pirsa.org/10010073>

Abstract: <span>This course begins with a thorough introduction to quantum field theory. Unlike the usual quantum field theory courses which aim at applications to particle physics, this course then focuses on those quantum field theoretic techniques that are important in the presence of gravity. In particular, this course introduces the properties of quantum fluctuations of fields and how they are affected by curvature and by gravitational horizons. We will cover the highly successful inflationary explanation of the fluctuation spectrum of the cosmic microwave background - and therefore the modern understanding of the quantum origin of all inhomogeneities in the universe (see these amazing visualizations from the data of the Sloan Digital Sky Survey. They display the inhomogeneous distribution of galaxies several billion light years into the universe: Sloan Digital Sky Survey).</span>



Recall: □ The QFT problem

$$\left( \frac{\partial^2}{\partial t^2} - \Delta + m^2 \right) \hat{\phi}(x,t) = 0 \text{ and } [\hat{\phi}(x,t), \hat{\phi}^\dagger(x',t')] = i \delta(x-x')$$

when Fourier-expanded into  $k$  modes (in a box of size  $L \times L \times L$ )

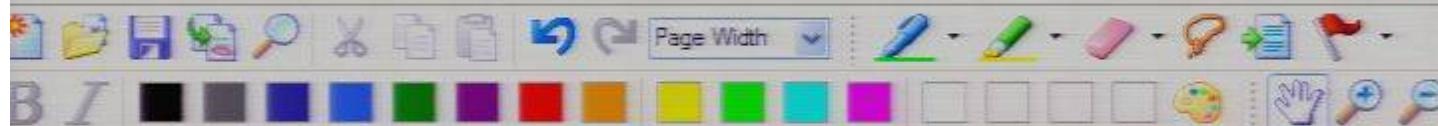
$$\hat{\phi}(x,t) = L^{-3/2} \sum_k \hat{\phi}_k(t) e^{ikx}$$

$$\sum k = \frac{2\pi}{L}(n_1, n_2, n_3), n_i \in \mathbb{Z}$$

becomes a harmonic oscillator for each mode:

$$\ddot{\hat{\phi}}_k(t) = -(\hbar^2 + m^2) \hat{\phi}_k(t) \text{ and } [\hat{\phi}_k, \hat{\phi}_{k'}] = i \delta_{k,k'}$$

□ Found interpretation: If QFT state  $|d\rangle$  is such that, e.g., the  $k$  mode oscillator is in  $n$ 'th energy eigenstate then  $n$  particles of momentum  $k$  are present.



Special case: States in QFT which describe a single particle.

□ Recall:

$$\hat{H} = \sum_k \hat{H}_k, \text{ with } \hat{H}_k = \frac{1}{2} \hat{\pi}_k^* \hat{\pi}_k + \frac{m^2}{2} \hat{\phi}_k^* \hat{\phi}_k$$

□ The state  $| \text{one particle of momentum } k \rangle$  is defined by:

$$\hat{H}_k | \text{one particle of momentum } k \rangle = \hbar \omega_k \left( \frac{1}{2} + 1 \right) | \text{one particle of momentum } k \rangle$$

and for  $k' \neq k$ :  $\hat{H}_{k'} | \text{one particle of momentum } k \rangle = \hbar \omega_{k'} \left( \frac{1}{2} + 0 \right) | \text{one particle of momentum } k \rangle$

□ By linear combination, the state

$|\text{1 particle w. prob. density } \psi(x)\rangle$

Recall:

$$\psi(x) = \int d^3x \psi(x) e^{ixk}$$

is given by:

Fourier transform of  $\psi(x)$

$$|\text{1 particle w. prob. density } \psi(x)\rangle = \frac{1}{\sqrt{2\pi}} \sum_k \tilde{\psi}_k | \text{one particle of momentum } k \rangle$$

$$= \frac{1}{\sqrt{2\pi}} \sum_k \left[ e^{-ikx} \psi(k) | \text{one particle of momentum } k \rangle \right]$$



But: Particle interpretation has limited applicability!

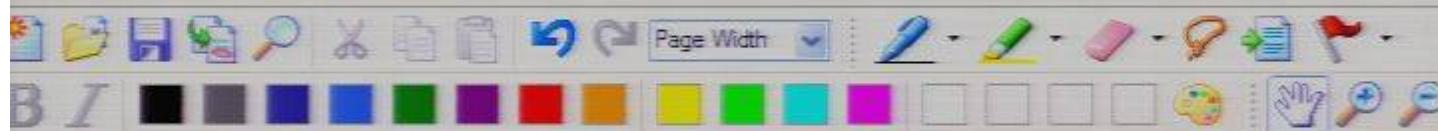
Why? □ Because the particle interpretation relies upon:

- Fourier decomposition (not covariant in general relativity)
- Harmonic oscillators have discrete spectrum (not if wavelength larger than "horizon")
- Vacuum state is lowest energy state (ill defined if time dependent: what is e.g. a momentum frequency?)

□ Each, a,b,c, have problems in presence of gravity, as we will see later!

□ Note: while the wave interpretation of  $\hat{\phi}(x,t)$  always applies,  
the field  $\hat{\phi}(x,t)$  is usually only indirectly measurable.

Notice: □ So far, we have independent harmonic oscillators - they do not get excited on their own.  
⇒ Waves do not increase in size and there is no particle creation.



## QFT for Cosmology, Achim Kempf, Winter 2010, Lecture 5

1/19/2006

Recall: □ The QFT problem



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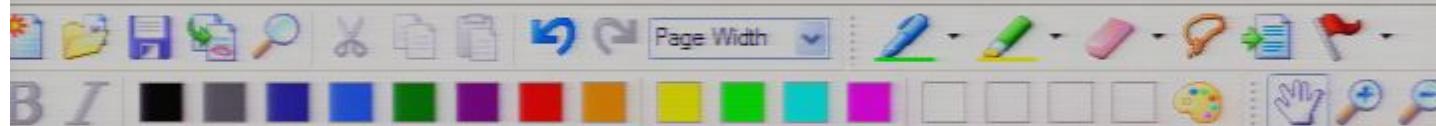
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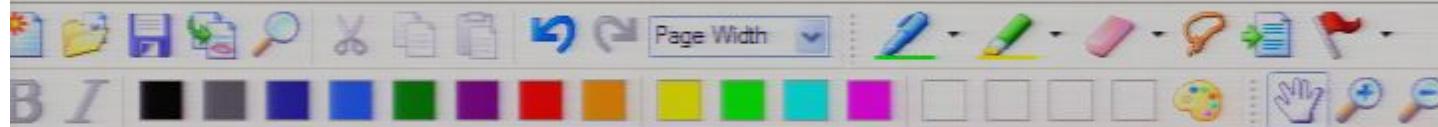
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Mechanisms for mode excitation / particle creation?

- Equivalent question:

What are mechanisms for exciting harmonic oscillators?



## Mechanisms for mode excitation/particle creation?

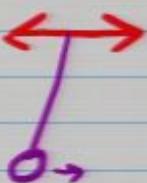
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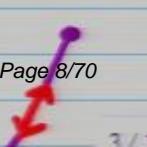
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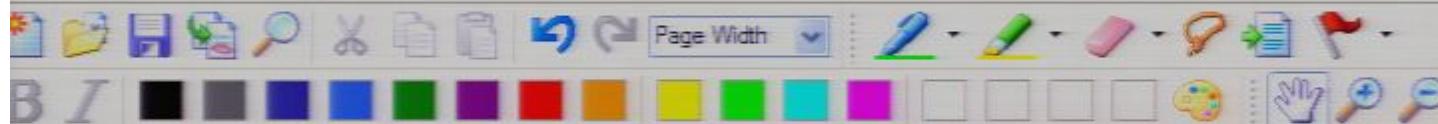
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e.g.:

