Title: Foundations of Quantum Mech. (PHYS 639) - Lecture 11

Date: Dec 14, 2009 11:00 AM

URL: http://pirsa.org/09120088

Abstract:

The deBroglie-Bohm interpretation

Louis deBroglie
(1892-1987)

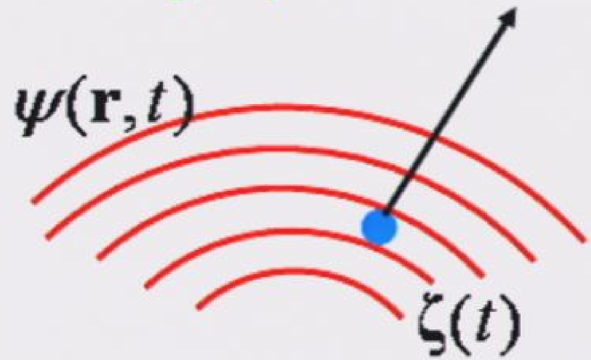
David Bohm
(1917-1992)

"I saw the impossible done..."

The deBroglie-Bohm interpretation for a single particle

The ontic state: $(\psi(\mathbf{r}), \zeta)$

Wavefunction ↗ ↖ Particle position



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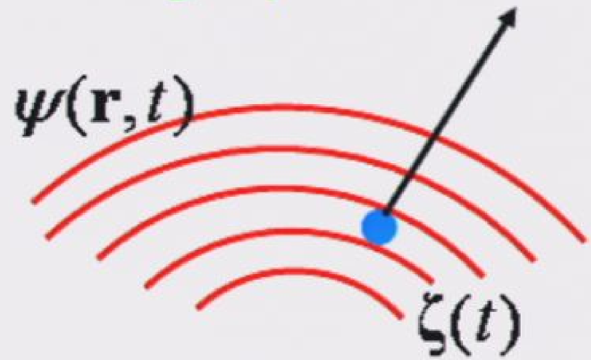

The evolution equations:

$$i\hbar \frac{\partial \psi(\mathbf{r},t)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2 \psi(\mathbf{r},t) + V(\mathbf{r})\psi(\mathbf{r},t)$$ Schrödinger's eq'n

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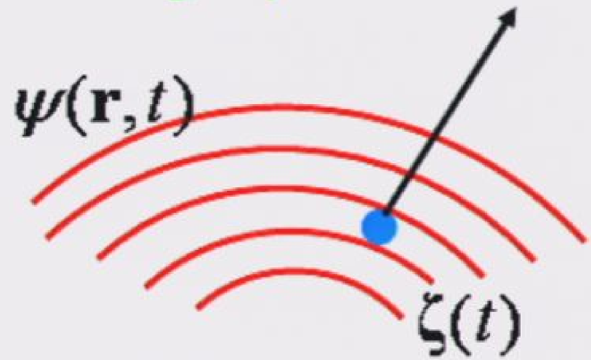

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$$\frac{d\zeta(t)}{dt} = \frac{1}{m}\left[\nabla S(\mathbf{r},t)\right]_{\mathbf{r}=\zeta(t)}$$

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where $\psi(\mathbf{r},t) = R(\mathbf{r},t)e^{iS(\mathbf{r},t)/\hbar}$

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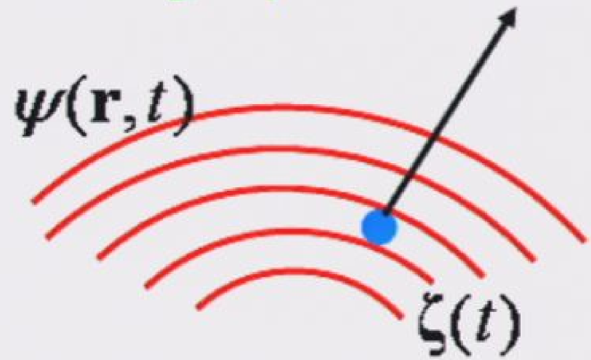

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The real part of the Schrodinger eq'n is:

$$\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} + Q + V = 0$$

where $Q(\mathbf{r},t) \equiv -\frac{\hbar^2}{2m}\frac{\nabla^2 R(\mathbf{r},t)}{R(\mathbf{r},t)}$ The "quantum potential"

The imaginary part of the Schrodinger eq'n is:

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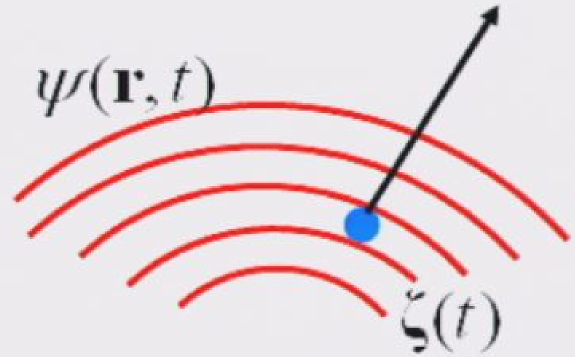
Newtonian form of the particle dynamics:



$$m\frac{d^2\zeta(t)}{dt^2} = -\left[\nabla V(\mathbf{r}) + \nabla Q(\mathbf{r},t)\right]_{\mathbf{r}=\zeta(t)}$$

where $$Q(\mathbf{r},t) \equiv -\frac{\hbar^2}{2m}\frac{\nabla^2 R(\mathbf{r},t)}{R(\mathbf{r},t)}$$ The "quantum potential"

(Note independence of quantum potential on amplitude)

Acting the ∇ operator on the real part of the Schrodinger eq'n gives:

$$\nabla\left[\frac{\partial S}{\partial t}+\frac{(\nabla S)^2}{2m}+Q+V\right]=0$$

$$\left(\frac{\partial}{\partial t}+\frac{\nabla S\cdot\nabla}{m}\right)\nabla S=-\nabla(Q+V)$$

Taking the time derivative of the guidance equation gives:

$$\frac{d\boldsymbol{\zeta}(t)}{dt}=\frac{1}{m}\left[\nabla S(\mathbf{r},t)\right]_{\mathbf{r}=\zeta(t)}$$

$$\frac{d^2\boldsymbol{\zeta}(t)}{dt^2}=\frac{1}{m}\left(\frac{\partial}{\partial t}+\frac{d\boldsymbol{\zeta}}{dt}\cdot\nabla\right)\nabla S$$

Thus

$$m\frac{d^2\zeta(t)}{dt^2}=-\left[\nabla V(\mathbf{r})+\nabla Q(\mathbf{r},t)\right]_{\mathbf{r}=\zeta(t)}$$

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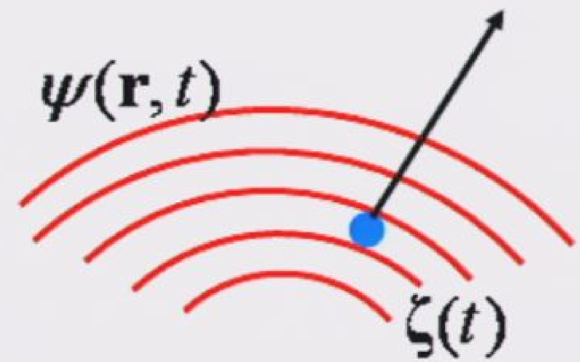

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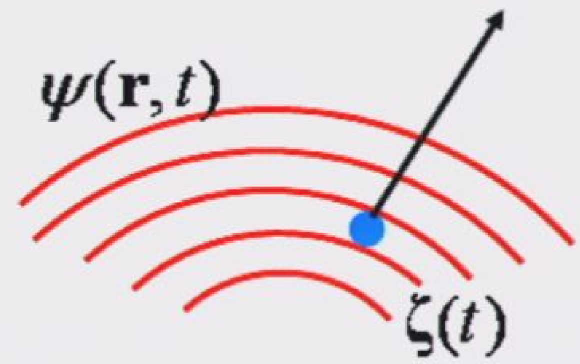

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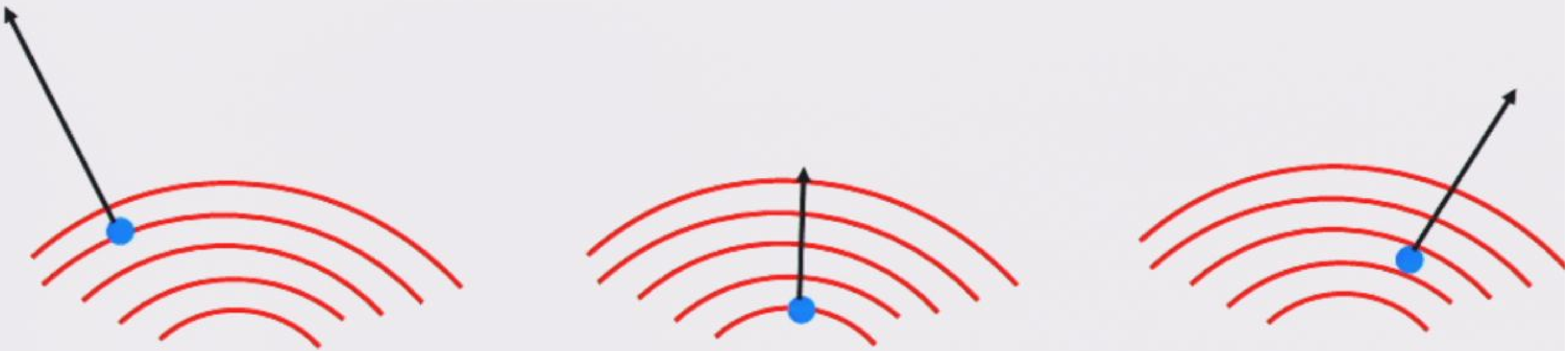
Nonetheless the dynamics are *fundamentally first order*

$$\frac{d\zeta(t)}{dt} = \frac{1}{m}\left[\nabla S(\mathbf{r},t)\right]_{\mathbf{r}=\zeta(t)}$$



Epistemic state (assuming perfect knowledge of $\psi(\mathbf{r},t)$)

$\rho(\zeta)d\zeta$ = the probability the particle is within $d\zeta$ of ζ.

