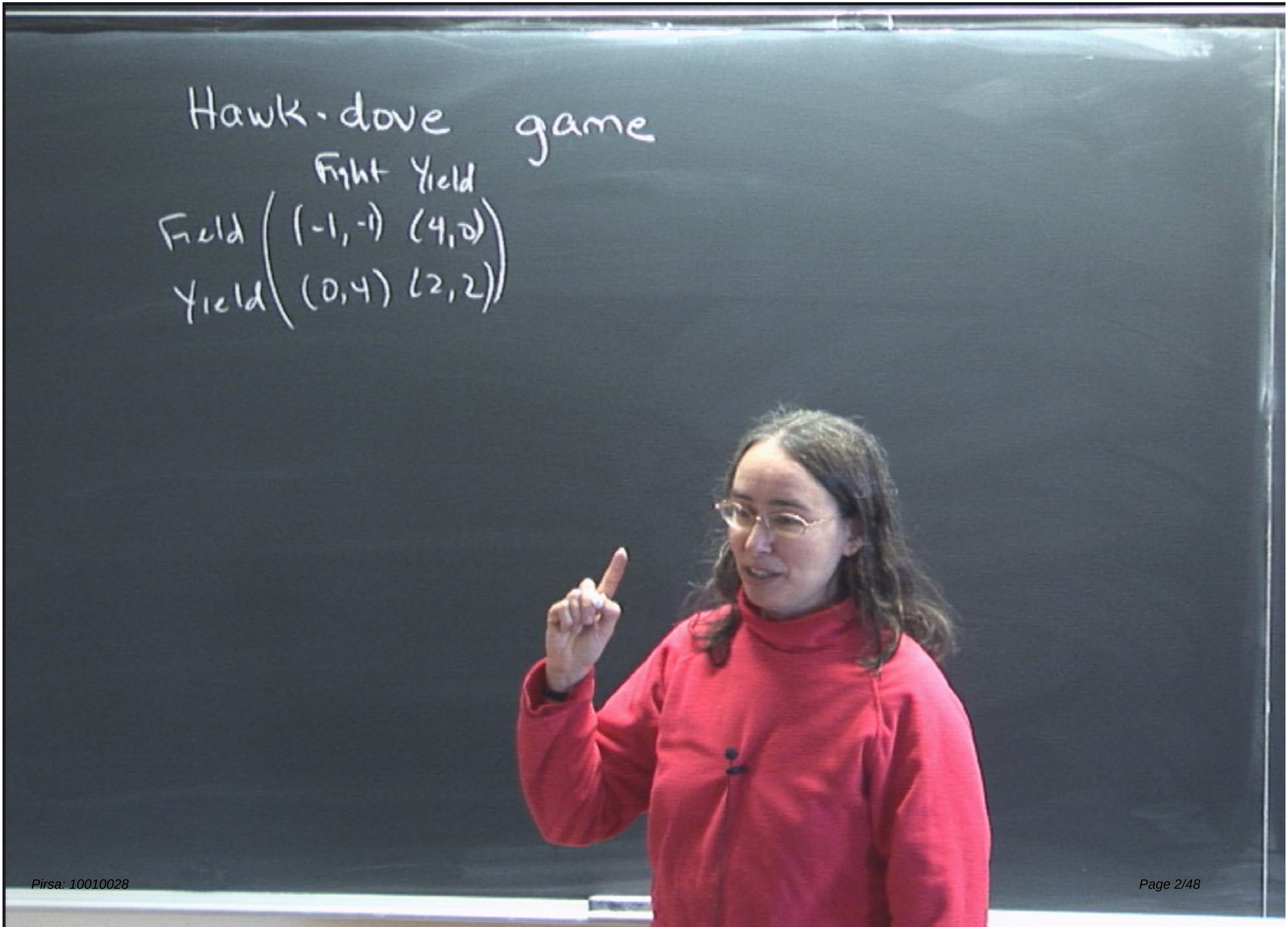
Title: Dynamical Systems - Review (PHYS 607) - Lecture 14

Date: Jan 21, 2010 09:00 AM

URL: http://pirsa.org/10010028

Abstract:


Hawk-dove game
$$\begin{array}{ccc} & \text{Fight} & \text{Yield} \\ \text{Field} & (-1,-1) & (4,0) \\ \text{Yield} & (0,4) & (2,2) \end{array}$$

Hawk-dove game

$$\begin{array}{c|cc} & \text{Fight} & \text{Yield} \\ \hline \text{Fight} & (-1,-1) & (4,0) \\ \text{Yield} & (0,4) & (2,2) \end{array}$$

pure Nash equilibria not symmetric

mixed Nash equilibrium

Hawk-dove game

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pure Nash equilibria not symmetric

mixed Nash equilibrium

x_1 = prob that 1st player fights

x_2 = prob that player 2 fights

$$u_1(x_1, x_2) = -x$$

Hawk-dove game

$$\begin{array}{rc} & \begin{array}{cc} \text{Fight} & \text{Yield} \end{array} \\ \begin{array}{r} \text{Fight} \\ \text{Yield} \end{array} & \begin{pmatrix} (-1,-1) & (4,0) \\ (0,4) & (2,2) \end{pmatrix} \end{array}$$

pure Nash equilibria not symmetric

ed Nash equilibrium

$$u_1(x_1, x_2) = -x_1 x_2 + 4x_1(1-x_2)$$

x_1 = prob that 1st player fights

x_2 = prob that player 2 fights

Hawk-dove game

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pure Nash equilibria not symmetric

mixed Nash equilibrium

$$u_1(x_1, x_2) = -x_1 x_2 + 4x_1(1-x_2) + 2(1-x_1)(1-x_2)$$

x_1 = prob that 1st player fights

x_2 = prob that player 2 fights

Hawk-dove game

$$\begin{array}{ccc} & \text{Fight} & \text{Yield} \\ \text{Fight} & (-1,-1) & (4,0) \\ \text{Yield} & (0,4) & (2,2) \end{array}$$

pure Nash equilibria not symmetric

mixed Nash equilibrium

x_1 = prob that 1st player fights

x_2 = prob that player 2 fights

$$u_1(x_1, x_2) = -x_1 x_2 + 4x_1(1-x_2) + 2(1-x_1)(1-x_2)$$

$$u_1(x_1,x_2) = -3x_1 x_2 + 2x_1 - 2x_2 + 2$$

$$u_2(x_1, x_2) = -3x_1 x_2 - 2x_1 + 2x_2 + 2$$

$$\frac{\partial u_1}{\partial x_1}$$

$$\frac{\partial u_2}{\partial x_2}$$

$$\frac{\partial u_1}{\partial x_1} = 0 \Rightarrow -3x_2 + 2 = 0 \Rightarrow x_2 = 2/3$$

$$x_1 = 2/3$$

$$\frac{\partial u_2}{\partial x_2} = 0 \Rightarrow$$

So mixed Nash eq $= (2/3, 2/3)$

$$\frac{\partial u_1}{\partial x_1} = 0 \Rightarrow -3x_2 + 2 = 0 \Rightarrow x_2 = 2/3$$

$$x_1 = 2/3$$

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So mixed Nash eq $= (2/3, 2/3)$

Is it ESS?

$$\frac{\partial u_1}{\partial x_1} = 0 \Rightarrow -3x_2 + 2 = 0 \Rightarrow x_2 = 2/3$$

$$x_1 = 2/3$$

$$\frac{\partial u_2}{\partial x_2} = 0 \Rightarrow$$

So mixed Nash eq $= (2/3, 2/3) = s^* \overset{?}{=} \tilde{s}$

Is it ESS?

any mixed strategy played against s^* gives same payoff

So given $y \neq x$ stability depends on what happens when you play against y.

