Title: Quantum Field Theory for Cosmology - Lecture 3

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Abstract: This course begins with a thorough introduction to quantum field theory. Unlike the usual quantum field theory courses which aim at applications to particle physics, this course then focuses on those quantum field theoretic techniques that are important in the presence of gravity. In particular, this course introduces the properties of quantum fluctuations of fields and how they are affected by curvature and by gravitational horizons. We will cover the highly successful inflationary explanation of the fluctuation spectrum of the cosmic microwave background - and therefore the modern understanding of the quantum origin of all inhomogeneities in the universe (see these amazing visualizations from the data of the Sloan Digital Sky Survey. They display the inhomogeneous distribution of galaxies several billion light years into the universe: Sloan Digital Sky Survey).

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QFT for Cosmology, Achim Kempf, Winter 10, Lecture 3

Recall:

- Considered the simplest relativistic generalization of the Schrödinger eqn, the Klein Gordon eqn:

$$\left(\frac{\partial^2}{\partial t^2} - \Delta + m^2\right)\phi = 0 \qquad (\hbar = 1 = c)$$

(unlike the Schrödinger equation)

Notice: The K.G. eqn does not couple $Re(\phi)$ to $Im(\phi)$: each separately fulfills the K.G. eqn.

$\Rightarrow$ It suffices to study real-valued ϕ.

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Quantization conditions:

analogous to:

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Quantization conditions:

analogous to:

$$[\hat{\phi}(x,t), \hat{\pi}(x',t)] = i\hbar\,\delta^3(x-x') \qquad [\hat{q}_a(t), \hat{p}_{a'}(t)] = i\hbar\,\delta_{aa'}$$

$$[\hat{\phi}(x,t), \hat{\phi}(x',t)] = 0 \qquad [\hat{q}_a(t), \hat{q}_{a'}(t)] = 0$$

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$$\left[\hat{\phi}(x,t), \hat{\phi}(x',t)\right] = 0$$

$$[\hat{q}_a(t), \hat{q}_{a'}(t)] = 0$$

$$\left[\hat{\pi}(x,t), \hat{\pi}(x',t)\right] = 0$$

$$[\hat{p}_a(t), \hat{p}_{a'}(t)] = 0$$

We keep the equations of motion:

E1) $\dot{\hat{\phi}}(x,t) = \hat{\pi}(x,t)$ $\qquad \dot{\hat{q}}_a(t) = \hat{p}_a(t)$

E2) $\dot{\hat{\pi}}(x,t) = -\left(-\Delta + m^2\right)\hat{\phi}(x,t)$ $\qquad \dot{\hat{p}}_a(t) = -k_a \hat{q}_a(t)$

Note: $\phi^*(x,t) = \phi(x,t)$ now implies hermiticity: $\hat{\phi}^+(x,t) = \hat{\phi}(x,t)$

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Proposition: $E1, E2$ follow from the Heisenberg eqns

analogous to:

$$i\hbar\,\dot{\hat{\phi}}(x,t) = [\hat{\phi}(x,t), \hat{H}] \qquad i\hbar\,\dot{\hat{q}}_a(t) = [\hat{q}_a(t), \hat{H}]$$

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Proposition: E1, E2 follow from the Heisenberg eqns

$$i\hbar \dot{\hat{\phi}}(x,t) = [\hat{\phi}(x,t), \hat{H}]$$

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analogous to:

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with this QFT Hamiltonian:

$$\hat{H} = \int_{\mathbb{R}^3} \frac{1}{2}\hat{\pi}^2(x',t) + \frac{1}{2}\hat{\phi}(x',t)\left(m^2 - \Delta\right)\hat{\phi}(x',t)\, d^3x'$$

$$\hat{H} = \sum_a \frac{\hat{p}_a^2}{2} + \frac{\omega_a^2}{2}\hat{q}_a^2$$

Indeed, e.g.:

$$i\hbar \dot{\hat{\phi}}(x,t) = [\hat{\phi}(x,t), H] = \left[\hat{\phi}(x,t), \int_{\mathbb{R}^3} \frac{1}{2}\hat{\pi}^2(x',t) + \text{something}(\hat{\phi})\, d^3x'\right]$$

$$= \frac{1}{2}\int \left[\hat{\phi}(x,t), \hat{\pi}(x',t)\right]\hat{\pi}(x',t) + \hat{\pi}(x',t)\left[\hat{\phi}(x,t), \hat{\pi}(x',t)\right] d^3x'$$

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analogous to:

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$$= \frac{1}{2}\int [\hat{\phi}(x,t), \hat{\pi}(x',t)]\hat{\pi}(x',t) + \hat{\pi}(x',t)[\hat{\phi}(x,t), \hat{\pi}(x',t)]\, d^3x'$$

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$$\left[\hat{\phi}(x,t), \hat{\phi}(x',t)\right] = 0 \qquad\qquad [\hat{q}_a(t), \hat{q}_{a'}(t)] = 0$$

$$\left[\hat{\pi}(x,t), \hat{\pi}(x',t)\right] = 0 \qquad\qquad [\hat{p}_a(t), \hat{p}_{a'}(t)] = 0$$

- We keep the equations of motion:

E1) $\dot{\hat{\phi}}(x,t) = \hat{\pi}(x,t)$ $\qquad\qquad \dot{\hat{q}}_a(t) = \hat{p}_a(t)$

