

Title: Foundations of Quantum Mech. (PHYS 639) - Lecture 10

Date: Dec 11, 2009 11:00 AM

URL: <http://pirsa.org/09120074>

Abstract:

Contextuality

Problems with the traditional definition of noncontextuality:

- applies only to sharp measurements
- applies only to deterministic hidden variable models
- applies only to models of quantum theory

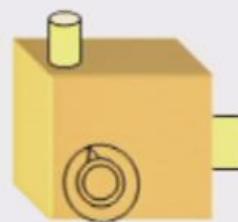
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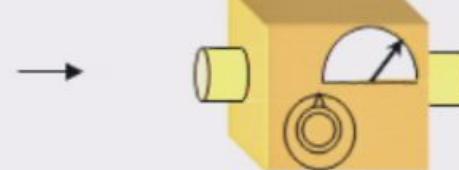
A better notion of noncontextuality would determine

- whether any given theory admits a noncontextual model
- whether any given experimental data can be explained by a noncontextual model

Operational Quantum Mechanics



Preparation
P



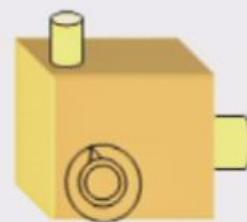
Measurement
M

Density operator
 ρ

Positive operator-valued
measure (POVM)
 $\{E_k\}$

$$Pr(k|P, M) = \text{Tr}[E_k \rho]$$

General Operational Theories



Preparation

P

Element of a
convex set

\vec{p}



Measurement

M

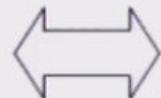
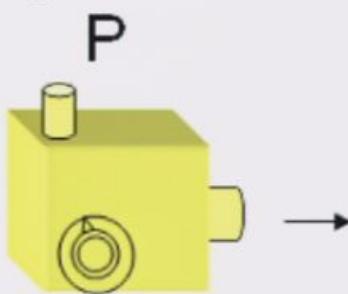
Set of elements
of a positive cone

$\{\vec{r}_k\}$

$$Pr(k|P, M) = \vec{r}_k \cdot \vec{p}$$

A realist model of an operational theory

Preparation

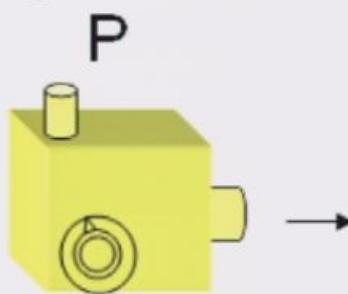


$$\int \mu_P(\lambda) d\lambda = 1$$

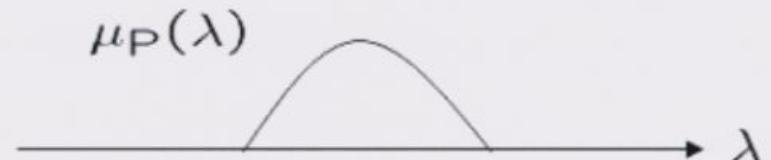


A realist model of an operational theory

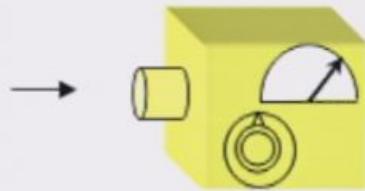
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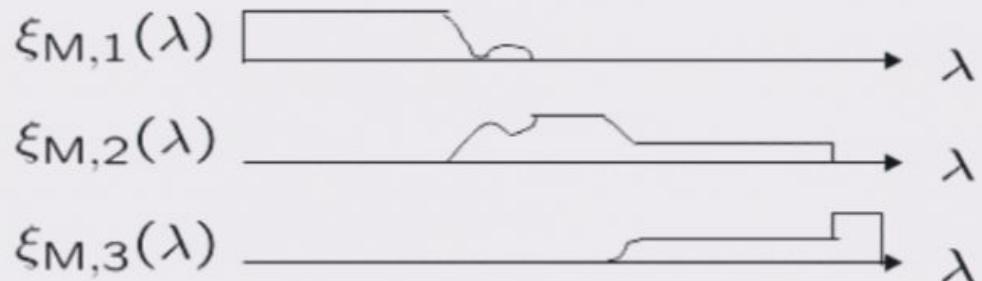


Measurement
M



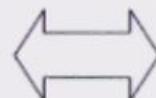
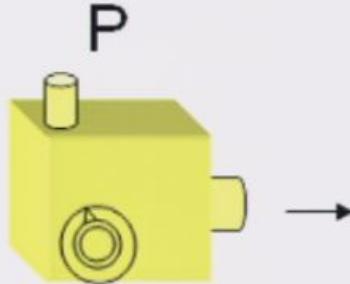
$$0 \leq \xi_{M,k} \leq 1$$

$$\sum_k \xi_{M,k}(\lambda) = 1 \text{ for all } \lambda$$



A realist model of an operational theory

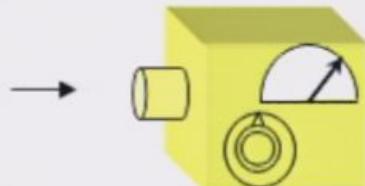
Preparation
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Measurement
M



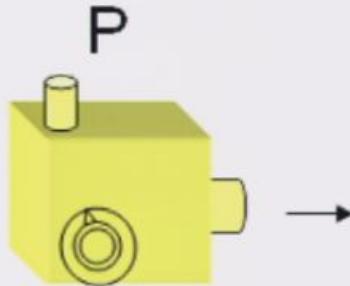
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A realist model of an operational theory

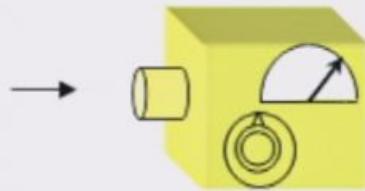
Preparation
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$$\int \mu_P(\lambda) d\lambda = 1$$



Measurement
M



$$0 \leq \xi_{M,k} \leq 1$$

$$\sum_k \xi_{M,k}(\lambda) = 1 \text{ for all } \lambda$$



$$p(k|P, M) = \int d\lambda \xi_{M,k}(\lambda) \mu_P(\lambda)$$

Generalized definition of noncontextuality:

A realist model of an operational theory is **noncontextual** if

Operational equivalence
of two experimental
procedures



Equivalent
representations
in the HV model

Generalized definition of noncontextuality:

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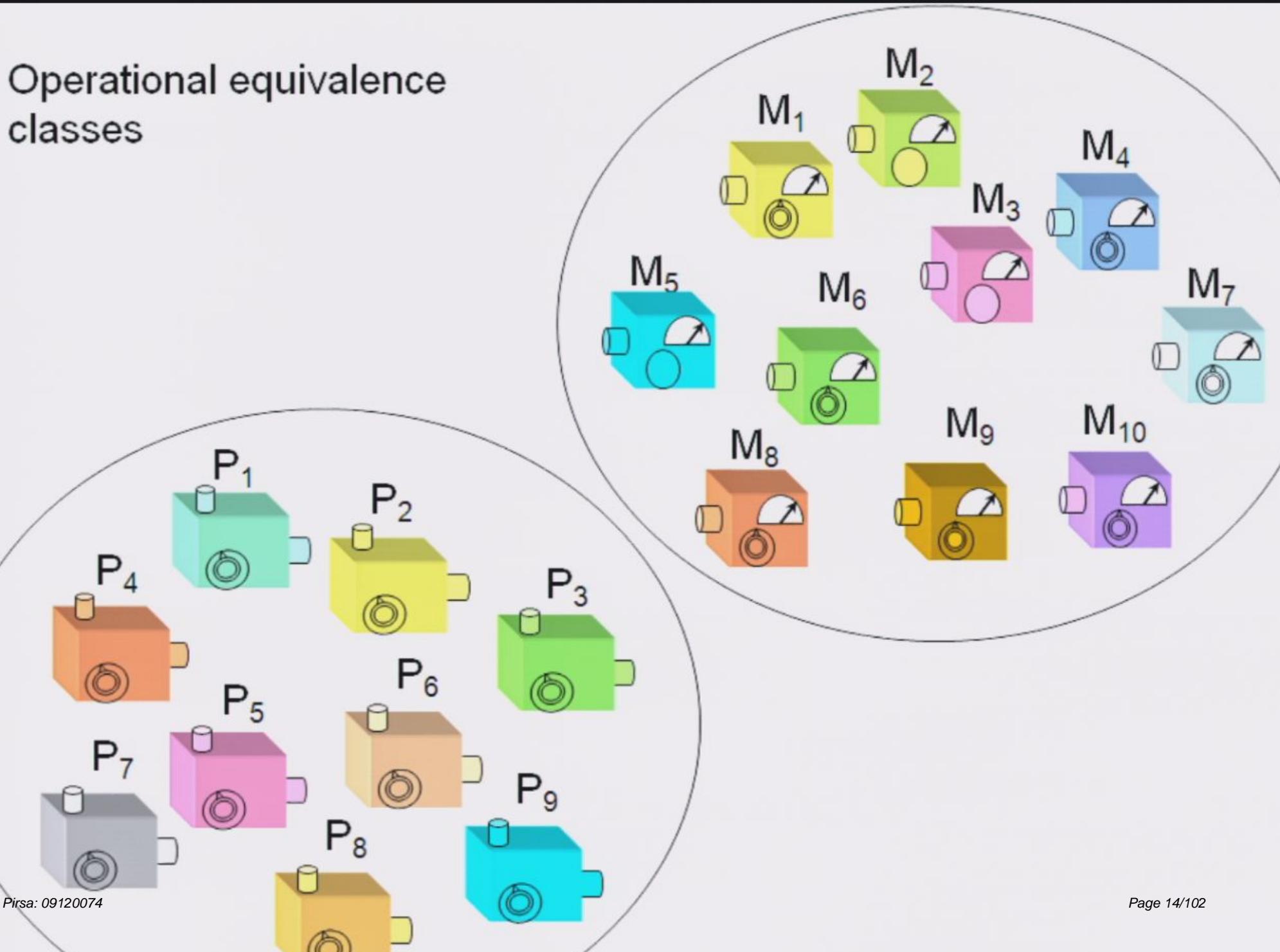
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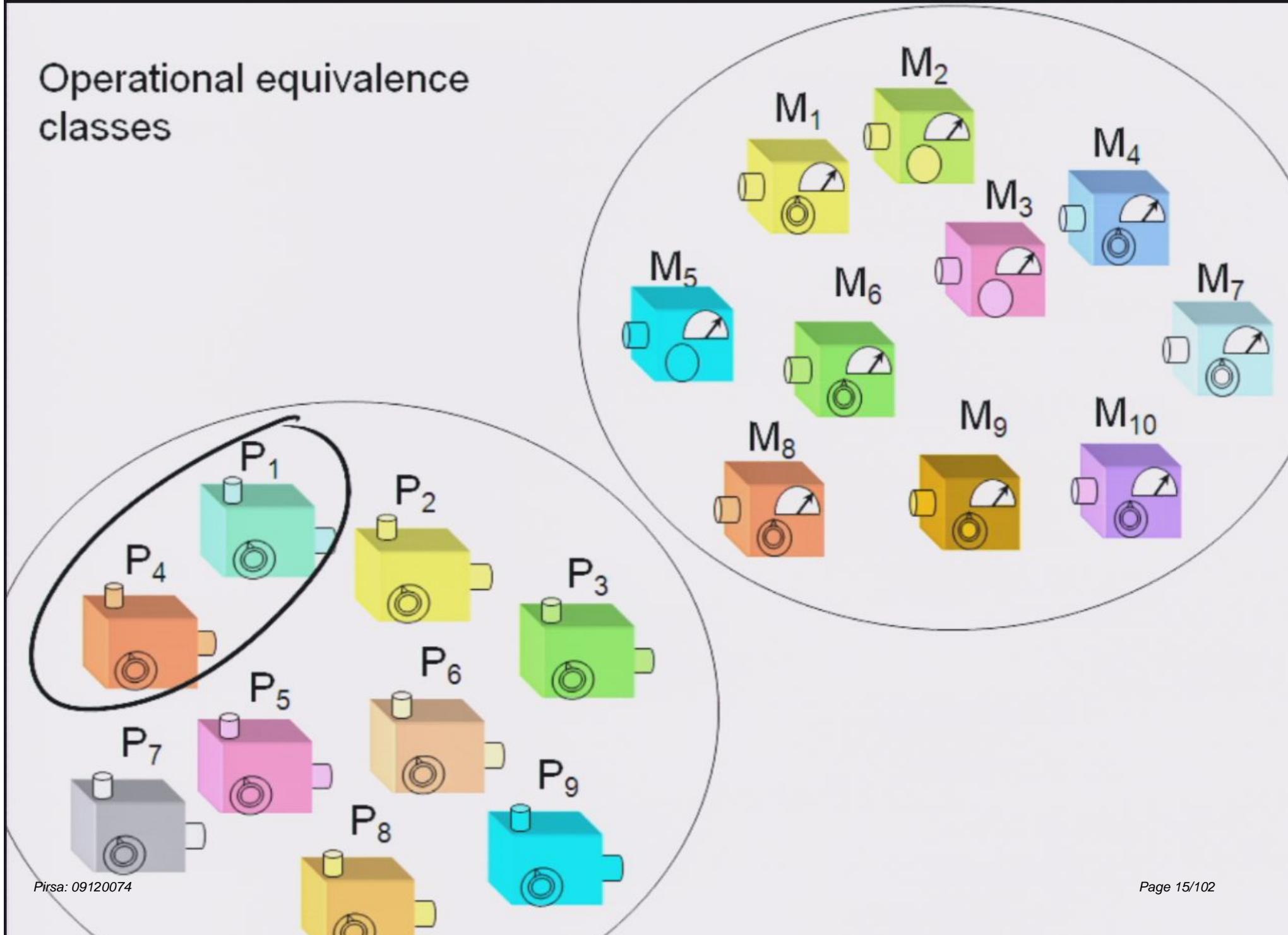


Equivalent
representations
in the HV model

Operational equivalence classes



Operational equivalence classes

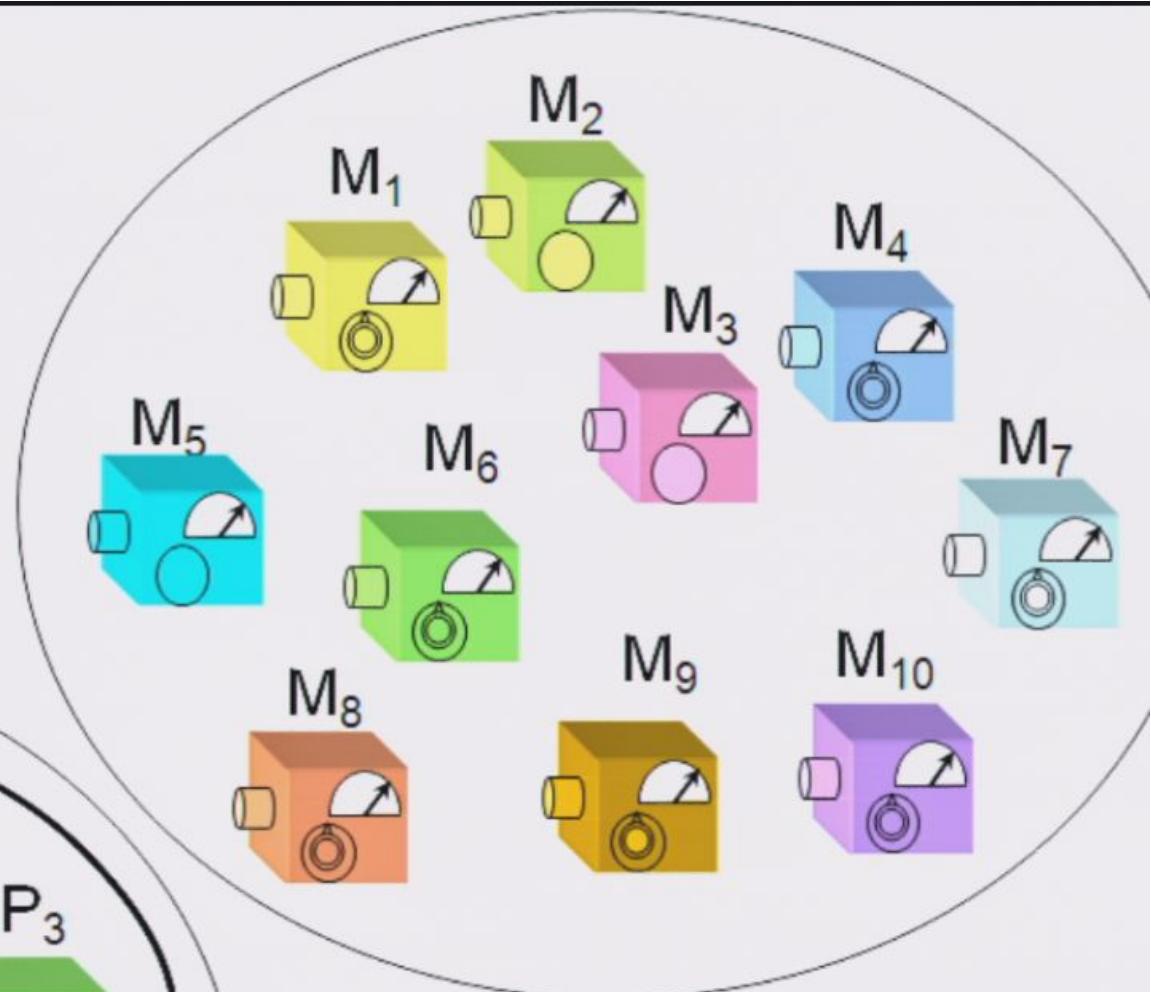
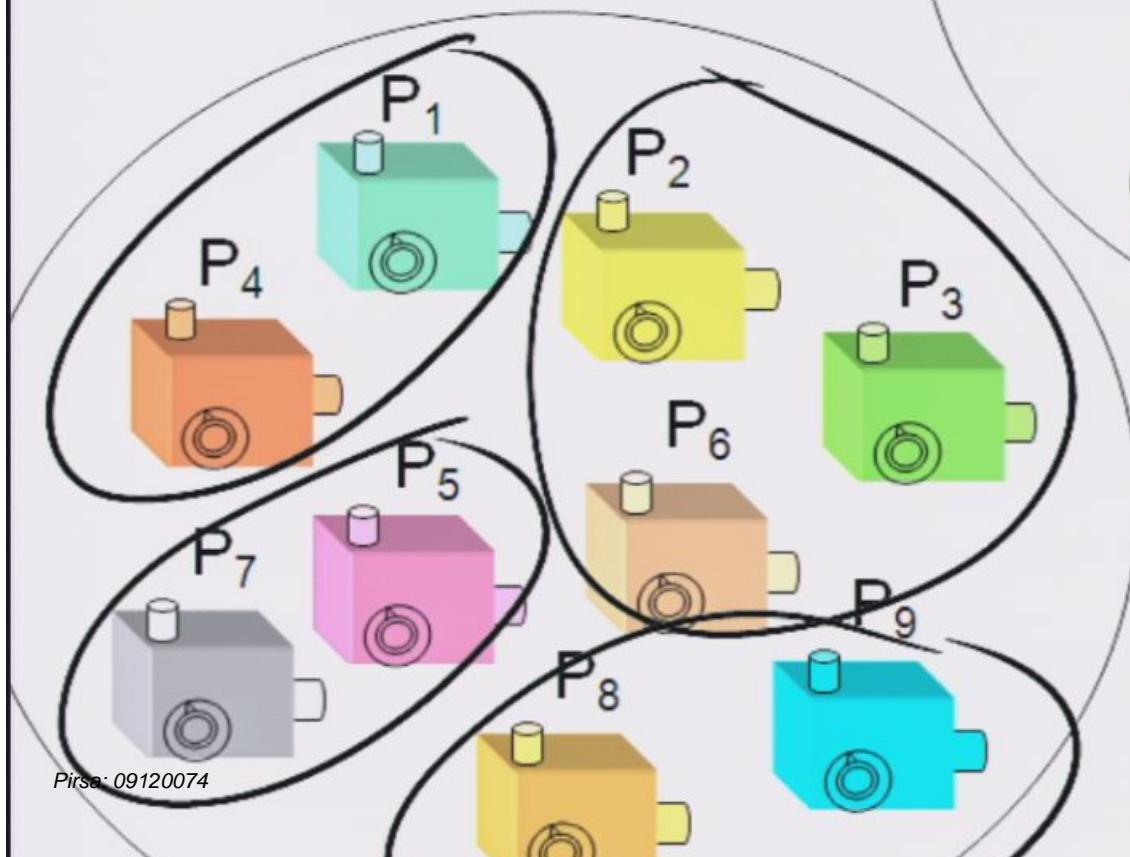