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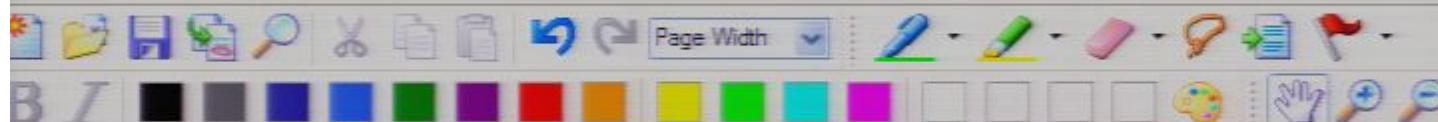
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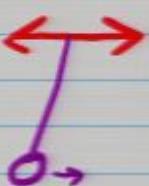
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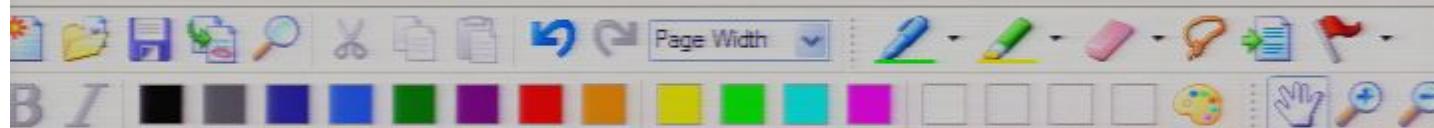
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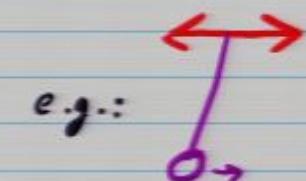
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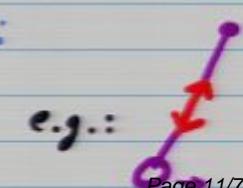
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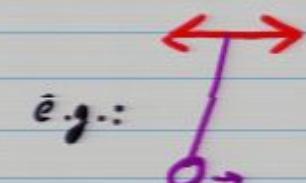
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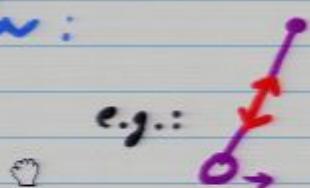
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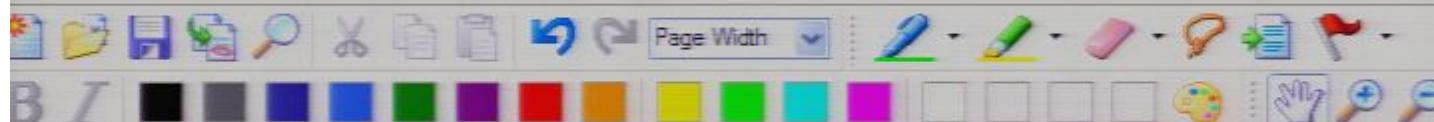
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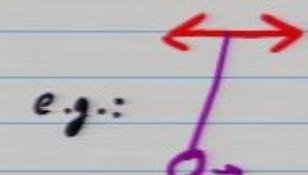
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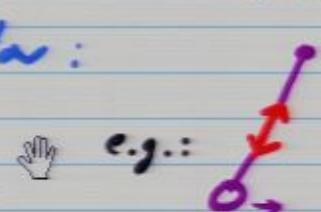
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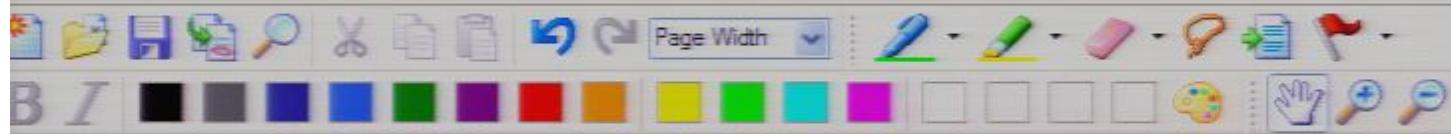


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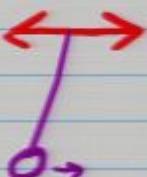


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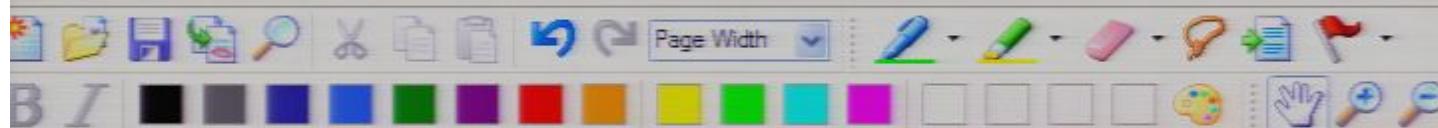


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Both occur in QFT:

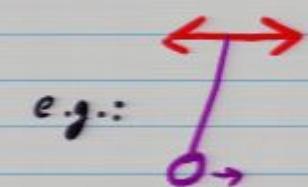
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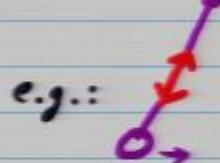
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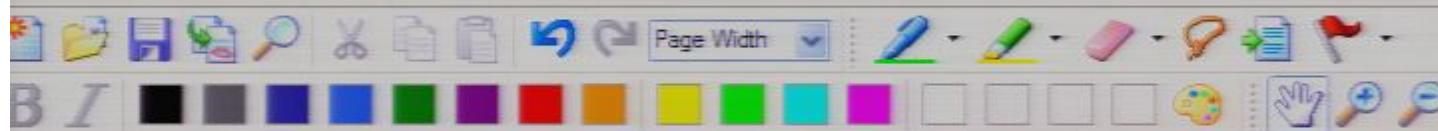
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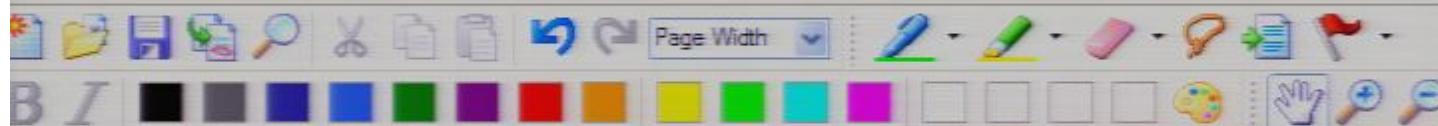
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□ Wave interpretation: Nontrivial interaction of waves of different types of fields

□ Particle interpretation: The collision of particles happens when their mode oscillators drive one another.  
→ Collisions can create and annihilate particles.

□ Strongest effects?

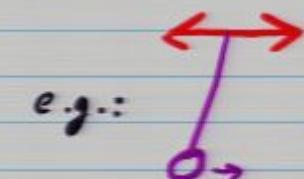
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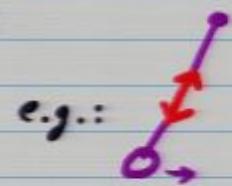
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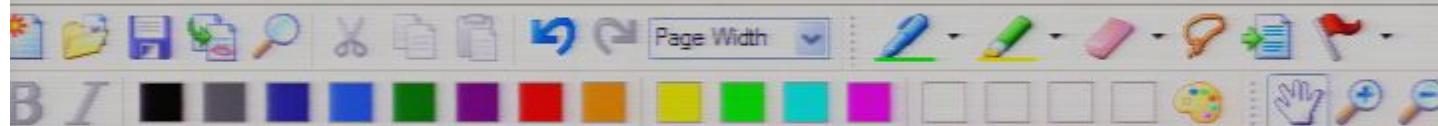
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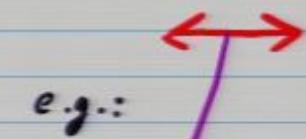
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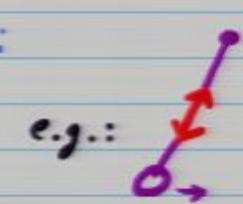
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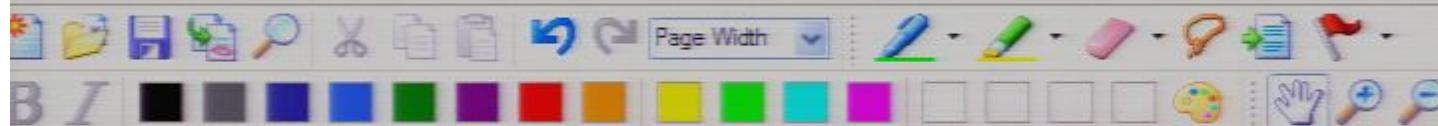
R. 11 . . . . .  $\Delta T$

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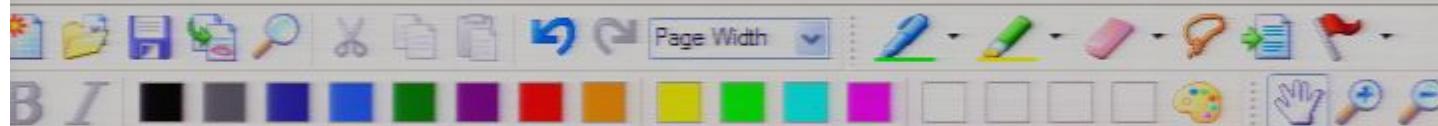
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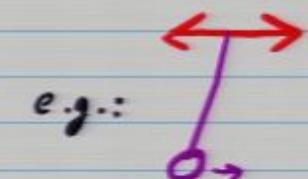
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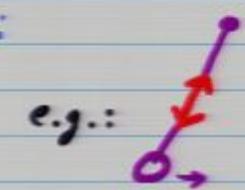
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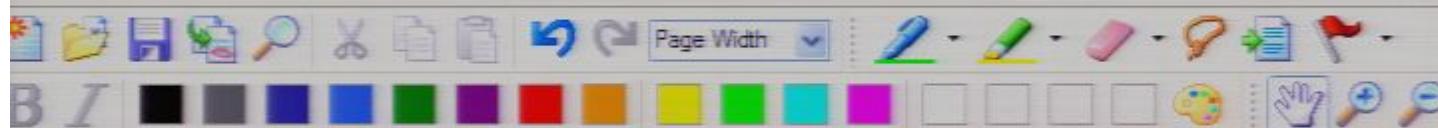
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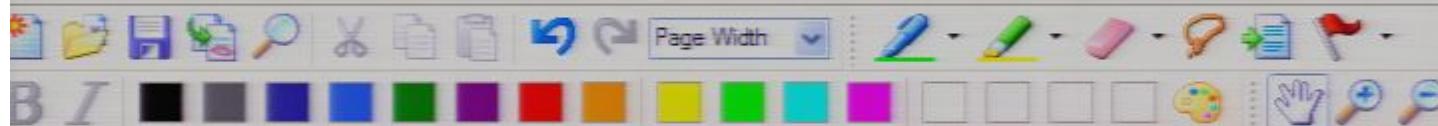
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⇒ expect  $\omega = \omega(t)$  decreases. True, and also...



