Title: Building Massless Tree Amplitudes without a Lagrangian

Date: Dec 02, 2009 11:00 AM

URL: http://pirsa.org/09120098

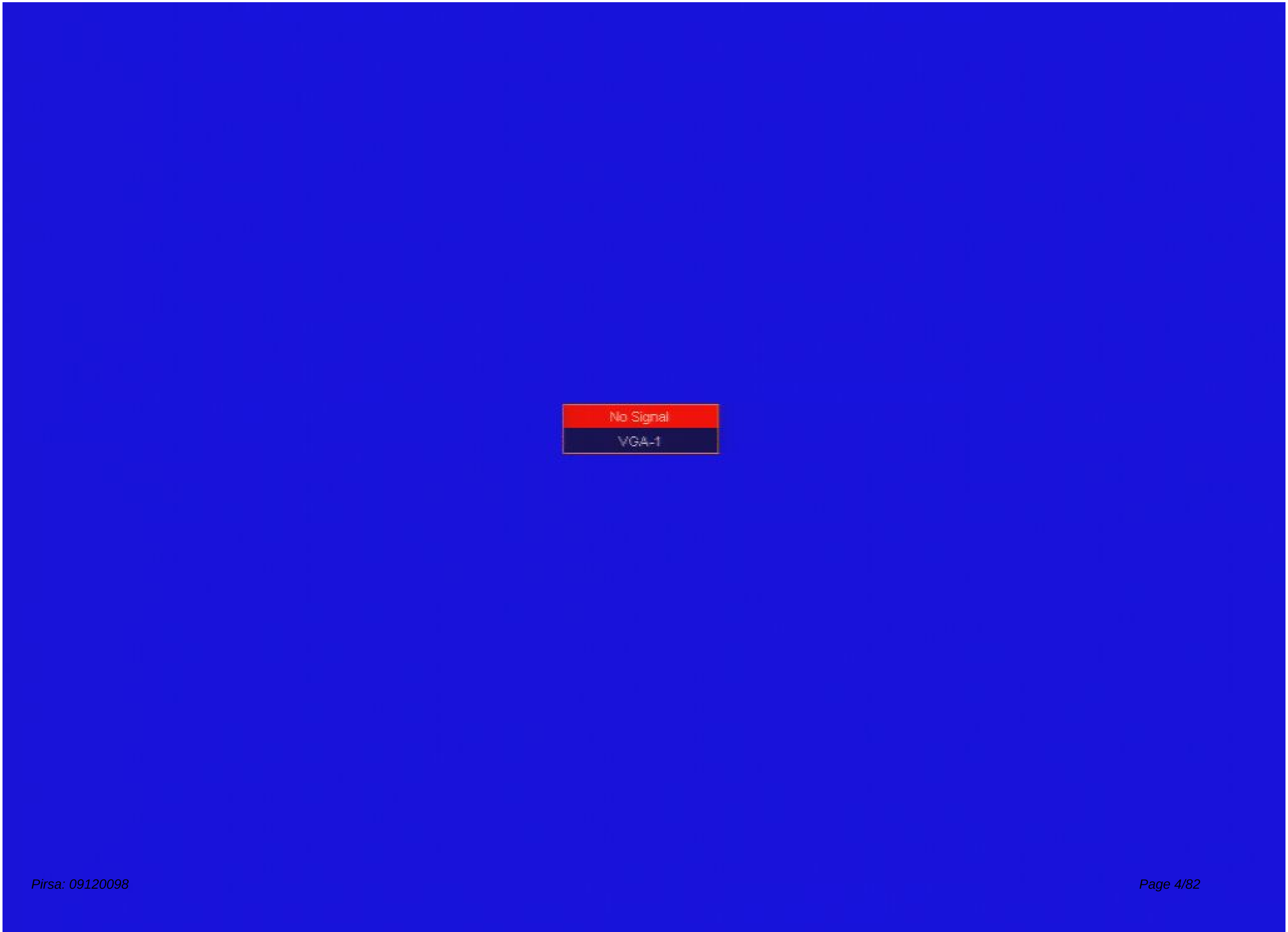
Abstract: The BCFW recursion relations define Yang-Mills and gravity amplitudes in terms of lower-point amplitudes. I will discuss several connections between the internal consistency of this recursive definition and the allowed interactions of massless, higher-spin particles.