Operational equivalence classes

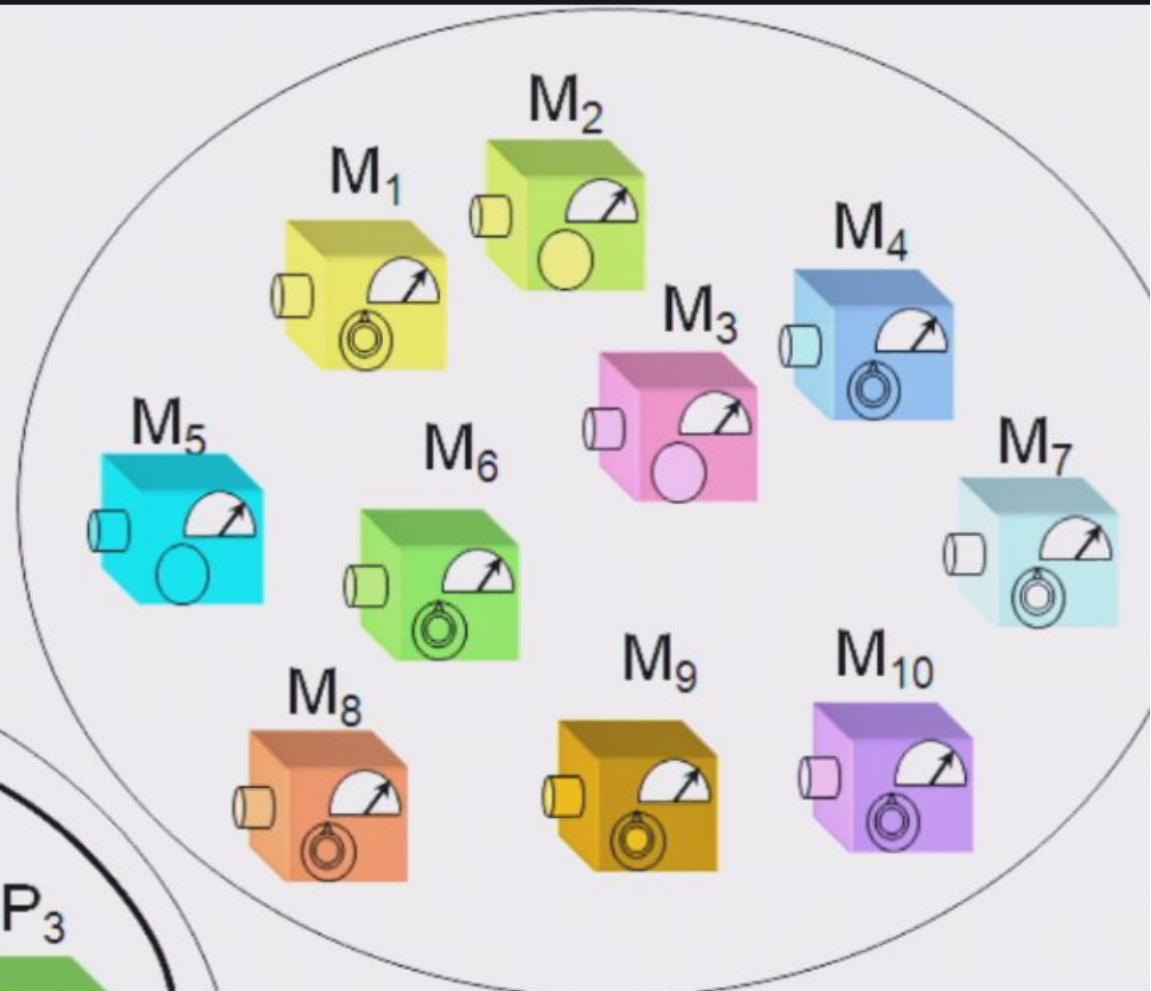
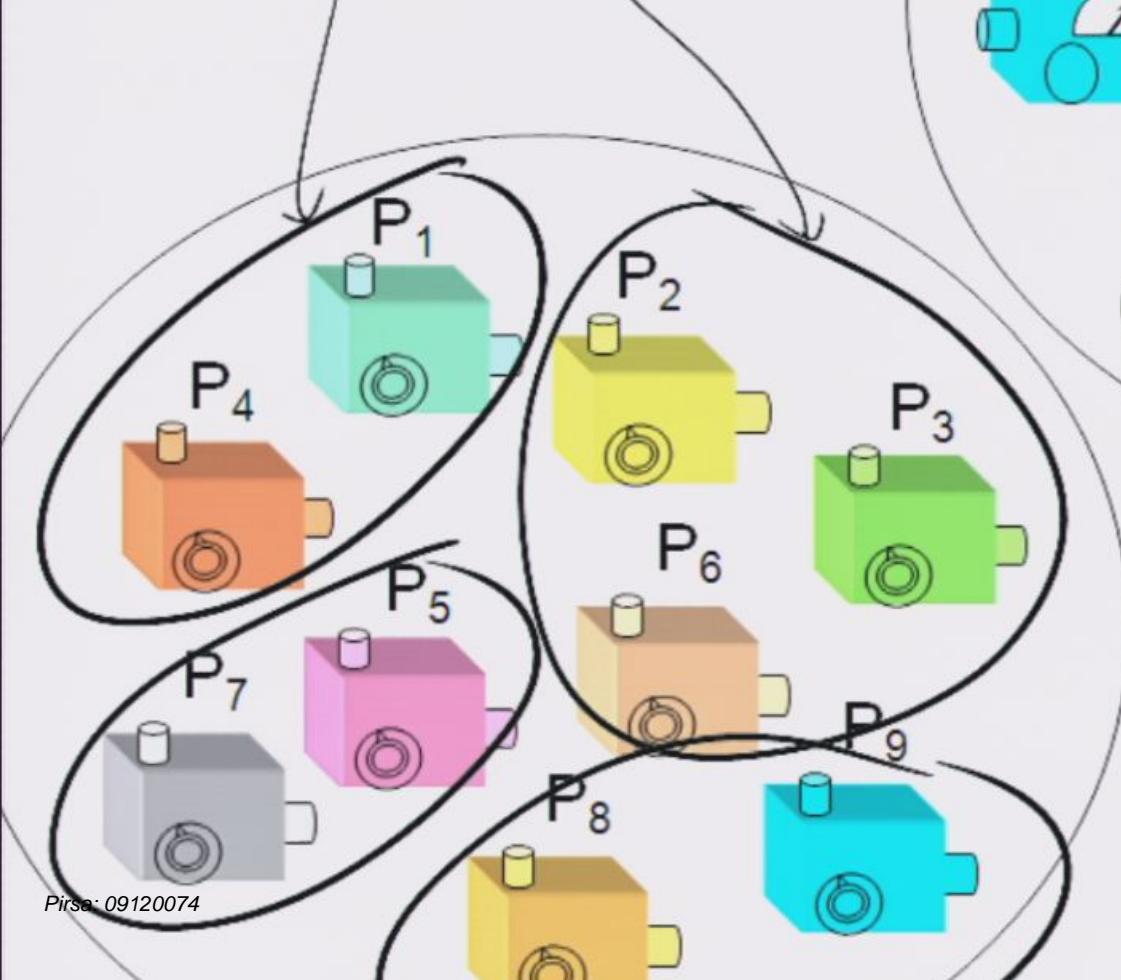
P is equivalent to P' if

