Title: On the Relation between Operator Constraint, Master Constraint, Reduced Phase Space, and Path Integral Quantisation, with Application to Quantum Gravity

Date: Dec 16, 2009 02:00 PM

URL: http://pirsa.org/09120114

Abstract: Path integral formulations for gauge theories must start from the canonical formulation in order to obtain the correct measure. A possible avenue to derive it is to start from the reduced phase space formulation. We review this rather involved procedure in full generality. Moreover, we demonstrate that the reduced phase space path integral formulation formally agrees with the Dirac's operator constraint quantisation and, more specifically, with the Master constraint quantisation for first class constraints. For first class constraints with non trivial structure functions the equivalence can only be established by passing to Abelian(ised) constraints which is always possible locally in phase space. With the above general considerations, we derive concretely the path integral formulations for GR from the canonical theory. We also show that there in principle exists a spin-foam model consistent with the canonical theory of GR.

On the Relation between Operator Constraint --, Master Constraint --, Reduced Phase Space --, and Path-integral Quantization

With Applications to Quantum Gravity

Muxin Han
(In collaboration with Thomas Thiemann)
Albert-Einstein-Institut, Potsdam-Golm

Outline:

- Motivation

- Reduced phase space quantization and path integral
  --- Application to Gravity (a continuum gauge fixed path-integral)

- Operator constraint quantization and path integral
  --- Application to Gravity (a path-integral for spin-foam construction)

- Master constraint quantization and path integral
  --- Application to Gravity (a discretized path-integral)

General idea:

⟶ Formally quantize a general reparametrization invariant system

⟶ Consider gravity as a special case

MH and T. Thiemann. arXiv:0911.3428, arXiv:0911.3431
MH. arXiv:0911.3432, arXiv:0911.3436
J. Engle, MH and T. Thiemann. arXiv:0911.3433

Motivation:

A path integral formulation consistent with canonical LQG

- The canonical interpretation of the path integral amplitude:

  ----- Physical inner product of physical states from

  - Reduced phase space quantization
  - Dirac quantization (operator constraint / Master constraint)

- Fixing the path integral measure ambiguity by the canonical framework

The path integral measure for a constrained system is usually not a formal Lebesgue measure $\mathcal{D}\phi$, but with a certain local measure factor.

$$\mathcal{D}\mu = \mathcal{D}\phi\ M(\phi) \qquad \left(\mathcal{D}\mu = \mathcal{D}g_{\mu\nu} \prod_{x\in M} \mathcal{V}^p V_s^q \quad \text{for gravity} \right)$$

- H. Leutwyler, Phys. Rev. 134 (1964) B1156
- E. S. Fradkin and G. A. Vilkovisky, Phys. Rev. D8 (1973) 4241
- E. S. Fradkin and G. A. Vilkovisky, CERN-TH-2332

For GR, the non-covariant local measure is un-avoidable in order to be consistent with canonical theory.

⟶ How to understand the local measure factor in Background Independent Quantization?

Reduced phase space quantization and path integral

The physical inner product for the physical states in reduced phase space quantization

↓

A path-integral expression for the physical inner product

- A general covariant (regular) system with the canonical coordinates

$$(p_a, P_I; q^a, T^I)$$

- T's are the clock variables. Choose the gauge fixings (τ^I phase space numbers)

$$\chi^I = T^I - \tau^I$$

- Abelianize the first class constraints

$$\tilde{C}_I = P_I + h_I(p_a, q^a, T^I)$$

guaranteed by the Abelization Theorem (M. Henneaux and C. Teitelboim 1992)

- Define the Dirac observables relationally

$$O_f(\tau) := [\alpha_\beta(f)]_{\alpha_\beta(T^I)=\tau^I}$$

a weak Poisson homomorphism from [the Poisson algebra of the functions on the gauge fixed constraint surface with Dirac bracket] to [the Possion algebra of Dirac observables]

- In particular $\{P_a(\tau), Q^a(\tau)\} = \delta_a^b$
- The physical Hamiltonians:

$$\{h_I\left(P_a(\tau), Q^a(\tau), \tau\right), O_f(\tau)\} = \partial_{\tau^I} O_f(\tau)$$

the algebra of the physical Hamiltonians is weakly commutative.

Remark:

For GR, A preferred algebra of observables and a preferred dynamics are defined from the gauge fixing conditions. In contrast to the gauge invariant approach of Yang-Mills theories.

- Quantum algebra $\mathfrak{A}$ of observables

$$\left[\hat{P}_a(\tau), \hat{Q}^b(\tau)\right] = -i\delta_a^b$$

- Choose a representation Hilbert space, e.g.

$$\mathcal{H}_{red} = L^2(\mathrm{d}Q)$$

- We assume the quantum dynamics exists and is anomaly-free, i.e.
  (1.) the physical Hamiltonians $h_I\left(\hat{P}_a, \hat{Q}^a, \tau\right)$ are self-adjoint operators.
  (2.) the physical Hamiltonians form a commutative algebra $[h_I, h_J] = 0$.
- Multi-finger unitary evolution operator:

$$U(\tau, \tau') := \prod_I U_I(\tau_I, \tau_I')$$

$$U_I(\tau_I, \tau_I') := 1 + \sum_{n=1}^{\infty}(-i)^n \int_{\tau_I'}^{\tau_I} \mathrm{d}\tau_{I,n} \int_{\tau_I'}^{\tau_{I,n}} \mathrm{d}\tau_{I,n-1} \cdots \int_{\tau_I'}^{\tau_{I,2}} \mathrm{d}\tau_{I,1}\, \hat{h}_I(\tau_{I,1}) \cdots \hat{h}_I(\tau_{I,n})$$

- Given two Heisenberg states $\Psi, \Psi' \in \mathcal{H}_{red}$

$$\begin{aligned} &\langle \Psi | \Psi' \rangle \\ = \; &\int \mathrm{d}Q^a(\tau_N)\mathrm{d}Q^a(\tau_{N-1}) \cdots \mathrm{d}Q^a(\tau_1)\mathrm{d}Q^a(\tau_0) \\ &\times \langle \Psi | Q^a(\tau_N) \rangle \langle Q^a(\tau_N) | Q^a(\tau_{N-1}) \rangle \cdots \langle Q^a(\tau_1) | Q^a(\tau_0) \rangle \langle Q^a(\tau_0) | \Psi' \rangle \end{aligned}$$

by the standard skeletonization procedure.

Theorem:

The reduced phase space physical inner product has a path-integral expression by a standard skeletonization procedure:

$$\langle\Psi|\Psi'\rangle$$

$$= \int \mathcal{D}p_a \mathcal{D}q^a \mathcal{D}P_I \mathcal{D}T^I \prod_{t,I}\left[\delta\left(P_I + h_I\right)\delta\left(T^I - \tau^I\right)\right] e^{i\int_{t_i}^{t_f} dt\left[\sum_a p_a(t)\dot{q}^a(t) + \sum_I P_I(t)\dot{T}^I(t)\right]}\, \overline{\Psi\left(Q^a(\tau_f)\right)}\, \Psi'\left(Q^a(\tau_i)\right)$$

Liouville measure — Abelianized constraint, gauge fixing conditions — Boundary physical states

$$= \int \mathcal{D}x^A(t) \prod_{t\in[t_i,t_f]} \sqrt{\det \omega[x^A(t)]} \prod_{t\in[t_i,t_f]} \left[\sqrt{|D_1[x^A(t)]|}\, \delta\left(C_I[x^A(t)]\right) \delta\left(\xi^I[x^A(t)]\right)\right] \prod_{t\in[t_i,t_f]} \left[\sqrt{|D_2[x^A(t)]|}\, \delta\left(\Phi_i[x^A(t)]\right)\right]$$

$$\times \exp\left(iS[x^A(t)]\right) \overline{\Psi\left(Q^a[x^A](t_f)\right)}\, \Psi'\left(Q^a[x^A](t_i)\right)$$

Dirac matrix, second-class constraint

Original constraint, general Faddeev-Popov gauge fixing

Boundary physical states

which is equivalent to the Liouville path-integral on the reduced phase space:

$$Z := \int \prod_{t\in[t_1,t_2]} \left[dy^j(t) \sqrt{\left|\det \omega_R[y^j(t)]\right|}\,\right] \exp(iS[y^j(t)])$$

This reduced phase space path-integral is suggested to be a starting point for the path-integral quantization of a general constrained system.

M. Henneaux and C. Teitelboim, Quantization of Gauge System (Princeton University Press, 1992)

Application to the case of gravity:

$$\mathcal{Z}_\pm = \int_{II\pm} \mathcal{D}\omega_\mu^{IJ} \mathcal{D}e_\mu^I \mathcal{V}^3 V_s \sqrt{|D_1|} \prod_\alpha \delta(\xi_\alpha) \cos\int e^I \wedge e^J \wedge \left(*F_{IJ} - \frac{1}{\gamma}F_{IJ}\right)[\omega]$$

$$\mathcal{Z}_\pm = \int_{II\pm} \mathcal{D}\omega_\alpha^{IJ} \mathcal{D}X_{\alpha\beta}^{IJ} \prod_{x\in M} \mathcal{V}^9 V_s \delta^{20}\left(\epsilon_{IJKL} X_{\alpha\beta}^{IJ} X_{\gamma\delta}^{KL} - \frac{1}{4!}\mathcal{V}\epsilon_{\alpha\beta\gamma\delta}\right) \sqrt{|D_1|} \prod_\alpha \delta(\xi_\alpha)$$

choice of sectors

$$\times \exp i \int (X - \frac{1}{\gamma} * X)_{IJ} \wedge F^{IJ}$$

The local measure factors appear in the path-integral measures:

- The path-integral measure is invariant under all the gauge transformations generated by the canonical constraints, i.e. invariant under dynamical Bergmann-Komar "group" BK(M). BK(M)=Diff(M) only when e.o.m. is imposed.
- But, non-invariant under non-spatial diffeomorphisms, i.e. non-spatial diffeomorphism symmetry is anomalous in the quantization!
- This fact is consistent with canonical theory. There is no notion of local symmetry on the phase space corresponding to the non-spatial diffeomorphism!

  Lee and Wald. J. Math. Phys. 31 (1990) 725

Questions:

(Conceptual) Can the reduced phase space path-integral gives the solutions to all the quantum constraints?

(Practical) How to remove the gauge fixings in the practical computation, when constructing a spin-foam model?