The "standard distribution"

$$\rho(\zeta,t)=|\psi(\zeta,t)|^2$$

Note: it is preserved by the dynamics:

if $\rho(\zeta,0)=|\psi(\zeta,0)|^2$ then $\rho(\zeta,t)=|\psi(\zeta,t)|^2$

Proof of the preservation of the standard distribution:

The velocity field is

$$\mathbf{v}(\mathbf{r},t) = \frac{1}{m}\left[\nabla S(\mathbf{r},t)\right]$$

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1s orbital of Hydrogen atom

$|\psi(\mathbf{r})|^2$

$\rho(\zeta)$

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If only the *k*th wave is occupied

Then the guidance equation depends only on the *k*th wave

Proof of ineffectiveness of empty waves

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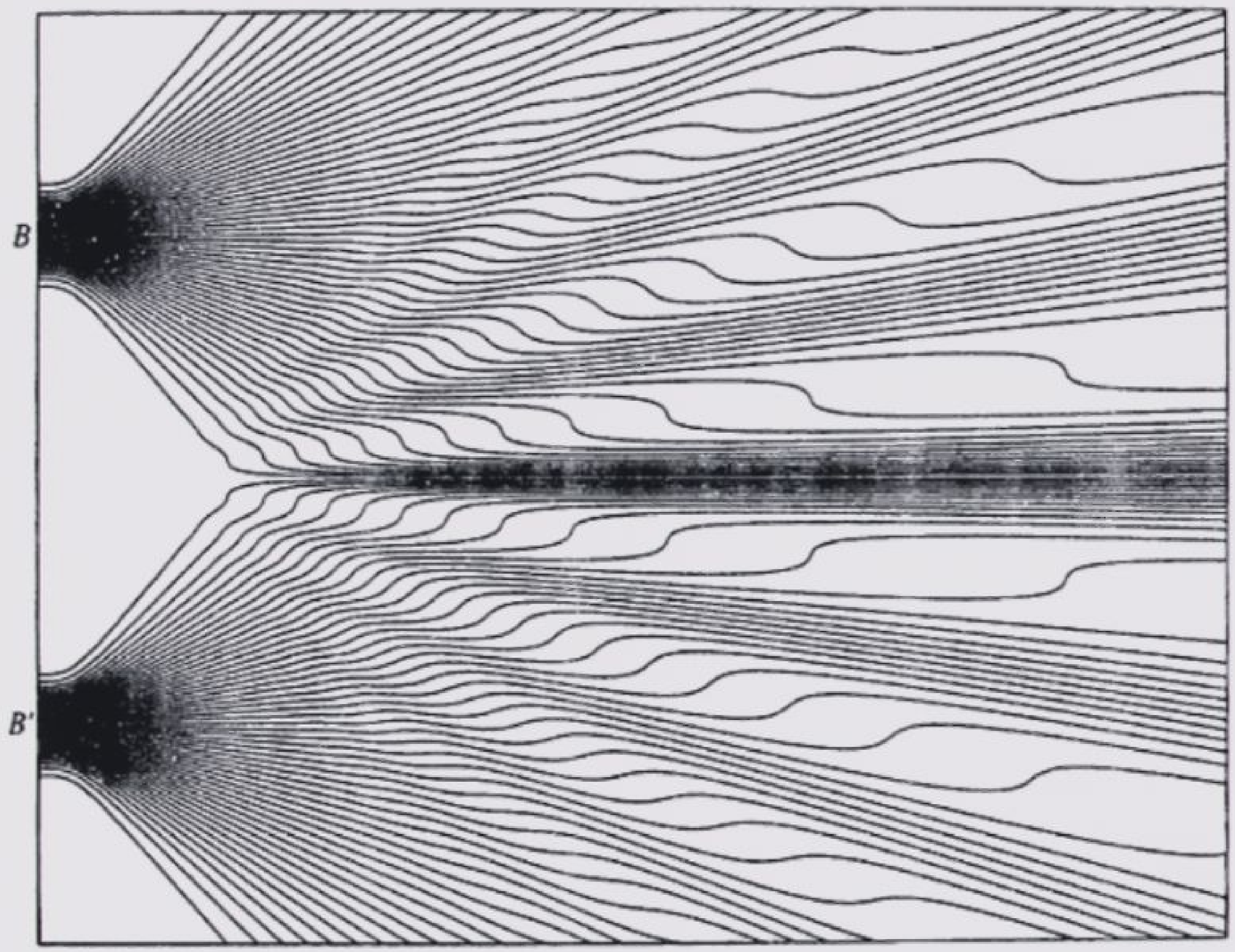
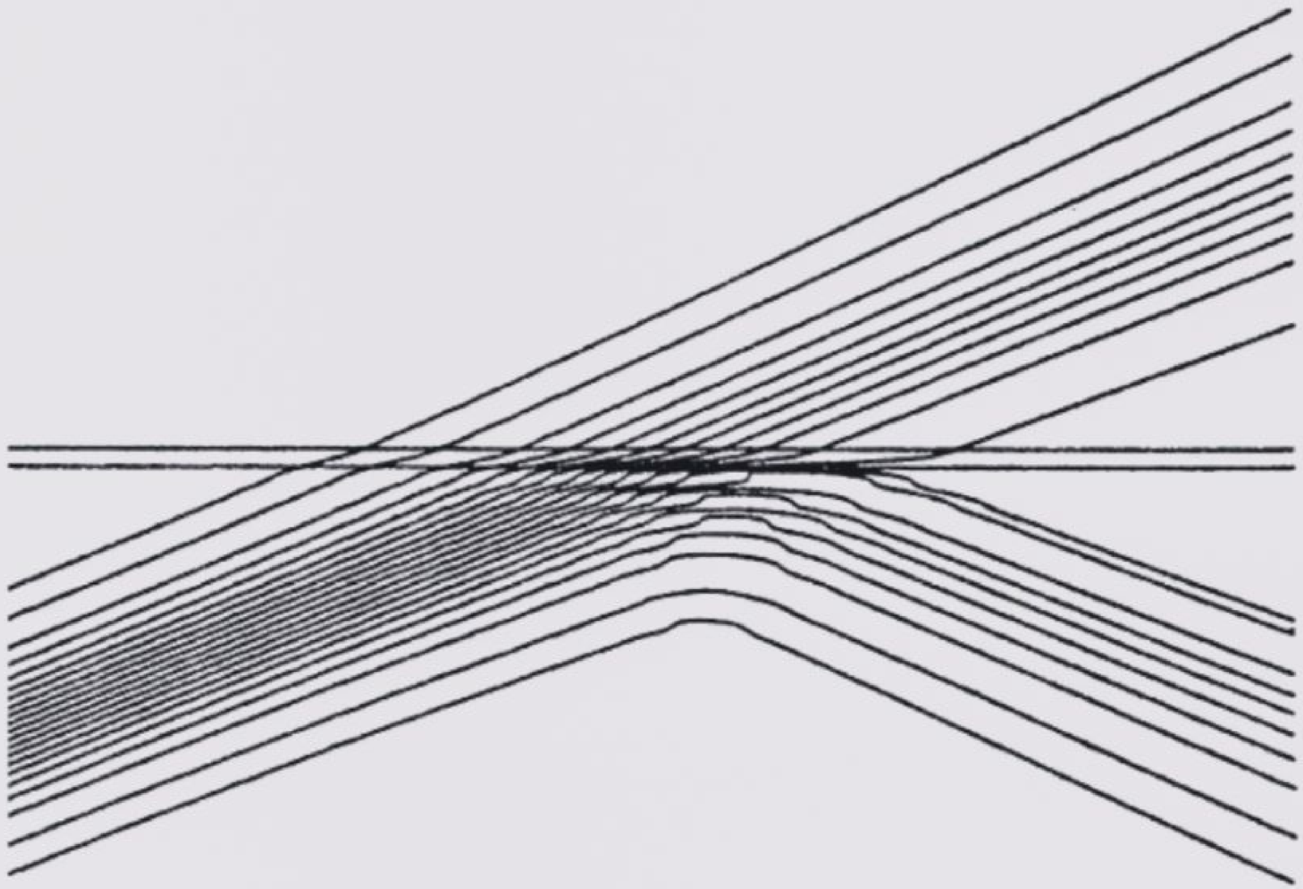
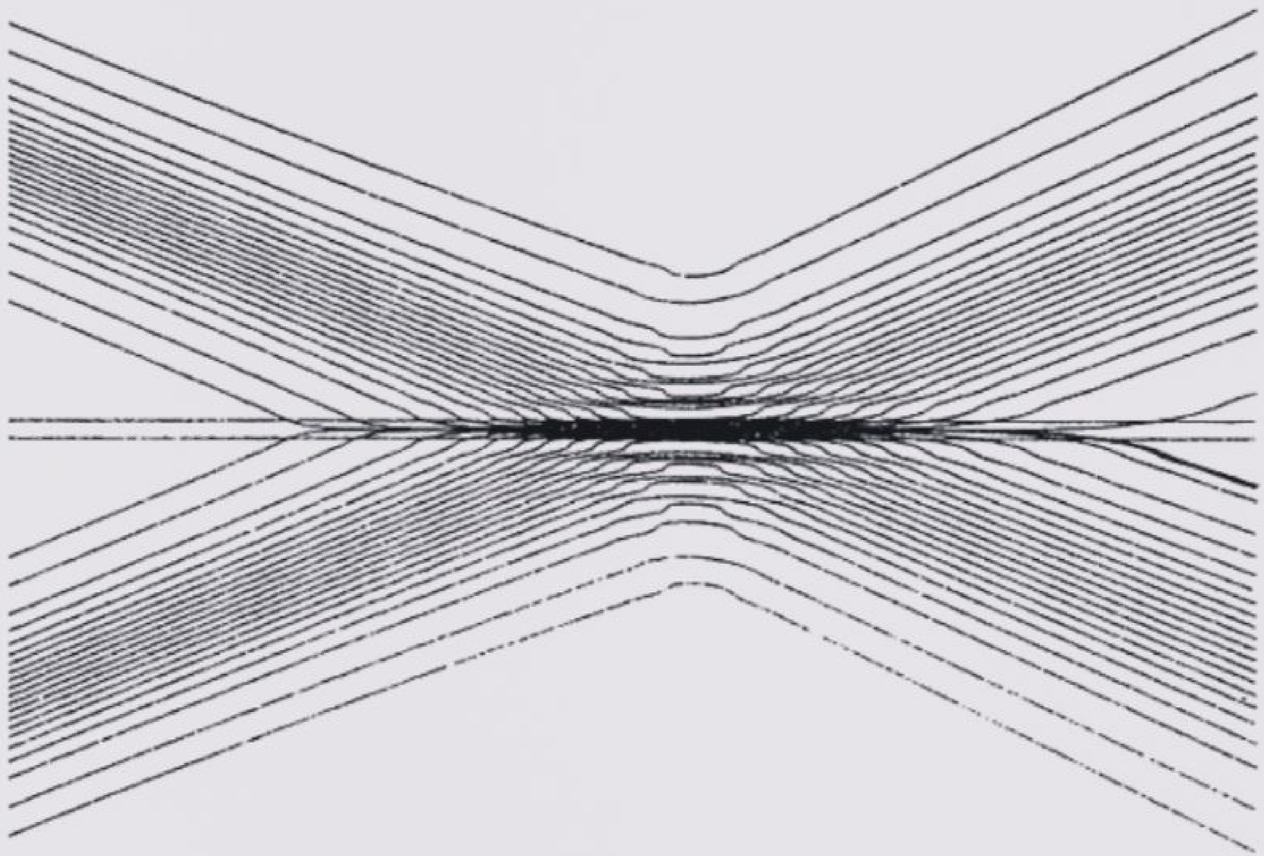
∇S