E2) $\dot{\hat{\pi}}(x,t) = -(-\Delta + m^2)\,\hat{\phi}(x,t)$ $\qquad\qquad \dot{\hat{p}}_a(t) = -k_a \hat{q}_a(t)$

- **Note:** $\phi^*(x,t) = \phi(x,t)$ now implies hermiticity: $\hat{\phi}^+(x,t) = \hat{\phi}(x,t)$

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$i\hbar\,\dot{\hat{\phi}}(x,t) = [\hat{\phi}(x,t), \hat{H}]$ $\qquad\qquad i\hbar\,\dot{\hat{q}}(t) = [\hat{q}(t), \ldots$

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$$\left[\hat\phi(x,t),\ \hat\phi(x',t)\right] = 0 \qquad [\hat q_a(t), \hat q_{a'}(t)] = 0$$

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with this QFT Hamiltonian:

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E2) $\dot{\hat{\pi}}(x,t) = -\left(-\Delta + m^2\right)\hat{\phi}(x,t)$ $\qquad \dot{\hat{p}}_a(t) = -k_a \hat{q}_a(t)$

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$$[\hat\phi(x,t), \hat\phi(x',t)] = 0$$ $[\hat q_a(t), \hat q_{a'}(t)] = 0$

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Indeed, e.g.:

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$$= \frac{1}{2}\int [\hat\phi(x,t), \hat\pi(x',t)]\hat\pi(x',t) + \hat\pi(x',t)[\hat\phi(x,t), \hat\pi(x',t)]\, d^3x'$$

$$= \frac{i\hbar}{2}\int \delta^3(x-x')\hat\pi(x',t) + \hat\pi(x',t)\delta^3(x-x')\, d^3x' = \hat\pi(x,t)\, i\hbar$$

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Proposition: E1, E2 follow from the Heisenberg eqns

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Indeed, e.g.:

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Plan:

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E1) $\dot{\hat{\phi}}(x,t) = \hat{\pi}(x,t)$ $\qquad$ $\dot{\hat{q}}_a(t) = \hat{p}_a(t)$

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Plan:

1. Recall harmonic oscillators ✓
2. Relativistic fields ✓
3. 2nd quantization ✓
4. Harmonic oscillators in fields $\Rightarrow$ vacuum fluctuations

4. Harmonic oscillators in quantum fields

From the above, we need to solve 2 equations:

$x = (x_1, x_2, x_3)$

a) The K.G. eqn: $\left(\frac{\partial^2}{\partial t^2} - \Delta + m^2\right)\hat{\phi}(x,t) = 0$

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Proposition: (Exercise: show this)

Analogous to:

$k^2 = \sum_{i=1}^{3} k_i^2$

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c) $$\dot{\hat{\phi}}_k(t) = \hat{\pi}_k(t)$$

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$\hat{H} = \sum_a \frac{1}{2}\hat{p}_a\hat{p}_a + \frac{1}{2}\omega_a \hat{q}_a\hat{q}_a$

b) $[\hat{\phi}_k(t), \hat{\pi}_{k'}(t)] = i\hbar\,\delta^3(k+k')$

$[\hat{q}_a, \hat{p}_{a'}] = i\hbar\,\delta_{a,a'}$

$[\hat{\phi}_k(t), \hat{\phi}_{k'}(t)] = 0$

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c) $\dot{\hat{\phi}}_k(t) = \hat{\pi}_k(t)$

$\dot{\hat{q}}_a(t) = \hat{p}_a(t)$

$\dot{\hat{\pi}}_k(t) = -\left(k^2+m^2\right)\hat{\phi}_k(t)$

$\dot{\hat{p}}_a(t) = -\omega_a^2\,\hat{\phi}_a(t)$

$\Rightarrow$ For each mode $\vec{k}$ we seem to have a harmonic oscillator with $\omega_k = \sqrt{k^2+m^2}$

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$$\dot{\hat{\pi}}(x,t) = -(-\Delta + m^2)\,\hat{\phi}(x,t) \quad \overset{\text{Analogy}}{\longleftrightarrow} \quad \dot{\hat{p}}_a(t) = -K_a\, \hat{q}_a(t)$$

we notice that $(-\Delta + m^2)$, unlike K_a, is not a number!

- Q: Can we "transform" $(-\Delta + m^2)$ into a number?

A: Yes: Fourier transform turns derivatives into numbers!

Fourier transform of the spatial variables x_i:

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$$= \frac{i\hbar}{2}\int \delta^3(x-x')\hat{\pi}(x',t) + \hat{\pi}(x',t)\,\delta^3(x-x')\,d^3x' = \hat{\pi}(x,t)\,i\hbar \checkmark$$

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Plan:

1. Recall harmonic oscillators ✓
2. Relativistic fields ✓
3. 2nd quantization ✓
4. Harmonic oscillators in fields ⇒ vacuum fluctuations