Operator constraint quantization, reduced phase space quantization and path integral

Reduced phase space quantization physical inner product

↔

The rigging inner product with Abelianized constraint

↓

A path-integral expression for the rigging map and rigging inner product
(answering both conceptual and practical questions)

Refined Algebraic Quantization (RAQ):

Solutions to the constraints are elements of the algebraic dual $\mathfrak{D}^\star_{Kin}$ of a dense domain $\mathfrak{D}_{Kin} \subset \mathcal{H}_{Kin}$. What we are looking at are states $\Psi \in \mathfrak{D}^\star$ such that:

$$\Psi\left[\hat{C}^\dagger_I f\right] := \hat{C}'_I \Psi[f] = 0, \qquad \forall f \in \mathfrak{D}$$

The space of solutions is denoted by $\mathfrak{D}^\star_{Phys}$. The physical Hilbert space will be a subspace of $\mathfrak{D}^\star_{Phys}$. Eventually, $\mathfrak{D}^\star_{Phys}$ will be the algebraic dual of a dense domain $\mathfrak{D}_{Phys} \in \mathcal{H}_{Phys}$, which is invariant under the algebra of operators corresponding to Dirac observables. Hence we obtain a Gel'fand triple:

$$\mathfrak{D}_{Phys} \hookrightarrow \mathcal{H}_{Phys} \hookrightarrow \mathfrak{D}^\star_{Phys}$$

A systematic construction of the physical Hilbert space is available if we have an anti-linear rigging map:

$$\eta : \mathfrak{D}_{Kin} \to \mathfrak{D}^\star_{Phys};\ f \mapsto \eta(f)$$

such that $\eta(f')[f]$ is a positive semi-definite sesquilinear form on $\mathfrak{D}_{Kin}$ If the quantum constraint algebra is generated by self-adjoint constraints $\hat{C}_I$ and their commutator algebra is a Lie algebra i.e. the structure functions are constant, then we can try to heuristically define the rigging map via the group averaging procedure:

$$\eta(f) := \int d\mu(t)\ < e^{it^I \hat{C}_I} f, . >$$

The physical inner product is defined by the rigging inner product

$$\langle \eta(f) | \eta(f') \rangle_{Phys} := \eta(f')[f],\ \forall f, f' \in \mathfrak{D}_{Kin}.$$

Then a null space $\mathfrak{N} \subset \mathfrak{D}^\star_{Phys}$ is defined by $\left\{\eta(f) \in \mathfrak{D}^\star_{Phys} \,\middle|\, \|\eta(f)\|_{Phys} = 0 \right\}$. Therefore

$$\mathfrak{D}_{Phys} := \eta\left(\mathfrak{D}_{Kin}\right)/\mathfrak{N} \qquad (0.1)$$

However, in general the constraint algebra is not a Lie algebra

$\{C_I, C_J\} = f_{IJ}{}^K C_K$ where $f_{IJ}{}^K$ is a structure function

The group averaging fails for C_I.

Fortunately we have the Abelianization Theorem (all the first-class constraint algebra can be locally Abelianized)

Equivalent constraints:

$$\tilde{C}_I = P_I + h_I(p_a, q^a, T^I) \qquad \left\{\tilde{C}_I, \tilde{C}_J\right\} = 0$$

If the constraint operators are self-adjoint and anomaly-free, we define the group averaging for the Abelianized constraints

$$\eta_\omega(f)[f'] = \frac{\int \prod_I \mathrm{d}t^I \left\langle f \left| \exp\left[i \sum_I t^I \tilde{C}_I\right] \right| f' \right\rangle}{\int \prod_I \mathrm{d}t^I \left\langle \omega \left| \exp\left[i \sum_I t^I \tilde{C}_I\right] \right| \omega \right\rangle}$$

where $f, f', \omega \in \mathfrak{D}_{Kin}$ are kinematical states.

Refined Algebraic Quantization (RAQ):

Solutions to the constraints are elements of the algebraic dual $\mathfrak{D}^\star_{Kin}$ of a dense domain $\mathfrak{D}_{Kin} \subset \mathcal{H}_{Kin}$. What we are looking at are states $\Psi \in \mathfrak{D}^\star$ such that:

$$\Psi\left[\hat{C}_I^\dagger f\right] := \hat{C}_I'\Psi[f] = 0, \qquad \forall f \in \mathfrak{D}$$

The space of solutions is denoted by $\mathfrak{D}^\star_{Phys}$. The physical Hilbert space will be a subspace of $\mathfrak{D}^\star_{Phys}$. Eventually, $\mathfrak{D}^\star_{Phys}$ will be the algebraic dual of a dense domain $\mathfrak{D}_{Phys} \in \mathcal{H}_{Phys}$, which is invariant under the algebra of operators corresponding to Dirac observables. Hence we obtain a Gel'fand triple:

$$\mathfrak{D}_{Phys} \hookrightarrow \mathcal{H}_{Phys} \hookrightarrow \mathfrak{D}^\star_{Phys}$$

A systematic construction of the physical Hilbert space is available if we have an anti-linear rigging map:

$$\eta : \mathfrak{D}_{Kin} \to \mathfrak{D}^\star_{Phys};\ f \mapsto \eta(f)$$

such that $\eta(f')[f]$ is a positive semi-definite sesquilinear form on $\mathfrak{D}_{Kin}$ If the quantum constraint algebra is generated by self-adjoint constraints $\hat{C}_I$ and their commutator algebra is a Lie algebra i.e. the structure functions are constant, then we can try to heuristically define the rigging map via the group averaging procedure:

$$\eta(f) := \int \mathrm{d}\mu(t)\ < e^{it^I\hat{C}_I} f, . >$$

The physical inner product is defined by the rigging inner product

$$\langle \eta(f) | \eta(f') \rangle_{Phys} := \eta(f')[f],\ \forall f, f' \in \mathfrak{D}_{Kin}.$$

Then a null space $\mathfrak{N} \subset \mathfrak{D}^\star_{Phys}$ is defined by $\left\{\eta(f) \in \mathfrak{D}^\star_{Phys}\ \middle|\ \|\eta(f)\|_{Phys} = 0\right\}$. Therefore

$$\mathfrak{D}_{Phys} := \eta\left(\mathfrak{D}_{Kin}\right)/\mathfrak{N} \qquad (0.1)$$

However, in general the constraint algebra is not a Lie algebra

$$\{C_I, C_J\} = f_{IJ}{}^K C_K \text{ where } f_{IJ}{}^K \text{ is a structure function}$$

The group averaging fails for C_I.

Fortunately we have the Abelianization Theorem (all the first-class constraint algebra can be locally Abelianized)

Equivalent constraints:

$$\tilde{C}_I = P_I + h_I(p_a, q^a, T^I) \qquad \left\{\tilde{C}_I, \tilde{C}_J\right\} = 0$$

If the constraint operators are self-adjoint and anomaly-free, we define the group averaging for the Abelianized constraints

$$\eta_\omega(f)[f'] = \frac{\int \prod_I \mathrm{d}t^I \left\langle f \middle| \exp\left[i \sum_I t^I \tilde{C}_I\right] \middle| f' \right\rangle}{\int \prod_I \mathrm{d}t^I \left\langle \omega \middle| \exp\left[i \sum_I t^I \tilde{C}_I\right] \middle| \omega \right\rangle}$$

where $f, f', \omega \in \mathfrak{D}_{Kin}$ are kinematical states.

Relation with reduced phase space (path-integral) quantization:

From $e^{i\tau^I P_I}\psi(T,Q) = \psi(T-\tau,Q)$

Rigging physical state

$$\begin{aligned}\eta : \psi \mapsto [\eta(\psi)](T,Q) &= \int \mathrm{d}\tau\, [e^{i\tau^I \tilde{C}_I}\psi](T,Q) \\ &= \int \mathrm{d}\tau\, V(\tau)\,\psi(T-\tau,Q) = -\int \mathrm{d}\tau\, V(T-\tau)\,\psi(\tau,Q) \\ &= -V(T)\int \mathrm{d}\tau\, V(\tau)^{-1}\,\psi(\tau,Q) = -V(T)\,[\eta'(\psi)](Q)\end{aligned}$$

Unitary map $V(\tau) = e^{i\tau^I[P_I+h_I]}\, e^{-i\tau^I P_I}$

physical states in reduced phase space quantization

<u>Theorem</u>

$$\langle \eta(\psi), \eta(\psi')\rangle = \langle \eta'(\psi), \eta'(\psi')\rangle_{\mathcal{H}_{red}=L_2(\mathrm{d}Q)}$$

1. we can identify $\mathcal{H}_{Phys}$ with $\mathcal{H}_{red}$ by the unitary map $V(T)$. There is a reduced phase space path integral directly relating to the rigging inner product.

2. All the Dirac observables on $\mathcal{H}_{Phys}$ can be defined by the Dirac observables on $\mathcal{H}_{red}$ via

$$O_{Phys} = V(T)\,O_{red}\,V(T)^{-1}$$

Theorem (answering conceptual & practical questions):

There is a path-integral formula for the rigging inner product (by skeletonization):

$$\langle \eta_\omega(f) | \eta_\omega(f') \rangle_{Phys} := \frac{\int \prod_I \mathrm{d}t^I \left\langle f \left| \exp\left[i \sum_I t^I \tilde{C}_I \right] \right| f' \right\rangle}{\int \prod_I \mathrm{d}t^I \left\langle \omega \left| \exp\left[i \sum_I t^I \tilde{C}_I \right] \right| \omega \right\rangle}$$

$$= \frac{\int \mathcal{D}p_a \mathcal{D}q^a \mathcal{D}P_I \mathcal{D}T^I \prod_{t,I} \delta\left(P_I + h_I\right) e^{i \int_{t_i}^{t_f} \mathrm{d}t \left[\sum_a p_a(t) \dot{q}^a(t) + \sum_I P_I(t) \dot{T}^I(t) \right]} \overline{f(q_f^a, T_f^I)} f'(q_i^a, T_i^I)}{\int \mathcal{D}p_a \mathcal{D}q^a \mathcal{D}P_I \mathcal{D}T^I \prod_{t,I} \delta\left(P_I + h_I\right) e^{i \int_{t_i}^{t_f} \mathrm{d}t \left[\sum_a p_a(t) \dot{q}^a(t) + \sum_I P_I(t) \dot{T}^I(t) \right]} \overline{\omega(q_f^a, T_f^I)} \omega(q_i^a, T_i^I)}$$

Abelianized constraints — Boundary kinematical states

1. The path integral formally solves all the quantum Abelianized constraints:

$$\eta_\omega(f)[\tilde{C}_I f'] = 0$$

thus η_ω is qualified as a rigging map (the conceptual question is answered).

2. All the gauge fixings are removed. The boundary states are kinematical states (the practical question is answered).

The path-integral formula has a direct canonical interpretation as a physical inner product, and can be used for practical computation.

