Title: Asymptotics of the Ponzano-Regge model for handlebodies

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Abstract: The asymptotic formula for the Ponzano-Regge model amplitude is given for non-tardis triangulations of handlebodies in the limit of large boundary spins. The formula produces a sum over all possible immersions of the boundary triangulation in three dimensional Euclidean space weighted by the cosine of the Regge action evaluated on these immersions. Furthermore the asymptotic scaling registers the existence of flexible immersions.

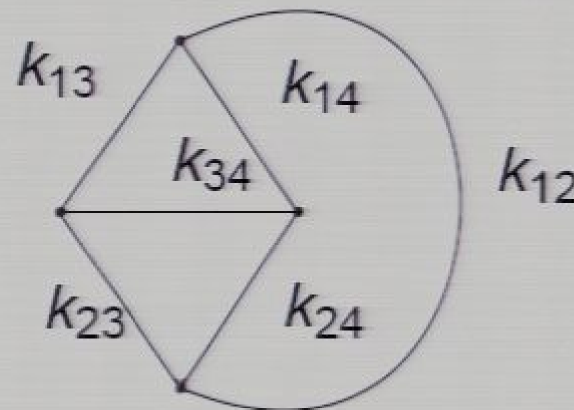
Plan

- 6j symbol and the Ponzano-Regge model
- Coherent intertwiners
- Asymptotic formula
- Flexible polyhedra

Asymptotics of the 6j symbol

6j symbol given by the $SU(2)$ spin network

$$\left\{ \begin{array}{ccc} k_{12} & k_{13} & k_{14} \\ k_{34} & k_{24} & k_{23} \end{array} \right\} = \qquad (1)$$



Ponzano and Regge gave an asymptotic formula

$$\left\{ \begin{array}{ccc} \lambda k_{12} & \lambda k_{13} & \lambda k_{14} \\ \lambda k_{34} & \lambda k_{24} & \lambda k_{23} \end{array} \right\} \to \frac{1}{\sqrt{12\pi \mathrm{Vol}}} \cos\left(\sum_{a<b} (\lambda k_{ab} + \frac{1}{2}) \Theta_{ab} + \frac{\pi}{4} \right)$$

For $\lambda \to \infty$

- Vol - volume of tetrahedron with edge lengths $\lambda k_{ab} + \frac{1}{2}$
- Θ_{ab} - dihedral angles of the tetrahedron

The Ponzano-Regge model
3d quantum gravity

On the triangulation $\mathcal{T}$ of 3-manifold-with-boundary Σ^3

$$\mathcal{Z}_{PR}(\Psi, \mathcal{T}) = \sum_{j_e} \prod_{\text{edges } e} \text{dim}(j_e) \prod_{\text{tetrahedra}} \{6j\} \quad (2)$$

Ψ is the boundary state - depends on boundary spins
Sum over internal spin label

- Spin foam model for 3d gravity
- Fourier transform of path integral for discrete $\mathrm{SU}(2)$ BF theory
- Topological invariant with regularization

One tetrahedron doesn't capture the dynamics (ie 4d BF)
What is the asymptotic formula for the whole model?

The Ponzano-Regge model

Spin network on the boundary

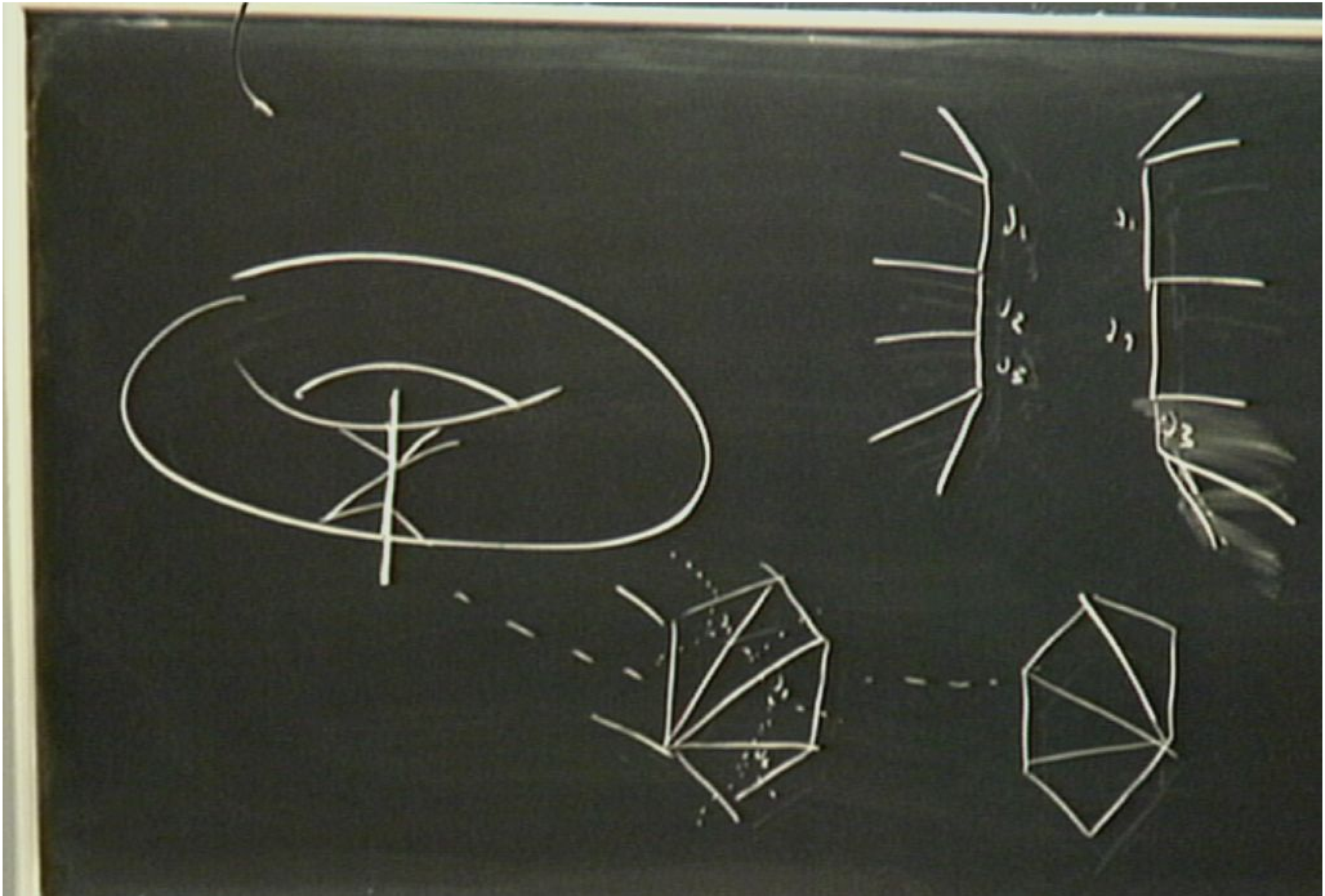
PR model was constructed to agree with boundary spin network on $\mathcal{S}^2$

Works for any handlebody of genus g

Reduce Σ^3 to B^3 with a set of cuts C

At each cut, replace the projector with a group integral

$$\int_{\mathrm{SU}(2)} dh\, D^{j_1}_{ab}(h) \otimes D^{j_2}_{cd}(h) \otimes D^{j_3}_{ef}(h) = \iota\iota^\dagger$$



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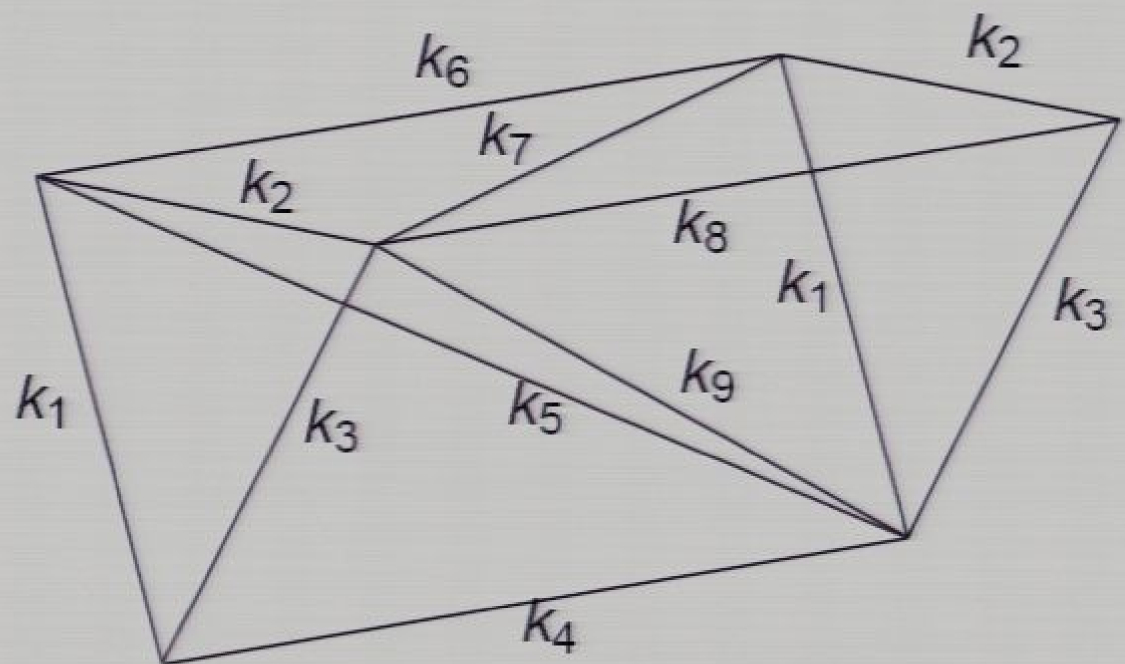
The Ponzano-Regge model