4. Harmonic oscillators in quantum fields

□ From the above, we need to solve 2 equation

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$$i\hbar\,\dot{\hat\phi}(x,t) = [\hat\phi(x,t), \hat H] \qquad\qquad i\hbar\,\dot{\hat q}_a(t) = [\hat q_a(t), \hat H]$$

$$i\hbar\,\dot{\hat\pi}(x,t) = [\hat\pi(x,t), \hat H] \qquad\qquad i\hbar\,\dot{\hat p}_a(t) = [\hat p_a(t), \hat H]$$

with this QFT Hamiltonian:

$$\hat H = \int_{\mathbb{R}^3} \frac{1}{2}\hat\pi^2(x',t) + \frac{1}{2}\hat\phi(x',t)(m^2-\Delta)\hat\phi(x',t)\, d^3x' \qquad\qquad \hat H = \sum_a \frac{\hat p_a^2}{2} + \frac{\omega_a^2}{2}\hat q_a^2$$

Indeed, e.g.:

$$i\hbar\,\dot{\hat\phi}(x,t) = [\hat\phi(x,t), H] = \left[\hat\phi(x,t), \int_{\mathbb{R}^3}\frac{1}{2}\hat\pi^2(x',t) + \text{something}(\hat\phi)\, d^3x'\right]$$

$$= \frac{1}{2}\int [\hat\phi(x,t), \hat\pi(x',t)]\hat\pi(x',t) + \hat\pi(x',t)[\hat\phi(x,t), \hat\pi(x',t)]\, d^3x'$$

$$= \frac{i\hbar}{2}\int \delta^3(x-x')\hat\pi(x',t) + \hat\pi(x',t)\delta^3(x-x')\, d^3x' = \hat\pi(x,t)\, i\hbar \checkmark$$

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Proposition: E1, E_2 follow from the Heisenberg eqns

$$i\hbar \dot{\hat\phi}(x,t) = [\hat\phi(x,t), \hat H]$$

$$i\hbar \dot{\hat\pi}(x,t) = [\hat\pi(x,t), \hat H]$$

analogous to:

$$i\hbar \dot{\hat q}_a(t) = [\hat q_a(t), \hat H]$$

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Recall:

- Considered the simplest relativistic generalization of the Schrödinger eqn, the Klein Gordon eqn:

$$\left(\frac{\partial^2}{\partial t^2} - \Delta + m^2\right)\phi = 0 \qquad (\hbar = 1 = c)$$

(unlike the Schrödinger equation)

Notice: The K.G. eqn does not couple $Re(\phi)$ to $Im(\phi)$: each separately fulfills the K.G. eqn.

$\Rightarrow$ It suffices to study real-valued ϕ. Making ϕ complex is then straightforward.

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analogous to:

$$\left[\hat\phi(x,t), \hat\pi(x',t)\right] = i\hbar\,\delta^3(x-x')$$ $\quad [\hat q_a(t), \hat p_{a'}(t)] = i\hbar\,\delta_{aa'}$

$$\left[\hat\phi(x,t), \hat\phi(x',t)\right] = 0$$ $\quad [\hat q_a(t), \hat q_{a'}(t)] = 0$

$$\left[\hat\pi(x,t), \hat\pi(x',t)\right] = 0$$ $\quad [\hat p_a(t), \hat p_{a'}(t)] = 0$

- We keep the equations of motion:

E1) $$\dot{\hat\phi}(x,t) = \hat\pi(x,t)$$ $\quad \dot{\hat q}_a(t) = \hat p_a(t)$

E2) $$\dot{\hat\pi}(x,t) = -\left(-\Delta + m^2\right)\hat\phi(x,t)$$ $\quad \dot{\hat p}_a(t) = -K_a \hat q_a(t)$

- **Note:** $\phi^*(x,t) = \phi(x,t)$ now implies hermiticity: $\hat\phi^+(x,t) = \hat\phi(x,t)$

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$$i\hbar\,\dot{\hat{\pi}}(x,t) = [\hat{\pi}(x,t), \hat{H}]$$

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$$= \frac{i\hbar}{2}\int \delta^3(x-x')\hat{\pi}(x',t) + \hat{\pi}(x',t)\delta^3(x-x')\, d^3x' = \hat{\pi}(x,t)\, i\hbar \checkmark$$

Plan:

1. Recall harmonic oscillators ✓

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A: Use similarity to harmonic oscillator problem after overcoming a few technical difficulties:

1st Difficulty: (in reducing the QFT problem to harmonic oscillators)

- In the K.G. equation,

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Proposition: (Exercise: show this)

Analogous to:

a) $$\hat{H} = \int_{\mathbb{R}^3} \frac{1}{2}\hat{\pi}_k^+(t)\hat{\pi}_k(t) + \frac{1}{2}\hat{\phi}_k^+(t)\left(k^2+m^2\right)\hat{\phi}_k(t)\, d^3k \qquad k^2 = \sum_{i=1}^{3} k_i^2$$

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Show that a) + b) + Heisenberg eqn $\dot{\hat{f}}(t) = \frac{1}{i\hbar}[\hat{f}(t), \hat{H}]$ yields c.)

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We notice that the commutation relations

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- Idea: If use Fourier series instead, should have

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- Strategy:

  1. Put system into a large box $[-L/2, L/2]^{\times 3}$
  2. Assume (for example) periodic boundary conditions.

     (If box large enough it should not matter here what happens at the boundary of the box)

  3. Instead of Fourier transform, we can now cast

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- **Idea:** If use Fourier series instead, should have discrete values of k, thus Kronecker δ for CCR!

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2. Assume (for example) periodic boundary conditions.
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Terminology: Putting a system in a box is called "Infrared regularization".

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3. Instead of Fourier transform, we can now use Fourier series.

Terminology: Putting a system in a box is called "Infrared regularization".

because "long" wavelengths are removed.

