Title: Gravitational Physics - Review (PHYS 636) - Lecture 3

Date: Jan 05, 2010 10:00 AM

URL: http://pirsa.org/10010031

Abstract:

perimeter scholars
international

A connection provides a way of linking tangent spaces at different points, it tells us how the bases transform or link together

$$\underset{\sim}{\nabla}\, \underset{\sim}{e}_b = \Gamma^{c}{}_{ab}\, \underset{\sim}{e}_c \otimes \underset{\sim}{\omega}^a$$

$$\underset{\sim}{\nabla}\, \underset{\sim}{e}_b = \Gamma_a{}^c{}_b \, \underset{\sim}{e}_c \otimes \underset{\sim}{\omega}^a$$

covector

basis vector

connection cpts.

$$\underset{\sim}{\nabla}\, \underset{\sim}{e}_b = \Gamma^{c}{}_{ab}\, \underset{\sim}{e}_c \otimes \underset{\sim}{\omega}^a$$

covector

basis vector

connection cpts.

or $$\Gamma^{a}{}_{bc} = \langle \underset{\sim}{\omega}^a | \nabla_b\, \underset{\sim}{e}_c \rangle$$

$$\underset{\sim}{\nabla}\, \underset{\sim}{e}_b = \Gamma_{a\,b}^{\;\;c}\, \underset{\sim}{e}_c \otimes \underset{\sim}{\omega}^a$$

covector

basis vector

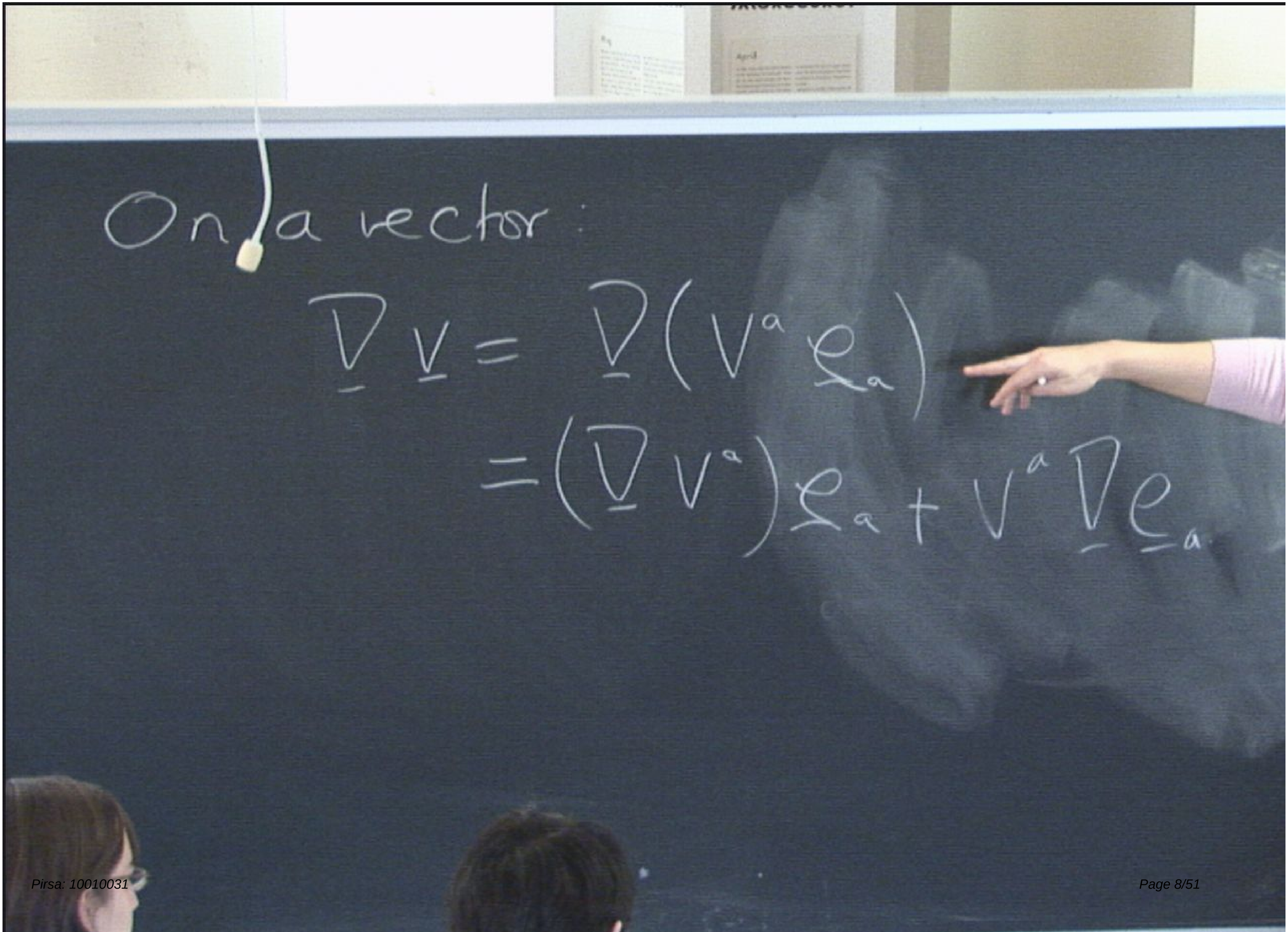
connection cpts.