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 $\Rightarrow$  expect  $\omega = \omega(t)$  decreases. Time, and also:

\* if wavelength > horizon then  $\omega^2(t) < 0$ !

$\Rightarrow$  runaway harmonic mode oscillators



(then: field amplification but no particle creation)

□ Particle interpretation:

Gravity can excite mode oscillators, i.e.  
it can create particles from the vacuum.

$$\omega_k = \sqrt{k^2 + m^2}$$



b.) The presence of gravity can effectively influence the  $\omega_r(t)$ .

□ Wave interpretation: \* E.g. cosmic expansion stretches the wavelength  
 $\Rightarrow$  expect  $\omega = \omega(t)$  decreases. Time, and also:

\* if wavelength > horizon then  $\omega^2(t) < 0$ !

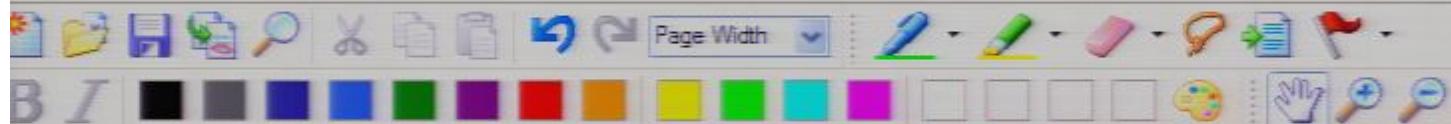
$\Rightarrow$  runaway harmonic mode oscillators



(then: field amplification but no particle creation)

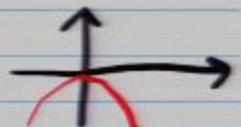
□ Particle interpretation:

Gravity can excite mode oscillators, i.e.  
it can create particles from the vacuum.



\* if wavelength > horizon then  $w(t) < 0$ !

⇒ runaway harmonic mode oscillators



(then: field amplification but no particle interpretation)

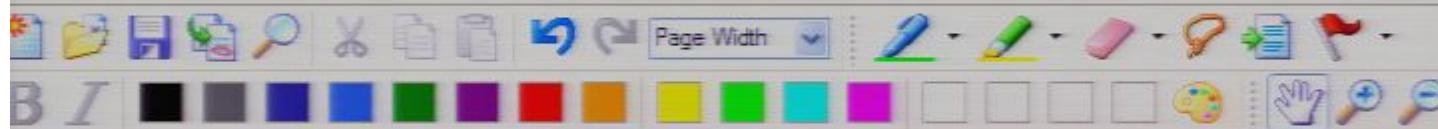
## Particle interpretation:

Gravity can excite mode oscillators, i.e.  
it can create particles from the vacuum.

Strongest effects? When oscillator resonates with  $w(t)$ . This effect is called parametric resonance.

Case a: Particle creation through external driving of mode oscillators.

Example: Production of photons by an antenna:



Case a: Particle creation through external driving of mode oscillators.

Example: Production of photons by an antenna:

□ We model the electromagnetic field as a Klein Gordon field.

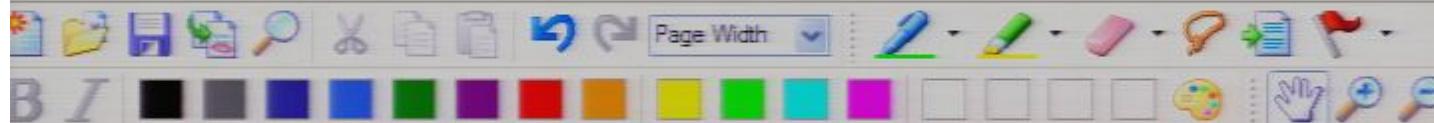
(The fact that EM fields have polarization and have  $m=0$  is not important here)

□ Consider an arbitrary mode of the electromagnetic field:

$$\hat{\phi}_k(t)$$

should really  
be quantized too

□ We model the electric current as a given classical field  $j(x,t)$  whose modes are  $j_k(t)$ .



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□ Consider an arbitrary mode of the electromagnetic field:

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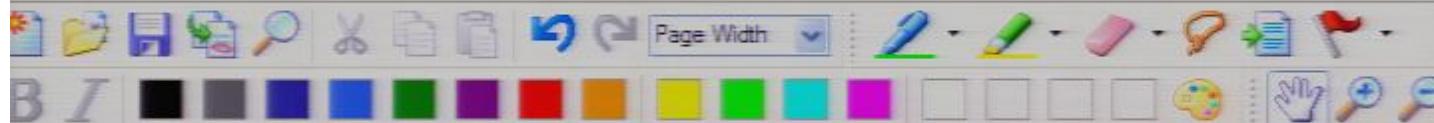
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□ We model the electric current as a given classical field  $j(x, t)$  whose modes are  $j_k(t)$ .  
should really be vector-valued

□ In a rough simplification, the EM **k** mode obeys:

$$\hat{H}_k = \frac{1}{2} \hat{\pi}_k^+(t) \hat{\pi}_k(t) + \frac{1}{2} \omega_k^2 \hat{\phi}_k^+(t) \hat{\phi}_k(t) + \hat{\phi}_k(t) j_k(t)$$

$\Rightarrow$  If the current  $j(t)$  varies in time it



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$\Rightarrow$  If the current  $j(t)$  varies in time it can excite the mode oscillators, thus creating photons.

$\Rightarrow$  Need to study the quantized driven harmonic oscillator!

$$\hat{H}(t) = \frac{1}{2} \hat{p}(t)^2 + \frac{\omega^2}{2} \hat{q}(t)^2 - J(t) \hat{q}(t)$$

for  $\hat{H}_k(t)$

for  $\hat{\pi}_k(t)$

stands for a field mode  $\hat{\phi}_k(t)$



stands for a mode  $j_k(t)$  of another classical (or better quantum) field.



$\Rightarrow$  Need to study the quantized driven harmonic oscillator!

$$\hat{H}(t) = \frac{1}{2} \hat{p}(t)^2 + \frac{\omega^2}{2} \hat{q}(t)^2 - J(t) \hat{q}(t)$$

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stands for a mode  $J_x(t)$   
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## I Preparation:

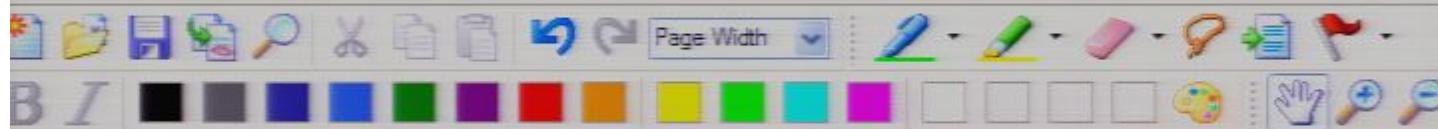
Recall that for all observables  $\hat{f}$ :

$$\hat{f}(t) = \langle \nu_0 | \hat{U}^+(t) \hat{f}_0 \hat{U}(t) | \nu_0 \rangle$$

state at initial time

operator at initial time

with the time-evolution operator obeying:



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state at initial time  
[operator at initial time]

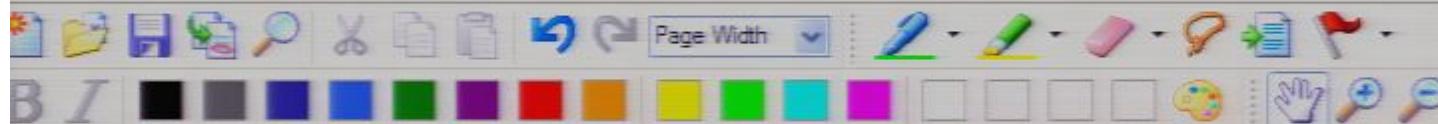
with the time-evolution operator obeying:

$$\hat{U}(t_0) = 1, \quad i \frac{d}{dt} \hat{U}(t) = \hat{U}(t) \hat{H}(t)$$

the original Hamiltonian  
[Heisenberg Hamiltonian]

□ Schrödinger picture? We write, equivalently:

$$\begin{aligned} \bar{f}(t) &= \left( \langle \psi_0 | \hat{U}^*(t) \right) \hat{f}_0 \underbrace{\left( \hat{U}(t) | \psi_0 \rangle \right)}_{= |\psi(t)\rangle} \\ &= \langle \psi(t) | \hat{f}_0 | \psi(t) \rangle \end{aligned}$$



$\Rightarrow$  Need to study the quantized driven harmonic oscillator!