Infrared regularization:

* $(k_1, k_2, k_3) = \frac{2\pi}{L}(n_1, n_2, n_3)$ with $n_1, n_2, n_3 \in \mathbb{Z}$

* $V = L^3$ (Volume of box)

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* Fourier series expansion coefficients:

$$\hat{\phi}_k(t) = V^{-1/2} \int_{-L/2}^{L/2}\int_{-L/2}^{L/2}\int_{-L/2}^{L/2} \hat{\phi}(x,t)\, e^{-ixk}\, d^3x$$

* The inverse is the Fourier series:

$$\hat{\phi}(x,t) = V^{-1/2} \sum_k \hat{\phi}_k(t)\, e^{ixk}$$

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k ← discrete set of vectors!

The QFT problem in the box:

analogous to

a) $\hat{H} = \sum_k \frac{1}{2} \hat{\pi}_k^\dagger \hat{\pi}_k + \frac{1}{2} \omega_k^2 \hat{\phi}_k^\dagger \hat{\phi}_k$

$\hat{H} = \sum \frac{1}{2} \hat{p}_a \hat{p}_a + \frac{1}{2} \omega \ldots$

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a) $$\hat{H} = \sum_k \frac{1}{2}\hat{\pi}_k^+ \hat{\pi}_k + \frac{1}{2}\omega_k^2 \hat{\phi}_k^+ \hat{\phi}_k$$

$\omega_k^2 = k^2 + m^2$

analogous to

$$\hat{H} = \sum_a \frac{1}{2}\hat{p}_a\hat{p}_a + \frac{1}{2}\omega_a^2 \hat{q}_a\hat{q}_a$$

Kronecker δ

b) $$[\hat{\phi}_k(t), \hat{\pi}_{k'}(t)] = i\hbar\, \delta_{k,-k'}$$

$[\hat{q}_a, \hat{p}_{a'}] = i\hbar\delta$

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b) $$[\hat{\phi}_k(t), \hat{\pi}_{k'}(t)] = i\hbar\, \delta_{k,-k'} \qquad [\hat{q}_a, \hat{p}_{a'}] = i\hbar\, \delta_{a,a'}$$

$$[\hat{\phi}_k(t), \hat{\phi}_{k'}(t)] = 0 \qquad [\hat{q}_a(t), \hat{q}_{a'}(t)] = 0$$

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c) $$\dot{\hat{\phi}}_k(t) = \hat{\pi}_k(t) \qquad \dot{\hat{q}}_a(t) = \hat{p}_a(t)$$

$$\dot{\hat{\pi}}_k(t) = -(k^2+m^2)\, \hat{\phi}_k(t) \qquad \dot{\hat{p}}_a(t) = -\omega_a^2\, \hat{\phi}_a(t)$$

Notice: Mukhanov here glosses over remaining mismatche

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$$\left[\omega_k^2 = k^2 + m^2 \right.$$

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3rd Difficulty: (in reducing the QFT problem to harmonic oscillators)

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a.) $\hat H = \sum_k \frac{1}{2}\hat\pi_k \hat\pi_k + \frac{1}{2}\omega_k^2 \hat\varphi_k \hat\varphi_k$ $\qquad\qquad$ $\hat H = \sum_a \frac{1}{2}\hat p_a \hat p_a + \frac{1}{2}\omega_a^2 \hat q_a \hat q_a$

$\omega_k^2 = k^2 + m^2$

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b) $[\hat\phi_k(t), \hat\pi_{k'}(t)] = i\hbar\,\delta_{k,-k'}$ $\qquad\qquad$ $[\hat q_a, \hat p_{a'}] = i\hbar\,\delta_{a,a'}$

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3rd Difficulty: (in reducing the QFT problem to harmonic oscillators)

- **Hermiticity:**

We notice that $\hat{\phi}^{+}(x,t) = \hat{\phi}(x,t)$, $\hat{\pi}^{+}(x,t) = \hat{\pi}(x,t)$ implies

$$\hat{\phi}_k^{+}(t) = \hat{\phi}_{-k}(t), \quad \hat{\pi}_k^{+}(t) = \hat{\pi}_{-k}(t) \qquad \text{(H)}$$

(Indeed:

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$$\dot{\hat{\phi}}_k(t) = \hat{\pi}_k(t) \qquad\qquad \dot{q}_a(t) = p_a(t)$$

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(Indeed:

$$\hat{\phi}_k^+(t) = (2\pi)^{-3/2}\int_{\mathbb{R}^3} e^{ixk}\,\hat{\phi}^+(x,t)\,d^3x = (2\pi)^{-3/2}\int_{\mathbb{R}^3} e^{ixk}\,\hat{\phi}(x,t)\,d^3x = \hat{\phi}_{-k}(t)$$)

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(Indeed:

$$\hat{\phi}_k^+(t) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{ixk}\, \hat{\phi}^+(x,t)\, d^3x = (2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{ixk}\, \hat{\phi}(x,t)\, d^3x = \hat{\phi}_{-k}(t) \Bigg)$$

But eqns **(H)** do not match:

$$\hat{q}_k^+(t) = \hat{q}_k(t) \qquad \hat{p}_k^+(t) = \hat{p}_k(t)$$

Namely: Our $\hat{\phi}$, $\hat{\pi}$ are not hermitean!