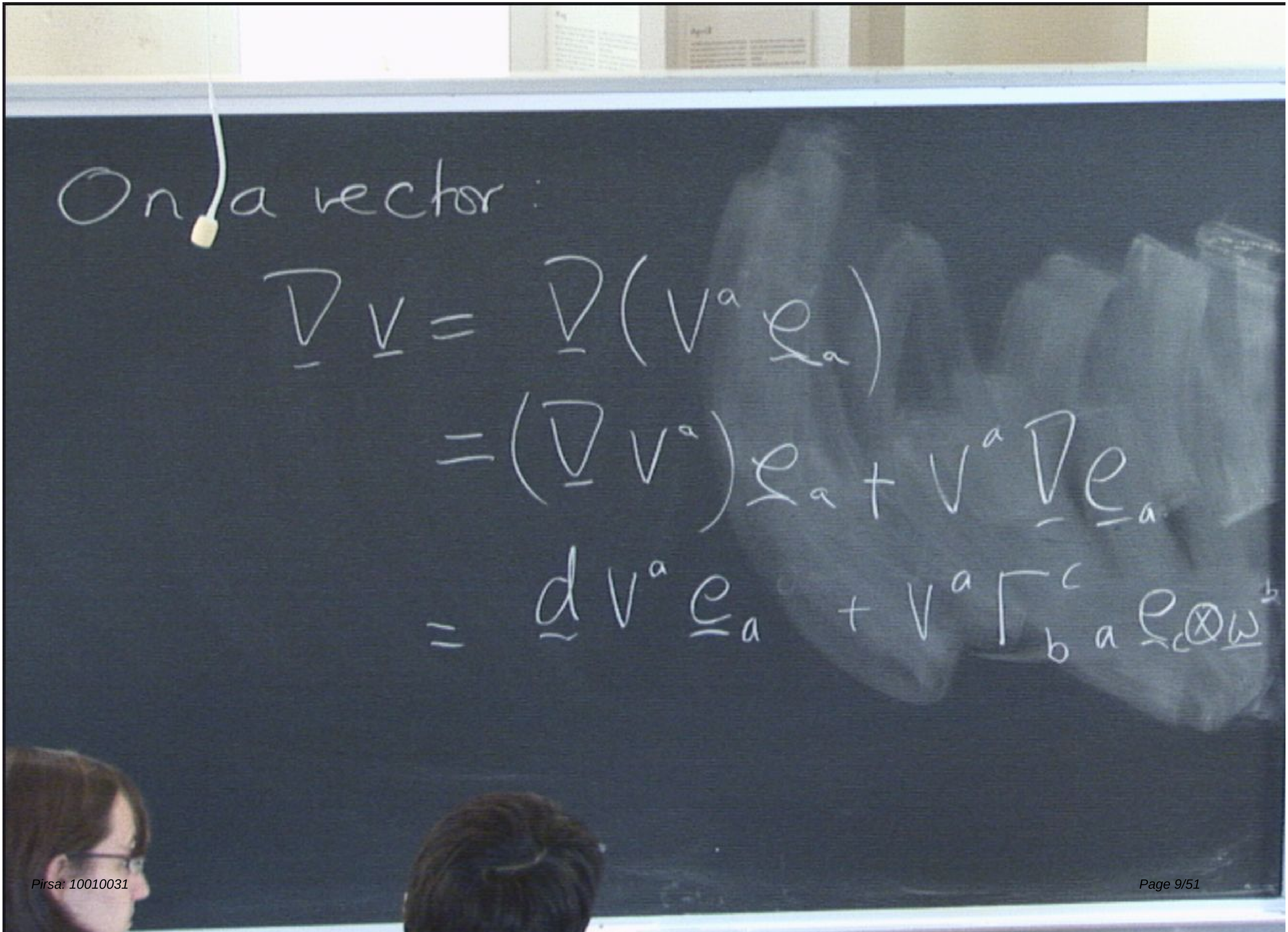
or $\Gamma^a_{\;bc} = \langle \underset{\sim}{\omega}^a | \nabla_b \underset{\sim}{e}_c \rangle$

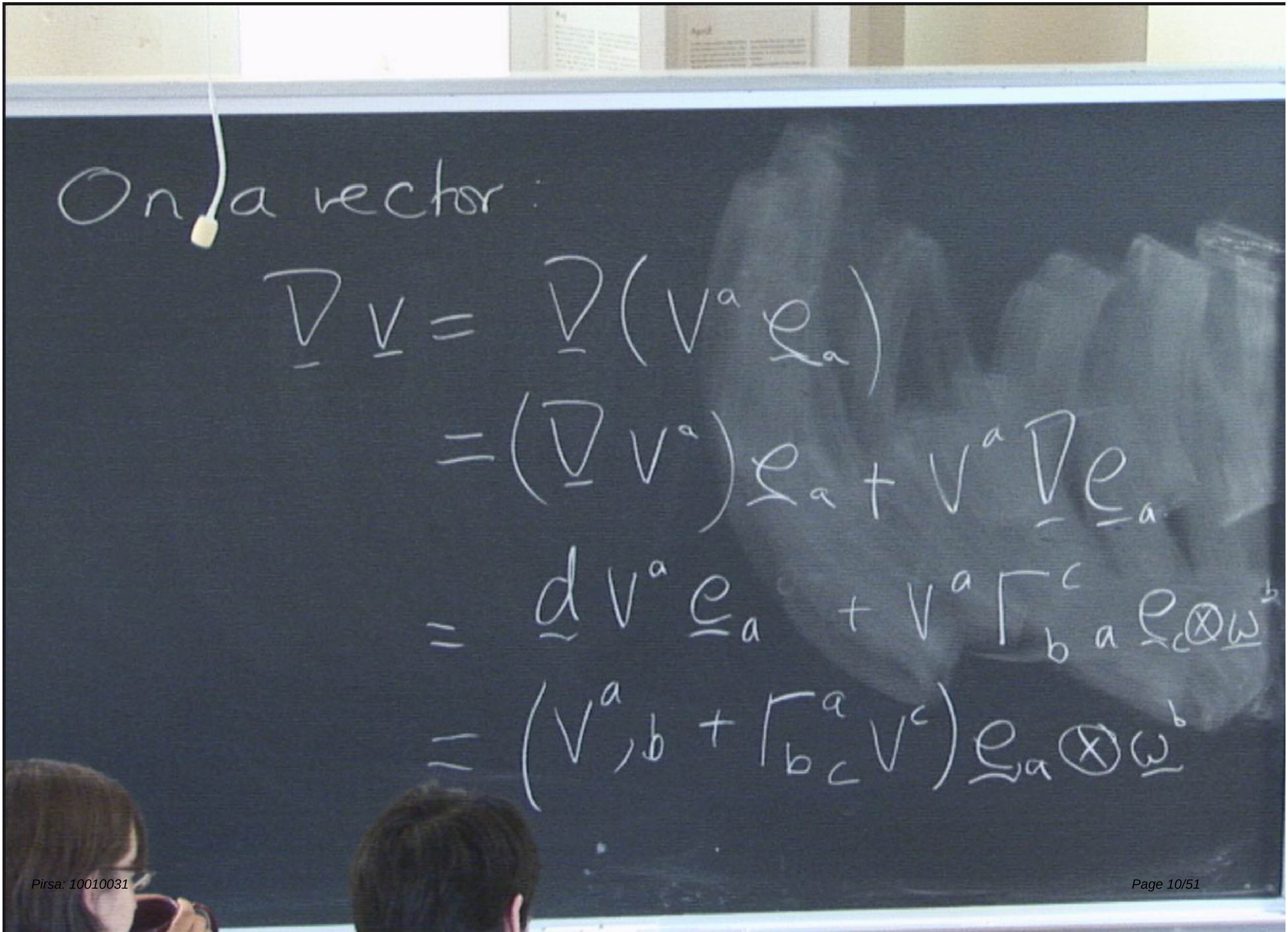
$\underset{\sim}{\nabla}$ is (i) Leibnitzian

(ii) Commutes with contractions

(iii) reduces to $\underset{\sim}{d}$ on fns.