Double slit experiment



Transmission through a barrier (probability $\frac{1}{2}$)



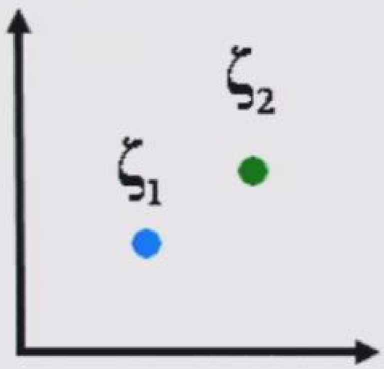
Beam splitter experiment

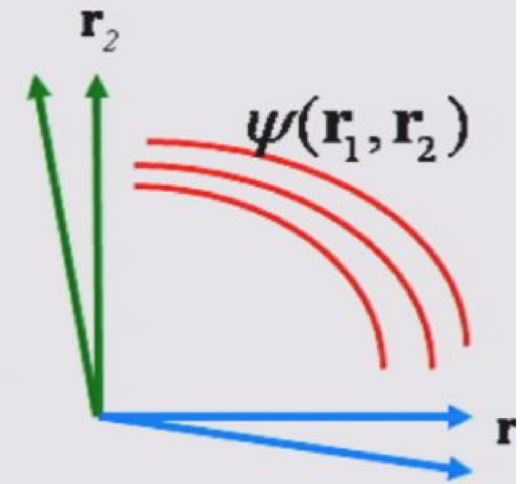
The deBroglie-Bohm interpretation for **many particles**

The ontic state: $(\psi(\mathbf{r}_1, \mathbf{r}_2), \zeta_1, \zeta_2)$

↗ Wavefunction on configuration space

↑ Particle positions





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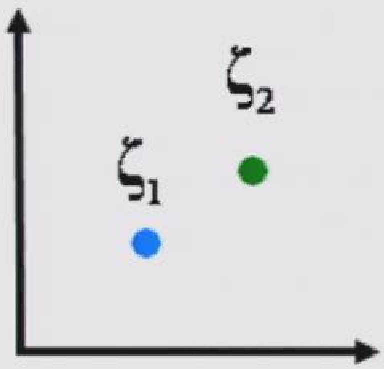
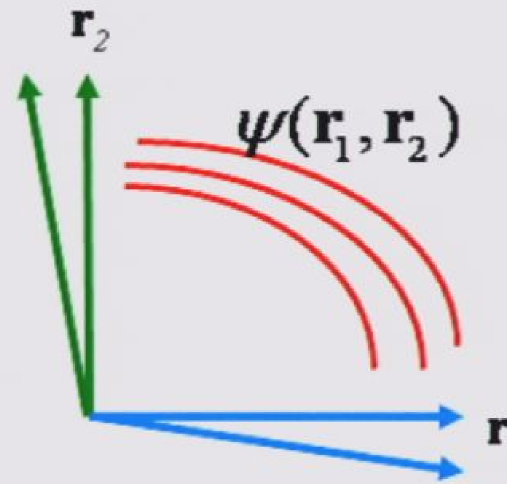
Wavefunction on configuration space — Particle positions





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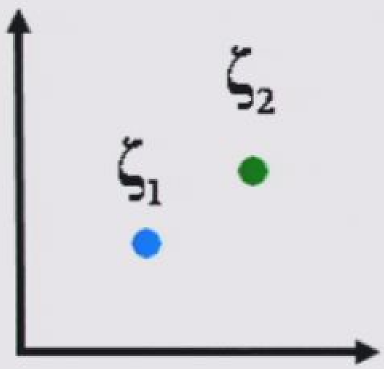
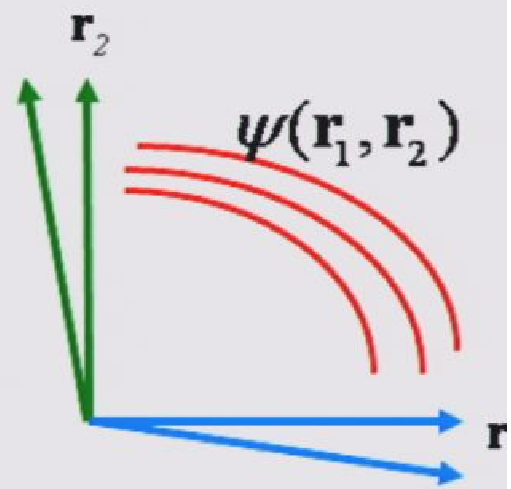
Wavefunction on configuration space ↗ (for $\psi(\mathbf{r}_1,\mathbf{r}_2)$)

Particle positions ↑ (for ζ_1,ζ_2)





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$$\left.\begin{aligned} \frac{d\zeta_1(t)}{dt} &= \frac{1}{m_1}\left[\nabla_1 S(\mathbf{r}_1,\mathbf{r}_2,t)\right]_{\mathbf{r}_1=\zeta_1(t),\mathbf{r}_2=\zeta_2(t)} \\ \frac{d\zeta_2(t)}{dt} &= \frac{1}{m_2}\left[\nabla_2 S(\mathbf{r}_1,\mathbf{r}_2,t)\right]_{\mathbf{r}_1=\zeta_1(t),\mathbf{r}_2=\zeta_2(t)} \end{aligned}\right\}$$

The guidance equation

$\psi(\mathbf{r}_1,\mathbf{r}_2,t) = R(\mathbf{r}_1,\mathbf{r}_2,t)\,e^{iS(\mathbf{r}_1,\mathbf{r}_2,t)/\hbar}$

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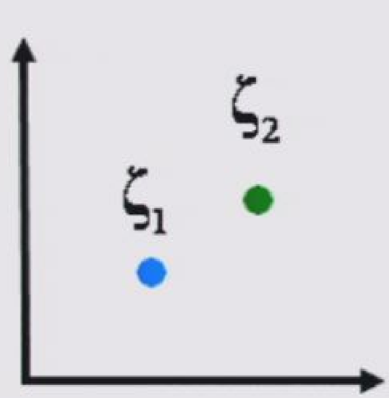

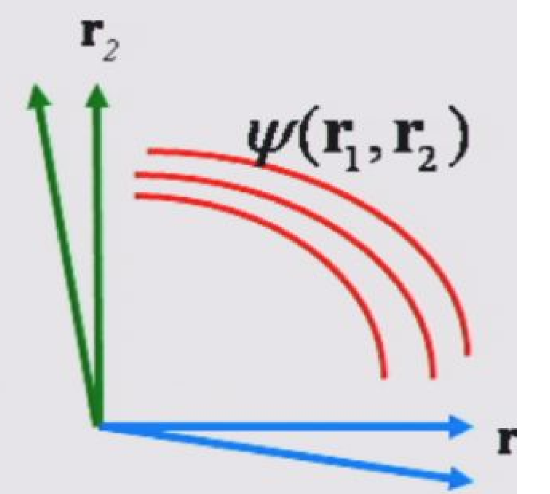

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$$\psi(\mathbf{r}_1, \mathbf{r}_2, t) = \phi^{(1)}(\mathbf{r}_1, t)\ \chi^{(2)}(\mathbf{r}_2, t)$$ Product state