Application to the case of gravity coupling to 4 scalar fields:

The action:

$$S_{KG}[T^I, g_{\mu\nu}] = -\frac{\alpha}{2}\sum_{I=0}^{3}\int d^4x\ \sqrt{|\det(g)|}\ g^{\alpha\beta}\partial_\alpha T^I \partial_\beta T^I$$

The diffeomorphism and Hamiltonian constraint:

$$C^{tot} = C + C^{KG}, \qquad C^{KG} = \frac{1}{\sqrt{\det q}}\left[\frac{\alpha(\det q)}{2}q^{ab}\partial_a T^I \partial_b T^I + \frac{1}{2\alpha}P_I^2\right]$$

$$C_a^{tot} = C_a + C_a^{KG}, \qquad C_a^{KG} = P_I \partial_a T^I$$

The Jacobian matrix:

$$\frac{\partial(C^{tot}, C_a^{tot})}{\partial(P_0, P_j)} = \begin{pmatrix} \frac{P_0}{\alpha\sqrt{\det q}} & \frac{P_j}{\alpha\sqrt{\det q}} \\ \partial_a T^0 & \partial_a T^j \end{pmatrix}$$

The constraints can be locally Abelianized

$$\tilde{C}^{tot} = P + h, \qquad h = h(P^{ab}, q^{ab}, T^I)$$

$$\tilde{C}_j^{tot} = P_j + h_j, \qquad h_j = h_j(P^{ab}, q^{ab}, T^I).$$

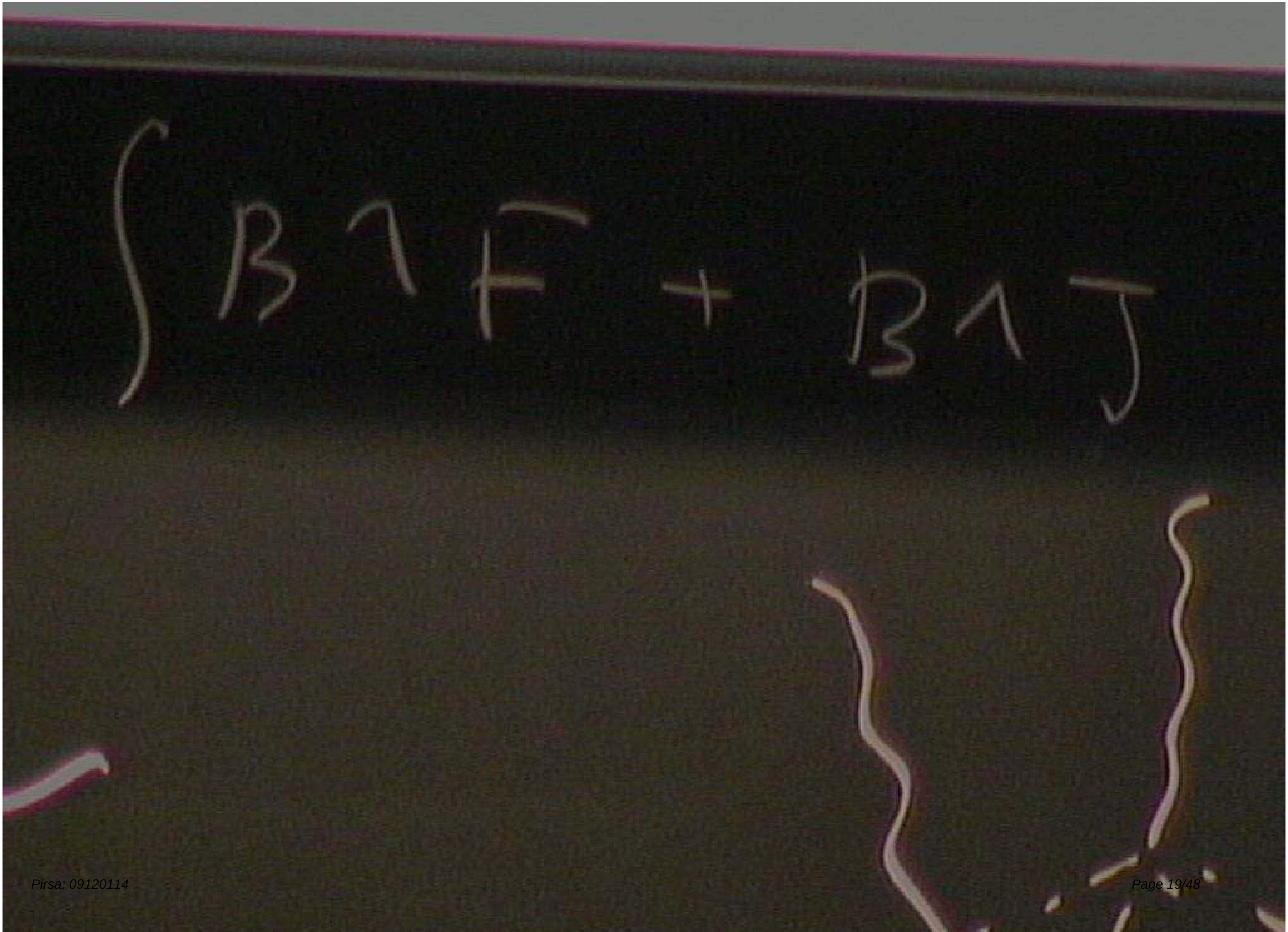
Local measure

Time-gauge

$$\langle \eta_\omega(f')|\eta_\omega(f)\rangle_{Phys} = \frac{Z_T(f, f')}{Z_T(\omega, \omega)}$$

$$Z_T(f, f') = \int_{II\pm} \mathcal{D}A_\alpha^{IJ}\mathcal{D}e_\alpha^I \mathcal{D}T^I \prod_{x\in M}\left|{}^{s}V^{1/2}V_s^6\right| \mathcal{J}_{KG}\left[\alpha, T^I, q^{ab}\right]\delta^3\left(e_a^0\right)\left[\cos\int_M e^I\wedge e^J\wedge\left(*F_{IJ}-\frac{1}{\gamma}F_{IJ}\right)\right]$$

$$\times\exp\left[-\frac{i\alpha}{2}\int_M d^4x\ \sqrt{|\det(g)|}\ g^{\alpha\beta}\partial_\alpha T^I\partial_\beta T^I\right]\overline{f\left(A_a^i, T^I\right)_{t_f}} f'\left(A_a^i, T^I\right)_{t_i}$$


$\int B \wedge F + B \wedge J$

Application to the case of gravity coupling to 4 scalar fields:

The action:

$$S_{KG}[T^I, g_{\mu\nu}] = -\frac{\alpha}{2}\sum_{I=0}^{3}\int d^4x\ \sqrt{|\det(g)|}\, g^{\alpha\beta}\partial_\alpha T^I \partial_\beta T^I$$

The diffeomorphism and Hamiltonian constraint:

$$C^{tot} = C + C^{KG}, \qquad C^{KG} = \frac{1}{\sqrt{\det q}}\left[\frac{\alpha(\det q)}{2} q^{ab}\partial_a T^I \partial_b T^I + \frac{1}{2\alpha}P_I^2\right]$$

$$C_a^{tot} = C_a + C_a^{KG}, \qquad C_a^{KG} = P_I \partial_a T^I$$

The Jacobian matrix:

$$\frac{\partial(C^{tot}, C_a^{tot})}{\partial(P_0, P_j)} = \begin{pmatrix} \frac{P_0}{\alpha\sqrt{\det q}} & \frac{P_j}{\alpha\sqrt{\det q}} \\ \partial_a T^0 & \partial_a T^j \end{pmatrix}$$

The constraints can be locally Abelianized

$$\tilde{C}^{tot} = P + h, \qquad h = h(P^{ab}, q^{ab}, T^I)$$

$$\tilde{C}_j^{tot} = P_j + h_j, \qquad h_j = h_j(P^{ab}, q^{ab}, T^I).$$

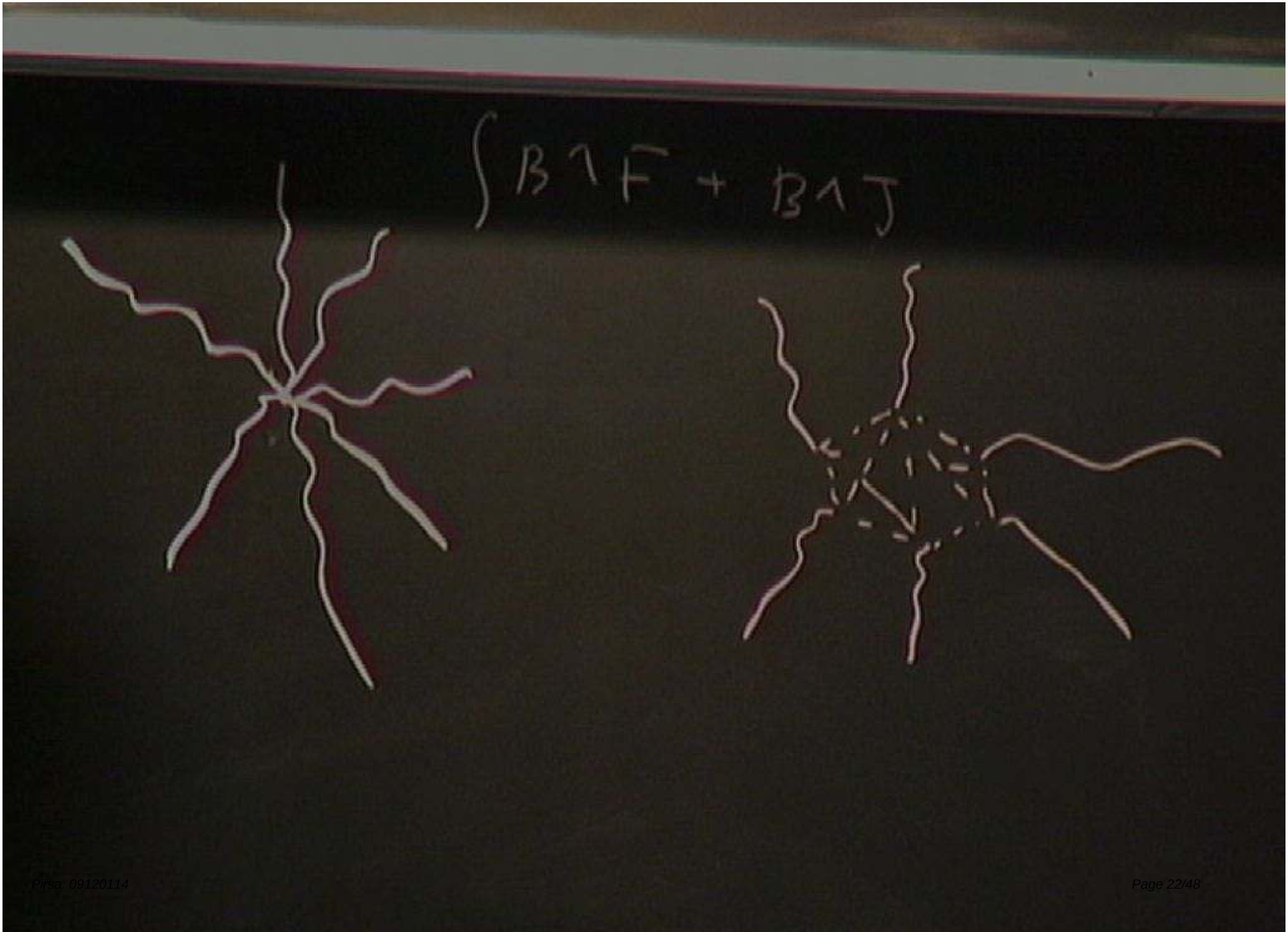
Local measure

Time-gauge

$$\langle \eta_\omega(f')|\eta_\omega(f)\rangle_{Phys} = \frac{Z_T(f, f')}{Z_T(\omega, \omega)}$$

$$Z_T(f, f') = \int_{II\pm} \mathcal{D}A_\alpha^{IJ}\mathcal{D}e_\alpha^I \mathcal{D}T^I \prod_{x\in M}\left|\mathcal{V}^{1/2}V_s^6\right| \mathcal{J}_{KG}\left[\alpha, T^I, q^{ab}\right]\delta^3\left(e_a^0\right)\left[\cos\int_M e^I\wedge e^J\wedge\left(*F_{IJ} - \frac{1}{\gamma}F_{IJ}\right)\right]$$

$$\times \exp\left[-\frac{i\alpha}{2}\int_M d^4x\ \sqrt{|\det(g)|}\, g^{\alpha\beta}\partial_\alpha T^I \partial_\beta T^I\right] \overline{f\left(A_a^i, T^I\right)_{t_f}} f'\left(A_a^i, T^I\right)_{t_i}$$


