Title: Non-critical string field theory and stochastic quantization

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Abstract: 15 years ago Ishibashi, Kawai and collaborators developed non-critical string field theory, starting with the formalism of dynamical triangulations. The same construction can be repeated using causal dynamical triangulations, and in this case one can actually sum explicitly over all genera. The theory can be viewed as stochastic quantization of space, proper (world sheet) time playing the role of stochastic time.

Time in quantum gravity is stochastic time

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PI, January 27, 2010

Time ?

What is time in GR ?

What is time in QG ?

Answer: (at least in 2d quantum gravity)

QG time is stochastic time

Toy models of QG might trigger new ways of thinking about time. 2d toy models are particular nice since they still maintain most of the conceptual problems with QG and can (sometimes) be solved, and in this way might highlight non-trivial aspects of time, in much the same way as the Ising model highlighted the fact that one could have non-trivial scaling exponents in a quantum field theory.

Recalling CDT (in 2d)

CDT gives time a special role: the starting assumption is that we have a global time foliation in our (quantum) universe.

We think of this time as proper time: ADM decomposition:

$$ds^2 = -dt^2 + h(x,t)(dx + N_1(x,t)dt)^2$$

Laps put to $N(x,t) = 1$ and if $h(x,t) = h(t)$ we call it temporal gauge, $N_1(x,t) = 0$ proper time gauge. CDT geometries are discretized in a way which allows the introduction of temporal gauge coordinates.

CDT virtues

- No background geometry put in by hand.
- Path integral a regularized sum over geometries. Piecewise linear geometries do not need coordinates.
- The Einstein-Hilbert action has a natural geometric realization on piecewise linear geometries (Regge).
- The cut-off a is geometric (diffeomorphism invariant)
- Each configuration allows a rotation to Euclidean geometry, corresponding to $t \to t_4 = it$.
- The regularization automatically simultaneously acts as a regularization of the unboundedness of Euclidean QG.
- The use of the path integral representation has the advantage that we can use the concept of a classical geometry even if we are trying to explore the meaning of "quantum" geometry.

$$G(\mathbf{g}_{\rm i}, \mathbf{g}_{\rm f}, t) := \int_{\text{geometries: } \mathbf{g}_{\rm i} \to \mathbf{g}_{\rm f}} e^{iS[\mathbf{g}_{\mu\nu}(t')]}$$

$$= \lim_{a \to 0} \sum_{T: T_i^{(d-1)} \to T_f^{(d-1)}} \frac{1}{C_T} e^{iS_T}$$

In the case $d = 2$ one can solve the discretized model explicitly and take the limit $a \to 0$ to obtain the continuum "propagator".

Transition amplitudes in CDT QG

Transition amplitude as a weighted sum over all possible trajectories, in ordinary QM and in QG.

$$G(\mathbf{x}_\mathrm{i}, \mathbf{x}_\mathrm{f}, t) := \int_{\text{trajectories: } \mathbf{x}_\mathrm{i} \longrightarrow \mathbf{x}_\mathrm{f}} e^{iS[\mathbf{x}(t)]}$$

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The QG amplitude between the two geometric states

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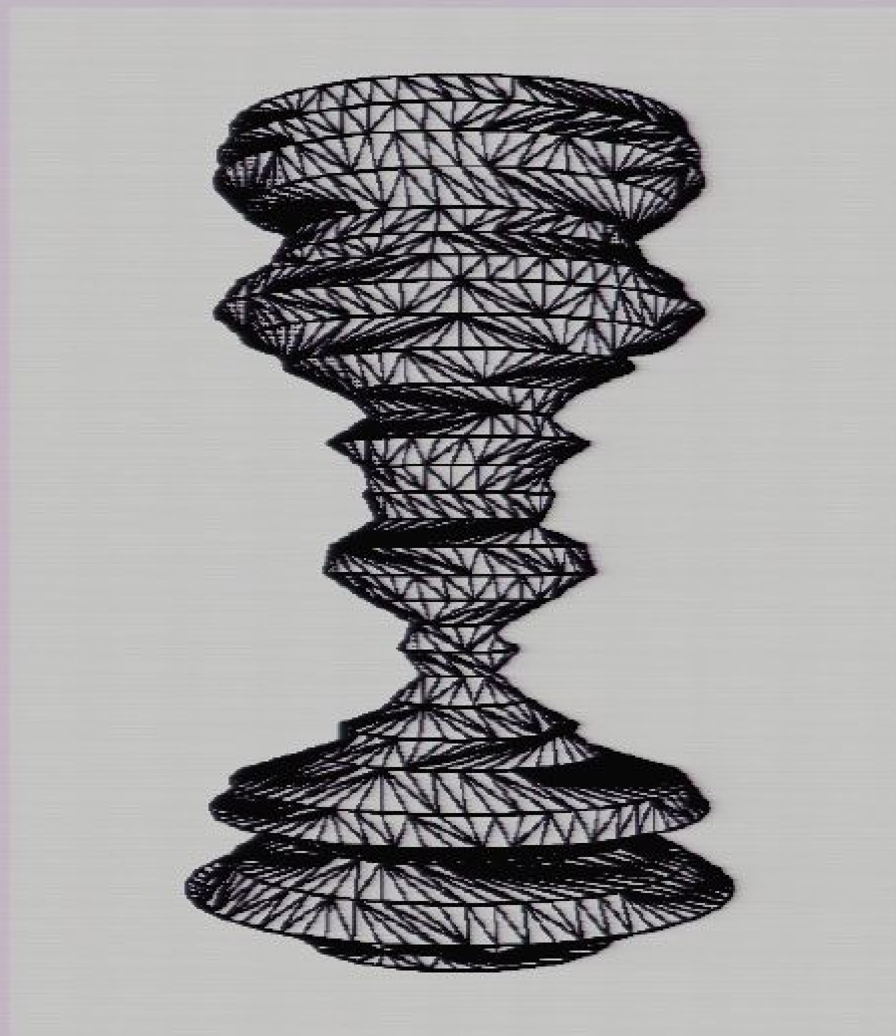
To define this path integral we need a geometric cut-off *a* and a definition of the class of geometries entering.

Discretized CDT geometry



A diffeomorphism invariant cut off: lattice spacing a.

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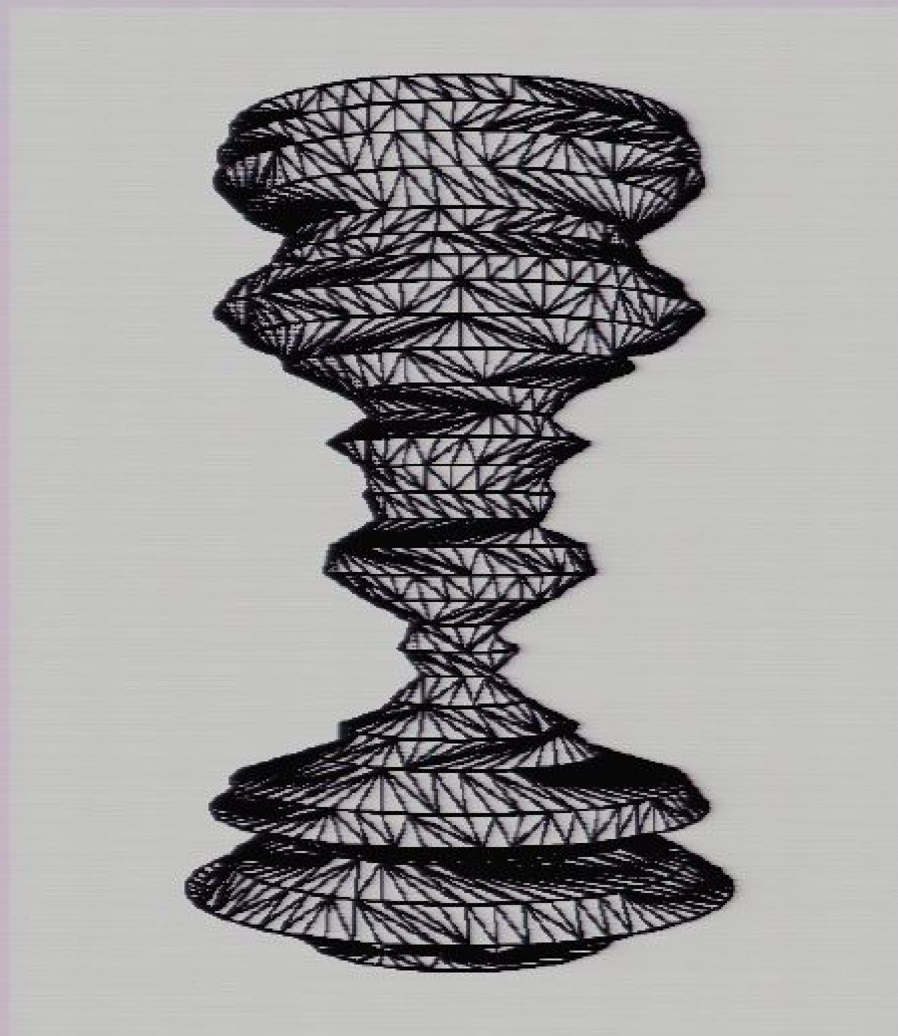