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$$= R_1(\mathbf{r}_1, t) e^{iS_1(\mathbf{r}_1, t)/\hbar}\ R_2(\mathbf{r}_2, t) e^{iS_2(\mathbf{r}_2, t)/\hbar}$$

$$S(\mathbf{r}_1, \mathbf{r}_2, t) = S_1(\mathbf{r}_1, t) + S_2(\mathbf{r}_2, t)$$

$$\psi(\mathbf{r}_1,\mathbf{r}_2,t)=\phi^{(1)}(\mathbf{r}_1,t)\;\chi^{(2)}(\mathbf{r}_2,t)$$ Product state

$$=R_1(\mathbf{r}_1,t)e^{iS_1(\mathbf{r}_1,t)/\hbar}\;R_2(\mathbf{r}_2,t)e^{iS_2(\mathbf{r}_2,t)/\hbar}$$

$$S(\mathbf{r}_1,\mathbf{r}_2,t)=S_1(\mathbf{r}_1,t)+S_2(\mathbf{r}_2,t)$$

$$\frac{d\boldsymbol{\zeta}_1(t)}{dt}=\frac{1}{m_1}\left[\nabla_1 S(\mathbf{r}_1,\mathbf{r}_2,t)\right]_{\mathbf{r}_1=\boldsymbol{\zeta}_1(t),\,\mathbf{r}_2=\boldsymbol{\zeta}_2(t)}=\frac{1}{m_1}\left[\nabla_1 S_1(\mathbf{r}_1,t)\right]_{\mathbf{r}_1=\boldsymbol{\zeta}_1(t)}$$

$$\psi(\mathbf{r}_1,\mathbf{r}_2,t)=\phi^{(1)}(\mathbf{r}_1,t)\;\chi^{(2)}(\mathbf{r}_2,t)$$ Product state

$$=R_1(\mathbf{r}_1,t)e^{iS_1(\mathbf{r}_1,t)/\hbar}\;R_2(\mathbf{r}_2,t)e^{iS_2(\mathbf{r}_2,t)/\hbar}$$

$$S(\mathbf{r}_1,\mathbf{r}_2,t)=S_1(\mathbf{r}_1,t)+S_2(\mathbf{r}_2,t)$$

$$\frac{d\boldsymbol{\zeta}_1(t)}{dt}=\frac{1}{m_1}\left[\nabla_1 S(\mathbf{r}_1,\mathbf{r}_2,t)\right]_{\mathbf{r}_1=\boldsymbol{\zeta}_1(t),\,\mathbf{r}_2=\boldsymbol{\zeta}_2(t)}=\frac{1}{m_1}\left[\nabla_1 S_1(\mathbf{r}_1,t)\right]_{\mathbf{r}_1=\boldsymbol{\zeta}_1(t)}$$

$$\frac{d\boldsymbol{\zeta}_2(t)}{dt}=\frac{1}{m_2}\left[\nabla_2 S(\mathbf{r}_1,\mathbf{r}_2,t)\right]_{\mathbf{r}_1=\boldsymbol{\zeta}_1(t),\,\mathbf{r}_2=\boldsymbol{\zeta}_2(t)}=\frac{1}{m_2}\left[\nabla_2 S_2(\mathbf{r}_2,t)\right]_{\mathbf{r}_2=\boldsymbol{\zeta}_2(t)}$$

$$\psi(\mathbf{r}_1,\mathbf{r}_2,t)=\phi^{(1)}(\mathbf{r}_1,t)\;\chi^{(2)}(\mathbf{r}_2,t)$$ Product state

$$=R_1(\mathbf{r}_1,t)e^{iS_1(\mathbf{r}_1,t)/\hbar}\;R_2(\mathbf{r}_2,t)e^{iS_2(\mathbf{r}_2,t)/\hbar}$$

$$S(\mathbf{r}_1,\mathbf{r}_2,t)=S_1(\mathbf{r}_1,t)+S_2(\mathbf{r}_2,t)$$

$$\frac{d\boldsymbol{\zeta}_1(t)}{dt}=\frac{1}{m_1}\left[\nabla_1 S(\mathbf{r}_1,\mathbf{r}_2,t)\right]_{\mathbf{r}_1=\boldsymbol{\zeta}_1(t),\,\mathbf{r}_2=\boldsymbol{\zeta}_2(t)}=\frac{1}{m_1}\left[\nabla_1 S_1(\mathbf{r}_1,t)\right]_{\mathbf{r}_1=\boldsymbol{\zeta}_1(t)}$$

$$\frac{d\boldsymbol{\zeta}_2(t)}{dt}=\frac{1}{m_2}\left[\nabla_2 S(\mathbf{r}_1,\mathbf{r}_2,t)\right]_{\mathbf{r}_1=\boldsymbol{\zeta}_1(t),\,\mathbf{r}_2=\boldsymbol{\zeta}_2(t)}=\frac{1}{m_2}\left[\nabla_2 S_2(\mathbf{r}_2,t)\right]_{\mathbf{r}_2=\boldsymbol{\zeta}_2(t)}$$

The two particles evolve independently

$$\psi(\mathbf{r}_1, \mathbf{r}_2, t) = \phi^{(1)}(\mathbf{r}_1, t)\ \chi^{(2)}(\mathbf{r}_2, t)$$ Product state

$$= R_1(\mathbf{r}_1, t) e^{iS_1(\mathbf{r}_1, t)/\hbar}\ R_2(\mathbf{r}_2, t) e^{iS_2(\mathbf{r}_2, t)/\hbar}$$

$$S(\mathbf{r}_1, \mathbf{r}_2, t) = S_1(\mathbf{r}_1, t) + S_2(\mathbf{r}_2, t)$$

$$\frac{d\boldsymbol{\zeta}_1(t)}{dt} = \frac{1}{m_1}\left[\nabla_1 S(\mathbf{r}_1, \mathbf{r}_2, t)\right]_{\mathbf{r}_1 = \boldsymbol{\zeta}_1(t),\, \mathbf{r}_2 = \boldsymbol{\zeta}_2(t)} = \frac{1}{m_1}\left[\nabla_1 S_1(\mathbf{r}_1, t)\right]_{\mathbf{r}_1 = \boldsymbol{\zeta}_1(t)}$$

$$\frac{d\boldsymbol{\zeta}_2(t)}{dt} = \frac{1}{m_2}\left[\nabla_2 S(\mathbf{r}_1, \mathbf{r}_2, t)\right]_{\mathbf{r}_1 = \boldsymbol{\zeta}_1(t),\, \mathbf{r}_2 = \boldsymbol{\zeta}_2(t)} = \frac{1}{m_2}\left[\nabla_2 S_2(\mathbf{r}_2, t)\right]_{\mathbf{r}_2 = \boldsymbol{\zeta}_2(t)}$$

The two particles evolve independently

$$\psi(\mathbf{r}_1, \mathbf{r}_2, t) = \sum_j c_j \phi_j^{(1)}(\mathbf{r}_1, t)\ \chi_j^{(2)}(\mathbf{r}_2, t)$$ Entangled state

$$\psi(\mathbf{r}_1, \mathbf{r}_2, t) = \sum_j c_j \phi_j^{(1)}(\mathbf{r}_1, t)\, \chi_j^{(2)}(\mathbf{r}_2, t)$$ Entangled state

$(\zeta_1, \zeta_2) \in$ support of $\phi_j^{(1)}(\mathbf{r}_1, t)\, \chi_j^{(2)}(\mathbf{r}_2, t)$ jth wave is **occupied**

$$\psi(\mathbf{r}_1, \mathbf{r}_2, t) = \sum_j c_j \phi_j^{(1)}(\mathbf{r}_1, t)\ \chi_j^{(2)}(\mathbf{r}_2, t)$$ Entangled state

$(\zeta_1, \zeta_2) \in$ support of $\phi_j^{(1)}(\mathbf{r}_1, t)\ \chi_j^{(2)}(\mathbf{r}_2, t)$ *j*th wave is **occupied**

$(\zeta_1, \zeta_2) \notin$ support of $\phi_j^{(1)}(\mathbf{r}_1, t)\ \chi_j^{(2)}(\mathbf{r}_2, t)$ *j*th wave is **empty**

$$\psi(\mathbf{r}_1, \mathbf{r}_2, t) = \sum_j c_j \phi_j^{(1)}(\mathbf{r}_1, t)\, \chi_j^{(2)}(\mathbf{r}_2, t)$$ Entangled state

$(\zeta_1, \zeta_2) \in$ support of $\phi_j^{(1)}(\mathbf{r}_1, t)\, \chi_j^{(2)}(\mathbf{r}_2, t)$ *j*th wave is **occupied**

$(\zeta_1, \zeta_2) \notin$ support of $\phi_j^{(1)}(\mathbf{r}_1, t)\, \chi_j^{(2)}(\mathbf{r}_2, t)$ *j*th wave is **empty**

$$\psi(\mathbf{r}_1,\mathbf{r}_2,t)=\sum_j c_j\phi_j^{(1)}(\mathbf{r}_1,t)\,\chi_j^{(2)}(\mathbf{r}_2,t)$$ Entangled state

$(\zeta_1,\zeta_2)\in$ support of $\phi_j^{(1)}(\mathbf{r}_1,t)\,\chi_j^{(2)}(\mathbf{r}_2,t)$ *j*th wave is **occupied**

$(\zeta_1,\zeta_2)\notin$ support of $\phi_j^{(1)}(\mathbf{r}_1,t)\,\chi_j^{(2)}(\mathbf{r}_2,t)$ *j*th wave is **empty**

If only the *k*th wave is occupied

$$\psi(\mathbf{r}_1, \mathbf{r}_2, t) = \sum_j c_j \phi_j^{(1)}(\mathbf{r}_1, t)\, \chi_j^{(2)}(\mathbf{r}_2, t)$$ Entangled state

$(\zeta_1, \zeta_2) \in$ support of $\phi_j^{(1)}(\mathbf{r}_1, t)\, \chi_j^{(2)}(\mathbf{r}_2, t)$ *j*th wave is **occupied**

$(\zeta_1, \zeta_2) \notin$ support of $\phi_j^{(1)}(\mathbf{r}_1, t)\, \chi_j^{(2)}(\mathbf{r}_2, t)$ *j*th wave is **empty**

If only the *k*th wave is occupied

Then the particles evolve independently

$$\psi(\mathbf{r}_1, \mathbf{r}_2, t) = \sum_j c_j \phi_j^{(1)}(\mathbf{r}_1, t)\, \chi_j^{(2)}(\mathbf{r}_2, t)$$ Entangled state

$(\zeta_1, \zeta_2) \in$ support of $\phi_j^{(1)}(\mathbf{r}_1, t)\, \chi_j^{(2)}(\mathbf{r}_2, t)$ — *j*th wave is **occupied**

$(\zeta_1, \zeta_2) \notin$ support of $\phi_j^{(1)}(\mathbf{r}_1, t)\, \chi_j^{(2)}(\mathbf{r}_2, t)$ — *j*th wave is **empty**

If only the *k*th wave is occupied

Then the particles evolve independently

But in general, they do not

$$\psi(\mathbf{r}_1,\mathbf{r}_2,t)=\sum_j c_j\phi_j^{(1)}(\mathbf{r}_1,t)\ \chi_j^{(2)}(\mathbf{r}_2,t)$$ Entangled state