given x $(= 2/3)$

look at y

$u_1(y, y)$

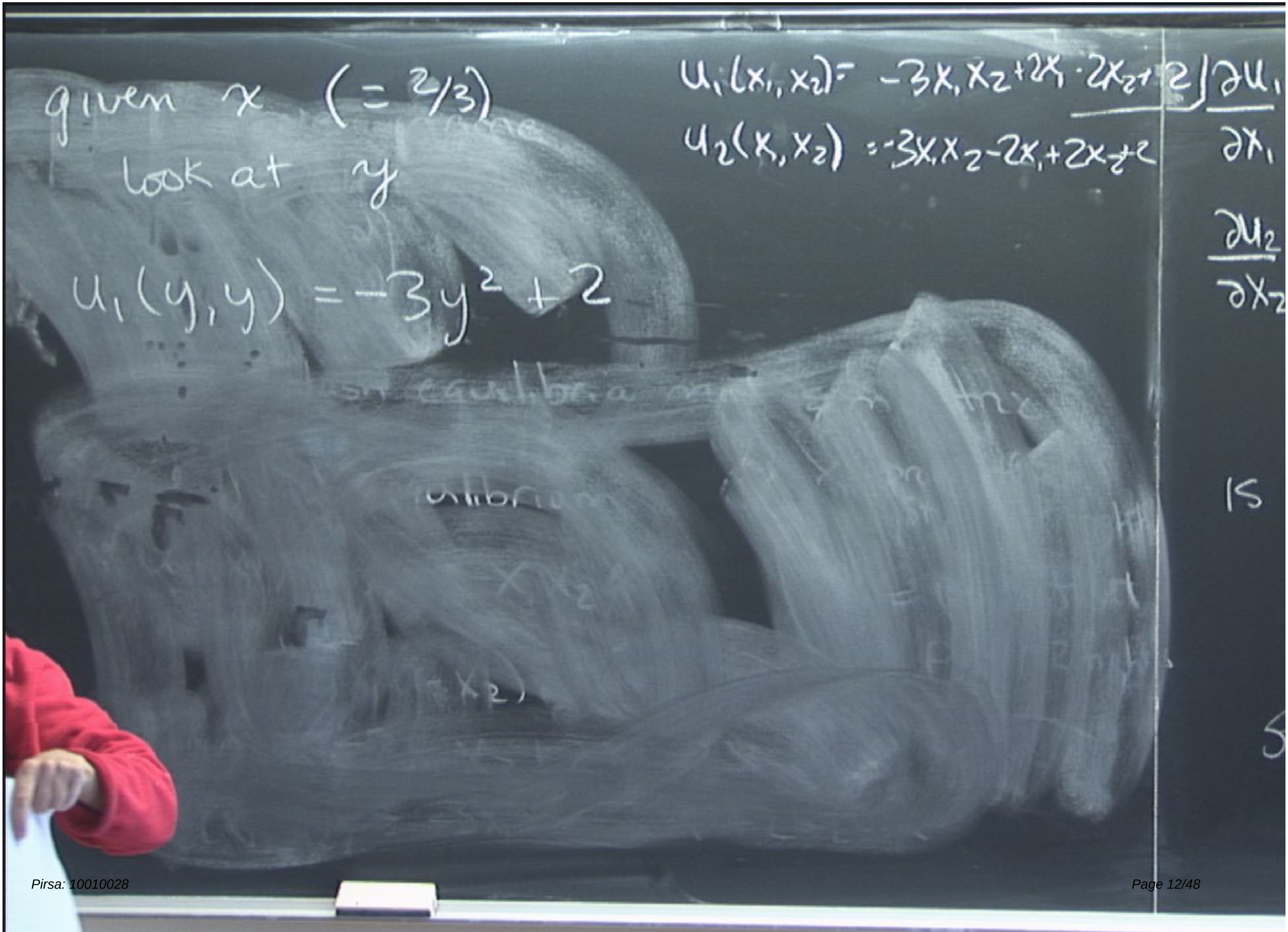
$$u_1(x_1, x_2) = -3x_1x_2 + 2x_1 - 2x_2 + 2$$

$$u_2(x_1, x_2) = -3x_1x_2 - 2x_1 + 2x_2 + 2$$

$\frac{\partial u_1}{\partial x_1}$

$\frac{\partial u_2}{\partial x_2}$

is


given x (= 2/3)
look at y
u1(y,y) = -3y2 + 2
u1(x1,x2) = -3x1x2+2x1-2x2+2
u2(x1,x2) = -3x1x2-2x1+2x2+2
∂u1/∂x1
∂u2/∂x2
is

given x $(= 2/3)$

look at y

$u_1(y,y) = -3y^2 + 2$

$u_1(y, 2/3)$

$u_2(y, 2/3)$

$u_1(x_1, x_2) = -3x_1x_2 + 2x_1 - 2x_2 + 2$

$u_2(x_1, x_2) = -3x_1x_2 - 2x_1 + 2x_2 + 2$

$u_1(y, 2/3)\ -3y\left(\frac{2}{3}\right)$

$\frac{\partial u_1}{\partial x_1}$

$\frac{\partial u_2}{\partial x_2}$

is

given x $(= 2/3)$

look at y

$u_1(y,y) = -3y^2 + 2$

$u_1(y, 2/3)$

$u_2(y, 2/3)$

$u_1(x_1,x_2) = -3x_1x_2 + 2x_1 \cdot 2x_2 + 2$

$u_2(x_1,x_2) = -3x_1x_2 - 2x_1 + 2x_2 + 2$

$u(y, 2/3) = -3y\left(\frac{2}{3}\right) + 2y \cdot \frac{4}{3} + 2$

$= -2y + 2y \cdot \frac{4}{3} + 2$

$\frac{2}{3}$

$\frac{\partial u_1}{\partial x_1}$

$\frac{\partial u_2}{\partial x_2}$

is

given x $(= 2/3)$

look at y

$$u_1(y,y) = -3y^2 + 2$$

$$u_1(y, 2/3) = \frac{2}{3}$$ ind of y, as expected

$$u_2(y, 2/3)$$

$$u_1(x_1, x_2) = -3x_1x_2 + 2x_1 \cdot 2x_2 + 2$$

$$u_2(x_1, x_2) = -3x_1x_2 - 2x_1 + 2x_2 + 2$$

$$u(y, 2/3) = -3y\left(\tfrac{2}{3}\right) + 2y \cdot \tfrac{4}{3} + 2$$

$$-2y + 2y \cdot \tfrac{4}{3} + 2$$

$$\frac{2}{3}$$

$$\frac{\partial u_1}{\partial x_1}$$

$$\frac{\partial u_2}{\partial x_2}$$

is

given x $(= 2/3)$

look at y

$u_1(y,y) = -3y^2 + 2$

$u_1(y, 2/3) = \frac{2}{3}$ ind of y, as expected

$u_2(y, 2/3) = -4y + \frac{4}{3} + 2$

$u_2(y,2/3) - u_2(y,y) = -4y + \frac{4}{3} + 2 - (-3y^2 + 2)$

$u_1(x_1, x_2) = -3x_1x_2 + 2x_1 \cdot 2x_2 + 2$

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$-2y + 2y - \frac{4}{3} + 2$

$= \frac{2}{3}$

$\frac{\partial u_1}{\partial x_1}$

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$$u_2(y,2/3) - u_2(y,y) = -4y + \frac{4}{3} + 2 - (-3y^2 + 2)$$
$$= 4y + \frac{4}{3} + 3y^2$$
$$= \frac{1}{3}(2-3y)^2 \geq 0$$

$$u_1(x_1, x_2) = -3x_1x_2 + 2x_1 \cdot 2x_2 + 2$$
$$u_2(x_1, x_2) = -3x_1x_2 - 2x_1 + 2x_2 + 2$$

$$u(y, 2/3) = -3y\left(\frac{2}{3}\right) + 2y \cdot \frac{4}{3} + 2$$
$$= 2y + 2y - \frac{4}{3} + 2$$
$$= \frac{2}{3}$$

$$\frac{\partial u_1}{\partial x_1}$$

$$\frac{\partial u_2}{\partial x_2}$$

is

given x $(= 2/3)$

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$$u_1(x_1, x_2) = -3x_1x_2 + 2x_1 \cdot 2x_2 + 2$$

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$$u_1(y, y) = -3y^2 + 2$$

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$$= -2y + 2y - \frac{4}{3} + 2$$