$\forall M \forall k :$

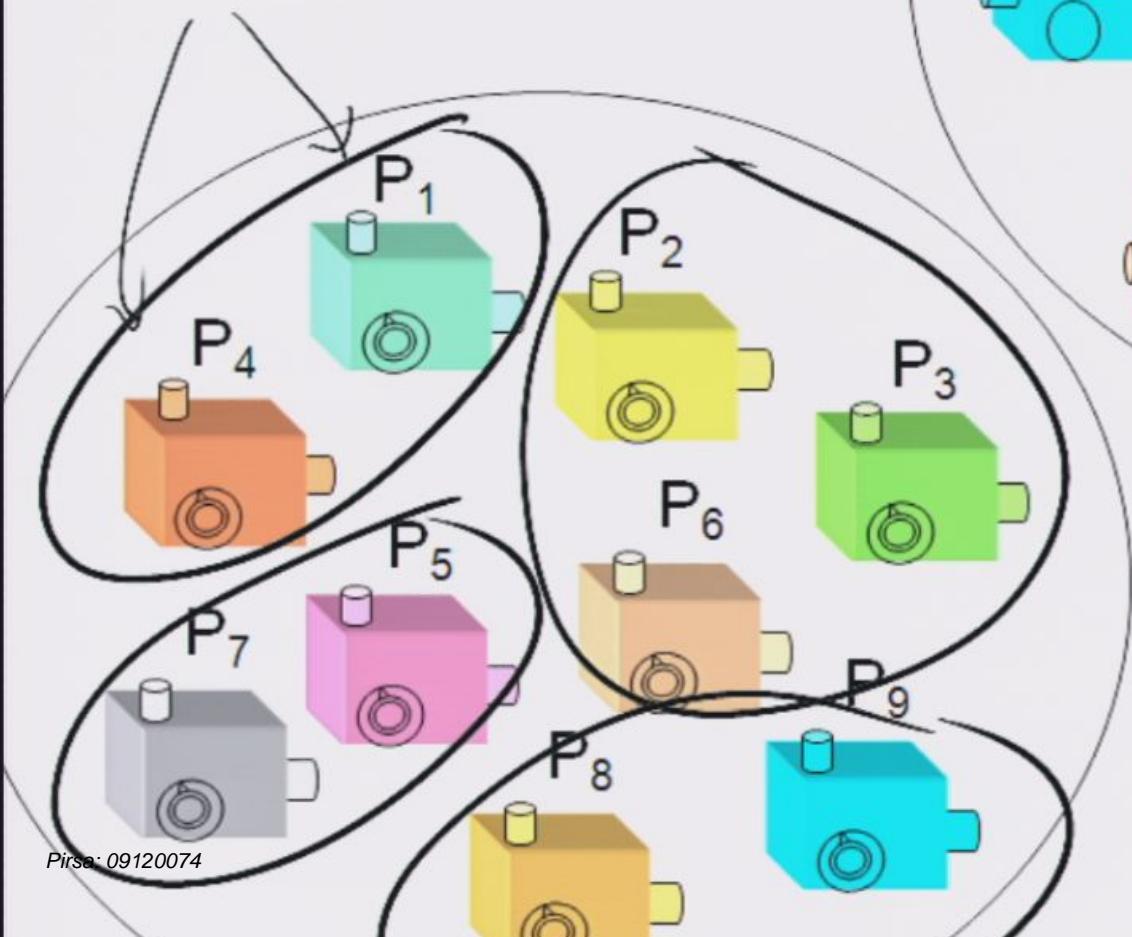
$$p(k|P, M) = p(k|P', M)$$



Difference of Equivalence class

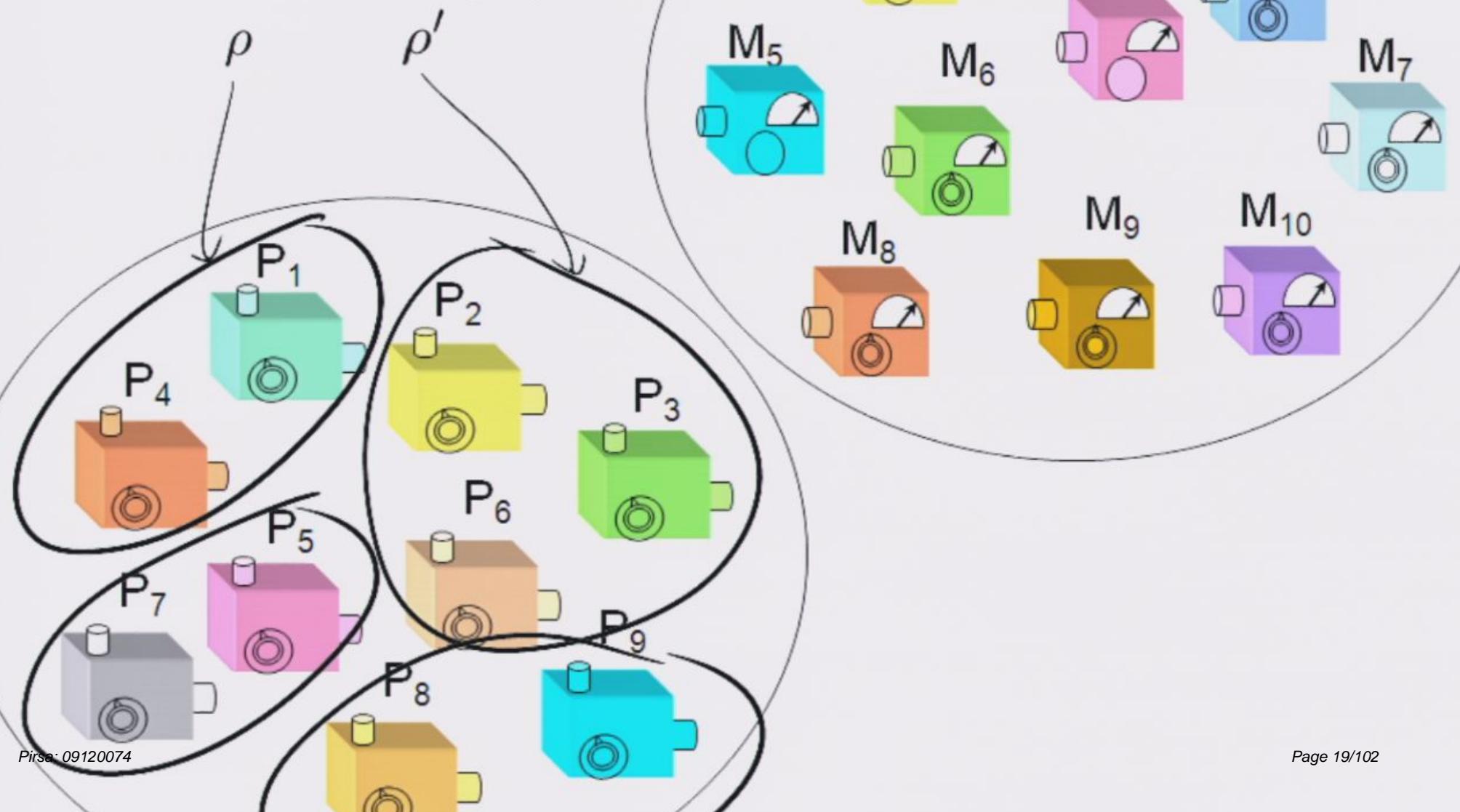


Difference of context



Example from quantum theory

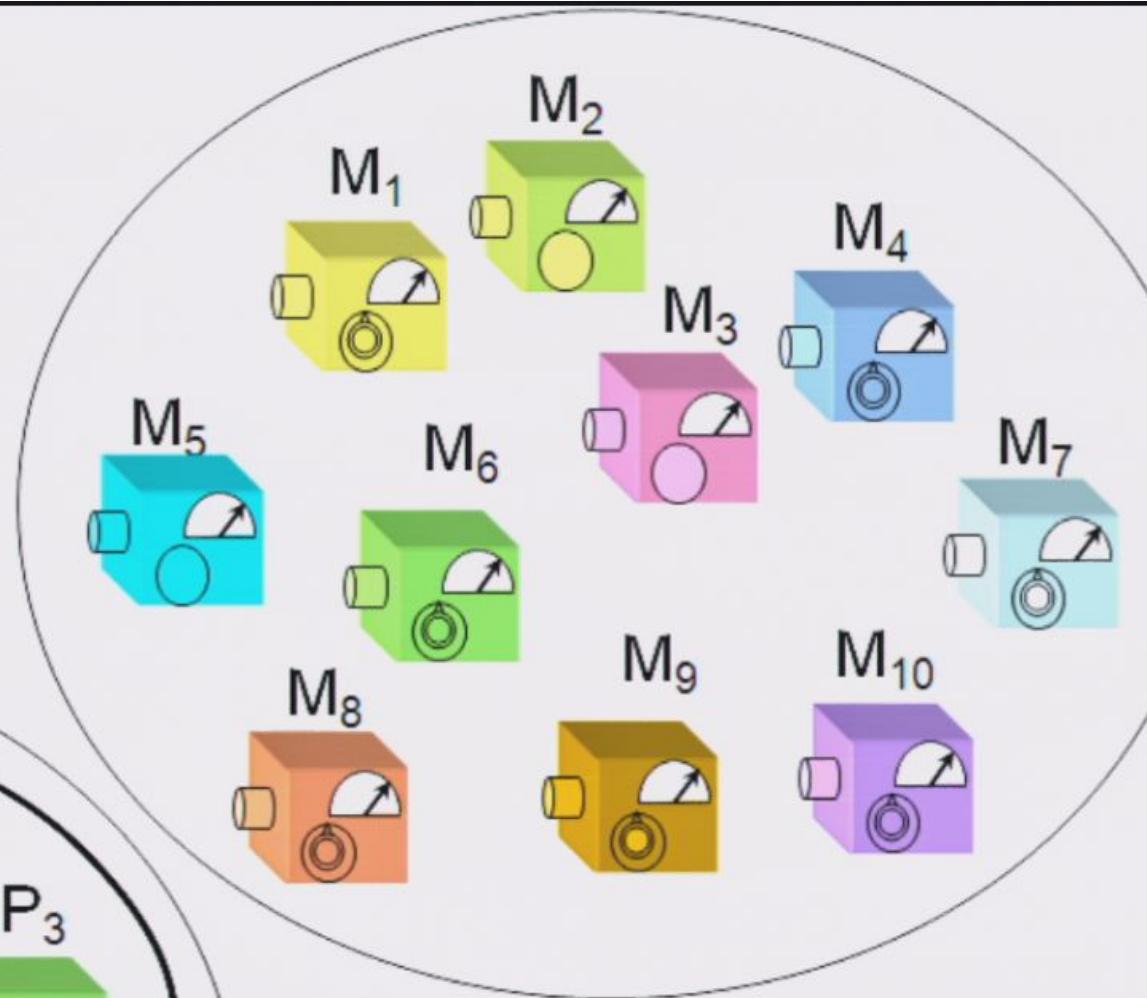
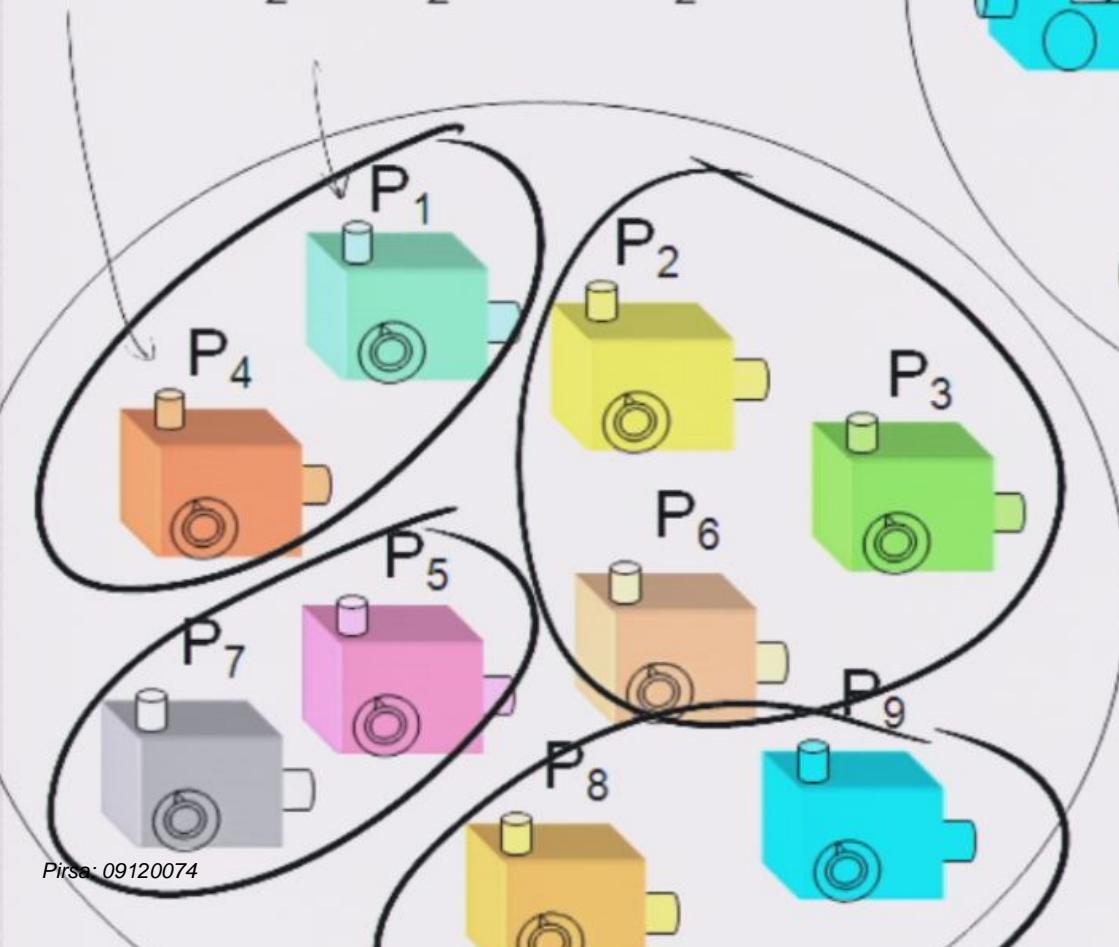
Different density op's



Example from quantum theory

$$I = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$$

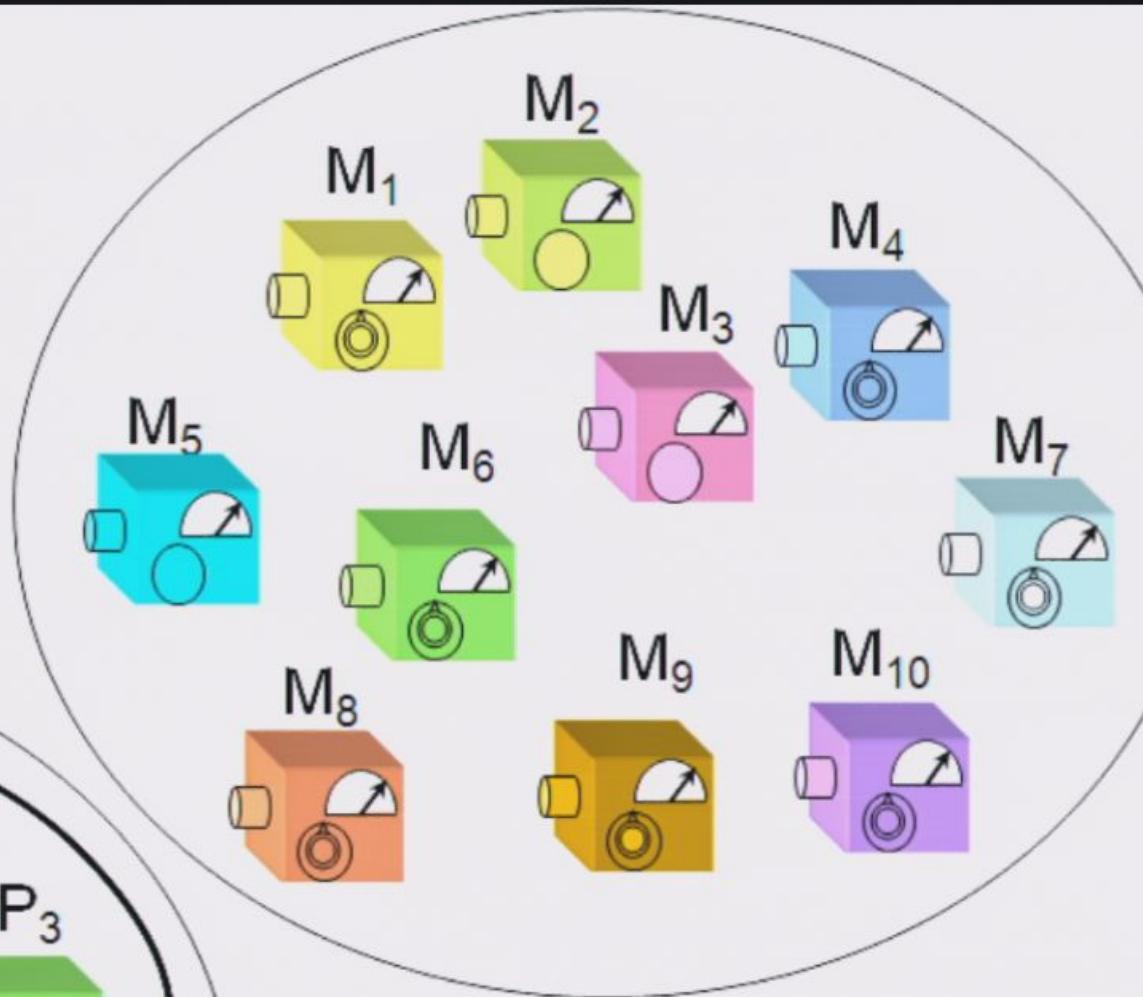
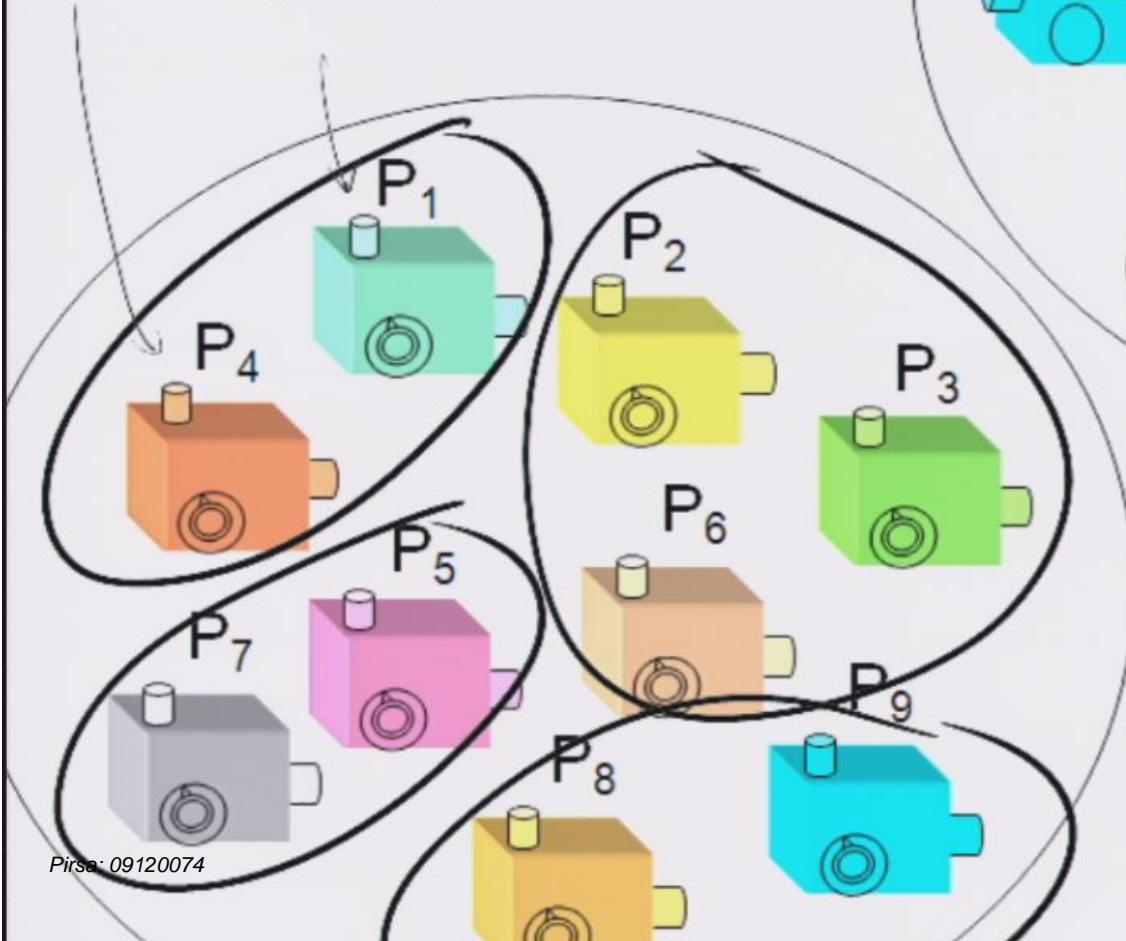
$$\frac{1}{2}I = \frac{1}{2}|+\rangle\langle +| + \frac{1}{2}|-\rangle\langle -|$$