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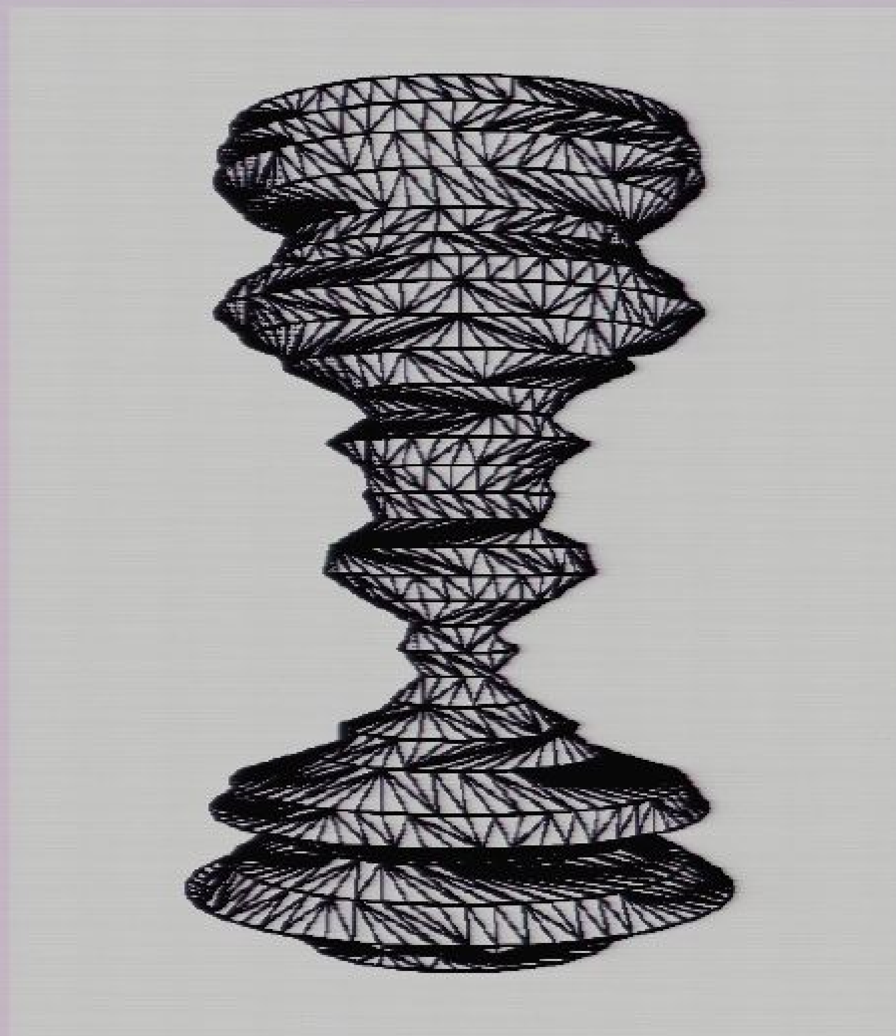

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Wick rotation

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- Thus quantum amplitude $\to$ partition function
- "Classical" trajectory is an average over quantum trajectories in the statistical ensemble of trajectories.
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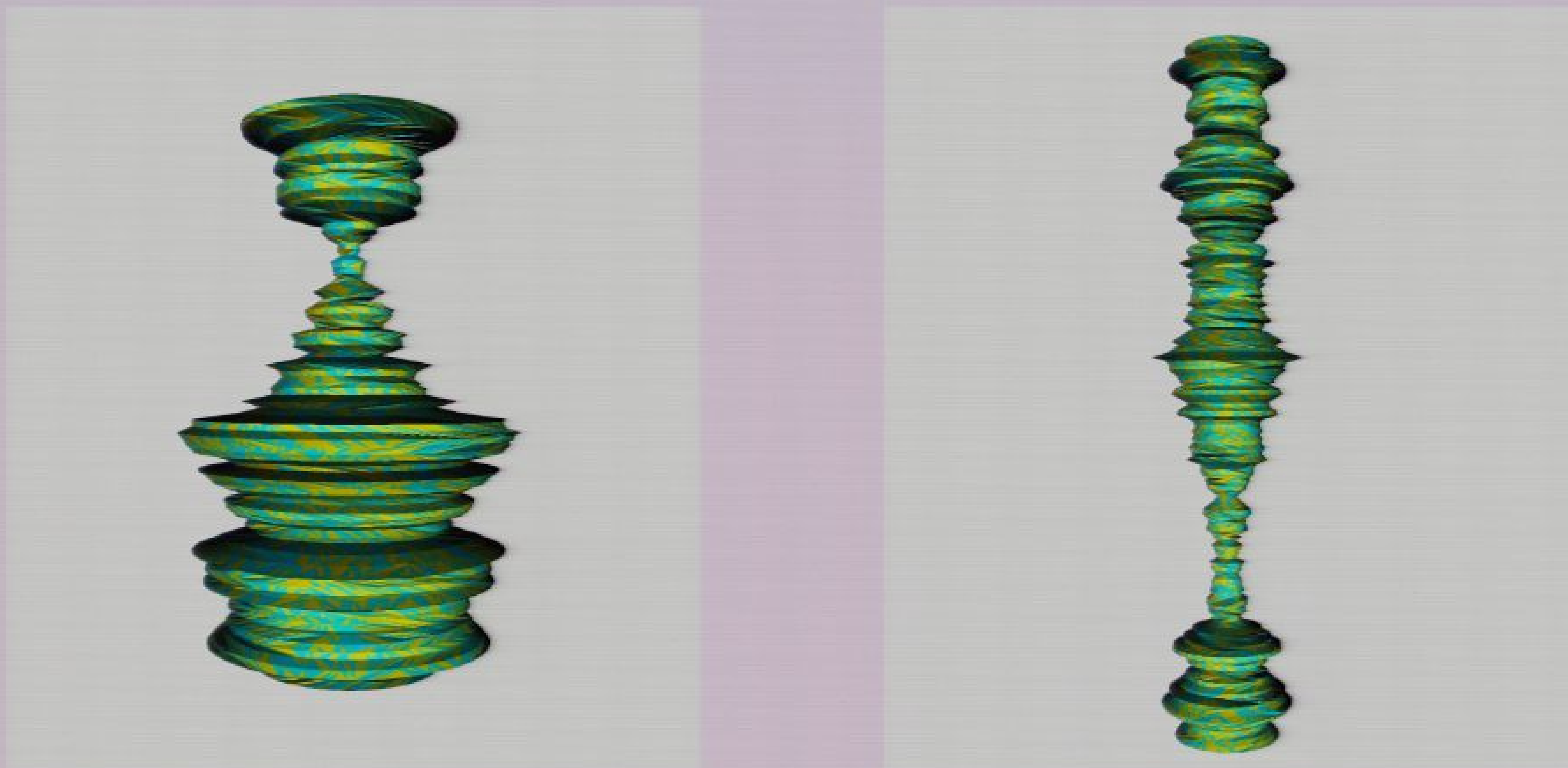

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The 2d solution

$$\frac{\partial G_0(\ell,\ell';t)}{\partial t} = \left(\ell \frac{d^2}{d\ell^2} - \lambda\ell\right) G_0(\ell,\ell';t)$$

$$G_0(\ell,\ell';t) = \langle \ell' | e^{-tH_0(\ell)} | \ell \rangle$$

$$H_0(\ell) = -\ell \frac{d^2}{d\ell^2} + \lambda\ell$$

$H_0(\ell)$ selfadjoint with measure $d\ell/\ell$. Eigenvectors, eigenvalues:

$$H_0(\ell)\psi_n(\ell) = E_n\psi_n(\ell), \quad E_n = 2n\sqrt{\lambda}, \quad n > 0.$$

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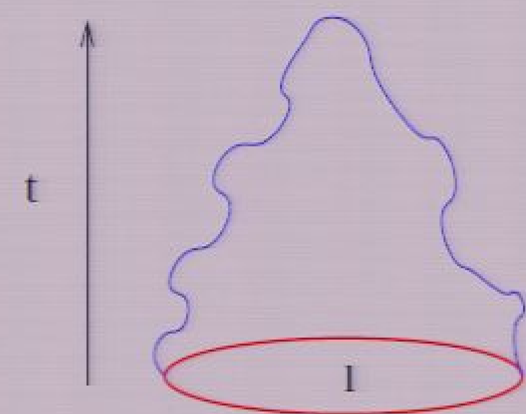
The CDT Hartle-Hawking wave function

The CDT Hartle-Hawking wave function is defined by:

$$w_0(\ell) = \int_0^\infty dt \; G_0(\ell, \ell' = 0; t) = e^{-\sqrt{\lambda}\ell}$$



$$H_0(\ell) w_0(\ell) = 0, \qquad \text{Wheeler-deWitt}$$

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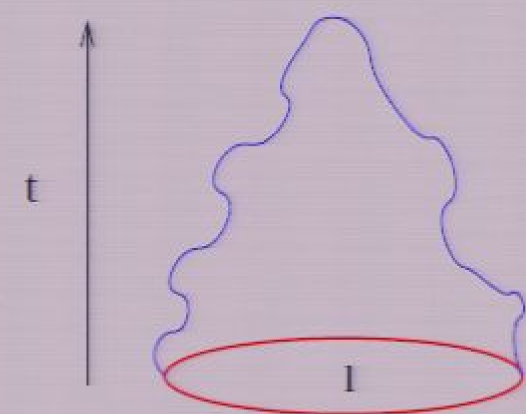

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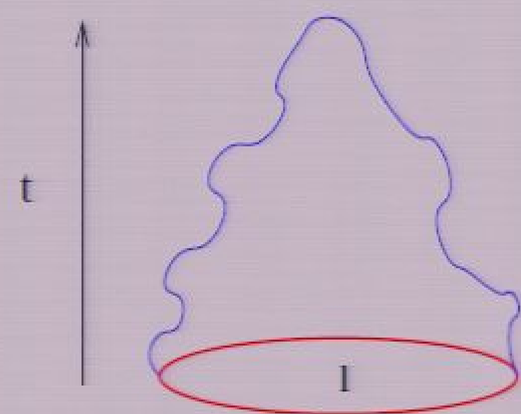

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Laplace transformation

The model was first solved by combinatorial methods, using the Laplace transformed representation:

$$\tilde{G}_0(x,y;t) = \int_0^\infty d\ell \int_0^\infty d\ell' \, e^{-x\ell - y\ell'} \, G_0(\ell,\ell';t)$$

The equations above now reads:

$$\frac{\partial \tilde{G}_0(x,y;t)}{\partial t} = -\frac{\partial}{\partial x}\left(\frac{\partial S(x)}{\partial x}\,\tilde{G}_0(x,y;t)\right),$$

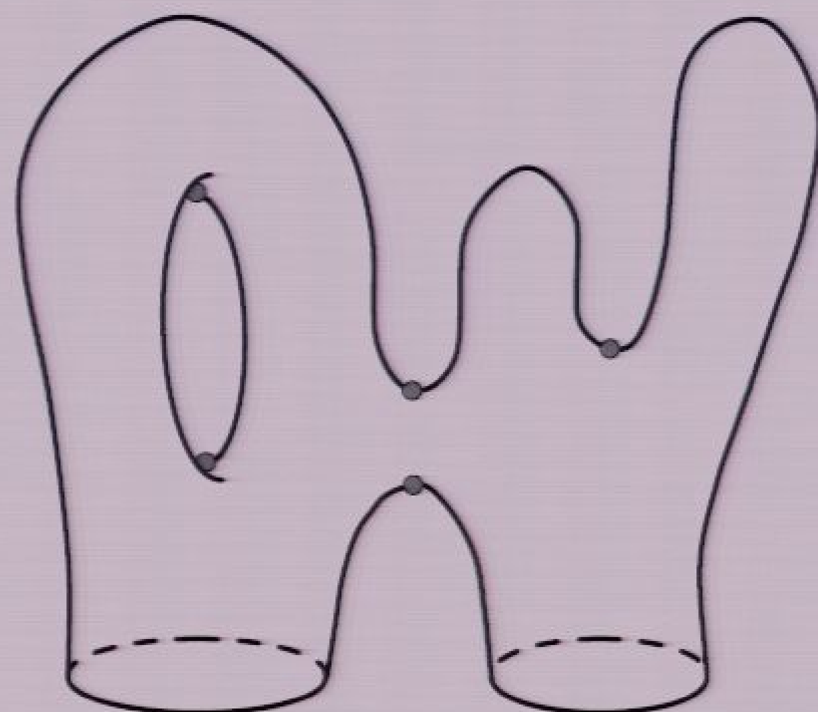
$$S(x) = \frac{x^3}{3} - \lambda x, \quad \tilde{H}_0(x) = \frac{\partial}{\partial x}\frac{\partial S(x)}{\partial x}$$

When x is real it has the physical interpretation as a boundary cosmological constant.