$$= \frac{2}{3}$$

$$u_1(y, 2/3) = \frac{2}{3}$$ ind of y, as expected

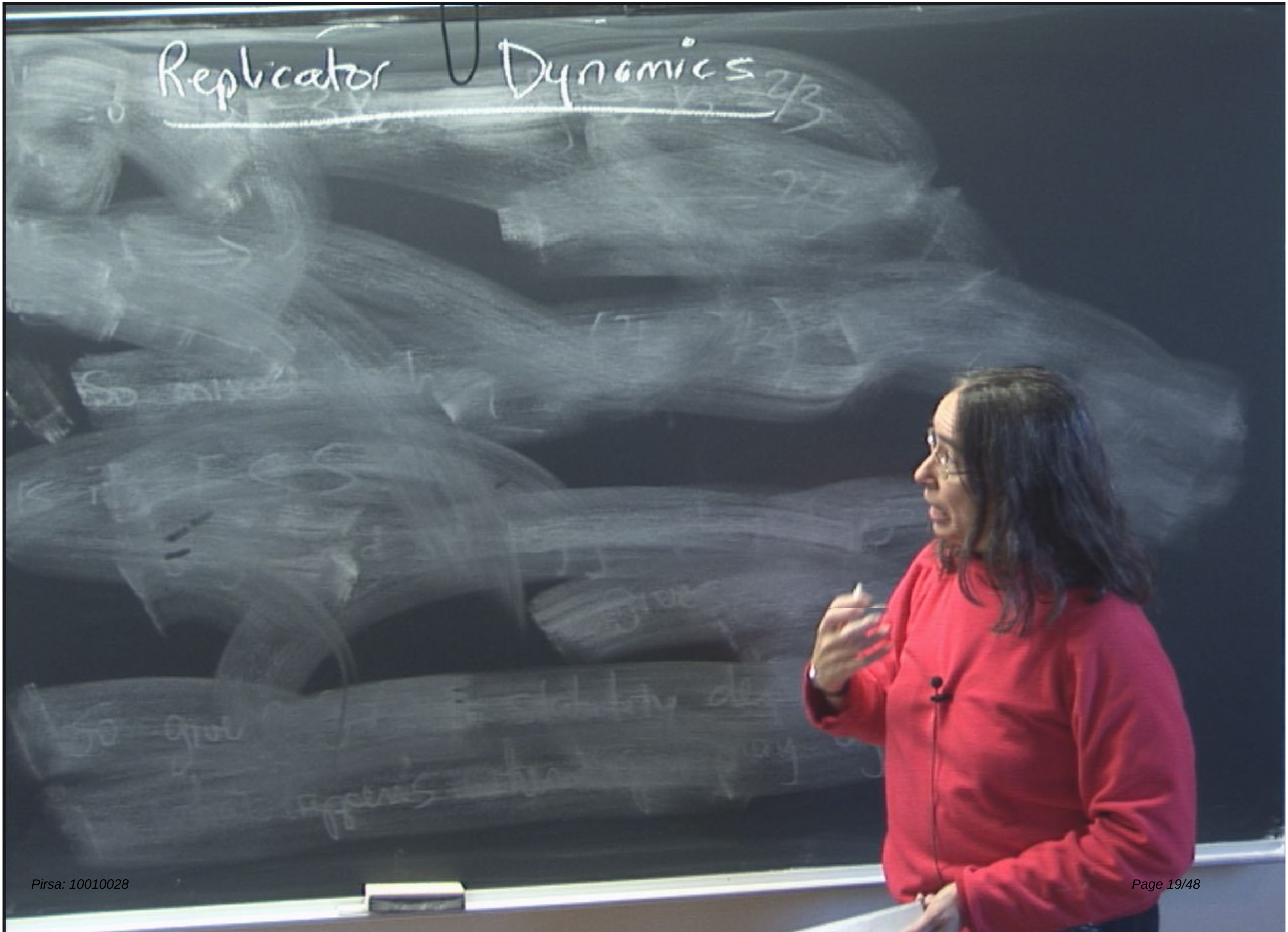
is

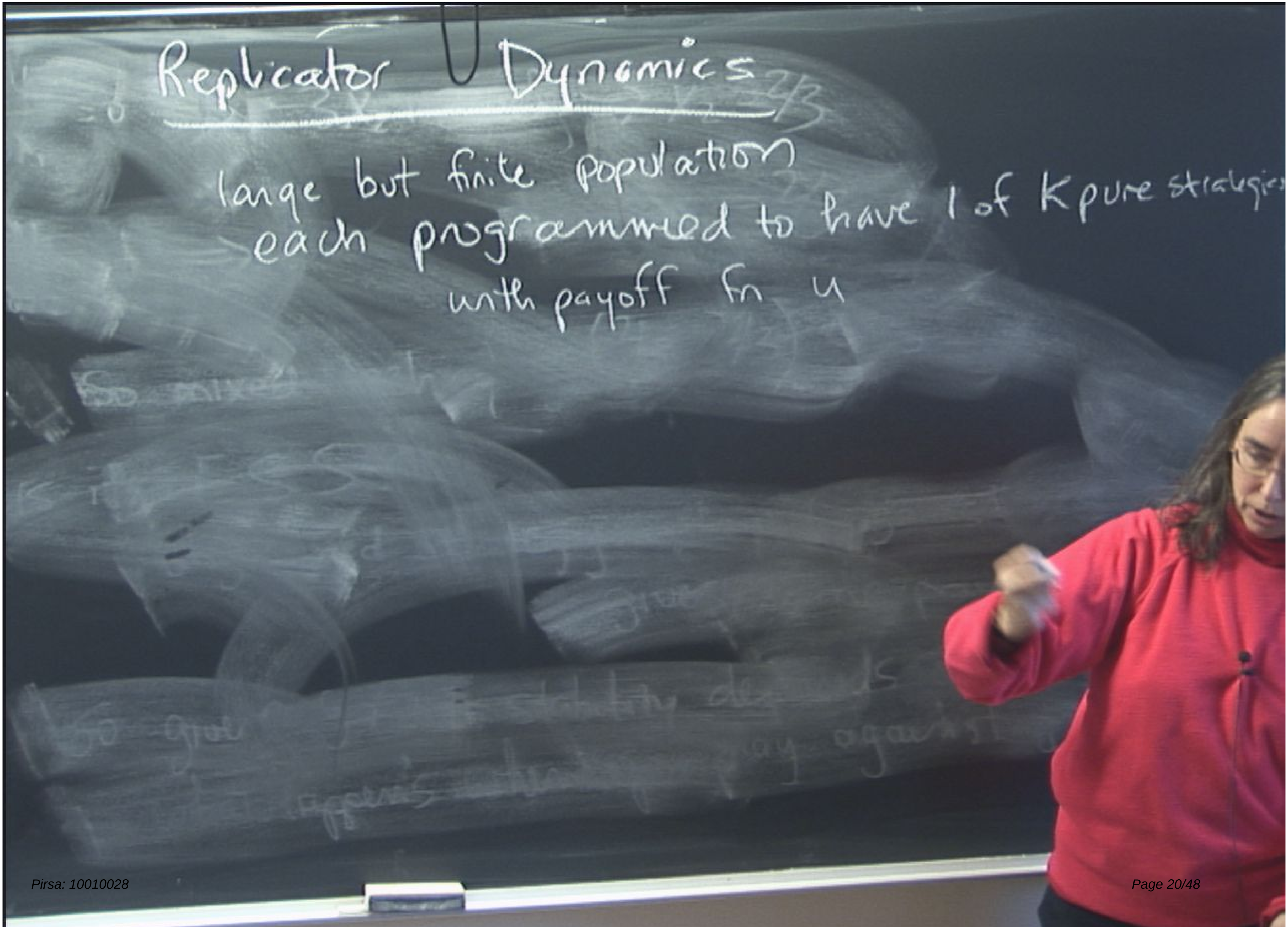
$$u_2(y, 2/3) = -4y + \frac{4}{3} + 2$$

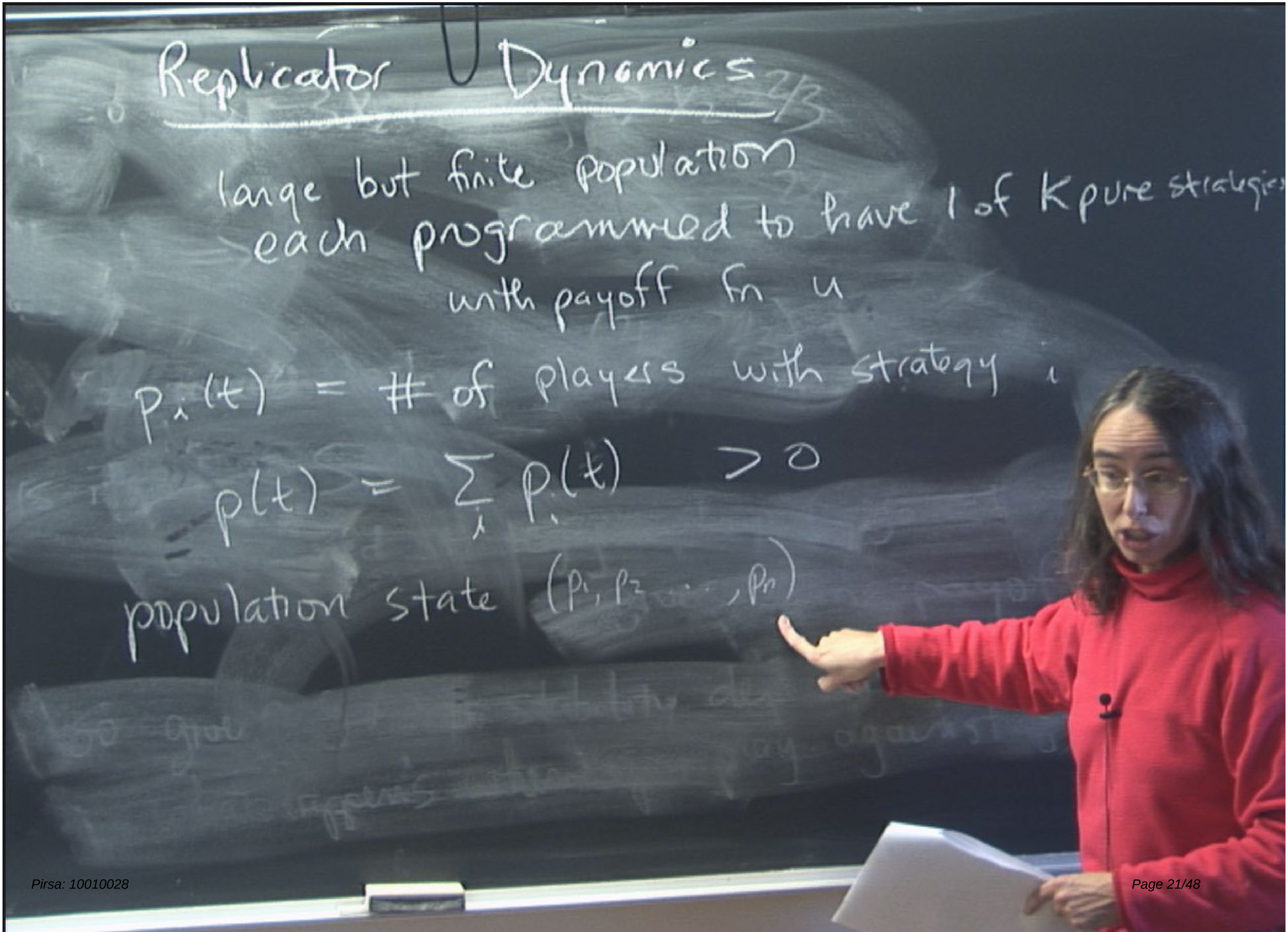
$$u_2(y, 2/3) - u_2(y, y) = -4y + \frac{4}{3} + 2 - (-3y^2 + 2)$$

$$= 4y + \frac{4}{3} + 3y^2$$

$$= \frac{1}{3}(2 - 3y)^2 \geq 0$$


Replicator Dynamics


Replicator Dynamics
large but finite population
each programmed to have 1 of K pure strategies
with payoff fn u