$(\zeta_1,\zeta_2)\in$ support of $\phi_j^{(1)}(\mathbf{r}_1,t)\ \chi_j^{(2)}(\mathbf{r}_2,t)$ *j*th wave is **occupied**

$(\zeta_1,\zeta_2)\notin$ support of $\phi_j^{(1)}(\mathbf{r}_1,t)\ \chi_j^{(2)}(\mathbf{r}_2,t)$ *j*th wave is **empty**

If only the *k*th wave is occupied

Then the particles evolve independently

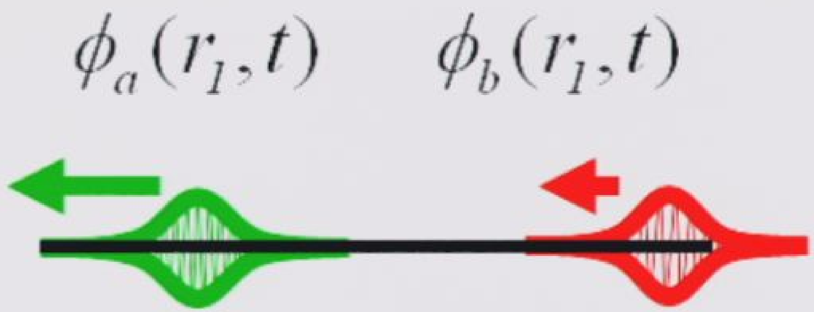
But in general, they do not

$$\psi(r_1,r_2;t)=$$
$$c_a\phi_a(r_1,t)\chi_a(r_2,t)+c_b\phi_b(r_1,t)\chi_b(r_2,t)$$

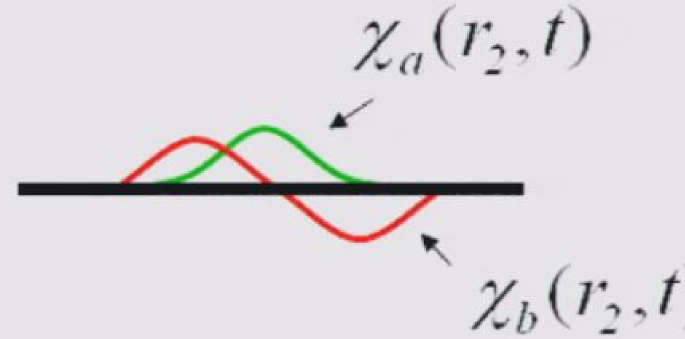
electron 1

electron 2

$$\psi(r_1,r_2;t) = c_a\phi_a(r_1,t)\chi_a(r_2,t) + c_b\phi_b(r_1,t)\chi_b(r_2,t)$$



electron 1

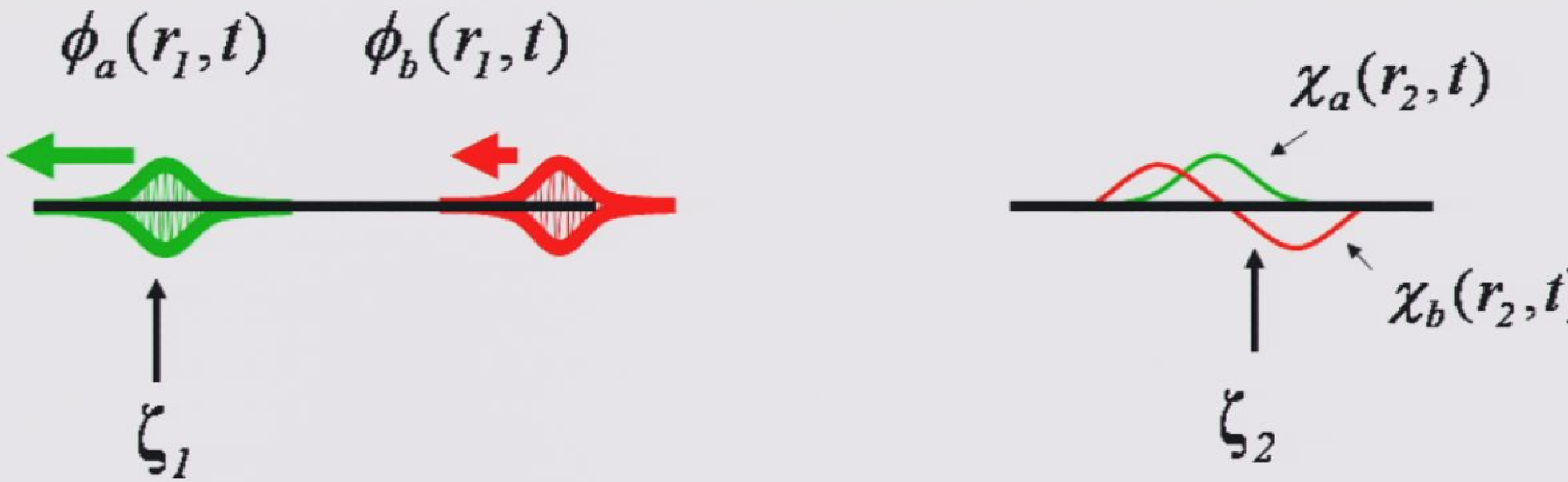


electron 2

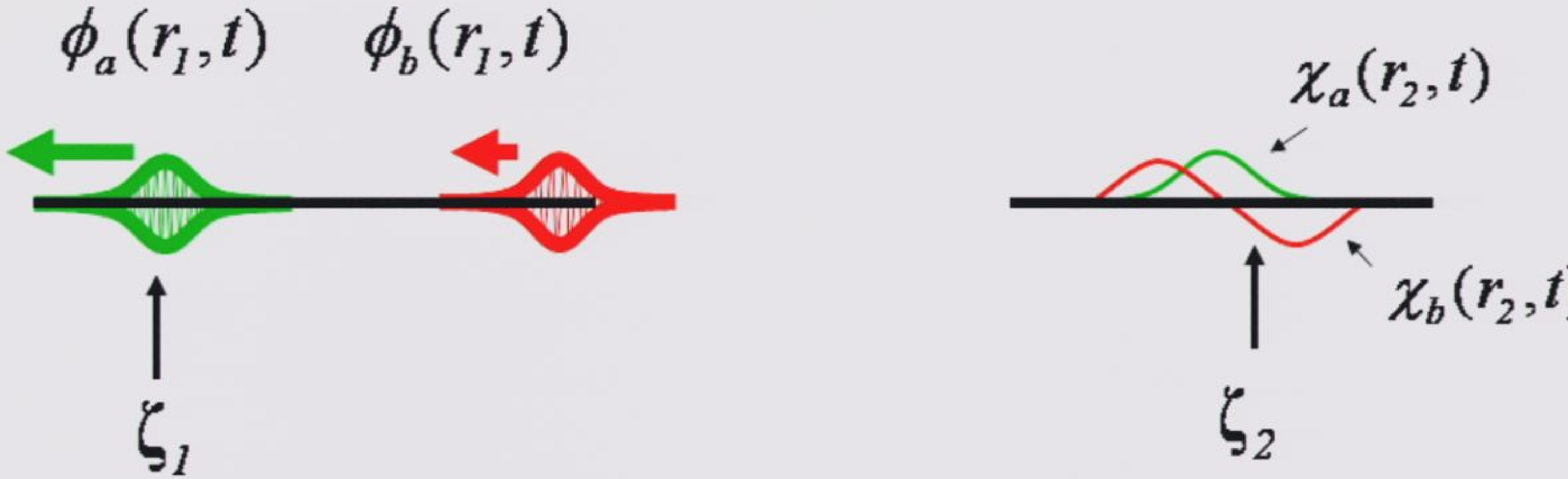
$$\psi(r_1,r_2;t)=$$

$$c_a\phi_a(r_1,t)\chi_a(r_2,t)+c_b\phi_b(r_1,t)\chi_b(r_2,t$$

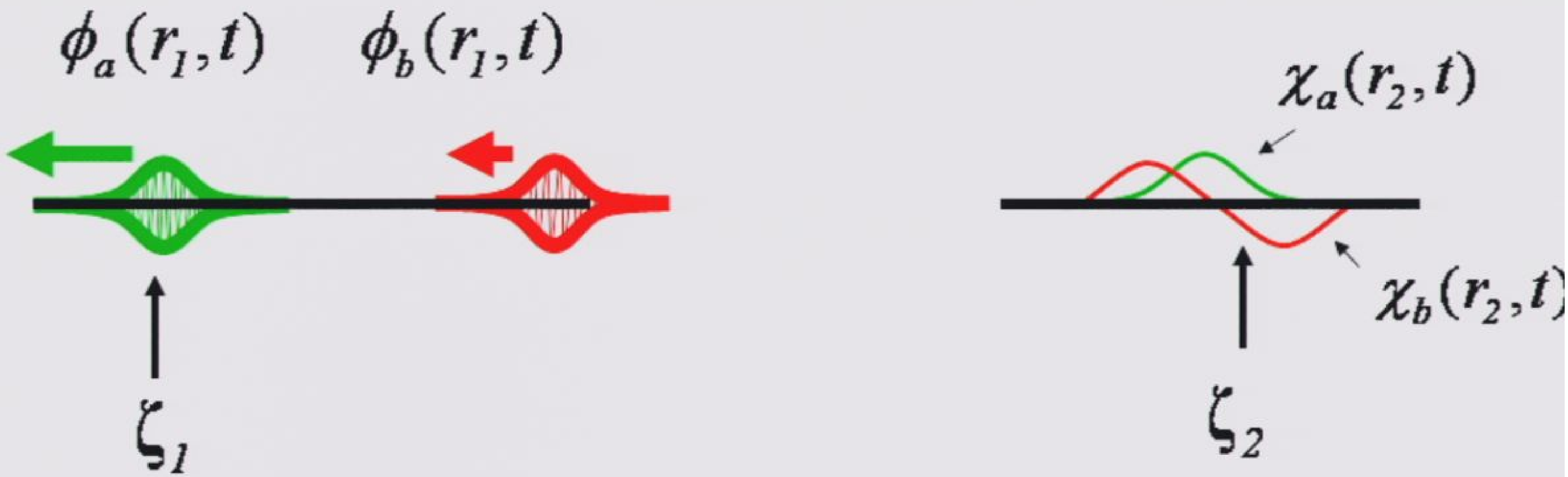
$$\psi(r_1,r_2;t) = c_a\phi_a(r_1,t)\chi_a(r_2,t) + c_b\phi_b(r_1,t)\chi_b(r_2,t)$$