Example from quantum theory

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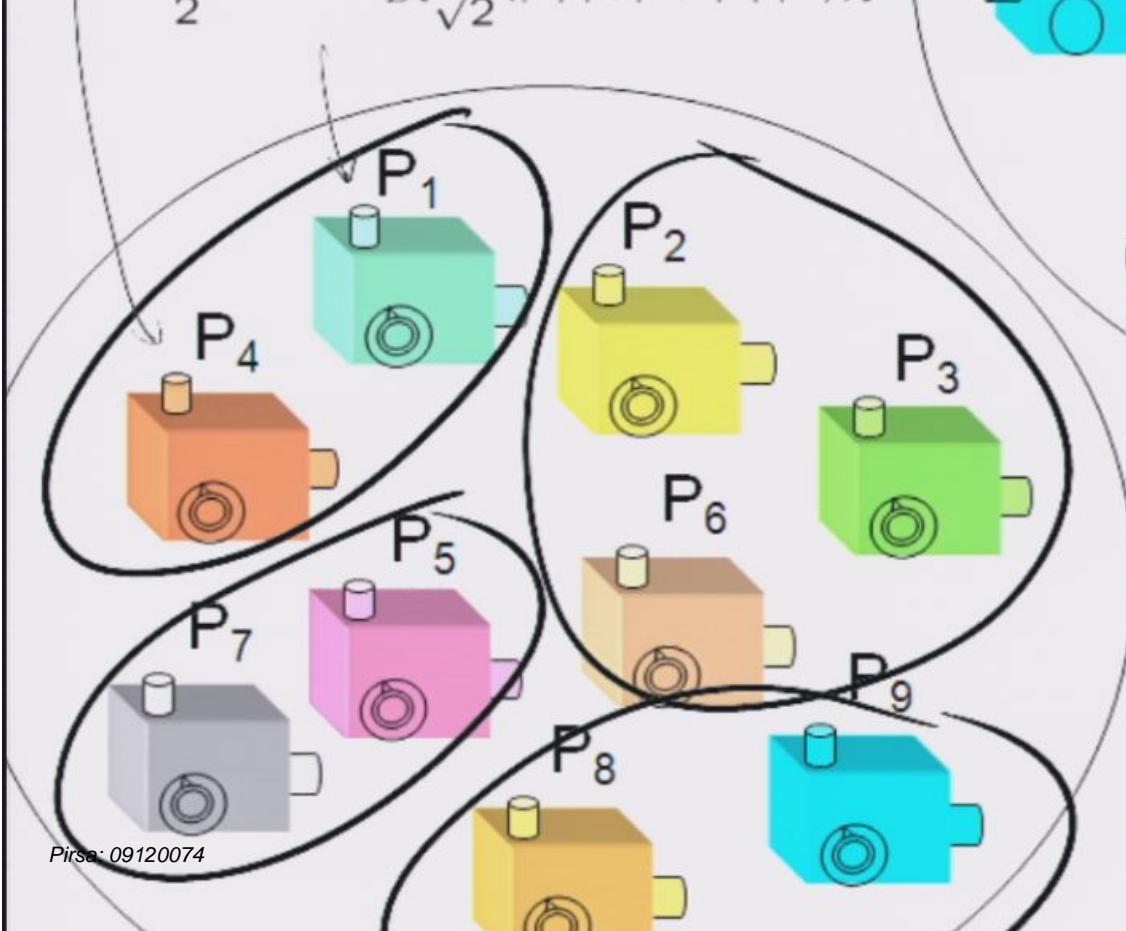
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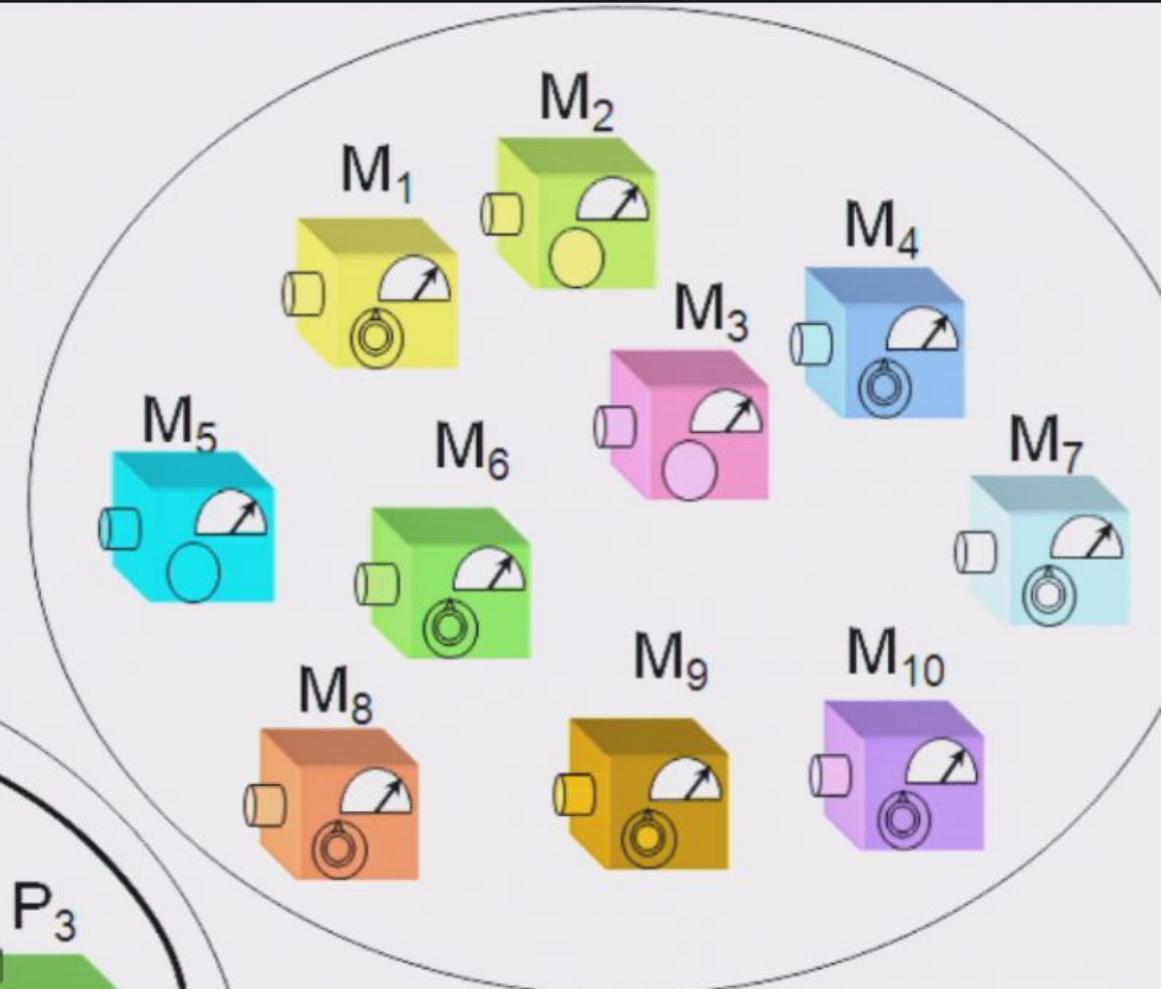
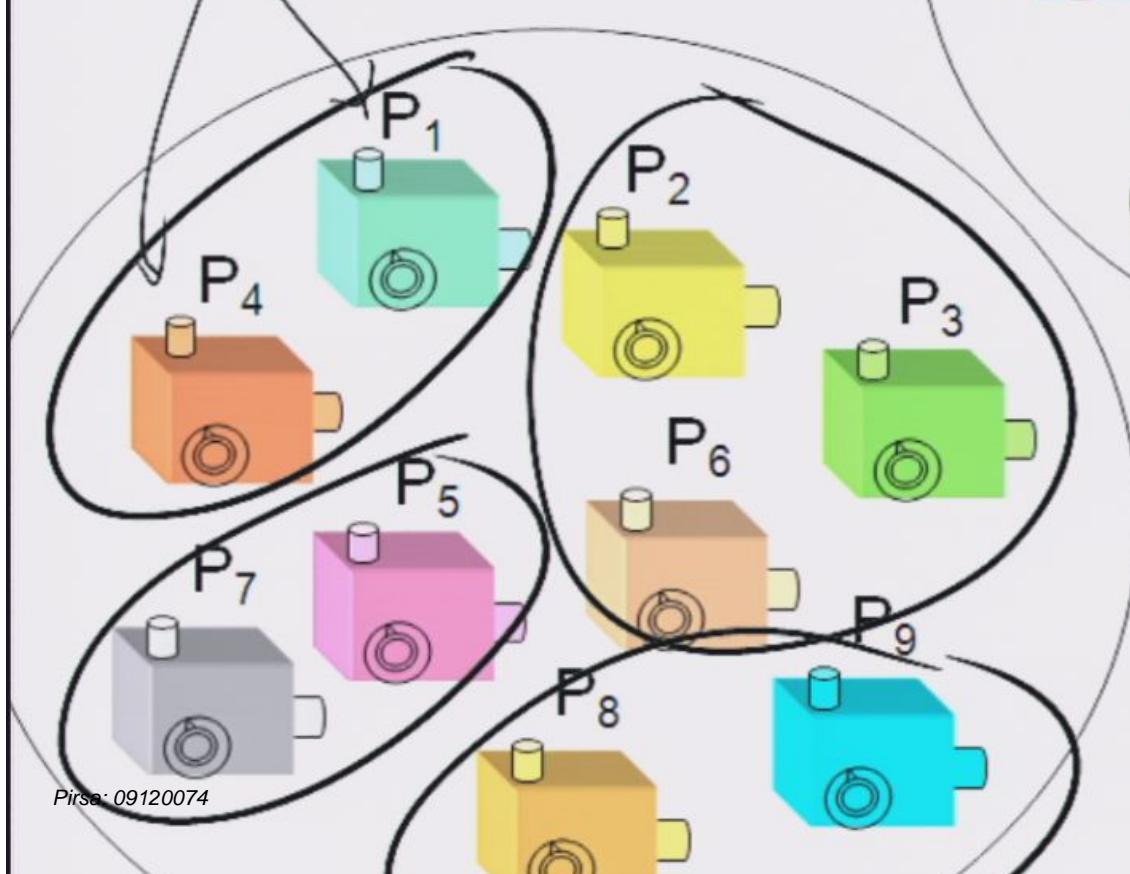
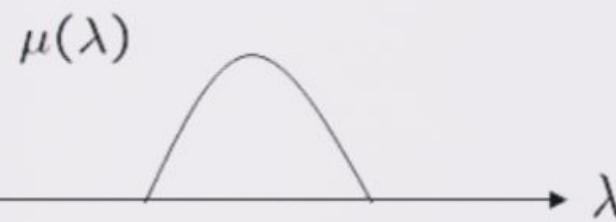
Example from quantum theory

$$I = \text{Tr}_B\left[\frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle)\right]$$

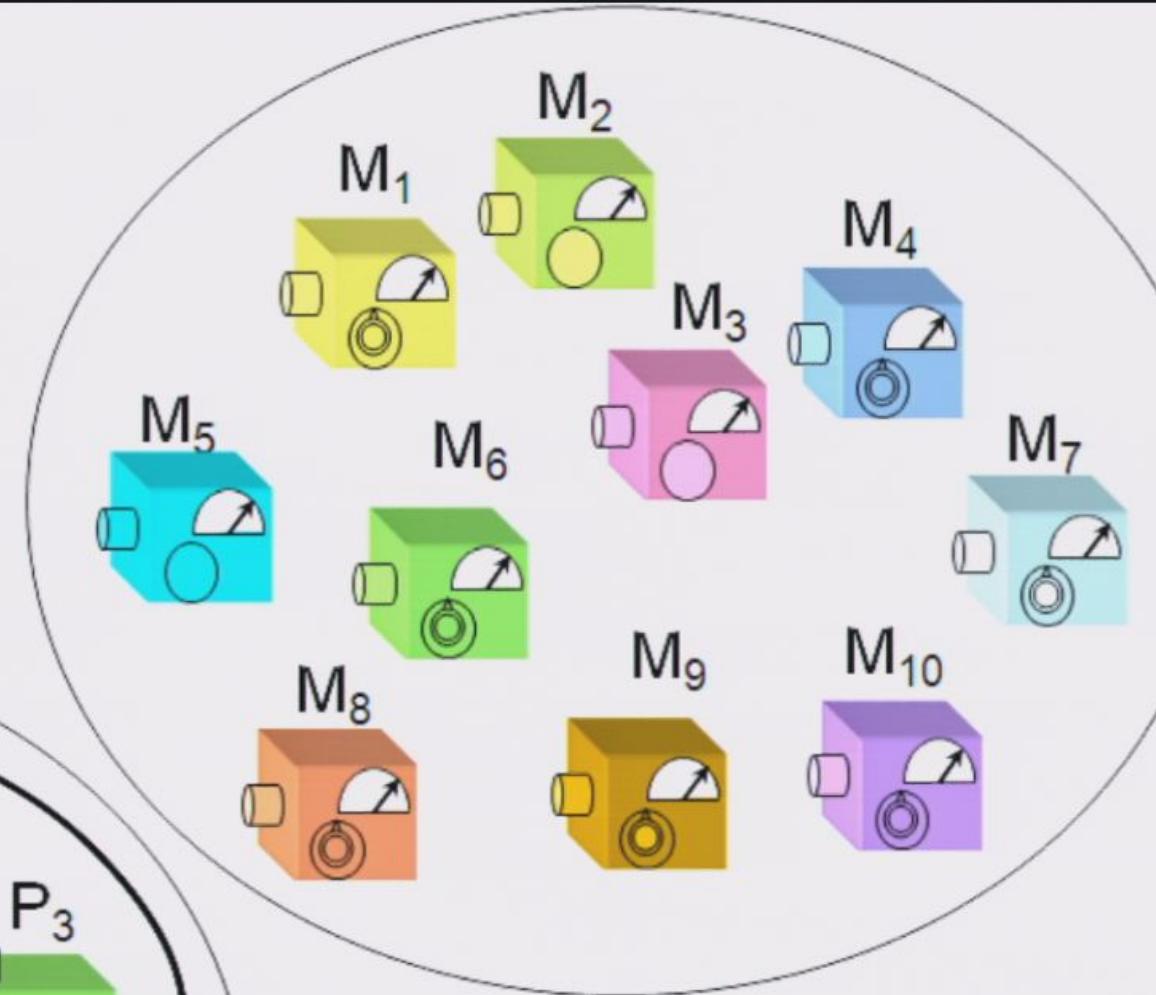
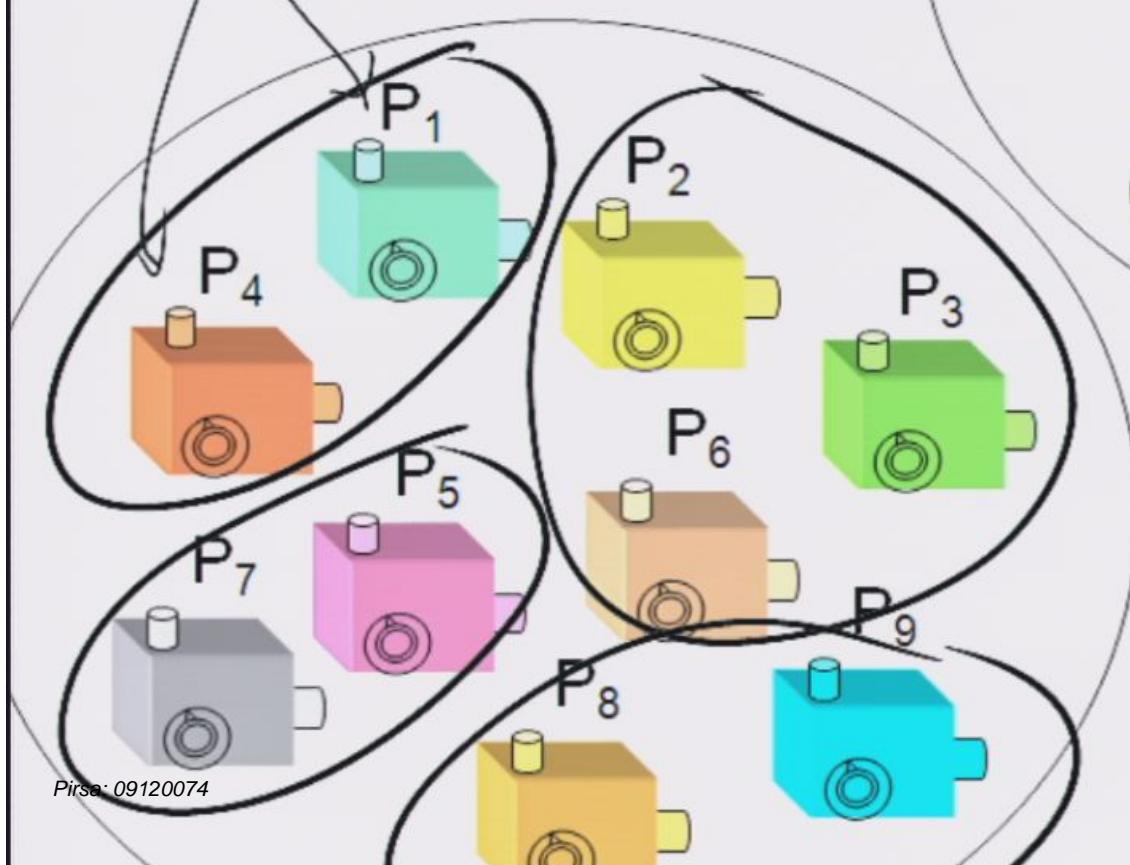
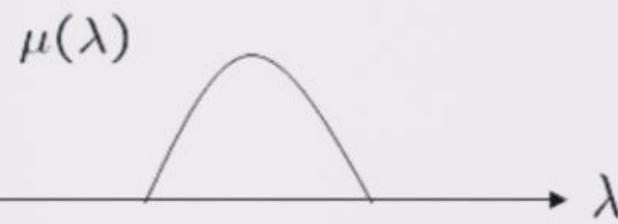
$$\frac{1}{2}I = \text{Tr}_B\left[\frac{1}{\sqrt{2}}(|0\rangle|+\rangle + |1\rangle|-\rangle)\right]$$