[for  $\hat{H}_0(t)$ ] [for  $\hat{\Pi}_x(t)$ ] [stands for a field mode  $\hat{\phi}_x(t)$ ]

$$\hat{H}(t) = \frac{1}{2} \hat{p}(t)^2 + \frac{\omega^2}{2} \hat{q}(t)^2 - J(t) \hat{q}(t)$$



[stands for a mode  $J_x(t)$  of another classical (or better quantum) field.]

## I Preparation:

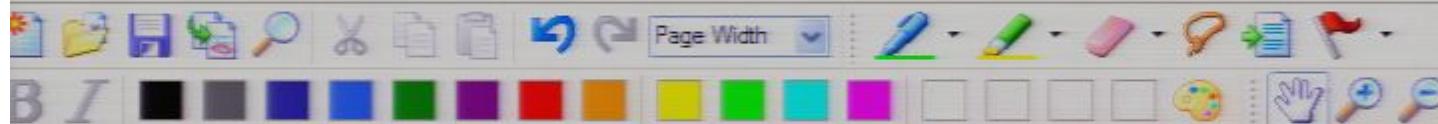
□ Recall that for all observables  $\hat{f}$ :

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↑ state at initial time  
↓ operator at initial time

with the time-evolution operator obeying:

$$\hat{U}(t_2) = 1, \quad i \frac{d}{dt} \hat{U}(t) = \hat{U}(t) \hat{H}(t)$$



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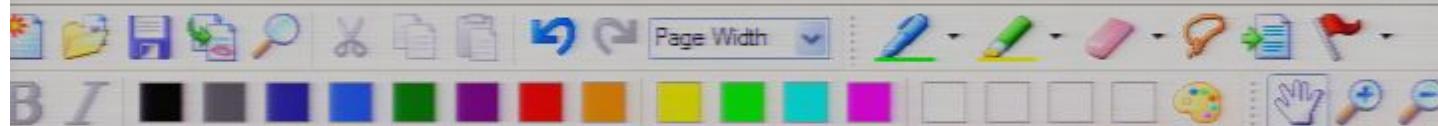
↑ the original Hamiltonian  
↑ "Heisenberg Hamiltonian"

□ Schrödinger picture? We write, equivalently:

$$= |\psi(t)\rangle$$

$$\bar{f}(t) = \left( \langle \psi_0 | \hat{U}^+(t) \right) \hat{f}_0 \left( \underbrace{\hat{U}(t)}_{\text{Exercise: check!}} | \psi_0 \rangle \right)$$

$$= \langle \psi(t) | \hat{f}_0 | \psi(t) \rangle$$



$$\hat{U}(t_0) = 1, \quad i \frac{d}{dt} \hat{U}(t) = \hat{U}(t) \hat{H}(t)$$

↑ "Heisenberg Hamiltonian"

↓ the original Hamiltonian

□ Schrödinger picture? We write, equivalently: Exercise: check!

$$\begin{aligned} &= |\psi(t)\rangle \\ \tilde{f}(t) &= (\langle \tilde{\psi}_0 | \hat{U}^+(t)) \tilde{f}_0 \underbrace{(\hat{U}(t) | \psi_0 \rangle)}_{\text{Hand icon}} \\ &= \langle \psi(t) | \tilde{f}_0 | \psi(t) \rangle \end{aligned}$$

↓ Recall:  $\tilde{A}_S(t) = \hat{H}(t)$  only if  $\frac{d}{dt} \hat{H}(t) = 0$

$$\text{The dynamics is } i \frac{d}{dt} |\psi(t)\rangle = \hat{H}_S(t) |\psi(t)\rangle$$

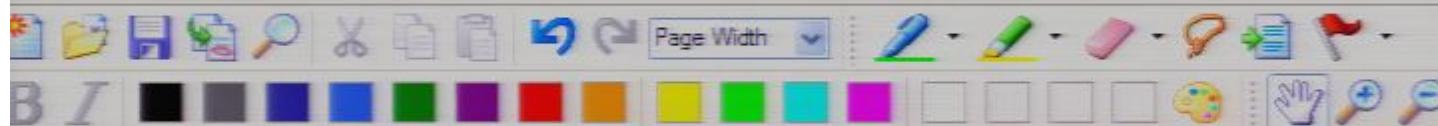
with Schrödinger Hamiltonian:  $\hat{H}_S(t) = \hat{U}(t) \hat{H}(t) \hat{U}^+(t)$

↓ Exercise: check

$$\omega_k = \sqrt{k^2 + m^2}$$

$$-i(t-t_0) \hat{H}$$

$$\hat{u}(t) = e$$



I Schrödinger picture? We write, equivalently: Exercise: check!

$$\begin{aligned} &= |\psi(t)\rangle \\ \tilde{\psi}(t) &= (\langle \psi_0 | \hat{U}^+(t)) \tilde{\psi}_0 \underbrace{(\hat{U}(t) | \psi_0 \rangle)}_{\text{Hand icon}} \\ &= \langle \psi(t) | \tilde{\psi}_0 | \psi(t) \rangle \end{aligned}$$

Recall:  $\dot{A}_S(t) = \dot{H}(t)$  only if  $\frac{d}{dt} \hat{H}(t) = 0$

The dynamics is  $i \frac{d}{dt} |\psi(t)\rangle = \dot{H}_S(t) |\psi(t)\rangle$

with Schrödinger Hamiltonian:  $\hat{H}_S(t) = \hat{U}(t) \hat{H}(t) \hat{U}^+(t)$

Exercise: check



Exercise: check

- We will use, equivalently, the Heisenberg picture:

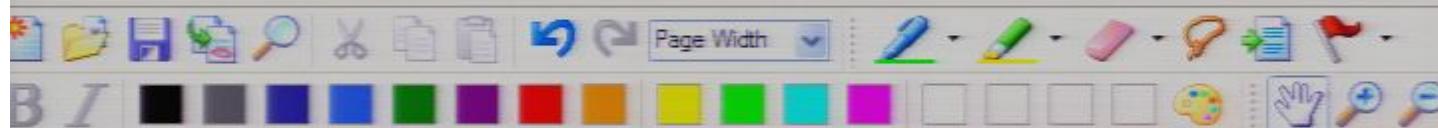
$$\begin{aligned}\bar{f}(t) &= \langle \nu_0 | \underbrace{\left( \hat{U}^\dagger(t) f_0 \hat{U}(t) \right)}_{\text{"}} | \nu_0 \rangle \\ &= \langle \nu_0 | \hat{f}(t) | \nu_0 \rangle\end{aligned}$$

with dynamics:

$i \frac{d}{dt} \hat{f}(t) = [\hat{f}(t), \hat{H}(t)]$

## II Aspects of the Heisenberg picture:

- The state of the quantum system stays the same



## II Aspects of the Heisenberg picture:

- The state of the quantum system stays the same Hilbert space vector, say  $|x\rangle \in \mathcal{H}$  (from measurement to measurement).
- The observables, say  $\hat{H}(t)$ ,  $\hat{f}(t)$ , etc, are time-dependent operators in Hilbert space.
- Important implication:

The eigenbases and the eigenvalues of observables, such as  $\hat{H}(t)$  and any  $\hat{f}(t)$  depend on time!