Spin network on the boundary. Example: the solid torus



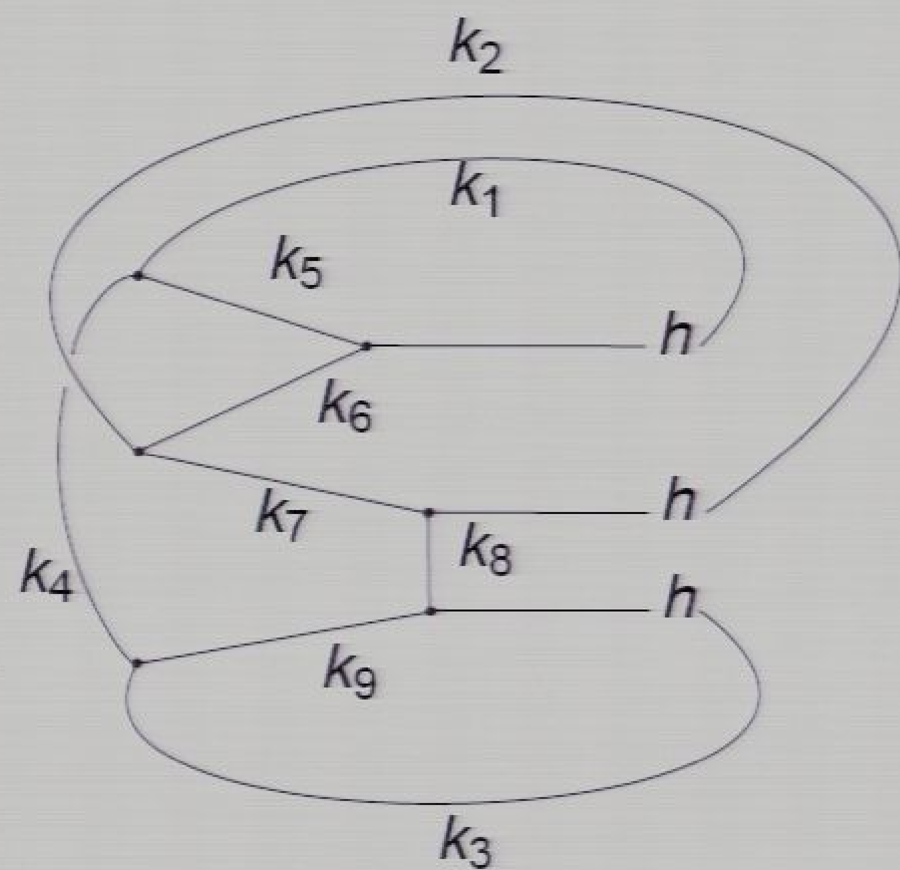
Identify k_1, k_2, k_3

$$\mathcal{Z}_{PR}(\Psi, \mathbb{T}) = \left\{ \begin{array}{ccc} k_1 & k_2 & k_3 \\ k_8 & k_9 & k_7 \end{array} \right\} \left\{ \begin{array}{ccc} k_1 & k_2 & k_3 \\ k_9 & k_4 & k_5 \end{array} \right\} \left\{ \begin{array}{ccc} k_1 & k_5 & k_6 \\ k_2 & k_7 & k_9 \end{array} \right\}. \quad (3)$$

The Ponzano-Regge model
Spin network on the boundary of the solid torus

PR amplitude can be expressed as spin network on the boundary

$$\mathcal{Z}_{PR}(\Psi, \mathbb{T}) = \int_{\mathrm{SU}(2)} dh$$

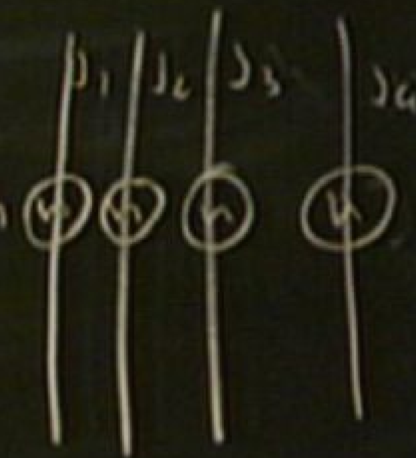


Non-planar diagram

$$\int dh\, D^{j_1}(h) \otimes D^{j_2}(h) \otimes D^{j_3}(h)$$

$$=$$

$$= \int dh$$



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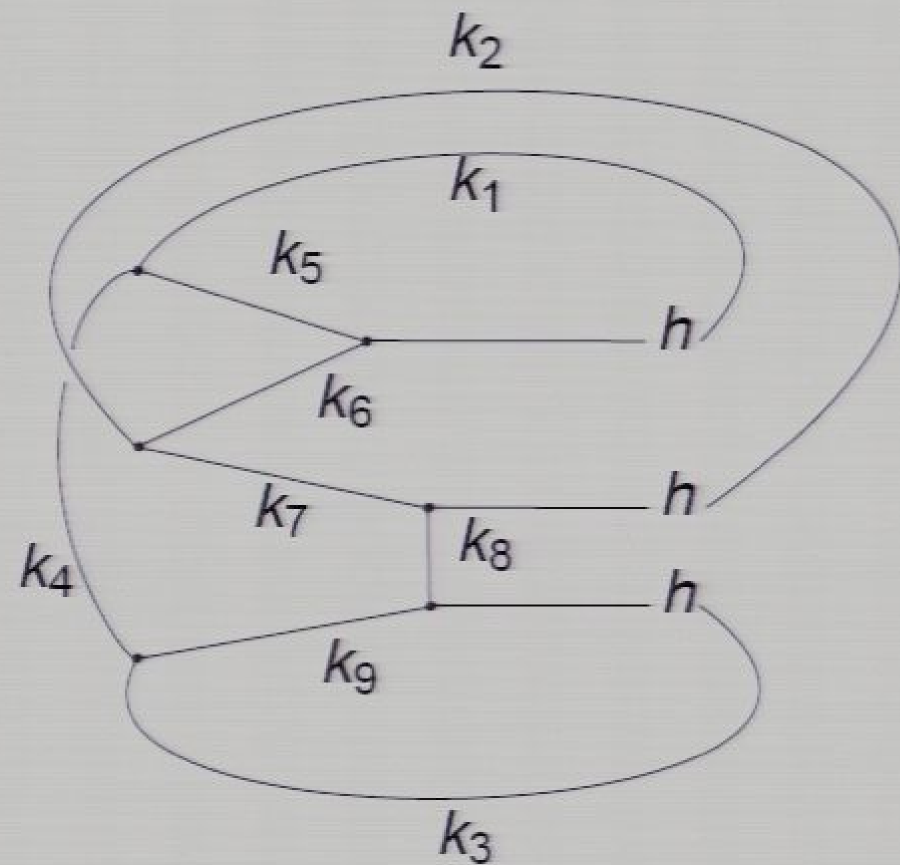


Non-planar diagram

Stationary phase approximation

We want to approximate the amplitude for large boundary k_{ab}
Use stationary phase approximation

$$\int dx\; a(x)\, e^{\lambda S(x)} \;\simeq\; \sum_{x_0} a(x_0) \left(\frac{2\pi}{\lambda}\right)^{n/2} \frac{1}{\sqrt{\det(-H)}}\, e^{\lambda S(x_0)}$$

- S complex valued function with $\mathrm{Re} S \leq 0$
- x_0 are critical points - $\delta S(x_0) = 0, \mathrm{Re} S(x_0) = 0$
- n is dimension of integral
- H is Hessian matrix

Need to rewrite the amplitude in a different basis.