On a vector:
$\underline{\nabla}\,\underline{V} = \underline{\nabla}(V^a \underline{e}_a)$
$= (\underline{\nabla} V^a)\underline{e}_a + V^a \underline{\nabla}\,\underline{e}_a$


On a vector:
$\underline{\nabla}\, \underline{V} = \underline{\nabla}(V^a \underline{e}_a)$
$= (\underline{\nabla} V^a)\underline{e}_a + V^a \underline{\nabla}\, \underline{e}_a$
$= \underline{d}V^a \underline{e}_a + V^a \Gamma^c{}_{ba}\, \underline{e}_c \otimes \underline{\omega}^b$


On a vector:
$\underset{\sim}{\nabla} \underset{\sim}{V} = \underset{\sim}{\nabla}(V^a \underset{\sim}{e}_a)$
$= (\underset{\sim}{\nabla} V^a) \underset{\sim}{e}_a + V^a \underset{\sim}{\nabla} \underset{\sim}{e}_a$
$= \underset{\sim}{d} V^a \underset{\sim}{e}_a + V^a \Gamma^c{}_{ba} \underset{\sim}{e}_c \otimes \underset{\sim}{\omega}^b$
$= (V^a{}_{,b} + \Gamma^a{}_{bc} V^c) \underset{\sim}{e}_a \otimes \underset{\sim}{\omega}^b$

On a vector:

$$\underline{\nabla}\, \underline{v} = \underline{\nabla}(V^a \underline{e}_a)$$

$$= (\underline{\nabla} V^a)\, \underline{e}_a + V^a\, \underline{\nabla}\, \underline{e}_a$$

$$= \underset{\sim}{d} V^a\, \underline{e}_a + V^a\, \Gamma^c{}_{ba}\, \underline{e}_c \otimes \underset{\sim}{\omega}^b$$

$$= (V^a{}_{,b} + \Gamma^a{}_{bc} V^c)\, \underset{\sim}{e}_a \otimes \underset{\sim}{\omega}^b$$

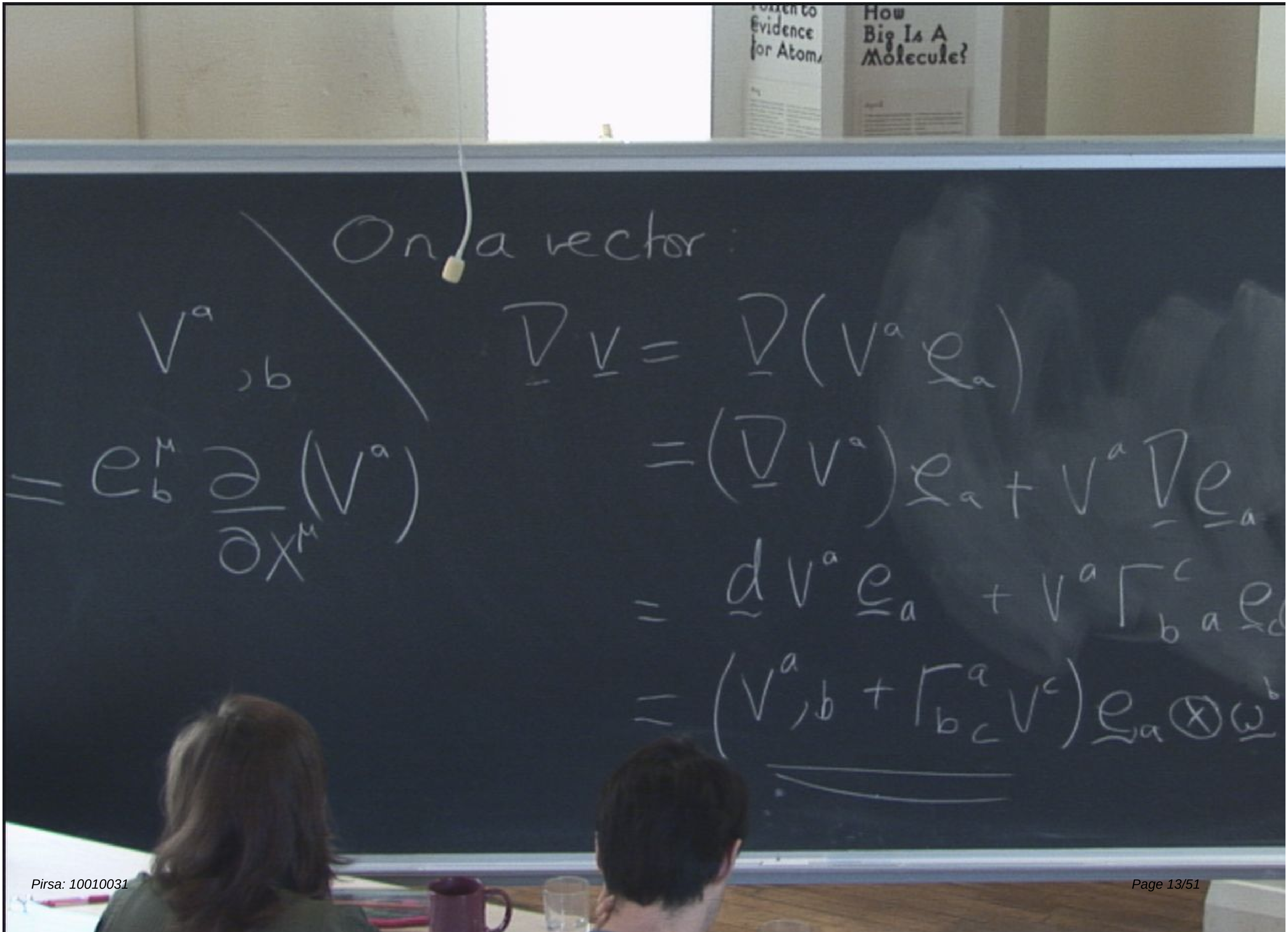
On a vector:

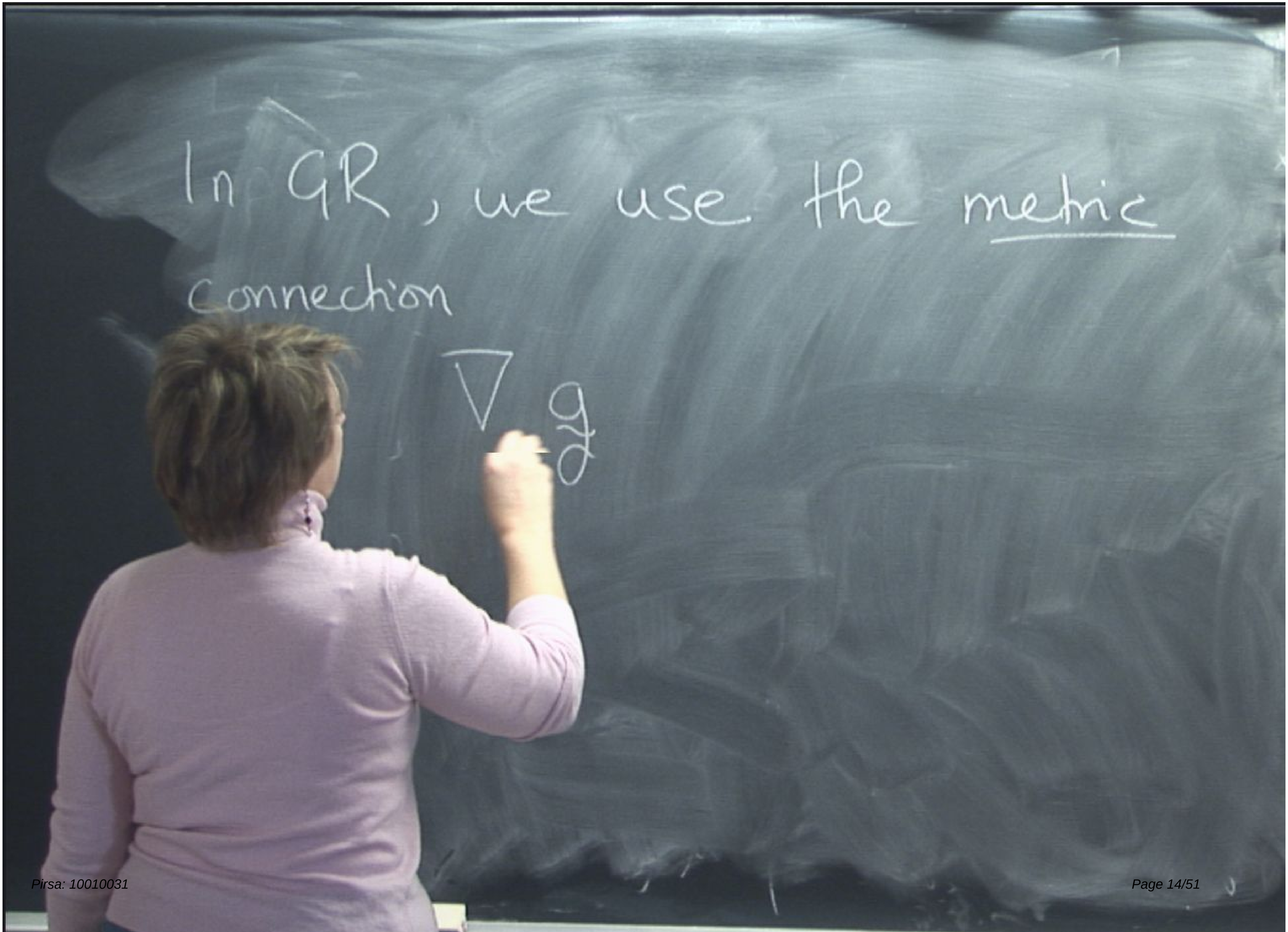
$$\underline{\nabla}\,\underline{v} = \underline{\nabla}(V^a \underline{e}_a)$$

$$= (\underline{\nabla} V^a)\,\underline{e}_a + V^a\, \underline{\nabla}\,\underline{e}_a$$

$$= \underline{d} V^a\, \underline{e}_a + V^a\, \Gamma^c{}_{ba}\, \underline{e}_c \otimes \underline{\omega}^b$$

$$= \left(V^a{}_{,b} + \Gamma^a{}_{bc} V^c\right) \underline{e}_a \otimes \underline{\omega}^b$$

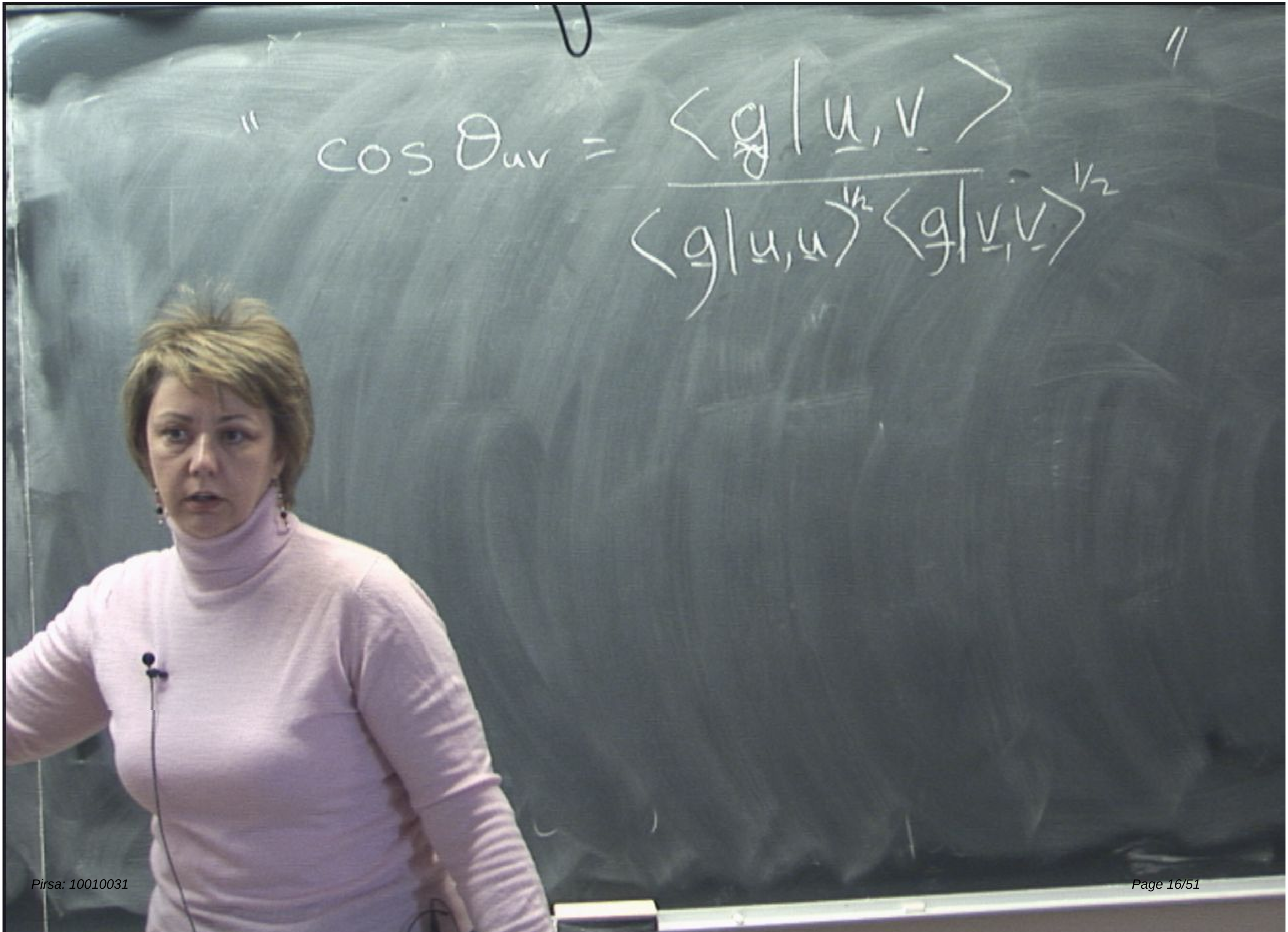

Evidence
for Atoms
How
Big Is A
Molecule?
On a vector:
$V^a{}_{;b}$
$= e_b^\mu \frac{\partial}{\partial x^\mu}(V^a)$
$\underline{\nabla}\,\underline{V} = \underline{\nabla}(V^a \underline{e}_a)$
$= (\underline{\nabla} V^a)\underline{e}_a + V^a \underline{\nabla}\,\underline{e}_a$
$= \underline{d}V^a \underline{e}_a + V^a \Gamma^c_{b\,a} \underline{e}_c$
$= (V^a{}_{;b} + \Gamma^a_{b\,c} V^c)\,\underline{e}_a \otimes \underline{\omega}^b$