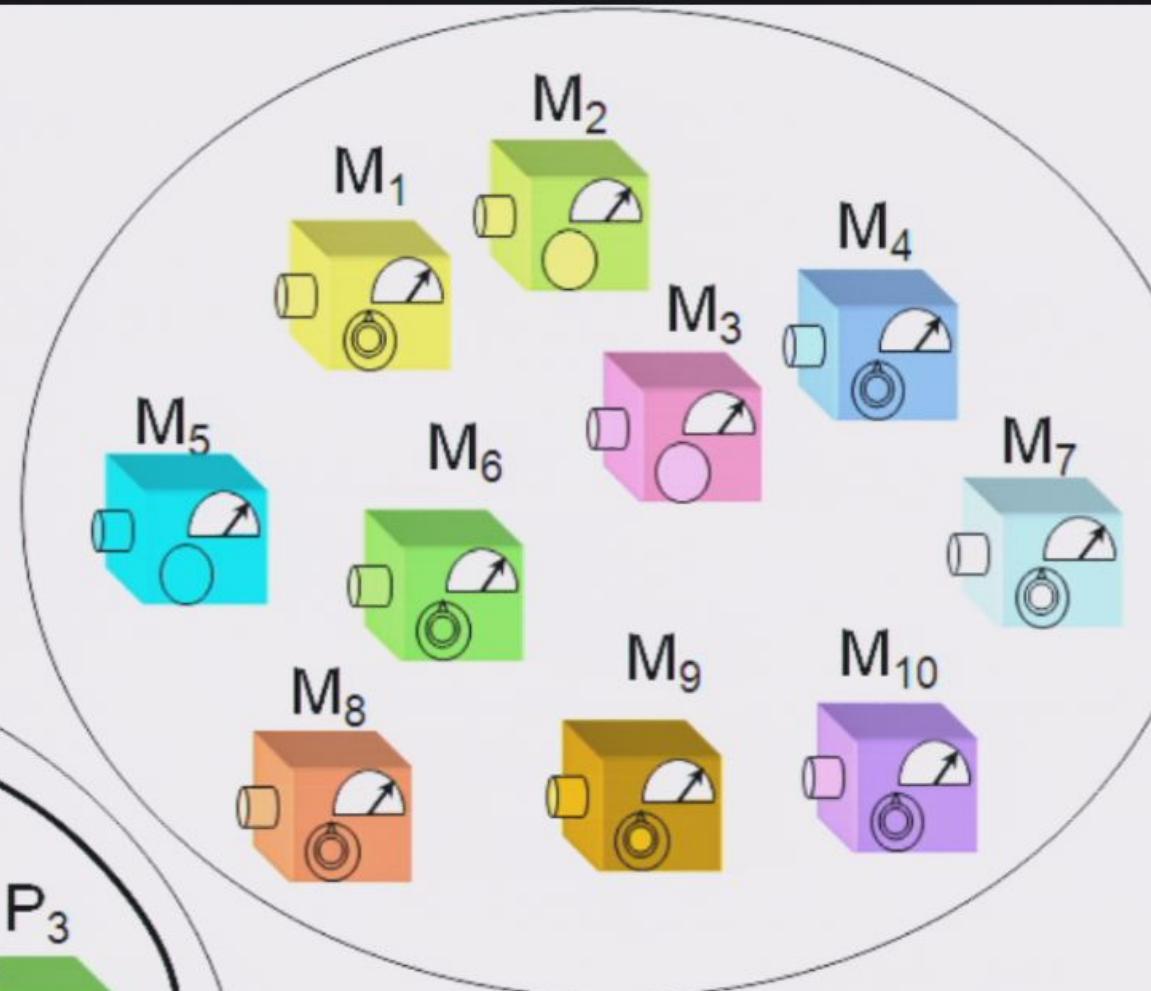
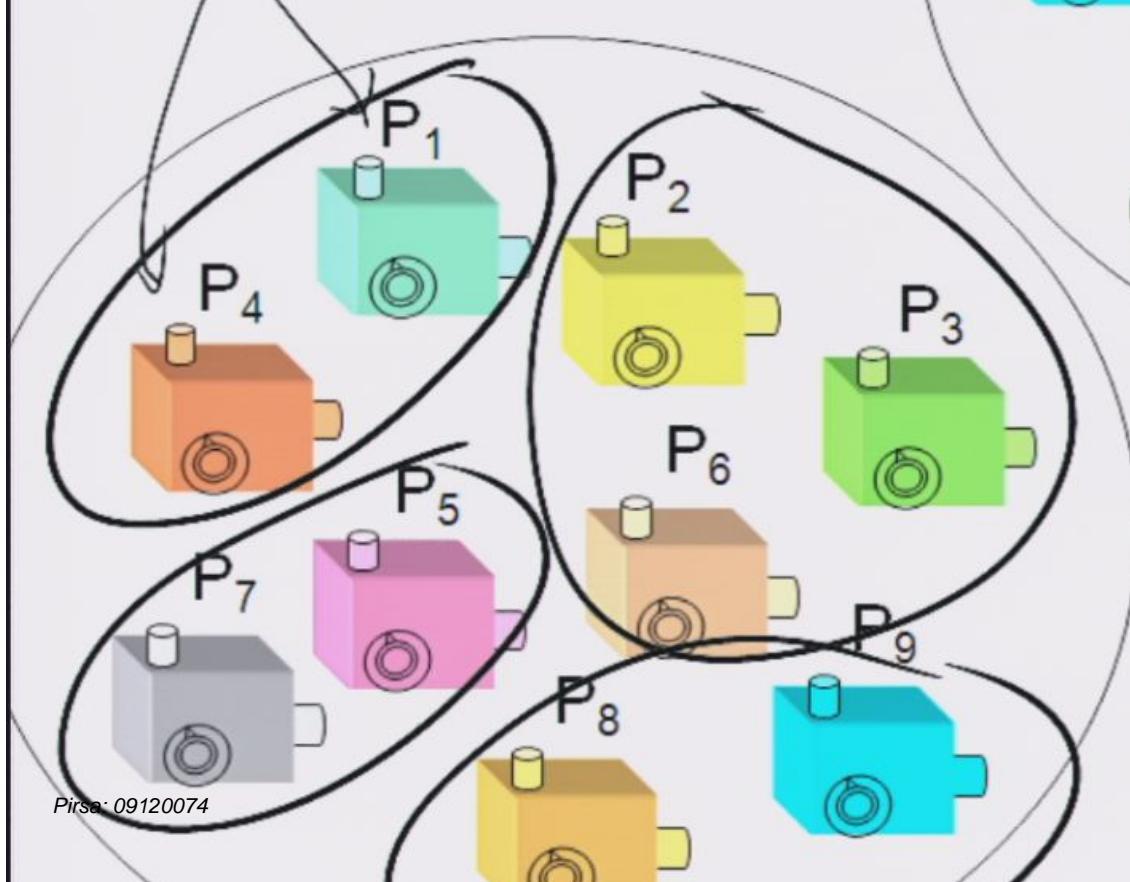
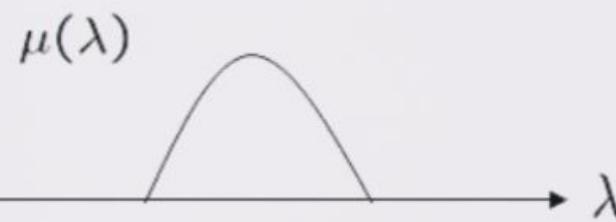
Preparation noncontextual model



Preparation noncontextual model



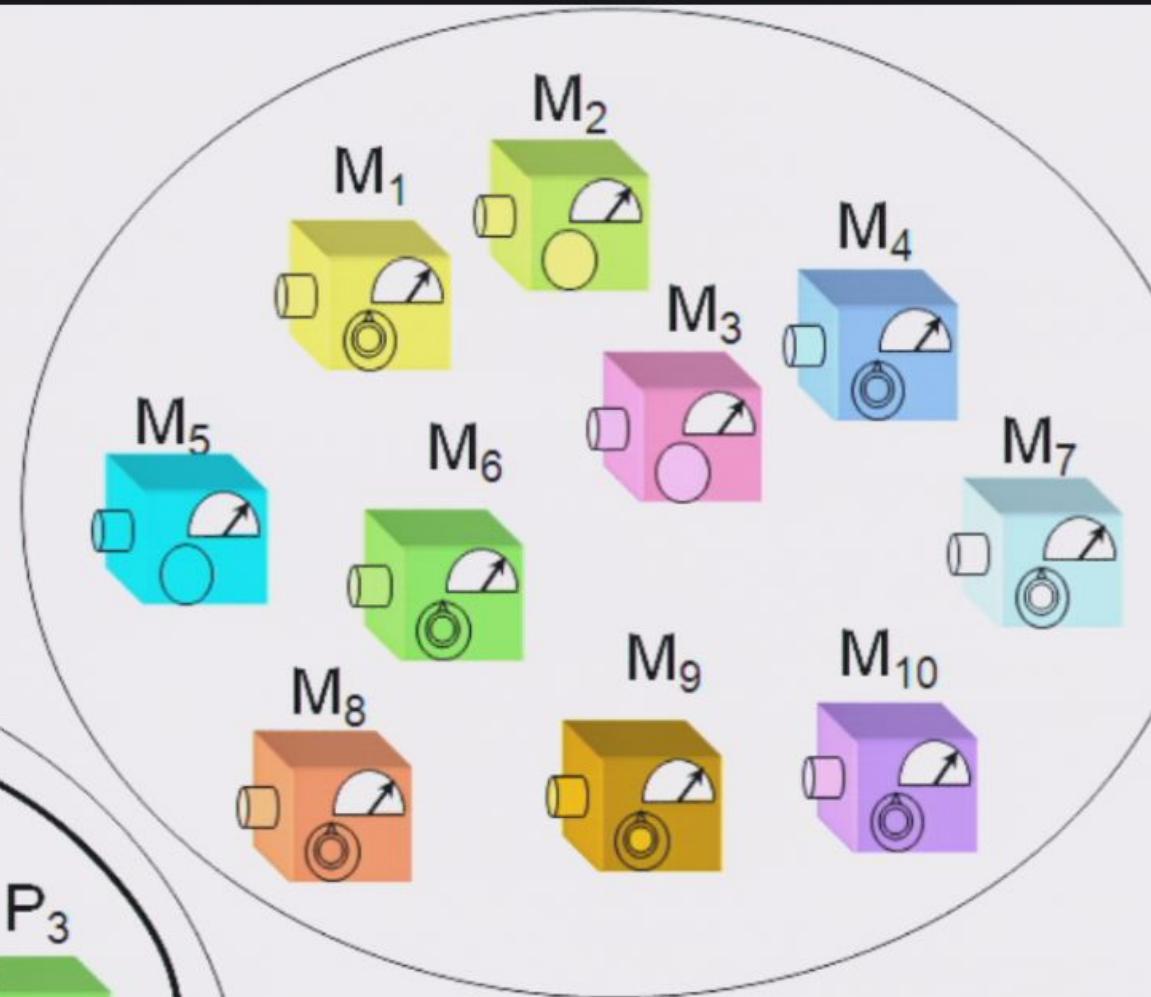
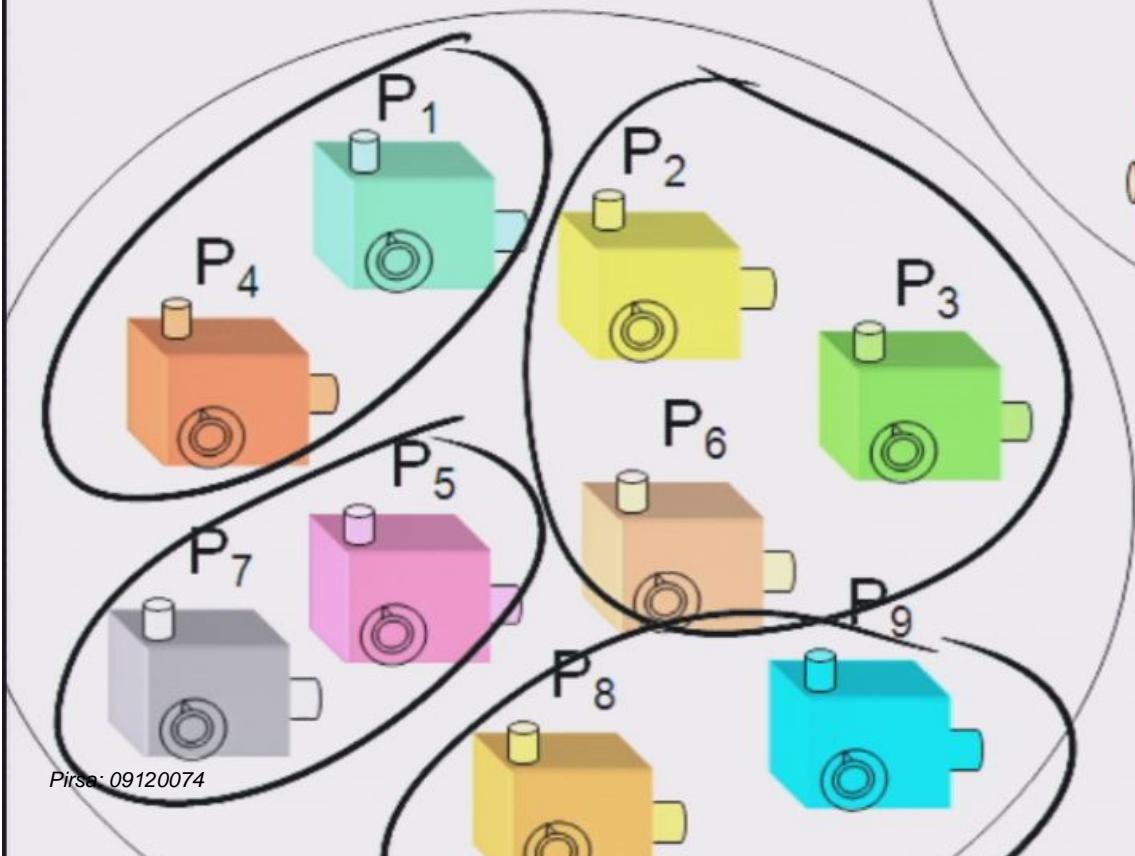
Preparation noncontextual model

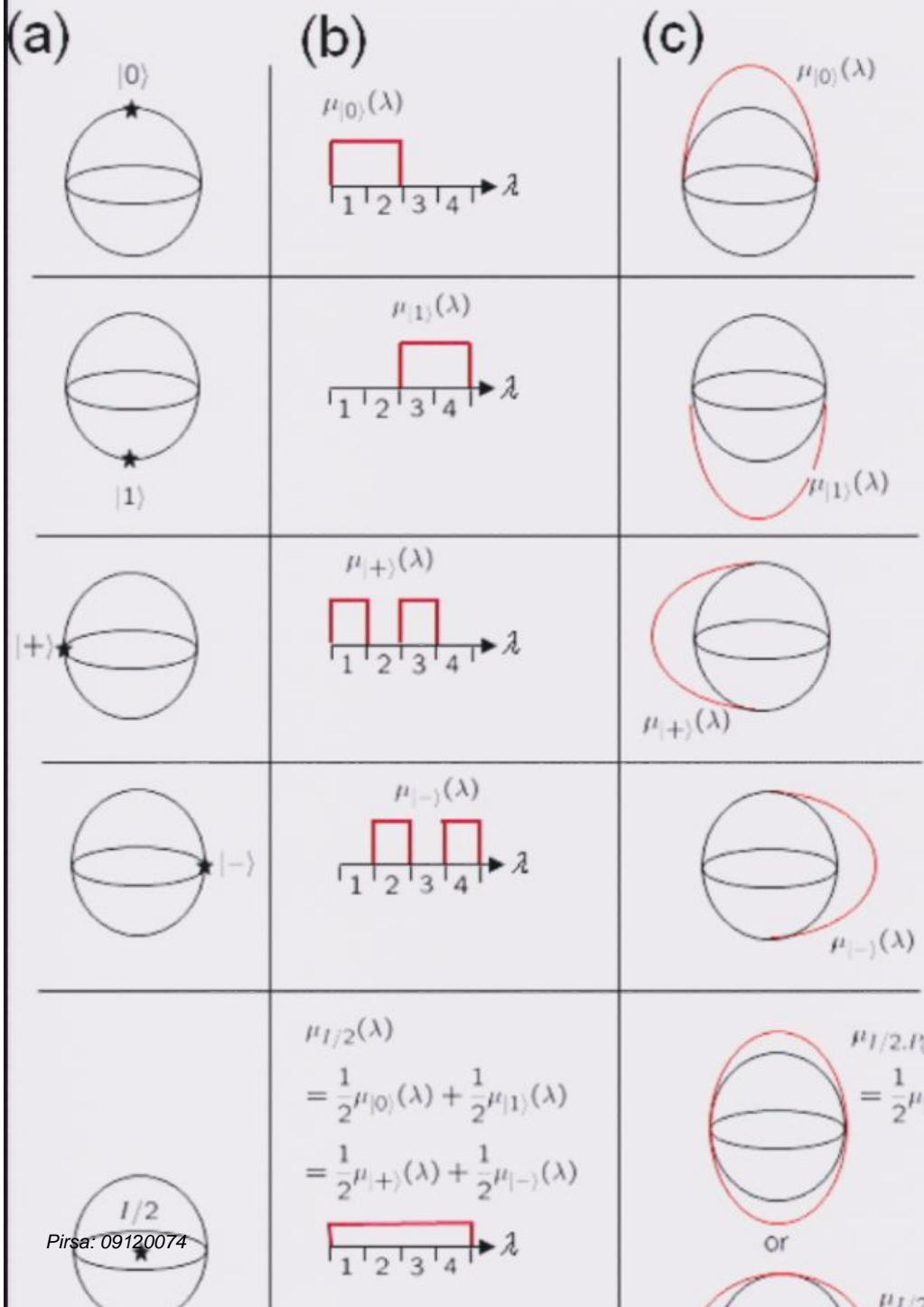


Definition of preparation noncontextual model:

$$\forall M : p(k|P, M) = p(k|P', M)$$

→ $p(\lambda|P) = p(\lambda|P')$



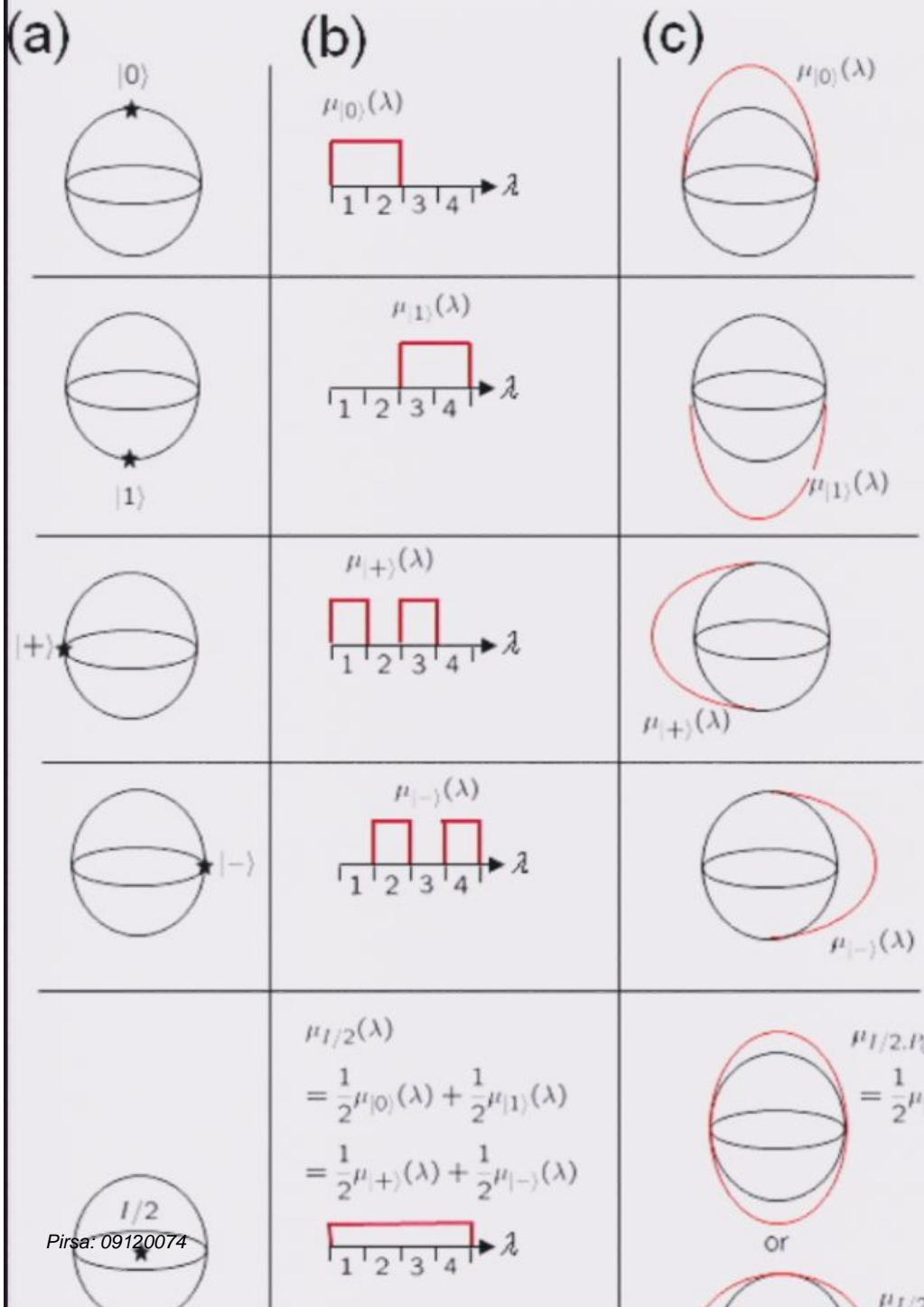


(a) Some states of a qubit

(b) A preparation **noncontextual** model of these
(RWS, PRA 75, 032110, 2007)