Beyond the ADM decomposition

Can one solve the model, if we allow space to split (and merge) as a function of proper time ?

The concept is diffeomorphism invariant in Lorentzian signature spacetimes, and we can thus introduce a new coupling constant g for splitting and merging.



String Field Theory

One can follow Ishibashi and Kawai and develop a SFT which automatically takes these changes into account.

$$[\Psi(\ell), \Psi^\dagger(\ell')] = \ell^{-1}\delta(\ell - \ell'), \quad \Psi(\ell)|0\rangle = \langle 0|\Psi^\dagger(\ell) = 0$$

$$\Psi(\ell) = \sum_n a_n \psi_n(\ell), \quad [a_n, a_m^\dagger] = \delta_{nm}$$

Thus second quantization of spatial universes leads to

$$\hat{H}_0 = \int_0^\infty \frac{d\ell}{\ell}\, \Psi^\dagger(\ell) H_0(\ell) \Psi(\ell),$$

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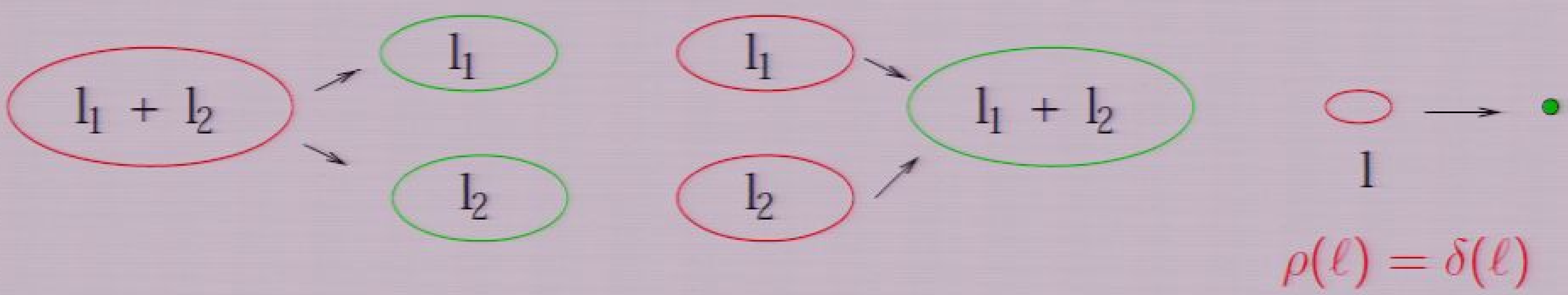
SFT

$\hat{H}_0$ is a single (spatial) universe Hamiltonian, and the full many-universe Hamiltonian is:

$$\begin{aligned}\hat{H} &= \hat{H}_0 - g\int d\ell_1 \int d\ell_2 \Psi^\dagger(\ell_1)\Psi^\dagger(\ell_2)\Psi(\ell_1+\ell_2) \\ &\quad -\alpha g \int d\ell_1 \int d\ell_2 \Psi^\dagger(\ell_1+\ell_2)\Psi(\ell_2)\Psi(\ell_1) - \int \frac{d\ell}{\ell}\,\rho(\ell)\Psi(\ell),\end{aligned}$$



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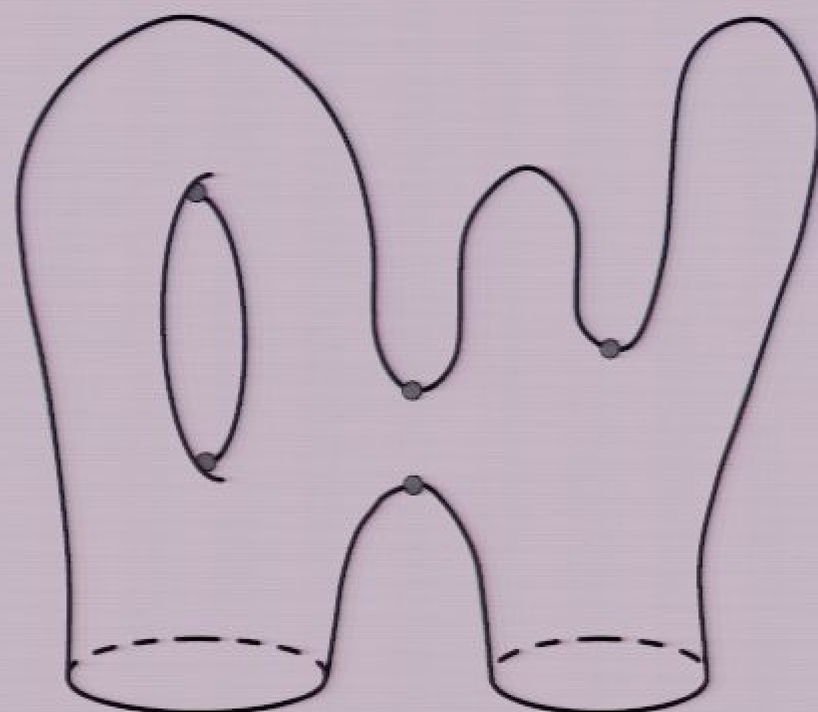

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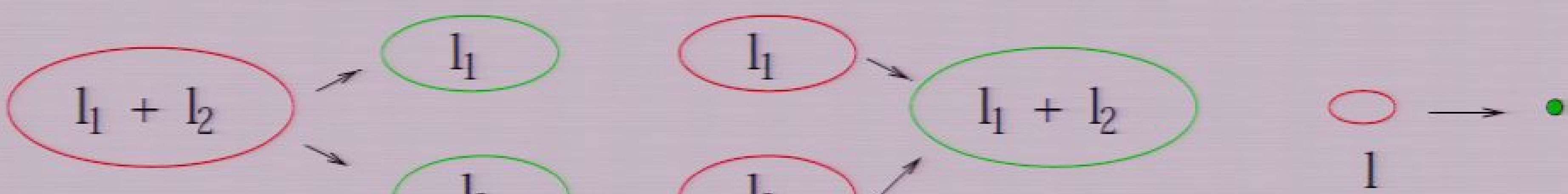

$$\rho(\ell) = \delta(\ell)$$

SFT (2)

The additional coupling constant α is introduced in order to distinguish between the action of the two terms proportional to g in $\hat{H}$ when expanding in powers of g. Thus the power of α counts the genus of the surface.

Note that the signs of all the interaction terms are negative: the terms represent the insertion of new geometric structures compared to the "free" propagation generated by $\hat{H}_0$. They should appear with positive weight when we expand $e^{-t\hat{H}}$.

Note also that the SFT is defined as a perturbation series. For fixed genus the series is convergent, but the complete series is only an asymptotic series. We will try to go beyond this asymptotic expension.

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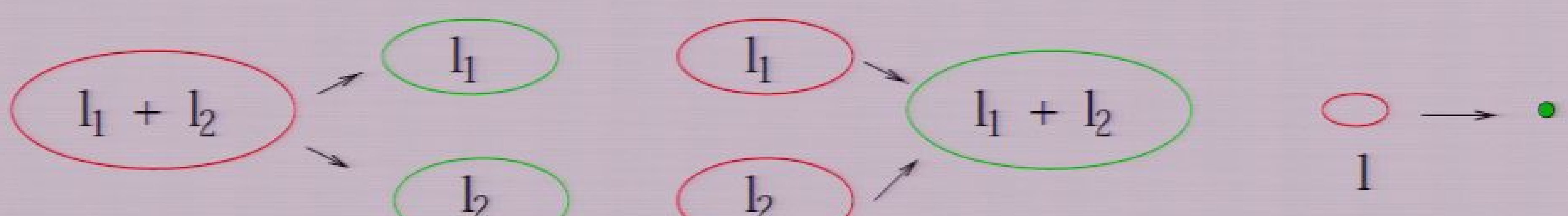

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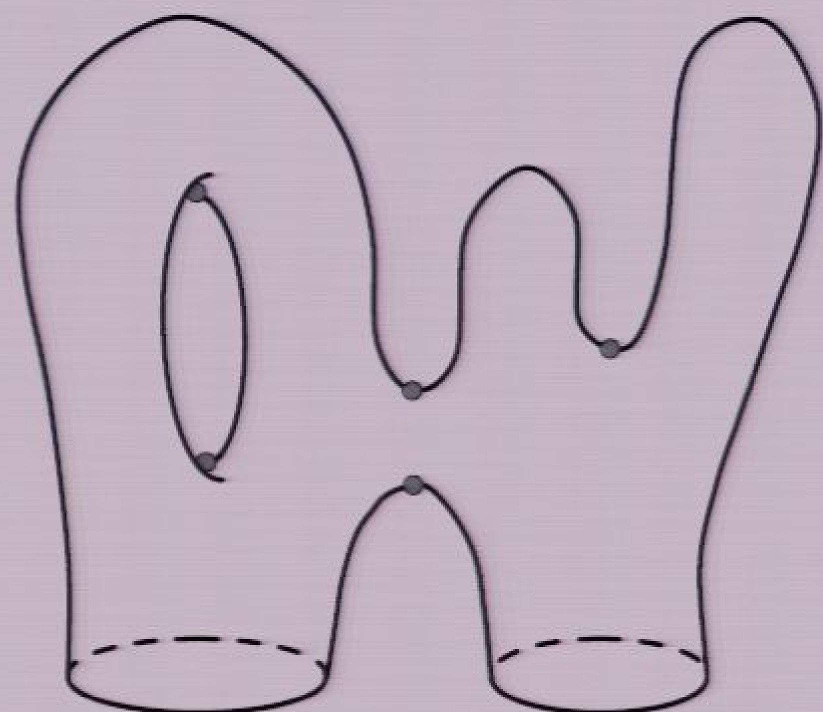

