Title: Gravitational Physics - Review (PHYS 636) - Lecture 6

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Abstract:

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$$\det \hat{g} = \det\left(\frac{\partial X}{\partial \hat{X}}\right)^2 \det g$$

hence $\sqrt{g}\, d^4X = \sqrt{\hat{g}}\, d^4\hat{X}$ is covariant

where $g$ is shorthand for $|\det g_{ab}|$

A useful identity is

$$\det M = \exp \operatorname{tr} \log M$$

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Here $\partial_a \sqrt{-g} = \frac{1}{2}\frac{1}{\sqrt{-g}} \cdot -g\, g^{cd} g_{cd,a}$

$= \sqrt{-g}\, \Gamma^{d}_{a\,d}$

For a massless scalar field

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$$\mathcal{L}_\phi = \frac{1}{2}(\partial\phi)^2 \longrightarrow S_\phi = \int \frac{1}{2}(\partial\phi)^2 \sqrt{-g}\, d^4x$$

$$\delta S_\phi = \int d^4x \left[ \partial_a\phi\, \partial_b \delta\phi\, g^{ab}\sqrt{-g} + \frac{1}{2}\partial_a\phi\, \partial_b\phi\, \delta g^{ab}\sqrt{-g} - \frac{1}{2} g_{ab}\delta g^{ab}\sqrt{-g}\left(\frac{1}{2}(\partial\phi)^2\right)\right]$$

$$= \int \sqrt{-g}\, d^4x \left[ \frac{1}{2}\delta g^{ab}\left(\phi_{,a}\phi_{,b} - \frac{1}{2}(\partial\phi)^2 g_{ab}\right) - \frac{1}{\sqrt{-g}}\partial_b\sqrt{-g}\, g^{ab}\partial_a\phi\,\delta\phi \right]$$

$$\frac{\delta S_\phi}{\delta \phi} = -\frac{1}{\sqrt{g}}\left(\partial_a \sqrt{-g}\, \partial^a \phi\right)$$

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$$\frac{\delta S_\phi}{\delta \phi} = -\frac{1}{\sqrt{g}}\left(\partial_a \sqrt{-g}\,\partial^a \phi\right) = 0$$

e.o.m for $\phi$.

$$\frac{\delta S_\phi}{\delta g^{ab}} = \frac{1}{2}\left(\phi_{,a}\phi_{,b} - \frac{1}{2}(\partial\phi)^2 g_{ab}\right) \Longleftarrow$$

$$\frac{1}{\sqrt{-g}}\partial_a \sqrt{-g}\,\partial^a \phi = \partial_a \partial^a \phi + \Gamma^a_{ad}\,\partial^d \phi$$

$$= \Box \phi = \nabla_a \nabla^a \phi$$

This useful identity generalises to antisymmetric tensors.

$$\nabla_a F^{ab} = \frac{1}{\sqrt{g}} \partial_a \sqrt{g} F^{ab}$$

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$$\delta R = \delta\left(R_{ab}\, g^{ab}\right) = \left(\delta R_{ab}\right) g^{ab} + R_{ab}\, \delta g^{ab}$$

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$$\delta R^a{}_{bcd} = \delta\left(\Gamma^a_{bd,c} - \Gamma^a_{bc,d}\right) \quad \text{in n.c}$$

In $\delta\Gamma^a_{bd,c}$ recall $g_{ab,c} = 0$

$$\delta\Gamma^a_{bd,c} = \frac{1}{2} g^{ae}\left(\delta g_{be,dc} + \delta g_{de,bc} - \delta g_{bd,ec}\right.$$

$$\delta R = \delta R_{ab}\, g^{ab} + R_{ab}\, \delta g^{ab}.$$

$$g^{bd}\,\delta R_{bd} = \delta\Gamma^{a}_{bd,a} - \delta\Gamma^{a}_{ba,d}$$

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$$= \frac{1}{2}\left(\delta g_{be,dc} + \delta g_{de,bc} - \delta g_{bd,ec}\right) g^{ec} g$$

$$- \frac{1}{2}\,\delta g_{ec,bd}\, g^{ec} g^{bd}$$

$$= \partial^{a}\partial^{b}\,\delta g_{ab} - g^{cd}\,\partial^{2}\,\delta g_{cd}$$

$$g^{bd}\,\delta R_{bd} = \delta\Gamma^{a}_{bd,a} - \delta\Gamma^{a}_{ba,d}$$

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To get a covariant expression $\partial \to \nabla$ or

"comma $\to$ semicolon" $\frac{\partial V^a}{\partial x^b} = \partial_b V^a = V^a{}_{,b} \to V^a{}_{;b}$
$= \nabla_b V^a$

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* 2nd deriv of $g$ - this means (strictly) we require $\delta g_{ab} = 0$ on $\partial V$, but also $\nabla_n \delta g = 0$

If write $S_{EH} = \int -R\sqrt{g}\, d^4x$

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then $\dfrac{\delta S_{EH}}{\delta g^{ab}} = -c\left[R_{ab} - \frac{1}{2} R g_{ab}\right]\delta g^{ab}$

If write $S_{EH} = c \int -R\sqrt{g}\, d^4x$

then $\dfrac{\delta S_{EH}}{\delta g^{ab}} = -c\left[R_{ab} - \frac{1}{2} R g_{ab}\right]$

Compare to $\dfrac{\delta S_\phi}{\delta g^{ab}} = \frac{1}{2}\left(\phi_{,a}\phi_{,b} - \frac{1}{2}(\partial\phi)^2 g_{ab}\right)$

$= \frac{1}{2} T_{\phi ab}$

For $C = \frac{1}{16\pi G}$ this gives $G_{ab} = 8\pi G T_{ab}$.

Alternative theories of gravity.

Could try $\Phi R$.

$$g^{ab}\Phi\,\delta R_{ab} \doteq -\Phi\,\nabla_a\nabla_b\,\delta g^{ab} + \Phi\,\Box(g_{ab}\,\delta g^{ab})$$

$$\text{"}=\text{"} \quad -\delta g^{ab}\,\nabla_a\nabla_b\,\Phi + \delta g^{ab}\,\Box(\Phi\, g_{ab})$$

Modify Einstein tensor:

$$\Phi G_{ab} - \nabla_a \nabla_b \Phi + g_{ab} \Box \Phi .$$

- This is Brans-Dicke gravity, with a kinetic term: $\frac{\omega (\partial \Phi)^2}{\Phi}$

- example of a <u>scalar-tensor theory</u>