Coherent triangles

SU(2) coherent states

SU(2) spin-k representation space denoted V_k:
A coherent state $|k, k\rangle$ has

- Minimum uncertainty
- Maximum projection in $(0, 0, 1)$ direction

A state $|k, \mathbf{n}\rangle := g(\mathbf{n})|k, k\rangle$ has maximum projection in $\mathbf{n}$ direction, ie

$$L.\mathbf{n}|k, \mathbf{n}\rangle = ik|k, \mathbf{n}\rangle$$

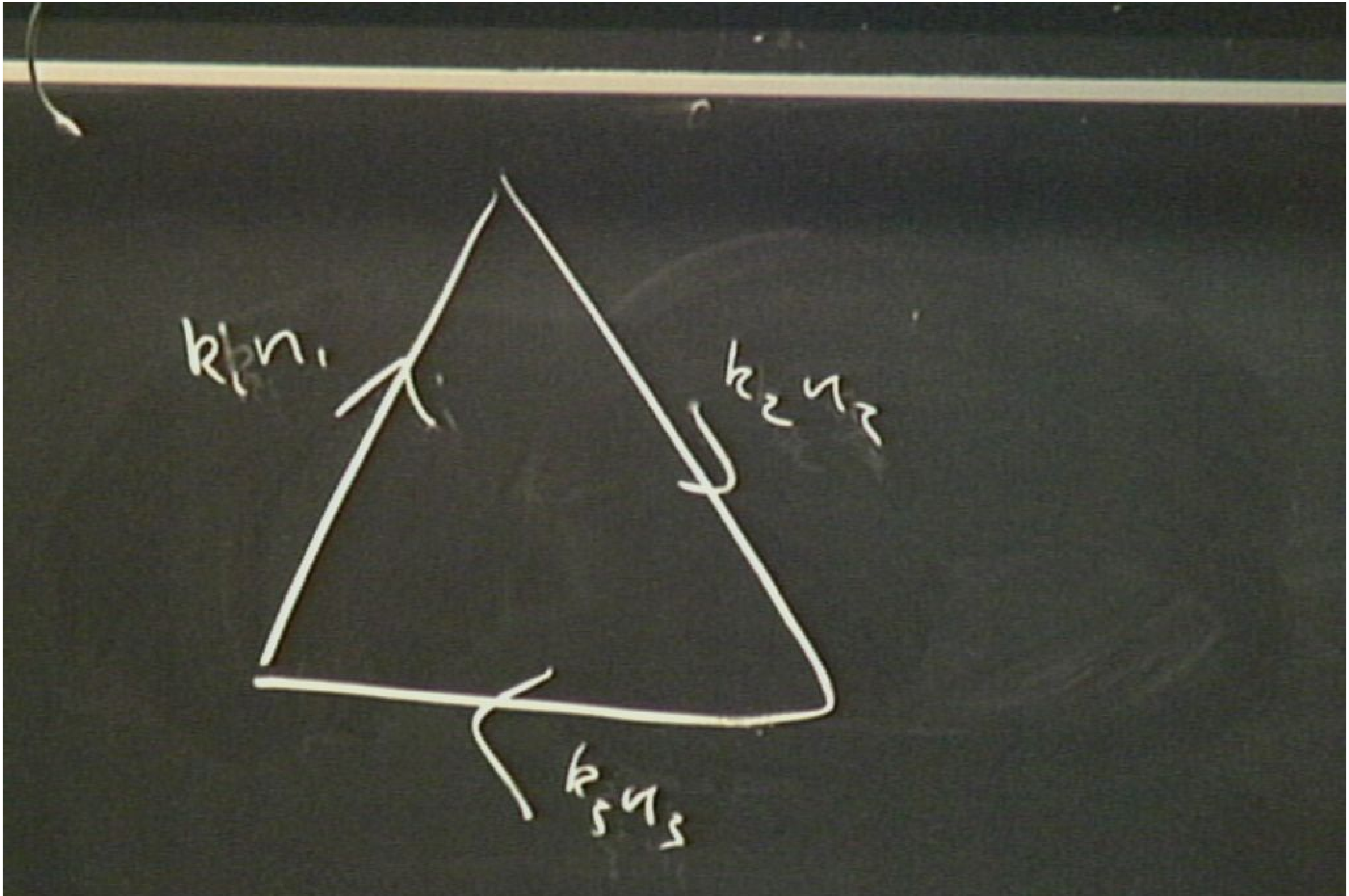
k and $\mathbf{n}$ specify $|k, \mathbf{n}\rangle$ up to a phase

Coherent triangles
Geometry of the boundary

We specify the geometry of a triangle, a, on the boundary with

- Lengths of edges k_{ab} (spins)
- Unit edge vectors $\mathbf{n}_{ab}$

This gives 3 coherent states $|k_{ab}, \mathbf{n}_{ab}\rangle \in V_{k_{ab}}$



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Coherent triangles
Coherent intertwiners

Up to phase specifies coherent intertwiner
$\iota_a \in \mathrm{Inv}(V_{k_{ab}} \otimes V_{k_{ac}} \otimes V_{k_{ad}})$

$$\iota_a = \int_{\mathrm{SU}(2)} \mathrm{d}X_a \bigotimes_{b:\, b\neq a} X_a|k_{ab}, \mathbf{n}_{ab}\rangle \tag{4}$$

How do we specify the phase?
Can pick the $\mathbf{n}_{ab}$ orthogonal to some vector, say $\mathcal{N} = (0, 0, 1)$
$\exists g_{ab} \in \mathrm{U}(1) \subset \mathrm{SU}(2)$ such that

$$\mathbf{n}_{ba} = -g_{ab} \triangleright \mathbf{n}_{ab}$$

So define

$$|k_{ab}, \mathbf{n}_{ba}\rangle := g_{ab}|k_{ab}, -\mathbf{n}_{ab}\rangle$$

With this choice, $\mathcal{Z}_{PR}$ does not depend on the phase choice

The Ponzano-Regge model
In the coherent state basis

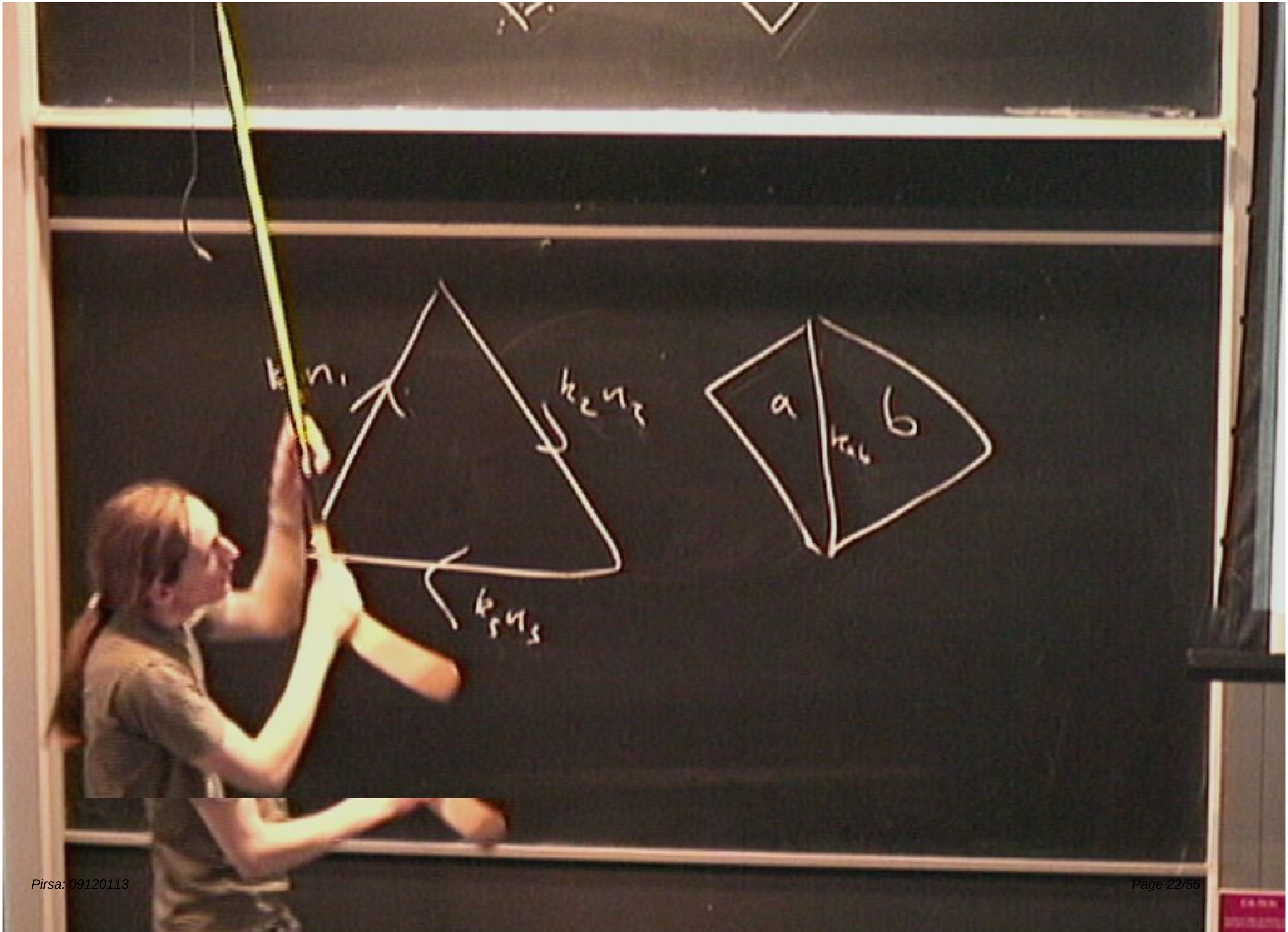
The amplitude now reads

$$\mathcal{Z}_{PR}(\Psi, \Sigma^3) = \int \prod_{\text{triangles } i} dX_i \prod_{\text{cuts } j} dh_j e^S \tag{5}$$

with the action

$$\begin{aligned} S \;=\; & \sum_{\text{edges } ab} 2k_{ab} \ln \langle -\mathbf{n}_{ab} | X_a^\dagger X_b | \mathbf{n}_{ba} \rangle \\ & + \sum_{\text{cuts } l} \sum_{\substack{\text{edges} \\ de \in l}} 2k_{de} \ln \langle -\mathbf{n}_{de} | X_d^\dagger h_l X_e | \mathbf{n}_{ed} \rangle . \end{aligned} \tag{6}$$