$$\hat{f}(t) |f_n(t)\rangle = f_n(t) |\tilde{f}_n(t)\rangle$$

$$\hat{H}(t) |E_m(t)\rangle = E_m(t) |E_m(t)\rangle$$

$$\omega_k = \sqrt{k} + \dots$$

$$\hat{v}(t) = e$$

$$i\bar{Q}u^+$$

$$\omega_k = \sqrt{k} + \dots$$

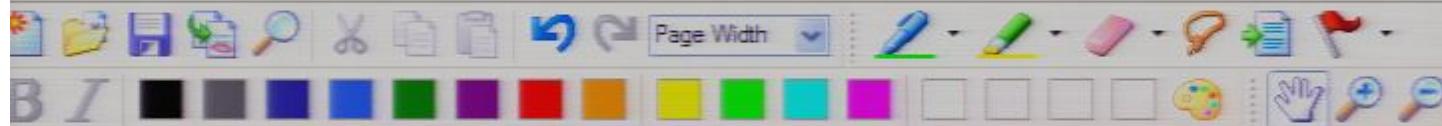
$$\hat{v}(t) = e$$

$$i\dot{Q}u^+ , \hat{Q}$$

$$\mathcal{U} \quad \dot{\mathcal{U}}(t) =$$
$$\mathcal{U}^1 \parallel +$$
$$\mathcal{U}Q\mathcal{U}^{-1}, Q$$

$$\mathcal{U}^{-1} \hat{u}(t) = e^{-it} \hat{Q} \mathcal{U}^+ u$$

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Example: \* Assume the driven harmonic oscillator starts out at time  $t_1$ , in a state, say  $|x\rangle = |E_n(t_1)\rangle$ :

$$\hat{H}(t_1) |E_n(t_1)\rangle = E_n(t_1) |E_n(t_1)\rangle$$



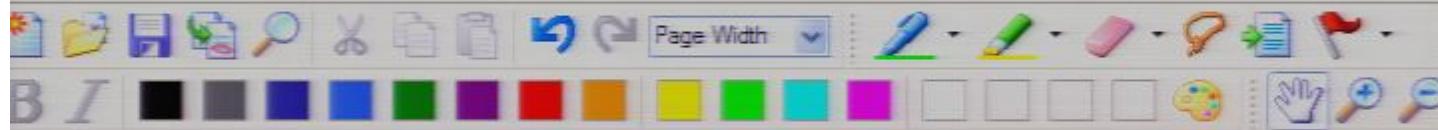
\* State vector of the system stays  $|x\rangle$  for  $t > t_1$ .

\* But at later times, say  $t > t_1$ , the Hamiltonian and its eigenvectors and eigenvalues are

$$\hat{H}(t) |E_n(t)\rangle = E_n(t) |E_n(t)\rangle$$

and we generally have

$$E_n(t) \neq E_n(t_1), |E_n(t)\rangle \neq |E_n(t_1)\rangle$$



$$\hat{H}(t_1) |E_n(t_1)\rangle = E_n(t_1) |E_n(t_1)\rangle$$

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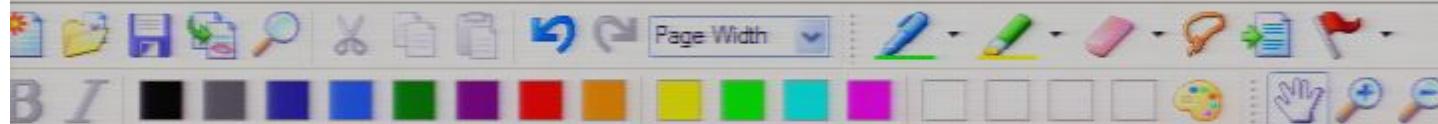
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$$E_n(t) \neq E_n(t_1), |E_n(t)\rangle \neq |E_n(t_1)\rangle$$

$\Rightarrow$  At time  $t_2$  system is still in state  $|x\rangle$  and still

$$|x\rangle = |E_n(t_1)\rangle$$

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\* But at later times, say  $t > t_0$ , the Hamiltonian and its eigenvectors and eigenvalues are

$$\hat{H}(t) |E_n(t)\rangle = E_n(t) |E_n(t)\rangle$$

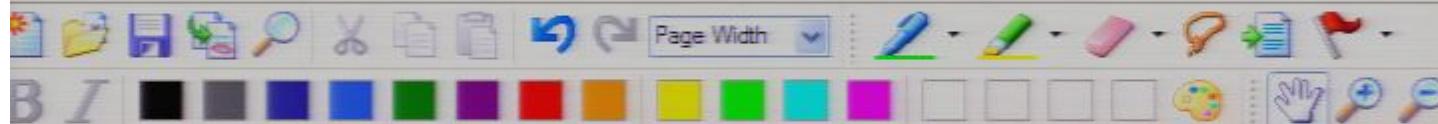
and we generally have

$$E_n(t) \neq E_n(t_0), |E_n(t)\rangle \neq |E_n(t_0)\rangle$$

$\Rightarrow$  At time  $t_0$ , system is still in state  $|x\rangle$  and still  
 $|x\rangle = |E_n(t_0)\rangle$

but  $|x\rangle$  is generally no longer with (or any other) energy eigenstate!

In particular:



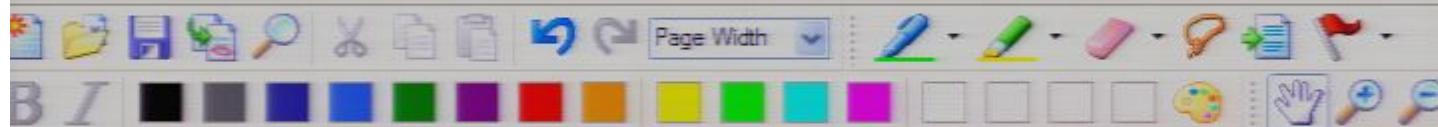
In particular:

\* Assume system starts out at  $t_1$  in lowest energy state (i.e. in vacuum):  $| \psi \rangle = | E_0(t_1) \rangle$

\* Then if  $| \psi \rangle = | E_0(t_1) \rangle \neq | E_0(t_2) \rangle$

$\Rightarrow$  At  $t_2$  the system's state  $| \psi \rangle$  is not the ground state i.e. not the vacuum state, i.e. particles (e.g. photons) exist at time  $t_2$ .

### III Strategy for solving quantized driven harmonic oscillator



### III Strategy for solving quantized driven harmonic oscillator

□ Problem: \* CCR:  $[\hat{q}(t), \hat{p}(t)] = i\hbar$

\* Hermiticity:  $\hat{q}^+(t) = \hat{q}(t)$ ,  $\hat{p}^+(t) = \hat{p}(t)$

\* Hamiltonian:  $\hat{H}(t) = \frac{1}{2}\hat{p}(t)^2 + \frac{\omega^2}{2}\hat{q}(t)^2 - J(t)\hat{q}(t)$

\* Heisenberg eqns  $i\dot{f}(t) = [f(t), \hat{H}(t)]$  yield:

$$\dot{\hat{q}}(t) = \hat{p}(t)$$

$$\dot{\hat{p}}(t) = -\omega^2 \hat{q}(t) + J(t)$$

This is a good strategy  
with and without a  
driving force

↓  
□ Strategy: \* Combine  $a(t) := \alpha \hat{q}(t) + i\beta \hat{p}(t)$

↓ is operator even though no "hat".

$\alpha(t) := \alpha \hat{q}(t) + i\beta \hat{p}(t)$   
[Mukhanov calls it  $a^-(t)$ ]

(analogous to "real" &  
"imaginary" parts)

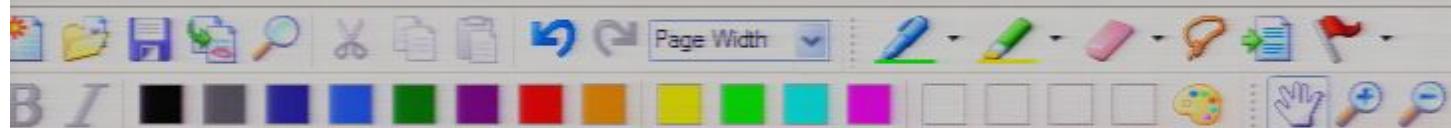
\* Choose  $\alpha, \beta$  so that  $\hat{H}(t)$  and eqn of motion simplify.

$$-i(t-t_0) \hat{H}$$

$$\hat{H} \cdot A = \int \cancel{A} \phi \psi$$

$$-i(t-\zeta) \hat{H}$$

$$\hat{H} \tilde{\psi} = \int \cancel{\mathcal{A}} \phi \tilde{\psi} d^3x$$



\* Hamiltonian:  $H(t) = \frac{1}{2} \hat{p}(t)^2 + \frac{\omega^2}{2} \hat{q}(t)^2 - J(t) \hat{q}(t)$

\* Heisenberg eqns  $i \dot{f}(t) = [\hat{f}(t), \hat{H}(t)]$  yield:

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This is a good strategy  
with and without a  
driving force

Strategy: \* Combine

$$\underline{a}(t) := \alpha \hat{q}(t) + i\beta \hat{p}(t)$$

↓ is operator even though no "hat".  
↓ Much easier to handle than  $\hat{q}(t)$  and  $\hat{p}(t)$

(analogous to "real" &  
"imaginary" parts)

\* Choose  $\alpha, \beta$  so that  $\hat{H}(t)$  and eqn of motion simplify.