In GR, we use the metric
connection
∇g

In GR, we use the metric connection

(1) $\underset{\sim}{\nabla}\, \underset{\sim}{g} = 0$

This tells us that $\nabla_T \underset{\sim}{V} = 0$ really corresponds to parallel transport.


" $\cos\theta_{uv} = \dfrac{\langle g | \underline{u}, \underline{v} \rangle}{\langle g | \underline{u}, \underline{u} \rangle^{1/2} \langle g | \underline{v}, \underline{v} \rangle^{1/2}}$ "

" $\cos\theta_{uv} = \dfrac{\langle g | \underline{u}, \underline{v} \rangle}{\langle g | \underline{u}, \underline{u} \rangle^{1/2} \langle g | \underline{v}, \underline{v} \rangle^{1/2}}$ "

② In addition, we take a torsion-free connec

$$T(\underline{u}, \underline{v}) = \nabla_u \underline{v} - \nabla_v \underline{u} - [\underline{u}, \underline{v}]$$

ly

s

" $\cos\theta_{uv} = \dfrac{\langle g | \underline{u}, \underline{v} \rangle}{\langle g | \underline{u}, \underline{u} \rangle^{1/2} \langle g | \underline{v}, \underline{v} \rangle^{1/2}}$ "

② In addition, we take a torsion-free co

$$T(\underline{u}, \underline{v}) = \nabla_u \underline{v} - \nabla_v \underline{u} - [\underline{u}, \underline{v}]$$

– almost the same as saying Γ is symme

$$T^a{}_{bc} = \Gamma^a{}_{bc} - \Gamma^a{}_{cb} - C^a{}_{bc}$$

where $C^{a}{}_{bc} = \langle \underline{\omega}^{a} | [\underline{e}_{b}, \underline{e}_{c}]$

are the structure constants of

the basis $\underline{e}_{a}$

where $C^a_{bc} = \langle \underline{\omega}^a | [\underline{e}_b, \underline{e}_c]$

are the structure constants of the basis $\underline{e}_a$

This gives the familiar Levi-Cinta connection

$$\Gamma^{a}_{bc} = \frac{1}{2} g^{ae} \left(g_{eb,c} + g_{ec,b} - g_{bc,e} \right)$$

$$\Gamma_{b}{}^{a}{}_{c} = \frac{1}{2} g^{ae}\left(g_{eb,c} + g_{ec,b} - g_{bc,e}\right)$$

Now define the connection 1-forms or spin connection as

$$\underline{\theta}^{a}{}_{c} = \Gamma_{b}{}^{a}{}_{c}\,\underline{\omega}^{b}$$

$$\Gamma_b{}^a{}_c = \frac{1}{2} g^{ae}\left(g_{eb,c} + g_{ec,b} - g_{bc,e}\right)$$

Now define the connection 1-forms or spin connection as

$$\underline{\theta}^a{}_c = \Gamma_b{}^a{}_c\, \underline{\omega}^b \qquad \left[\begin{array}{c}\text{alt} \\ \omega_\mu{}^a{}_b\end{array}\right].$$

then $\underset{\sim}{\Theta}_{ac} + \underset{\sim}{\Theta}_{ca} = d\, g_{ac}$

then $\underset{\sim}{\Theta}_{ac} + \underset{\sim}{\Theta}_{ca} = \underset{\sim}{d}\, g_{ac}$

<u>Proof</u> $\underset{\sim}{d}(g_{ab}) = d \langle g \,|\, \underset{\sim}{e}_a, \underset{\sim}{e}_b \rangle$

then $\underset{\sim}{\Theta}_{ac} + \underset{\sim}{\Theta}_{ca} = \underset{\sim}{d}\, g_{ac}$

Proof $\underset{\sim}{d}(g_{ab}) = d \langle g \mid \underset{\sim}{e}_a, \underset{\sim}{e}_b \rangle$

$= \underset{\sim}{\nabla} \langle g \mid \underset{\sim}{e}_a, \underset{\sim}{e}_b \rangle$

$= \langle g \mid \nabla \underset{\sim}{e}_a, \underset{\sim}{e}_b \rangle + \langle g \mid \underset{\sim}{e}_a \nabla \underset{\sim}{e}_b \rangle$

$= \underset{\sim}{\Theta}^c{}_a \langle g \mid \underset{\sim}{e}_c, \underset{\sim}{e}_b \rangle + \underset{\sim}{\Theta}^c{}_b \langle g \mid \underset{\sim}{e}_a \underset{\sim}{e}_c \rangle$

then $\underset{\sim}{\theta}_{ac} + \underset{\sim}{\theta}_{ca} = \underset{\sim}{d}\, g_{ac}$

Proof

$$\underset{\sim}{d}(g_{ab}) = d\langle g \,|\, \underset{\sim}{e}_a, \underset{\sim}{e}_b \rangle$$

$$= \underset{\sim}{\nabla}\langle g \,|\, \underset{\sim}{e}_a, \underset{\sim}{e}_b \rangle$$

$$= \langle g \,|\, \nabla \underset{\sim}{e}_a, \underset{\sim}{e}_b \rangle + \langle g \,|\, \underset{\sim}{e}_a \nabla \underset{\sim}{e}_b \rangle$$

$$= \underset{\sim}{\theta}^c{}_a \langle g \,|\, \underset{\sim}{e}_c, \underset{\sim}{e}_b \rangle + \underset{\sim}{\theta}^c{}_b \langle g \,|\, \underset{\sim}{e}_a \underset{\sim}{e}_c \rangle$$

$$= g_{cb}\, \underset{\sim}{\theta}^c{}_a + g_{ac}\, \underset{\sim}{\theta}^c{}_b$$

$$= \underset{\sim}{\theta}_{ba} + \underset{\sim}{\theta}_{ab}$$

then $\underset{\sim}{\Theta}_{ac} + \underset{\sim}{\Theta}_{ca} = \underset{\sim}{d}\, g_{ac}$ //