Can now use stationary phase analysis


a
b

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Can now use stationary phase analysis

Stationary phase

Amplitude dominated by the critical points

$$\delta S = 0, \quad \mathrm{Re} S = 0$$

This gives the conditions:

$$\hat{X}_a \mathbf{n}_{ab} = -\hat{X}_b \mathbf{n}_{ba} \quad \text{Gluing}$$

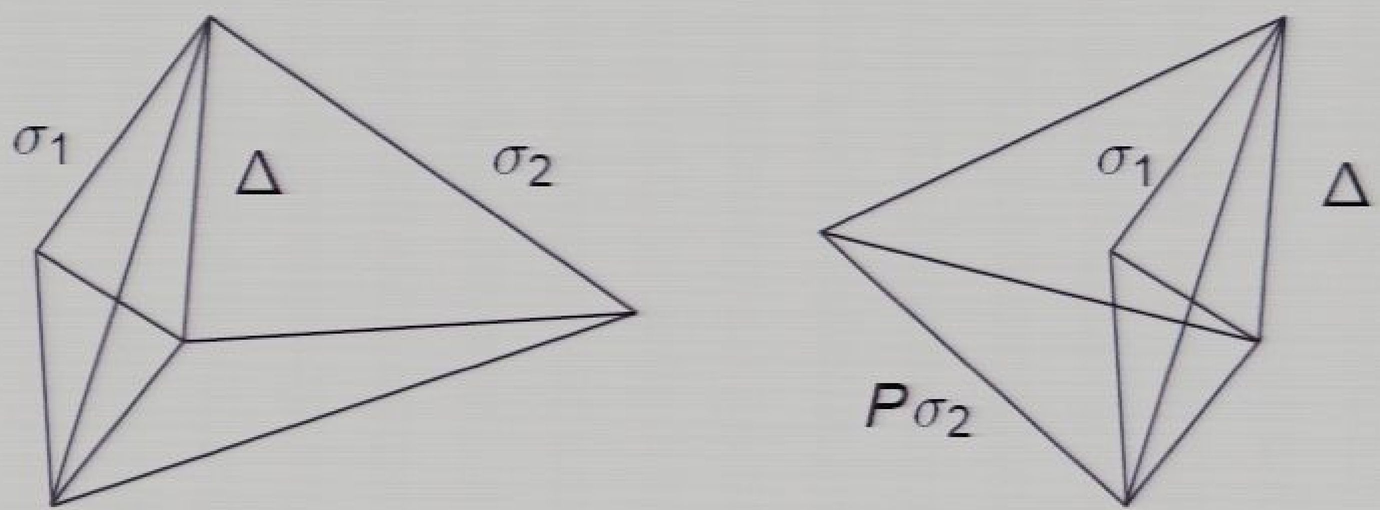
$$\sum_{b:\, b \neq a} k_{ab}\, \mathbf{n}_{ab} = 0 \quad \text{Closure}$$

$$\sum_{ab \in C_i} k_{ab}\, \hat{X}_a \mathbf{n}_{ab} = 0 \quad \text{Closure on cuts}$$

Each critical point gives an immersion of boundary $k_{ab}\mathbf{n}_{ab}$ in $\mathbb{R}^3$

Stationary points
Example: 3-ball

Triangulate $\mathcal{B}^3$ with two tetrahedra σ_1, σ_2

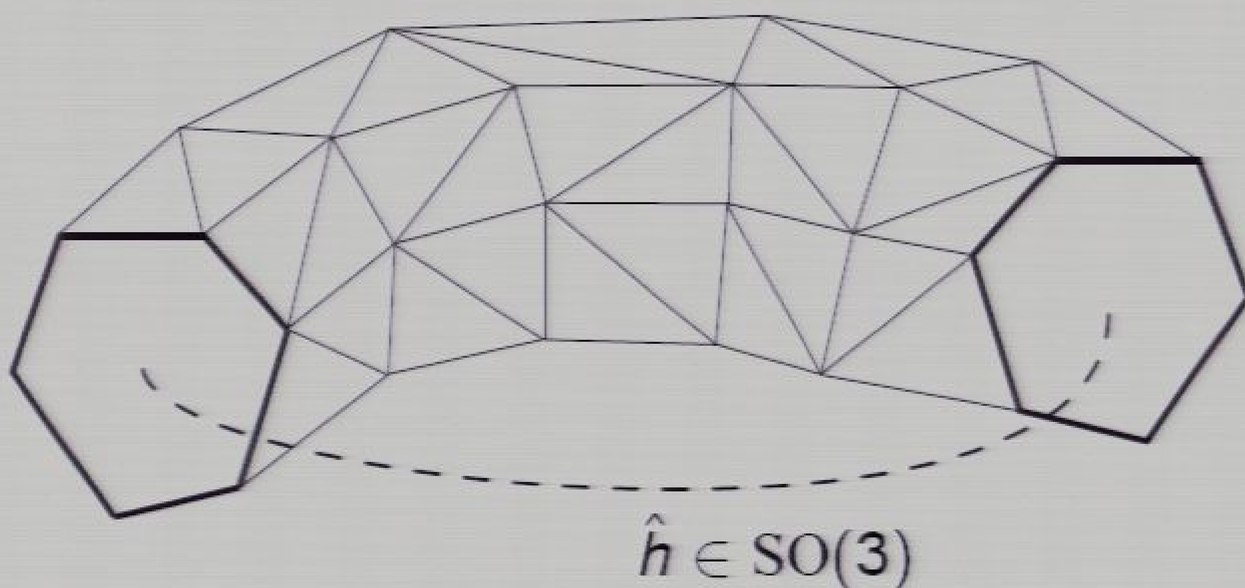


Two different possible immersions in the asymptotic formula
Parity related immersions also contribute

Stationary points

Example: $g > 0$ The solid torus

Cut immersions of the type shown below contribute



Stationary points

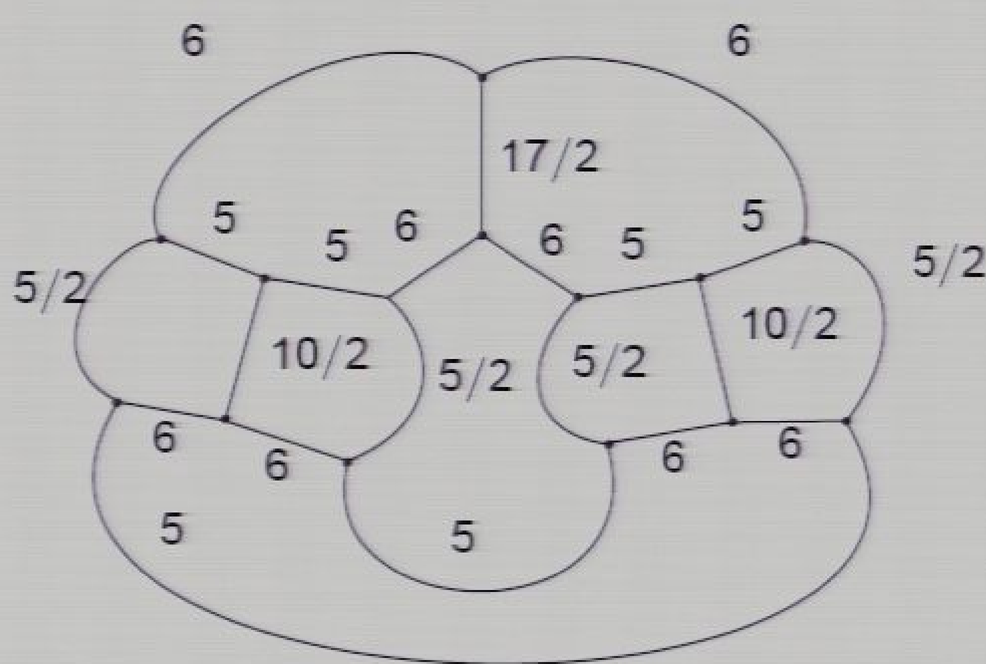
Non-isolated critical points and flexible polyhedra

Critical points may not be isolated - ie manifold of critical points

Corresponds to a flexible immersion

Example: Steffen's flexible polyhedron

http://demonstrations.wolfram.com/SteffensFlexiblePolyhedron/



Spin network for Steffen's polyhedron

Stationary Phase

The action at the critical points

At critical points

$$\begin{aligned} S &= \sum_{\text{edges } ab} k_{ab} \Theta^{\mathrm{i}}_{ab} \\ &= S_{\mathrm{Regge}}(\mathrm{i}) \end{aligned}$$