∫ B∧F + B∧J

Application to the case of gravity coupling to 4 scalar fields:

The action:

$$S_{KG}[T^I, g_{\mu\nu}] = -\frac{\alpha}{2}\sum_{I=0}^{3}\int d^4x \sqrt{|\det(g)|}\, g^{\alpha\beta}\partial_\alpha T^I \partial_\beta T^I$$

The diffeomorphism and Hamiltonian constraint:

$$C^{tot} = C + C^{KG}, \qquad C^{KG} = \frac{1}{\sqrt{\det q}}\left[\frac{\alpha(\det q)}{2} q^{ab}\partial_a T^I \partial_b T^I + \frac{1}{2\alpha}P_I^2\right]$$

$$C_a^{tot} = C_a + C_a^{KG}, \qquad C_a^{KG} = P_I \partial_a T^I$$

The Jacobian matrix:

$$\frac{\partial(C^{tot}, C_a^{tot})}{\partial(P_0, P_j)} = \begin{pmatrix} \frac{P_0}{\alpha\sqrt{\det q}} & \frac{P_j}{\alpha\sqrt{\det q}} \\ \partial_a T^0 & \partial_a T^j \end{pmatrix}$$

The constraints can be locally Abelianized

$$\tilde{C}^{tot} = P + h, \qquad h = h(P^{ab}, q^{ab}, T^I)$$

$$\tilde{C}_j^{tot} = P_j + h_j, \qquad h_j = h_j(P^{ab}, q^{ab}, T^I).$$

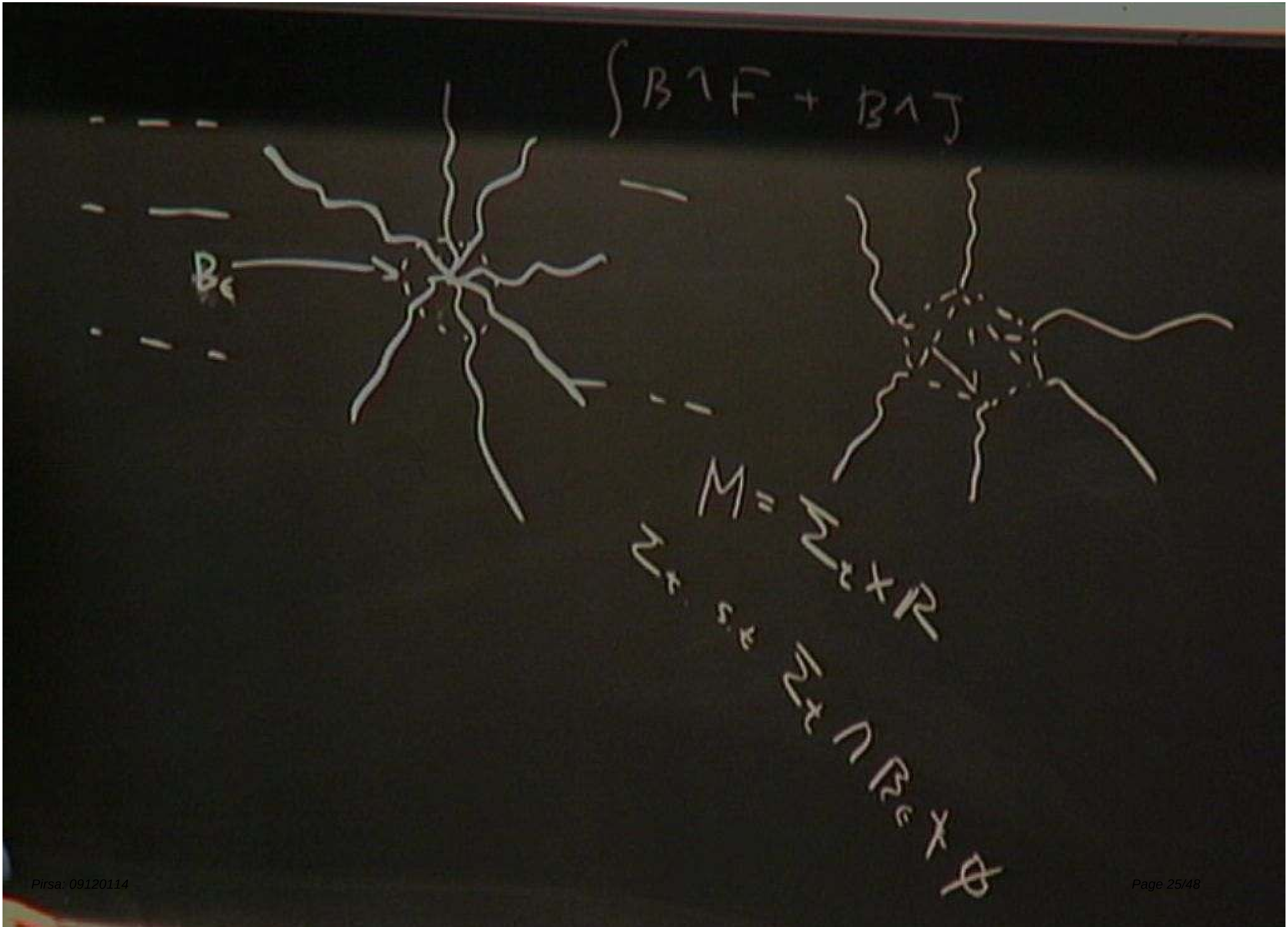
Local measure

Time-gauge

$$\langle \eta_\omega(f') | \eta_\omega(f) \rangle_{Phys} = \frac{Z_T(f, f')}{Z_T(\omega, \omega)}$$

$$Z_T(f, f') = \int_{II\pm} \mathcal{D}A_\alpha^{IJ} \mathcal{D}e_\alpha^I \mathcal{D}T^I \prod_{x\in M} \left|\mathcal{V}^{1/2}V_s^6\right| \mathcal{J}_{KG}\left[\alpha, T^I, q^{ab}\right] \delta^3\left(e_a^0\right) \left[\cos\int_M e^I \wedge e^J \wedge \left(*F_{IJ} - \frac{1}{\gamma}F_{IJ}\right)\right]$$

$$\times \exp\left[-\frac{i\alpha}{2}\int_M d^4x \sqrt{|\det(g)|}\, g^{\alpha\beta}\partial_\alpha T^I \partial_\beta T^I\right] \overline{f\left(A_a^i, T^I\right)_{t_f}} f'\left(A_a^i, T^I\right)_{t_i}$$


∫ B∧F + B∧J
Bε
M = Σt × R
Σt s.t. Σt ∩ Bε × Φ

Application to the case of gravity coupling to 4 scalar fields:

The action:

$$S_{KG}[T^I, g_{\mu\nu}] = -\frac{\alpha}{2}\sum_{I=0}^{3}\int \mathrm{d}^4x\ \sqrt{|\det(g)|}\ g^{\alpha\beta}\partial_\alpha T^I \partial_\beta T^I$$

The diffeomorphism and Hamiltonian constraint:

$$C^{tot} = C + C^{KG}, \qquad C^{KG} = \frac{1}{\sqrt{\det q}}\left[\frac{\alpha(\det q)}{2} q^{ab}\partial_a T^I \partial_b T^I + \frac{1}{2\alpha}P_I^2\right]$$

$$C_a^{tot} = C_a + C_a^{KG}, \qquad C_a^{KG} = P_I \partial_a T^I$$

The Jacobian matrix:

$$\frac{\partial(C^{tot}, C_a^{tot})}{\partial(P_0, P_j)} = \begin{pmatrix} \frac{P_0}{\alpha\sqrt{\det q}} & \frac{P_j}{\alpha\sqrt{\det q}} \\ \partial_a T^0 & \partial_a T^j \end{pmatrix}$$

The constraints can be locally Abelianized

$$\tilde{C}^{tot} = P + h, \qquad h = h(P^{ab}, q^{ab}, T^I)$$

$$\tilde{C}_j^{tot} = P_j + h_j, \qquad h_j = h_j(P^{ab}, q^{ab}, T^I).$$

Local measure

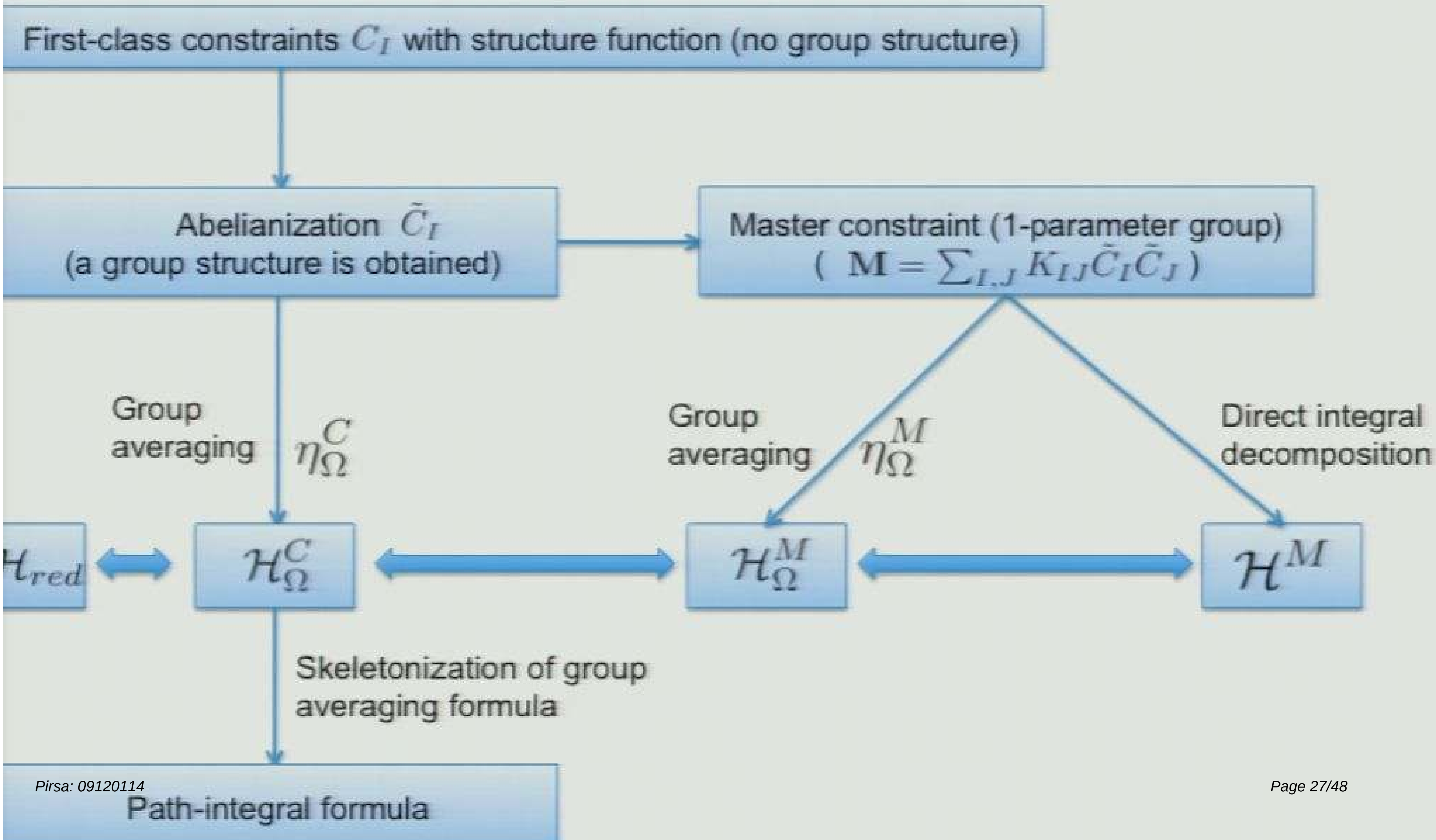
Time-gauge

$$\langle \eta_\omega(f')|\eta_\omega(f)\rangle_{Phys} = \frac{Z_T(f, f')}{Z_T(\omega, \omega)}$$