(c) A preparation **contextual** mode of these
(Kochen-Specker, 1967)

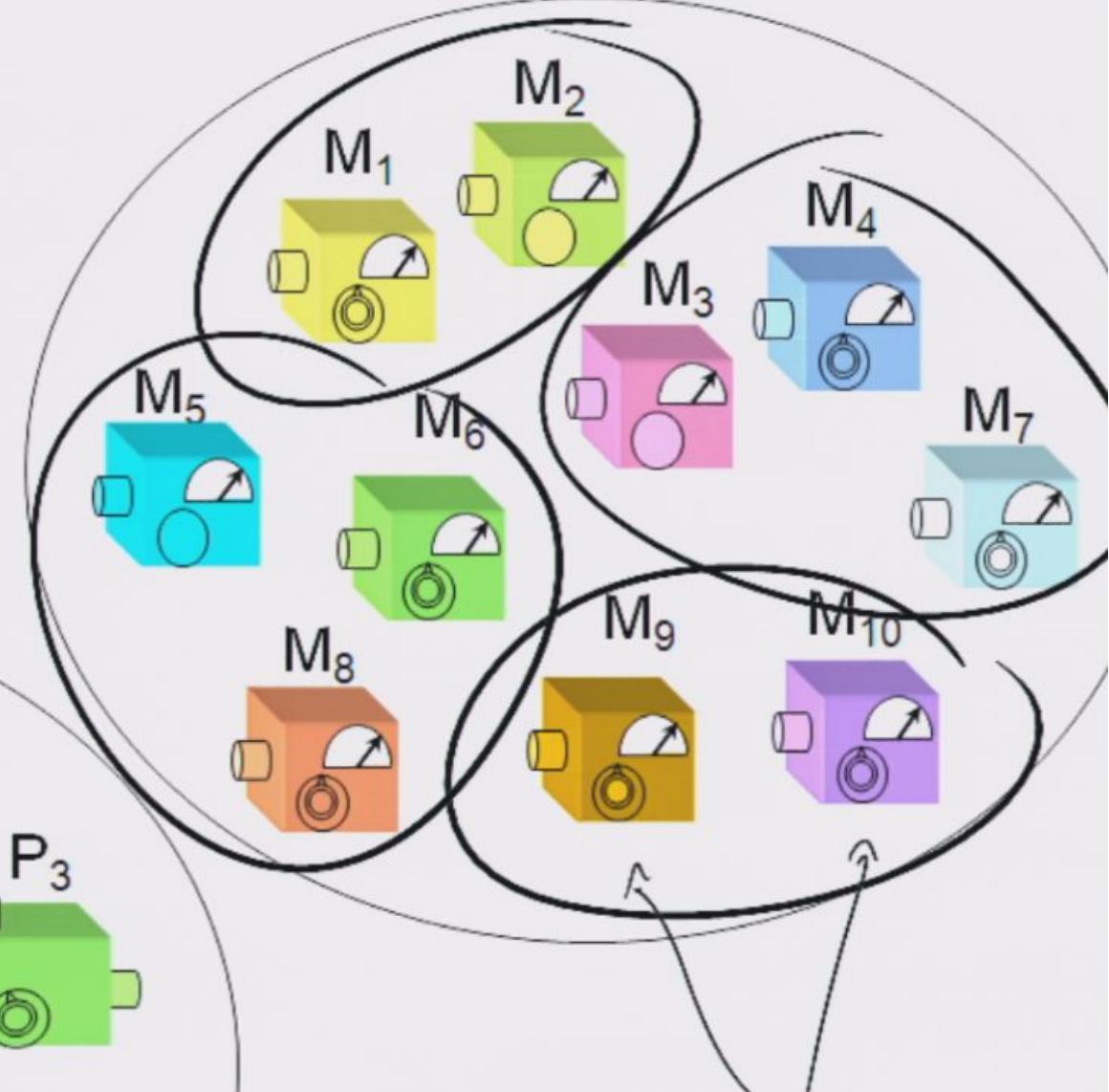
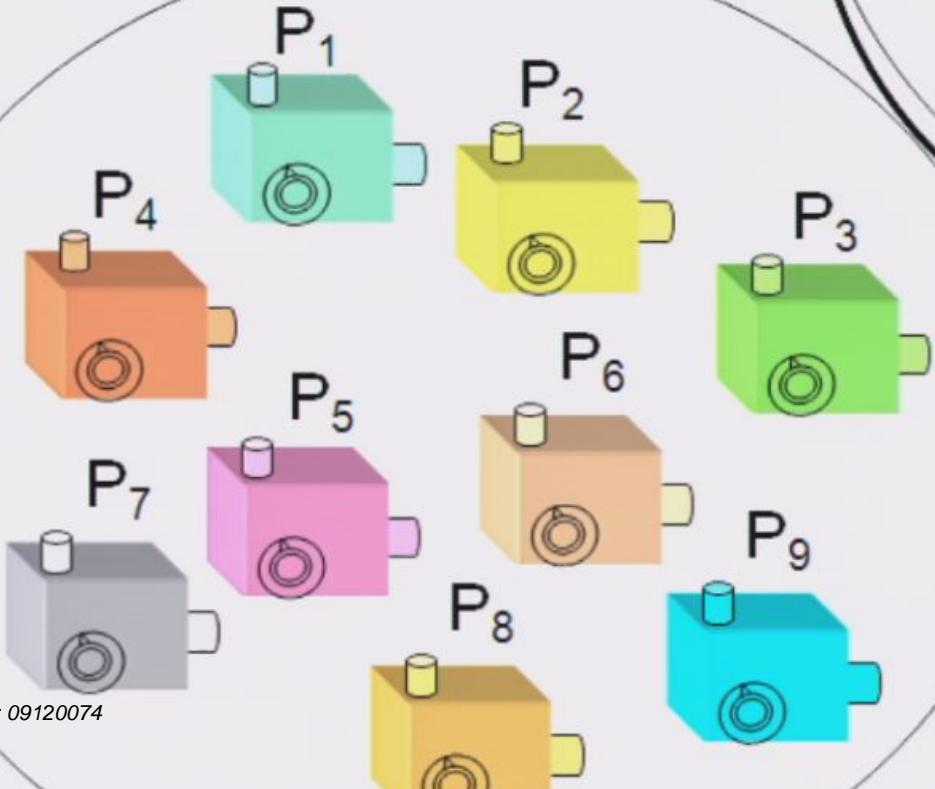
$$\begin{aligned}
 & \frac{1}{2} \left(\frac{1}{2}, \frac{1}{2}, 0, 0 \right) \\
 + \frac{1}{2} \left(0, 0, \frac{1}{2}, \frac{1}{2} \right) &= \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4} \right) \\
 \left(\frac{1}{2}, 0, \frac{1}{2}, 0 \right) \\
 \left(0, \frac{1}{2}, 0, \frac{1}{2} \right)
 \end{aligned}$$