Regge action of the immersion

Use choice of spin structure to fix some minus signs

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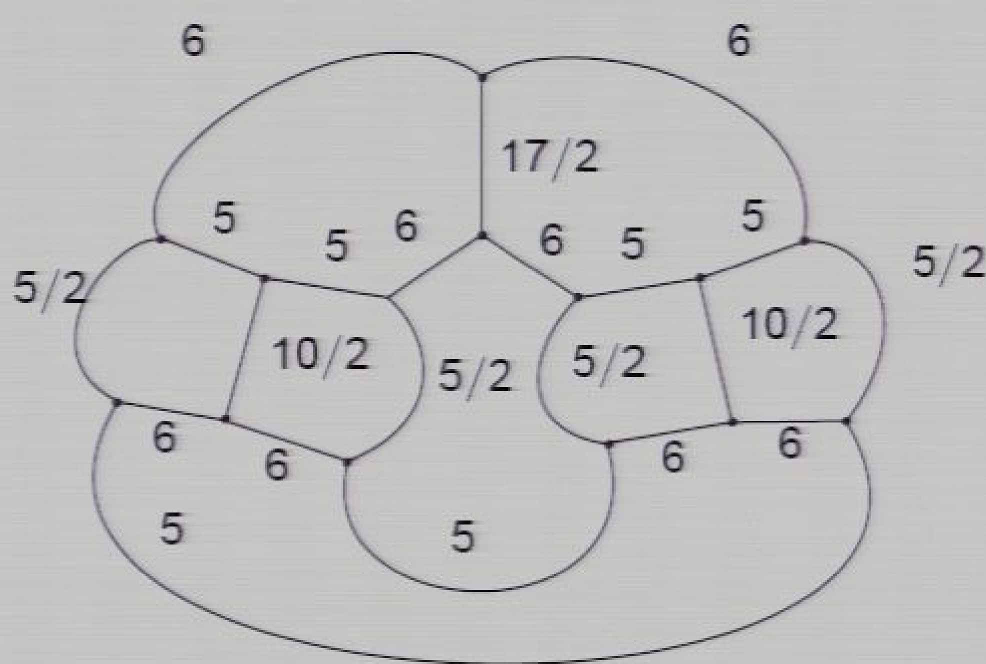

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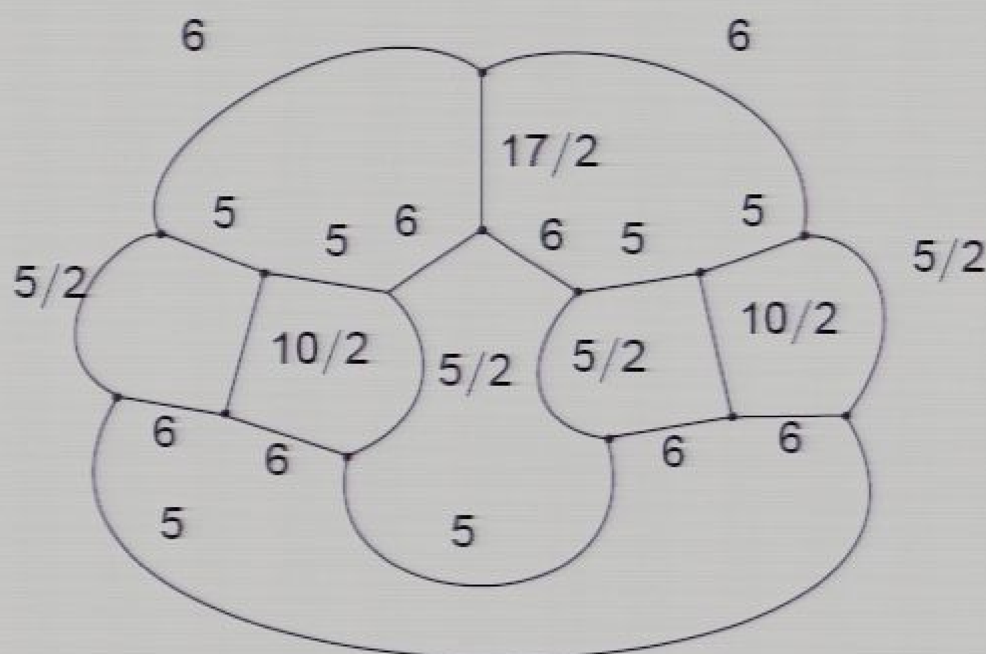

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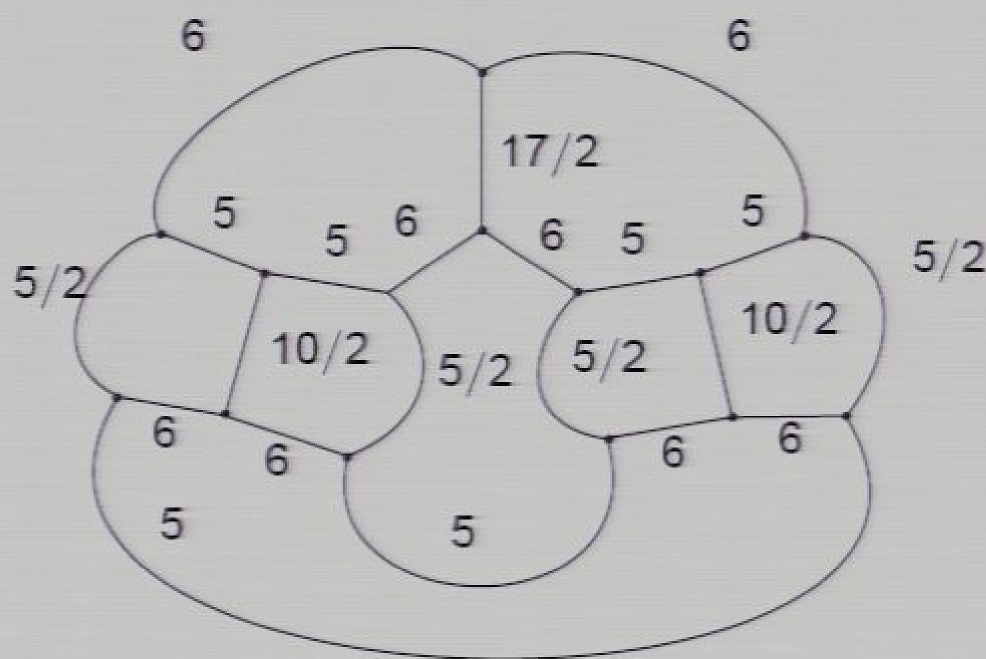

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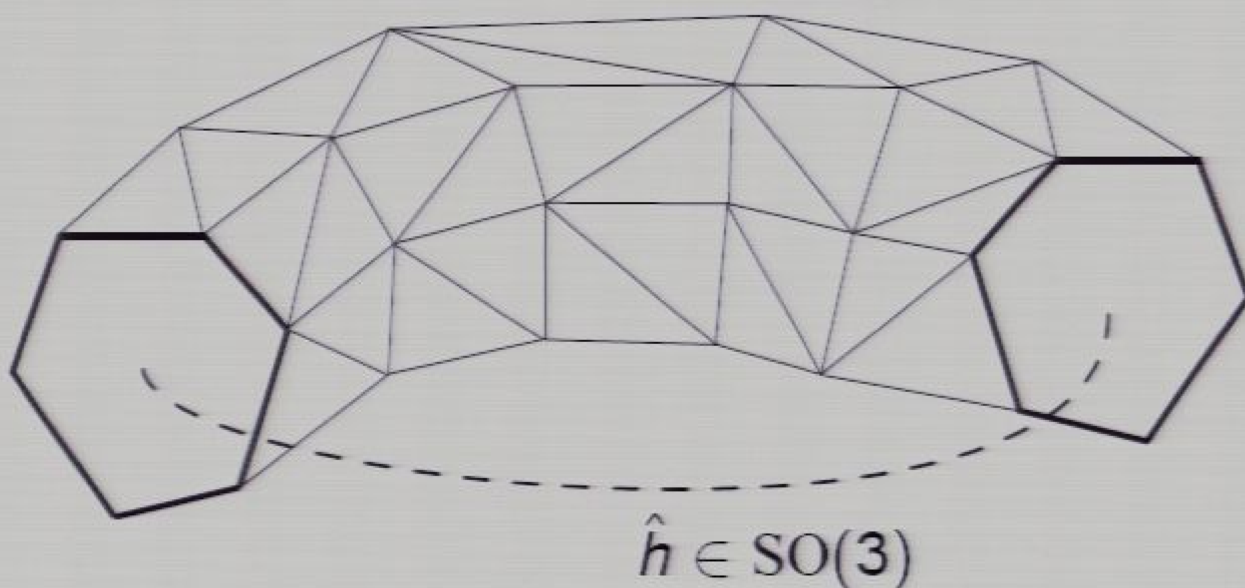
Example: $g > 0$ The solid torus

Cut immersions of the type shown below contribute



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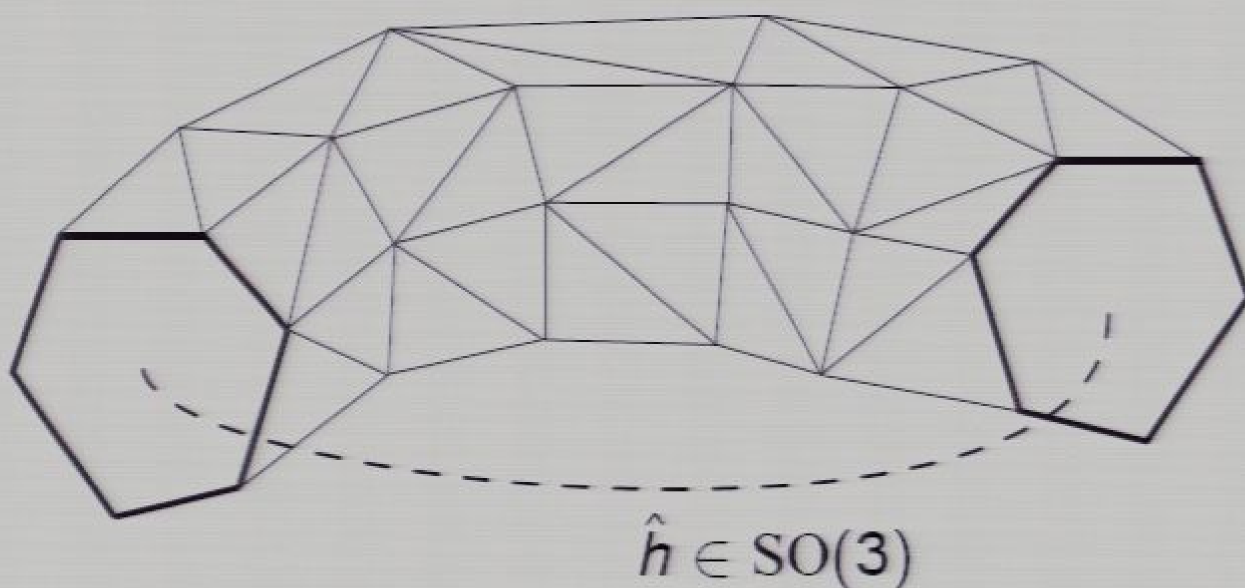