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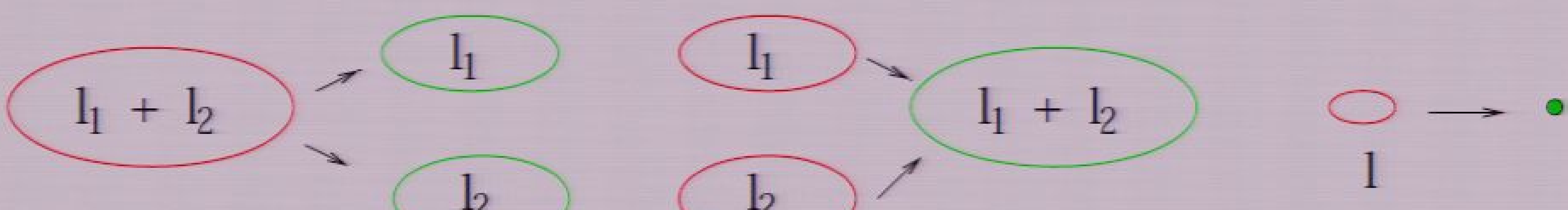

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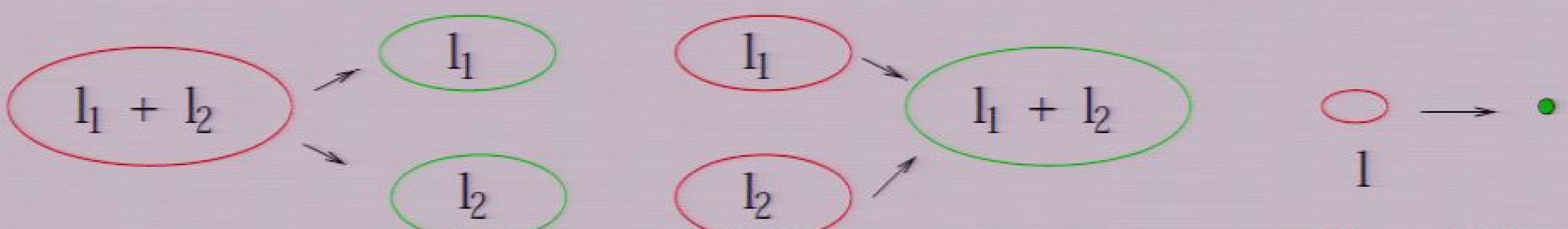

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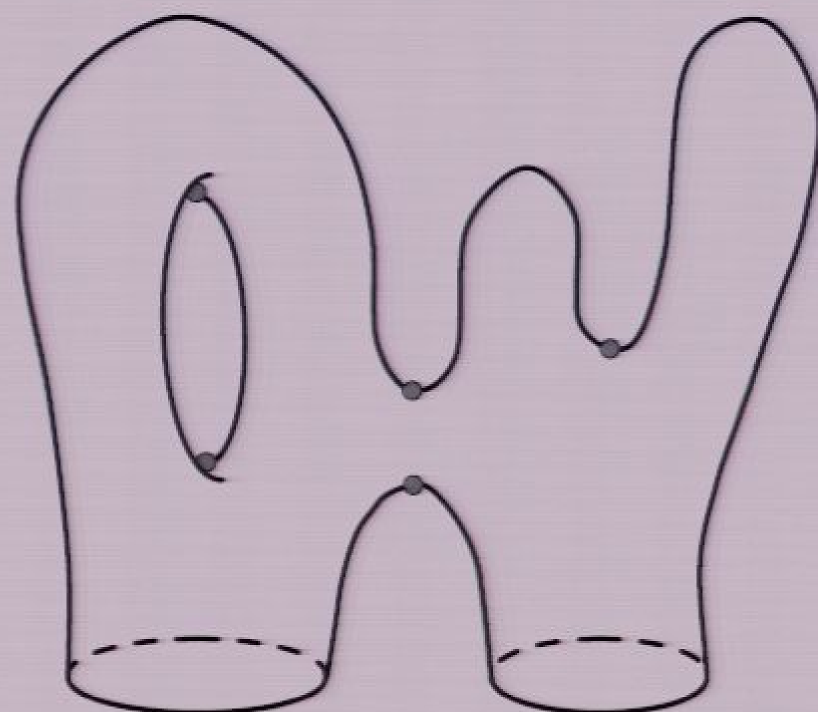

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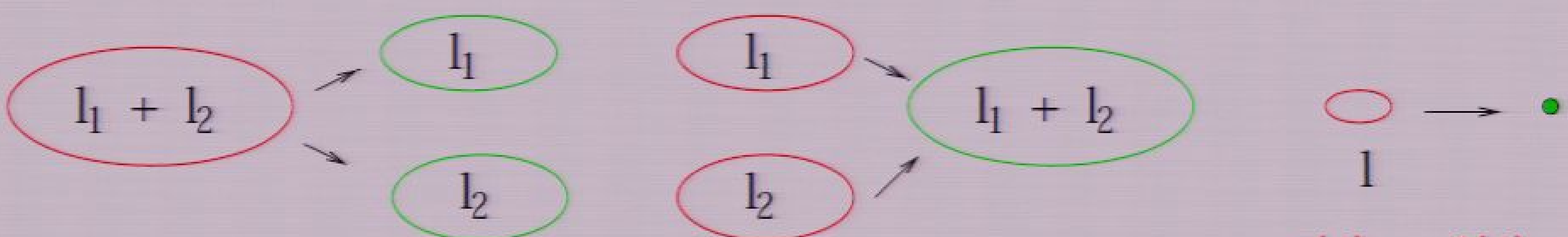