Replicator Dynamics
large but finite population
each programmed to have 1 of K pure strategies
with payoff fn u
$p_i(t)$ = # of players with strategy i
$p(t) = \sum_i p_i(t) > 0$
population state $(p_1, p_2, \ldots, p_n)$

Replicator Dynamics

large but finite population
each programmed to have 1 of K pure strategies
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population state $(p_1^{(t)}, p_2^{(t)}, \ldots, p_K) \to p(t)\,(x_1(t), \ldots, x_n(t))$

Replicator Dynamics

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population state $(p_1^{(t)}, p_2^{(t)}, \ldots, p_K) \to p(t)\,(x_1(t), \ldots, x_n(t))$

$$p(t)\vec{x} \equiv p(t)\,x$$

$$x_i(t) = \frac{p_i(t)}{p(t)}$$

Replicator Dynamics

large but finite population
each programmed to have 1 of K pure strategies
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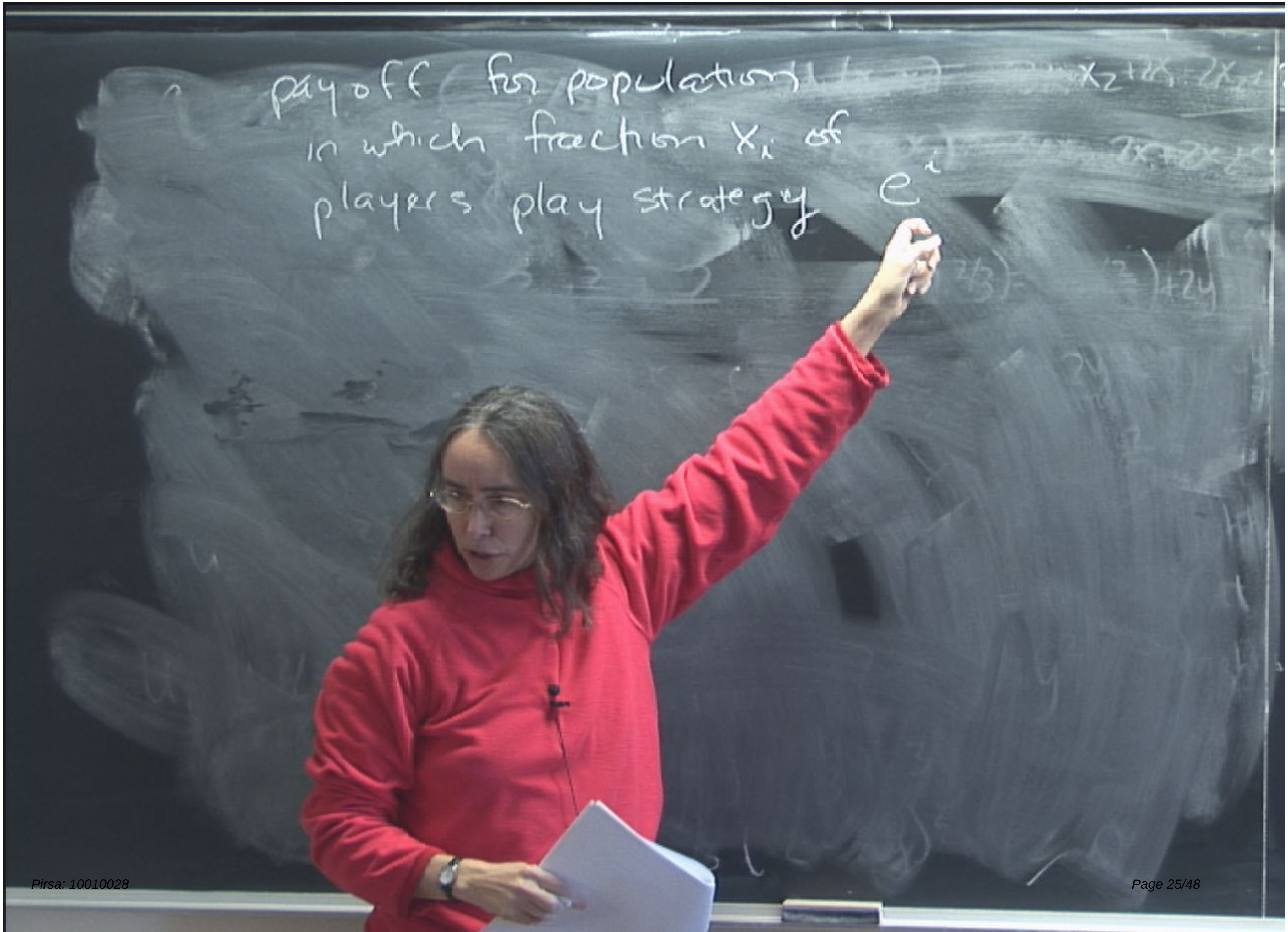
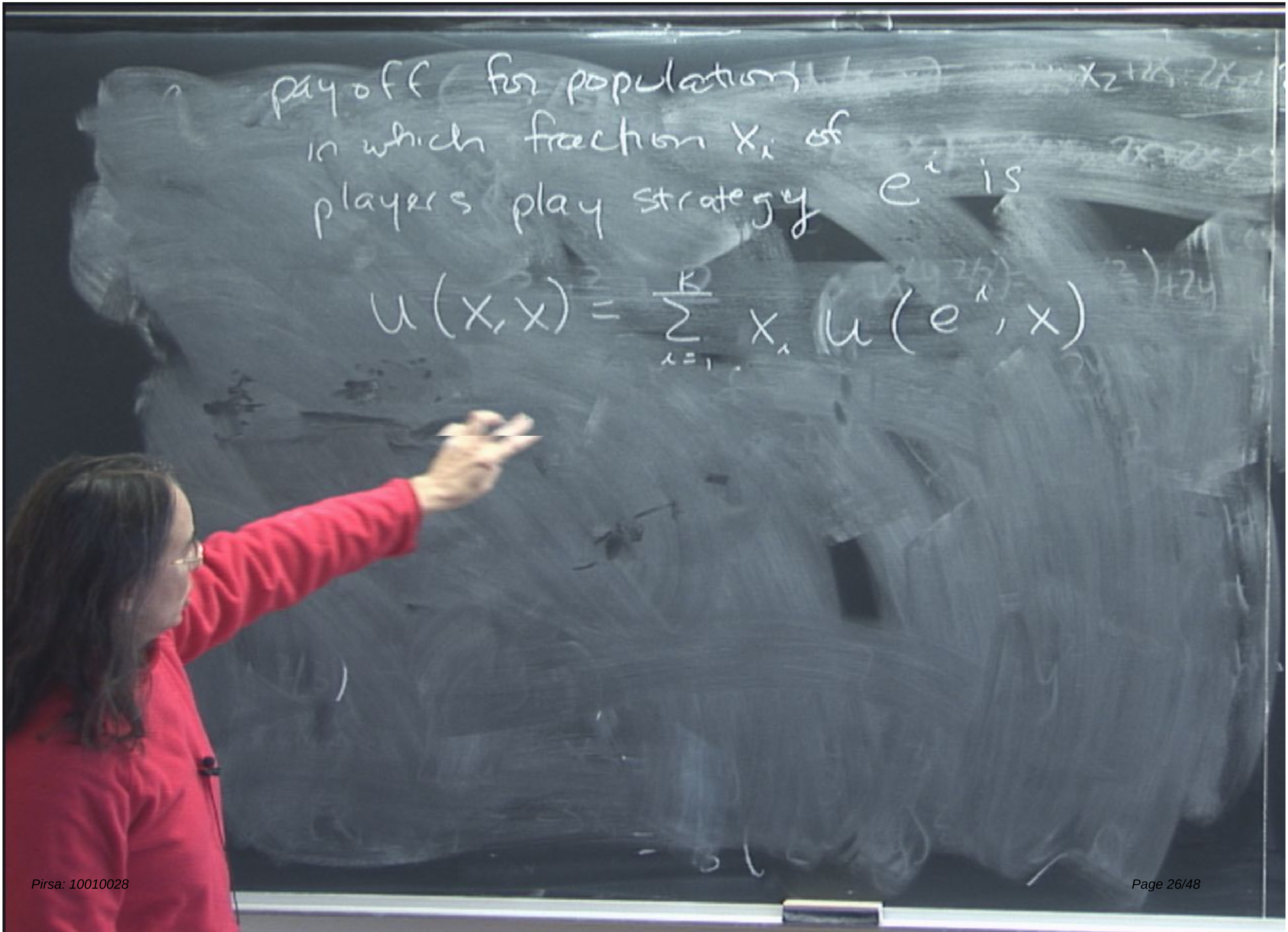
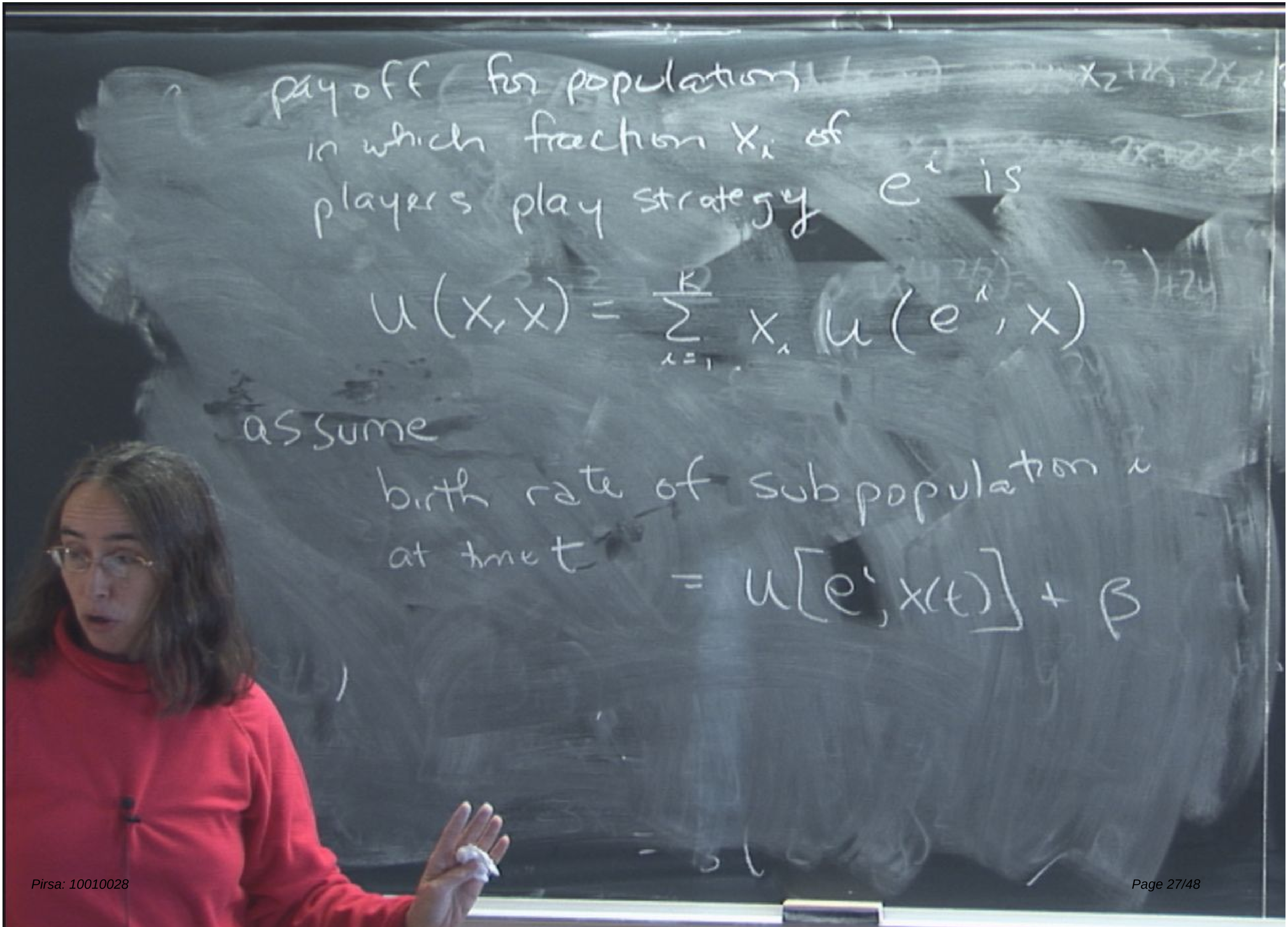