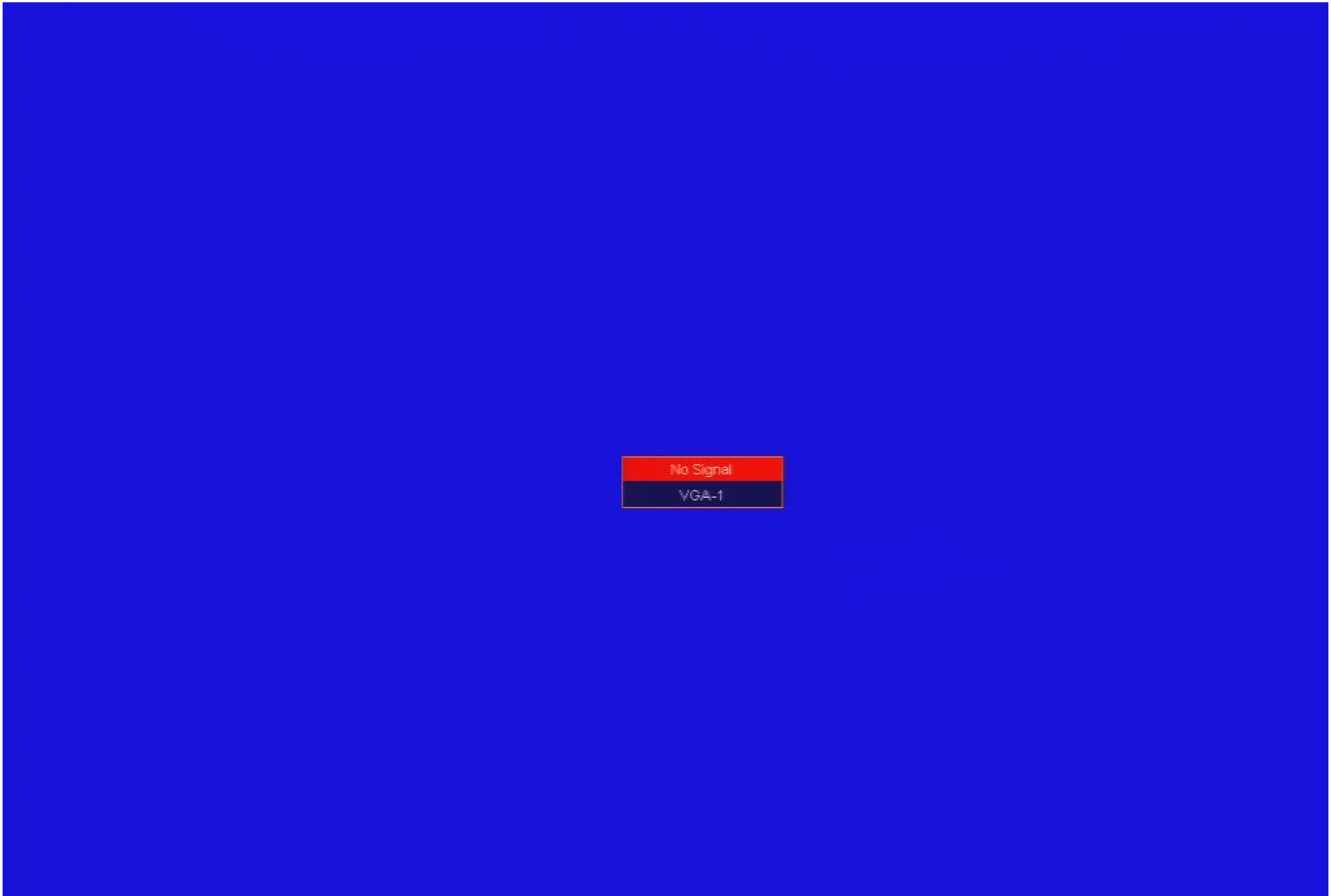
No Signal

VGA-1

No Signal

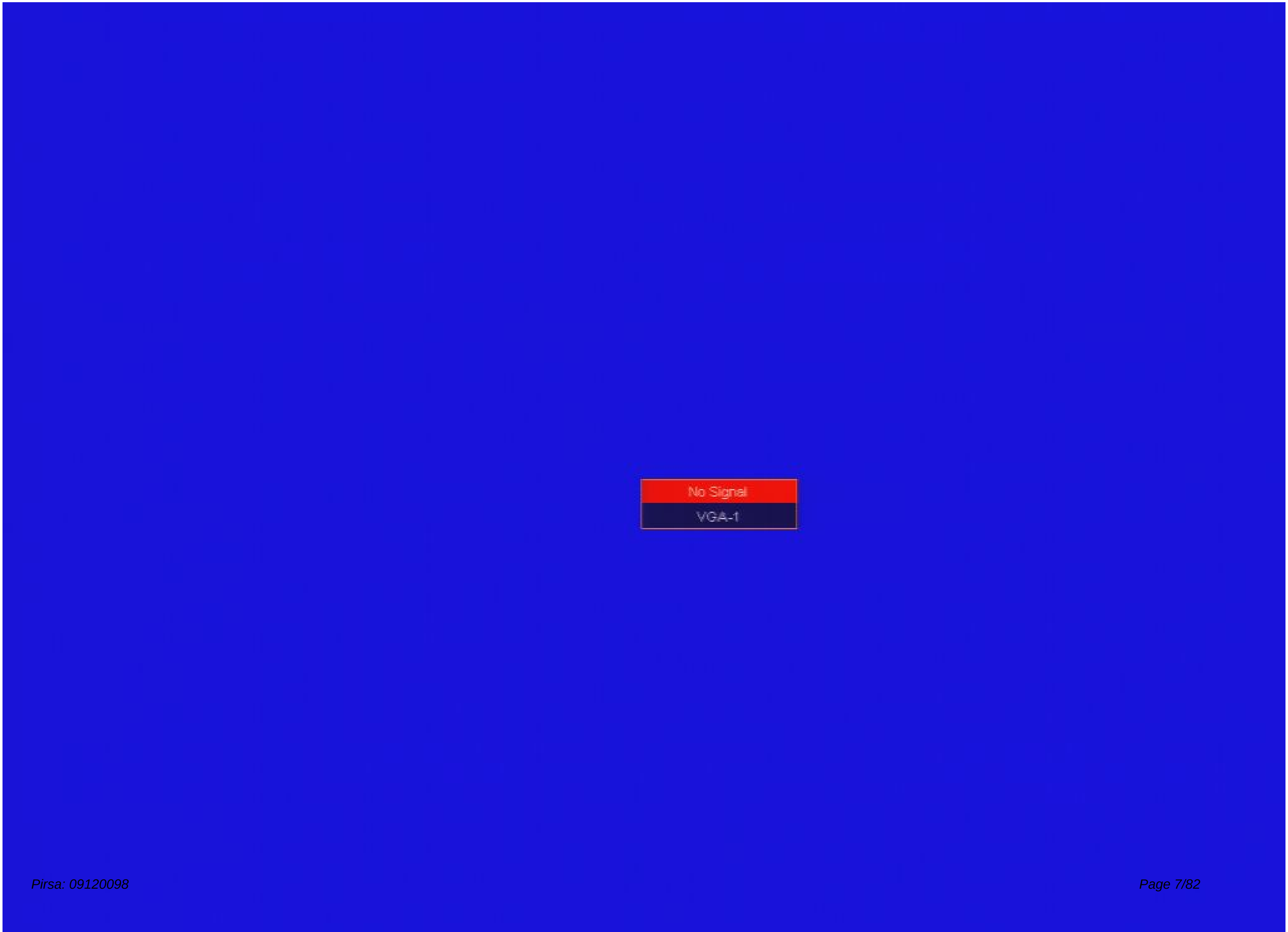
VGA-1

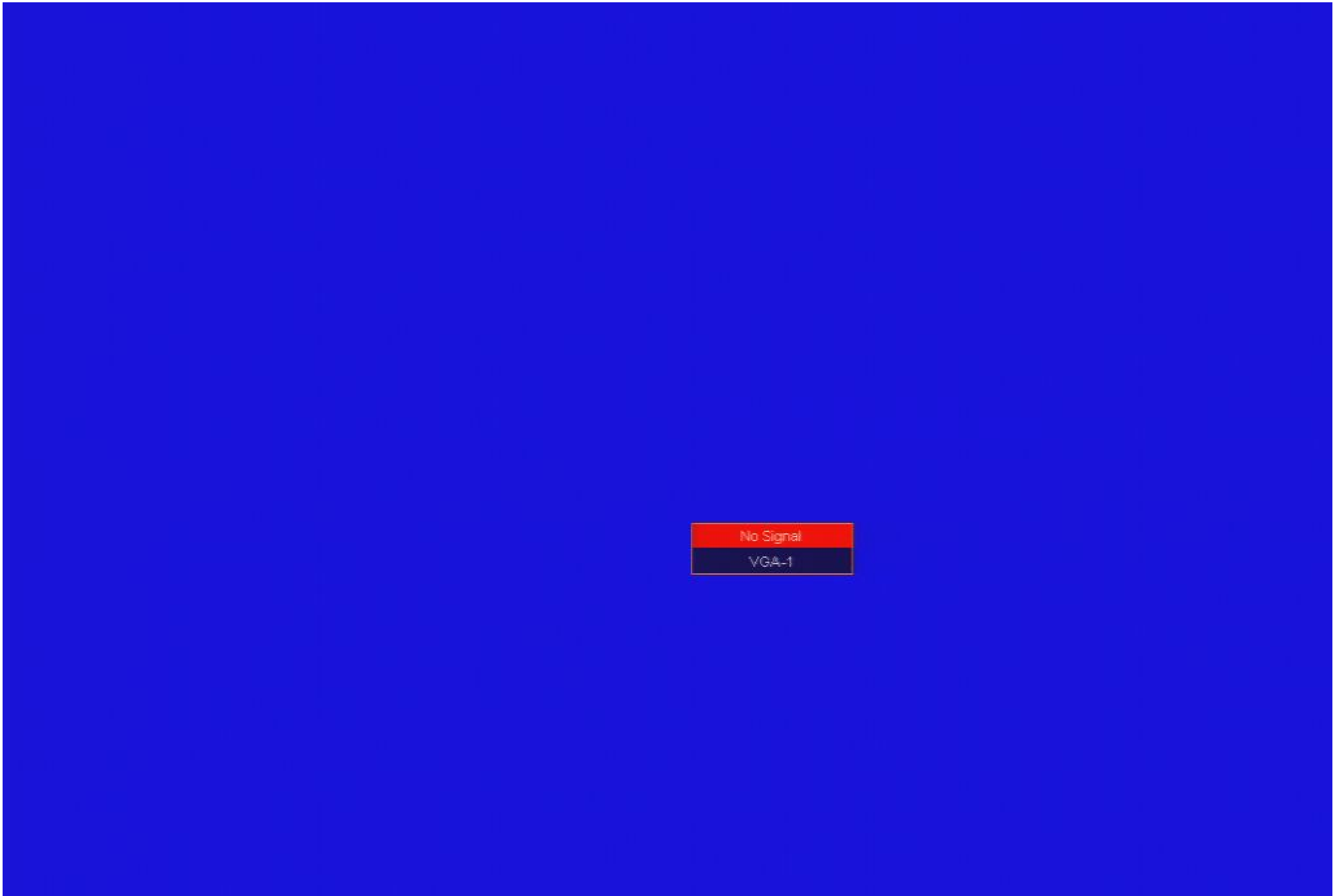

No Signal
VGA-1


No Signal
VGA-1

No Signal

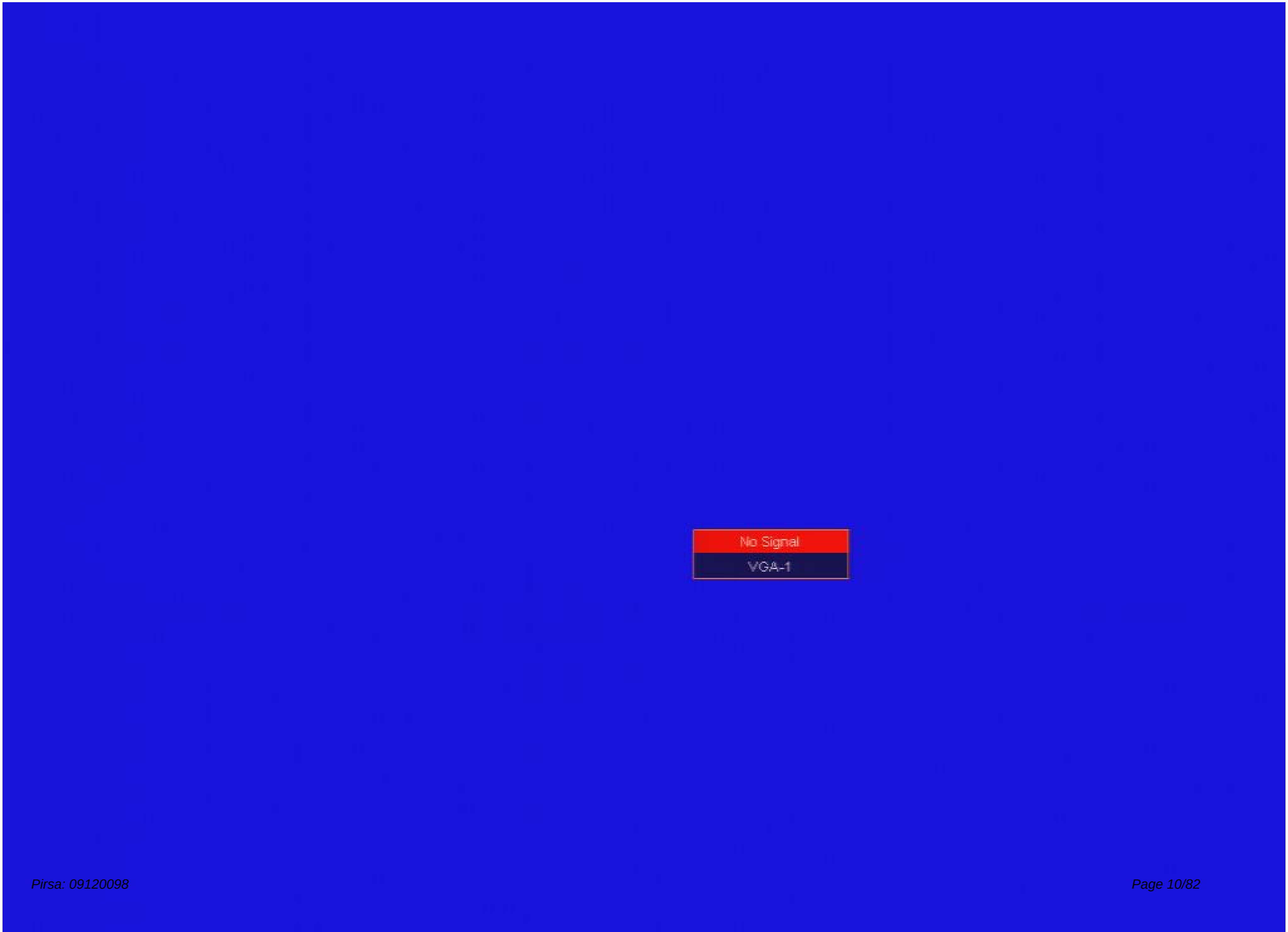
VGA-1

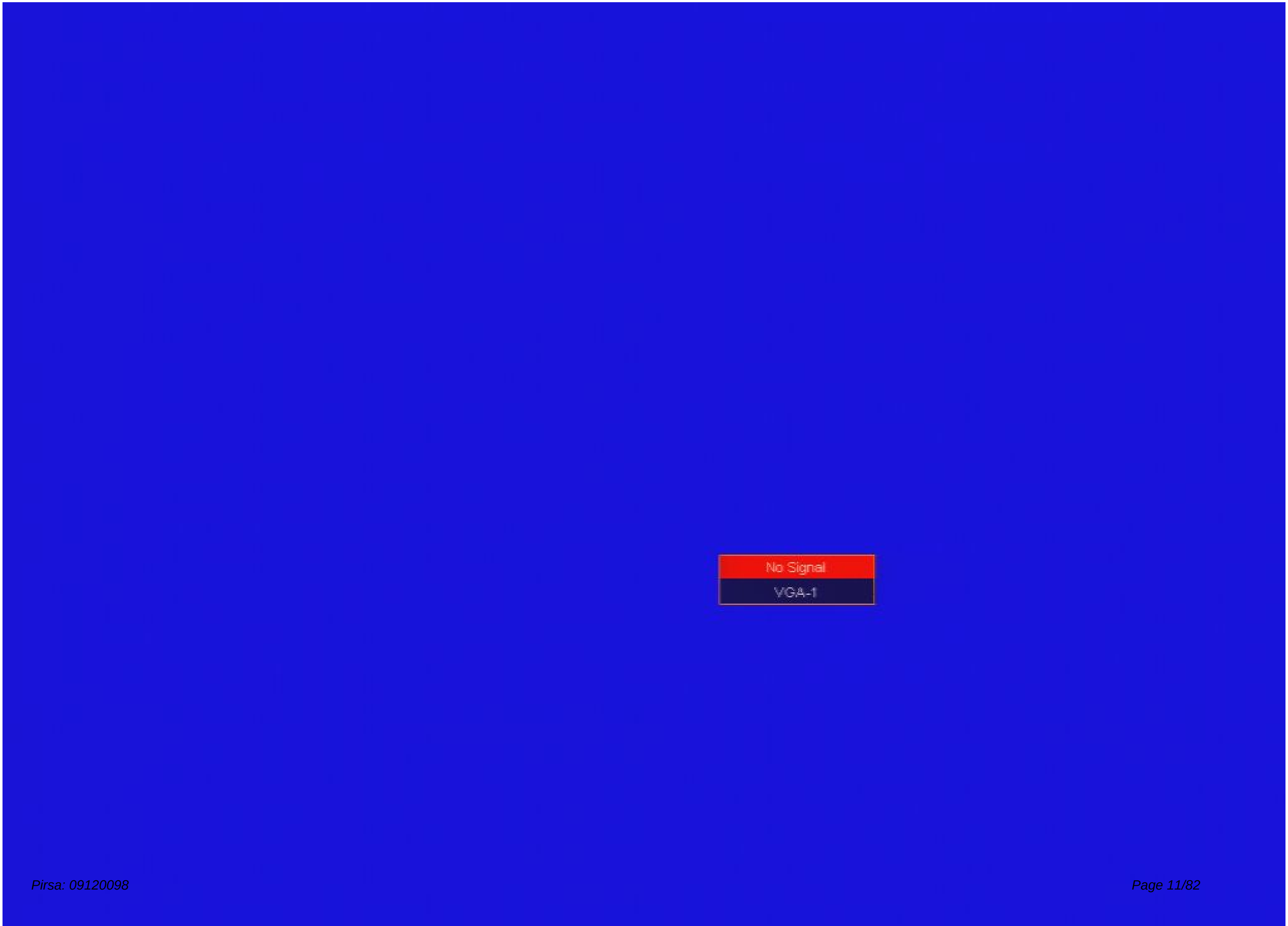

No Signal
VGA-1

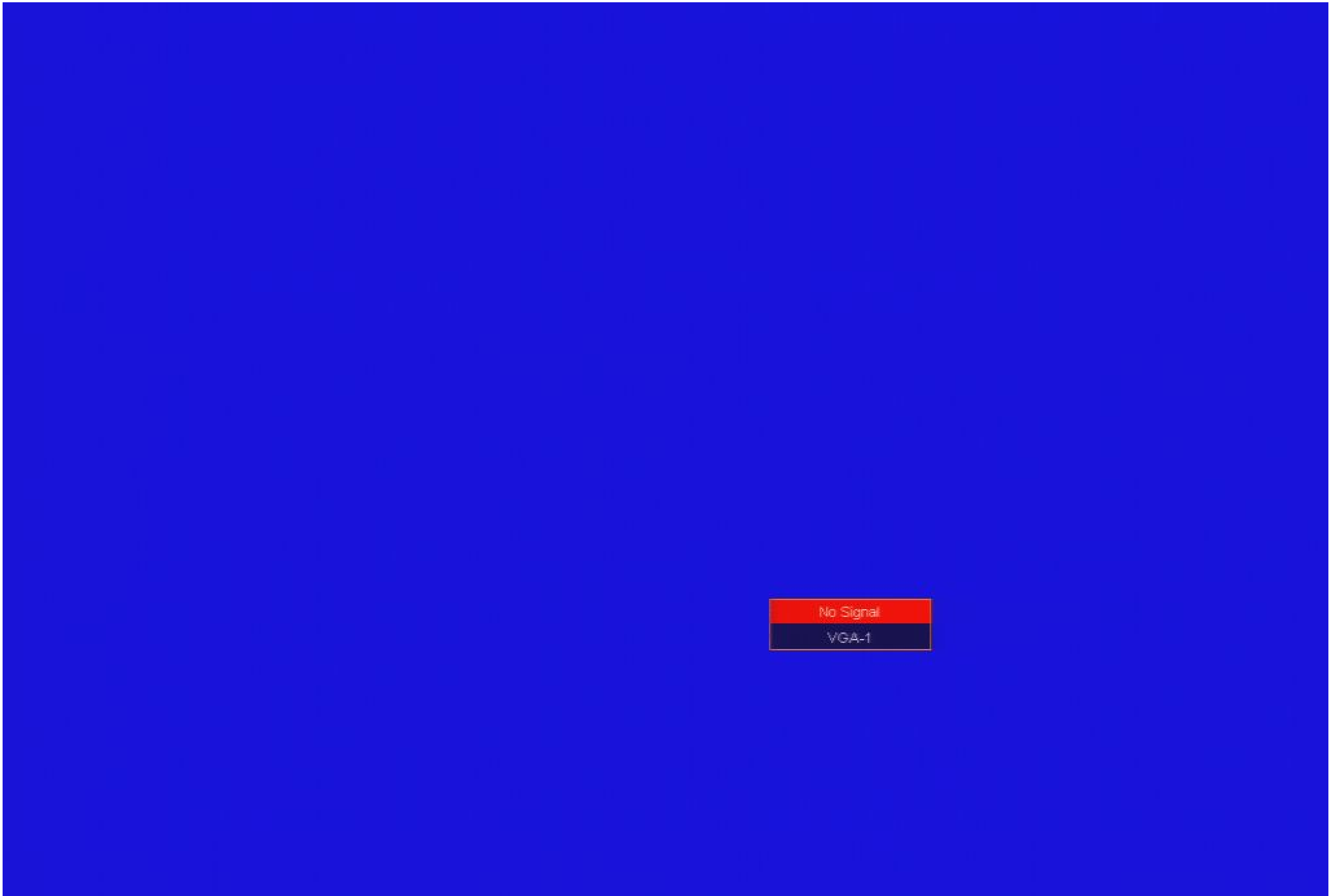

No Signal
VGA-1

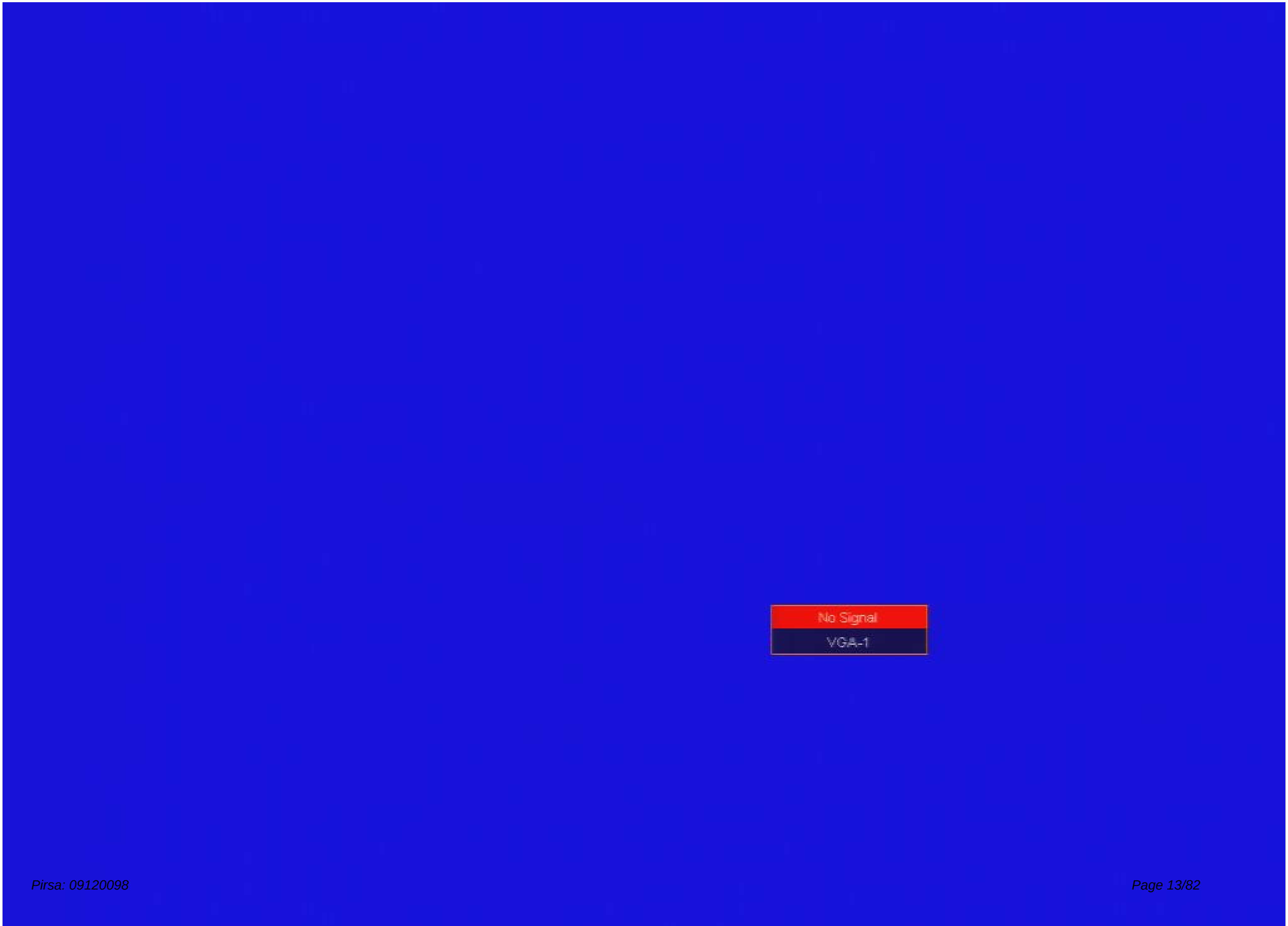
No Signal

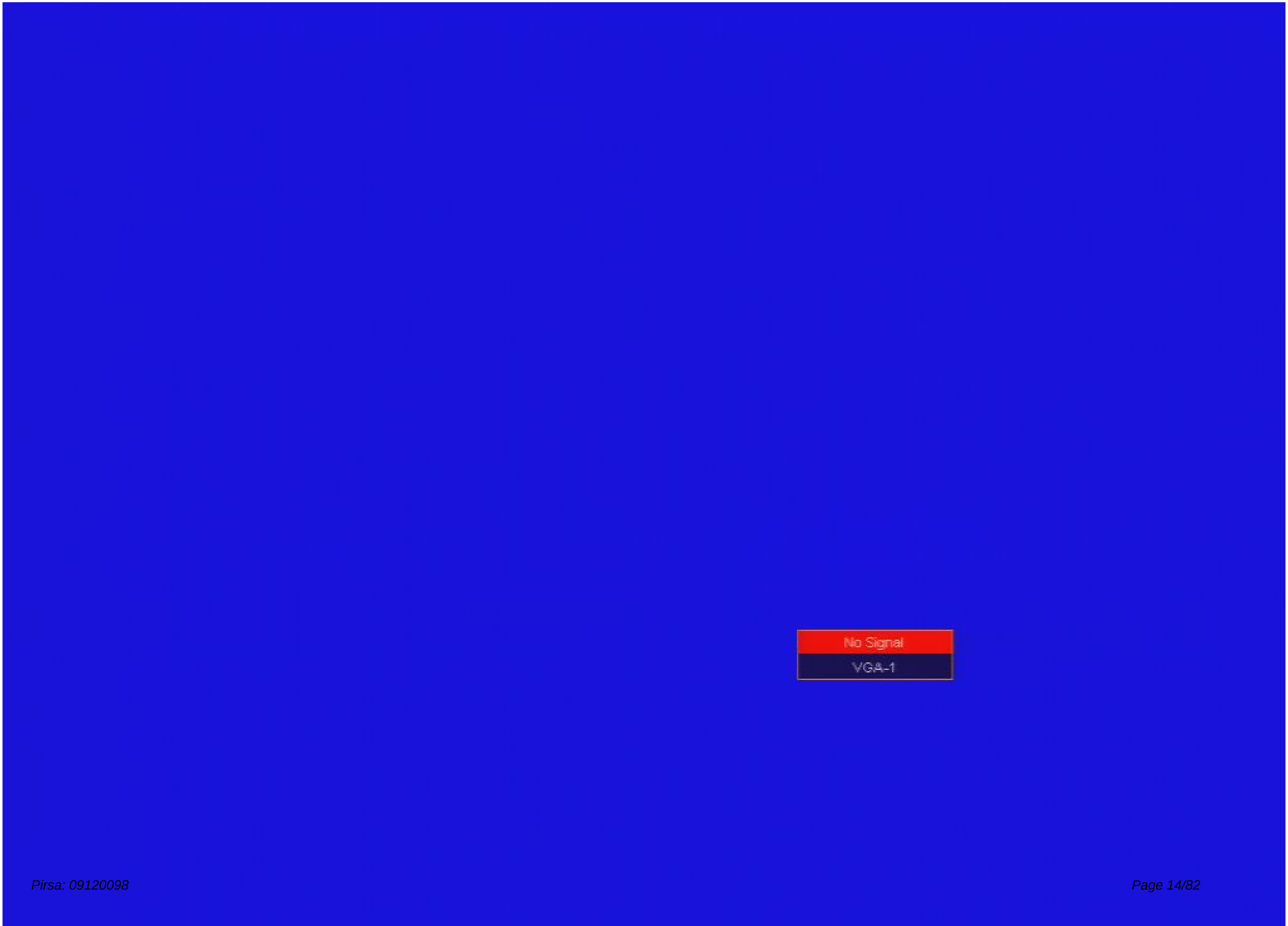
VGA-1

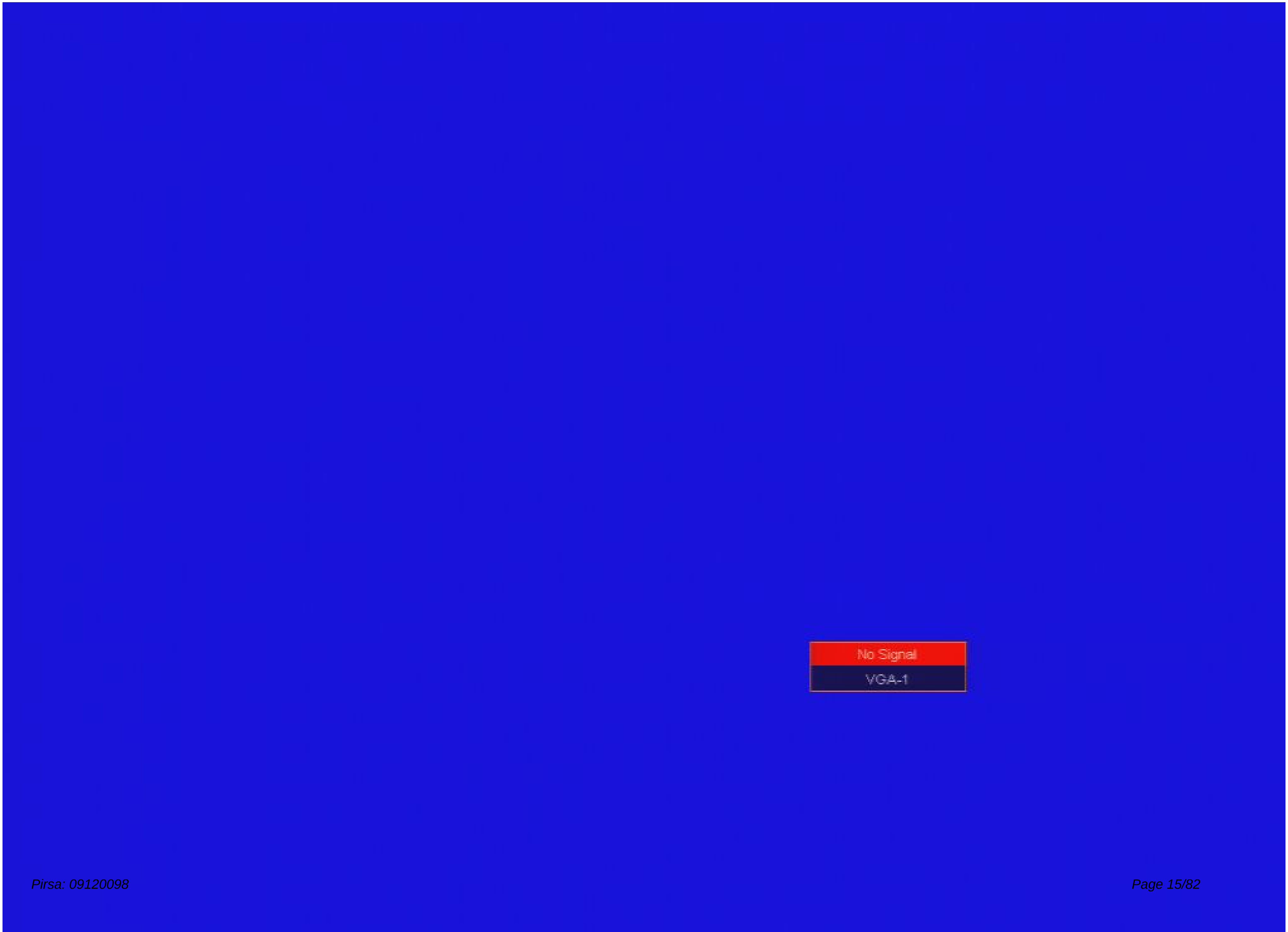

No Signal
VGA-1

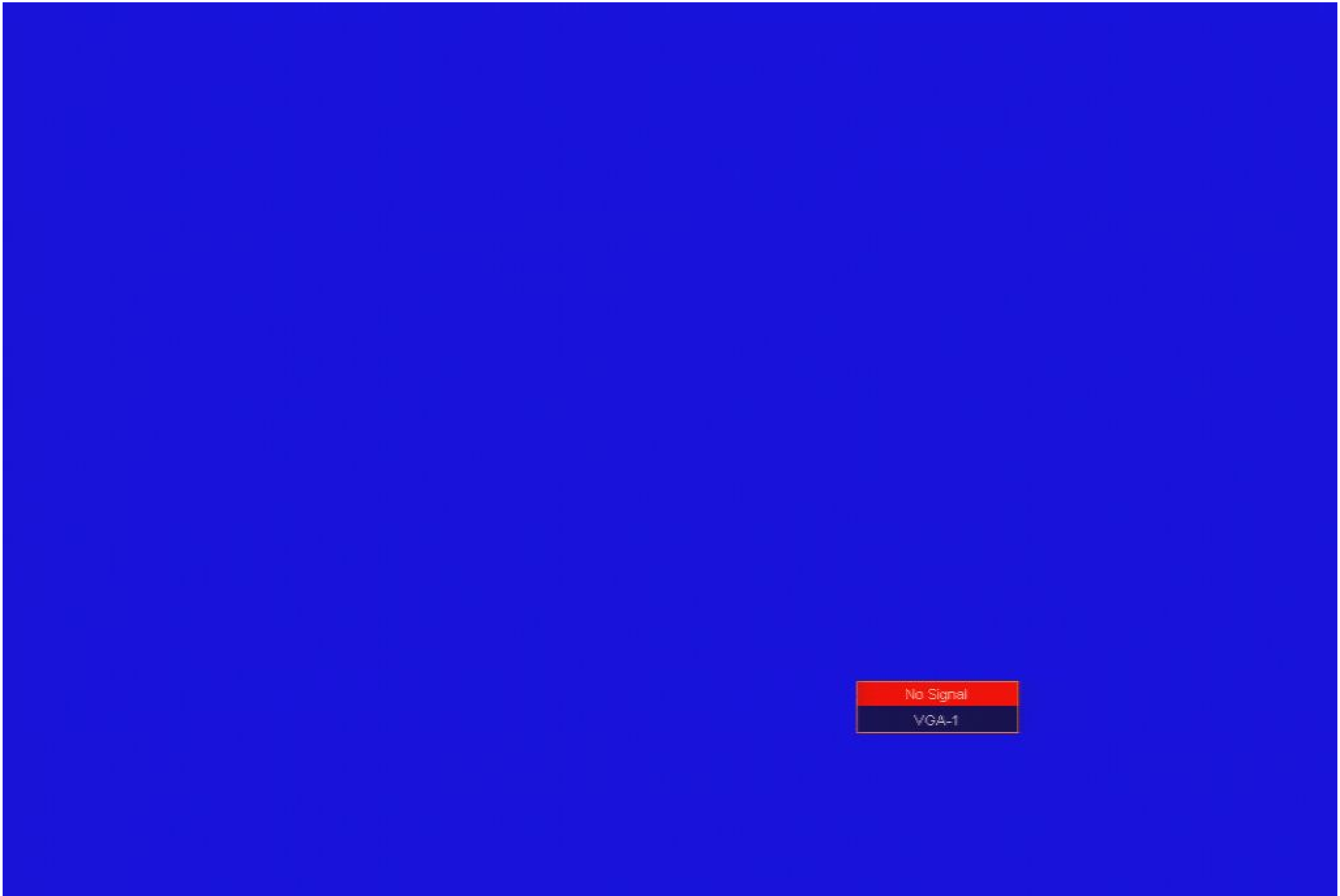

No Signal
VGA-1