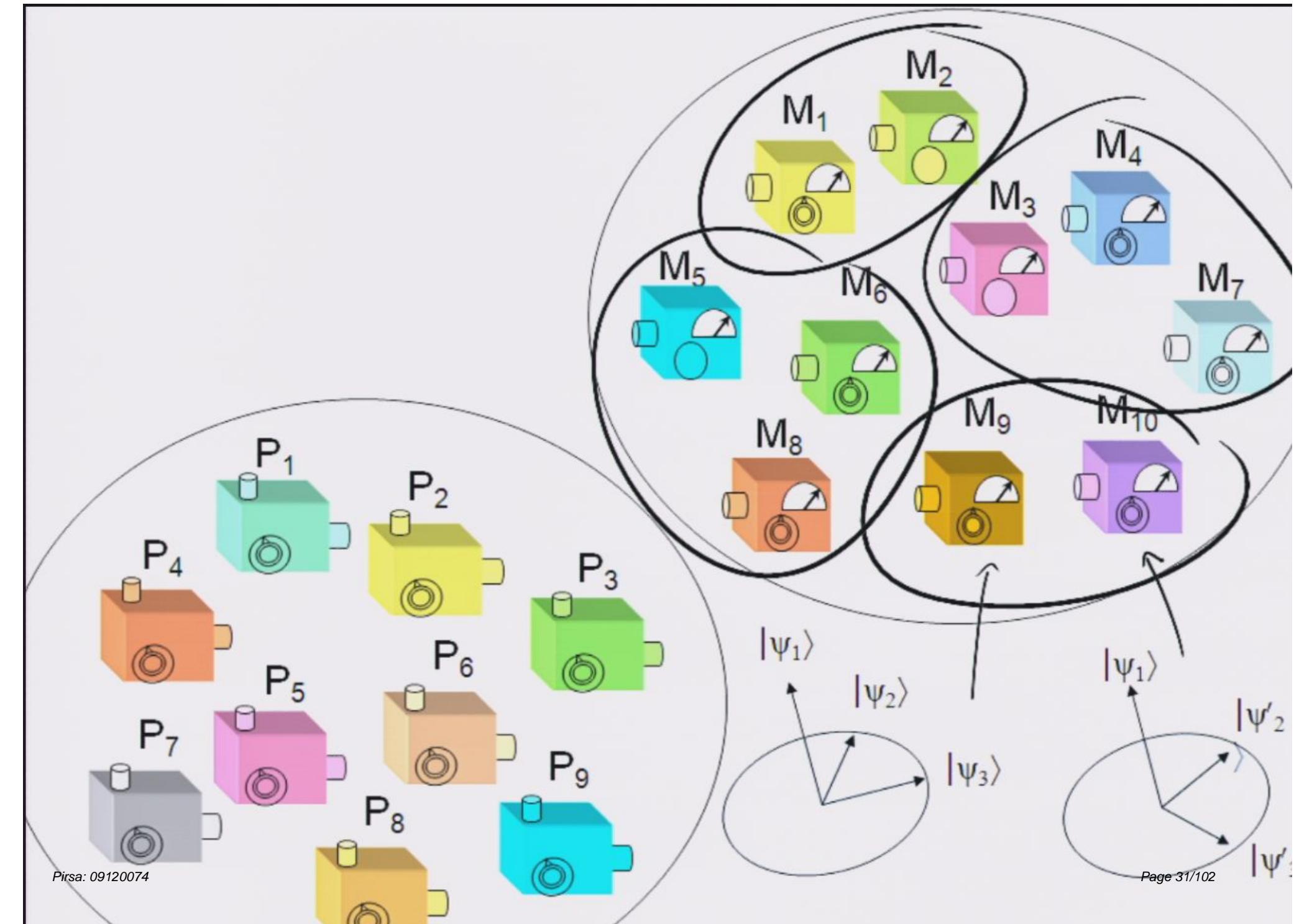
(a) Some states of a qubit

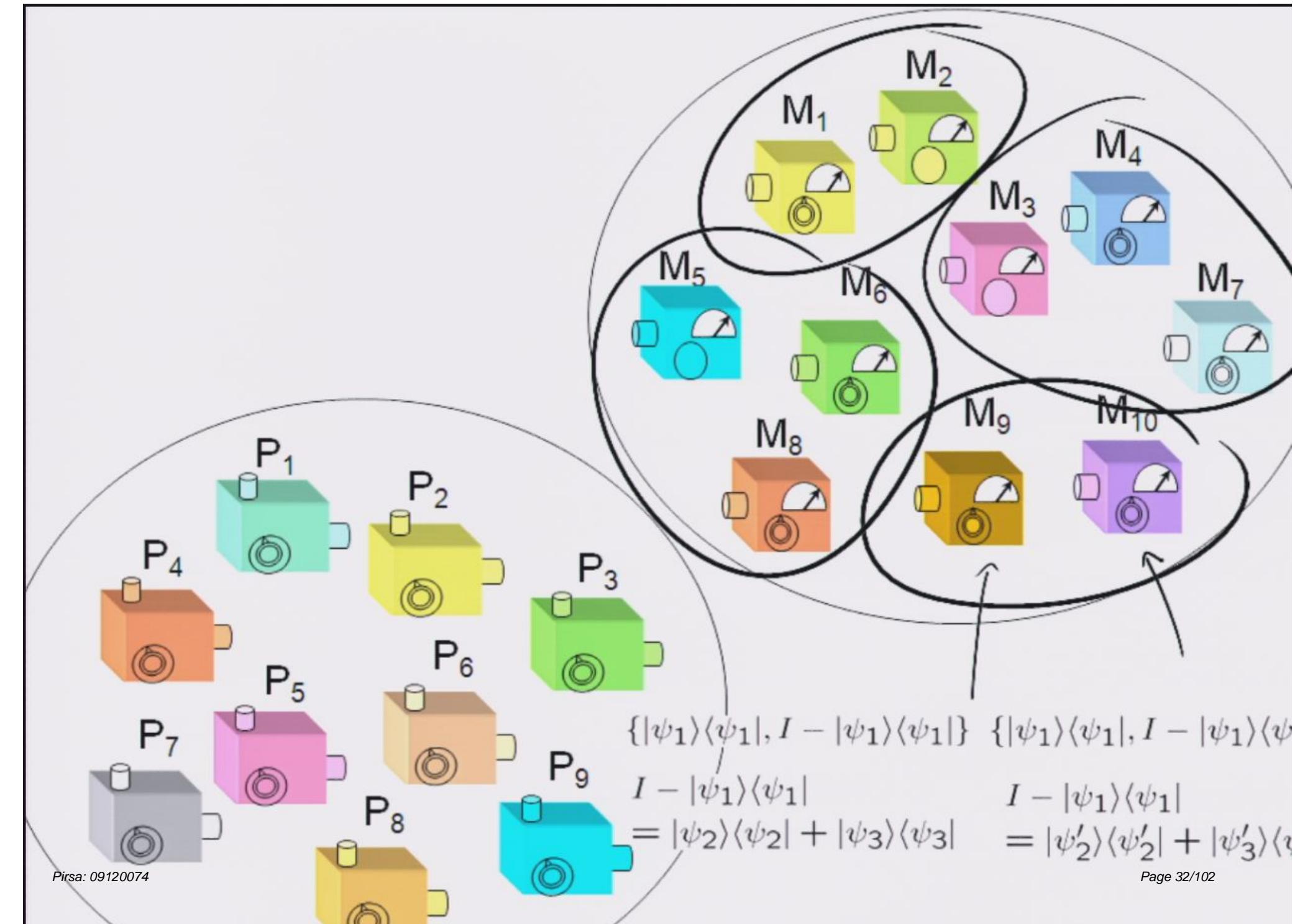
(b) A preparation **noncontextual** model of these
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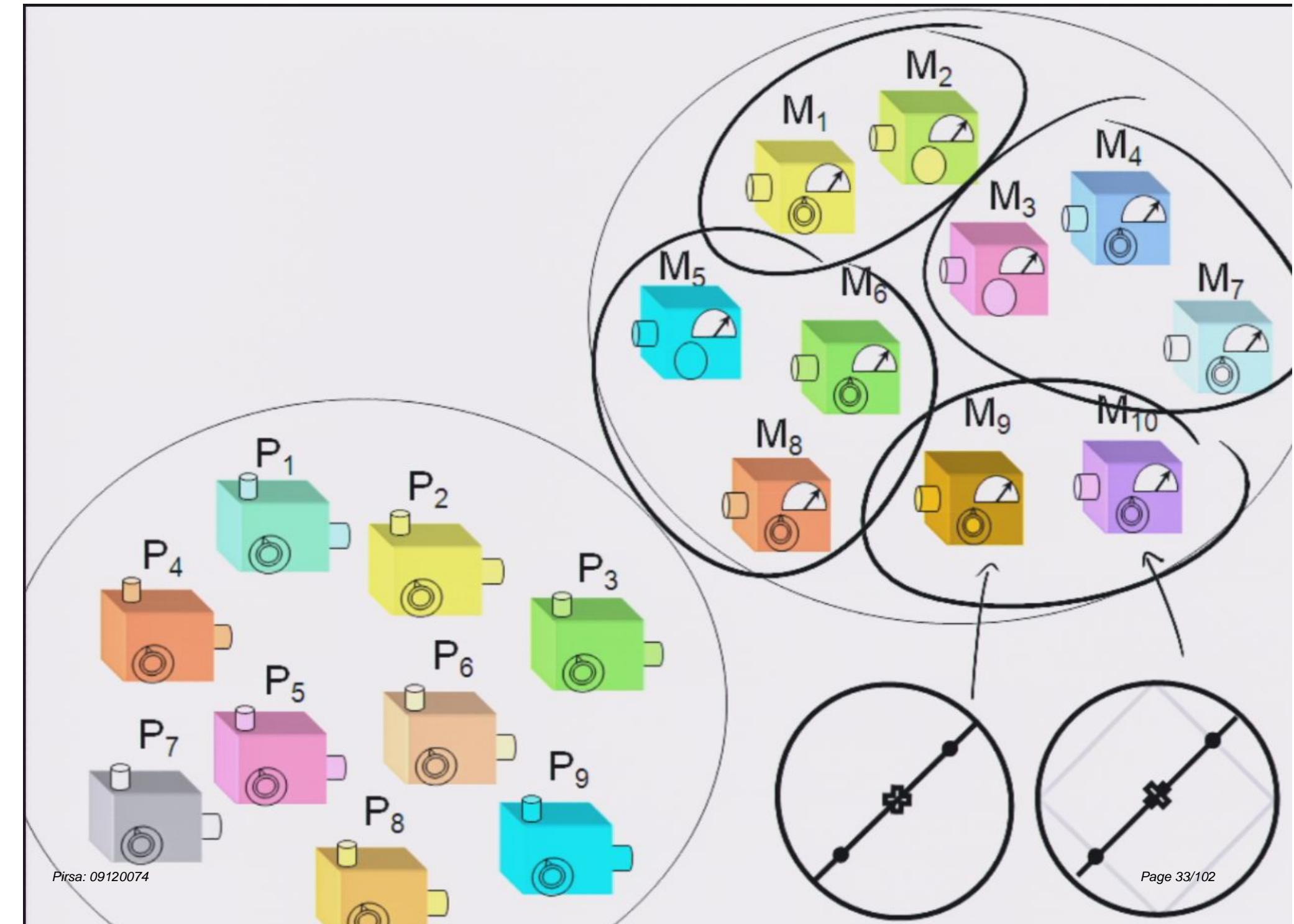
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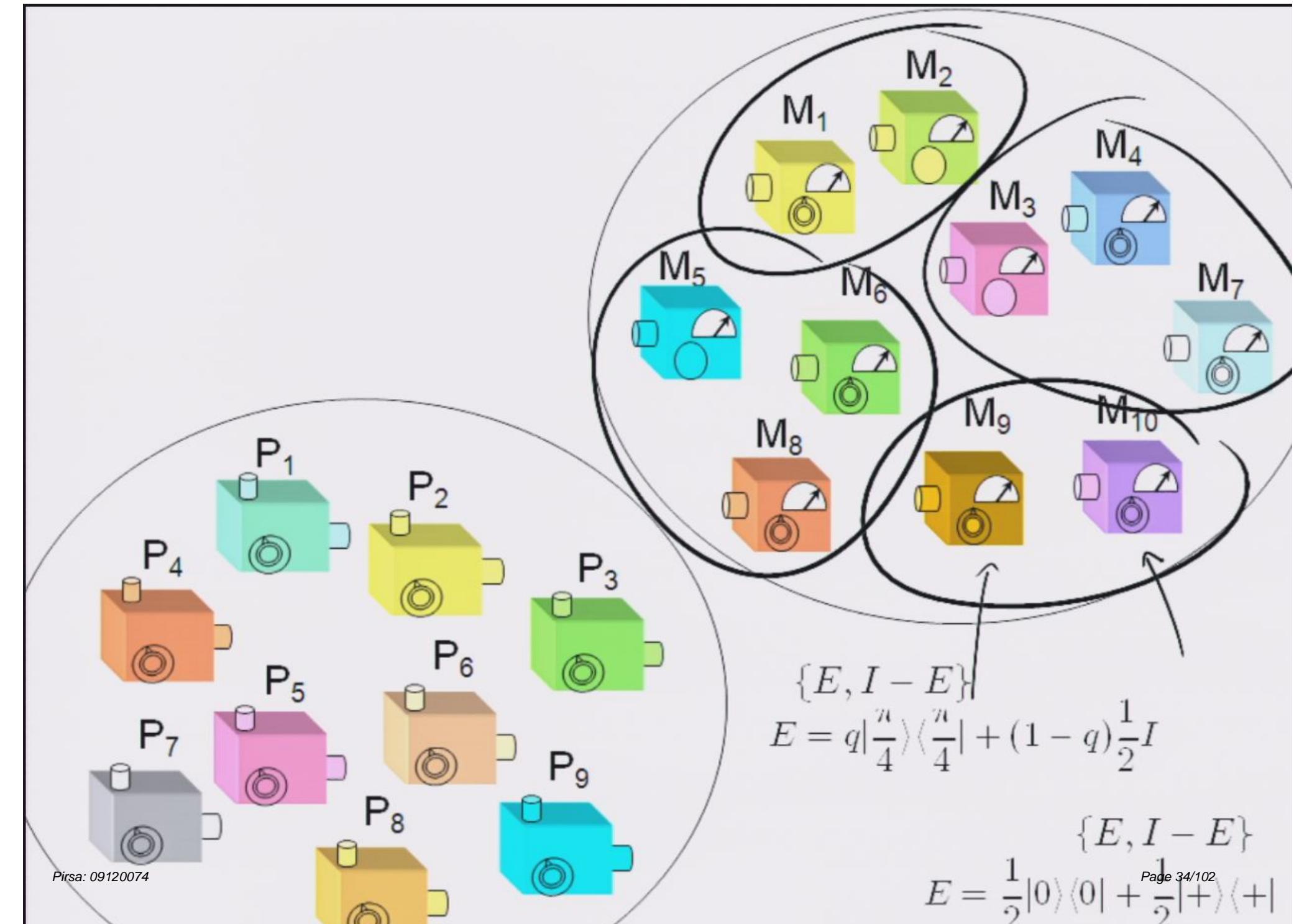