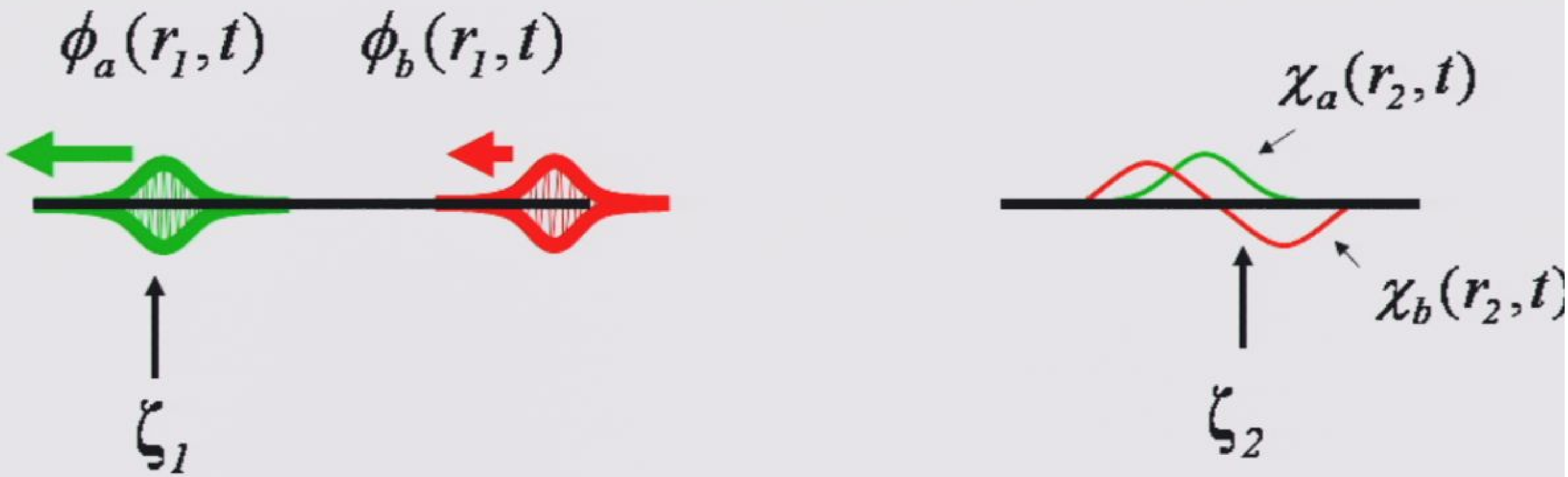
$$\psi(r_1,r_2;t)= c_a\phi_a(r_1,t)\chi_a(r_2,t)+c_b\phi_b(r_1,t)\chi_b(r_2,t)$$



$$\psi(r_1,r_2;t) = c_a\phi_a(r_1,t)\chi_a(r_2,t) + c_b\phi_b(r_1,t)\chi_b(r_2,t)$$

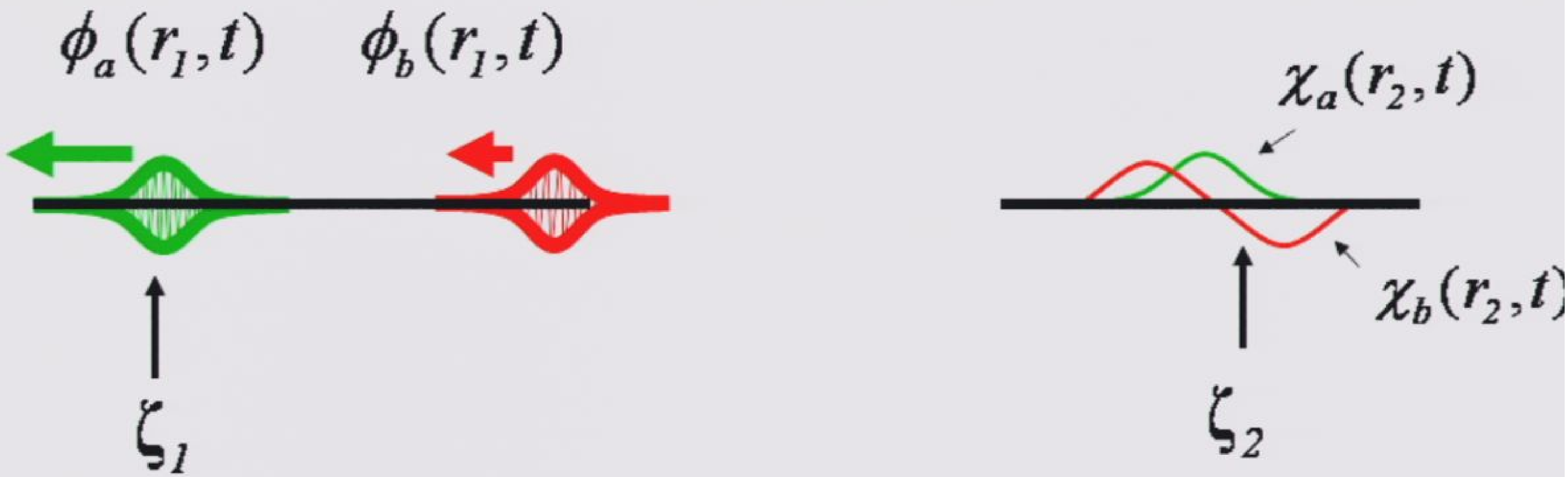


$$\psi(r_1,r_2;t) = c_a\phi_a(r_1,t)\chi_a(r_2,t) + c_b\phi_b(r_1,t)\chi_b(r_2,t)$$



$$\psi(r_1,r_2;t) = c_a\phi_a(r_1,t)\chi_a(r_2,t) + c_b\phi_b(r_1,t)\chi_b(r_2,t)$$

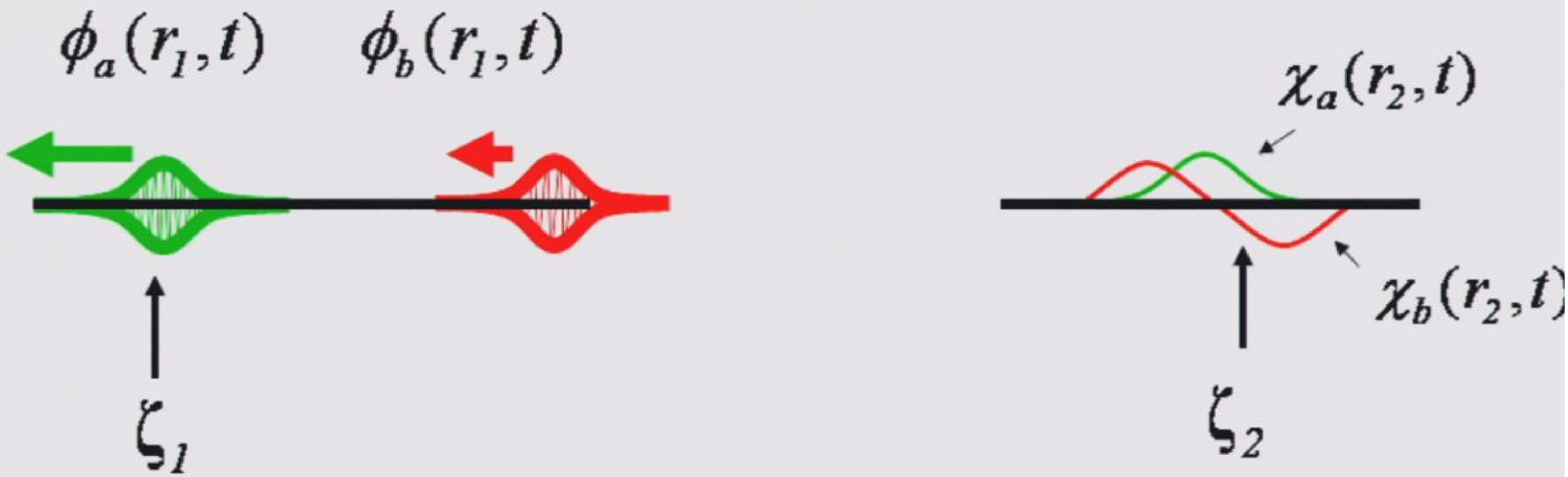
occupied wave



$$\psi(r_1,r_2;t)= c_a\phi_a(r_1,t)\chi_a(r_2,t)+c_b\phi_b(r_1,t)\chi_b(r_2,t)$$

occupied wave

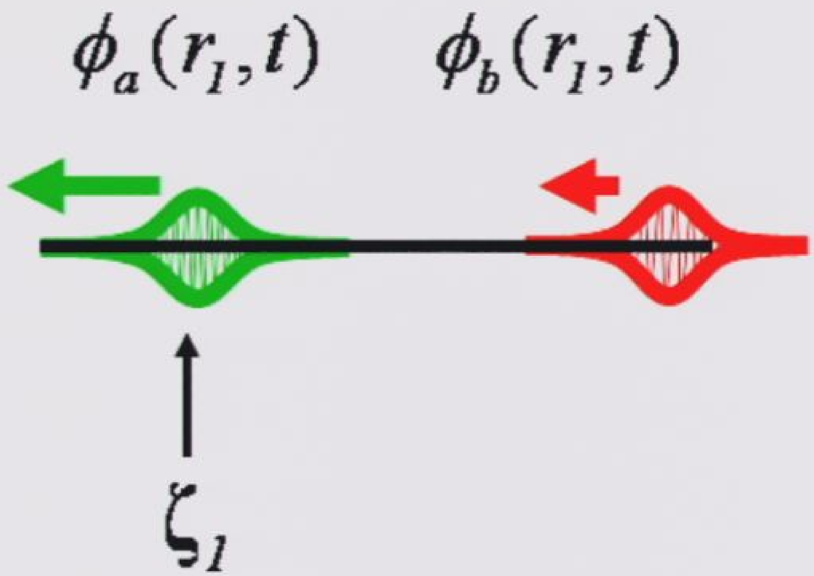
$$\frac{d\zeta_2(t)}{dt}=\frac{1}{m_2}\left[\nabla_2 S(\mathbf{r}_1,\mathbf{r}_2,t)\right]_{\mathbf{r}_1=\zeta_1(t),\mathbf{r}_2=\zeta_2(t)}$$