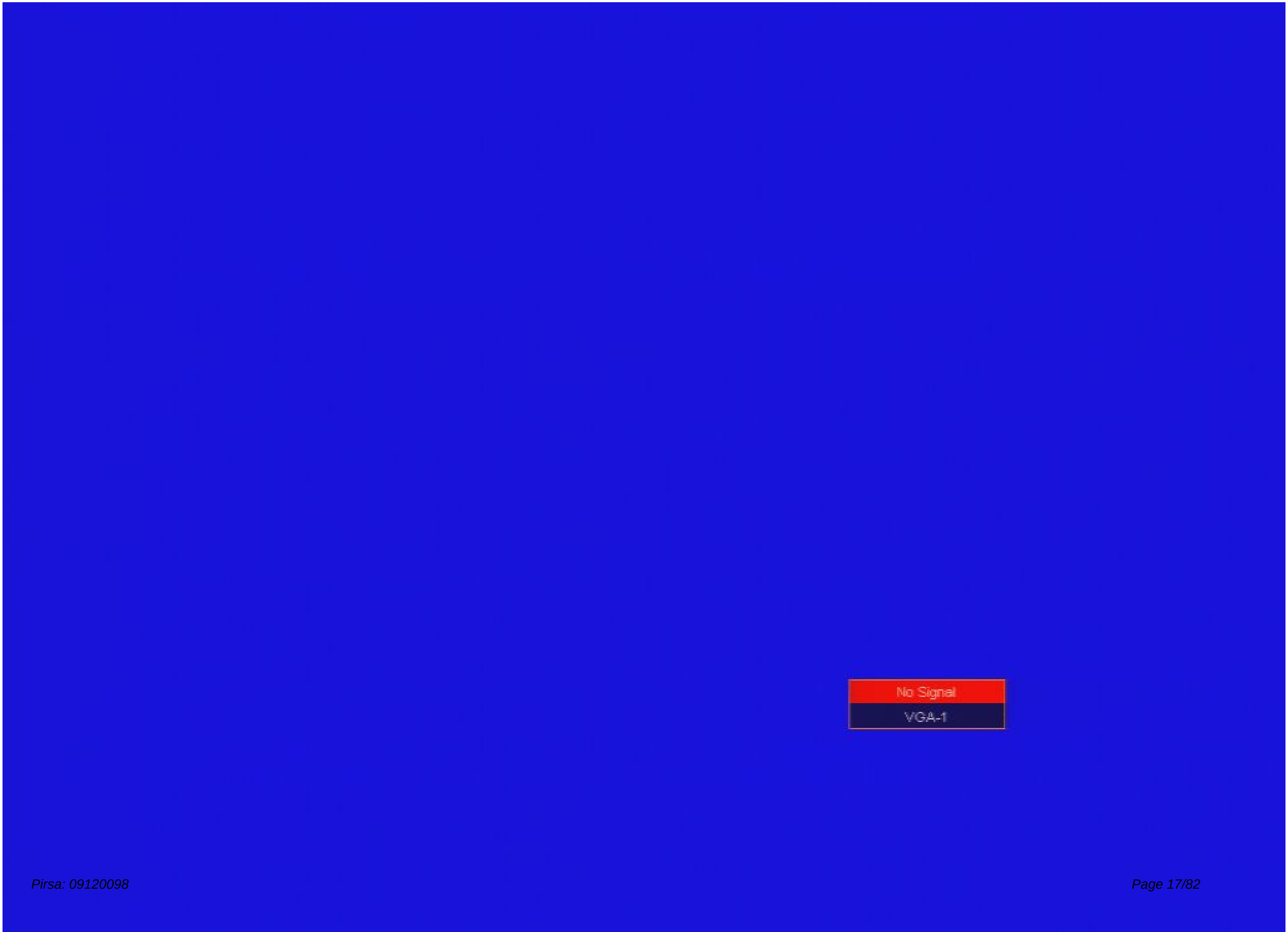

No Signal
VGA-1

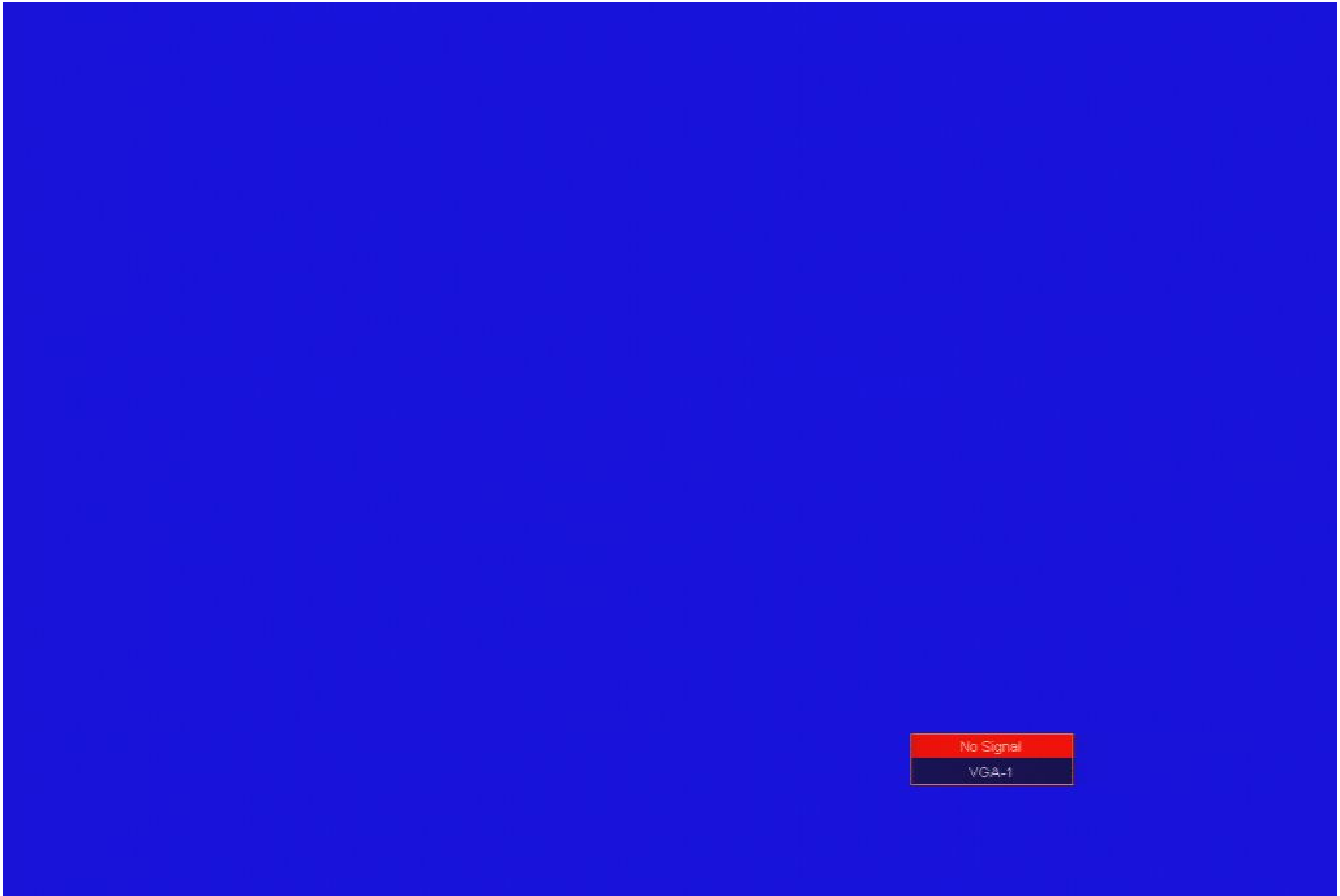

No Signal
VGA-1

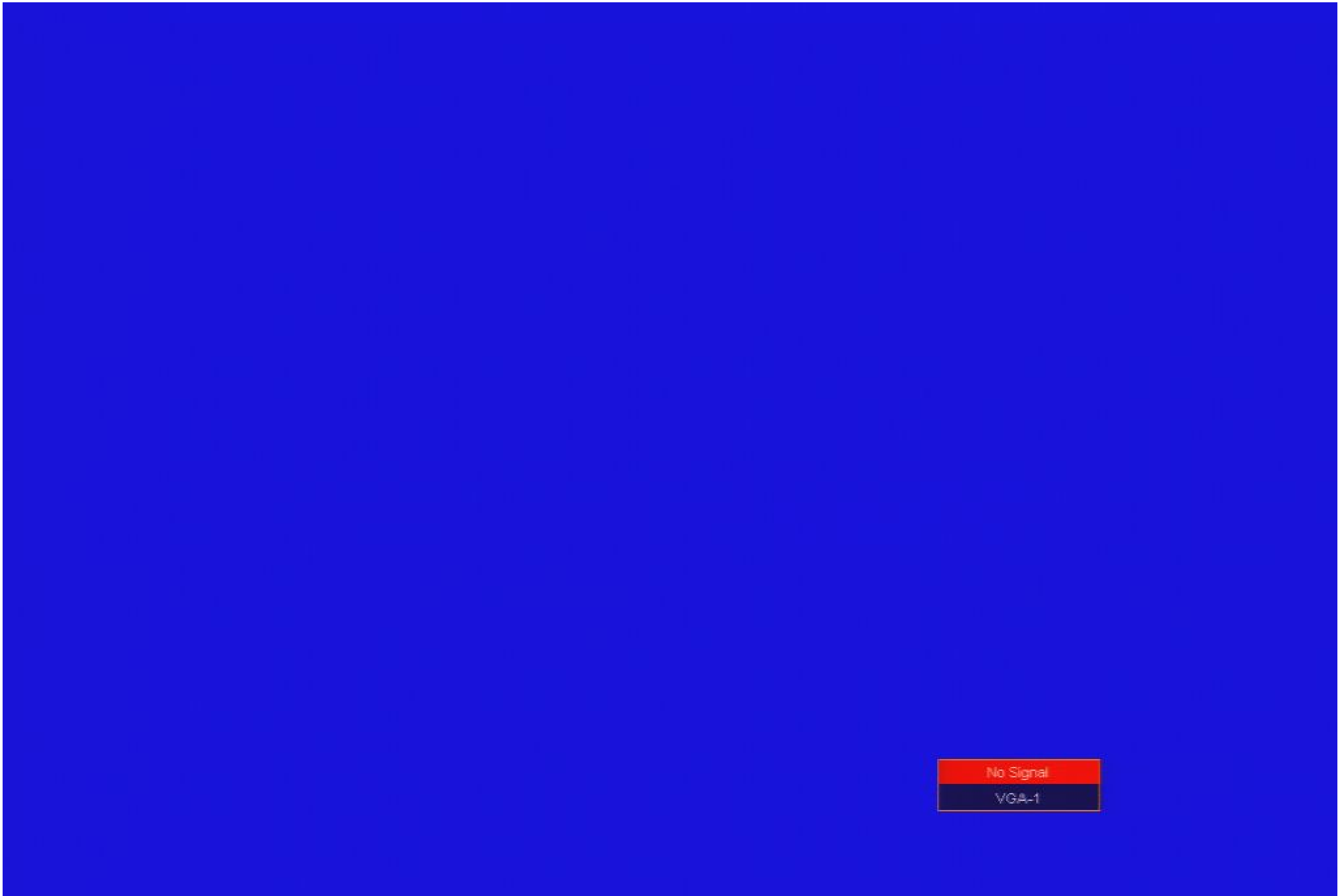

No Signal
VGA-1


No Signal
VGA-1


No Signal
VGA-1


No Signal
VGA-1


No Signal
VGA-1

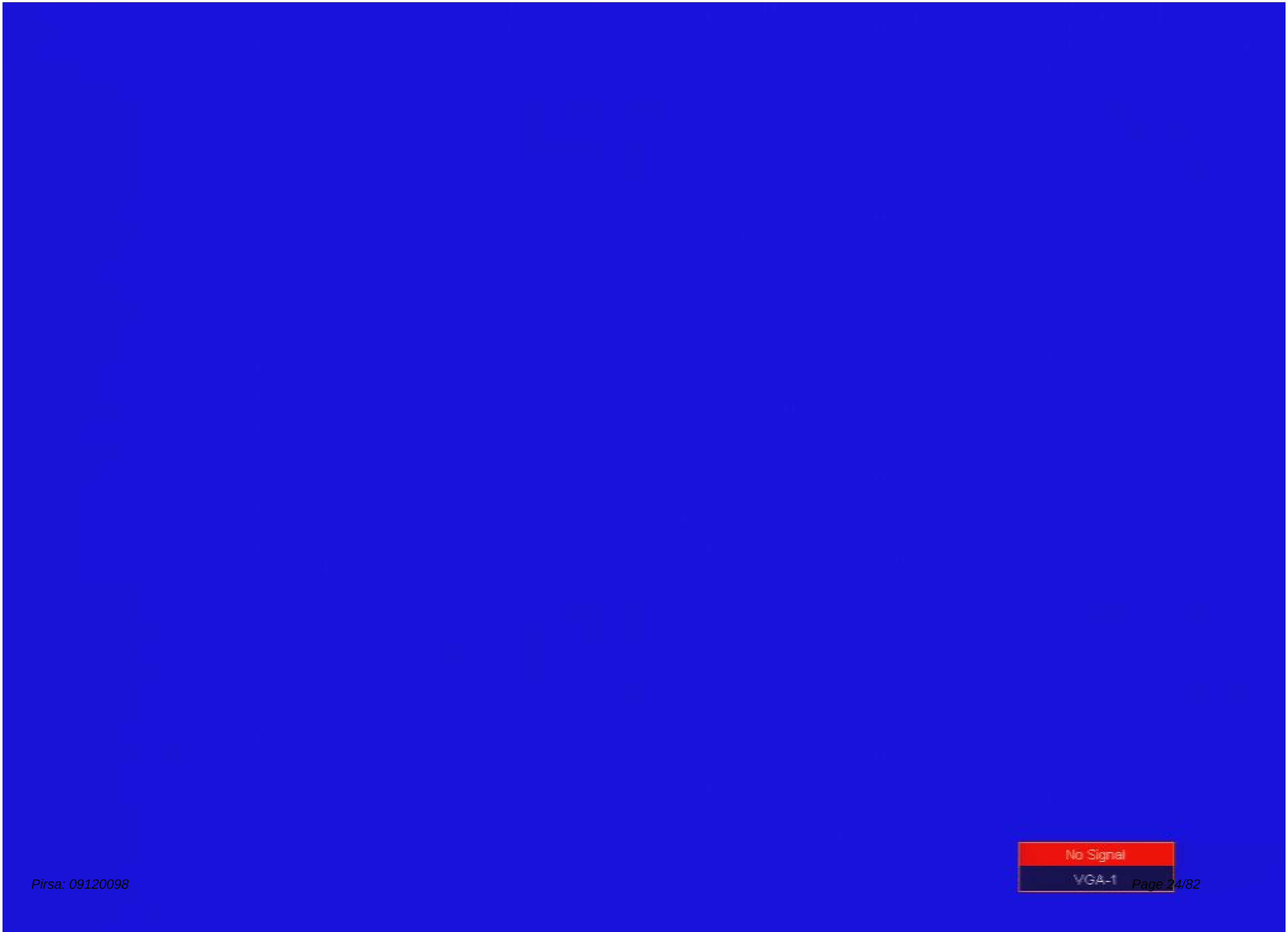

No Signal
VGA-1

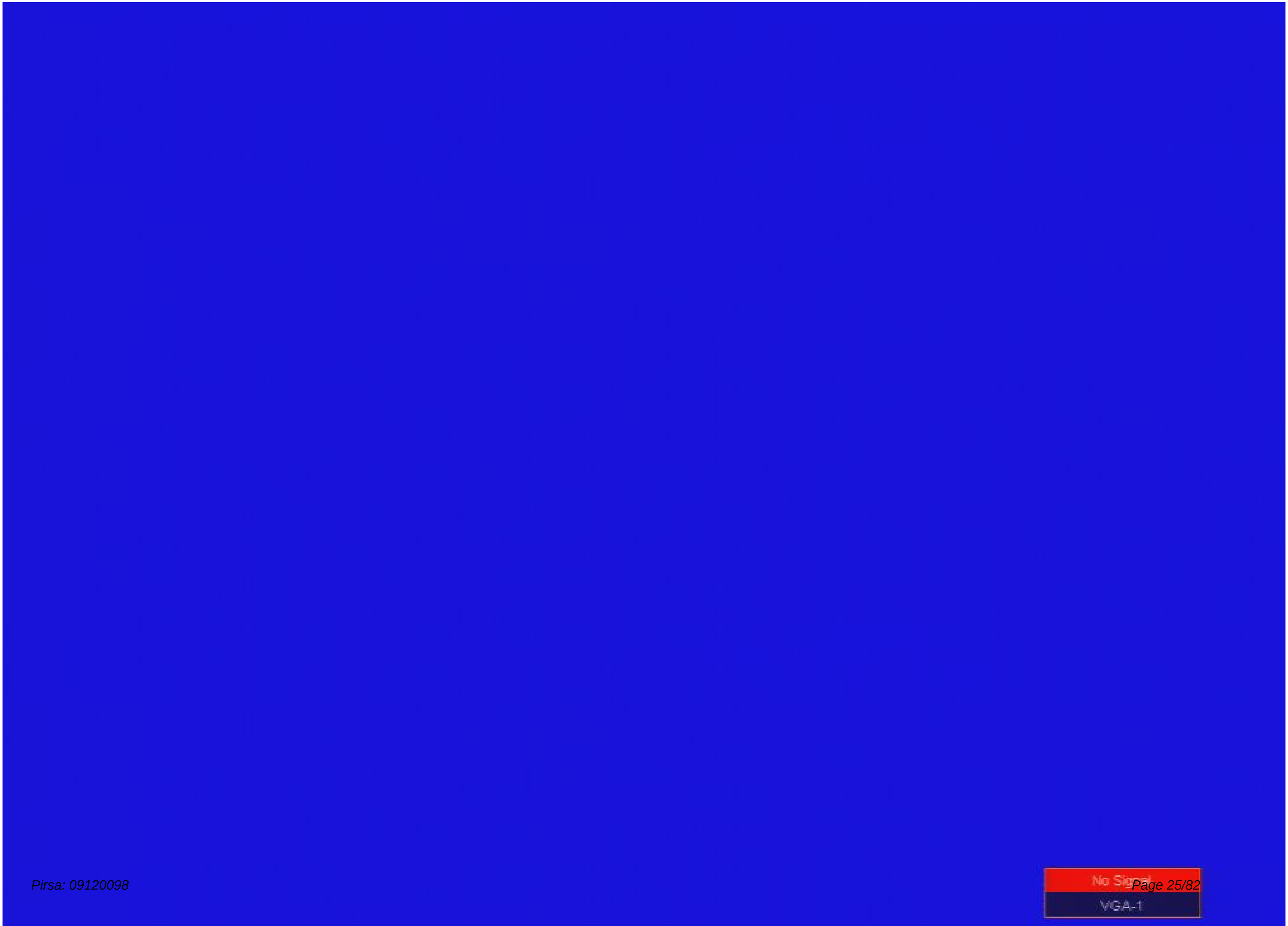
No Signal
VGA-1


No Signal
VGA-1


No Signal
VGA-1


No Signal
VGA-1


No Signal
VGA-1


No Signal
VGA-1

Outline

- Motivation (then and now)
  - Simplicity of Massless Scattering Amplitudes
  - Checking BCFW recursion without a QFT
  - Why are massless S-matrices simple?