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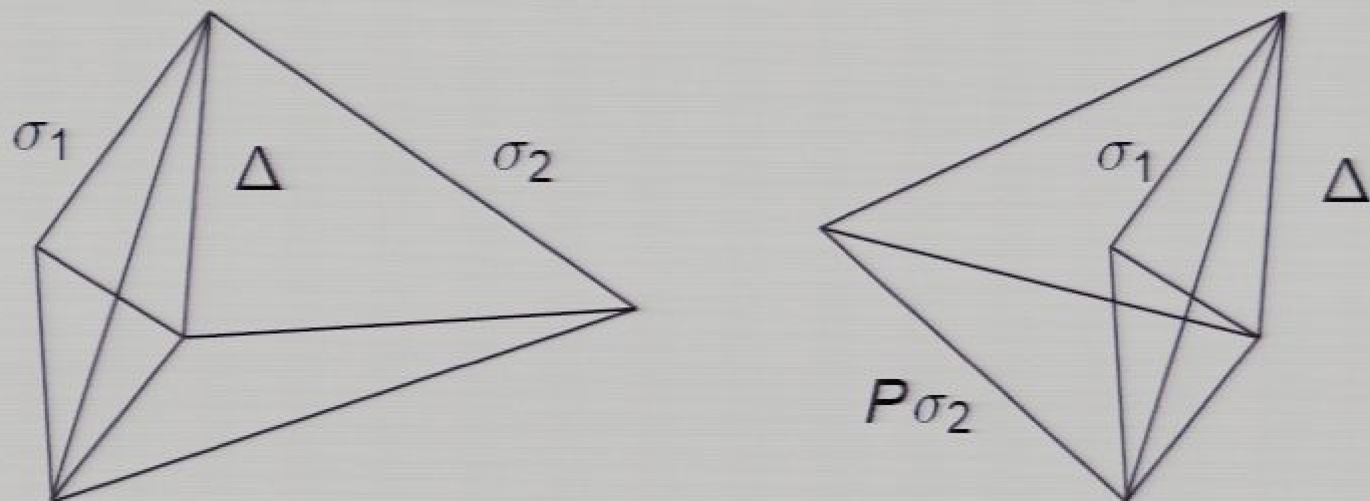

Two different possible immersions in the asymptotic formula

Parity related immersions also contribute

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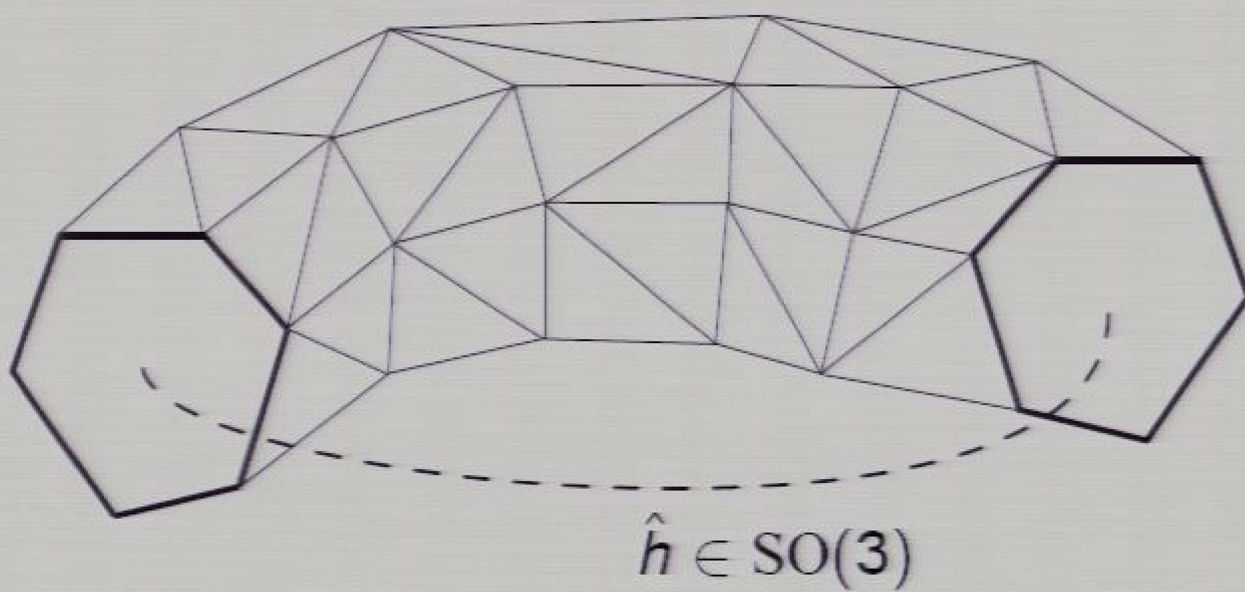

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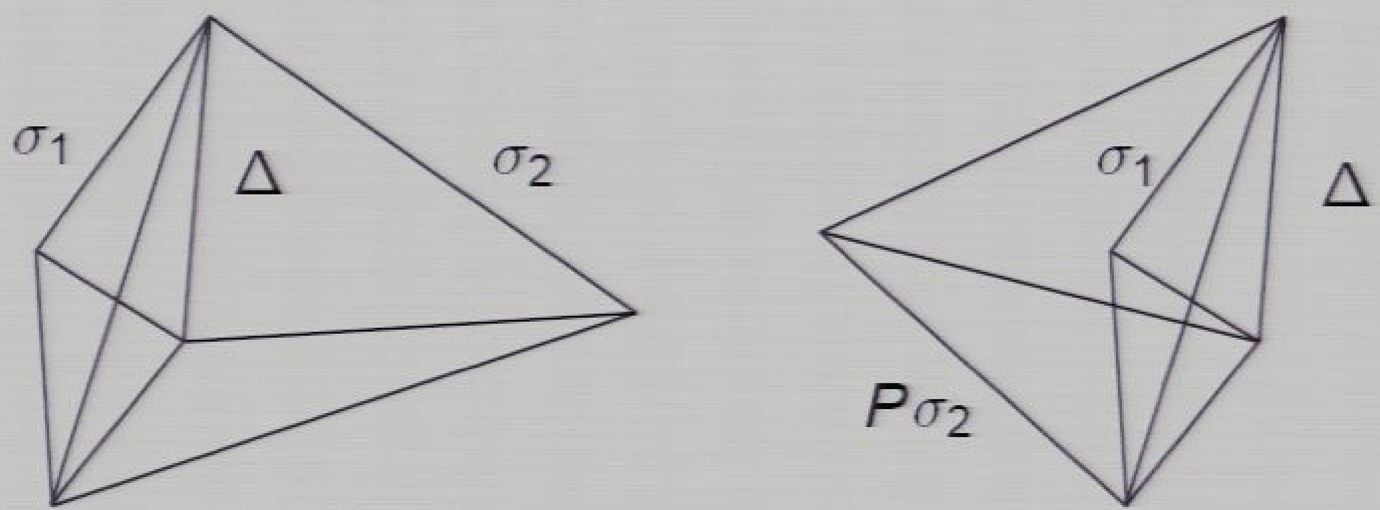