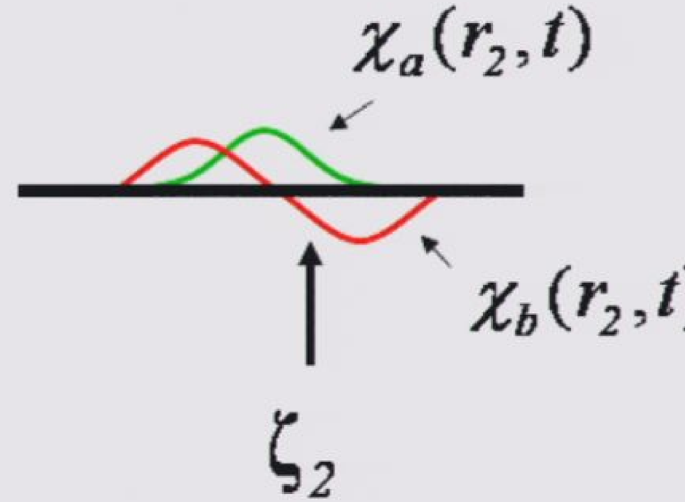


$$\psi(r_1,r_2;t) = c_a\phi_a(r_1,t)\chi_a(r_2,t) + c_b\phi_b(r_1,t)\chi_b(r_2,t)$$

occupied wave

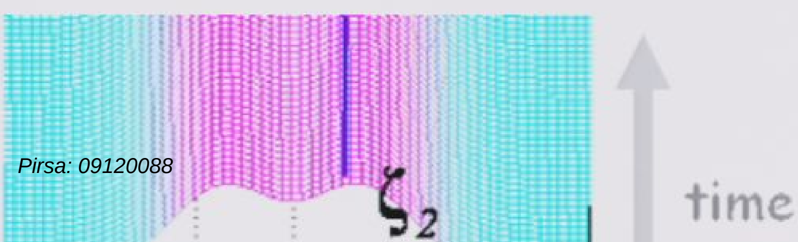
$$\frac{d\zeta_2(t)}{dt} = \frac{1}{m_2}\left[\nabla_2 S(\mathbf{r}_1,\mathbf{r}_2,t)\right]_{\mathbf{r}_1=\zeta_1(t),\,\mathbf{r}_2=\zeta_2(t)}$$



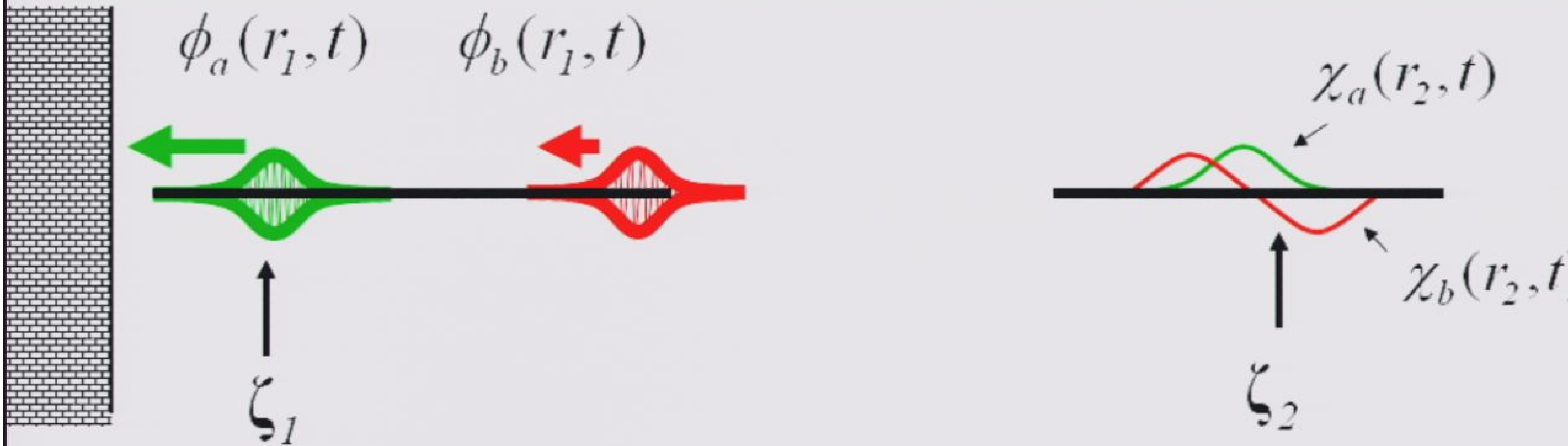


$$\psi(r_1,r_2;t) = c_a\phi_a(r_1,t)\chi_a(r_2,t) + c_b\phi_b(r_1,t)\chi_b(r_2,t)$$

occupied wave

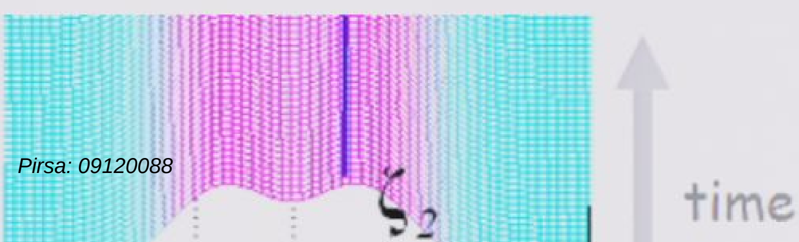


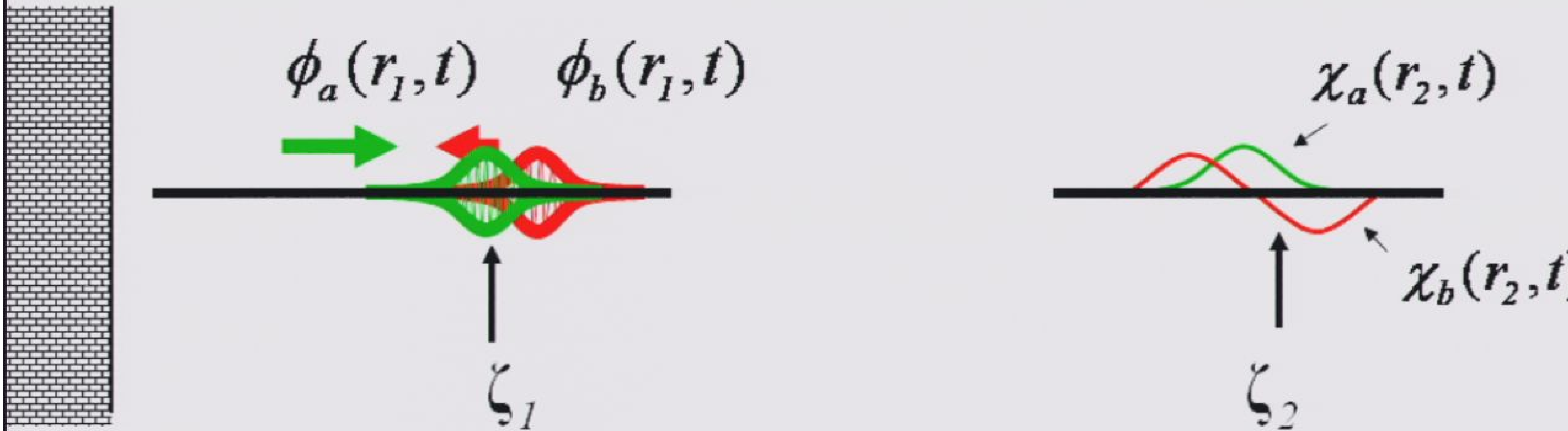
$$\frac{d\zeta_2(t)}{dt} = \frac{1}{m_2}\left[\nabla_2 S(\mathbf{r}_1,\mathbf{r}_2,t)\right]_{\mathbf{r}_1=\zeta_1(t),\mathbf{r}_2=\zeta_2(t)}$$

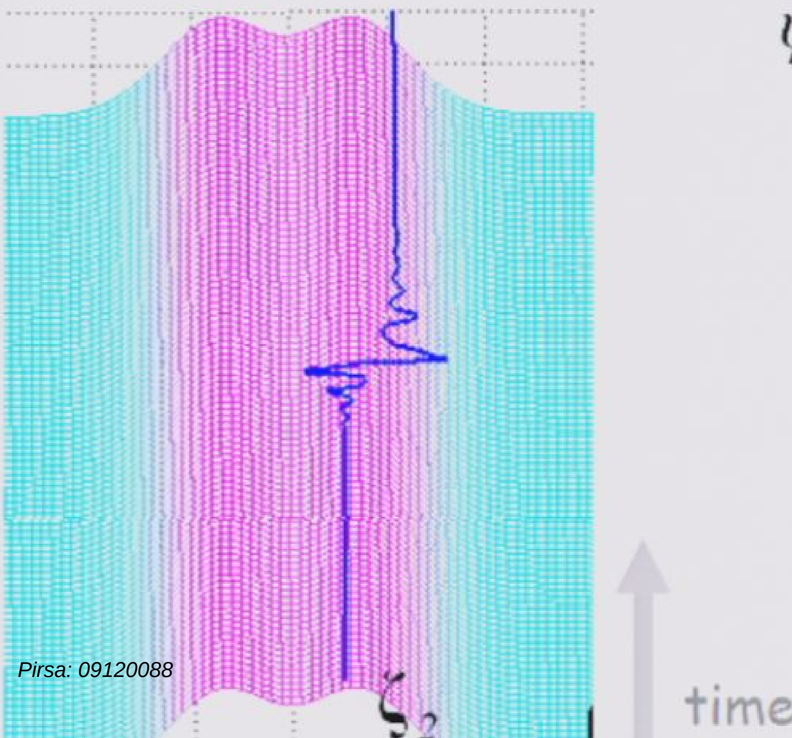


$$\psi(r_1,r_2;t)=$$
$$c_a\phi_a(r_1,t)\chi_a(r_2,t)+c_b\phi_b(r_1,t)\chi_b(r_2,t)$$