$$Z_T(f, f') = \int_{II\pm} \mathcal{D}A_\alpha^{IJ}\mathcal{D}e_\alpha^I \mathcal{D}T^I \prod_{x\in M}\left|\mathcal{V}^{1/2}V_s^6\right| \mathcal{J}_{KG}\left[\alpha, T^I, q^{ab}\right]\delta^3\left(e_a^0\right)\left[\cos\int_M e^I\wedge e^J \wedge\left(*F_{IJ} - \frac{1}{\gamma}F_{IJ}\right)\right]$$

$$\times \exp\left[-\frac{i\alpha}{2}\int_M \mathrm{d}^4x\ \sqrt{|\det(g)|}\ g^{\alpha\beta}\partial_\alpha T^I \partial_\beta T^I\right]\overline{f\left(A_a^i, T^I\right)_{t_f}} f'\left(A_a^i, T^I\right)_{t_i}$$

Operator constraint quantization, Master constraint quantization and path integral



Application to the case of gravity coupling to 4 scalar fields:

The action:

$$S_{KG}[T^I, g_{\mu\nu}] = -\frac{\alpha}{2}\sum_{I=0}^{3}\int d^4x\ \sqrt{|\det(g)|}\, g^{\alpha\beta}\partial_\alpha T^I \partial_\beta T^I$$

The diffeomorphism and Hamiltonian constraint:

$$C^{tot} = C + C^{KG}, \qquad C^{KG} = \frac{1}{\sqrt{\det q}}\left[\frac{\alpha(\det q)}{2} q^{ab}\partial_a T^I \partial_b T^I + \frac{1}{2\alpha}P_I^2\right]$$

$$C_a^{tot} = C_a + C_a^{KG}, \qquad C_a^{KG} = P_I \partial_a T^I$$

The Jacobian matrix:

$$\frac{\partial(C^{tot}, C_a^{tot})}{\partial(P_0, P_j)} = \begin{pmatrix} \frac{P_0}{\alpha\sqrt{\det q}} & \frac{P_j}{\alpha\sqrt{\det q}} \\ \partial_a T^0 & \partial_a T^j \end{pmatrix}$$

The constraints can be locally Abelianized

$$\tilde{C}^{tot} = P + h, \qquad h = h(P^{ab}, q^{ab}, T^I)$$

$$\tilde{C}_j^{tot} = P_j + h_j, \qquad h_j = h_j(P^{ab}, q^{ab}, T^I).$$

Local measure

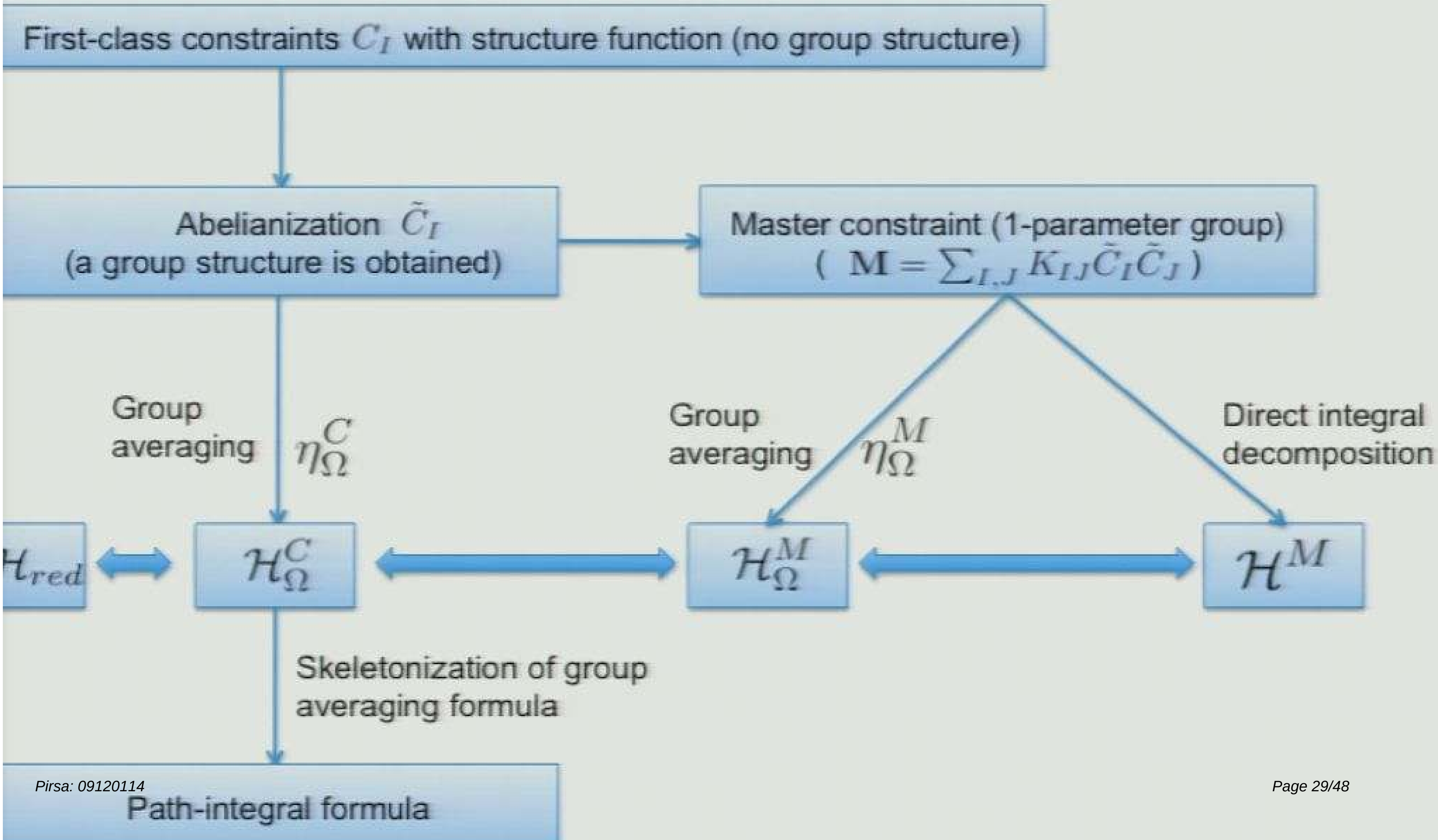
Time-gauge

$$\langle \eta_\omega(f')|\eta_\omega(f)\rangle_{Phys} = \frac{Z_T(f, f')}{Z_T(\omega, \omega)}$$

$$Z_T(f, f') = \int_{II\pm} \mathcal{D}A_\alpha^{IJ}\mathcal{D}e_\alpha^I \mathcal{D}T^I \prod_{x\in M}\left|{}^4V^{1/2}V_s^6\right| \mathcal{J}_{KG}\left[\alpha, T^I, q^{ab}\right]\delta^3\left(e_a^0\right)\left[\cos\int_M e^I\wedge e^J \wedge\left(*F_{IJ} - \frac{1}{\gamma}F_{IJ}\right)\right]$$

$$\times \exp\left[-\frac{i\alpha}{2}\int_M d^4x\ \sqrt{|\det(g)|}\, g^{\alpha\beta}\partial_\alpha T^I \partial_\beta T^I\right] \overline{f\left(A_a^i, T^I\right)_{t_f}} f'\left(A_a^i, T^I\right)_{t_i}$$

Operator constraint quantization, Master constraint quantization and path integral



Recall that:

1. We consider a set of Abeleanized constraints $\tilde{C}_I$. We don't lose any generality because of Abeleanization theorem (all first-class constraint algebra can in principle be locally Abeleanized i.e. $\tilde{C}_I = R_{IJ}C_J$ such that $\{\tilde{C}_I, \tilde{C}_J\} = 0$).

2. We quantize the Abeleanized constraints $\tilde{C}_I$ to be anomaly-free self-adjoint operators s.t. $[\tilde{C}_I, \tilde{C}_J] = 0$

3. If the number of the constraints N is finite, a regularized group averaging (RGA) is defined via

   For each state ψ in a dense subset $\mathcal{D}$ of $\mathcal{H}_{Kin}$, a linear functional $\eta^C_\Omega(\psi)$ in the algebra dual of $\mathcal{D}$ can be defined such that $\forall \phi \in \mathcal{D}$

$$\eta^C_\Omega(\psi)[\phi] := \lim_{\epsilon_I \to 0} \frac{\int_{\mathbb{R}} \prod_{I=1}^N \mathrm{d}t_I \ \langle\psi| \prod_{I=1}^N e^{it_I(\tilde{C}_I - \epsilon_I)} |\phi\rangle_{Kin}}{\int_{\mathbb{R}} \prod_{I=1}^N \mathrm{d}t_I \ \langle\Omega| \prod_{I=1}^N e^{it_I(\tilde{C}_I - \epsilon_I)} |\Omega\rangle_{Kin}}$$

   where $\Omega \in \mathcal{H}_{Kin}$ is a reference vector. Therefore we can define the group averaging inner product on the linear space of $\eta^C_\Omega(\psi)$ via $\langle\eta^C_\Omega(\psi)|\eta^C_\Omega(\phi)\rangle_\Omega := \eta^C_\Omega(\psi)[\phi]$. The resulting Hilbert space is denoted by $\mathcal{H}^C_\Omega$

Recall that this group averaging rigging inner product is related directly to the reduced phase space quantization physical inner product.