payoff for population
in which fraction x_i of
players play strategy e^i


payoff for population
in which fraction x_i of
players play strategy e^i is
$u(x,x) = \sum_{i=1}^{k} x_i \, u(e^i, x)$


payoff for population
in which fraction x_i of
players play strategy e^i is
$u(x,x) = \sum_{i=1}^{k} x_i \, u(e^i, x)$
assume
birth rate of subpopulation i
at time t
$= u[e^i, x(t)] + \beta$

payoff for population in which fraction x_i of players play strategy e^i is

$$u(x,x) = \sum_{i=1}^{k} x_i \, u(e^i, x)$$

assume

birth rate of subpopulation i at time t $= u[e^i, x(t)] + \beta$

death rate $= \delta$

$$\frac{d}{dt} p_i(t) = \left[\beta + u(e^i, x) - \delta\right] p_i(t)$$

payoff for population in which fraction x_i of players play strategy e^i is

$$u(x,x) = \sum_{i=1}^{k} x_i \, u(e^i, x)$$

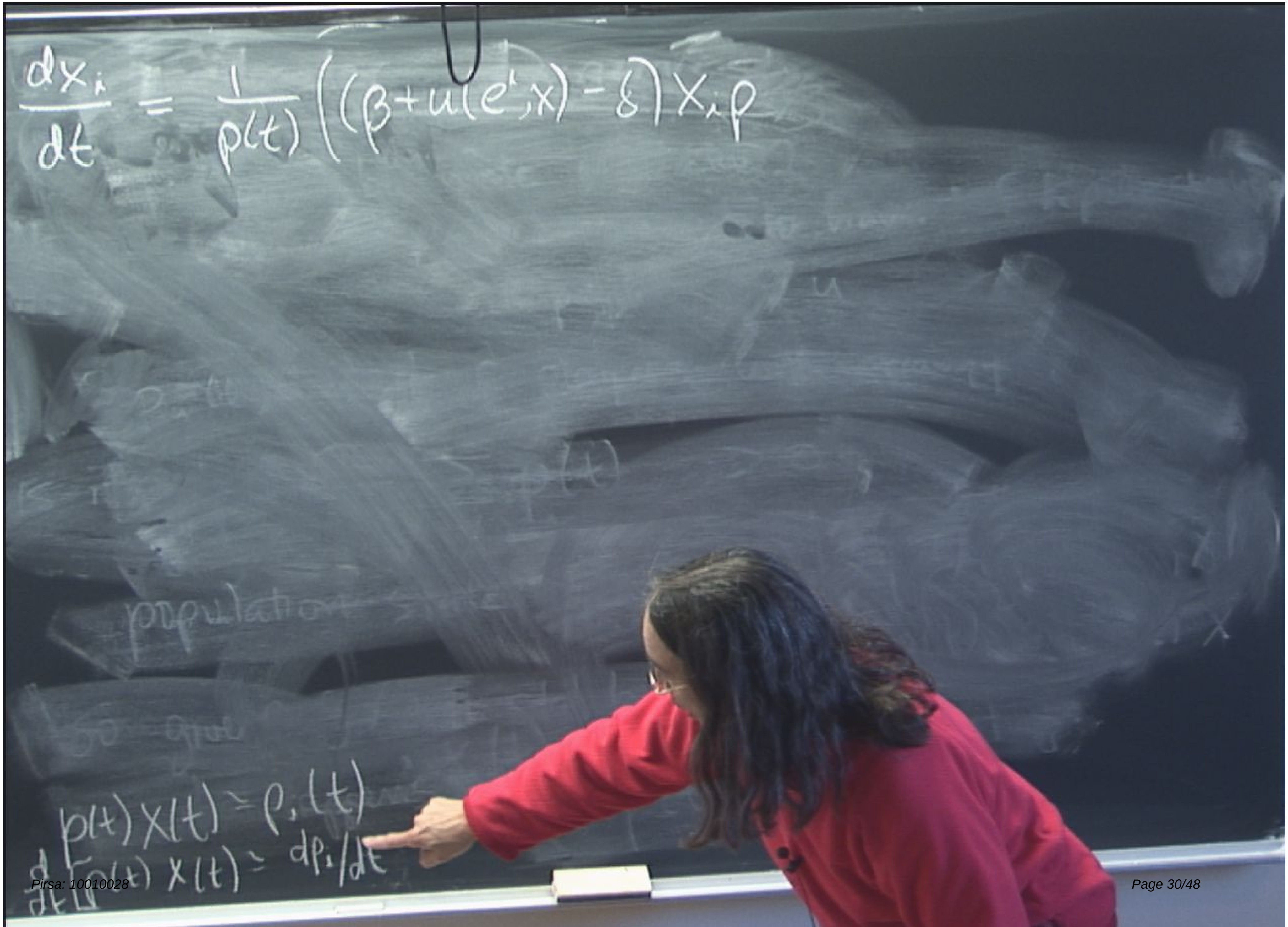
assume

birth rate of subpopulation i at time t $= u[e^i, x(t)] + \beta$

death rate $= \delta$

$$\frac{d}{dt} p_i(t) = \left[\beta + u(e^i, x) - \delta\right] p_i(t)$$

$$x(t) = p_i(t)/p(t)$$


$\frac{dx_i}{dt} = \frac{1}{\rho(t)}\left((\beta + u(e^i, x) - \delta) x_i \rho\right.$
$\rho(t) x(t) = \rho_i(t)$
$\frac{d}{dt}[\rho(t) x(t)] = d\rho_i/dt$