occupied wave

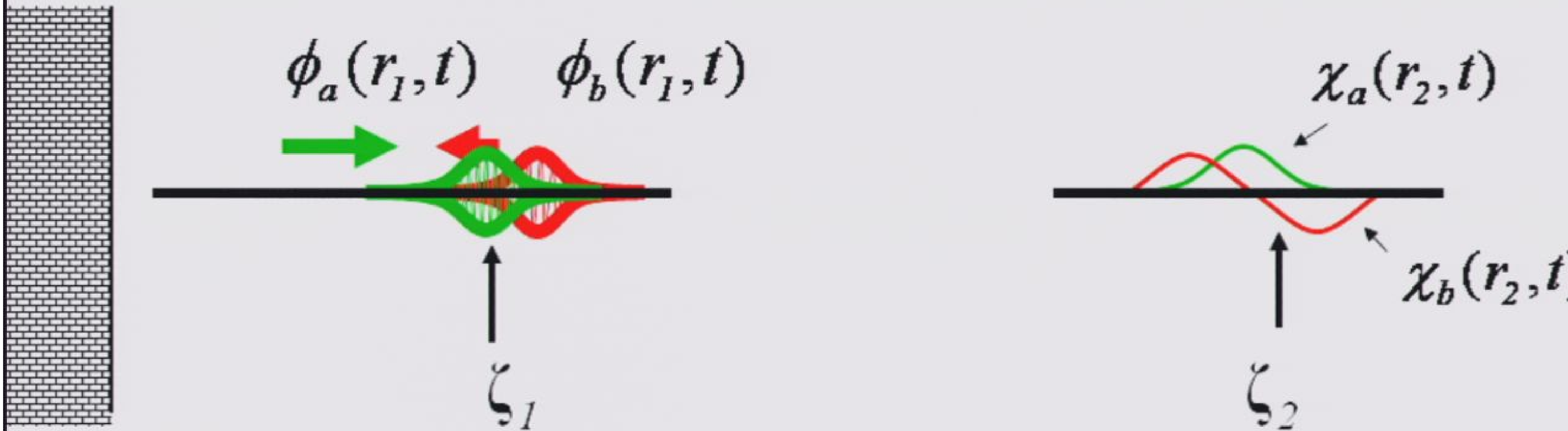






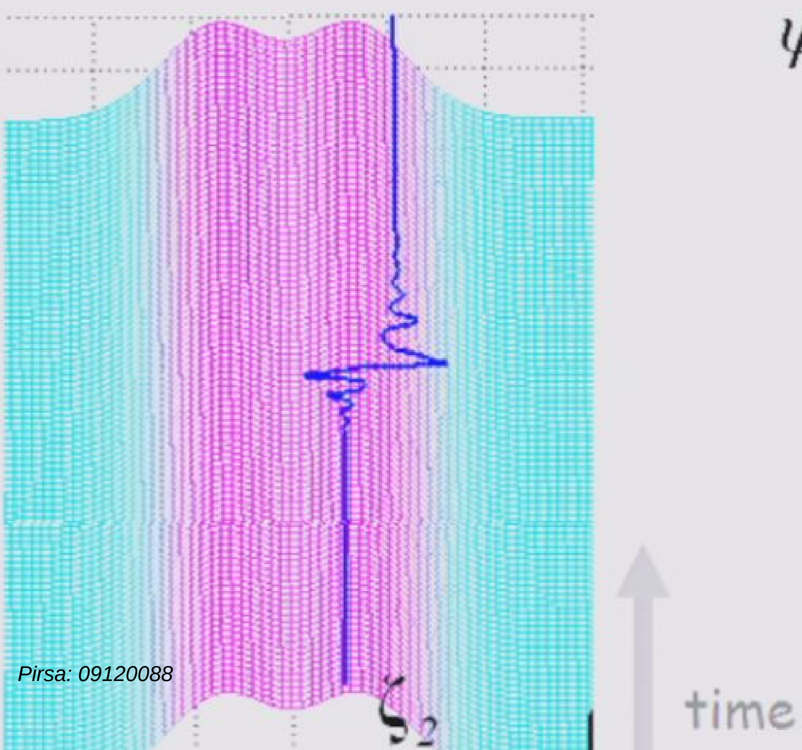
$$\psi(r_1,r_2;t)=$$
$$c_a\phi_a(r_1,t)\chi_a(r_2,t)+c_b\phi_b(r_1,t)\chi_b(r_2,t)$$

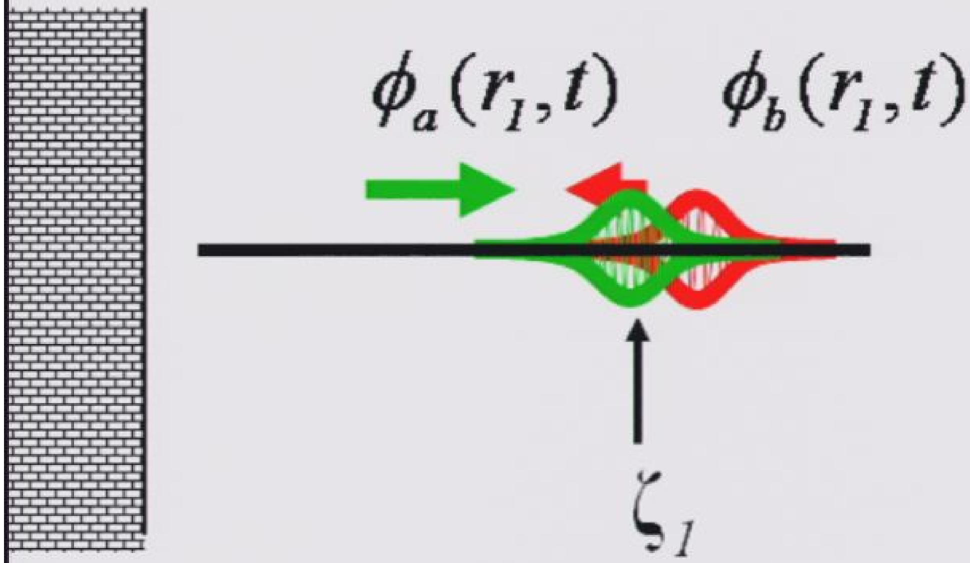
both waves occupied

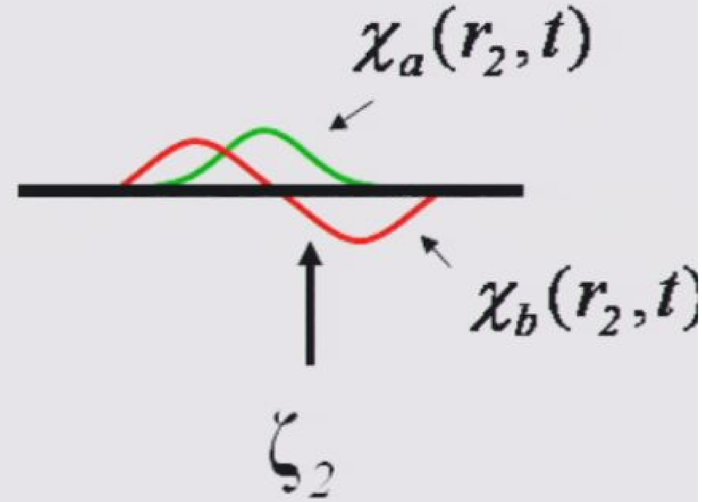


$$\psi(r_1,r_2;t) = c_a\phi_a(r_1,t)\chi_a(r_2,t) + c_b\phi_b(r_1,t)\chi_b(r_2,t)$$

both waves occupied







$$\psi(r_1,r_2;t)=$$
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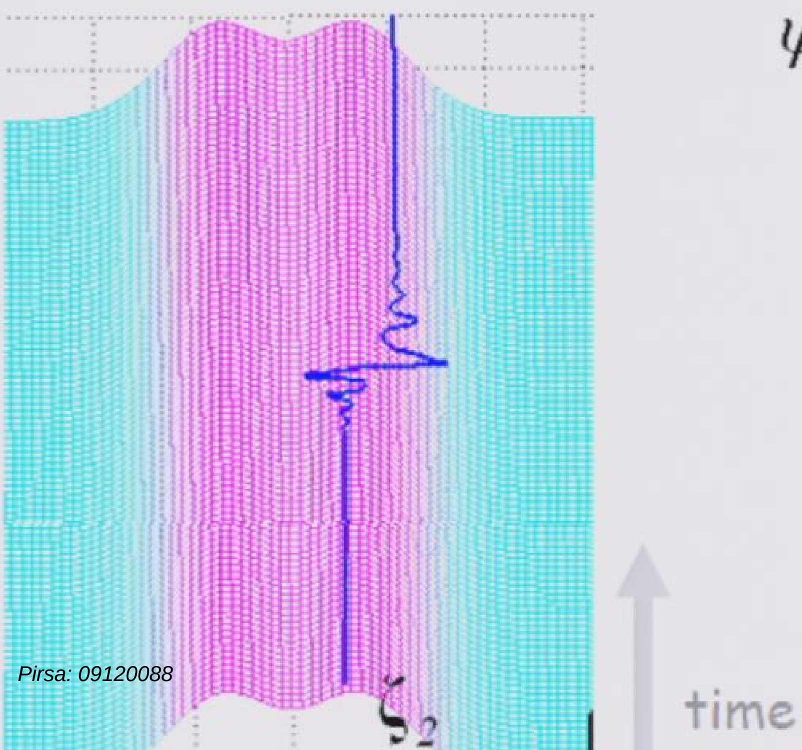

Failure of local causality