- The Four-Particle Test
- A Spin-1 Tree S-Matrix from BCFW
- Gravity's Hidden Relations

See also: Benincasa and Cachazo,
He and Zhang 0811.3210

Scattering Amplitudes are *simple*

$$A = c_n Tr[T_1 T_2 \cdots T_n] \underbrace{A_{\text{c.o.}}(1,2,\ldots,n)}_{\text{color-ordered amplitude}} + \text{perm's}$$

Yang Mills n-gluon amplitudes are zero if they contain <2 gluons of helicity +1.

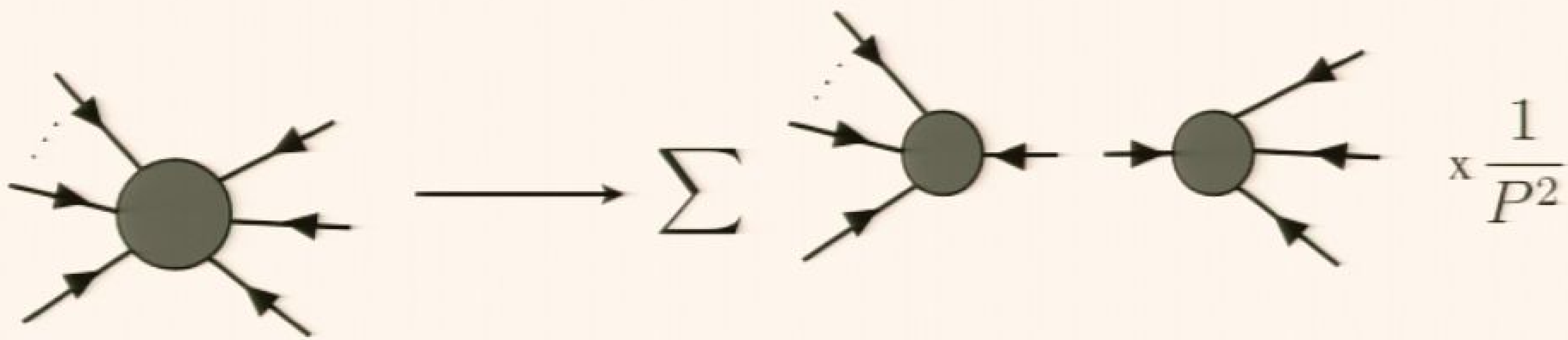
For 2 helicity +1 gluons i and j (<u>M</u>aximal <u>H</u>elicity <u>V</u>iolating).

$$|A_{\text{c.o.}}|^2 = \frac{(p_i . p_j)^4}{(p_1 . p_2)(p_2 . p_3) \ldots (p_n . p_1)}$$

Parke, Taylor '86

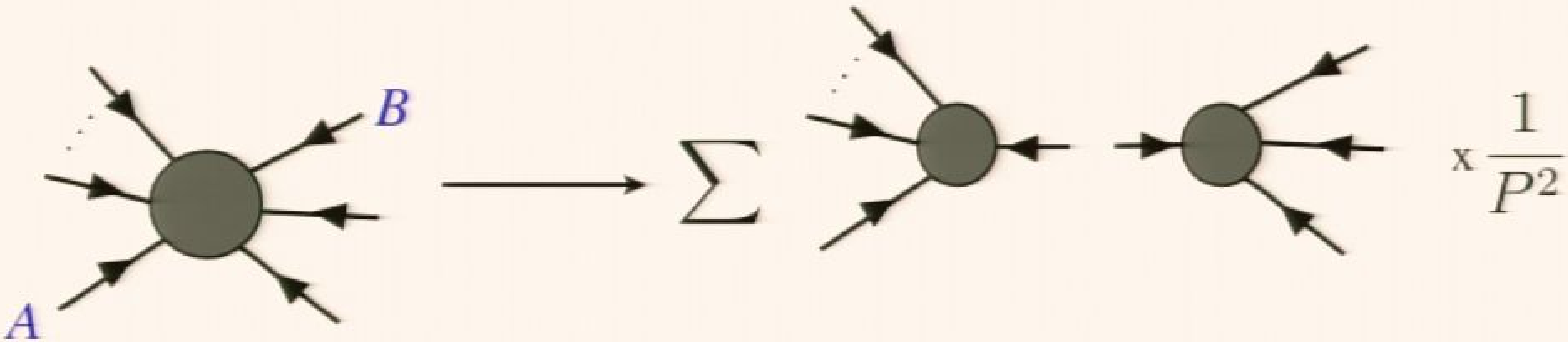
Simple Amplitudes: BCFW Recursion

Any *n*-gluon scattering amplitude can be written in terms of lower-point **scattering amplitudes**.



Simple Amplitudes: BCFW Recursion

Any n-gluon scattering amplitude can be written in terms of lower-point **scattering amplitudes**.



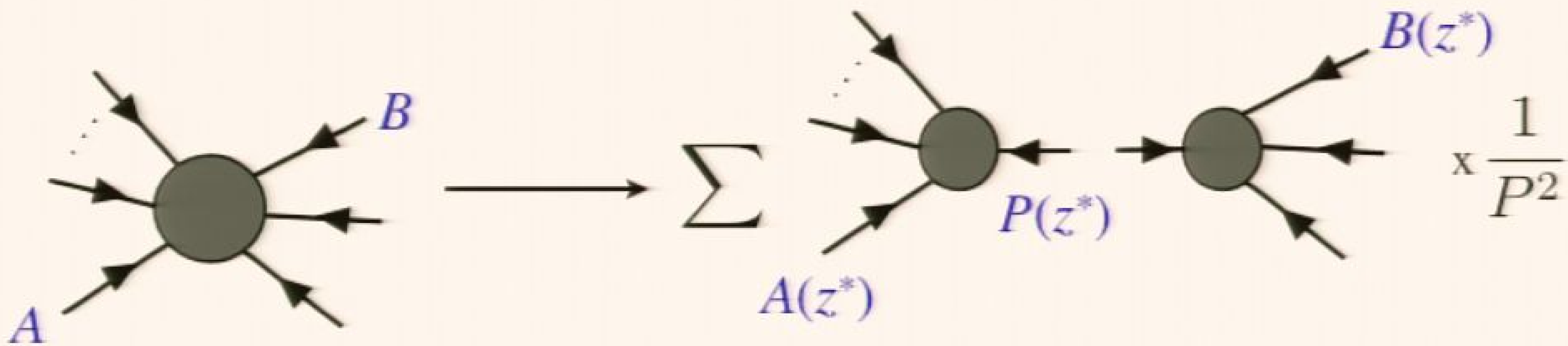
Two complex q's such that

$$q^2 = p_A.q = p_B.q = 0$$

e.g. if $p_{A/B}$=(1,±1,0,0), q=(0,0,1,±i)

Simple Amplitudes: BCFW Recursion

Any *n*-gluon scattering amplitude can be written in terms of lower-point **scattering amplitudes**.



Two complex q's such that

$$q^2 = p_A . q = p_B . q = 0$$

e.g. if $p_{A/B}$=(1,±1,0,0), q=(0,0,1,±i)

Then

$$p_A^\mu(z) = p_A + zq^\mu$$

$$p_B^\mu(z) = p_B - zq^\mu$$

are null and

$(p_A(z)+P)^2$ is linear in z

Britto, Cachazo, Feng

Simple Amplitudes: BCFW Recursion

$$p_A^\mu(z) = p_A + zq^\mu$$
$$p_B^\mu(z) = p_B - zq^\mu$$

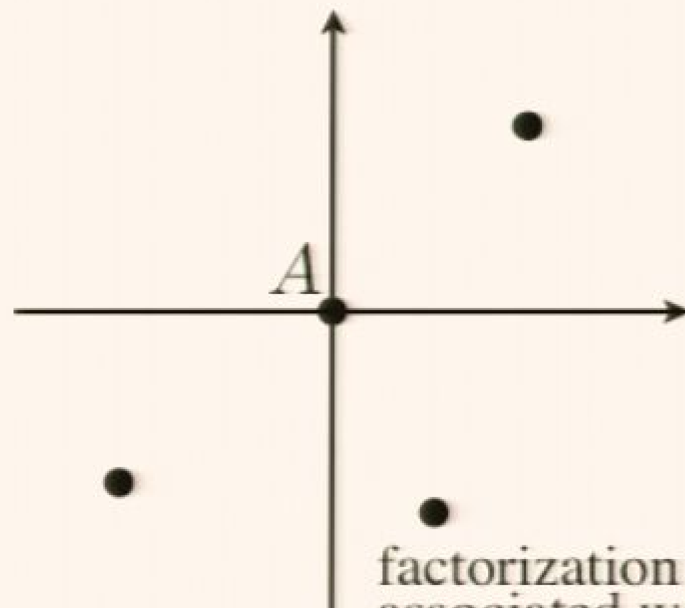
$$q^2 = p_A . q = p_B . q = 0$$

$$A(p_1, \ldots, p_A, \ldots, p_B) \rightarrow$$
$$A(z) \equiv A(p_1, \ldots, p_A(z), \ldots, p_B(z))$$

$$\oint \frac{A(z)}{z} = 0$$

if A falls as $1/z$ or faster at large z

=> solve for $A(0)$ as sum of other poles.



factorization poles associated w/ on-shell intermediate line

Britto, Cachazo, Feng, Witten

Simple Amplitudes: BCFW Recursion

Proofs of BCFW:

- Show that $A(z) \sim 1/z$:
  - **Diagrammatic:** build collections of Feynman diagrams where $1/z$ or faster fall-off is manifest
  - **Background fields:** Determine z-scaling of $M_{\mu\nu}$ from symmetry in convenient gauge, contract with $\varepsilon^{\mu}(z)\varepsilon^{\nu}(z)$

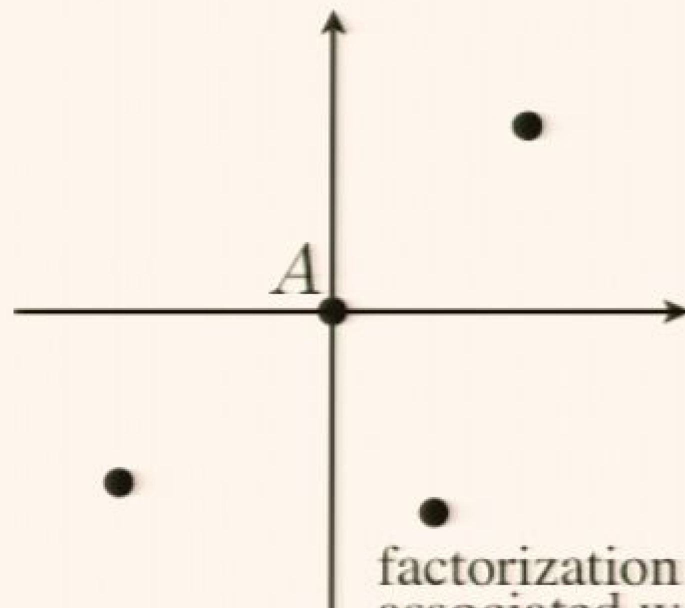
Simple Amplitudes: BCFW Recursion

$$p_A^\mu(z) = p_A + zq^\mu$$
$$p_B^\mu(z) = p_B - zq^\mu$$

$$q^2 = p_A.q = p_B.q = 0$$

$$A(p_1, \ldots, p_A, \ldots, p_B) \rightarrow$$
$$A(z) \equiv A(p_1, \ldots, p_A(z), \ldots, p_B(z))$$

$$\oint \frac{A(z)}{z} = 0$$



factorization poles
associated w/ on-shell
intermediate line

if A falls as $1/z$ or faster at large z

=> solve for $A(0)$ as sum of other poles.

Britto, Cachazo, Feng, Witten

Simple Amplitudes: BCFW Recursion

Proofs of BCFW:

- Show that $A(z) \sim 1/z$:
  - **Diagrammatic:** build collections of Feynman diagrams where $1/z$ or faster fall-off is manifest
  - **Background fields:** Determine z-scaling of $M_{\mu\nu}$ from symmetry in convenient gauge, contract with $\varepsilon^{\mu}(z)\varepsilon^{\nu}(z)$

Simple Amplitudes: BCFW Recursion

Proofs of BCFW:

Simple Amplitudes: BCFW Recursion

Proofs of BCFW:

- Show that $A(z) \sim 1/z$:
  - **Diagrammatic:** build collections of Feynman diagrams where $1/z$ or faster fall-off is manifest
  - **Background fields:** Determine z-scaling of $M_{\mu\nu}$ from symmetry in convenient gauge, contract with $\varepsilon^{\mu}(z)\varepsilon^{\nu}(z)$

Simple Amplitudes: BCFW Recursion

Proofs of BCFW:

- Show that $A(z) \sim 1/z$:
  - **Diagrammatic:** build collections of Feynman diagrams where $1/z$ or faster fall-off is manifest
  - **Background fields:** Determine z-scaling of $M_{\mu\nu}$ from symmetry in convenient gauge, contract with $\varepsilon^{\mu}(z)\varepsilon^{\nu}(z)$

$$\oint \frac{A(z)}{z} = 0$$

- All poles of tree amplitudes correspond to factorization channels – BCFW form of $A(0)$ follows from integral above.

A wide range of theories have BCFW recursion relations:

- Gauge theory:
  - Valid if h_A=-1 or h_B=+1, for pure gauge theory [Britto, Cachazo, Feng, Witten]
  - Same conditions, where the other marked leg is matter [Cheung]

Can reduce any amplitude with gauge bosons to lower-point amplitudes

A wide range of theories have BCFW recursion relations:

- Gauge theory:
  - Valid if h_A=-1 or h_B=+1, for pure gauge theory
    [Britto, Cachazo, Feng, Witten]
  - Same conditions, where the other marked leg is matter
    [Cheung]

- Gravity: analogous
  [Benincasa, Cachazo, Verroneau; Arkani-Hamed, Kaplan; Cheung]
  - In fact, all of these amplitudes $\sim 1/z^2$, so both

$$\oint \frac{A(z)}{z} = 0 \qquad \text{and} \qquad \oint A(z) = 0$$

- Generalizations when $A(z) \to const.$ at infinity.
  [Benincasa, Cachazo;]

Can reduce any amplitude with gauge bosons (gravitons) to lower-point amplitudes

Simple Amplitudes: BCFW Recursion

Proofs of BCFW:

- Show that $A(z) \sim 1/z$:
  - **Diagrammatic:** build collections of Feynman diagrams where $1/z$ or faster fall-off is manifest
  - **Background fields:** Determine z-scaling of $M_{\mu\nu}$ from symmetry in convenient gauge, contract with $\varepsilon^{\mu}(z)\varepsilon^{\nu}(z)$

A wide range of theories have BCFW recursion relations:

- Gauge theory:
  - Valid if h_A=-1 or h_B=+1, for pure gauge theory [Britto, Cachazo, Feng, Witten]
  - Same conditions, where the other marked leg is matter [Cheung]

- Gravity: analogous [Benincasa, Cachazo, Verroneau; Arkani-Hamed, Kaplan; Cheung]
  - In fact, all of these amplitudes ~$1/z^2$, so both

$$\oint \frac{A(z)}{z} = 0 \qquad \text{and} \qquad \oint A(z) = 0$$

- Generalizations when *A(z) -> const*. at infinity. [Benincasa, Cachazo;]

Can reduce any amplitude with gauge bosons (gravitons) to lower-point amplitudes

Motivation

- 3-point amplitudes & BCFW define (tree) S-matrices for YM and gravity without reference to a Lagrangian.
- Logical completion: Show consistency, again without reference to Lagrangians!
- Can we define other "S-matrix theories" that have no gauge-inv. Lagrangian description? (e.g. anti-self-dual 3-form in 6d)

N=4/8 Amplitudes *even simpler*

- Generalized BCFW for all diagrams (involves SUSY transf. as well as *p*-shift)

- Simple loop expansion – entirely in terms of "box" diagrams

- General formulas for N=4 amplitudes in twistor space.
- Conformal & dual superconformal invariance

Simple S-Matrices:

- Accidentally inherited from SUSY theories?

  Pure gauge/gravity tree amplitudes ~ SUSY amplitudes (other states appear in pairs)

- Or general properties, with extra simplification in SUSY?

  BCFW with matter – not obviously derived from SUSY

  Connection between BCFW at 4-point and elementary consistency conditions on interactions (Jacobi identity, equivalence...).

Outline

- Motivation (then and now)
- The Four-Particle Test
  "4-point amplitudes alone constrain interactions and BCFW shifts" [Benincasa, Cachazo]
- A Spin-1 Tree S-Matrix from BCFW
  "Simple arguments and identities from 4-point ensure that BCFW amplitudes have <u>all</u> factorization poles."
- Gravity's Hidden Relations
  "$1/z^2$ fall-off is needed to see that BCFW amplitudes have all factorization poles."