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Correspondingly:

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Correspondingly:

The analogy between

$$[\hat{\phi}_k(t), \hat{\pi}_{k'}(t)] = i\hbar\,\delta_{k,-k'} \quad \text{and} \quad [\hat{q}_a, \hat{p}_{a'}] = i\hbar\,\delta_{a}$$

$$\left(\hat{\phi}_k^{+}(t) = (2\pi)^{-3/2}\int_{\mathbb{R}^3} e^{ikx}\,\hat{\phi}^{+}(x,t)\,d^3x = (2\pi)^{-3/2}\int_{\mathbb{R}^3} e^{ikx}\,\hat{\phi}(x,t)\,d^3x = \hat{\phi}_{-k}(t) \right)$$

But eqns (H) do not match:

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suffers from $\delta_{k,-k'}$ instead of $\delta_{k,k'}$. (we do have $[\hat{\phi}_k(t), \hat{\pi}_{k'}^{+}(t)] = i\hbar\,\delta_{..}$ ..)

$$\hat{q}_\alpha^+(t) = \hat{q}_\alpha(t) \qquad \hat{p}_\alpha^+(t) = \hat{p}_\alpha(t)$$

Namely: Our $\hat{\phi}_k, \hat{\pi}_k$ are not hermitean!

□ **Correspondingly:**

The analogy between

$$[\hat{\phi}_k(t), \hat{\pi}_{k'}(t)] = i\hbar\, \delta_{k,-k'} \quad \text{and} \qquad [\hat{q}_\alpha, \hat{p}_{\alpha'}] = i\hbar\, \delta_{\alpha,\alpha'}$$

suffers from $\delta_{k,-k'}$ instead of $\delta_{k,k'}$. (we do have $[\hat{\phi}_k(t), \hat{\pi}_{k'}^+(t)] = i\hbar\, \delta_{k,k'}$)

□ **Mukhanov:**

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Namely: But ϕ_k, π_k are not hermitian.

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Mukhanov:

Neglects hermiticity issue and treats the field's oscillators just like ordinary quantum oscillators but with complex, i.e., non-hermitian amplitudes

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Neglects hermiticity issue and treats the field's oscillators just like ordinary quantum oscillators but with complex, i.e., non hermitean amplitudes.

Proper treatment:

Define new variables $\hat{q}_k, \hat{p}_k$, which are proper oscillators:

Eqns of motion: $\hat{p}_k = \dot{\hat{q}}_k$, $\dot{\hat{p}}_k = -\omega_k^2 q_k$

Canon. com. rels: $[\hat{q}_k, \hat{p}_{k'}] = i\,\delta_{k,k'}$

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  Hermiticity: $\hat{q}_k^{\dagger} = \hat{q}_k \quad \hat{p}_k^{\dagger} = \hat{p}_k$

- Then, try ansatz:

$$\hat{\phi}_k = \frac{1}{\sqrt{2}}\left(\hat{q}_k + \hat{q}_{-k}\right) + \frac{i}{\omega_k \sqrt{2}}\left(\hat{p}_k - \hat{p}_{-k}\right) \qquad \text{(A)}$$

Remark: With $a_k := \sqrt{\omega_k}\,\hat{x}_k + \frac{i}{\sqrt{\omega_k}}\,\hat{p}_k$ this ansatz reads:

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Exercise!

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Eqns of motion: $\dot{p}_k = q_k$, $\dot{p}_k = -\omega_k^2 q_k$

Canon. com. rels: $[\hat{q}_k, \hat{p}_{k'}] = i\,\delta_{k,k'}$

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Exercise!

- Now, show that ansatz (A) succeeds, i.e., that indeed:

Hamiltonian $\hat{H} = \sum \frac{1}{2}\hat{p}_k + \frac{1}{2}\omega_k^2\hat{q}_k$ (H)

$$q + ip = a$$

$$[A, BC] = B[A, C] + [A, B$$

$$\phi^*(x) = \phi(x) \iff \phi_k^* = \phi_{-k}$$

$$\phi^\dagger(x) = \phi(x) \iff \boxed{\phi_k^\dagger = \phi_{-k}}$$

$\cancel{x} q \cancel{+x} i p = a^{(1)}$

$$[A, BC] = B[A, C] + C[A, B$$

$$\phi^*(x) = \phi(x) \iff \phi^*_k = \phi_{-k}$$

$$\phi^\dagger(x) = \phi(x) \iff \boxed{\phi^\dagger_k = \phi_{-k}}$$

Eqns of motion: $p_k = q_k$, $p_k = -\omega_k^2 q_k$

Canon. com. rels: $[\hat{q}_k, \hat{p}_{k'}] = i\,\delta_{k,k'}$

Hermiticity: $\hat{q}_k^{+} = \hat{q}_k \qquad \hat{p}_k^{+} = \hat{p}_k$

- Then, try ansatz:

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Hermiticity cond.: $\qquad \hat{\phi}_k^+ = \hat{\phi}_{-k} \,, \quad \hat{\pi}_k^+ = \hat{\pi}_{-k}$

- Finally, via inverse Fourier series show that:

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Remark: With $a_k := \sqrt{\omega_k}\,\hat{x}_k + \frac{i}{\sqrt{\omega_k}}\hat{p}_k$ this ansatz reads:

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- Now, show that ansatz **(A)** succeeds, i.e., that indeed:

Hamiltonian $\hat{H} = \sum_k \frac{1}{2}\hat{p}_k + \frac{1}{2}\omega_k^2\hat{q}_k$ **(H)**

Eqns of motion: $\hat{\pi}_k = \dot{\hat{\phi}}_k\ ,\quad \dot{\hat{\pi}}_k = -\omega_k^2\hat{\phi}_k$

Canon. com. rels: $[\hat{\phi}_k, \hat{\pi}_{k'}] = i\,\delta_{k,-k'}$

Hermiticity cond.: $\hat{\phi}_k^+ = \hat{\phi}_{-k}\ ,\quad \hat{\pi}_k^+ = \hat{\pi}_{-k}$

- Finally, via inverse Fourier series, show that:

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Remark: With $a_k := \sqrt{\omega_k}\,\hat{x}_k + \frac{i}{\sqrt{\omega_k}}\,\hat{p}_k$ this ansatz reads:

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Exercise!

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Hermiticity: $q_k^* = q_{-k}$ $\quad p_k^* = p_{-k}$

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Proper treatment:

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$$\hat\phi_k = \frac{1}{\sqrt{2}}\left(\hat q_k + \hat q_{-k}\right) + \frac{i}{\omega_k\sqrt{2}}\left(\hat p_k - \hat p_{-k}\right) \qquad \text{(A)}$$

Remark: With $a_k := \sqrt{\omega_k}\,\hat x_k + \frac{i}{\sqrt{\omega_k}}\hat p_k$ this ansatz reads:

$$\hat\phi_k = \frac{1}{\sqrt{2\omega_k}}\left(a_k + a^+_{-k}\right)$$

Exercise!

Now, show that ansatz (A) succeeds, i.e., that indeed:

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Eqns of motion: $\hat\pi_k = \dot{\hat\phi}_k\,,\quad \dot{\hat\pi}_k = -\omega_k^2\hat\phi_k$

Canon. com. rels: $[\hat\phi_k, \hat\pi_{k'}] = i\,\delta_{k,-k'}$

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$$\hat{\phi}(x) = \sqrt{\frac{2}{V}} \sum_k \left\{ \cos(xk)\,\hat{q}_k - \frac{1}{\omega_k} \sin(xk)\,\hat{p}_k \right\} \qquad (B)$$

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* Ground state of q_k, p_k oscillators ⇝ Vacuum

* This need not be lowest energy state of the QFT Hamiltonian

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Finally, via inverse Fourier series, show that:

$$\hat{\phi}(x) = \sqrt{\frac{2}{L}} \sum_k \left\{ \cos(xk)\, \hat{q}_k - \frac{1}{\omega_k} \sin(xk)\, \hat{p}_k \right\} \qquad \text{(B)}$$

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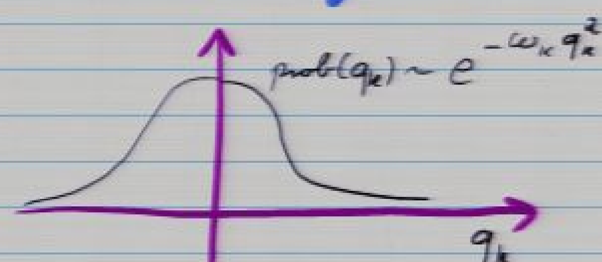
- 

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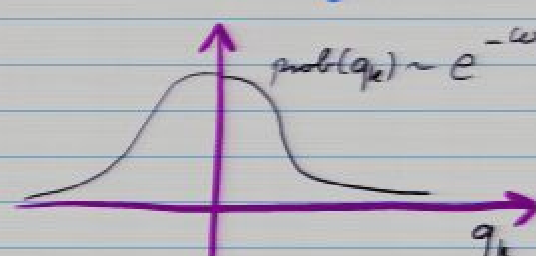
- 