## IV Determine $\alpha$ and $\beta$ :

Notice first that once we have  $a(t)$  we immediately obtain  $\dot{q}(t), \dot{p}(t)$ : Use of  $\dot{a}^*(t) = \alpha \dot{q}(t) - i\beta \dot{p}(t)$  yields:



\* Heisenberg eqns  $[f(t), H(t)]$  yield:

This is a good strategy  
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$$\dot{\hat{q}}(t) = \hat{p}(t)$$

$$\dot{\hat{p}}(t) = -\omega^2 \hat{q}(t) + J(t)$$

$\downarrow$  is operator even though no "hat".

$$\alpha(t) := \alpha^*(t) + i\beta\hat{p}(t)$$

$\downarrow$  Muthann calls it  $\alpha^*(t)$

(analogous to "real" &  
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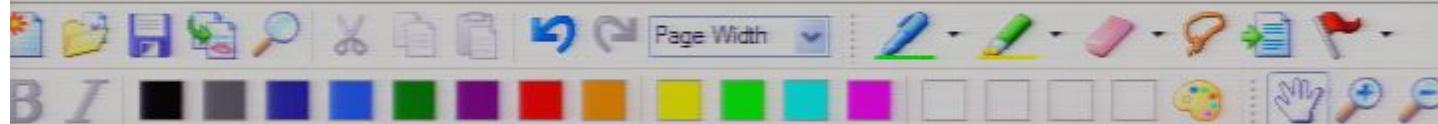
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$$\dot{\hat{q}}(t) = \frac{1}{2\omega} (\alpha^*(t) + \alpha(t))$$



## IV Determine $\alpha$ and $\beta$ :

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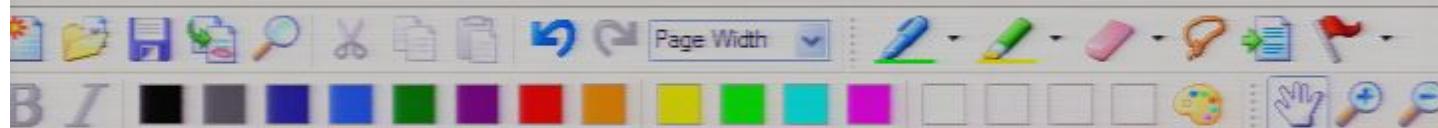
$$\dot{p}(t) = \frac{i}{2\beta} (a^+(t) - a(t))$$

□ Use this to express  $[q, p] = i$  in terms of new variable  $a(t)$ :

$$\Rightarrow [a(t), a^+(t)] = 2\alpha\beta$$

For simplicity, we choose  $\beta = \frac{1}{2\alpha}$  so that:

$$[a(t), a^+(t)] = 1$$



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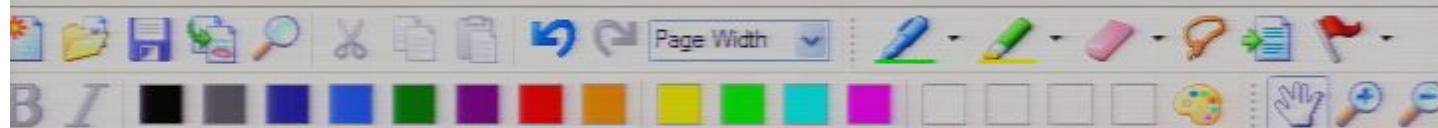
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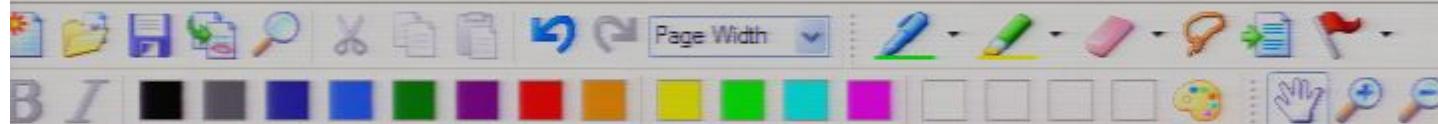
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□ Notice first that once we have  $a(t)$  we immediately obtain  $\dot{q}(t)$ ,  $\dot{p}(t)$ : Use of  $a^+(t) = \omega q(t) - i\beta p(t)$  yields:

$$\dot{q}(t) = \frac{1}{2\omega} (a^+(t) + a(t))$$

$$\dot{p}(t) = \frac{i}{2\beta} (a^+(t) - a(t))$$

□ Use this to express  $[q, p] = i$  in terms of new variable  $a(t)$ :

$$\Rightarrow [a(t), a^+(t)] = 2\omega\beta$$

For simplicity, we choose  $\beta = \frac{1}{2\omega}$  so that:

$$\boxed{[a(t), a^+(t)] = 1}$$



□ Now express  $\hat{H}(t)$  in terms of new variable  $a(t)$ :

$$\begin{aligned}\hat{H}(t) = & -\frac{1}{2} \omega^2 (a^+(t) - a(t))^2 + \frac{\omega^2}{2} \frac{1}{4\omega^2} (a^+(t) + a(t))^2 \\ & - J(t) \frac{1}{2\omega} (a^+(t) + a(t))\end{aligned}$$

We notice that the terms  $\sim a^+(t)^2$  and  $\sim a(t)^2$  drop out if we choose:

$$-\frac{1}{2} \omega^2 + \frac{\omega^2}{2} \frac{1}{4\omega^2} = 0$$

Thus, we choose:  $\omega = \sqrt{\frac{\omega}{2}}$  and therefore  $\beta = \frac{1}{\sqrt{2\omega}}$

□ Thus, by definition:

$$a(t) = \sqrt{\frac{\omega}{2}} \hat{q}(t) + i \frac{1}{\sqrt{2\omega}} \hat{p}(t)$$



1 Notice first that once we have  $a(t)$  we immediately obtain  $\dot{q}(t)$ ,  $\dot{p}(t)$ : Use of  $a^+(t) = \omega q(t) - i\beta p(t)$  yields:

$$\dot{q}(t) = \frac{1}{2\omega} (a^+(t) + a(t))$$

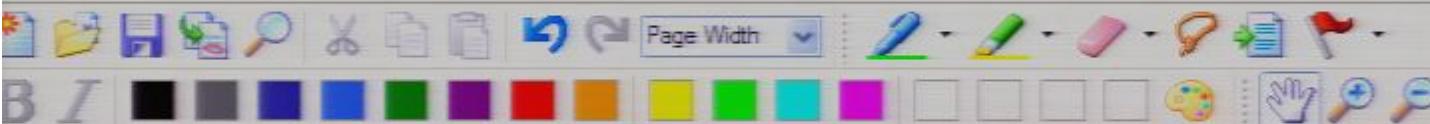
$$\dot{p}(t) = \frac{i}{2\beta} (a^+(t) - a(t))$$

2 Use this to express  $[q, p] = i$  in terms of new variable  $a(t)$ :

$$\Rightarrow [a(t), a^+(t)] = 2\omega\beta$$

For simplicity, we choose  $\beta = \frac{1}{2\omega}$  so that:

$$[a(t), a^+(t)] = 1$$



□ Now express  $\hat{H}(t)$  in terms of new variable  $a(t)$ :

$$\begin{aligned}\hat{H}(t) = & -\frac{1}{2} \omega^2 (a^*(t) - a(t))^2 + \frac{\omega^2}{2} \frac{1}{4\omega^2} (a^*(t) + a(t))^2 \\ & - J(t) \frac{1}{2\omega} (a^*(t) + a(t))\end{aligned}$$



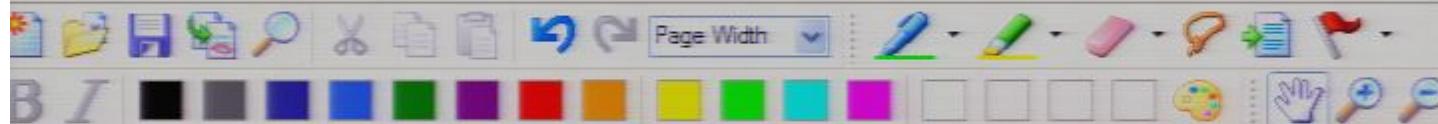
We notice that the terms  $\sim a^*(t)^2$  and  $\sim a(t)^2$  drop out if we choose:

$$-\frac{1}{2} \omega^2 + \frac{\omega^2}{2} \frac{1}{4\omega^2} = 0$$

Thus, we choose:  $\omega = \sqrt{\frac{\omega}{2}}$  and therefore  $\beta = \frac{1}{\sqrt{2\omega}}$

□ Thus, by definition:

$$a(t) = \sqrt{\frac{\omega}{2}} \hat{q}(t) + i \frac{1}{\sqrt{2\omega}} \hat{p}(t)$$



□ The Hamiltonian simplifies to become:

$$\hat{H}(t) = \omega (a^\dagger(t)a(t) + \frac{1}{2}) - J(t)\frac{1}{\sqrt{2\omega}}(a^\dagger(t) + a(t))$$

II Solve for  $a(t)$ :

□ The Heisenberg equation  $i\dot{f}(t) = [f(t), \hat{H}(t)]$  reads for  $a(t)$ :