Symmetry Properties of the S-Matrix

- Lorentz Invariance
- Little-Group Covariance

Spinor-Helicity: $P_{\alpha\dot{\alpha}} = \sigma^{\mu}_{\alpha\dot{\alpha}} p_{\mu}$

$p^2 = 0 : \det P = 0 \rightarrow P_{\alpha\dot{\alpha}} = \underbrace{\lambda_{\alpha}}_{-1/2}\underbrace{\tilde{\lambda}_{\dot{\alpha}}}_{+1/2}$

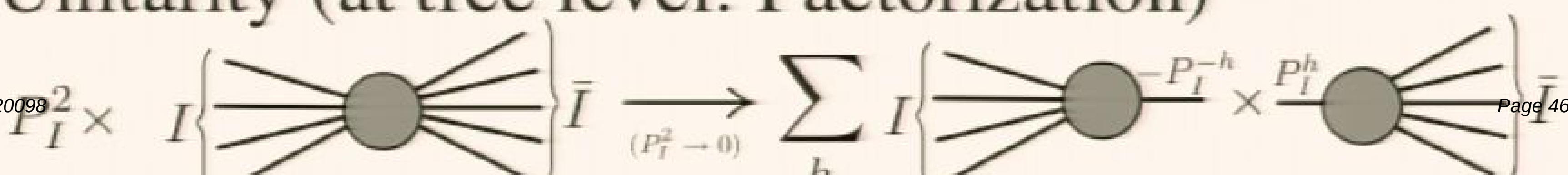
Real $p_{\mu} \leftrightarrow \lambda = \tilde{\lambda}^{*}$

$$P \rightarrow P \qquad \lambda \rightarrow e^{i\phi/2}\lambda \qquad \tilde{\lambda} \rightarrow e^{-i\phi/2}\tilde{\lambda}$$

$$\epsilon^{+} = \left(\frac{\mu_{\alpha}\tilde{\lambda}_{\dot{\alpha}}}{\mu.\lambda}\right)^{s} \rightarrow e^{is\phi}\epsilon^{+} \qquad \epsilon^{-} = \left(\frac{\lambda_{\alpha}\tilde{\mu}_{\dot{\alpha}}}{\tilde{\mu}.\tilde{\lambda}}\right)^{s} \rightarrow e^{-is\phi}\epsilon^{-}$$

$$\mathcal{M}(\lambda_i, \tilde{\lambda}_i, h_i) \sim \lambda_i^{p}\,\tilde{\lambda}_i^{(2h_i+p)}$$

- Unitarity (at tree level: Factorization)



3-Point Amplitudes

[Benincasa and Cachazo]

- Exist for complex momenta (indep. $\lambda, \tilde{\lambda}$)
- Two degenerate momentum configs:

$$\sum P_i = 0, P_i^2 = 0 \rightarrow \begin{cases} \lambda_i \cdot \lambda_j = 0 & \forall i,j \\ \tilde{\lambda}_i \cdot \tilde{\lambda}_j = 0 & \forall i,j \end{cases} \text{ w/ invariants } \begin{matrix} [ij] \equiv \tilde{\lambda}_i \cdot \tilde{\lambda}_j \\ \langle ij \rangle \equiv \lambda_i \cdot \lambda_j \end{matrix}$$

- Helicity+finite real-p limit fixes amplitudes (no scalar invariants)

$$A(1_a^{+1}, 2_b^{-1}, 3_c^{-1}) = \kappa_{abc} \frac{\langle 23 \rangle^3}{\langle 12 \rangle \langle 31 \rangle} \text{ or } \kappa'_{abc} \frac{[12][31]}{[23]^3}$$

$$A(1_a^{+1}, 2_b^{+1}, 3_c^{+1}) = \lambda_{abc} \langle 12 \rangle \langle 13 \rangle \langle 23 \rangle$$

Symmetry Properties of the S-Matrix

- Lorentz Invariance
- Little-Group Covariance

Spinor-Helicity: $P_{\alpha\dot{\alpha}} = \sigma^{\mu}_{\alpha\dot{\alpha}} p_{\mu}$

$p^2 = 0 : \det P = 0 \rightarrow P_{\alpha\dot{\alpha}} = \underbrace{\lambda_{\alpha}}_{-1/2} \underbrace{\tilde{\lambda}_{\dot{\alpha}}}_{+1/2}$

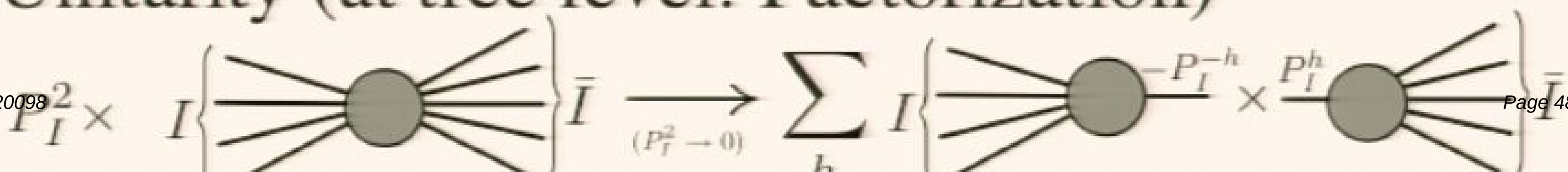
Real $p_{\mu} \leftrightarrow \lambda = \tilde{\lambda}^{*}$

$$P \rightarrow P \qquad \lambda \rightarrow e^{i\phi/2}\lambda \qquad \tilde{\lambda} \rightarrow e^{-i\phi/2}\tilde{\lambda}$$

$$\epsilon^{+} = \left(\frac{\mu_{\alpha}\tilde{\lambda}_{\dot{\alpha}}}{\mu.\lambda}\right)^{s} \rightarrow e^{is\phi}\epsilon^{+} \qquad \epsilon^{-} = \left(\frac{\lambda_{\alpha}\tilde{\mu}_{\dot{\alpha}}}{\tilde{\mu}.\tilde{\lambda}}\right)^{s} \rightarrow e^{-is\phi}\epsilon^{-}$$

$$\mathcal{M}(\lambda_i, \tilde{\lambda}_i, h_i) \sim \lambda_i^{p}\, \tilde{\lambda}_i^{(2h_i+p)}$$

- Unitarity (at tree level: Factorization)



3-Point Amplitudes

[Benincasa and Cachazo]

- Exist for complex momenta (indep. $\lambda, \tilde{\lambda}$)
- Two degenerate momentum configs:

$$\sum P_i = 0, P_i^2 = 0 \rightarrow \begin{cases} \lambda_i \cdot \lambda_j = 0 & \forall i,j \\ \tilde{\lambda}_i \cdot \tilde{\lambda}_j = 0 & \forall i,j \end{cases} \text{ w/ invariants } \begin{matrix} [ij] \equiv \tilde{\lambda}_i \cdot \tilde{\lambda}_j \\ \langle ij \rangle \equiv \lambda_i \cdot \lambda_j \end{matrix}$$

- Helicity+finite real-p limit fixes amplitudes (no scalar invariants)

$$A(1_a^{+1}, 2_b^{-1}, 3_c^{-1}) = \kappa_{abc} \frac{\langle 23 \rangle^3}{\langle 12 \rangle \langle 31 \rangle} \text{ or } \kappa'_{abc} \frac{[12][31]}{[23]^3}$$

$$A(1_a^{+1}, 2_b^{+1}, 3_c^{+1}) = \lambda_{abc} \langle 12 \rangle \langle 13 \rangle \langle 23 \rangle$$

Gauge 3-Point Amplitudes

$$A^{(h)}(1_a^{+1}, 2_b^{-1}, 3_c^{-1}) = \kappa_{abc} \frac{\langle 23 \rangle^3}{\langle 12 \rangle \langle 31 \rangle} \quad ([ij] = 0)$$

$$A^{(a)}(1_a^{-1}, 2_b^{+1}, 3_c^{+1}) = \bar{\kappa}_{abc} \frac{[23]^3}{[12][31]} \quad (\langle ij \rangle = 0)$$

Scalar matter:

$$A^{(h)}(1_a^{-1}, 2_b^0, 3_c^0) = k^a_{bc} \frac{\langle 12 \rangle \langle 13 \rangle}{\langle 23 \rangle}$$

$$A^{(a)}(1_a^{+1}, 2_b^0, 3_c^0) = \bar{k}^a_{bc} \frac{[12][13]}{[23]}$$

(k must satisfy Jacobi identities,
κ must form representation)

All others zero

4-Point Amplitudes

$$A(1^{h_1}, 2^{h_2}, 3^{h_3}, 4^{h_4}) = \mathcal{H}(1,2,3,4) \;\times\; f(s,t,u)$$

particular solution with correct helicity transformations

e.g. $\mathcal{H}(1-, 2-, 3+, 4+) = \langle 12\rangle^2[34]^2$

f isn't constrained by little group (scalar) or LI, but restricted by factorization at complex momenta:

$$\lim_{s\to 0} s \times A(1,2,3,4) = \sum_{h,a} A(1,2,-P_{12}^{-h})A(3,4,P_{12}^{h})$$

In fact, $\langle 12\rangle \to 0$ *and* $[12] \to 0$ *are distinct configurations, should* ***both*** *satisfy this limit.*

Jacobi from Factorization

- Impose *t* and *u*-channel factorization
  - the individual 3-point amplitudes are singular!

$$A(1^-,2^-,3^+,4^+) = \langle 12\rangle^2[34]^2\left[\frac{\kappa_{\beta 13}\bar{\kappa}_{\beta 42}}{st} - \frac{\kappa_{\beta 14}\bar{\kappa}_{\beta 32}}{su} + \ldots\right]$$

- When [12] –>0 :

$$sA(1^-,2^-,3^+,4^+) \rightarrow \langle 12\rangle^2[34]^2\left[\frac{\kappa_{\beta 13}\bar{\kappa}_{\beta 42}}{t} + \frac{\kappa_{\beta 14}\bar{\kappa}_{\beta 32}}{t} + \ldots\right]$$

Compare to factorization limit:

$$sA(1^-,2^-,3^+,4^+) \rightarrow \langle 12\rangle^2[34]^2\frac{\kappa_{12\alpha}\bar{\kappa}_{\alpha 34}}{t}$$

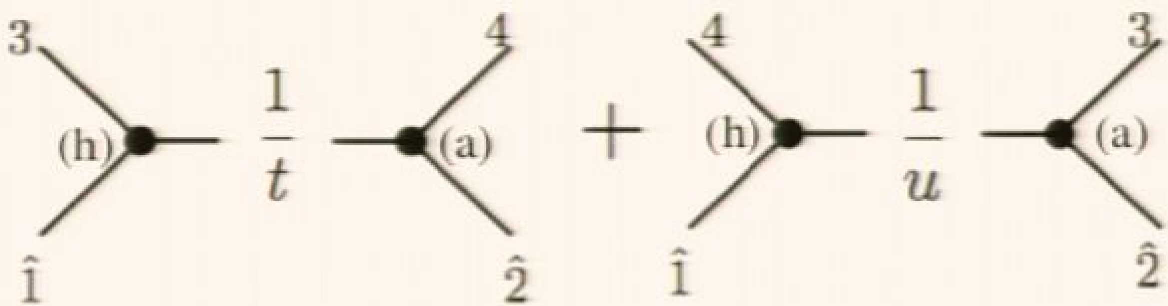
Requires $\kappa_{12\alpha}\bar{\kappa}_{34\alpha} + \kappa_{13\alpha}\bar{\kappa}_{24\alpha} + \kappa_{14\alpha}\bar{\kappa}_{32\alpha} = 0$

Factorization in BCFW

Consider BCFW, where we shift p_1 and p_2 by zq with $q = |1\rangle|2]$, i.e.

$$|1] \to |\hat{1}](z) = |1] + z|2]$$
$$|2\rangle \to |\hat{2}\rangle(z) = |2\rangle - z|1\rangle$$

Two terms:



Controlled by factorization as $[13], \langle 24\rangle \to 0$ and $[14], \langle 23\rangle \to 0$ – generates ansatz on previous slide.

Momentum-dependence of 3-point amplitudes (s>0) allows BCFW to work <u>and</u> imposes consistency conditions on 3-point couplings.

Jacobi from Factorization

- Impose *t* and *u*-channel factorization
  - the individual 3-point amplitudes are singular!

$$A(1^-,2^-,3^+,4^+) = \langle 12\rangle^2[34]^2\left[\frac{\kappa_{\beta 13}\bar{\kappa}_{\beta 42}}{st} - \frac{\kappa_{\beta 14}\bar{\kappa}_{\beta 32}}{su} + \ldots\right]$$

- When [12] –>0 :

$$sA(1^-,2^-,3^+,4^+) \rightarrow \langle 12\rangle^2[34]^2\left[\frac{\kappa_{\beta 13}\bar{\kappa}_{\beta 42}}{t} + \frac{\kappa_{\beta 14}\bar{\kappa}_{\beta 32}}{t} + \ldots\right]$$

Compare to factorization limit:

$$sA(1^-,2^-,3^+,4^+) \rightarrow \langle 12\rangle^2[34]^2\frac{\kappa_{12\alpha}\bar{\kappa}_{\alpha 34}}{t}$$

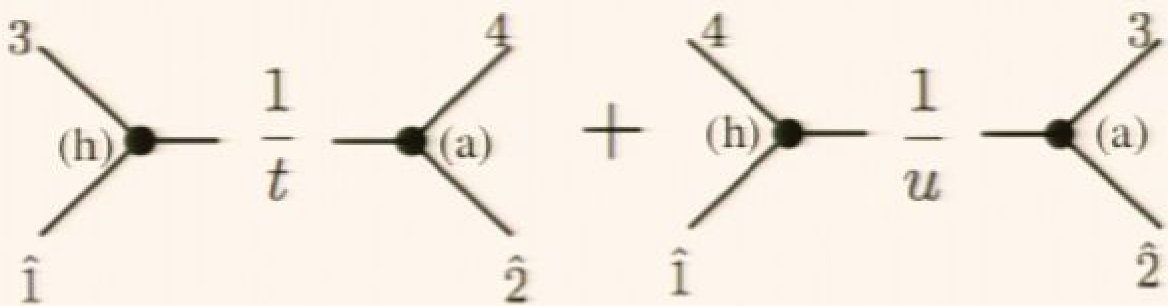
Requires $\kappa_{12\alpha}\bar{\kappa}_{34\alpha} + \kappa_{13\alpha}\bar{\kappa}_{24\alpha} + \kappa_{14\alpha}\bar{\kappa}_{32\alpha} = 0$

Factorization in BCFW

Consider BCFW, where we shift p_1 and p_2 by zq with $q = |1\rangle|2]$, i.e.

$$|1] \to |\hat{1}](z) = |1] + z|2]$$
$$|2\rangle \to |\hat{2}\rangle(z) = |2\rangle - z|1\rangle$$

Two terms:

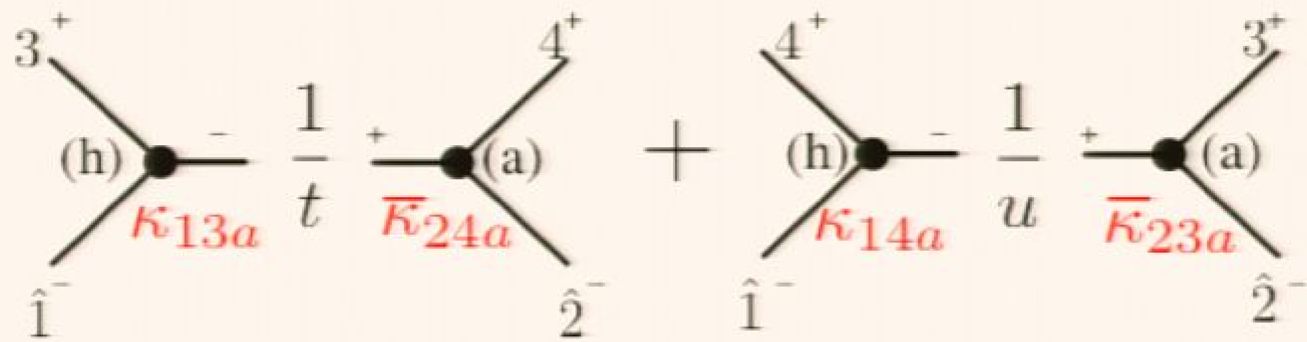


Controlled by factorization as $[13], \langle 24\rangle \to 0$ and $[14], \langle 23\rangle \to 0$ – generates ansatz on previous slide.

Momentum-dependence of 3-point amplitudes (s>0) allows BCFW to work <u>and</u> imposes consistency conditions on 3-point couplings.

Factorization from BCFW

Schematically (for gauge theory)



$$\lim_{[12]\to 0} \langle 12\rangle[12]A_{BCF}(1^-,2^-,3^+,4^+) =$$
$$\langle 12\rangle^2[34]^2\tfrac{1}{t}(\kappa_{13a}\overline{\kappa}_{24a}+\kappa_{14a}\overline{\kappa}_{32a})$$

Factorization: $\langle 12\rangle^2[34]^2\dfrac{1}{t}(\kappa_{12a}\overline{\kappa}_{34a})$

Requires $\kappa_{12a}\overline{\kappa}_{34a}+\kappa_{13a}\overline{\kappa}_{24a}+\kappa_{14a}\overline{\kappa}_{32a}=0$!

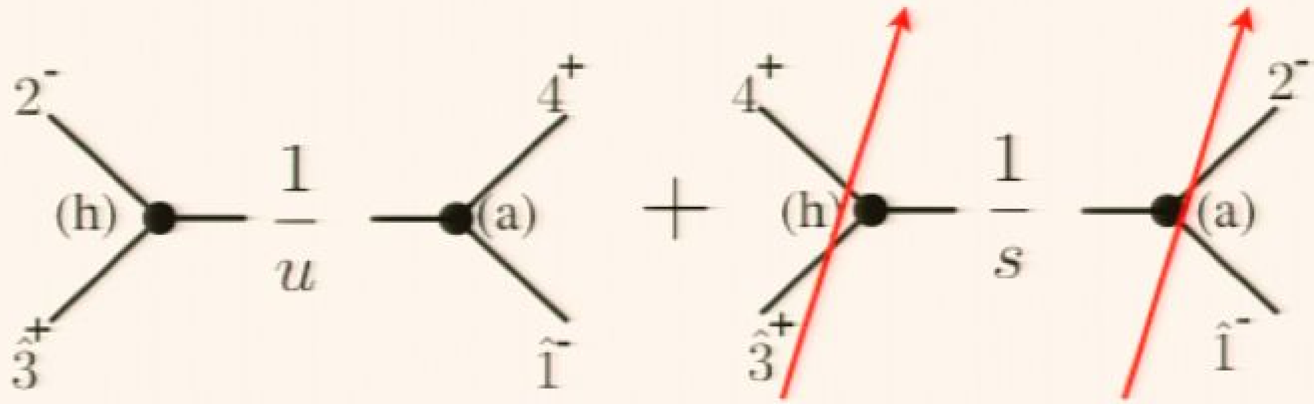
Similar arguments: interaction vertices of matter w/ spin-1 furnish representations; charge conservation

Gauge 4-Point Amplitudes: Good and Bad BCFW

- This procedure works (if Jacobi is satisfied) for $|-], |-\rangle$; $|+], |+\rangle$; and $|-], |+\rangle$ shifts

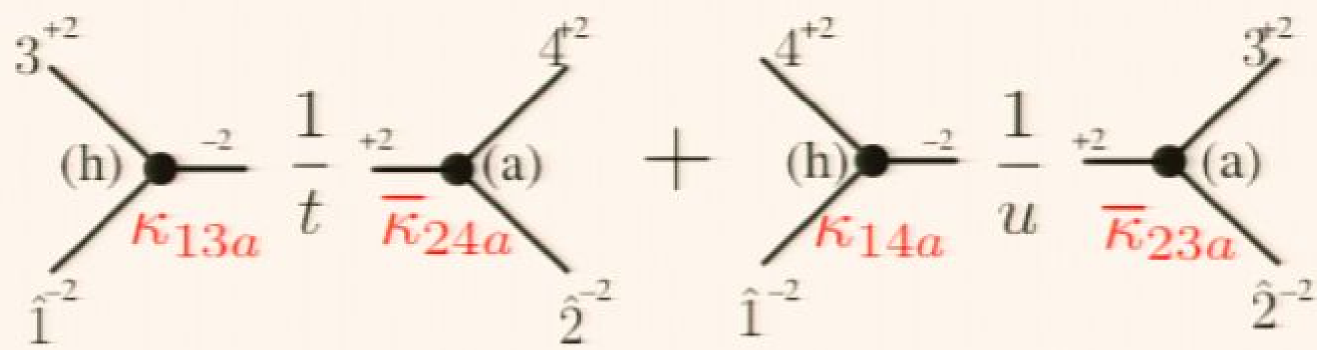
  These are the shifts for which 3-point amplitudes vanish identically or approach 0 at large z!