$$\frac{dx_i}{dt} = \frac{1}{\rho(t)}\Big(\big(\beta + u(e^i, x) - \delta\big)x_i\rho - x_i \sum_j \big(\beta + u(e^j, x) - \delta\big)x_j\rho\Big)$$

$$\frac{d}{dt}x_i(t) = x_i\Big[\big(\beta + u(e^i, x) - \delta\big) - \sum_j \big(\beta + u(e^j, x) - \delta\big)x_j$$

$$\rho(t)\,x(t) = \rho_i(t)$$

$$\frac{d}{dt}\big[\rho(t)\,x(t)\big] = d\rho_i/dt$$

$$\frac{dx_i}{dt} = \frac{1}{\rho(t)}\left(\left(\beta + u(e^i, x) - \delta\right) x_i \rho - x_i \sum_j \left(\beta + u(e^j, x) - \delta\right) x_j \rho\right)$$

$$\frac{d}{dt} x_i(t) = x_i \left[\left(\beta + u(e^i, x) - \delta\right) - \sum_j \left(\beta + u(e^j, x) - \delta\right) x_j \right]$$

$$\rho(t)\, x(t) = \rho_i(t)$$

$$\frac{d}{dt}\left[\rho(t)\, x(t)\right] = d\rho_i/dt$$

$$\frac{dx_i}{dt} = \frac{1}{p(t)}\left(\left(\beta+u(e^i,x)-\delta\right)x_i p - x_i \sum_j \left(\beta+u(e^j,x)-\delta\right)x_j p\right)$$

$$\frac{d}{dt}x_i(t) = x_i\left[(\beta+u(e^i,x)-\delta) - \sum_j (\beta+u(e^j,x)-\delta)x_j\right]$$

$$\frac{dx_i(t)}{dt} = \left[u(e^i,x) - u(x,x)\right]x_i$$

$$p(t)\,x(t) = p_i(t)$$

$$\frac{d}{dt}\left[p(t)\,x(t)\right] = dp_i/dt$$

$$\frac{dx_i}{dt} = \frac{1}{p(t)}\left(\left(\beta + u(e^i,x) - \delta\right) x_i p - x_i \sum_j \left(\beta + u(e^j,x) - \delta\right) x_j p\right)$$

$$\frac{d}{dt} x_i(t) = x_i \left[(\beta + u(e^i, x) - \delta) - \sum_j (\beta + u(e^j,x) - \delta) x_j \right]$$

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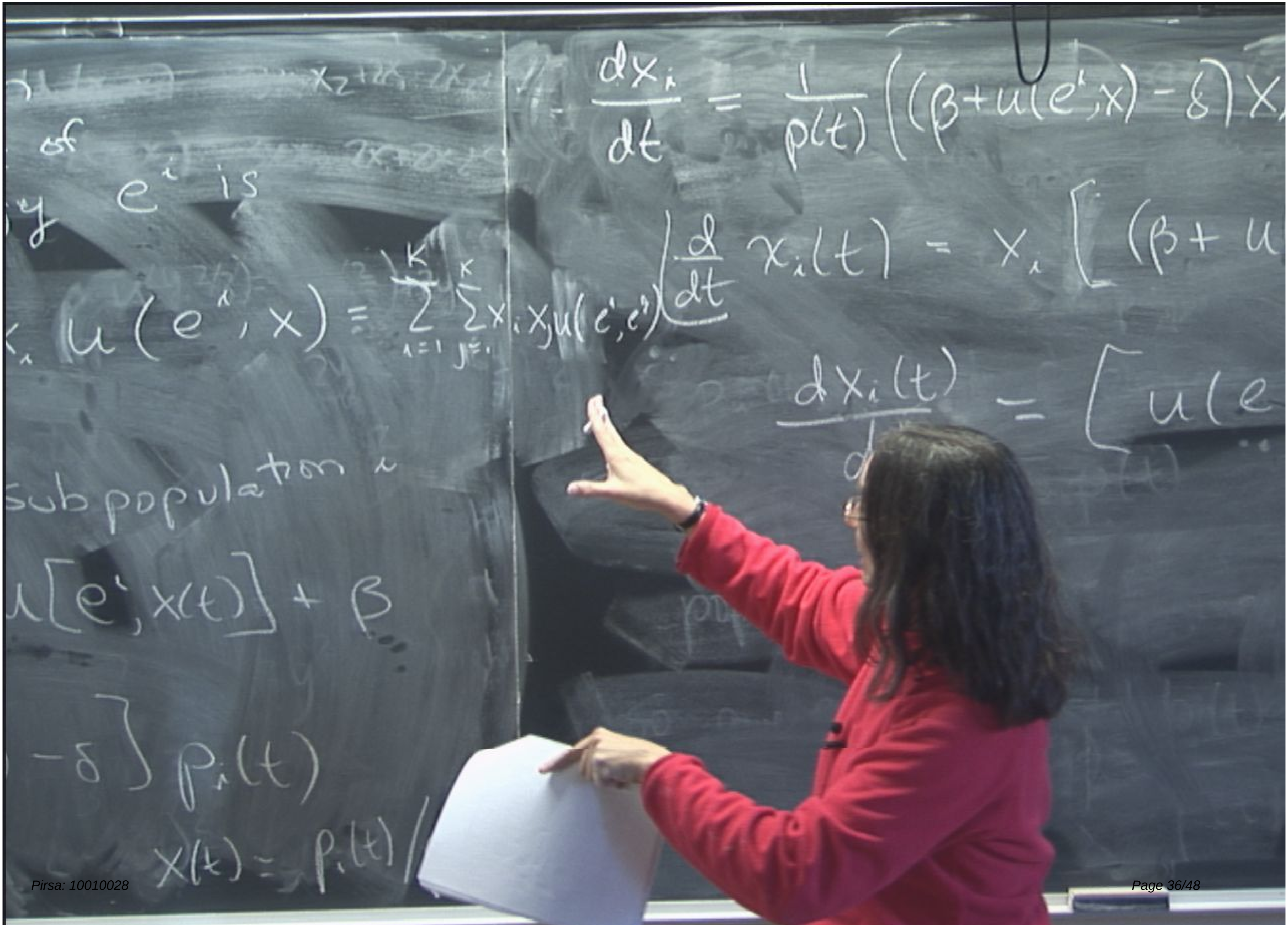
$$\frac{d}{dt}x_i(t) = x_i\left[\left(\beta+u(e^i,x)-\delta\right)-\sum_j\left(\beta+u(e^j,x)-\delta\right)x_j\right]$$

$$\frac{dx_i(t)}{dt} = \left[u(e^i,x)-u(x,x)\right]x_i$$

[ind of β & δ]


of
e^i is
dx_i/dt = 1/p(t) ((β+u(e^i,x) − δ) x
d/dt x_i(t) = x_i [(β + u
x_i u(e^i, x) = Σ_{i=1}^K Σ_{j=1}^K x_i x_j u(e^i, e^j)
dx_i(t)/dt = [u(e
subpopulation i
u[e^i, x(t)] + β
−δ] p_i(t)
x(t) = p_i(t) /

$$\frac{dx_i}{dt} = \frac{1}{\rho(t)}\left(\left(\beta+u(e^i,x)-\delta\right)x_i\rho - x_i\sum_j\left(\beta+u(e^j,x)-\delta\right)x_j\rho\right)$$

$$\frac{d}{dt}x_i(t) = x_i\left[(\beta+u(e^i,x)-\delta) - \sum_j(\beta+u(e^j,x)-\delta)x_j\right]$$