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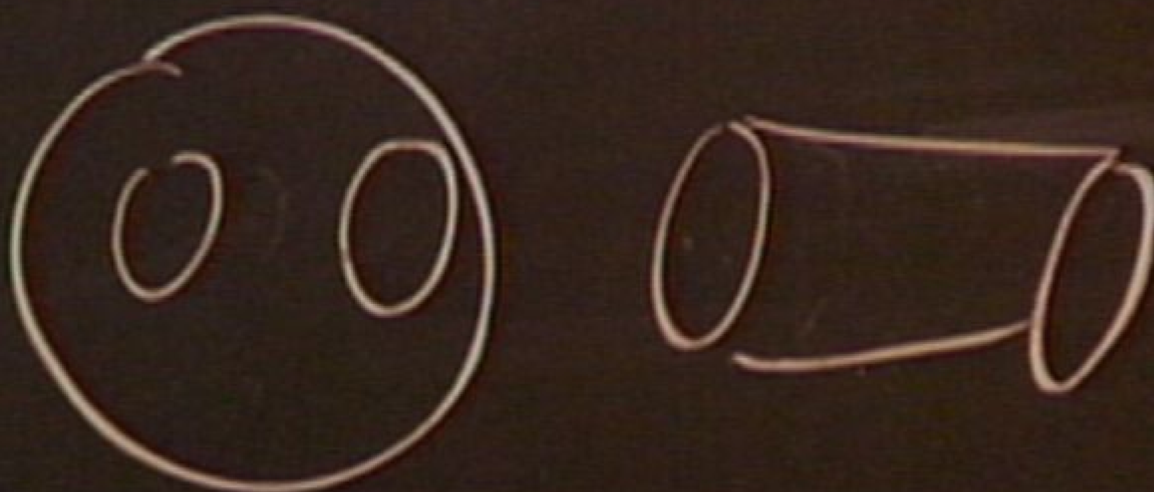

$$X_a n_{ab} = - X_b h n_{ba}$$

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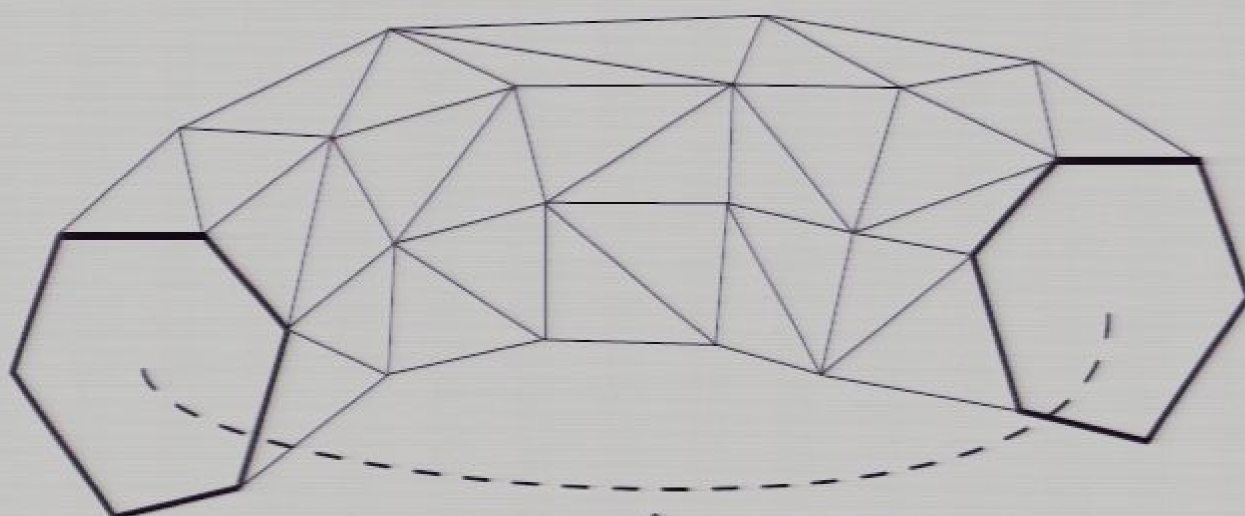