- BCF shift $|3^{+}], |1^{-}\rangle$ gives clearly unphysical answer



$$A_{BCF;[31\rangle}(1^{-}, 2^{-}, 3^{+}, 4^{+}) \propto \langle 12\rangle^2[34]^2 \frac{t^3}{s^4 u}$$

Gravity 4-Point Amplitudes: Commutation!



$$A_{BCF}(1^{-2},2^{-2},3^{+2},4^{+2}) = [34]^4\langle 12\rangle^4\left(\frac{\kappa_{13\alpha}\overline{\kappa}_{24\alpha}}{s^2t}+\frac{\kappa_{14\alpha}\overline{\kappa}_{23\alpha}}{s^2u}\right)$$

*Must **cancel double pole** and reproduce correct single pole as s–>0, for any kinematics*

$s+t=-u$: consistent if $\kappa_{13a}\overline{\kappa}_{24a} = \kappa_{14a}\overline{\kappa}_{23a} = \kappa_{12a}\overline{\kappa}_{34a}$

(Commutative, associative algebra
=> interactions can be diagonalized)

A Remarkably Powerful Condition

1. Pick particles & non-zero 3-point vertices
2. Consider BCFW shift of given-helicity legs
   - Bad shifts: BCFW produces clear nonsense (can never reproduce other poles)
   - Good shifts: BCFW can reproduce the "missing" pole, *if* 3-point amplitudes satisfy conditions.

→ These conditions mimic most of the known constraints on higher-spin interactions!
(**spin-1:** Jacobi, matter reps & charge conservation;
spin-2: commutative, equivalence; spins 2&3/2: supergravity)

3. Good shifts + 3-point => ansatz for a "theory" (set of constructible amplitudes)

Outline

- Motivation (then and now)

- The Four-Particle Test

  "4-point amplitudes alone constrain interactions and BCFW shifts"

- A Spin-1 Tree S-Matrix from BCFW

  "Simple arguments and Jacobi identities ensure that n-point BCFW amplitudes have <u>all</u> factorization poles."

- Gravity's Hidden Relations

  "$1/z^2$ fall-off is needed to see that BCFW amplitudes have all factorization poles."

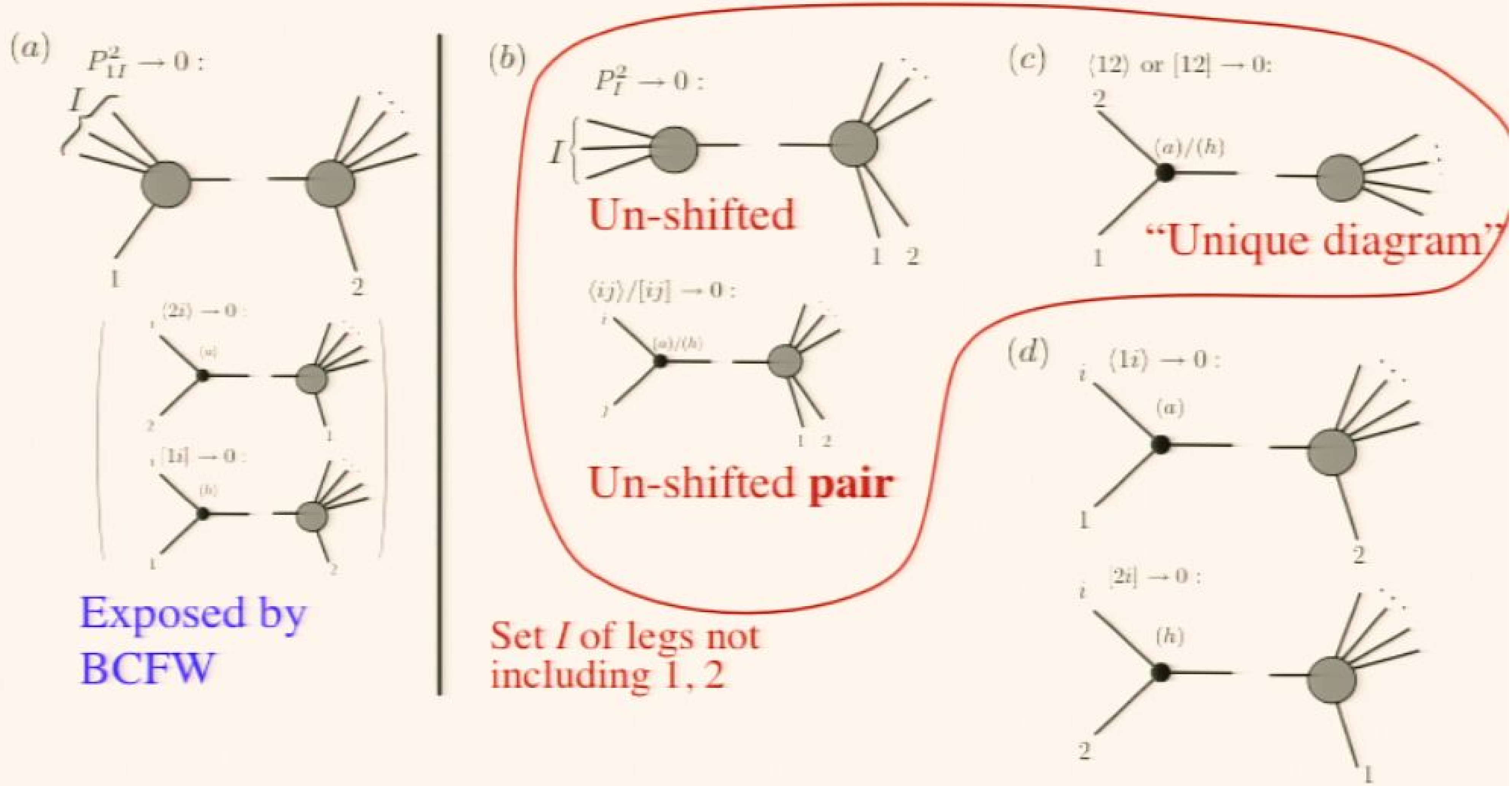
Consistency of n-Point Amplitudes

- Factorization on all physical poles
- No unphysical double poles

- No spurious poles, that do not correspond to intermediate particles propagating on-shell.
  - BCFW produces these poles, they always cancel.
  - This cancellation is also non-trivial from S-matrix perspective

What poles must we consider?

For definiteness, consider BCF where legs $|1]$ and $|2\rangle$ shift



"easy"

"hard"

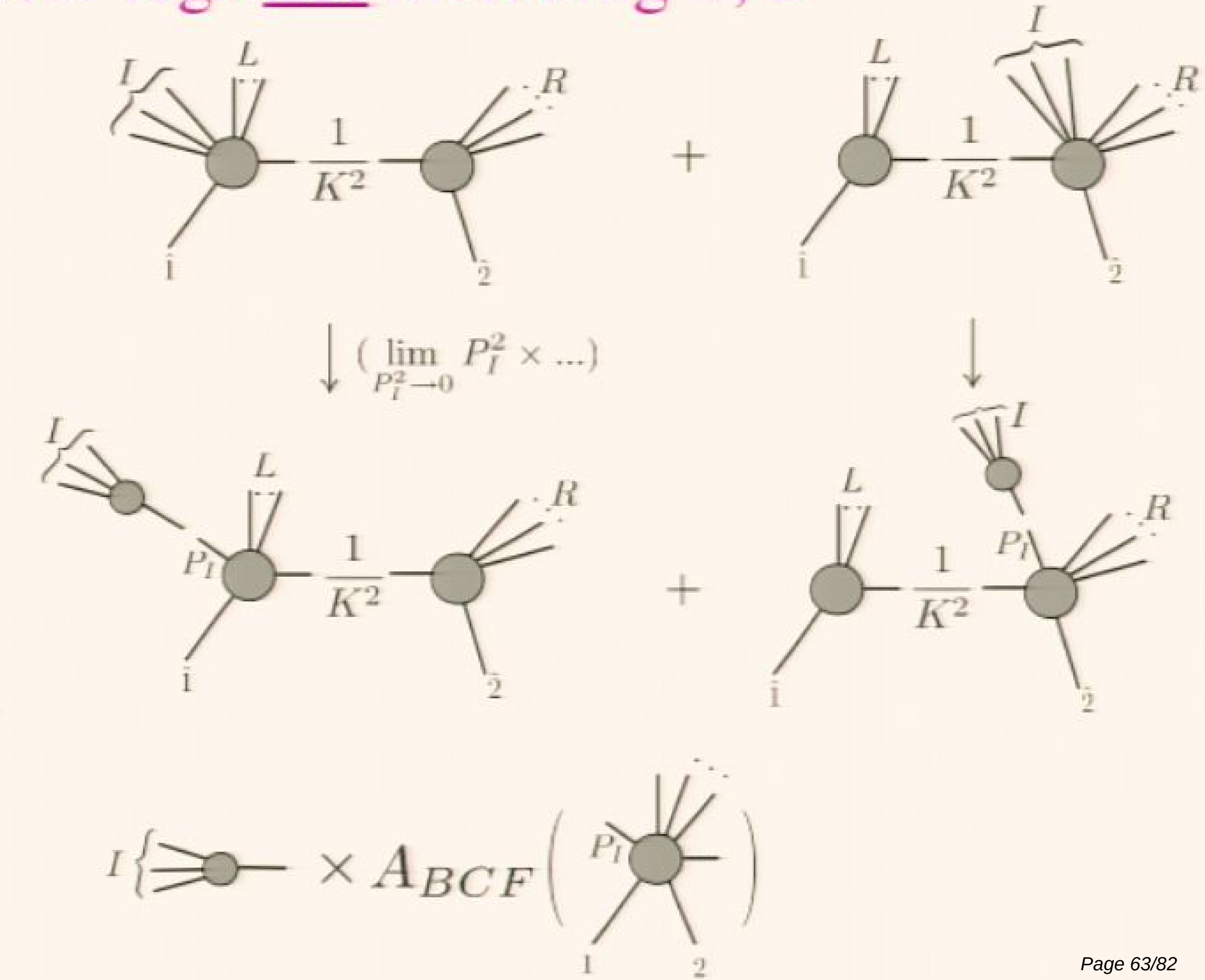
Wrong-helicity

(subtle for gravity**)

n-Point Unshifted Poles

I = set of 3 or more legs **not** including 1, 2

$$A_{BCF} \supset \sum_{L/R}$$

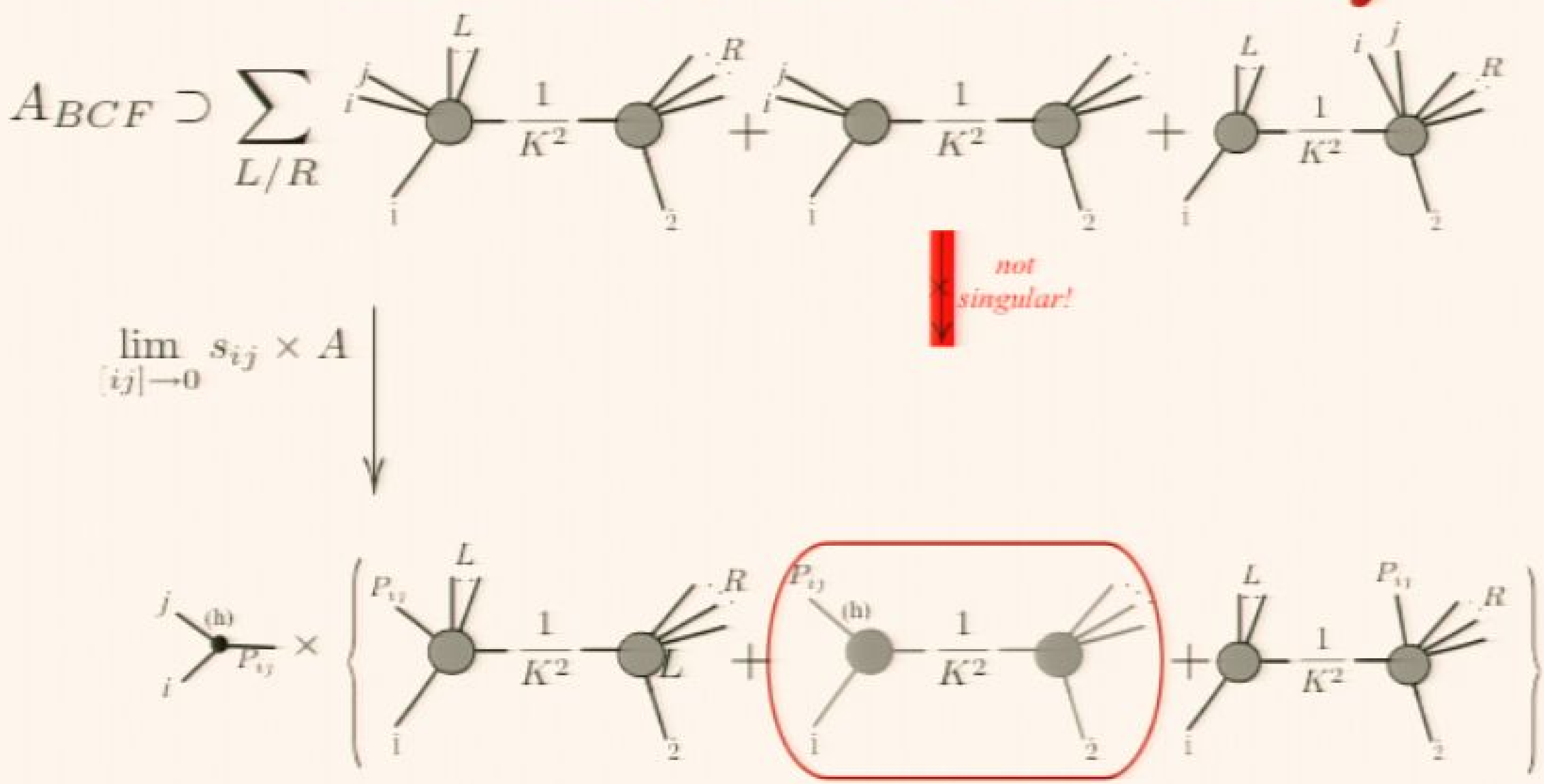


Pole(BCF sum)
= BCF sum(Pole)

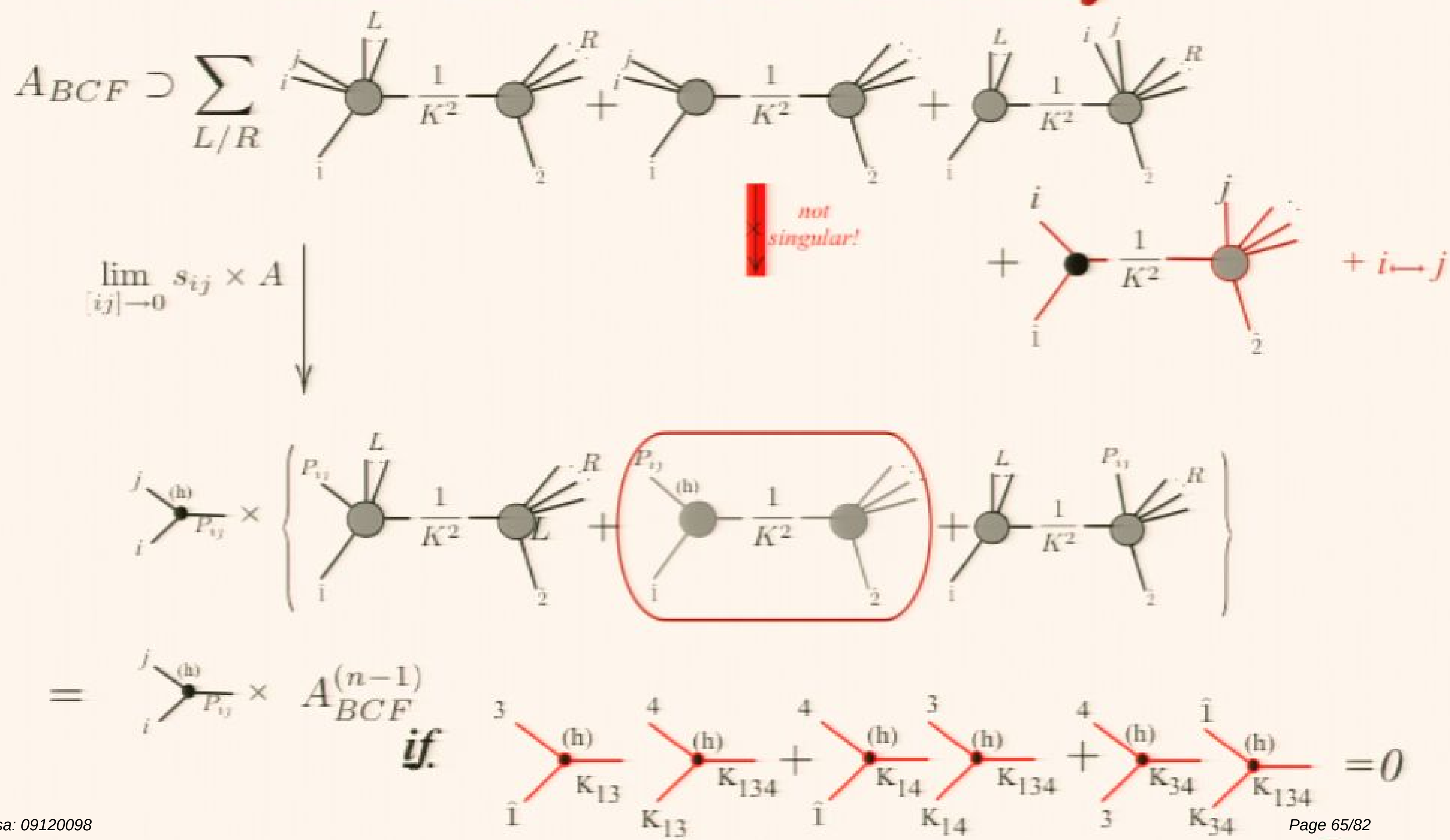
=

what if I=only 2 legs?

n-Point Unshifted Pair Poles and the Jacobi Identity



n-Point Unshifted Pair Poles and the Jacobi Identity

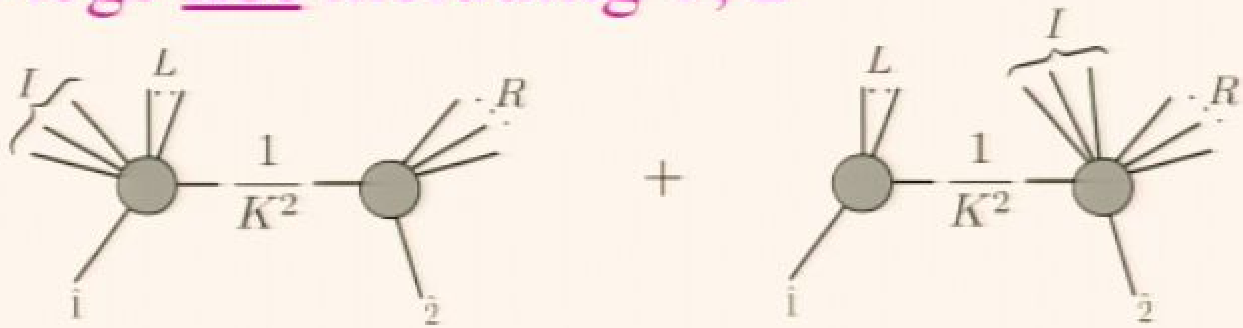


(Jacobi: $\kappa_{12a}\kappa_{34a} + \kappa_{13a}\kappa_{24a} + \kappa_{14a}\kappa_{32a} = 0$)