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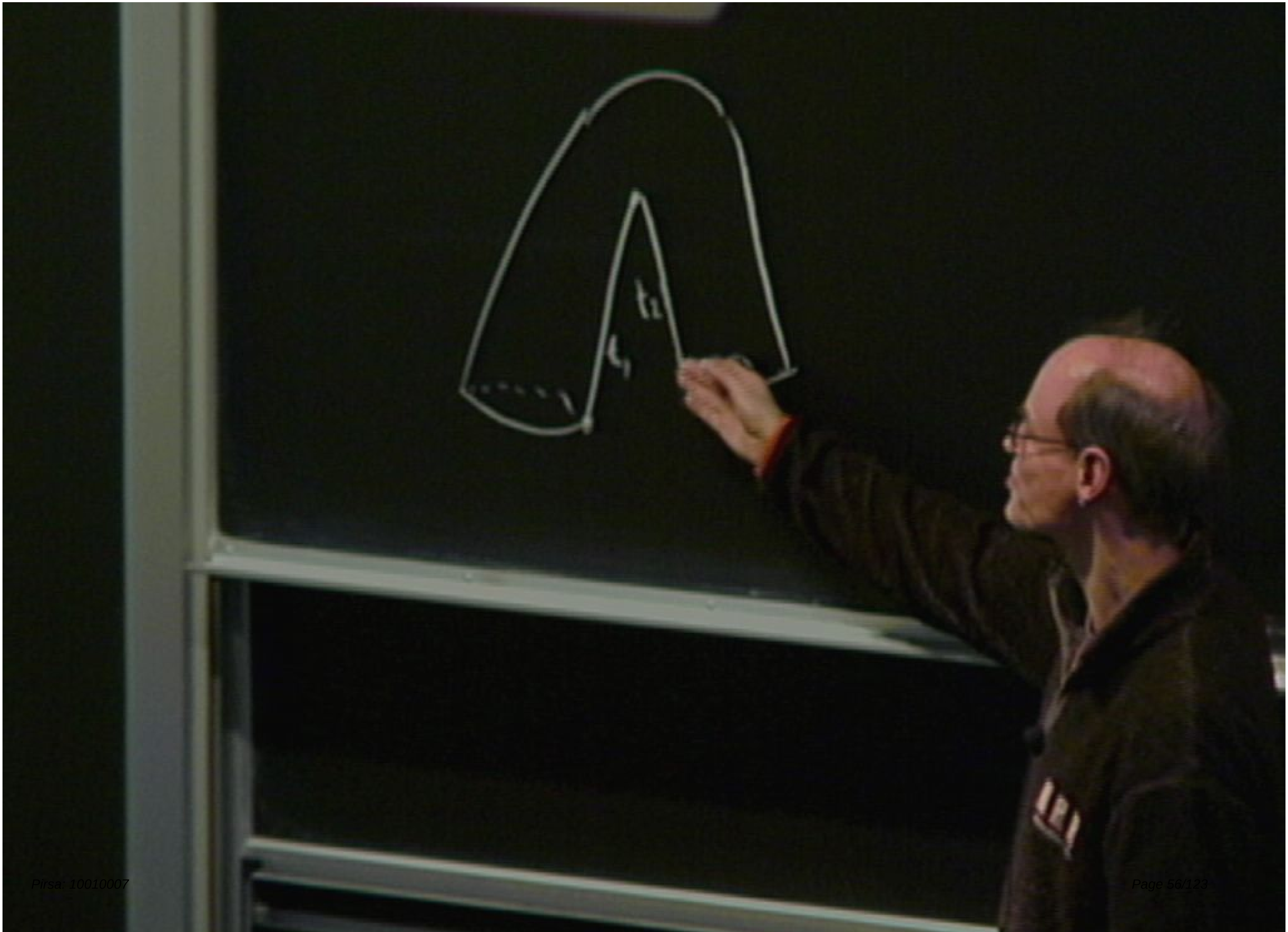
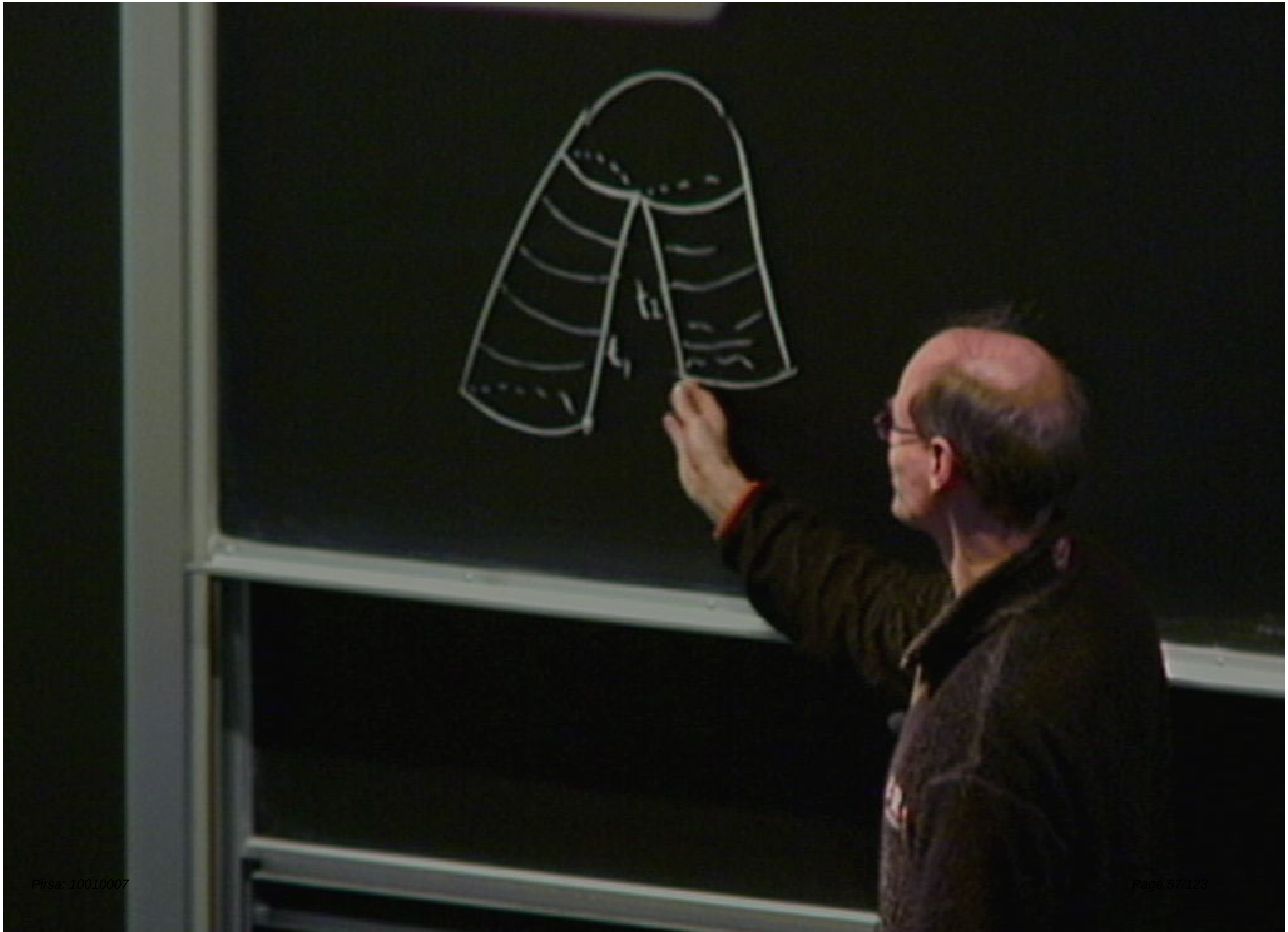
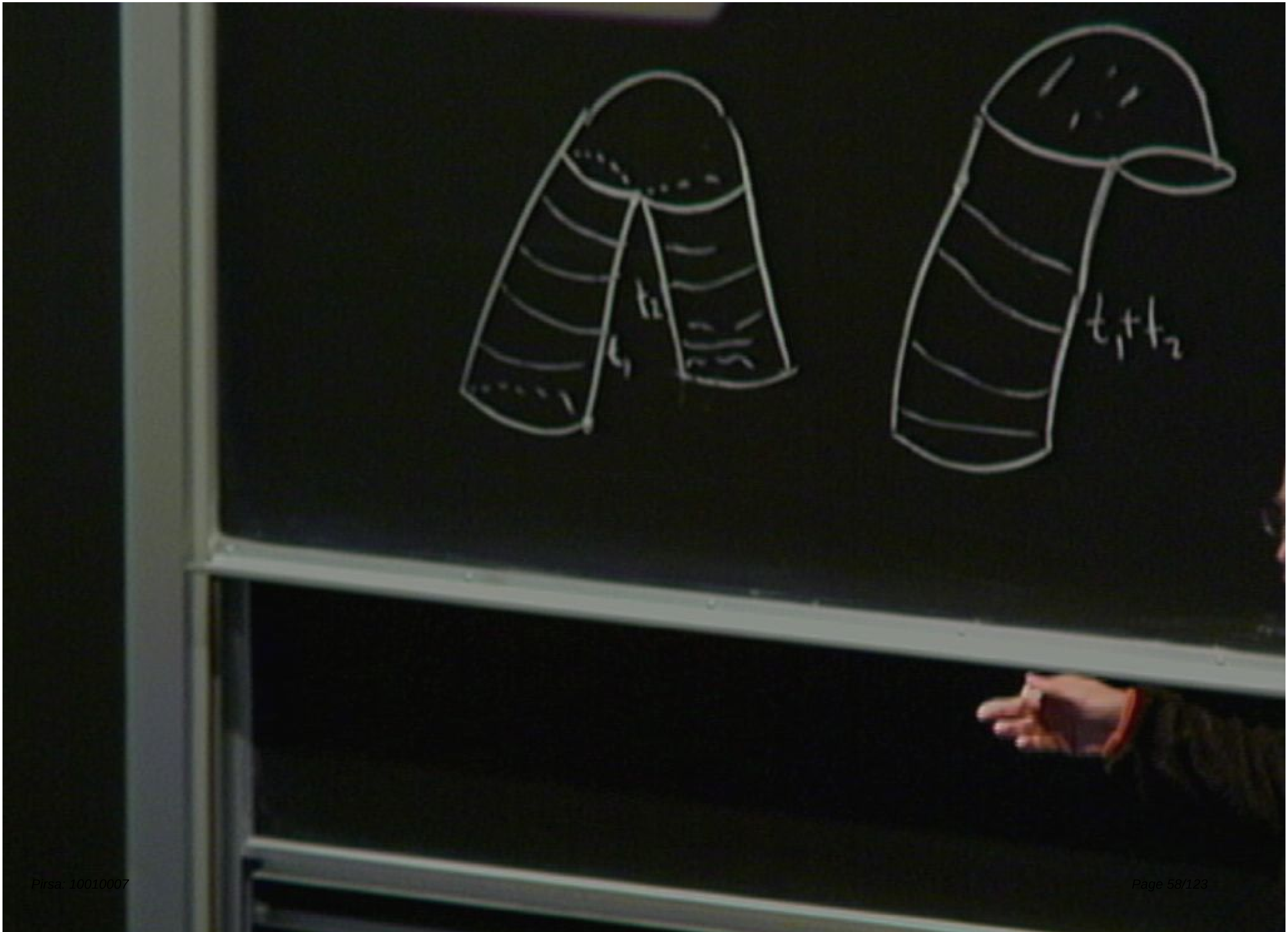
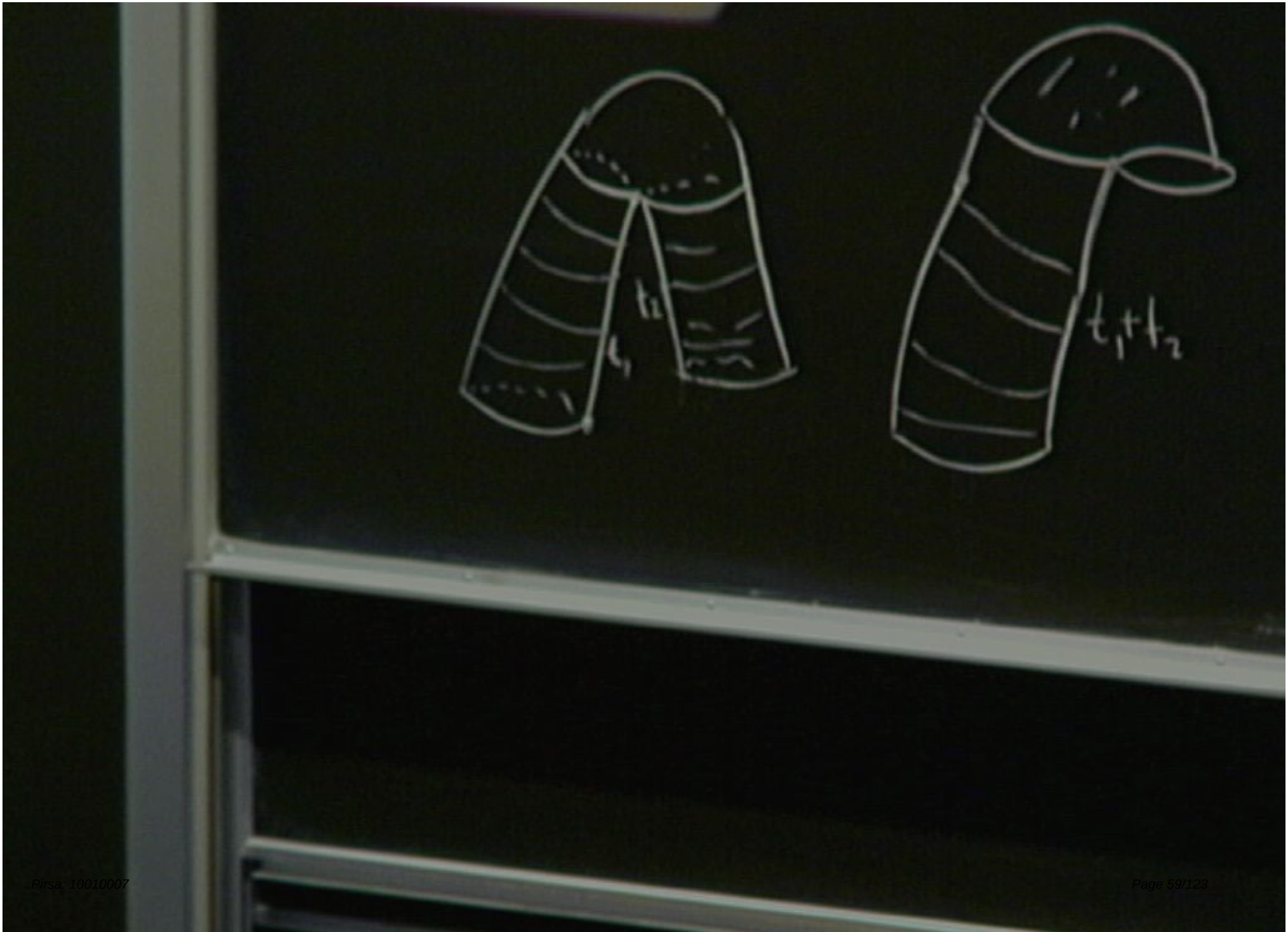
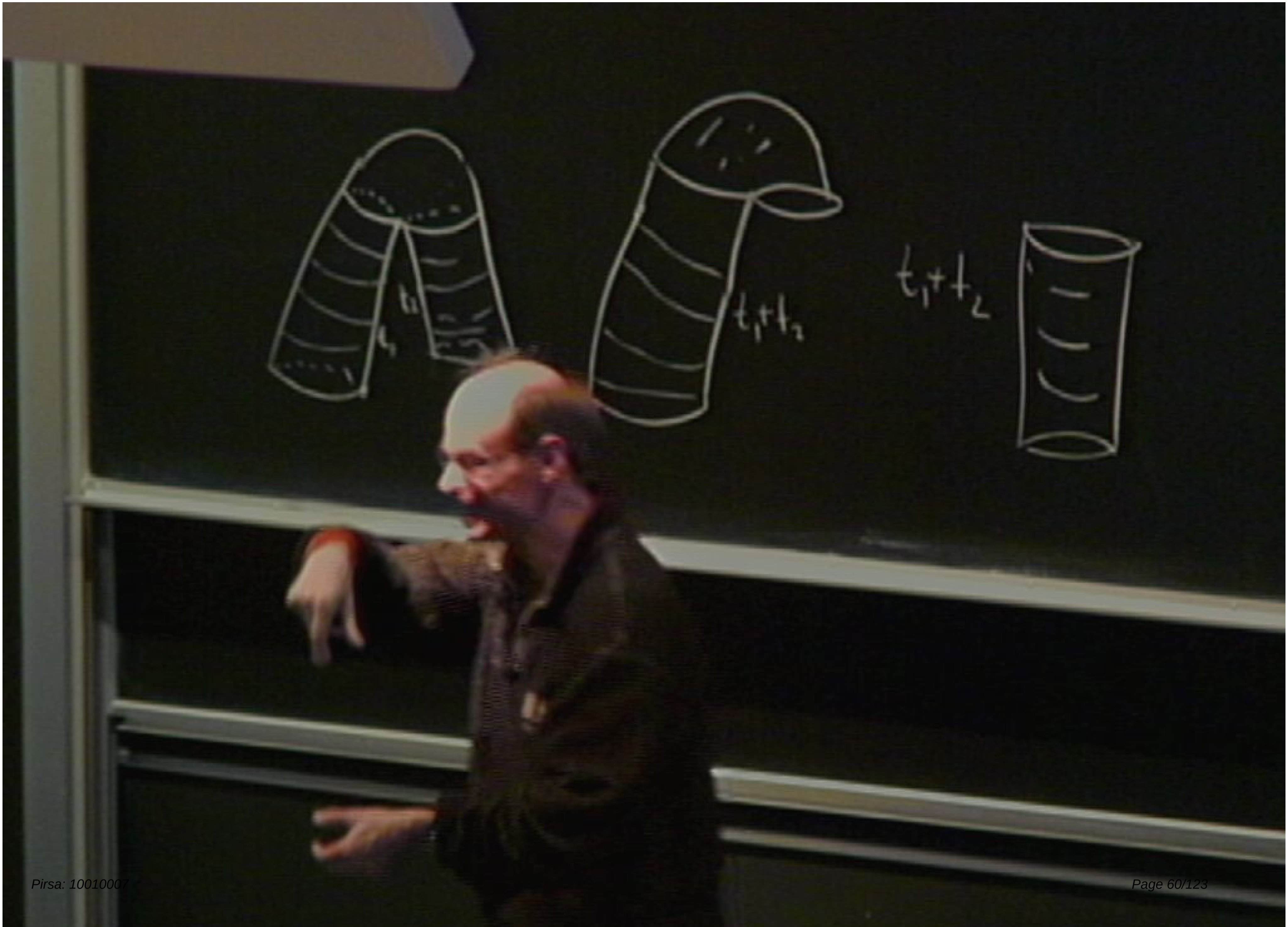