Master constraint approach:

1. We define a single master constraint $\mathbf{M} = \sum_{I,J}^{N} K_{IJ}\tilde{C}_I\tilde{C}_J$, with K_{IJ} a positive definite constant matrix and $\tilde{C}_I$ the Abeleanized constraints.
2. We quantize the master constraint to be a positive self-adjoint operator.
3. A regularized group averaging (RGA) is defined in the same way:

For each state ψ in the same dense subset $\mathcal{D}$ of $\mathcal{H}_{Kin}$, a linear functional $\eta_\Omega^M(\psi)$ in the algebra dual of $\mathcal{D}$ can be defined such that $\forall \phi \in \mathcal{D}$

$$\eta_\Omega^M(\psi)[\phi] := \lim_{\epsilon \to 0} \frac{\int_{\mathbb{R}} \mathrm{d}t\ \langle \psi | e^{it(\mathbf{M}-\epsilon)} | \phi \rangle_{Kin}}{\int_{\mathbb{R}} \mathrm{d}t\ \langle \Omega | e^{it(\mathbf{M}-\epsilon)} | \Omega \rangle_{Kin}}$$

where $\Omega \in \mathcal{H}_{Kin}$ is the reference vector. Therefore we can define another group averaging inner product on the linear space of $\eta_\Omega^M(\psi)$ via $\langle \eta_\Omega^M(\psi) | \eta_\Omega^M(\phi) \rangle_\Omega := \eta_\Omega^M(\psi)[\phi]$. The resultant Hilbert space is denoted by $\mathcal{H}_\Omega^M$.

It is expected that the two prescriptions are consistent in certain sense. There are two cases:

- For $N < \infty$: $\mathcal{H}_\Omega^M = \mathcal{H}_\Omega^C$
- For $N = \infty$: take the limit $N \to \infty$ in a specific way to obtain the consistency.

When the consistency is established, the group averaging of the Master constraint obtains the relations to both the reduced phase space quantization and path-integral quantization.

A finite number of Abeleanized constraints

Theorem:

In the case that there is only a finite number of constraints. We suppose $\vec{x}=0$ is not contained in the continuously singular spectrum and is not a limit point in pure point spectrum. We also assume that there exists a neighborhood $\mathcal{N}_0$ of $\vec{x}=0$ such that each $\rho^{ac}_{\Omega_m}$ is continuous at $\vec{x}=0$ and is differentiable on $\mathcal{N}_0 - \{\vec{x}=0\}$. With above assumptions, the group averaging Hilbert spaces from these two approaches, $\mathcal{H}^C_\Omega$ and $\mathcal{H}^M_\Omega$, are unitarily equivalent with each other.

An infinite number of Abeleanized constraints

The formula $\eta^C_{\Omega,N}(\psi)[\phi] := \lim_{\epsilon_I \to 0} \frac{\int_{\mathbb{R}} \prod_{I=1}^N \mathrm{d}t_I \, \langle\psi| \prod_{I=1}^N e^{it_I(\tilde{C}_I - \epsilon_I)} |\phi\rangle_{Kin}}{\int_{\mathbb{R}} \prod_{I=1}^N \mathrm{d}t_I \, \langle\Omega| \prod_{I=1}^N e^{it_I(\tilde{C}_I - \epsilon_I)} |\Omega\rangle_{Kin}}$

is ill-defined when $N \longrightarrow \infty$.

However $\eta^M_{\Omega}(\psi)[\phi] = \lim_{\epsilon \to 0} \frac{\int_{\mathbb{R}} \mathrm{d}t \, \langle\psi| e^{it(\mathbf{M}-\epsilon)} |\phi\rangle_{Kin}}{\int_{\mathbb{R}} \mathrm{d}t \, \langle\Omega| e^{it(\mathbf{M}-\epsilon)} |\Omega\rangle_{Kin}}$ is well-defined.

We define a truncated master constraint $\mathbf{M}_N = \sum_{I,J=1}^{N} K_{IJ} \tilde{C}_I \tilde{C}_J \;\; (\, N < \infty)$

Theorem:

Suppose $\mathbf{M}_N$ converges to $\mathbf{M}$ in the strong resolvent sense as $N \to \infty$, and $\mathbf{M}_N$ satisfies the conditions of previous theorem. For any ψ, ψ' and Ω in a dense domain $\mathcal{D} \subset \mathcal{H}_{Kin}$, there exists sequences $\{\psi_N\}_N$, $\{\psi'_N\}_N$ and $\{\Omega_N\}_N$ with $\lim_{N\to\infty} \psi_N = \psi$, $\lim_{N\to\infty} \psi'_N = \psi'$ and $\lim_{N\to\infty} \Omega_N = \Omega$ such that $\lim_{N\to\infty} \langle \eta^C_{\Omega_N,N}(\psi_N) | \eta^C_{\Omega_N,N}(\psi'_N) \rangle = \langle \eta^M_\Omega(\psi) | \eta^M_\Omega(\psi') \rangle$.

Remark:

We have obtained the consistency between the group averaging of Abeleanized constraints and Master constraint. The group averaging of the Master constraint is (indirectly) linked to both the reduced phase space quantization and path-integral quantization via the operator constraint quantization, but only for the Master constraint defined by the Abelianized constraints.

General Master constraint and path integral

Direct Integral Decomposition physical inner product ⟷ The rigging inner product with a general Master constraint

↓

A path-integral expression for the rigging map and rigging inner product

A general Master constraint

For a general Master constraint $\mathbf{M} = K^{IJ} C_I C_J$

1. C_I are not the Abelianized constraints, but more conveniently, the original constraints.
2. K^{IJ} may depend on the phase space coordinates.
3. But the group averaging can be defined since the constraint algebra is trivially a Lie algebra.

For example, in LQG/AQG

$$\mathbf{M} = \sum_{v \in V(\gamma)} \frac{G_{j,v}}{V_v^{1/2}} \frac{G_{j,v}}{V_v^{1/2}} + \frac{D_{j,v}}{V_v^{1/2}} \frac{D_{j,v}}{V_v^{1/2}} + \frac{H_v}{V_v^{1/2}} \frac{H_v}{V_v^{1/2}}$$

- Its group averaging loses the manifest (indirect) link with the reduced phase space quantization and the reduced phase space path-integral.
- However there is another way to justify the group averaging, and there is another way to directly obtain a path-integral formulation from the Master constraint.

Direct integral decomposition (DID)

(B. Dittrich and T. Thiemann in CQG23 (2006) 1025)

1. Given a kinematical Hilbert space $\mathcal{H}_{Kin}$, and a self-adjoint constraint operator (e.g. the Master constraint $\mathbf{M} = K^{IJ} C_I C_J$, C_I first class)
2. We split the kinematical Hilbert space into three mutually orthogonal sectors with respect to different spectral types $\mathcal{H}_{Kin} = \mathcal{H}^{pp} \oplus \mathcal{H}^{ac} \oplus \mathcal{H}^{cs}$ (where $\mathcal{H}^* = \{\Psi \in \mathcal{H}_{Kin};\ \mu_\Psi = \mu_\Psi^*\},\ * \in \{pp, ac, cs\},\ \mu_\Psi(B) = \langle \Psi, E(B)\Psi \rangle$)
3. The direct integral decomposition of each sector

$$\mathcal{H}^* = \int^{\oplus} d\mu^*(\lambda)\mathcal{H}^*_\lambda,\ * \in \{pp, ac, cs\}$$

4. The physical Hilbert space is defined by a direct sum of three fiber Hilbert spaces

$$\mathcal{H}^M = \mathcal{H}^{pp}_{\lambda=0} \oplus \mathcal{H}^{ac}_{\lambda=0} \oplus \mathcal{H}^{cs}_{\lambda=0}$$

Remark:

- DID is considered as a rigorous definition of the physical Hilbert space, and less ambiguities than RAQ.
- $\mathcal{H}^{cs}$ is often absent in physical models.
- DID requires full knowledge of the operator spectrum (e.g. step 2.), thus is not practical for complicated systems

Regularized group averaging (RGA)

Given a kinematical Hilbert space $\mathcal{H}_{Kin}$, and a self-adjoint constraint operator $\mathbf{M}$ (e.g. the master constraint $\mathbf{M} = K^{IJ}C_I C_J$)

Definition:

For each state ψ in a dense subset $\mathcal{D}$ of $\mathcal{H}_{Kin}$, a linear functional $\eta_\Omega^M(\psi)$ in the algebraic dual of $\mathcal{D}$ can be defined such that $\forall \phi \in \mathcal{D}$

$$\eta_\Omega^M(\psi)[\phi] := \lim_{\epsilon \to 0} \frac{\int_{\mathbb{R}} \mathrm{d}t \ \langle \psi | e^{it(\mathbf{M}-\epsilon)} | \phi \rangle_{Kin}}{\int_{\mathbb{R}} \mathrm{d}t \ \langle \Omega | e^{it(\mathbf{M}-\epsilon)} | \Omega \rangle_{Kin}}$$

where $\Omega \in \mathcal{H}_{Kin}$ is called a reference vector. Therefore we can define a new inner product on the linear space of $\eta_\Omega^M(\psi)$ via $\langle \eta_\Omega^M(\psi) | \eta_\Omega^M(\phi) \rangle_\Omega := \eta_\Omega^M(\psi)[\phi]$. The resultant Hilbert space is denoted by $\mathcal{H}_\Omega^M$

Remark:

- The proposal of group averaging requires less knowledge on the operator spectrum than the programme of direct integral decomposition.
- The reason to introducing the regularization parameter ϵ and take the limit after the integration is to make a consistency relation between the group averaging Hilbert space and the "ac"-sectors in physical Hilbert space from DID.

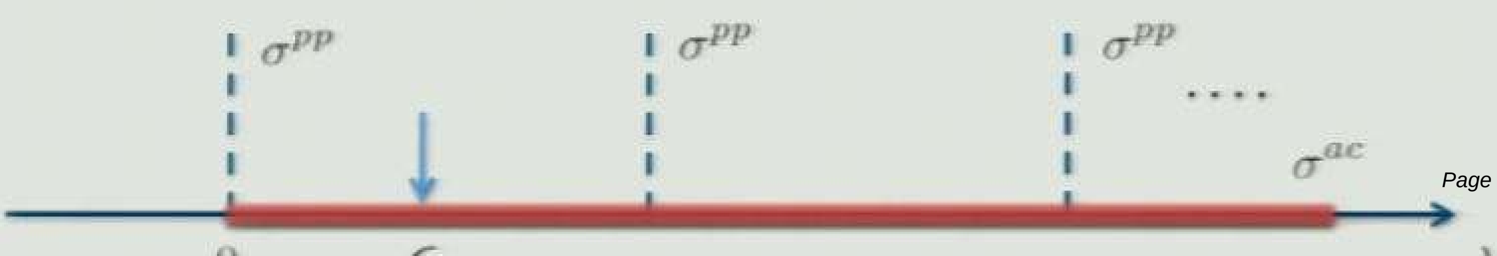
Theorem: We suppose zero is not a limit point in $\sigma^{pp}(\mathbf{M})$ and $\sigma^{cs}(\mathbf{M}) = \emptyset$ (which relates the argument that there is no state without physical interpretation). In addition, if we have any one of the following conditions

1. there exists $\delta > 0$ such that each μ_m^{ac} ($\mathrm{d}\mu_m^{ac} = \mu_m^{ac}\mathrm{d}\lambda$) is continuous on the closed interval $[0, \delta]$.
2. there exists $\delta > 0$ such that each ρ_m^{ac} is continuous at $\lambda = 0$ and is differentiable on the open interval $(0, \delta)$.
3. there exists $\delta > 0$ such that N^{ac} is constant on the neighborhood $[0, \delta)$.