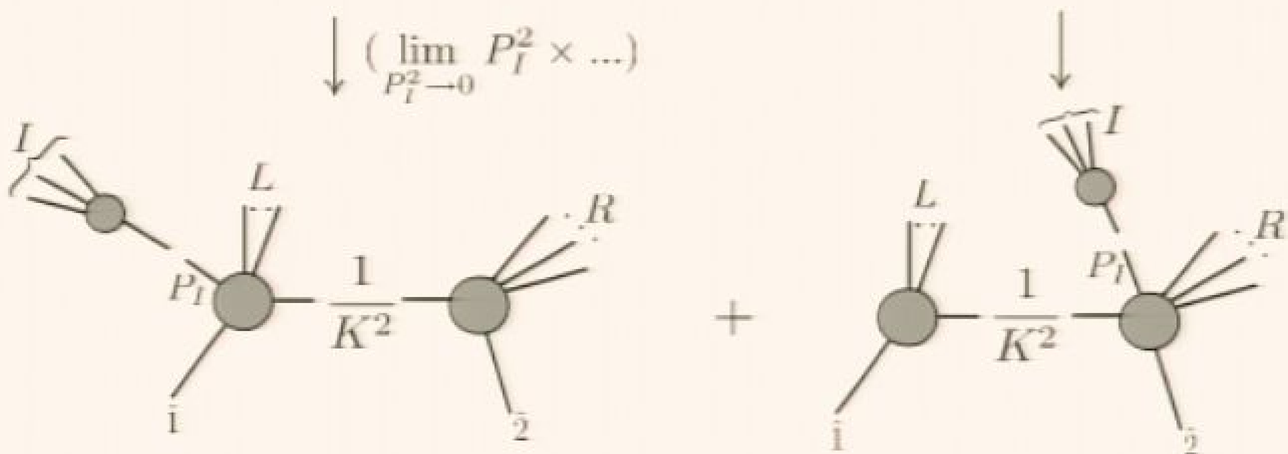
n-Point Unshifted Poles

I = set of 3 or more legs **not** including 1, 2

$$A_{BCF} \supset \sum_{L/R}$$



$\downarrow \left(\lim_{P_I^2 \to 0} P_I^2 \times ...\right)$



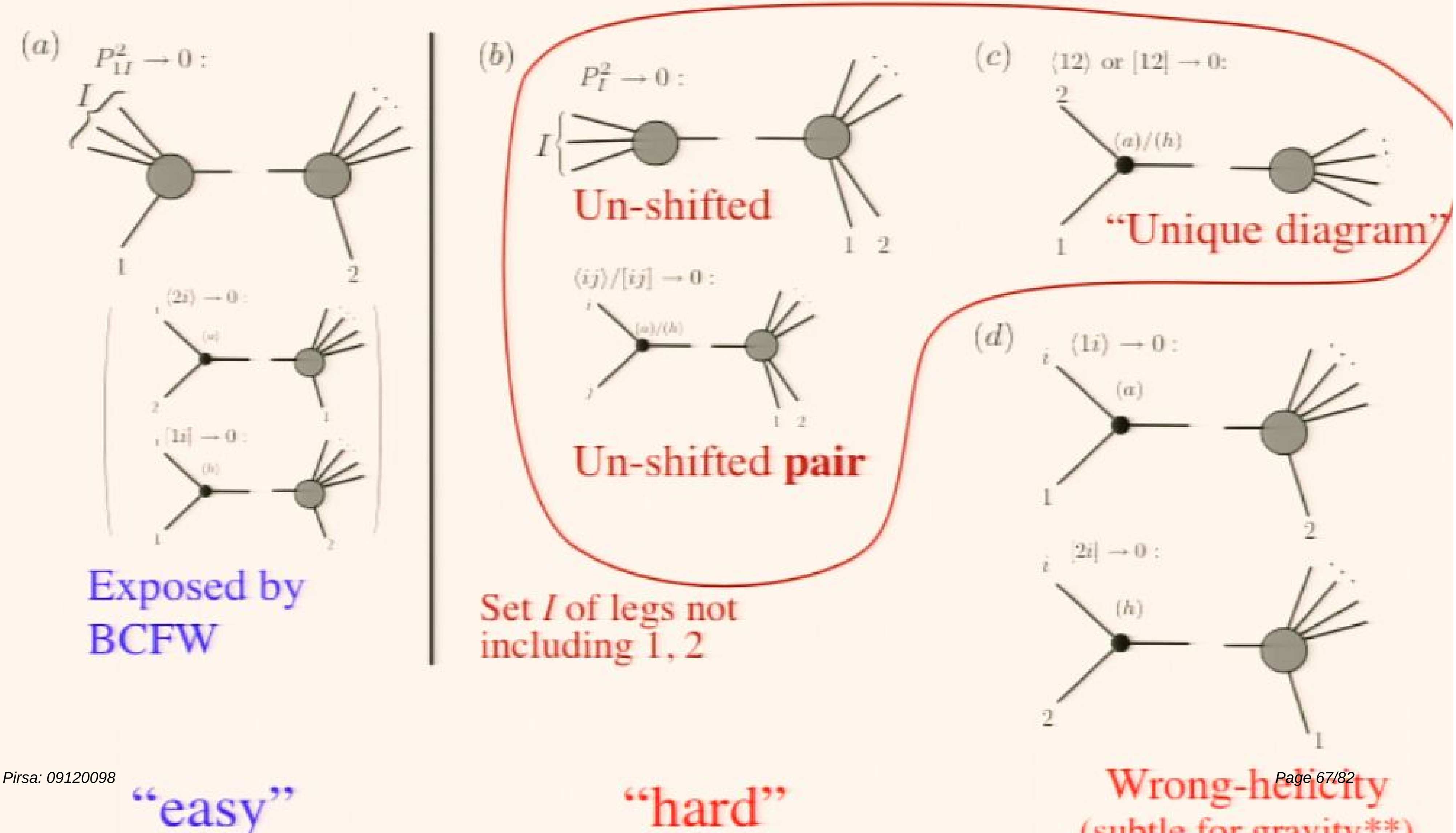
Pole(BCF sum)
= BCF sum(Pole)

=

what if I=only 2 legs?

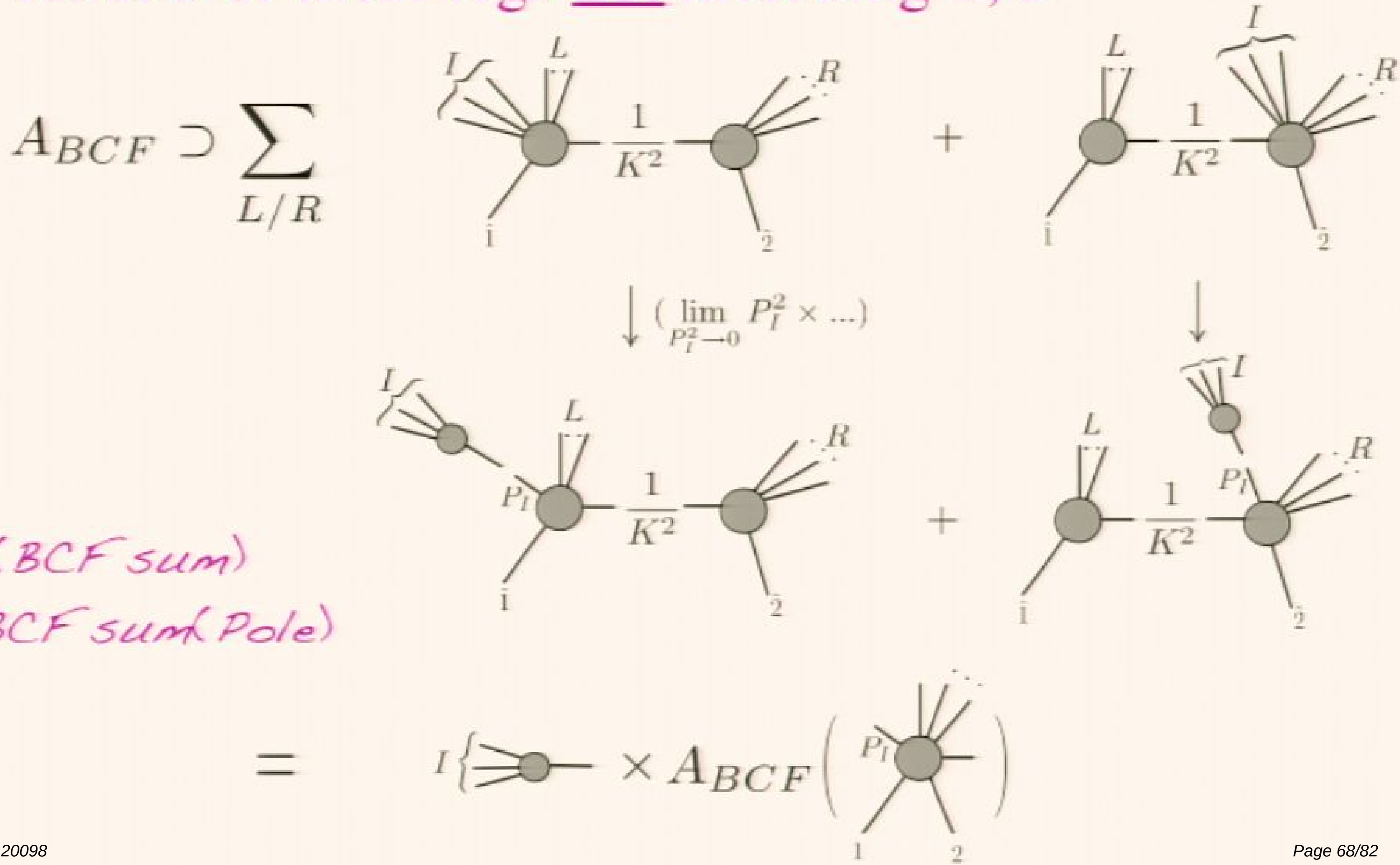
What poles must we consider?

For definiteness, consider BCF where legs $|1]$ and $|2\rangle$ shift



n-Point Unshifted Poles

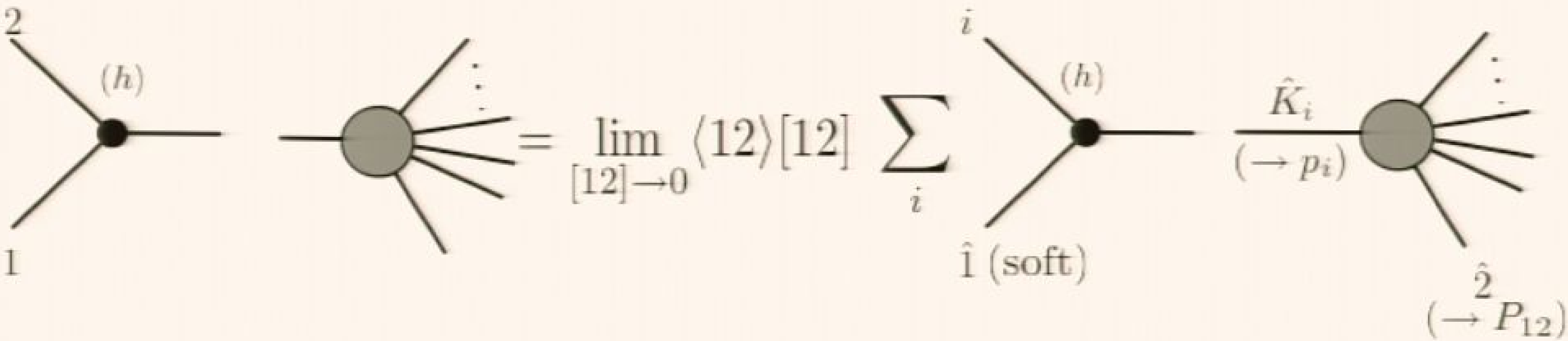
I = set of 3 or more legs **not** including 1, 2



Pole(BCF sum)
= BCF sum(Pole)

what if I=only 2 legs?

n-Point Unique Diagram Poles



<u>NO</u> BCFW terms have factorization limits with 1 & 2 on same side! *Something non-trivial must happen.*

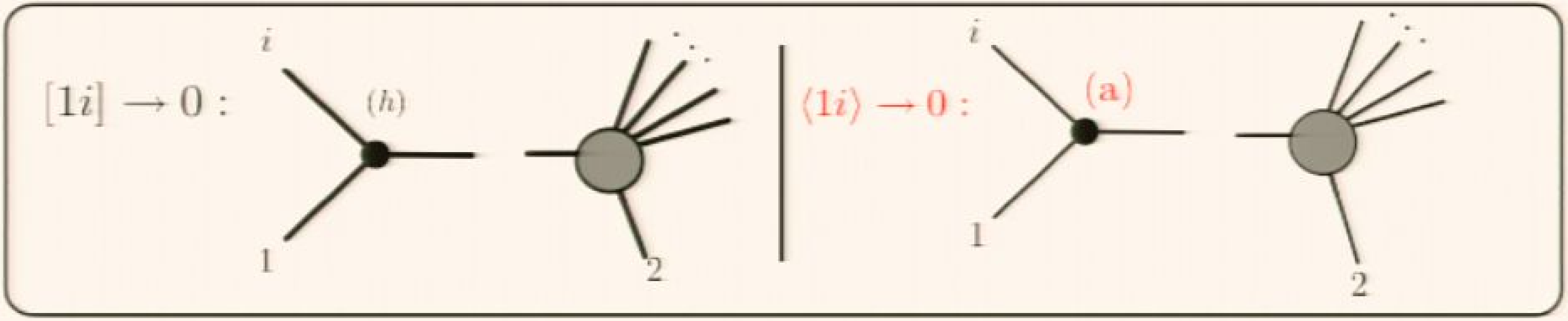
Right-hand amplitudes approach same kinematics as [12]->0
Singularities from **soft** limits

Very reminiscent of Weinberg's soft photon and graviton arguments – equality follows from gauge invariance/ conservation of charge.

(The Feynman diagrams w/ this pole dominate at large z, play a key role
n background-field proof of BCFW recursion [Arkani-Hamed,Kaplan].)

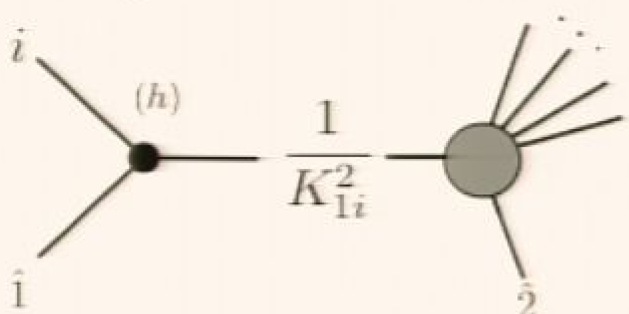
Wrong-Helicity Poles: Connecting Large-z Scaling & Factorization

For BCF shifting $|1]$:



Different 3-point kinematics <u>and different amplitude</u> –
factorization in 2nd case is not automatic!

Obvious diagram:

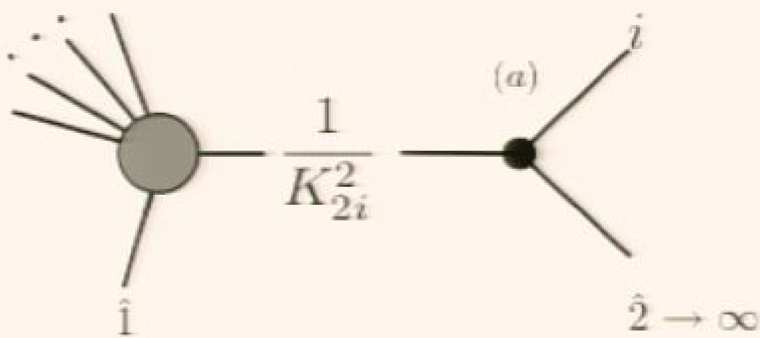


Not singular in this limit for spins s>0 because left amplitude scales as positive power of $\langle 1i \rangle$!

Wrong-Helicity Poles: Connecting Large-z Scaling & Factorization

Two possible contributions as $\langle 1i\rangle \to 0$:

A)



$$\langle i| \left(|2\rangle - z^* |1\rangle\right) = 0$$
$$\Rightarrow z^* = \langle 2i\rangle / \langle 1i\rangle \to \infty \text{ as } \langle 1i\rangle \to 0.$$

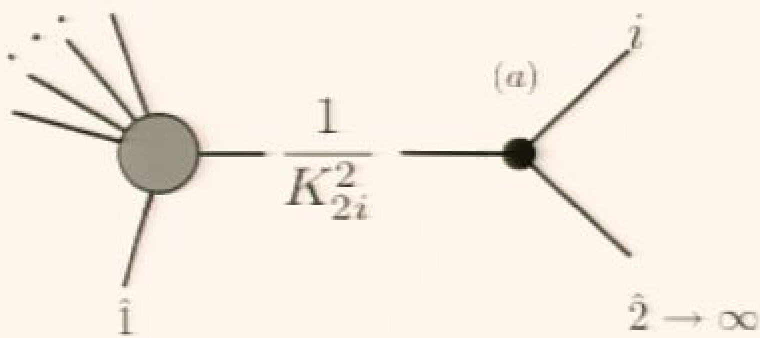
Pole if product of amplitudes grows at large-z.

B)

Wrong-Helicity Poles: Connecting Large-z Scaling & Factorization

Two possible contributions as $\langle 1i \rangle \to 0$:

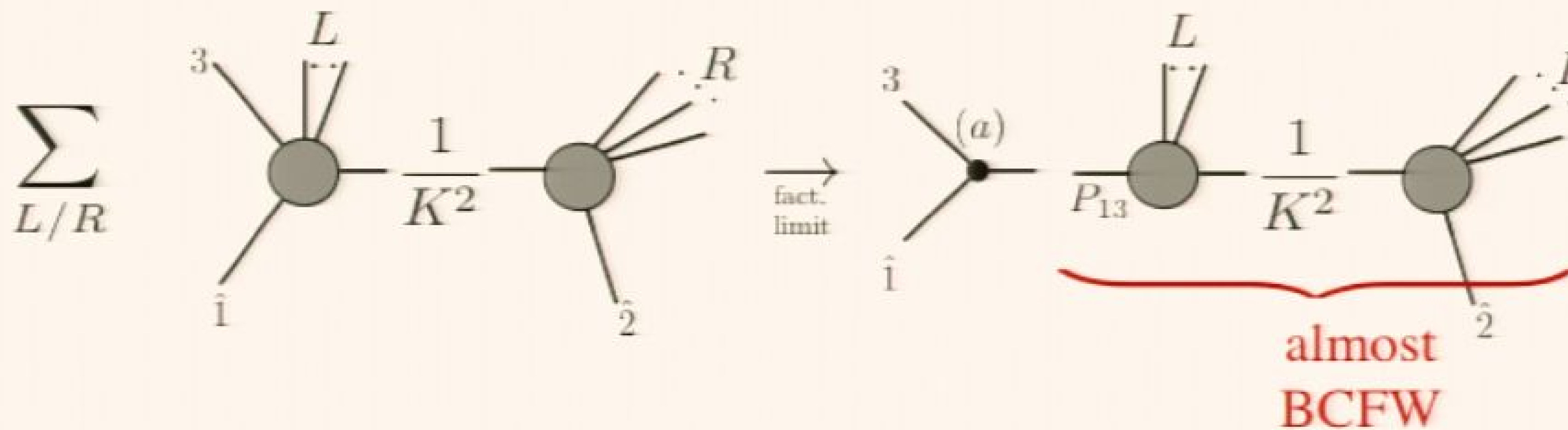
A)



$$\langle i | \left(|2\rangle - z^* |1\rangle \right) = 0$$
$$\Rightarrow z^* = \langle 2i \rangle / \langle 1i \rangle \to \infty \text{ as } \langle 1i \rangle \to 0.$$

Pole if product of amplitudes grows at large-z.