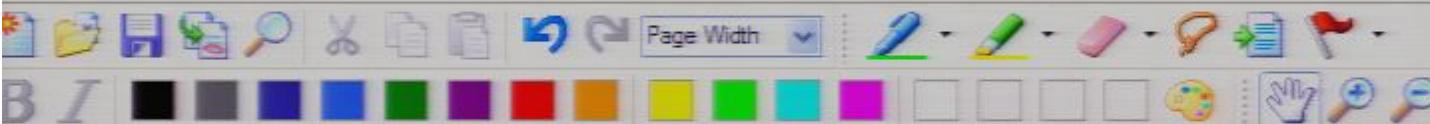
$$i\dot{a}(t) = \omega a(t) - \frac{i}{\sqrt{2\omega}} J(t)$$

□ Let us give  $a(t=0)$  a name:  $a_{in} = a(0)$ . Then:

Exercise:

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$$a(t) = a_{in} e^{-i\omega t} + 1 \frac{i}{\sqrt{1}} \int_0^t J(t') e^{i\omega(t'-t)} dt'$$



□ The Hamiltonian simplifies to become:

$$\hat{H}(t) = \omega (a^*(t)a(t) + \frac{1}{2}) - J(t) \frac{1}{\sqrt{2\omega}} (a^*(t) + a(t))$$

IV Solve for  $a(t)$ :

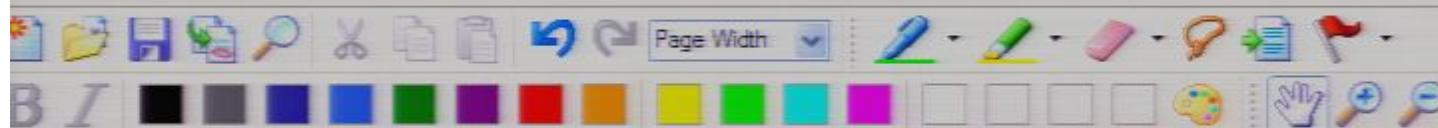
□ The Heisenberg equation  $i\dot{\hat{f}}(t) = [\hat{f}(t), \hat{H}(t)]$  reads for  $a(t)$ :

$$i\dot{a}(t) = \omega a(t) - \frac{i}{\sqrt{2\omega}} J(t)$$

□ Let us give  $a(t=0)$  a name:  $a_{in} = a(0)$ . Then:

Exercise:  
verify.

$$a(t) = a_{in} e^{-i\omega t} + \frac{i}{\sqrt{2\omega}} \int_0^t J(t') e^{i\omega(t'-t)} dt'$$



$$i \dot{\alpha}(t) = \omega_0 \alpha(t) - \frac{i}{\sqrt{2}\omega} J(t)$$

□ Let us give  $\alpha(t=0)$  a name:  $\alpha_{in} = \alpha(0)$ . Then:

Exercise:

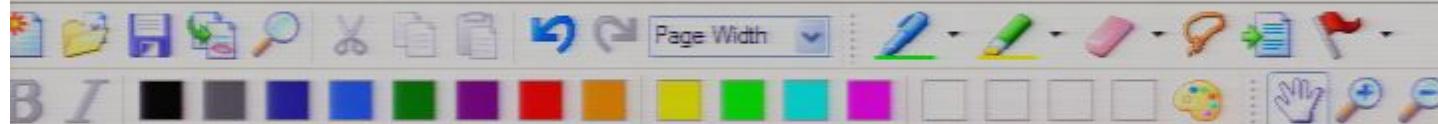
verify.

$$\alpha(t) = \alpha_{in} e^{-i\omega t} + \frac{1}{\sqrt{2}\omega} \int_0^t J(t') e^{i\omega(t'-t)} dt'$$

## VII Case of force of finite duration

□ Assume  $J(t) = 0$  for all  $t \notin [0, T]$





## II Solve for $a(t)$ :

□ The Heisenberg equation  $i\dot{f}(t) = [\hat{f}(t), \hat{H}(t)]$  reads for  $a(t)$ :

$$i\dot{a}(t) = \omega a(t) - \frac{i}{\sqrt{2\omega}} J(t)$$

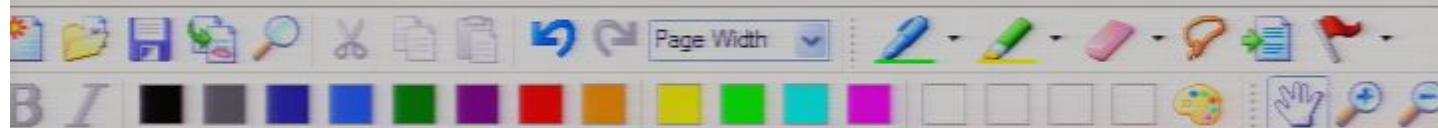
□ Let us give  $a(t=0)$  a name:  $a_{in} = a(0)$ . Then:

Exercise:  
verify.

$$a(t) = a_{in} e^{-i\omega t} + \frac{i}{\sqrt{2\omega}} \int_0^t J(t') e^{i\omega(t'-t)} dt'$$

## VII Case of force of finite duration

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□ Let us give  $a(t=0)$  a name:  $a_{in} = a(0)$ . Then:

Exercise:

verify:

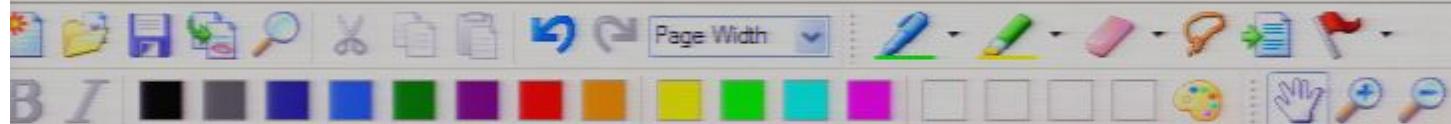
$$a(t) = a_{in} e^{-i\omega t} + \frac{1}{\sqrt{2\omega}} \int_0^t J(t') e^{i\omega(t'-t)} dt'$$



## VII Case of force of finite duration

□ Assume  $J(t) = 0$  for all  $t \notin [0, T]$

↑  $J(t)$

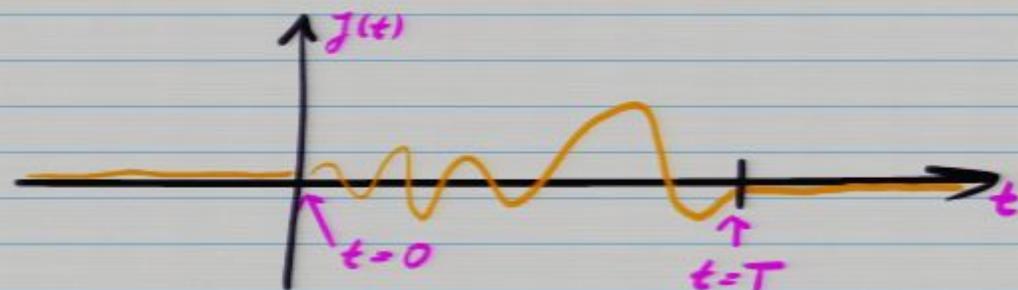


Exercise:  
verify.

$$a(t) = a_m e^{-i\omega t} + \frac{i}{\sqrt{2}\omega} \int_0^t J(t') e^{i\omega(t'-t)} dt'$$

## VII Case of force of finite duration

□ Assume  $J(t) = 0$  for all  $t \notin [0, T]$

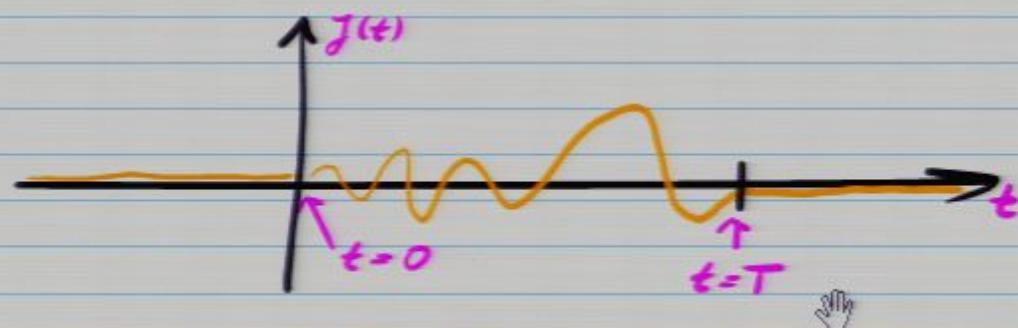


□ Define  $I_a := \frac{i}{\sqrt{2}\omega} \int_0^T J(t') e^{i\omega t'} dt'$



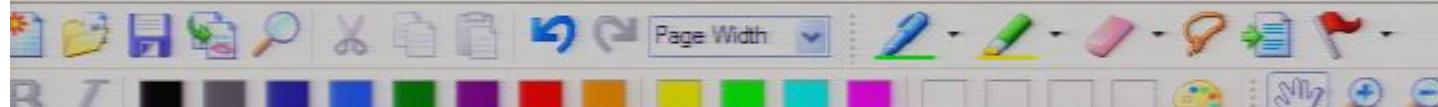
## VII Case of force of finite duration

□ Assume  $J(t) = 0$  for all  $t \notin [0, T]$



□ Define  $J_0 := \frac{1}{\sqrt{2\pi}} \int_0^T J(t') e^{-i\omega t'} dt'$

□ Then:  $a(t) = \begin{cases} a_m e^{-i\omega t} & \text{for } t < 0 \\ \text{see above} & \text{for } t \in [0, T] \\ (a_m + J_0) e^{-i\omega t} & \text{for } t > T \end{cases}$