Proof

$$\underset{\sim}{d}(g_{ab}) = d\langle g \,|\, \underset{\sim}{e}_a, \underset{\sim}{e}_b\rangle$$

$$= \underset{\sim}{\nabla}\langle \underset{\sim}{g} \,|\, \underset{\sim}{e}_a, \underset{\sim}{e}_b\rangle$$

$$= \langle \underset{\sim}{g} \,|\, \nabla \underset{\sim}{e}_a, \underset{\sim}{e}_b\rangle + \langle g \,|\, \underset{\sim}{e}_a \nabla \underset{\sim}{e}_b\rangle$$

$$= \underset{\sim}{\Theta}^c{}_a \langle g \,|\, \underset{\sim}{e}_c, \underset{\sim}{e}_b\rangle + \underset{\sim}{\Theta}^c{}_b \langle g \,|\, \underset{\sim}{e}_a \underset{\sim}{e}_c\rangle$$

$$= g_{cb}\, \underset{\sim}{\Theta}^c{}_a + g_{ac}\, \underset{\sim}{\Theta}^c{}_b$$

$$= \underset{\sim}{\Theta}_{ba} + \underset{\sim}{\Theta}_{ab}$$

In an o/n basis, the spin connection is anti-symmetric.

In an o/n basis, the spin connection is anti-symmetric

Finally

$$\Theta^a{}_c \wedge \underline{\omega}^c = \Gamma^a{}_{bc}\, \omega^b \wedge \underline{\omega}^c$$

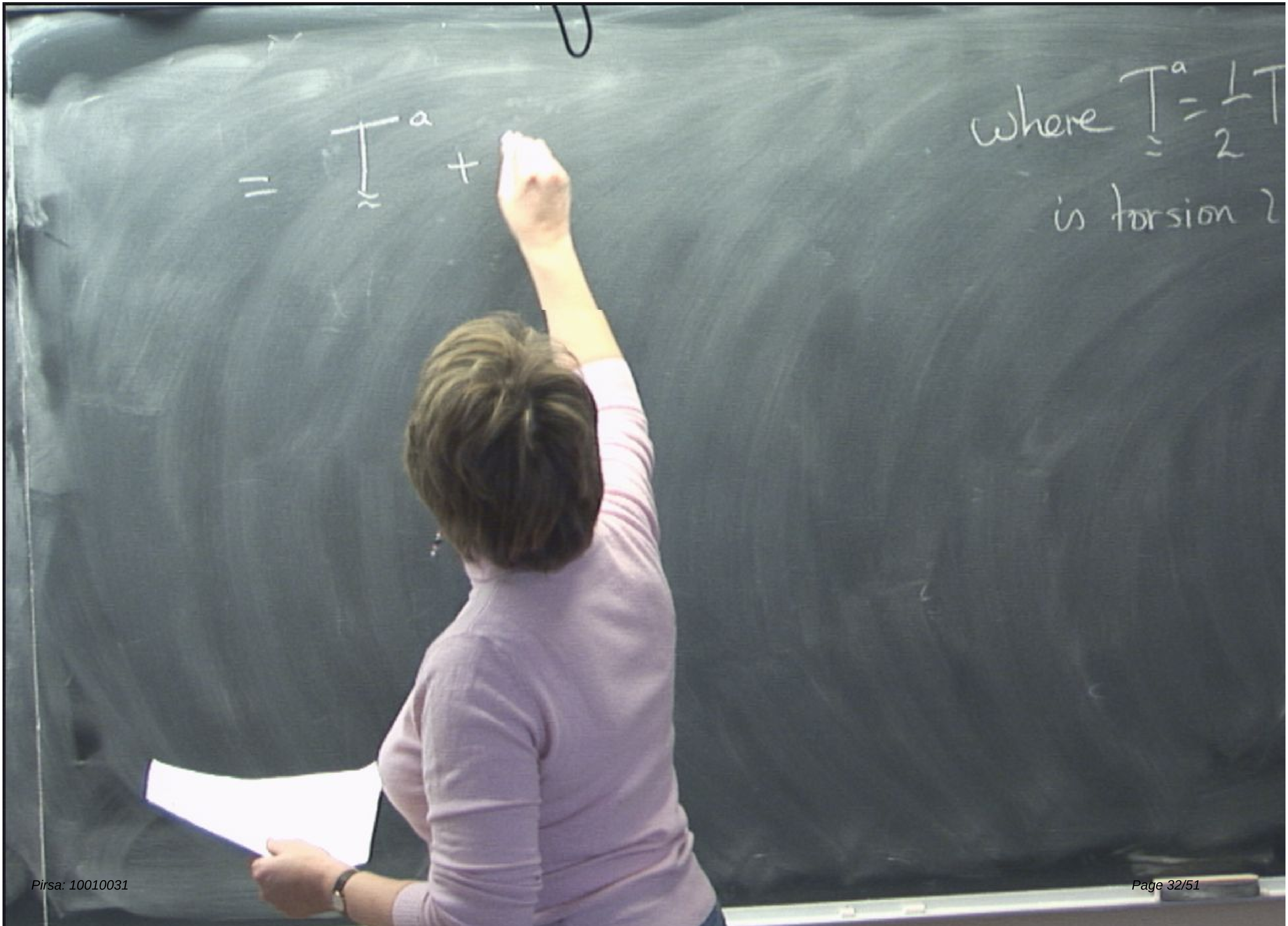
In an o/n basis, the spin connection is anti-symmetric

Finally,

$$\underline{\Theta}^a{}_c \wedge \underline{\omega}^c = \Gamma^a{}_{bc}\, \underline{\omega}^b \wedge \underline{\omega}^c$$

$$= \frac{1}{2}\left(\Gamma^a{}_{bc} - \Gamma^a{}_{cb}\right) \underline{\omega}^b \wedge \underline{\omega}^c$$

$$= \frac{1}{2}\left(T^a{}_{bc} + C^a{}_{bc}\right) \underline{\omega}^b \wedge \underline{\omega}^c$$


= T a +
where T a = 1 2 T
is torsion

$$= \underset{\sim}{T}^a + \frac{1}{2} \langle \underline{\omega}^a | [\underline{e}_b, \underline{e}_c] \rangle \, \underline{\omega}^b \wedge \underline{\omega}^c$$

(where $\underset{\sim}{T}^a = \frac{1}{2} T^a_{bc} \, \underline{\omega}^b \wedge \underline{\omega}^c$ is torsion 2 form)

$$= \underset{\sim}{T}^a + \frac{1}{2}\langle \underline{\omega}^a | [\underline{e}_b, \underline{e}_c] \rangle \underline{\omega}^b \wedge \underline{\omega}^c$$

(where $\underset{\sim}{T}^a = \frac{1}{2} T^a_{bc}\, \underline{\omega}^b \wedge \underline{\omega}^c$ is torsion 2 form