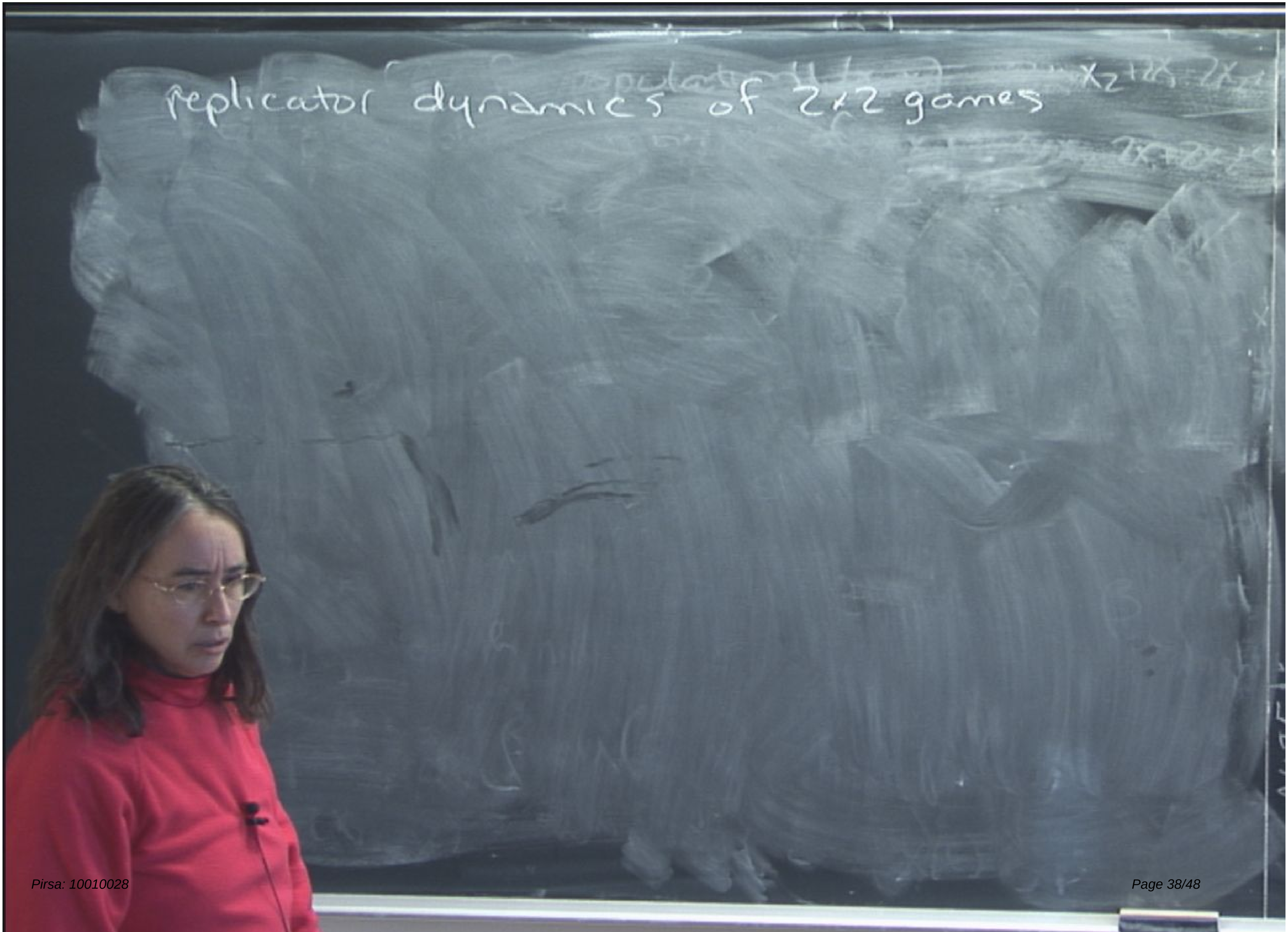
(e^i,e^j)

$$\frac{dx_i(t)}{dt} = \left[u(e^i,x) - u(x,x)\right]x_i$$

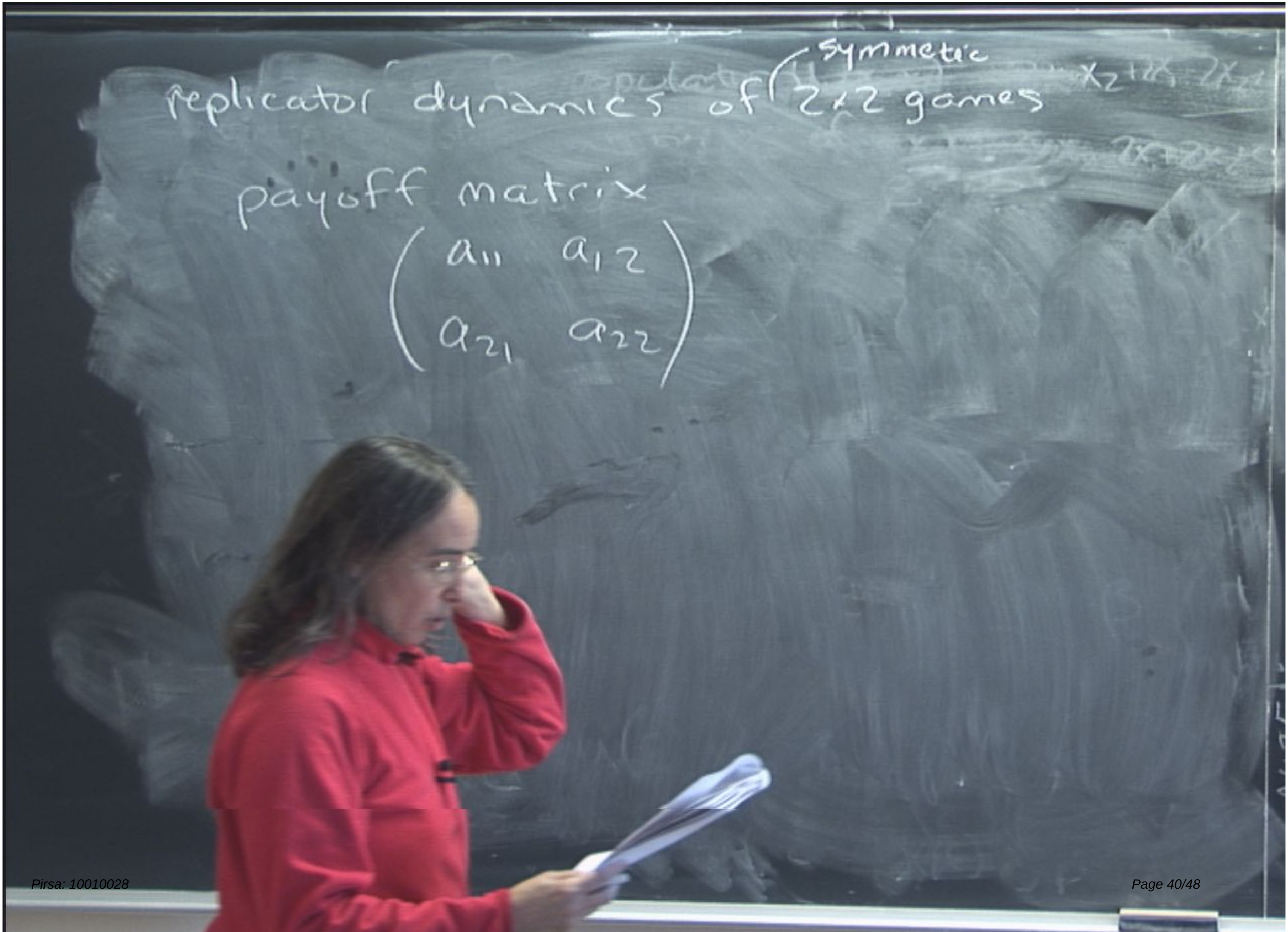
[ind of β & δ]

$$= u(e^i - x, x)\,x_i$$

$$\frac{d}{dt}\left[\frac{x_i}{x_j}\right] = \frac{\dot{x}_i}{x_j} - \frac{x_i}{x_j}\frac{\dot{x}_j}{x_j} = \left[u(e^i,x) - u(e^j,x)\right]\frac{x_i}{x_j}$$


replicator dynamics of 2x2 games

replicator dynamics of symmetric 2×2 games


symmetric
replicator dynamics of 2x2 games
payoff matrix
$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

replicator dynamics of 2×2 games

symmetric

payoff matrix

$$\begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} \\ \tilde{a}_{21} & \tilde{a}_{22} \end{pmatrix} \rightarrow \begin{pmatrix} a'_{11} & 0 \\ a'_{21} & a'_{22} \end{pmatrix}$$