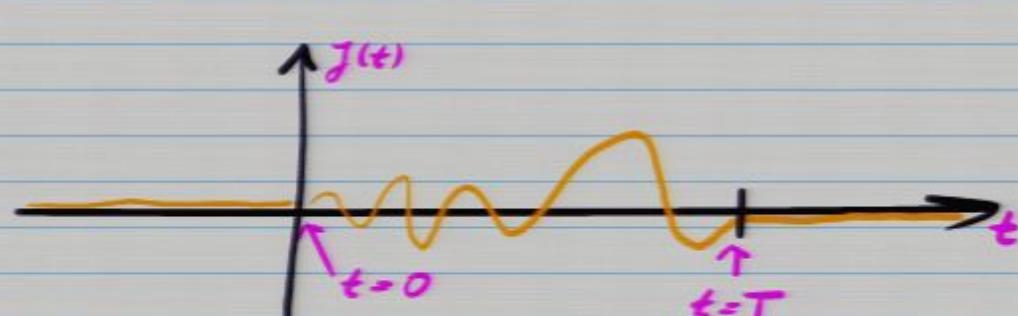
Let us give  $a(t=0)$  a name:  $a_{in} = a(0)$ . Then:

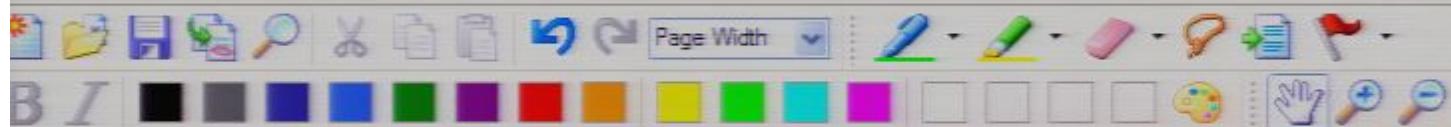
Exercise:  
verify.

$$a(t) = a_{in} e^{-i\omega t} + \frac{i}{\sqrt{2}\omega} \int_0^t J(t') e^{i\omega(t'-t)} dt'$$

## VII Case of force of finite duration

Assume  $J(t) = 0$  for all  $t \notin [0, T]$



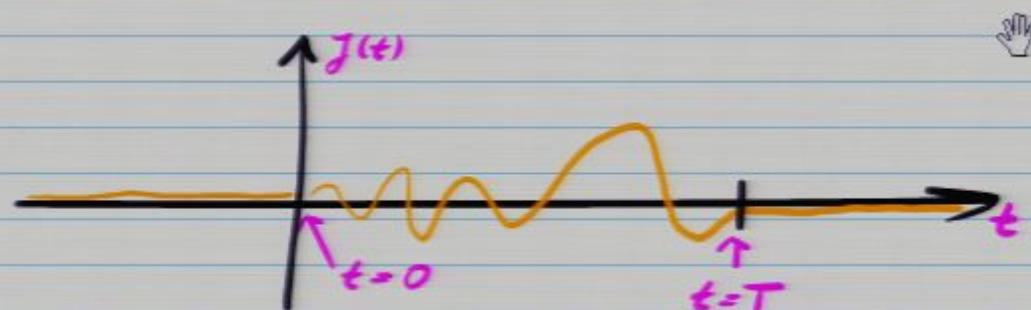


Exercise:  
verify.

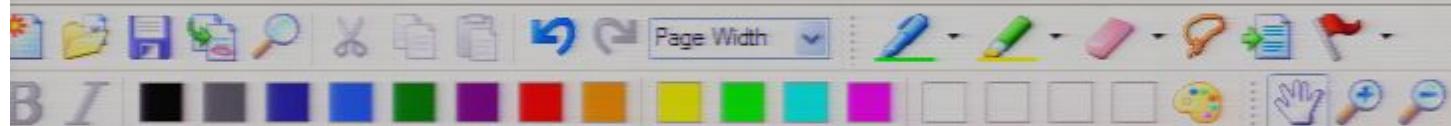
$$a(t) = a_m e^{-i\omega t} + \frac{i}{\sqrt{2\omega}} \int_0^t J(t') e^{i\omega(t'-t)} dt'$$

## VII Case of force of finite duration

□ Assume  $J(t) = 0$  for all  $t \notin [0, T]$



□ Define  $J_0 := \frac{i}{\sqrt{2\omega}} \int_0^T J(t') e^{i\omega t'} dt'$



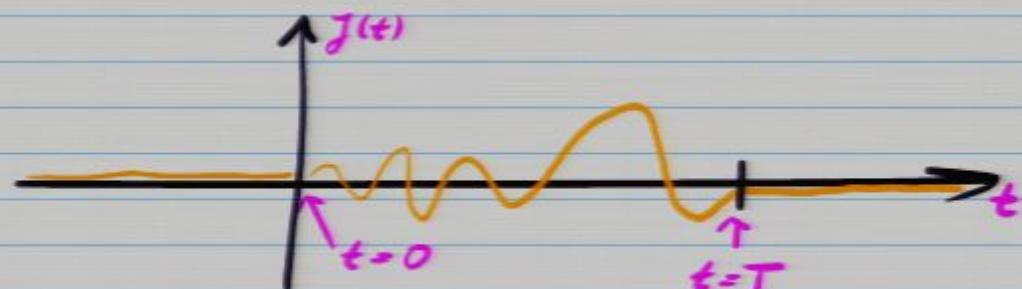
Exercise:

verify.

$$a(t) = a_m e^{-i\omega t} + \frac{i}{\sqrt{2}\omega} \int_0^t j(t') e^{i\omega(t'-t)} dt'$$

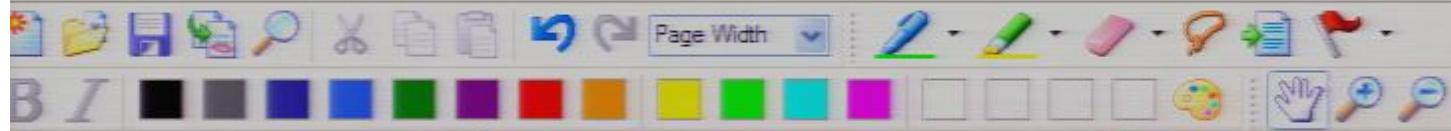
## VII Case of force of finite duration

□ Assume  $j(t) = 0$  for all  $t \notin [0, T]$

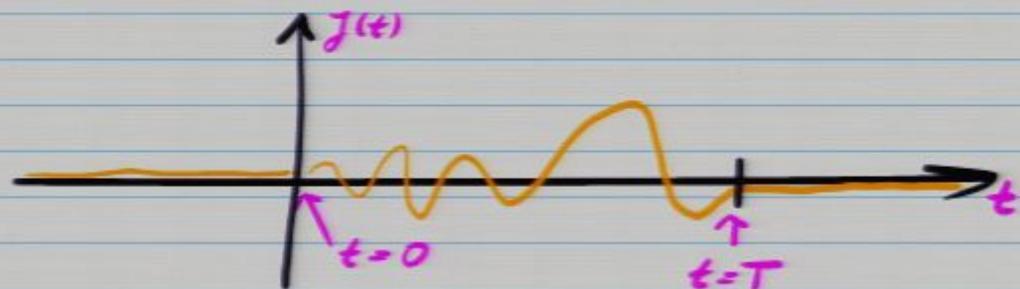


□ Define  $J_0 := \frac{i}{\sqrt{2}\omega} \int_0^T j(t') e^{i\omega t'} dt'$

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□ Assume  $J(t) = 0$  for all  $t \notin [0, T]$



□ Define  $J_0 := \frac{i}{\gamma_{2w}} \int_0^T J(t') e^{i\omega t'} dt'$

□ Then:  $a(t) = \begin{cases} a_0 e^{-i\omega t} & \text{for } t < 0 \\ \text{see above} & \text{for } t \in [0, T] \\ (a_0 + J_0) e^{-i\omega t} & \text{for } t > T \end{cases}$

$$\alpha_R = \cancel{\alpha} \beta_L + \cancel{\alpha} \beta_L$$