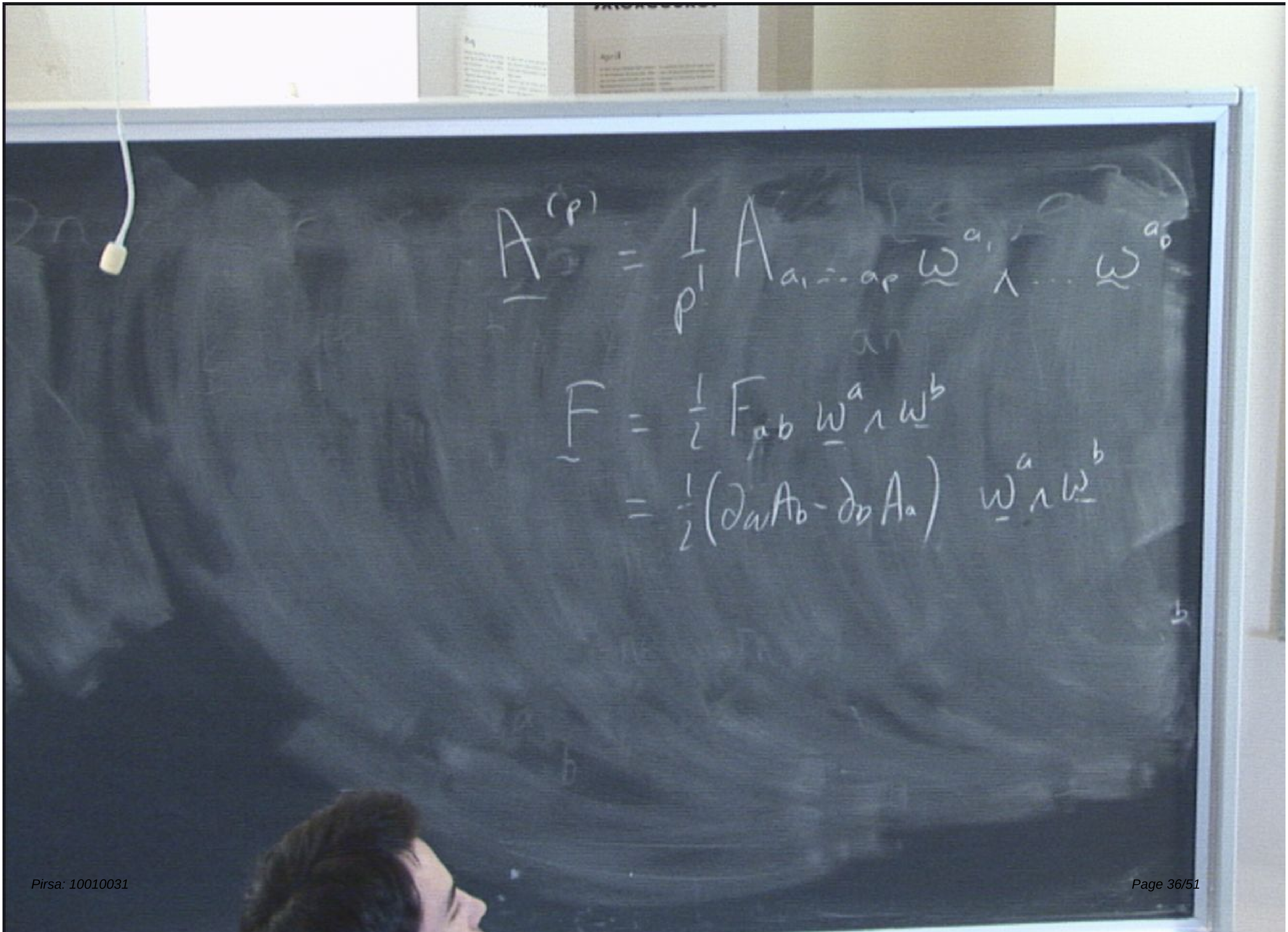
$$\underset{\sim}{T}^a - \frac{1}{2}\langle \underline{d}\underline{\omega}^a | \underline{e}_b \underline{e}_c \rangle \underline{\omega}^b \wedge \underline{\omega}^c$$

$$\underset{\sim}{T}^a - \underline{d}\underline{\omega}^a$$

$$\boxed{\underset{\sim}{T}^a = \underline{d}\underline{\omega}^a + \underline{\theta}^a{}_b \wedge \underline{\omega}^b}$$

Cartan's 1st structural eqn

Curvature



$\underset{\sim}{A}^{(p)} = \frac{1}{p!} A_{a_1 \cdots a_p} \underset{\sim}{\omega}^{a_1} \wedge \cdots \underset{\sim}{\omega}^{a_p}$
$\underset{\sim}{F} = \frac{1}{2} F_{ab} \underset{\sim}{\omega}^{a} \wedge \underset{\sim}{\omega}^{b}$
$= \frac{1}{2} (\partial_a A_b - \partial_b A_a) \underset{\sim}{\omega}^{a} \wedge \underset{\sim}{\omega}^{b}$

Curvature

Recall in gauge field theory the gauge pot^l

Curvature

Recall in gauge field theory the gauge pot^l is reqd to make a local symmetry so that the deriv transforms covariantly