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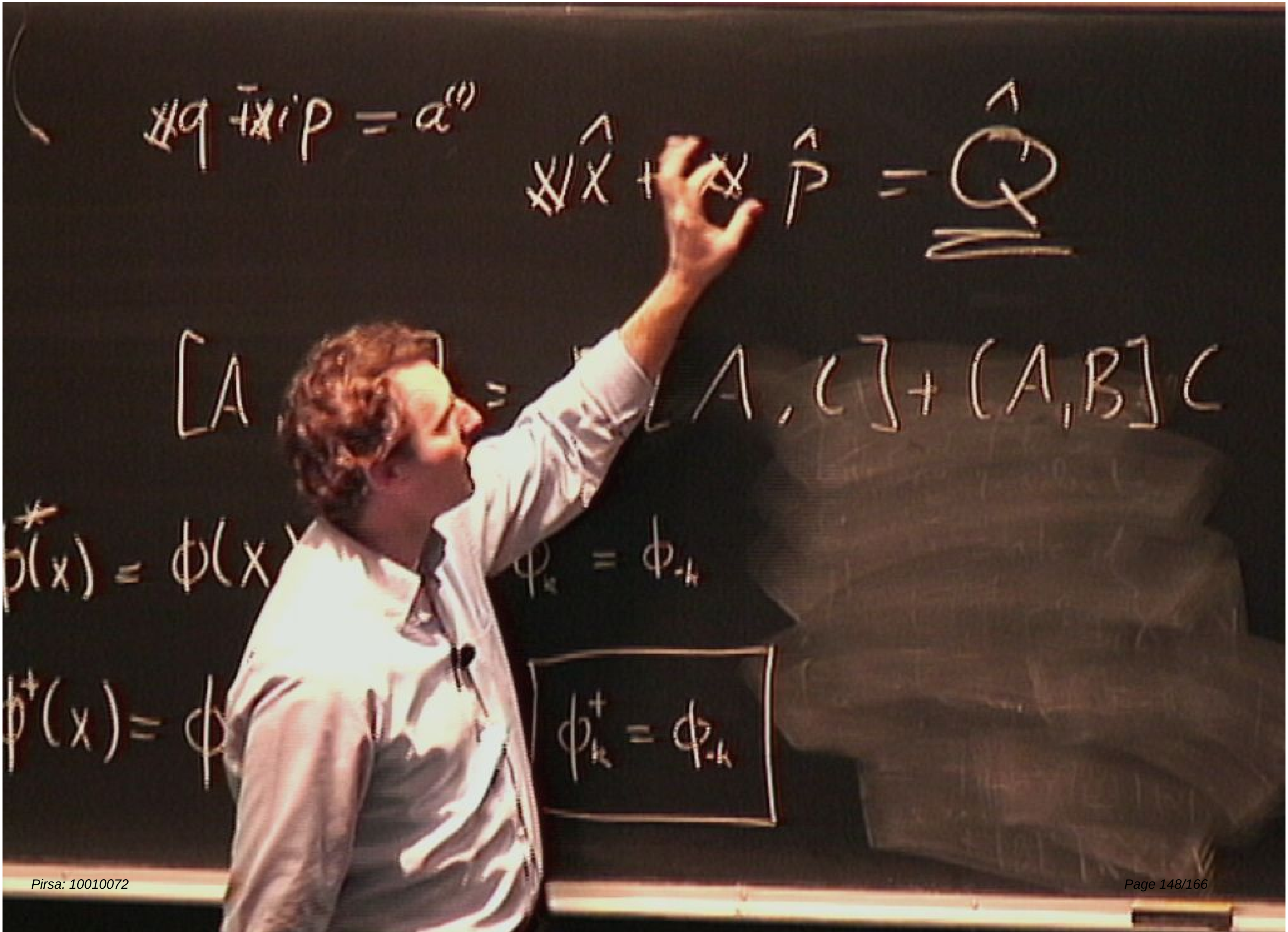
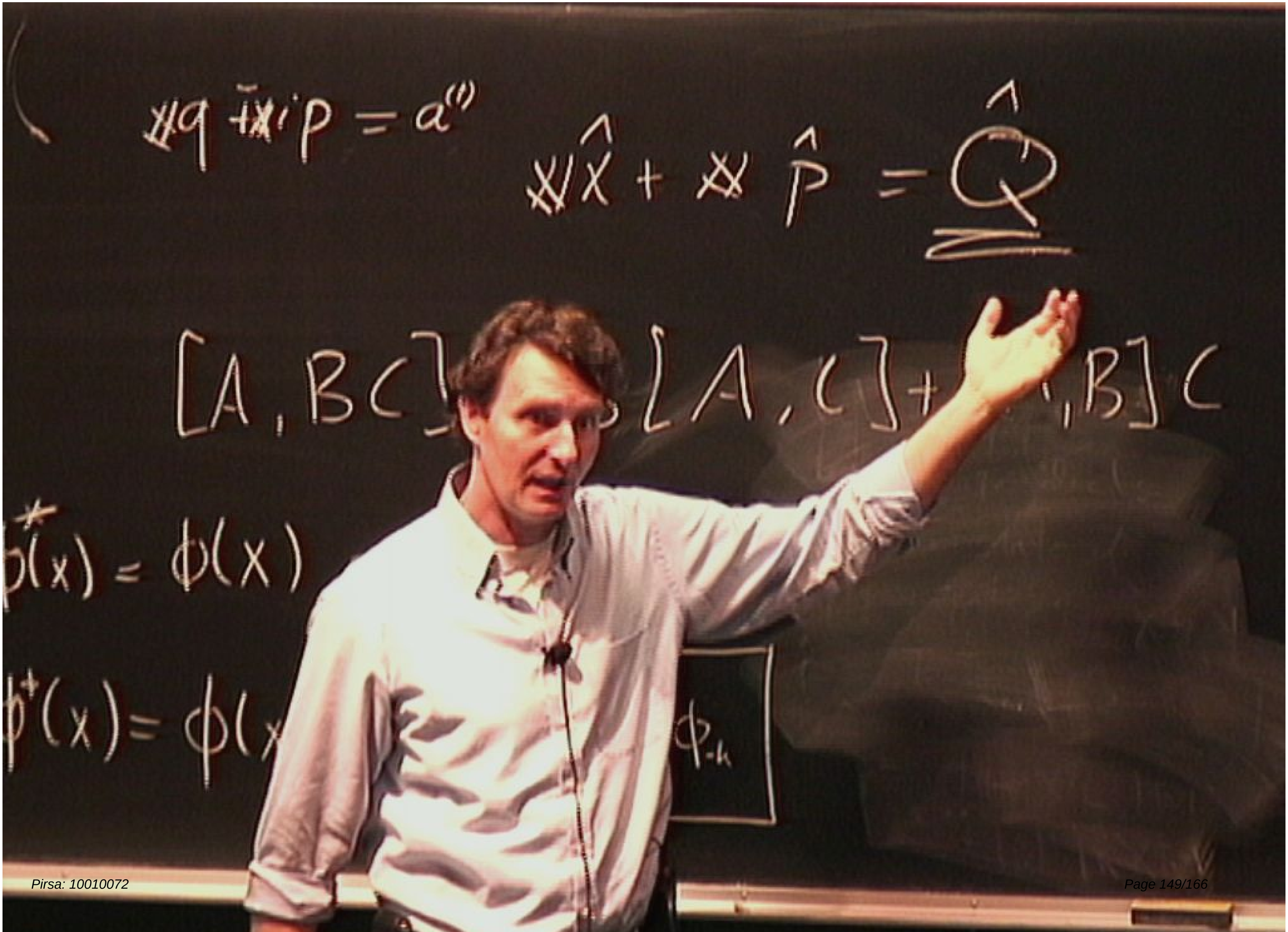