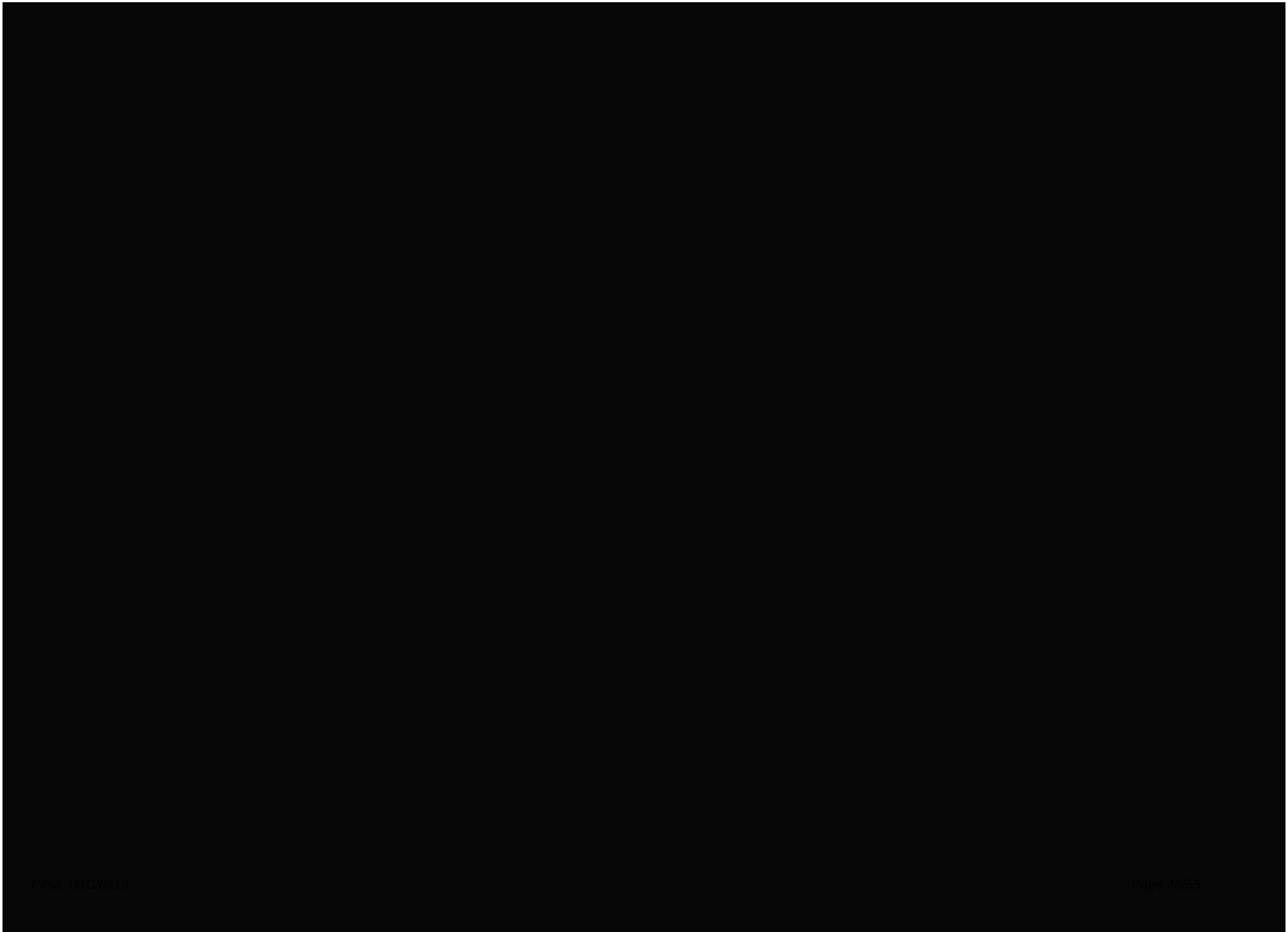

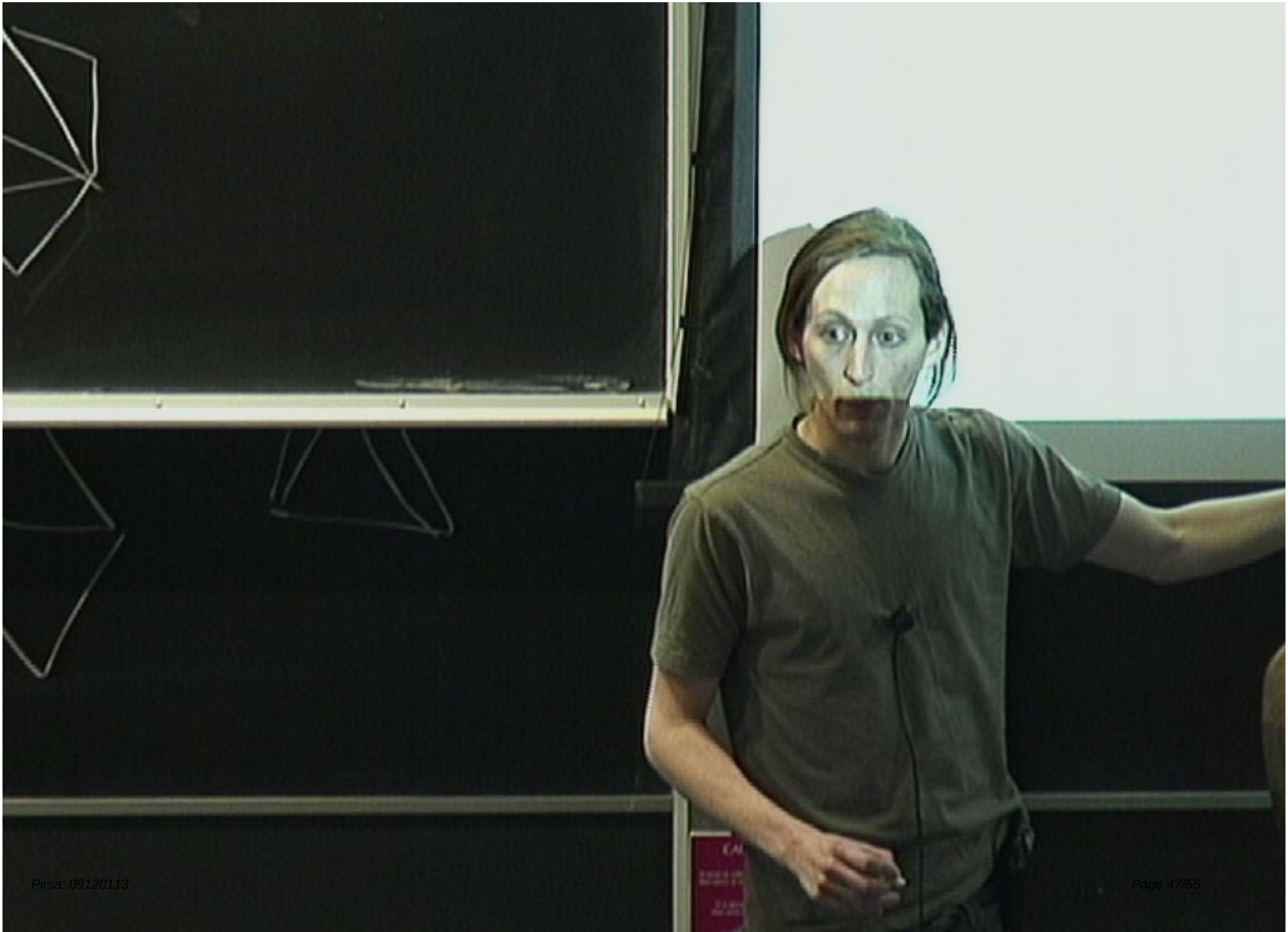

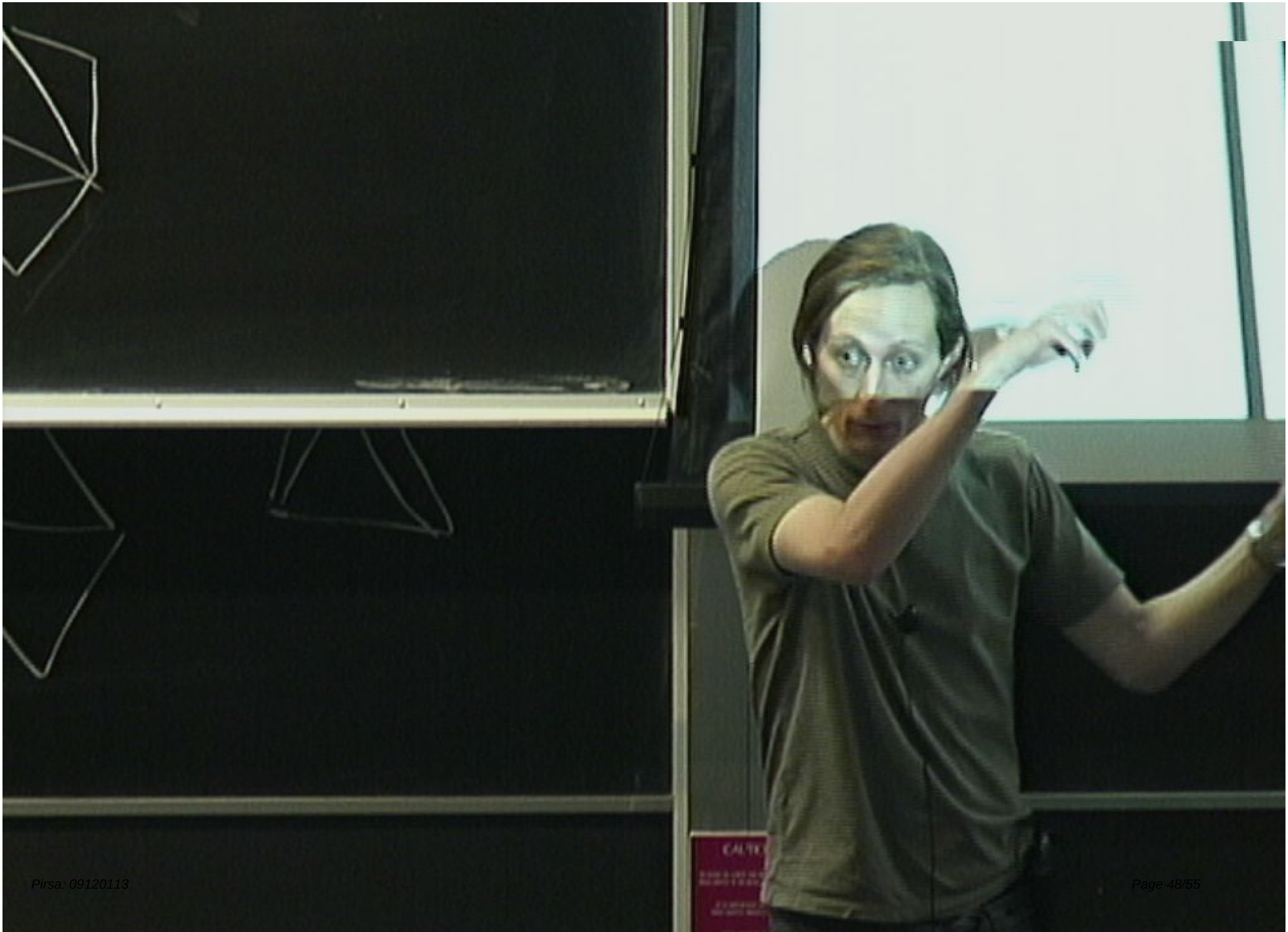

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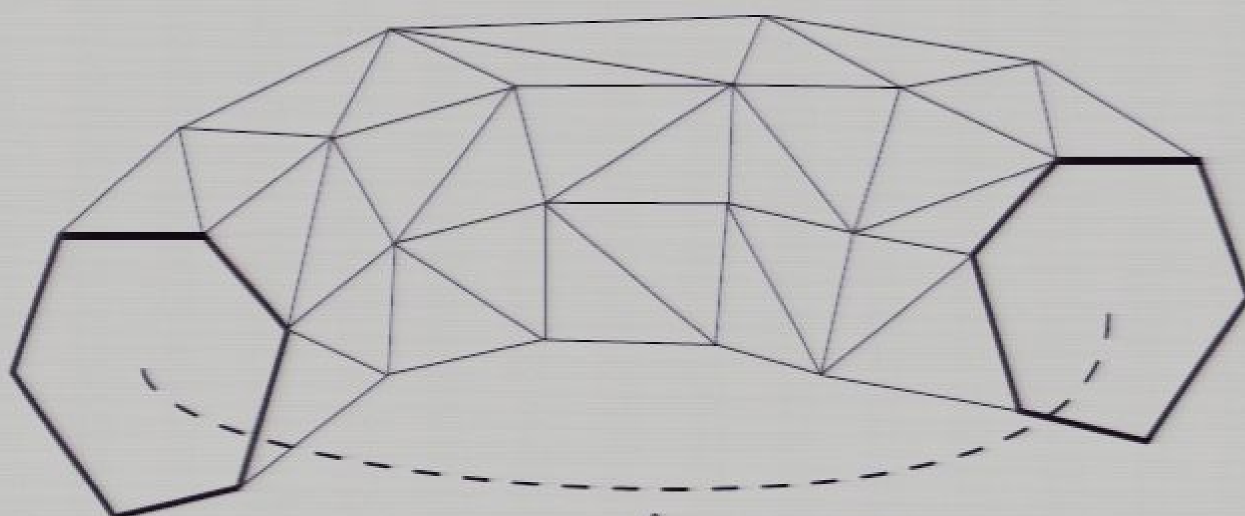

$\hat{h} \in SO(3)$

Asymptotic formula

For isolated critical points
When $\lambda \to \infty$

$$Z_{PR}(\psi_\lambda, \Sigma^3) = \left(\frac{2\pi}{\lambda}\right)^{\frac{3(|V|+|C|-1)}{2}} \sum_{\text{Immersions } \mathfrak{i}} N_{\mathfrak{i}} \cos\left(\lambda S_{\text{Regge}}(\mathfrak{i}) + \phi^{\mathfrak{i}}_{ab}\right)$$

- $\Theta^{\mathfrak{i}}_{ab}$ are the dihedral angles of $\mathfrak{i}$
- $|V|$ is number of triangles, $|C|$ is number of cuts

Stationary Phase

The action at the critical points

At critical points

$$S = \sum_{\text{edges } ab} k_{ab} \Theta^{\mathrm{i}}_{ab}$$

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Regge action of the immersion

Use choice of spin structure to fix some minus signs

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Asymptotic formula

Non-isolated critical points

$$\mathcal{Z}_{PR}(\psi_\lambda, \Sigma^3) = \left(\frac{2\pi}{\lambda}\right)^{\frac{3(|V|+|C|-1)-d}{2}} \sum_{\substack{\text{flexible} \\ \text{immersions } \mathfrak{f}}} N_{\mathfrak{f}} \cos\left(\lambda S_{\text{Regge}}(\mathfrak{f}) + \phi^{\mathfrak{f}}_{ab}\right)$$

- d is dimension of critical manifold
- Flexible immersions dominate
- Will flexibility be an issue in 4 dimensions?

Outlook

Refinement of the triangulation

What happens when we refine the boundary triangulation?

$$S_{Regge} \to S_{Einstein}$$

- Many more terms in the sum over immersions
- Related to immersions of manifolds with smooth metrics?

Stationary Phase
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Regge action of the immersion
Use choice of spin structure to fix some minus signs