replicator dynamics of symmetric 2×2 games

payoff matrix

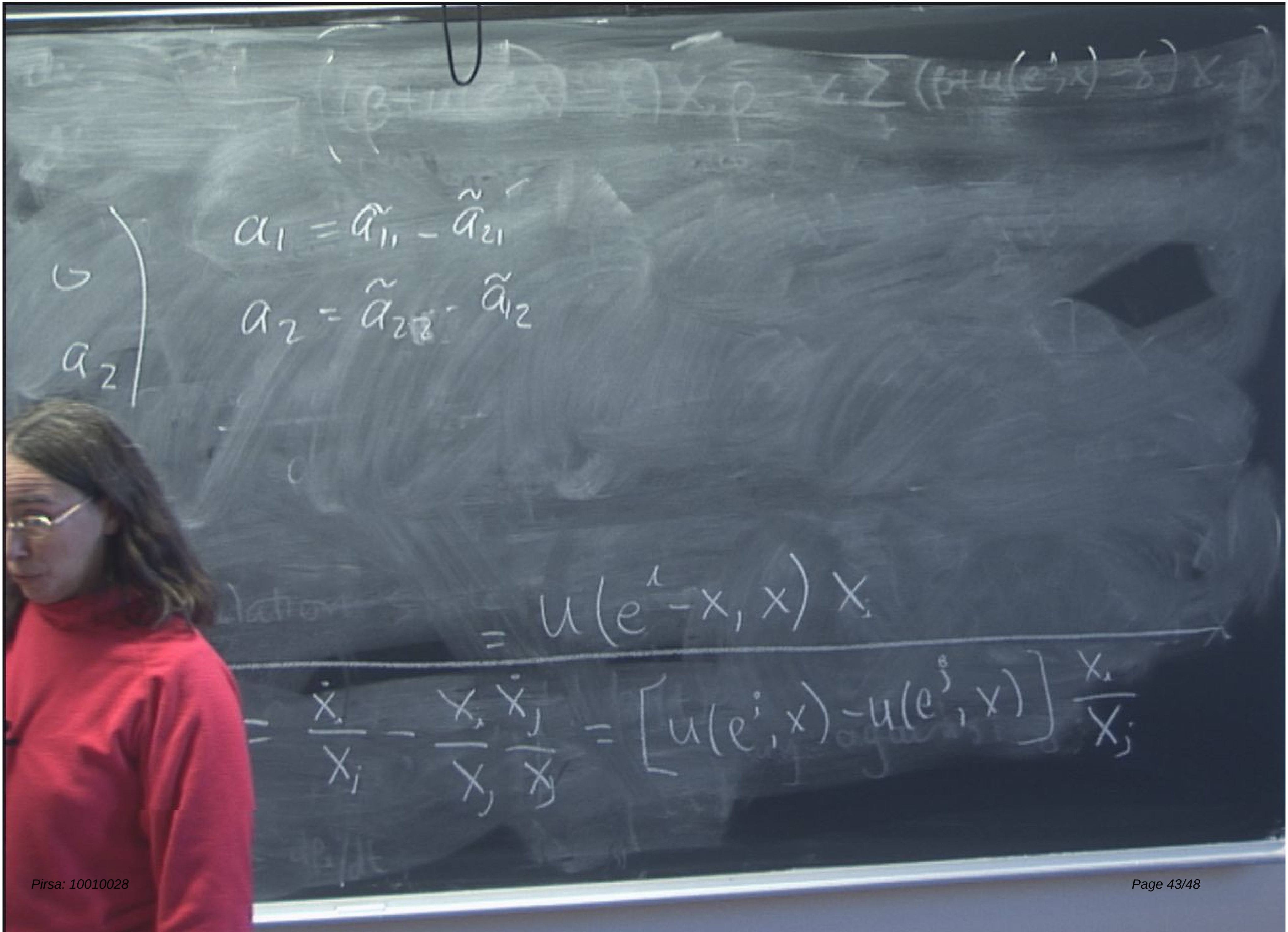
$$\begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} \\ \tilde{a}_{21} & \tilde{a}_{22} \end{pmatrix} \rightarrow \begin{pmatrix} a_{11}' & 0 \\ a_{21}' & a_{22}' \end{pmatrix}$$

if other player plays strategy 1

my choice depends only on $a_{11} - a_{12}$

if other play 2, my choice

depends only on $a_{21} - a_{22}$


$a_1 = \tilde{a}_{11} - \tilde{a}_{21}$
$a_2 = \tilde{a}_{22} - \tilde{a}_{12}$
a_2
$= u(e^i - x, x)\, x_i$
$= \frac{\dot{x}_i}{x_j} - \frac{x_i}{x_j}\frac{\dot{x}_j}{x_j} = [u(e^i, x) - u(e^j, x)] \frac{x_i}{x_j}$

replicator dynamics of (symmetric) 2×2 games

payoff matrix

$$\begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} \\ \tilde{a}_{21} & \tilde{a}_{22} \end{pmatrix} \to \begin{pmatrix} a'_{11} & 0 \\ a'_{21} & a'_{22} \end{pmatrix} \to ($$

if other player plays strategy 1

my choice depends only on $a_{11} - a_{21}$

if other play 2, my choice

depends only on $a_{12} - a_{22}$

symmetric

replicator dynamics of 2×2 games

payoff matrix

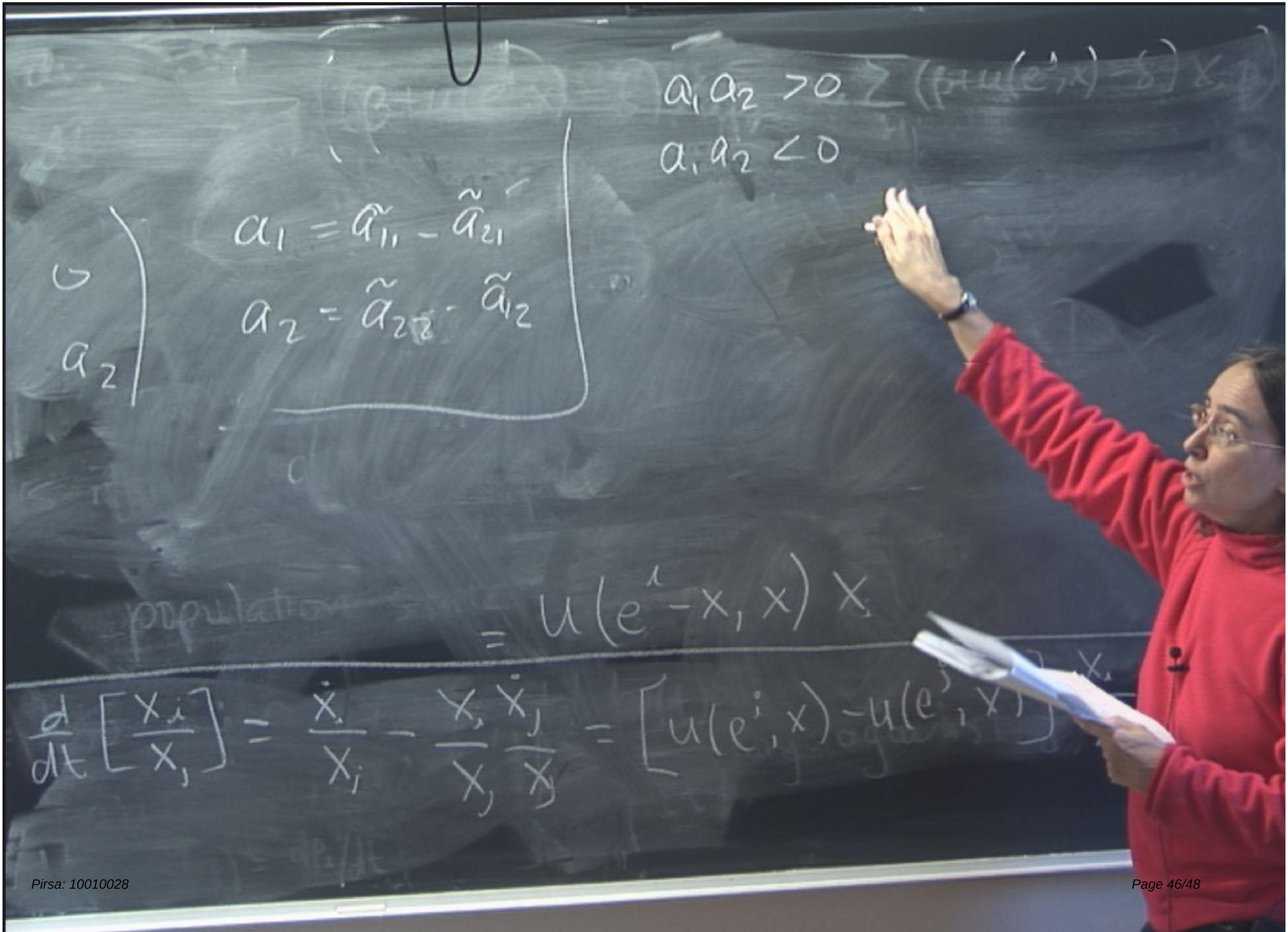
$$\begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} \\ \tilde{a}_{21} & \tilde{a}_{22} \end{pmatrix} \to \begin{pmatrix} a_{11}' & 0 \\ a_{21}' & a_{22}' \end{pmatrix} \to ($$

if other player plays strategy 1

my choice depends only on $a_{11} - a_{21}$

if other play 2, my choice

depends only on $a_{12} - a_{22}$


a1 a2 > 0
a1 a2 < 0
a1 = ã11 − ã21
a2 = ã22 − ã12
a2
= u(e^i − x, x) x
d/dt [xi/xj] = ẋi/xi − xi ẋj/(xj xj) = [u(e^i, x) − u(e^j, x)]

$a_1 a_2 > 0$

$a_1 a_2 < 0$

$$a_1 = \tilde{a}_{11} - \tilde{a}_{21}$$

$$a_2 = \tilde{a}_{22} - \tilde{a}_{12}$$

a_2

$$\dot{x}_1 = [a_1 x_1 - a_2 x_2]\, x_1 x_2$$

$$= u(e^i - x, x)\, x_i$$

$$\frac{d}{dt}\left[\frac{x_i}{x_j}\right] = \frac{\dot{x}_i}{x_j} - \frac{x_i}{x_j}\frac{\dot{x}_j}{x_j} = \left[u(e^i, x) - u(e^j, x)\right]\frac{x_i}{x_j}$$

$$a_1 a_2 > 0$$

$$a_1 a_2 < 0$$

$$a_1 = \tilde{a}_{11} - \tilde{a}_{21}$$

$$a_2 = \tilde{a}_{22} - \tilde{a}_{12}$$

$$\dot{x}_1 = [a_1 x_1 - a_2 x_2]\, x_1 x_2$$

$$= u(e^i - x, x)\, x_i$$

$$\frac{d}{dt}\left[\frac{x_i}{x_j}\right] = \frac{\dot{x}_i}{x_j} - \frac{x_i}{x_j}\frac{\dot{x}_j}{x_j} = \left[u(e^i, x) - u(e^j, x)\right]\frac{x_i}{x_j}$$