B)

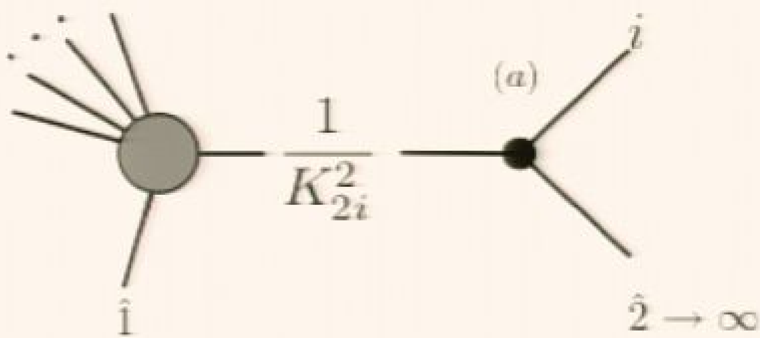


Consider general spin s (finish proof for spin-1, subtlety for spin-2)

Wrong-Helicity Poles: Connecting Large-z Scaling & Factorization

Two possible contributions as $\langle 1i\rangle \to 0$:

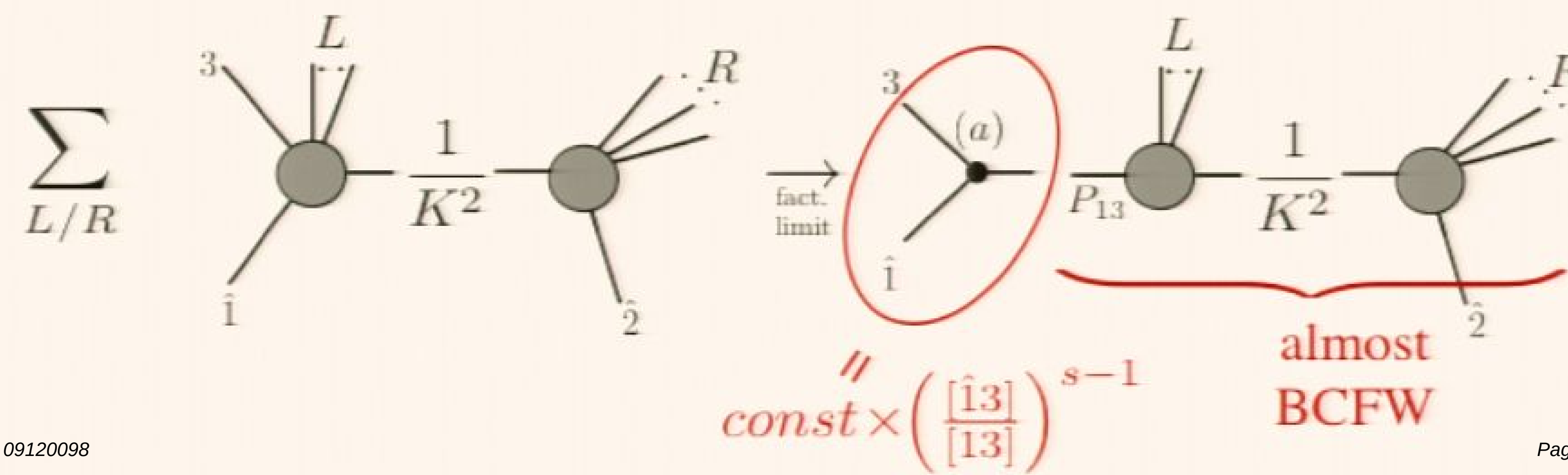
A)



$$\langle i|\left(|2\rangle - z^*|1\rangle\right) = 0$$
$$\Rightarrow z^* = \langle 2i\rangle / \langle 1i\rangle \to \infty \text{ as } \langle 1i\rangle \to 0.$$

Pole if product of amplitudes grows at large-z.

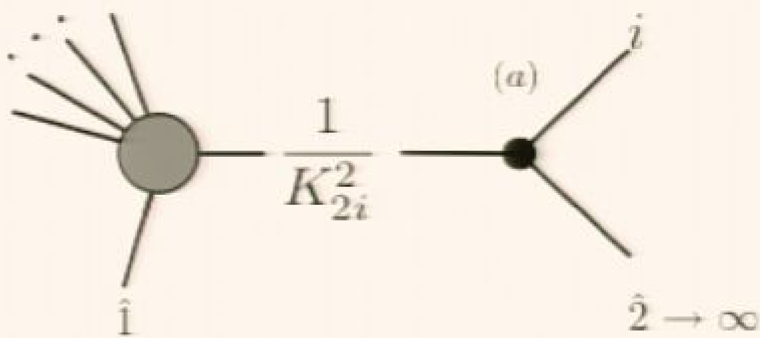
B)



Wrong-Helicity Poles: Connecting Large-z Scaling & Factorization

Two possible contributions as $\langle 1i\rangle \to 0$:

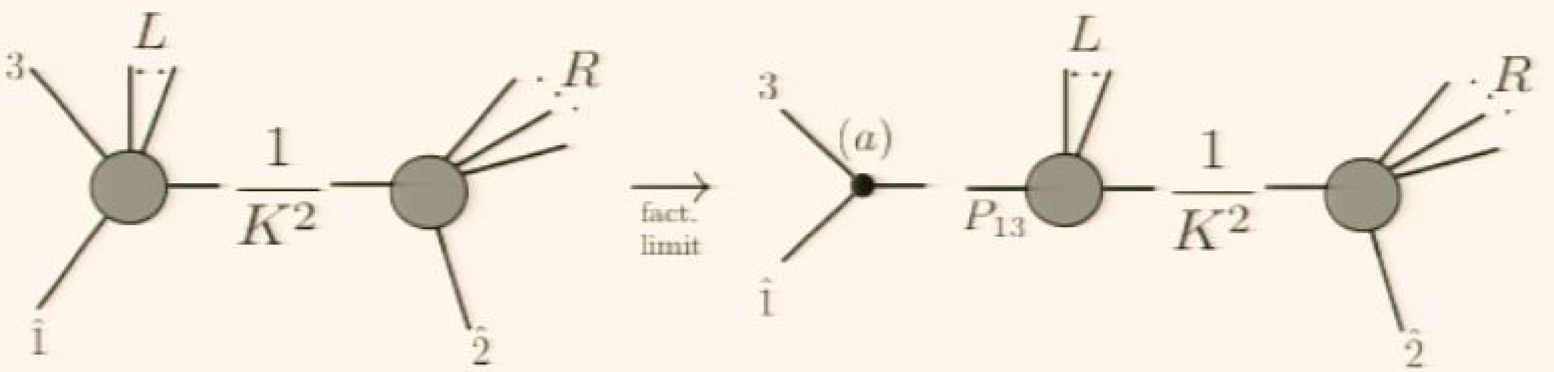
A)



$$\langle i|\left(|2\rangle - z^*|1\rangle\right) = 0$$
$$\Rightarrow z^* = \langle 2i\rangle/\langle 1i\rangle \to \infty \text{ as } \langle 1i\rangle \to 0.$$

Pole if product of amplitudes grows at large-z.

B) $\sum_{L/R}$



$$\longrightarrow \oint \left(1 + z\frac{[23]}{[13]}\right)^{s-1} \frac{A(z)}{z}$$

Wrong-Helicity Poles: Gauge Theory

Two possible contributions as $\langle 1i \rangle \to 0$:

A) Pole if product of amplitudes grows at large-z ($\epsilon \sim 1/z$)

h_1	h_2	h_3	h_K	$(n-1)$-point	Three-point	Total scaling
+	+	+	-	ϵ	$1/\epsilon$	1
+	+	-	+	$1/\epsilon^3$	ϵ^3	1
-	+	+	-	ϵ	$1/\epsilon$	1
-	+	-	+	ϵ	ϵ^3	ϵ^4
-	-	+	+	ϵ	$1/\epsilon$	1
-	-	-	X	(c) vanishes identically		
+	-	+	+	$1/\epsilon^3$	$1/\epsilon$	ϵ^{-4}
+	-	-	X	(c) vanishes identically		

good shifts: no contribution

(Scaling $1/z$ for "good shifts" z^3 for "bad shift" follows from helicity & power-counting – can derive without QFT!)

B) $\oint \left(1 + z\frac{[23]}{[13]}\right)^{s-1} \frac{A^{(n-1)}(z)}{z} = A^{(n-1)}(0)_{BCF}$ for $s=1$

gives correct factorization limit ✔

Wrong-Helicity Poles: Gravity

Two possible contributions as $\langle 1i \rangle \to 0$:

A) Pole if product of amplitudes grows at large-z ($\epsilon \sim 1/z$)

No contribution if large-z scaling of n-point amplitudes is like 3-point scaling (i.e. square of gauge theory scalings)

B) $$\oint \left(1 + z\frac{[23]}{[13]}\right)^{s-1} \frac{A(z)}{z} = A_{\rm BCF}^{(n-1)}(0) + \frac{[23]}{[13]} \oint A(z)$$

vanishes by $1/z^2$ scaling

Unlike $1/z$ in YM, the gravity $1/z^2$ is not obvious from BCFW, power counting, or any other arguments.

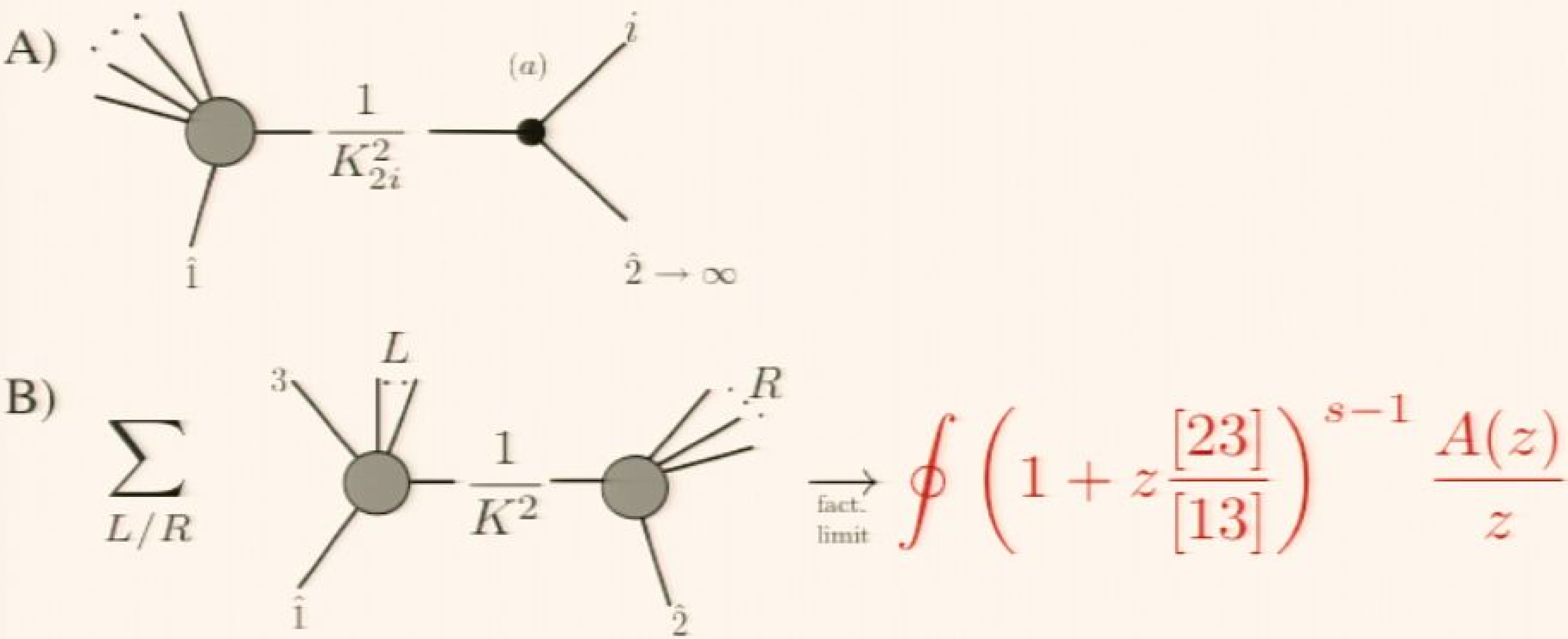
$1/z^2$

- Very opaque in direct Feynman diagrams (even $1/z$ requires summing many diagrams)
- Discovered in background field gauge
  [Arkani-Hamed and Kaplan]
  Consider hard graviton in background metric $h_{\mu\nu} = e^a_\mu \tilde{e}^{\tilde{a}}_\nu h_{a\tilde{a}}$

  "left" and "right" vielbein indices $a, \tilde{a}$ don't mix–
  two separate approximate Lorentz "spin" symmetries
  together constrain amplitudes to fall as $1/z^2$
- No known analogue of the twofold "spin Lorentz" symmetries in amplitudes
- Follows KLT relations: A_{GR} = "$(A_{YM})^2$"
- We'd like to understand origin of 1/z2 directly in S-matrix language – in fact it's **necessary** for BCFW to give sensible amplitudes

One More Possibility

- In YM and gravity, extra terms associated with $z \to \infty$ vanish



- Are there theories where, instead, BCFW gives well-behaved amplitudes because they cancel?

Summary

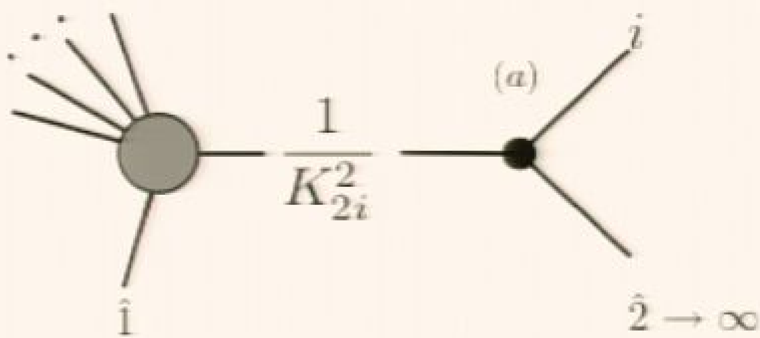
Hints at much more general structure to be understood:

- Consistency conditions for higher-spin interactions can be obtained from 4-point BCFW
- In known examples, BCFW's that work at 4-point construct consistent *n*-particle amplitudes
  - Spin-1: Guaranteed by simple arguments
  - Spin-2: Crucially relies on $1/z^2$ scaling

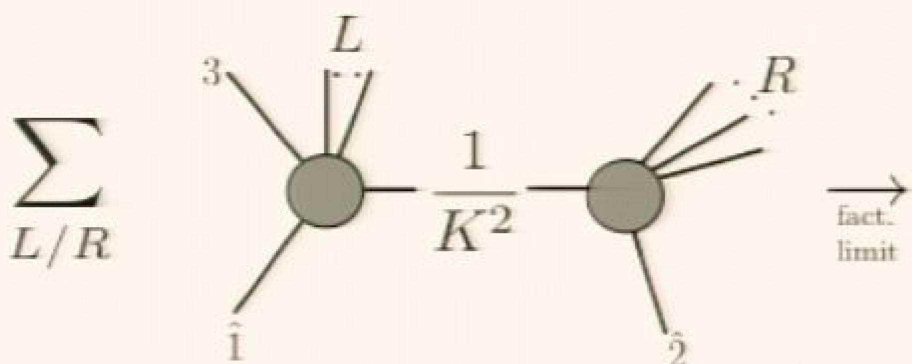
One More Possibility

- In YM and gravity, extra terms associated with $z \rightarrow \infty$ vanish

A)



B)

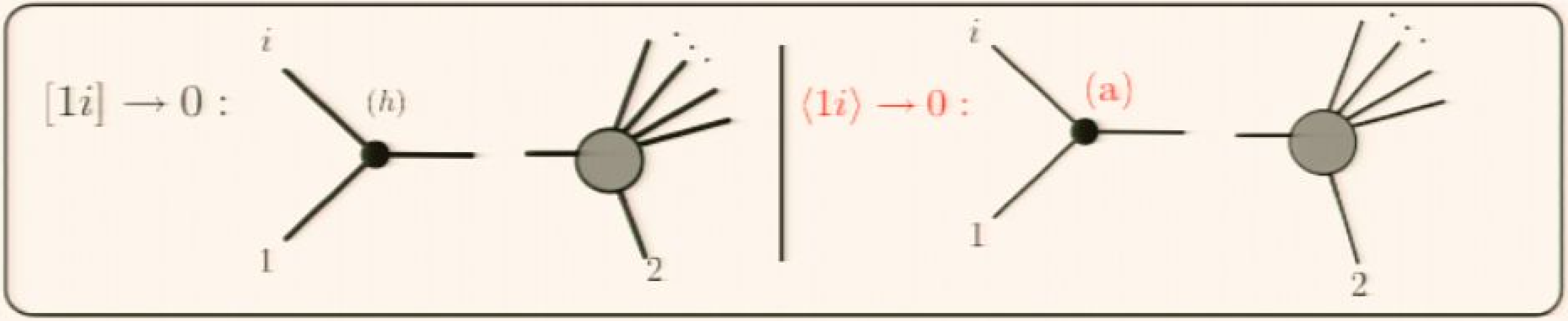


$$\oint \left(1 + z\frac{[23]}{[13]}\right)^{s-1} \frac{A(z)}{z}$$

- Are there theories where, instead, BCFW gives well-behaved amplitudes because they cancel?

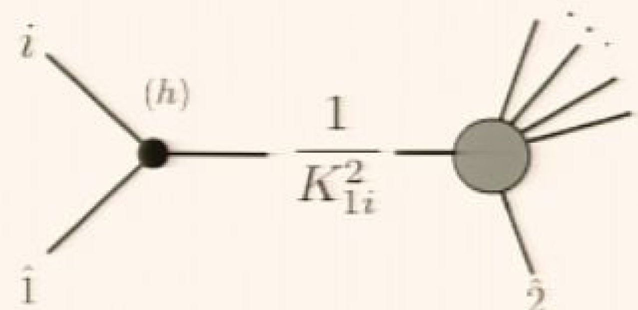
Wrong-Helicity Poles: Connecting Large-z Scaling & Factorization

For BCF shifting $|1]$:



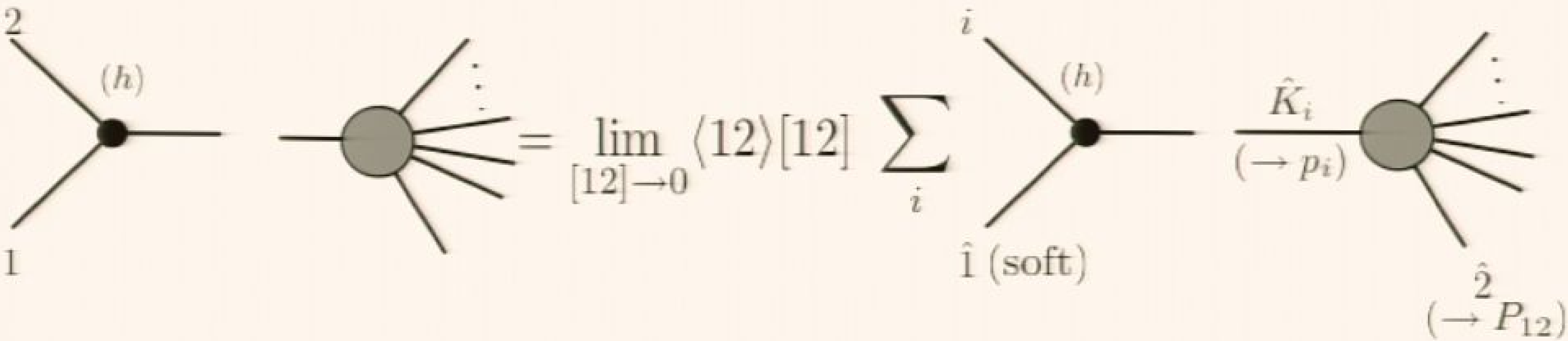
Different 3-point kinematics and different amplitude –
factorization in 2nd case is not automatic!

Obvious diagram:



Not singular in this limit for spins $s>0$ because left amplitude scales as positive power of $\langle 1i \rangle$!

n-Point Unique Diagram Poles



<u>NO</u> BCFW terms have factorization limits with 1 & 2 on same side!
Something non-trivial must happen.

Right-hand amplitudes approach same kinematics as [12]->0
Singularities from **soft** limits

Very reminiscent of Weinberg's soft photon and graviton arguments – equality follows from gauge invariance/ conservation of charge.

(The Feynman diagrams w/ this pole dominate at large z, play a key role
n background-field proof of BCFW recursion [Arkani-Hamed,Kaplan].)