Then there exists a dense domain $\mathcal{D}$ in $\mathcal{H}_{Kin}$, such that for some certain choices of reference vector Ω the group averaging Hilbert space $\mathcal{H}_\Omega^M$ is unitarily equivalent to the absolute continuous sector of physical Hilbert space $\mathcal{H}_{\lambda=0}^{ac}$.

Remark:

- The regularization scheme selects the "ac"-spectrum exclusively.
- All the conditions in the theorem are satisfied in the physical models tested so far (B. Dittrich and T. Thiemann in CQG23 (2006) 1067-1162).



A path-integral from Master constraint in LQG/AQG

The non-graph-changing Master constraint operator for LQG/AQG

$$\hat{\mathbf{M}} := \sum_{v \in V(\gamma)} [\hat{G}_j^{(1/2)}(v)^\dagger \hat{G}_j^{(1/2)}(v) + \hat{D}_j^{(1/2)}(v)^\dagger \hat{D}_j^{(1/2)}(v) + \hat{H}^{(1/2)}(v)^\dagger \hat{H}^{(1/2)}(v)]$$

- Gauss constraint

$$\hat{G}_j^{(1/2)}(v) := \hat{Q}_v^{(1/2)} \left[\sum_{b(e)=v} \hat{E}_j(e) - \sum_{f(e)=v} O_{jk}\left[\hat{A}(e)\right] \hat{E}_k(e) \right];$$

- Spatial diffeomorphism constraint

$$\hat{D}_j^{(1/2)}(v) := \frac{1}{E(v)} \sum_{e_1 \cap e_2 \cap e_3 = v} \frac{\epsilon_v(e_1, e_2, e_3)}{|L(v, e_1, e_2)|} \sum_{\beta \in L(v, e_1, e_2)} \mathrm{tr}\left(\tau_j \left[\hat{A}(\beta) - \hat{A}(\beta)^{-1}\right] \hat{A}(e_3) \left[\hat{A}(e_3)^{-1}, \sqrt{\hat{V}_v}\right]\right);$$

- Euclidean Hamiltonian constraint (up to an overall factor)

$$\hat{H}_E^{(r)}(v) := \frac{1}{E(v)} \sum_{e_1 \cap e_2 \cap e_3 = v} \frac{\epsilon_v(e_1, e_2, e_3)}{|L(v, e_1, e_2)|} \sum_{\beta \in L(v, e_1, e_2)} \mathrm{tr}\left(\left[\hat{A}(\beta) - \hat{A}(\beta)^{-1}\right] \hat{A}(e_3) \left[\hat{A}(e_3)^{-1}, \hat{V}_v^{(r)}\right]\right);$$

- Lorentzian Hamiltonian constraint (up to an overall factor)

$$\hat{T}(v) := \frac{1}{E(v)} \sum_{e_1 \cap e_2 \cap e_3 = v} \epsilon_v(e_1, e_2, e_3)\, \mathrm{tr}\left(\hat{A}(e_1) \left[\hat{A}(e_1)^{-1}, \left[\hat{H}_E^{(1)}, \hat{V}\right]\right] \hat{A}(e_2) \left[\hat{A}(e_2)^{-1}, \left[\hat{H}_E^{(1)}, \hat{V}\right]\right] \hat{A}(e_3) \left[\hat{A}(e_3)^{-1}, \sqrt{\hat{V}_v}\right]\right),$$

$$\hat{H}^{(1/2)}(v) = \hat{H}_E^{(1/2)}(v) + \hat{T}(v);$$

where the matrix $O_{jk}[g]$ is the adjoint representation of $g \in G$ on the Lie algebra $\mathfrak{g}$, $\hat{V} := \sum_v \hat{V}_v$, $\hat{H}_E^{(1)} := \sum_v \hat{H}_E^{(1)}(v)$ and

$$\hat{Q}_v^{(r)} := \frac{1}{E(v)} \sum_{e_1 \cap e_2 \cap e_3 = v} \epsilon_v(e_1, e_2, e_3)\, \mathrm{tr}\left(\hat{A}(e_1) \left[\hat{A}(e_1)^{-1}, \hat{V}_v^{(r)}\right] \hat{A}(e_2) \left[\hat{A}(e_2)^{-1}, \hat{V}_v^{(r)}\right] \hat{A}(e_3) \left[\hat{A}(e_3)^{-1}, \hat{V}_v^{(r)}\right]\right).$$

The Master constraint operator has correct semiclassical limit

K. Giesel and T. Thiemann, CQG24 (2007) 2465-2565

The semiclassical tool: complexifier coherent states (of Thiemann & Winkler)

On one edge:

$$\psi^{t_e}_{g(e)}(h(e)) := \left[e^{-\frac{C_e}{\hbar}}\delta_{h'(e)}(h(e))\right]_{h'(e)\to g(e)} = \left[e^{t_e\Delta_e/2}\delta_{h'(e)}(h(e))\right]_{h'(e)\to g(e)}$$

$$= \sum_{j_e}(2j_e+1)\, e^{-t_e j_e(j_e+1)/2}\chi_{j_e}\left(g(e)h(e)^{-1}\right)$$

where $g(e)$ is the complexified holonomy $g(e) = e^{-ip_j(e)\tau_j/2}h(e) \in G^{\mathbb{C}} = T^*G$.

Overcompleteness:

$$\int_{G^{\mathbb{C}}} dg(e)\, |\tilde{\psi}^{t_e}_{g(e)}\rangle\langle\tilde{\psi}^{t_e}_{g(e)}| = 1, \qquad dg = \frac{1}{t^3} d\mu_H(h) d^3p + o(t^\infty)$$

On a graph γ:

$$\tilde{\psi}^t_g = \prod_{e\in E(\gamma)} \tilde{\psi}^{t_e}_{g(e)}$$

For any polynomial of holonomies and fluxes

$$\left\langle \tilde{\psi}^t_g \middle| \mathrm{Pol}\left(\{\hat{h}(e)\}_{e\in E(\gamma)}, \{\hat{p}_j(e)\}_{e\in E(\gamma)}\right) \middle| \tilde{\psi}^t_g \right\rangle = \mathrm{Pol}\left(\{h(e)\}_{e\in E(\gamma)}, \{p_j(e)\}_{e\in E(\gamma)}\right) + o(t)$$

For non-polynomial operators, on a cubic graph

$$\left\langle \tilde{\psi}^t_g \middle| \hat{V}(R) \middle| \tilde{\psi}^t_g \right\rangle = V(R)[g] + o(t)$$

$$\left\langle \tilde{\psi}^t_g \middle| \hat{\mathbf{M}} \middle| \tilde{\psi}^t_g \right\rangle = \mathbf{M}[g] + o(t)$$

K. Giesel and T. Thiemann, CQG24 (2007) 2465-2565

C. Flori and T. Thiemann, arXiv:0812.1537, arXiv:0904.1303

Skeletonization of the group averaging with Master constraint operator

Suppose $f, f' \in \mathcal{H}_{Kin}$ have non-trivial excitations only on a finite number of edges.

$$\left\langle \eta(f) \middle| \eta(f') \right\rangle := \lim_{\epsilon \to 0} \frac{1}{\ell_p^2} \int_{-\infty}^{\infty} \frac{d\tau}{2\pi} \left\langle f \middle| \exp\left[\frac{i}{\ell_p^2}\tau(\hat{\mathbf{M}} - \epsilon)\right] \middle| f' \right\rangle_{Kin}$$

approximation

$$\left\langle f \middle| \left[1 + \frac{i\tau}{\ell_p^2 N}(\mathbf{M} - \epsilon)\right]^N \middle| f' \right\rangle_{Kin}$$

$$= \int dg_N \cdots dg_1 dg_0 \left\langle \tilde{\psi}^t_{g_N} \middle| 1 + \frac{i\tau}{\ell_p^2 N}(\mathbf{M} - \epsilon) \middle| \tilde{\psi}^t_{g_{N-1}} \right\rangle_{Kin} \left\langle \tilde{\psi}^t_{g_{N-1}} \middle| 1 + \frac{i\tau}{\ell_p^2 N}(\mathbf{M} - \epsilon) \middle| \tilde{\psi}^t_{g_{N-2}} \right\rangle_{Kin} \cdots \left\langle \tilde{\psi}^t_{g_1} \middle| 1 + \frac{i\tau}{\ell_p^2 N}(\mathbf{M} - \epsilon) \middle| \tilde{\psi}^t_{g_0} \right\rangle_{Kin}$$

$$\times \left\langle f \middle| \tilde{\psi}^t_{g_N} \right\rangle_{Kin} \left\langle \tilde{\psi}^t_{g_0} \middle| f' \right\rangle_{Kin}$$

where the measure $dg = \prod_{e \in E(\gamma)} d^3 p(e) dh(e)/t^3 + O(t^\infty)$ up to an overall constant.

Semiclassical property:

$$\left\langle \tilde{\psi}^t_{g_i} \middle| 1 + \frac{i\tau}{\ell_p^2 N}(\mathbf{M} - \epsilon) \middle| \tilde{\psi}^t_{g_{i-1}} \right\rangle_{Kin} = \left[1 + \frac{i\tau}{\ell_p^2 N}\left(\mathbf{M}[g_i] - \epsilon + tF^t(g_i, g_{i-1})\right)\right] \langle \tilde{\psi}^t_{g_i} | \tilde{\psi}^t_{g_{i-1}} \rangle_{Kin}$$

Fluctuation

will give the kinetic term

We define

$$h_k = e^{\theta_k \cdot \tau/2}$$

A path integral expression of the matrix element (γ is a finite cubic grap)

$$\left\langle f \middle| \exp\left[\frac{i}{\ell_p^2}\tau(\mathbf{M}-\epsilon)\right] \middle| f' \right\rangle_{Kin}$$

$$= \int \prod_{e\in E(\gamma)} \prod_{i=0}^{N} \frac{\mathrm{d}h_i \mathrm{d}^3 p_i}{t^3} \frac{\sinh(p_i)}{p_i} e^{-p_i^2/t} \prod_{k=1}^{N} \frac{z_{k,k-1}}{\sinh(z_{k,k-1})} e^{z_{k,k-1}^2/t}\, \overline{f(g_N)} f'(g_0)$$

$$\times \exp\left\{-i\frac{(p_k+p_{k-1})}{2t}\cdot(\theta_k-\theta_{k-1}) - \frac{1}{4t}\left[(p_k-p_{k-1})^2+(\theta_k-\theta_{k-1})^2\right] + i\frac{\tau}{\ell_p^2 N}\left[\mathbf{M}[g_k]-\epsilon+tF^t(g_k,g_{k-1})\right]\right\}$$

It might be interesting to derive a (highly interacting) hyper-cubic spin-foam model from this path-integral.