t2
t1


t₂
t₁


t₂
t₁
t₁+t₂


t_2
t_1
t_1+t_2


t_1
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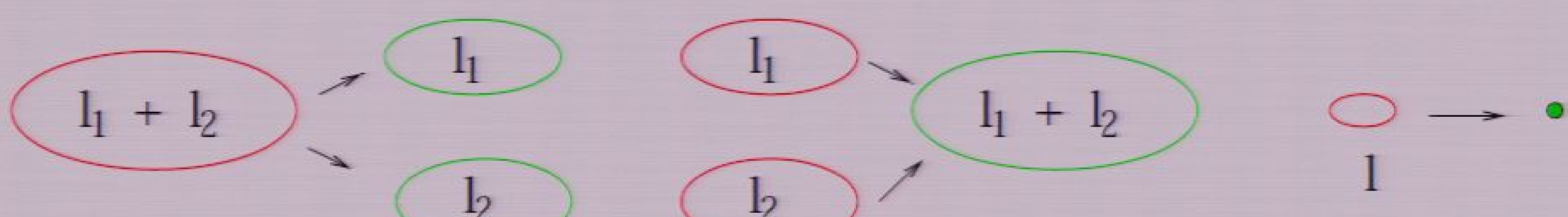

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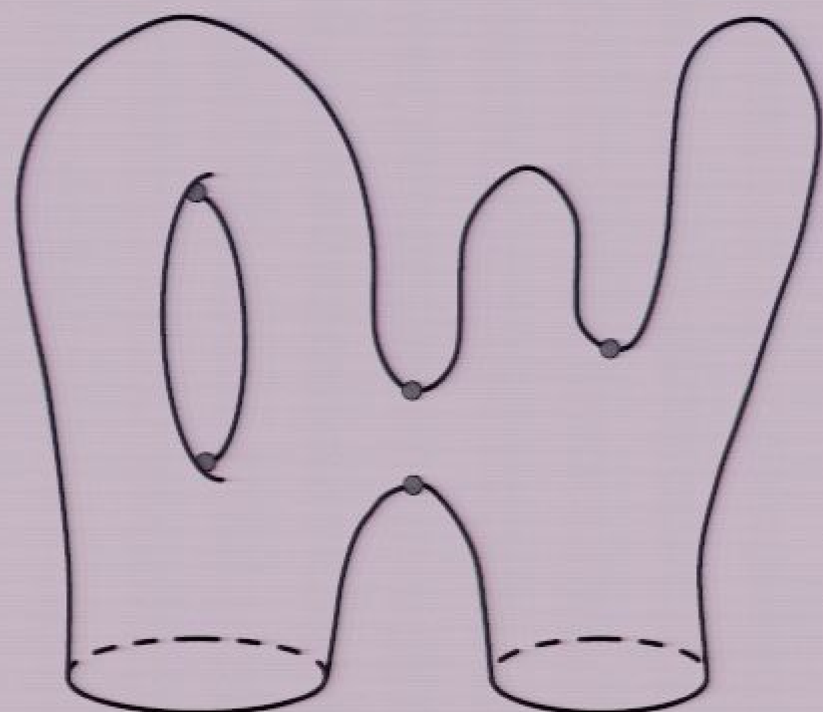

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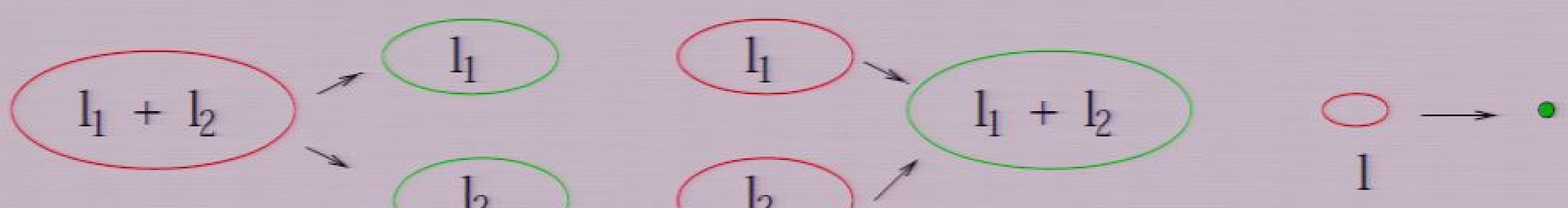

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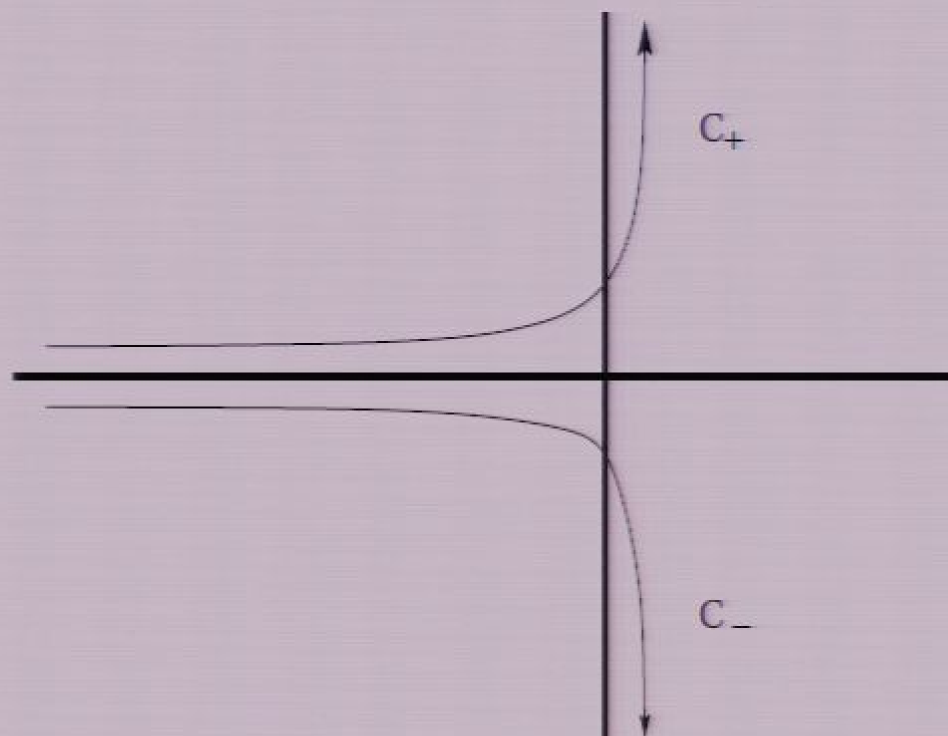