$\phi_k^+ = \phi_{-k}$


p = a^(1)
x̂ + p̂ = Q̂
[A, BC]
[A, C] +
,B] C
φ(x) = φ(x)
φ⁺(x) = φ(x
φ₋ₖ

$\cancel{\alpha} q \cancel{+\alpha} i p = a^{(1)}$

$\cancel{\alpha} \hat{x} + \cancel{\beta} \hat{p} = \hat{Q}$

$[A, BC] = B[A, C] + [A, B]C$

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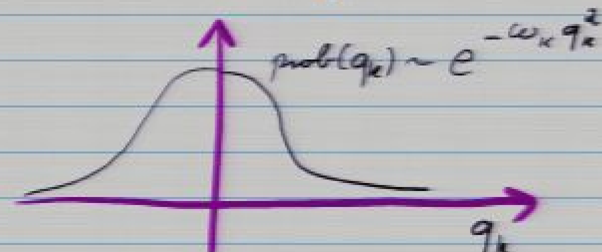
- 

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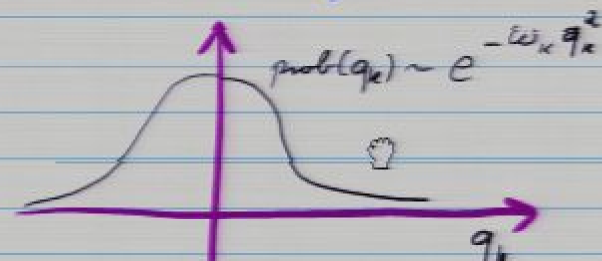
- 

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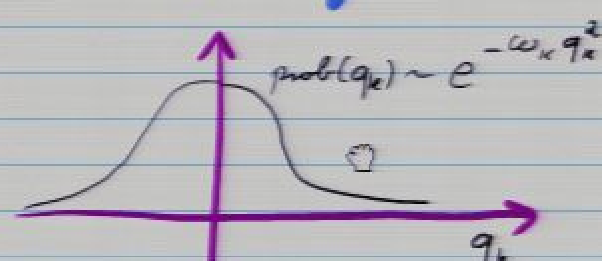

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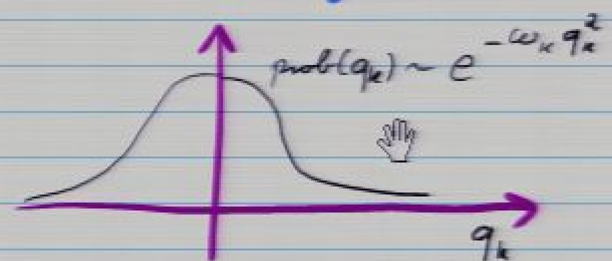

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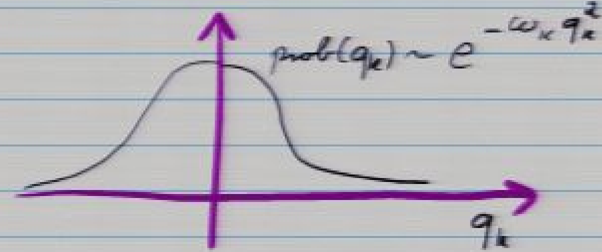

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(straightforward but tedious to show)

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$\hat{x} + \hat{p} = \hat{Q}$

$a = x + ip$

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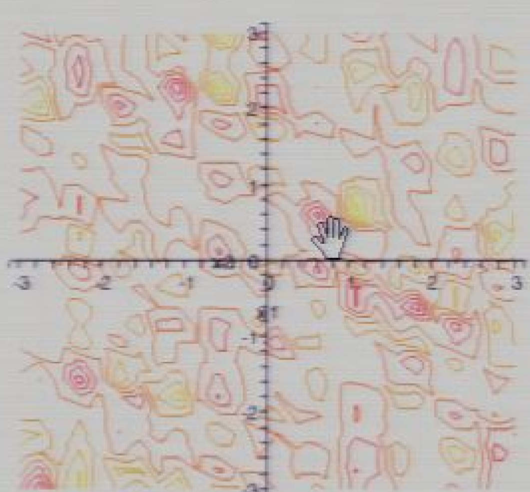

Actual draw from that distribution.

The fluctuations trace back to the Fourier coefficients and to the $\hat{q}_k$, $\hat{P}_k$ which fluctuate even in lowest energy state.