Make the following approximations in order to have the classical action on the exponential

- We assume the fluctuation F^t is small and negligible. It is a non-trivial assumption for the property of the master constraint operator $\hat{\mathbf{M}}$, i.e. we should design a certain operator ordering in the definition of the self-adjoint master constraint operator $\hat{\mathbf{M}}$, such that the fluctuation F^t is small and negligible. For simple system like the ordinary free quantum field theory, such an operator ordering is nothing but the normal ordering of the creation and annihilation operator, which results in $F^t = 0$.

- We only count the paths such that the second order terms $(\Delta p)^2$ and $(\Delta\theta)^2$ on the exponential is negligible. Note that the overlap function $\langle \tilde{\psi}^t_{g_i} | \tilde{\psi}^t_{g_{i-1}} \rangle_{Kin}$ is sharply peaked at the point $g_i = g_{i-1}$ with width $\sqrt{t}$. Thus $(\Delta p)^2, (\Delta\theta)^2 \sim t$ and have to be neglected if we want to have a classical action on the exponential. It is consistent if we want to have

Finally we obtain a discretized path integral for the rigging inner product:

$$\langle \eta(f) | \eta(f') \rangle_{\Omega} = \lim_{\epsilon \to 0} \frac{\int_{\mathbb{R}} dt \left\langle f \left| e^{it(\mathbf{M}-\epsilon)} \right| f' \right\rangle_{Kin}}{\int_{\mathbb{R}} dt \left\langle \Omega \left| e^{it(\mathbf{M}-\epsilon)} \right| \Omega \right\rangle_{Kin}}$$

$$= \frac{\int \mathcal{D}\mu \prod_{k=1}^{N} \exp\left\{ -\frac{i}{\ell_p^2} \left[\sum_{e \in E(\gamma)} a^2 \langle p \rangle_k \frac{(\theta_k - \theta_{k-1})^e}{\Delta T_k} \Delta T_k + \sum_{v \in V(\gamma)} \left(\Lambda^i_{v,k} G_{i,v} + N^i_{v,k} D_{i,v} + N_{v,k} H_v \right) \right] \right\} \overline{f(g_N)} f'(g_0)}{\int \mathcal{D}\mu \prod_{k=1}^{N} \exp\left\{ -\frac{i}{\ell_p^2} \left[\sum_{e \in E(\gamma)} a^2 \langle p \rangle_k \frac{(\theta_k - \theta_{k-1})^e}{\Delta T_k} \Delta T_k + \sum_{v \in V(\gamma)} \left(\Lambda^i_{v,k} G_{i,v} + N^i_{v,k} D_{i,v} + N_{v,k} H_v \right) \right] \right\} \overline{\Omega(g_N)} \Omega(g_0)}$$

$$\mathcal{D}\mu[g, \Lambda, N^a, N]$$

$$= \prod_{e \in E(\gamma)} \left[\prod_{i=0}^{N} \frac{dh_i d^3 p_i}{t^3} \right] \left[\prod_{k=1}^{N} \prod_{v \in V(\gamma)} d^3\Lambda_{v,k} d^3 N_{v,k} dN_{v,k} \right] \left[\prod_{k=1}^{N} \prod_{v \in V(\gamma)} V_v^{7/2}[g_k] \right] \left[\prod_{i=0}^{N} \frac{\sinh(p_i)}{p_i} e^{-p_i^2/t} \prod_{k=1}^{N} \frac{z_{k,k-1}}{\sinh(z_{k,k-1})} e^{z_{k,k-1}^2/t} \right]$$

local measure from the phase space dependence of K

Remarks

1. In contrast to the reduced phase space continuum path integral, this path integral, derived from the discrete setting, is defined on a cubulation.

2. By the disceteness, this path integral formula is a mathematical well-defined quantity, similar to the spin-foam models (the pair (N, γ) labels the triangulation-dependence).

3. In contrast to the reduced phase space quantization, this approach suggests a direct link from canonical LQG to a well-defined path-integral formulation (hopefully a spin-foam model).

We define

$$h_k = e^{\theta_k \cdot \tau/2}$$

A path integral expression of the matrix element (γ is a finite cubic grap)

$$\left\langle f \left| \exp\left[\frac{i}{\ell_p^2}\tau(\mathbf{M}-\epsilon)\right] \right| f' \right\rangle_{Kin}$$

$$= \int \prod_{e \in E(\gamma)} \prod_{i=0}^{N} \frac{\mathrm{d}h_i \mathrm{d}^3 p_i}{t^3} \frac{\sinh(p_i)}{p_i} e^{-p_i^2/t} \prod_{k=1}^{N} \frac{z_{k,k-1}}{\sinh(z_{k,k-1})} e^{z_{k,k-1}^2/t}\, \overline{f(g_N)} f'(g_0)$$

$$\times \exp\left\{-i\frac{(p_k+p_{k-1})}{2t}\cdot(\theta_k-\theta_{k-1}) - \frac{1}{4t}\left[(p_k-p_{k-1})^2+(\theta_k-\theta_{k-1})^2\right] + i\frac{\tau}{\ell_p^2 N}\left[\mathbf{M}[g_k]-\epsilon+tF^t(g_k,g_{k-1})\right]\right\}$$

It might be interesting to derive a (highly interacting) hyper-cubic spin-foam model from this path-integral.

Make the following approximations in order to have the classical action on the exponential

We assume the fluctuation F^t is small and negligible. It is a non-trivial assumption for the property of the master constraint operator $\hat{\mathbf{M}}$, i.e. we should design a certain operator ordering in the definition of the self-adjoint master constraint operator $\hat{\mathbf{M}}$, such that the fluctuation F^t is small and negligible. For simple system like the ordinary free quantum field theory, such an operator ordering is nothing but the normal ordering of the creation and annihilation operator, which results in $F^t = 0$.

We only count the paths such that the second order terms $(\Delta p)^2$ and $(\Delta\theta)^2$ on the exponential is negligible. Note that the overlap function $\langle \tilde{\psi}^t_{g_i} | \tilde{\psi}^t_{g_{i-1}} \rangle_{Kin}$ is sharply peaked at the point $g_i = g_{i-1}$ with width $\sqrt{t}$. Thus $(\Delta p)^2, (\Delta\theta)^2 \sim t$ and have to be neglected if we want to have a classical action on the exponential. It is consistent if we want to have

Finally we obtain a discretized path integral for the rigging inner product:

$$\langle \eta(f) | \eta(f') \rangle_\Omega = \lim_{\epsilon \to 0} \frac{\int_{\mathbb{R}} \mathrm{d}t \, \left\langle f \left| e^{it(\mathbf{M}-\epsilon)} \right| f' \right\rangle_{Kin}}{\int_{\mathbb{R}} \mathrm{d}t \, \left\langle \Omega \left| e^{it(\mathbf{M}-\epsilon)} \right| \Omega \right\rangle_{Kin}}$$

$$= \frac{\int \mathcal{D}\mu \prod_{k=1}^N \exp\left\{ -\frac{i}{\ell_p^2} \left[\sum_{e \in E(\gamma)} a^2 \langle p \rangle_k \frac{(\theta_k - \theta_{k-1})^a}{\Delta T_k} \Delta T_k + \sum_{v \in V(\gamma)} \left(\Lambda^i_{v,k} G_{i,v} + N^i_{v,k} D_{i,v} + N_{v,k} H_v \right) \right] \right\} \overline{f(g_N)} f'(g_0)}{\int \mathcal{D}\mu \prod_{k=1}^N \exp\left\{ -\frac{i}{\ell_p^2} \left[\sum_{e \in E(\gamma)} a^2 \langle p \rangle_k \frac{(\theta_k - \theta_{k-1})^a}{\Delta T_k} \Delta T_k + \sum_{v \in V(\gamma)} \left(\Lambda^i_{v,k} G_{i,v} + N^i_{v,k} D_{i,v} + N_{v,k} H_v \right) \right] \right\} \overline{\Omega(g_N)} \Omega(g_0)}$$

$$\mathcal{D}\mu[g, \Lambda, N^a, N]$$

$$= \prod_{e \in E(\gamma)} \left[\prod_{i=0}^N \frac{\mathrm{d}h_i \mathrm{d}^3 p_i}{t^3} \right] \left[\prod_{k=1}^N \prod_{v \in V(\gamma)} \mathrm{d}^3 \Lambda_{v,k} \mathrm{d}^3 N_{v,k} \mathrm{d} N_{v,k} \right] \left[\prod_{k=1}^N \prod_{v \in V(\gamma)} V_v^{7/2}[g_k] \right] \left[\prod_{i=0}^N \frac{\sinh(p_i)}{p_i} e^{-p_i^2/t} \prod_{k=1}^N \frac{z_{k,k-1}}{\sinh(z_{k,k-1})} e^{z_{k,k-1}^2/t} \right]$$

local measure from the phase space dependence of K

Remarks

1. In contrast to the reduced phase space continuum path integral, this path integral, derived from the discrete setting, is defined on a cubulation.

2. By the disceteness, this path integral formula is a mathematical well-defined quantity, similar to the spin-foam models (the pair (N, γ) labels the triangulation-dependence).

3. In contrast to the reduced phase space quantization, this approach suggests a direct link from canonical LQG to a well-defined path-integral formulation (hopefully a spin-foam model).

Summary and Outlook

- Reduced phase space -, operator constraint -, and Master constraint quantizations and their relations suggest a direct link relating canonical quantization to a path-integral formulation for a general constrained system.

- (Reduced phase space and operator constraint quantizations) → (a formal path-integral on the continuum) → (choose a triangulation) → (a certain spin-foam model, directly linking to the canonical physical inner product)

- (Non-graph-changing Master constraint quantization) → (a discrete path integral expression of the physical inner product, which is a mathematical well-defined quantity)

- The analysis for the spin-foam vertex consistent with the canonical theory is a research in progress.

- The graph-changing Master constraint quantization might hopefully relate canonical LQG (a background independent non-Abelian gauge theory) with a certain GFT directly, which is a future research project.
  (1) both theories are triangulation independent
  (2) both theories describe the interactions between spin-network vertices

- The relation between non-Abelian gauge theory and GFT might be universal.

Thanks !

Summary and Outlook

- Reduced phase space -, operator constraint -, and Master constraint quantizations and their relations suggest a direct link relating canonical quantization to a path-integral formulation for a general constrained system.
- (Reduced phase space and operator constraint quantizations) → (a formal path-integral on the continuum) → (choose a triangulation) → (a certain spin-foam model, directly linking to the canonical physical inner product)
- (Non-graph-changing Master constraint quantization) → (a discrete path integral expression of the physical inner product, which is a mathematical well-defined quantity)
- The analysis for the spin-foam vertex consistent with the canonical theory is a research in progress.
- The graph-changing Master constraint quantization might hopefully relate canonical LQG (a background independent non-Abelian gauge theory) with a certain GFT directly, which is a future research project.

  (1) both theories are triangulation independent

  (2) both theories describe the interactions between spin-network vertices
- The relation between non-Abelian gauge theory and GFT might be universal.