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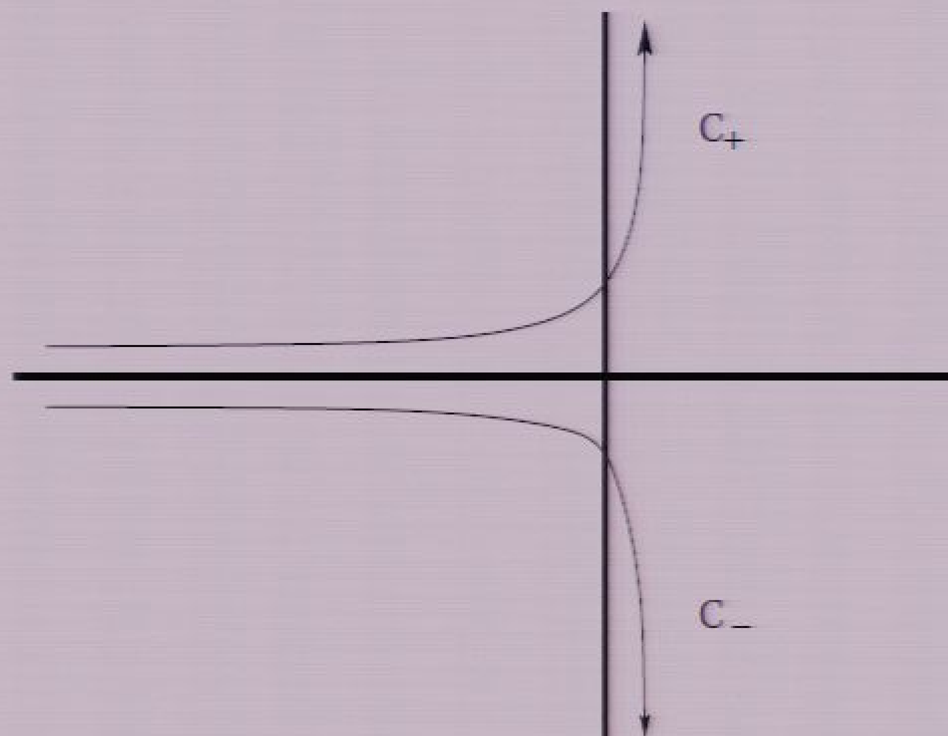

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$$\tilde{w}(x) = \int_0^\infty \mathrm{d}\ell \, e^{-x\ell} \left\{ \int_C \mathrm{d}m \, \frac{\exp\left[-\frac{1}{g}\left(\lambda(m - \kappa\sqrt{\lambda}\ell) - \frac{1}{3} m^3\right)\right]}{Z(g,\lambda)} \right\}$$

$$w(\ell) = \frac{\mathrm{Bi}(\kappa^{-2/3} - \kappa^{1/3}\sqrt{\lambda}\ell)}{\mathrm{Bi}(\kappa^{-2/3})}$$

$$\frac{\partial}{\partial x}\left(g \frac{\partial}{\partial x} + \lambda - x^2\right) \tilde{w}(x) = -1.$$

Stochastic quantization

Langevin equation (Gaussian noise $\nu(t)$ of unit strength):

$$\dot{x}^{(\nu)}(t) = -\frac{\partial S(x)}{\partial x}\Big|_{x^{(\nu)}(t)} + \sqrt{\Omega}\,\nu(t)$$

Corresponding probability distribution

$$P(x,y;t) = \left\langle \delta(y - x^{(\nu)}(t;x)) \right\rangle_{\nu},$$

satisfies a Fokker-Planck equation:

$$\frac{\partial P'_x(x,y;t)}{\partial t} = \frac{\partial}{\partial x}\left(\frac{1}{2}\Omega\frac{\partial}{\partial x} - \frac{\partial S(x)}{\partial x}\right) P'_x(x,y;t).$$

SFT as stochastic quantization (1)

If the noise term is absent we have a "classical" system:

$$\frac{\partial P'_x(x,y;t)}{\partial t} = -\frac{\partial}{\partial x}\left(\frac{\partial S(x)}{\partial x}\, P'_x(x,y;t)\right)$$

The formal ADM CDT analogy, with $S(x) = \frac{x^3}{3} - \lambda x$,

$$\frac{\partial \tilde{G}_0(x,y;t)}{\partial t} = -\frac{\partial}{\partial x}\left(\frac{\partial S(x)}{\partial x}\, G_0(x,y;t)\right)$$

$$P'_x(x,y;t) = \frac{\partial}{\partial x}\delta(y - x_{cl}(t,x)), \quad \tilde{G}_0(x,y;t) = \frac{\partial}{\partial x}\frac{1}{x_{cl}(t,x)+y},$$

where $x_{cl}(t)$ is the solution to $\dot{x}_{cl} = -\frac{\partial S(x_{cl})}{\partial x_{cl}}$ such that $x_{cl}(t=0) = x$ and where $1/(x+y)$ is the Laplace transformed of $\delta(\ell - \ell')$.

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Stochastic time = proper time

This observation was first made by Ishibashi and Kawai in Euclidean 2d QG. But it is seemingly also valid in CDT 2d QG.

How can intrinsic time (PT) become extrinsic (ST) ?

Extrinsic nature of space and time maybe inherent in QG ?
Example (Euclidean QG): define a diffeomorphism invariant correlator:

$$\langle \phi(R)\phi(0)\rangle = \int \mathcal{D}[g_{\mu\nu}(\xi)] \int \mathcal{D}\phi(\xi)\, \mathrm{e}^{-S(g_{\mu\nu},\phi)}$$
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Clearly the “proper time” R becomes an external parameter referring to no specific geometry even if it at the same time is the geodesic distance in *all* geometries entering in the functional integral.

Is this a general message send to us from special models ?

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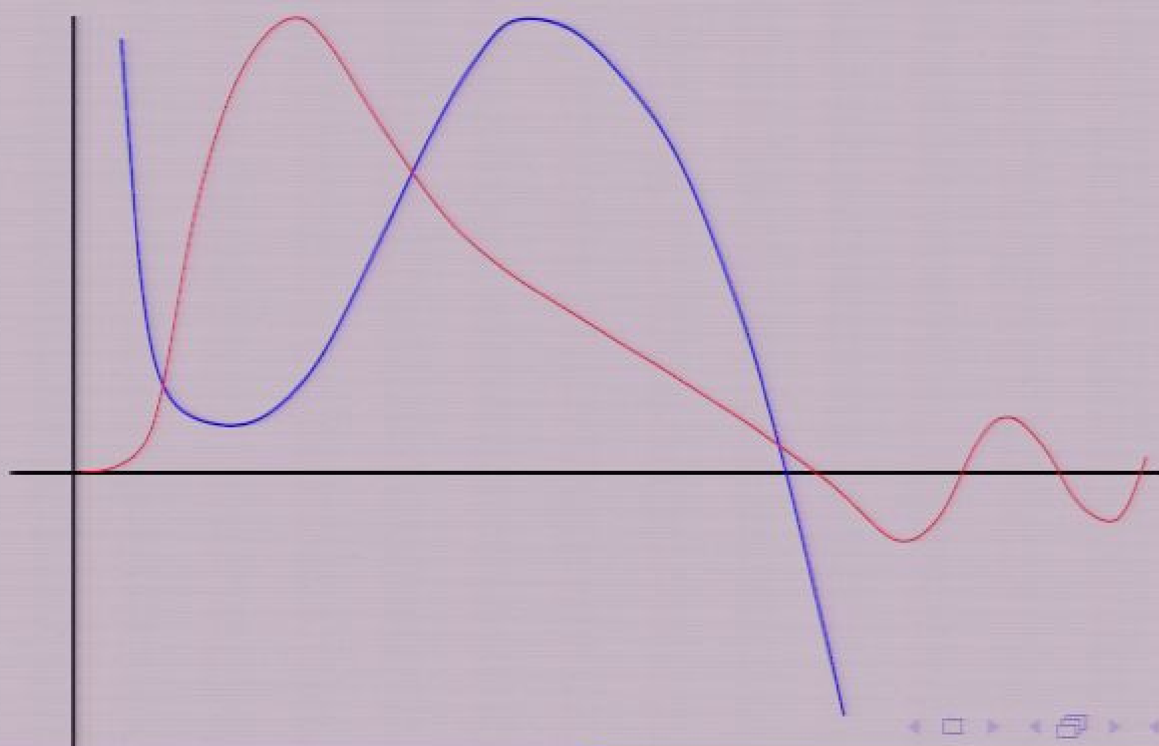

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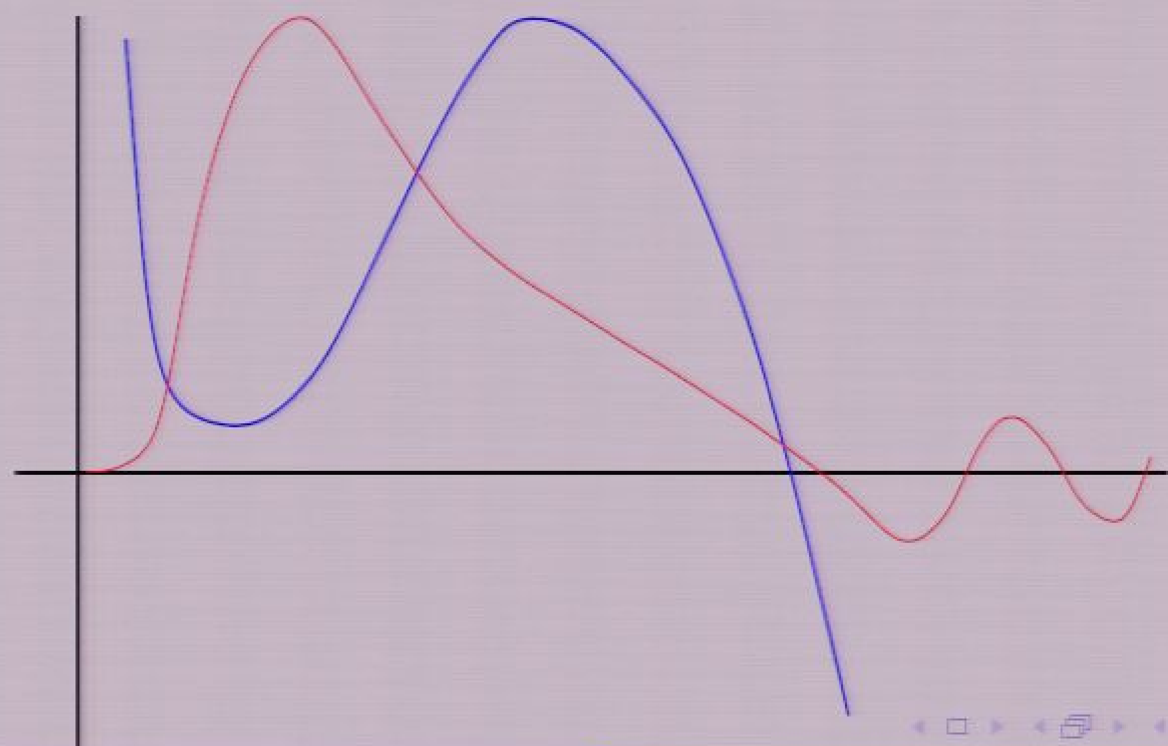