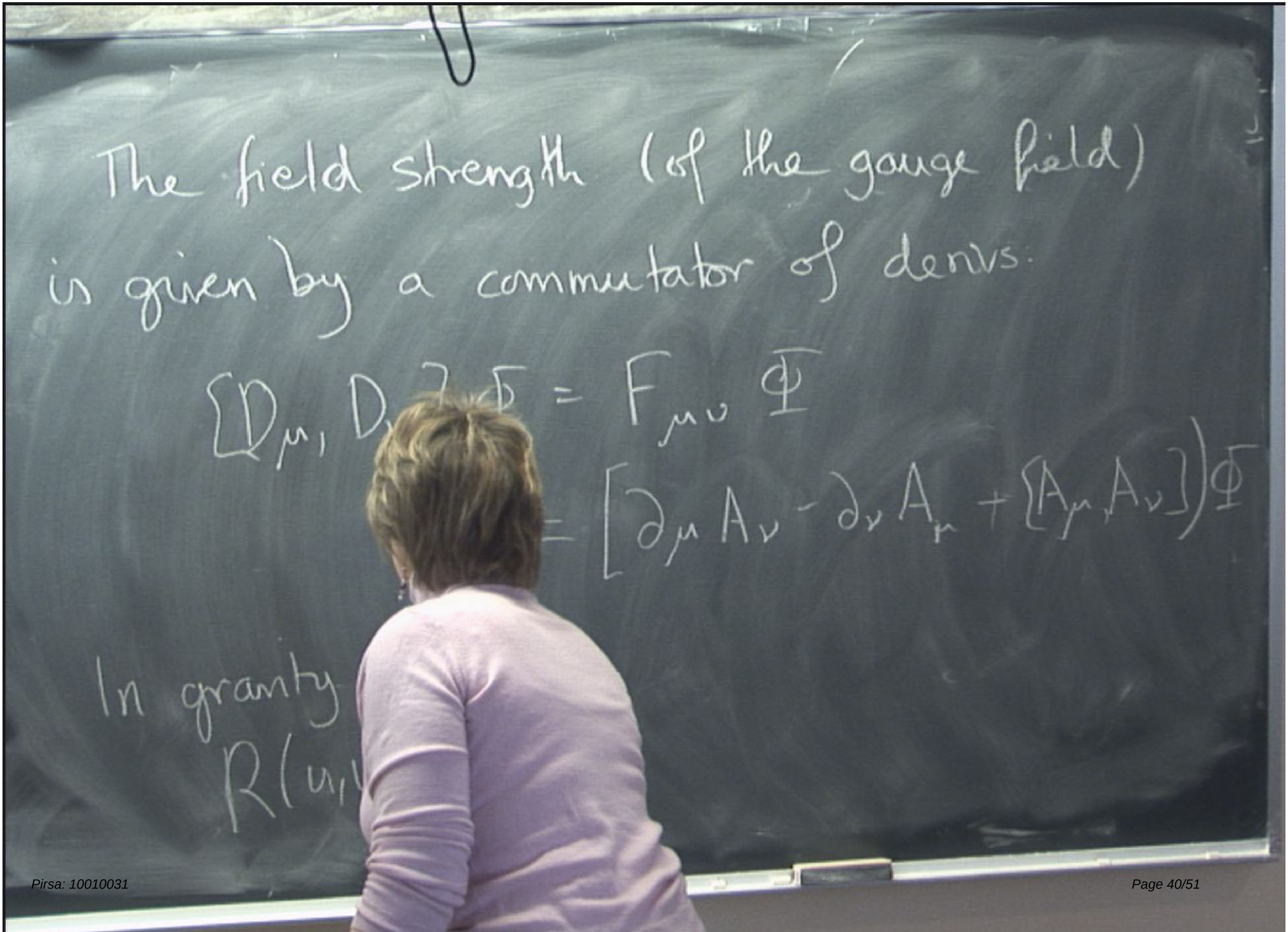
$$D_\mu \Phi = (\nabla_\mu + ieA_\mu)\,\Phi$$

Curvature

Recall in gauge field theory the gauge pot^{l} is reqd to make a local symmetry so that the deriv transforms covariantly

$$D_\mu \Phi = (\nabla_\mu + ieA_\mu)\,\Phi$$

$$\left(\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\sigma} V^\sigma \right)$$


The field strength (of the gauge field)
is given by a commutator of derivs:
$[D_\mu, D_\nu]\Phi = F_{\mu\nu}\Phi$
$= (\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu])\Phi$
In gravity
$R(u,$

The field strength (of the gauge field) is given by a commutator of derivs:

$$[D_\mu, D_\nu]\,\Phi = F_{\mu\nu}\,\Phi$$

$$= \left[\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]\right)\Phi$$

In gravity we define

$$R(u,v)W = \left(\nabla_u \nabla_v - \nabla_v \nabla_u - \nabla_{[u,v]}\right) W$$

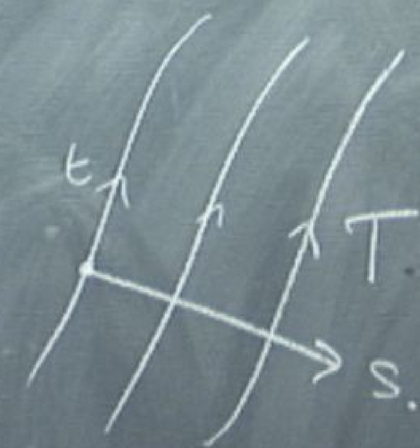
cpts:

$$R^a{}_{bcd} = \Gamma^a{}_{bd,c} - \Gamma^a{}_{bc,d} + \Gamma^a{}_{ce}\Gamma^e{}_{db} - \Gamma^a{}_{de}\Gamma^e{}_{cb} - C^e{}_{cd}\Gamma^a{}_{eb}$$

Physically, curvature tells us about tidal forces.

Physically, curvature tells us about tidal forces.

Take a set of geodesics, tgt vector T : $\nabla_T T = 0$



Label geodesics by s, & let $N = \dfrac{\partial}{\partial s}$.

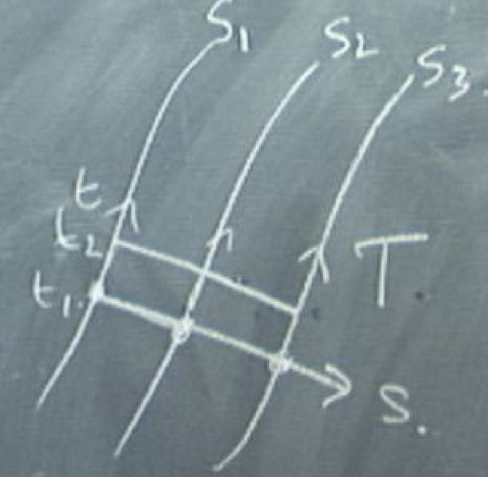
Physically, curvature tells us about tidal forces.

Take a set of geodesics, tgt vector T : $\nabla_T T = 0$

Label geodesics by s, & let $N = \frac{\partial}{\partial s}$

by construction $[T, N] = 0$.

Physically, curvature tells us about tidal forces.



Take a set of geodesics, tgt vector T : $\nabla_T T = 0$

Label geodesics by s, & let $N = \frac{\partial}{\partial s}$

by construction $[T, N] = 0$.

Then $R(N,T)T = \left(D_N T - D_T N\right)$

Then $R(N,T)T = (\nabla_N \nabla_T - \nabla_T \nabla_N)$

$$= - \nabla_T \nabla_N T$$

$$= - \nabla_T \nabla_T N.$$

Then $R(N,T)T = (\nabla_N \nabla_T - \nabla_T \nabla_N)$

$$= -\nabla_T \nabla_N T$$

$$= -\nabla_T \nabla_T N$$

$$\frac{D^2}{dt^2} N^a + R^a{}_{bcd} T^b N^c T^d = 0$$

Then $R(N,T)T = (\nabla_N \nabla_T - \nabla_T \nabla_N)$

$= -\nabla_T \nabla_N T$

$= -\nabla_T \nabla_T N$

$$\frac{D^2}{dt^2} N^a + R^a{}_{bcd} T^b N^c T^d = 0$$

— tells us how geodesics are squeezed toge

Then, defining the curvature 2-forms as

$$\underline{R}^a{}_b = \frac{1}{2} R^a{}_{bcd}\, \underline{\omega}^c \wedge \underline{\omega}^d$$

Cartan's 2nd structural eqn is

$$\underline{R}^a{}_b = \underline{d}\,\underline{\theta}^a{}_b + \underline{\theta}^a{}_c \wedge \underline{\theta}^c{}_b$$