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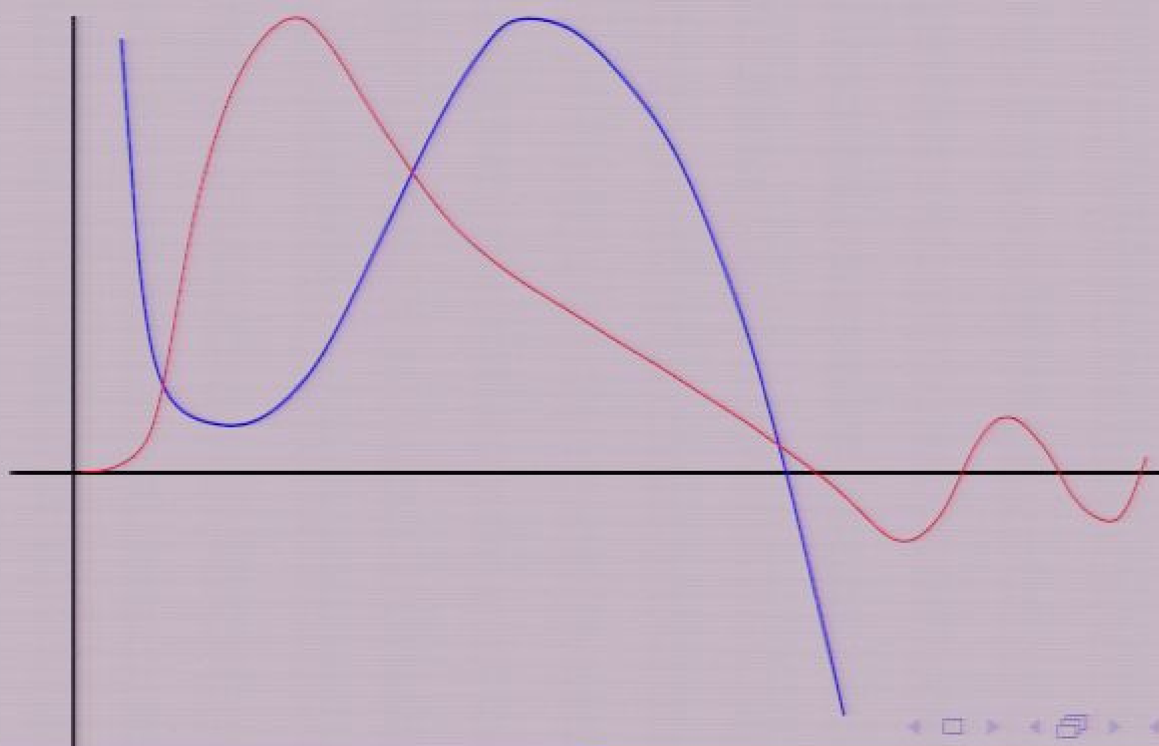

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The matrix integral becomes an ordinary integral !

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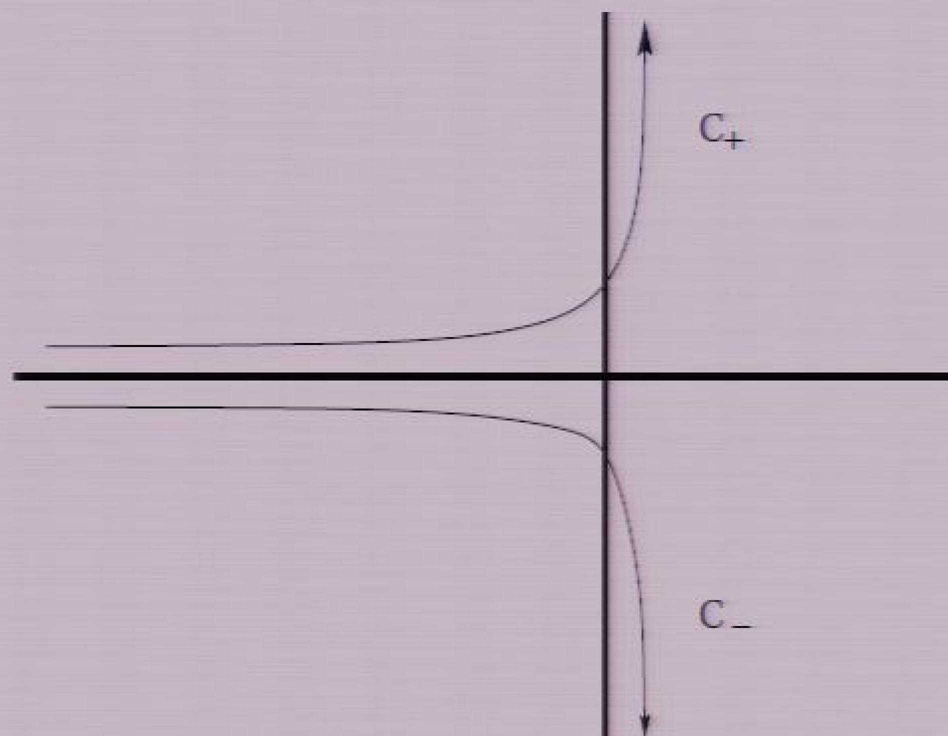

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SFT as stochastic quantization (1)

If the noise term is absent we have a "classical" system:

$$\frac{\partial P'_x(x,y;t)}{\partial t} = -\frac{\partial}{\partial x}\left(\frac{\partial S(x)}{\partial x}\, P'_x(x,y;t)\right)$$

The formal ADM CDT analogy, with $S(x) = \frac{x^3}{3} - \lambda x$,

$$\frac{\partial \tilde{G}_0(x,y;t)}{\partial t} = -\frac{\partial}{\partial x}\left(\frac{\partial S(x)}{\partial x}\, G_0(x,y;t)\right)$$

$$P'_x(x,y;t) = \frac{\partial}{\partial x}\delta(y - x_{cl}(t,x)), \quad \tilde{G}_0(x,y;t) = \frac{\partial}{\partial x}\frac{1}{x_{cl}(t,x)+y},$$

where $x_{cl}(t)$ is the solution to $\dot{x}_{cl} = -\frac{\partial S(x_{cl})}{\partial x_{cl}}$ such that $x_{cl}(t=0) = x$ and where $1/(x+y)$ is the Laplace transformed of $\delta(\ell - \ell')$.

SFT as stochastic quantization (2)

Stochastic quantization of space: include average over noise.

$$\frac{\partial \tilde{G}(x,y;t)}{\partial t} = \frac{\partial}{\partial x}\left(g\frac{\partial}{\partial x} + \lambda - x^2\right)\tilde{G}(x,y;t)$$

Choice $\Omega = 2g$ ensures perturbative agreement as well as same non-perturbative equation for $\tilde{w}(x)$ as in CDT SFT.

By inverse Laplace transformation:

$$\frac{\partial G(\ell,\ell';t)}{\partial t} = -H(\ell)\,G(\ell,\ell';t),$$

$$H(\ell) = H_0(\ell) - g\ell^2 = -\ell\frac{\partial^2}{\partial \ell^2} + \lambda\ell - g\ell^2.$$

$H(\ell)w(\ell) = 0$ Wheeler-deWitt equation

SFT as stochastic quantization (3)

$$\ell = \frac{1}{2}z^2, \quad \psi(\ell) = \sqrt{z}\phi(z), \qquad H(z)\phi(z) = E\phi(z),$$

$$H(z) = -\frac{1}{2}\frac{d^2}{dz^2} + \frac{1}{2}\lambda z^2 + \frac{3}{8z^2} - \frac{g}{4}z^4.$$

SFT as stochastic quantization (3)

We have a “double-well” potential since the time it takes a classical particle to reach infinity is finite (and equal the time a classical particle use between turning points when in the potential well, i.e. “symmetric” for *g* small.

Energy Spectrum discrete as before.But drastic change in wave-functions:

$$\langle \ell \rangle \sim \frac{1}{\sqrt{\lambda}} \quad \to \quad \langle \ell \rangle = \infty$$

Stochastic time = proper time

This observation was first made by Ishibashi and Kawai in Euclidean 2d QG. But it is seemingly also valid in CDT 2d QG.

How can intrinsic time (PT) become extrinsic (ST) ?

Extrinsic nature of space and time maybe inherent in QG ?
Example (Euclidean QG): define a diffeomorphism invariant correlator:

$$\langle \phi(R)\phi(0)\rangle = \int \mathcal{D}[g_{\mu\nu}(\xi)] \int \mathcal{D}\phi(\xi)\, e^{-S(g_{\mu\nu},\phi)} \int d^2\xi_1 \sqrt{g(\xi_1)} \int d^2\xi_2 \sqrt{g(\xi_2)}\;\; \phi(\xi_1)\phi(\xi_2)\; \delta(D_g(\xi_1,\xi_2) - R)$$

One way to understand better the properties of the Hamiltonian is to return to x-representation

$$\tilde{H}(x) = \tilde{H}_0(x) - g\frac{d^2}{dx^2}, \quad \tilde{H}_0(x) = \frac{\partial}{\partial x}\frac{\partial S(x)}{\partial x}.$$

A similarity transformation brings $\tilde{H}(x)$ to a standard form:

$$H(x) = e^{-S(x)/2g}\,\tilde{H}(x)\,e^{S(x)/2g},$$

$$H(x) = -g\frac{d^2}{dx^2} + V(x), \quad V(x) = \frac{1}{4g}(\lambda - x^2)^2 + x.$$

The spectrum of $H(x)$ is discrete, unambiguous and positive:

$$H(x) = g\hat{R}^\dagger\hat{R}, \quad \hat{R} = -\frac{d}{dx} + \frac{1}{2g}\frac{dS(x)}{dx},$$

Thus stochastic quantization has provided us with a non-perturbative definition of the CDT string field theory.