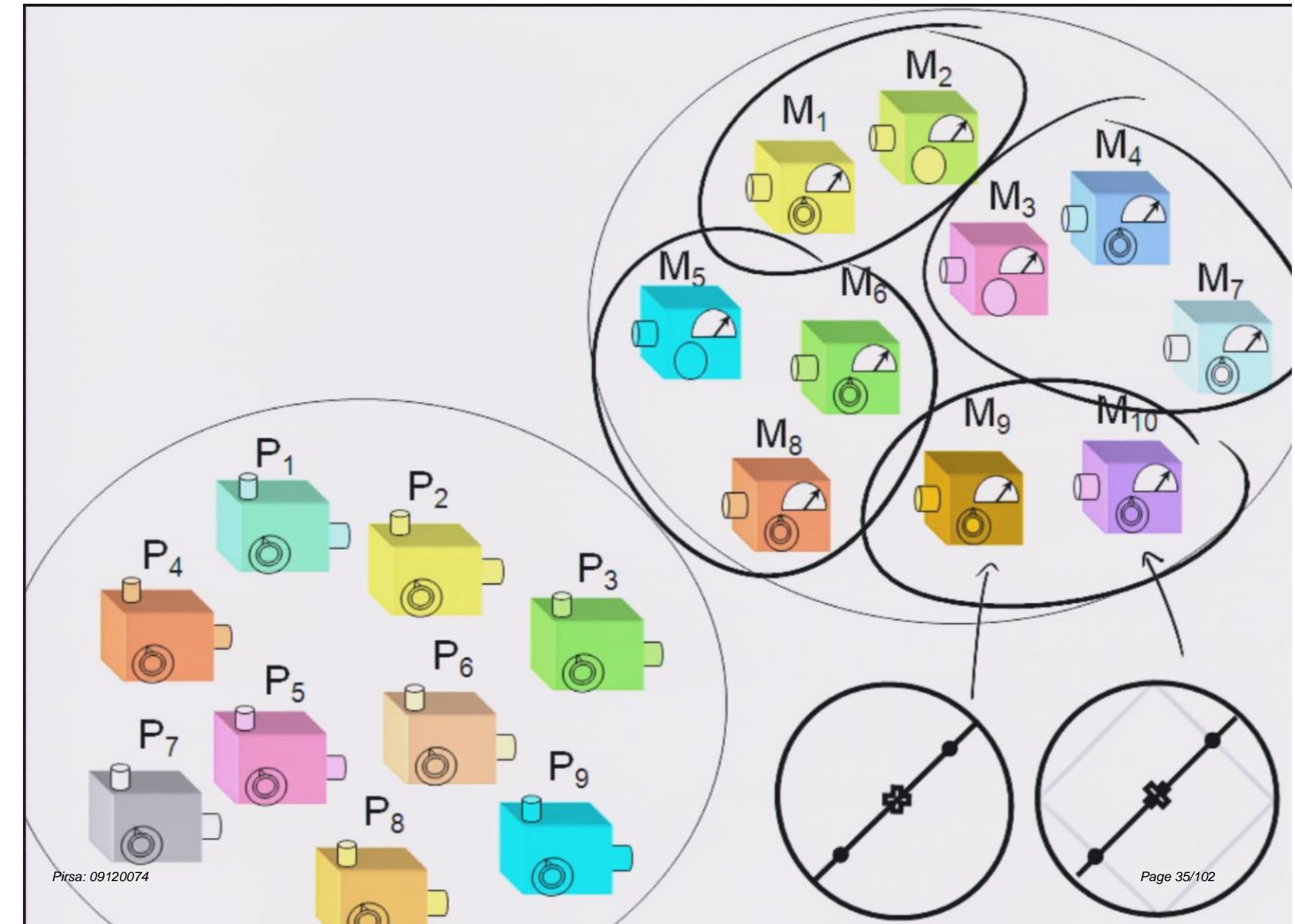
Difference of context

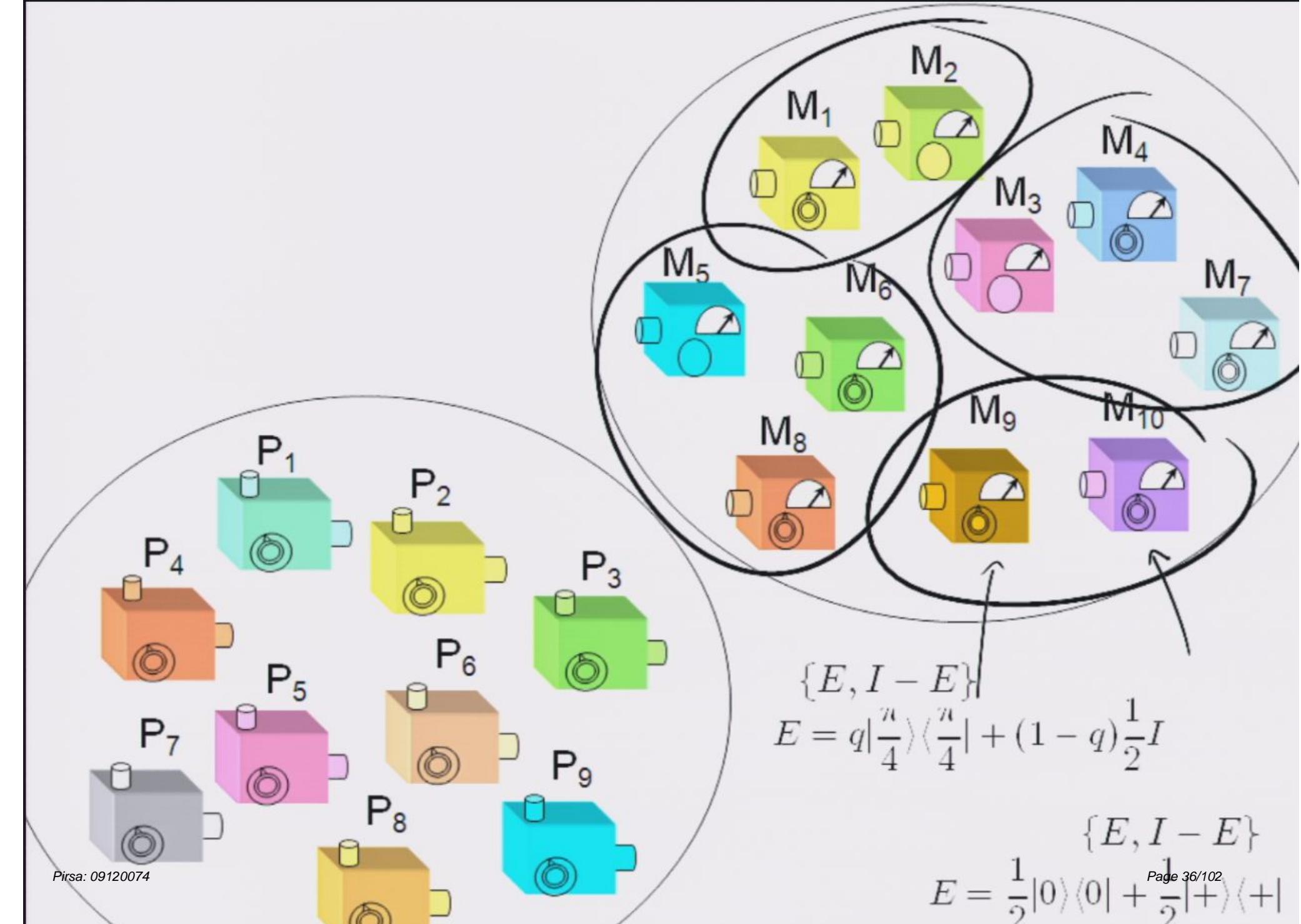


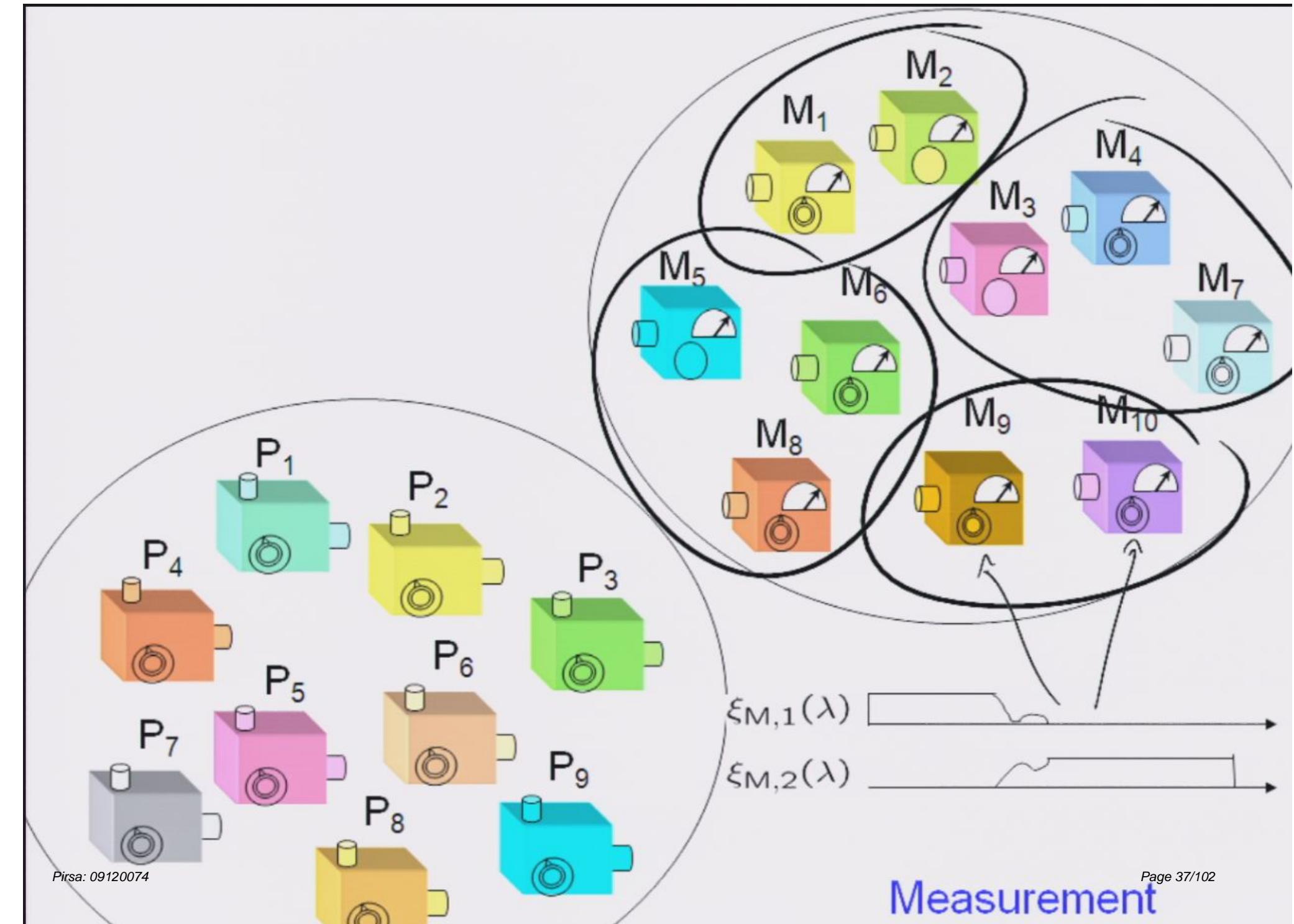


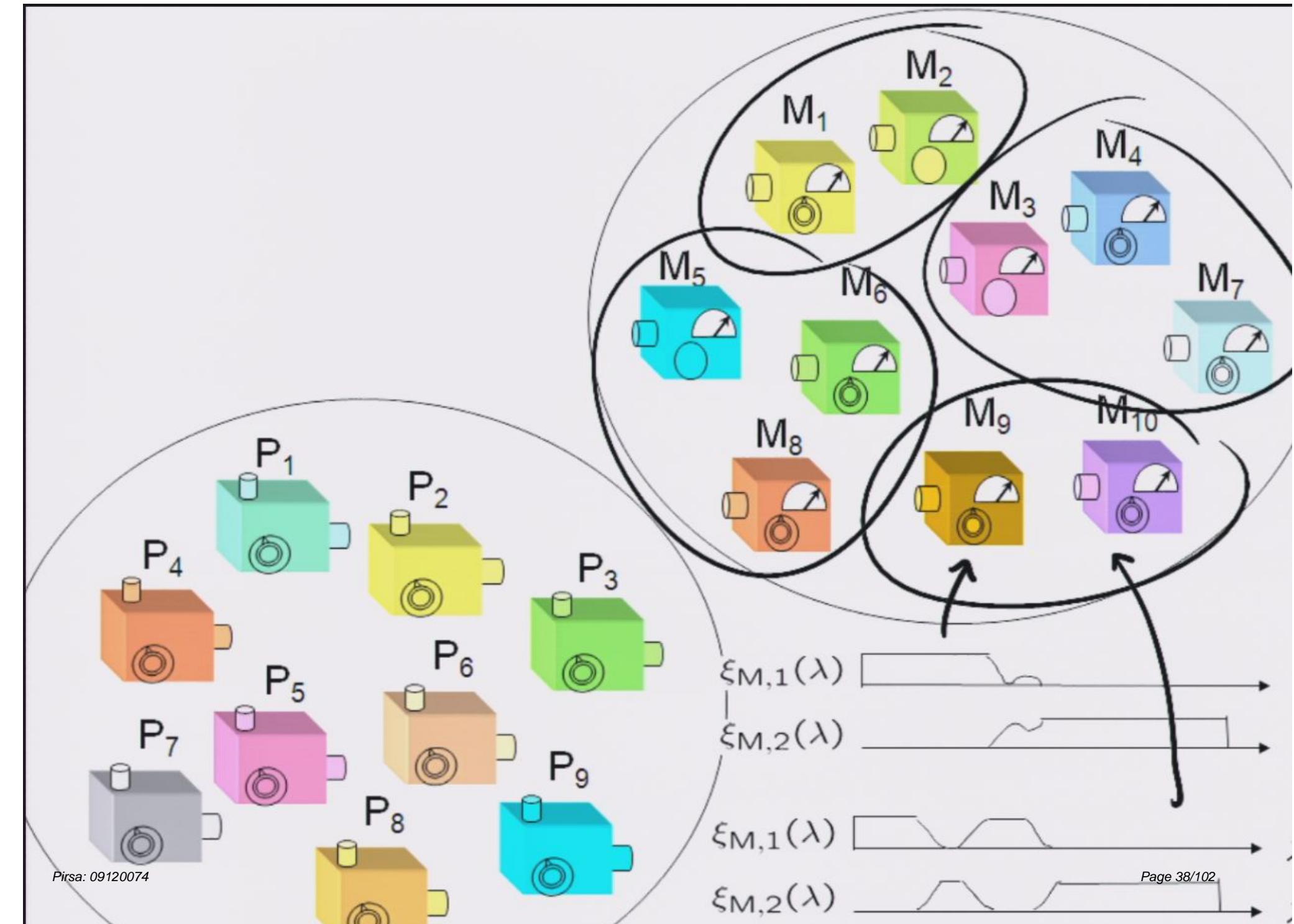


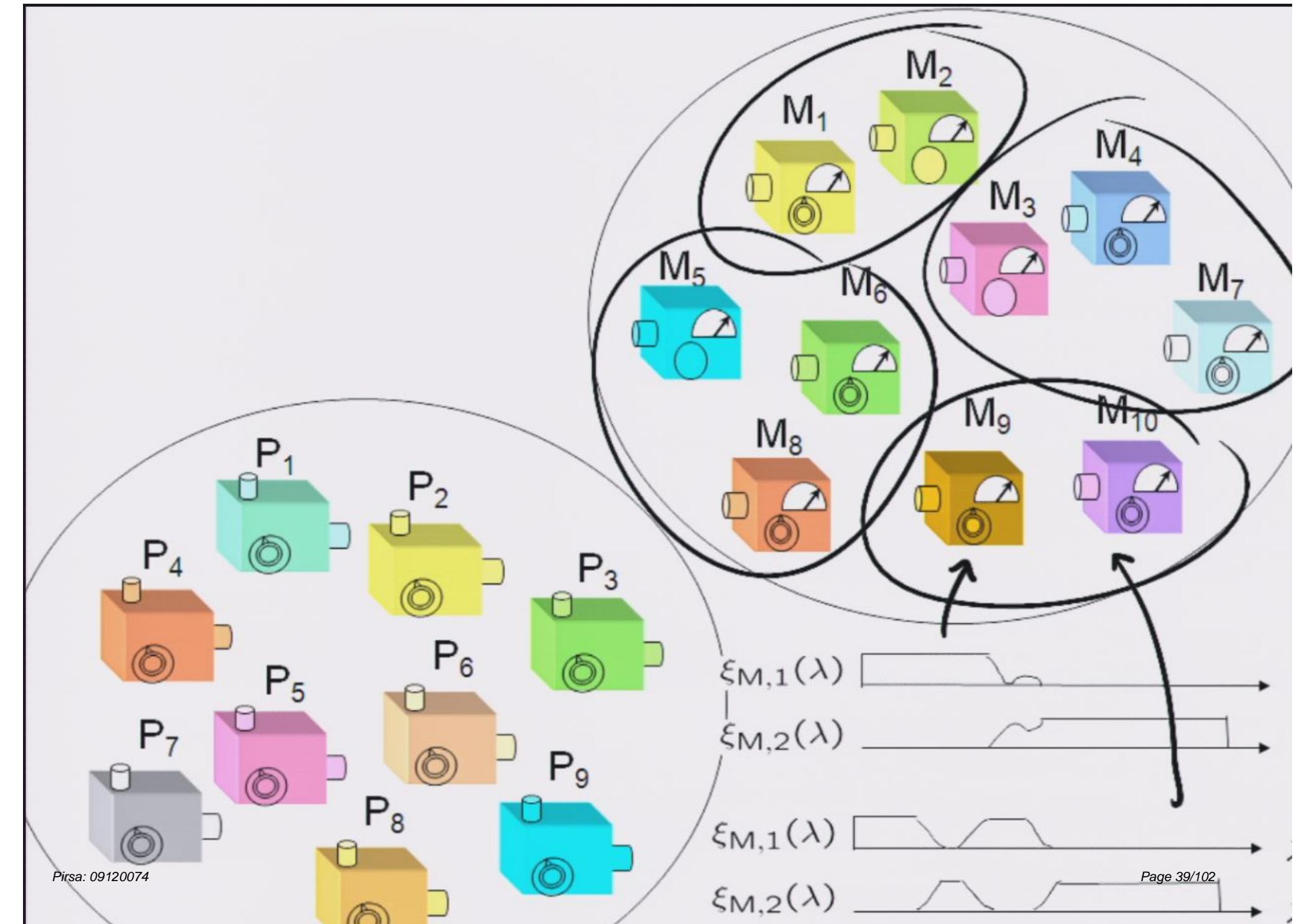












universal noncontextuality

= noncontextuality for preparations *and* measurements

Preparation-based proof of contextuality

(i.e. of the impossibility of a noncontextual
realist model of quantum theory)

Important features of realist models

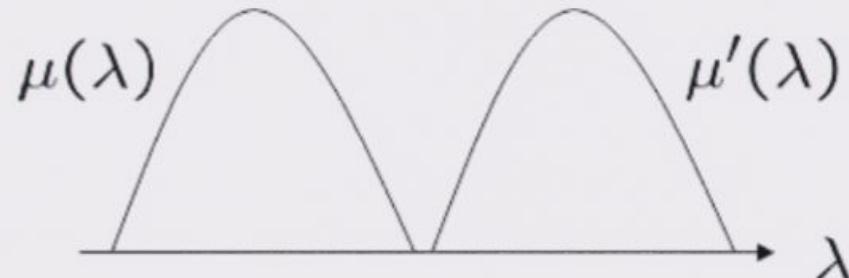
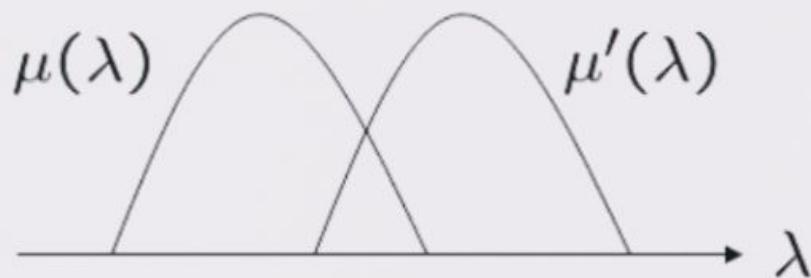
Let $P \leftrightarrow \mu(\lambda)$

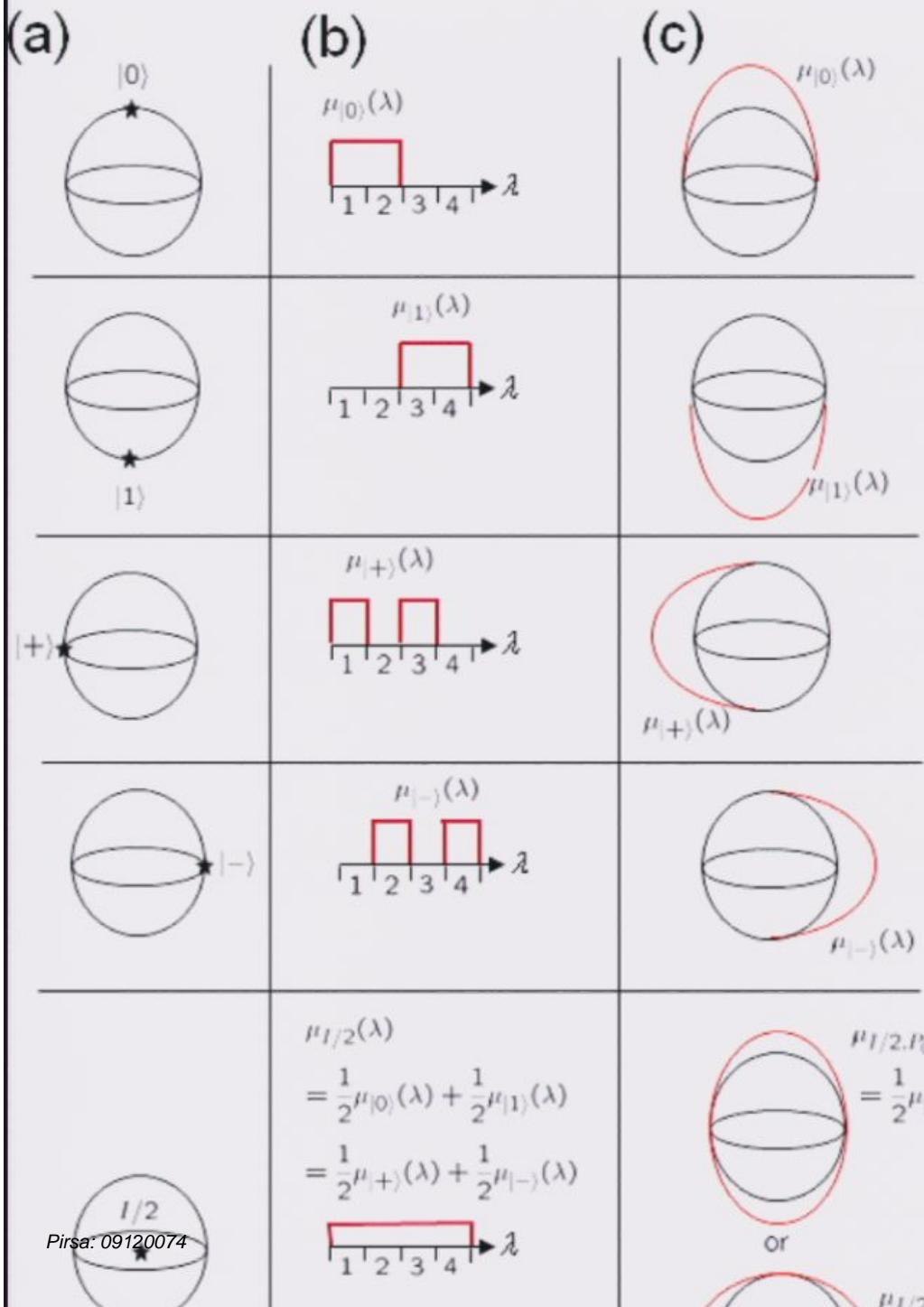
$P' \leftrightarrow \mu'(\lambda)$

Representing one-shot distinguishability:

If P and P' are distinguishable with certainty

then $\mu(\lambda) \mu'(\lambda) = 0$

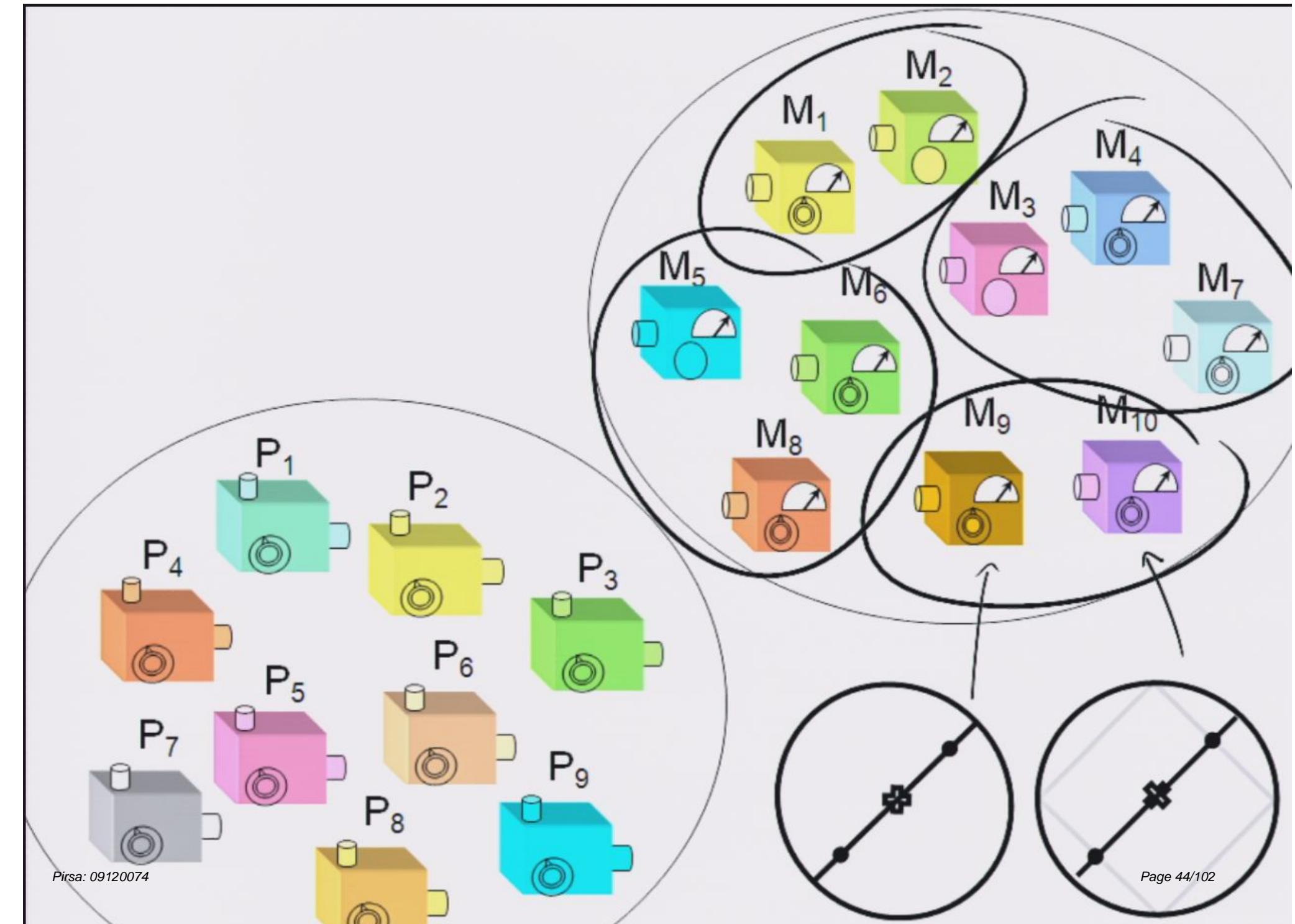


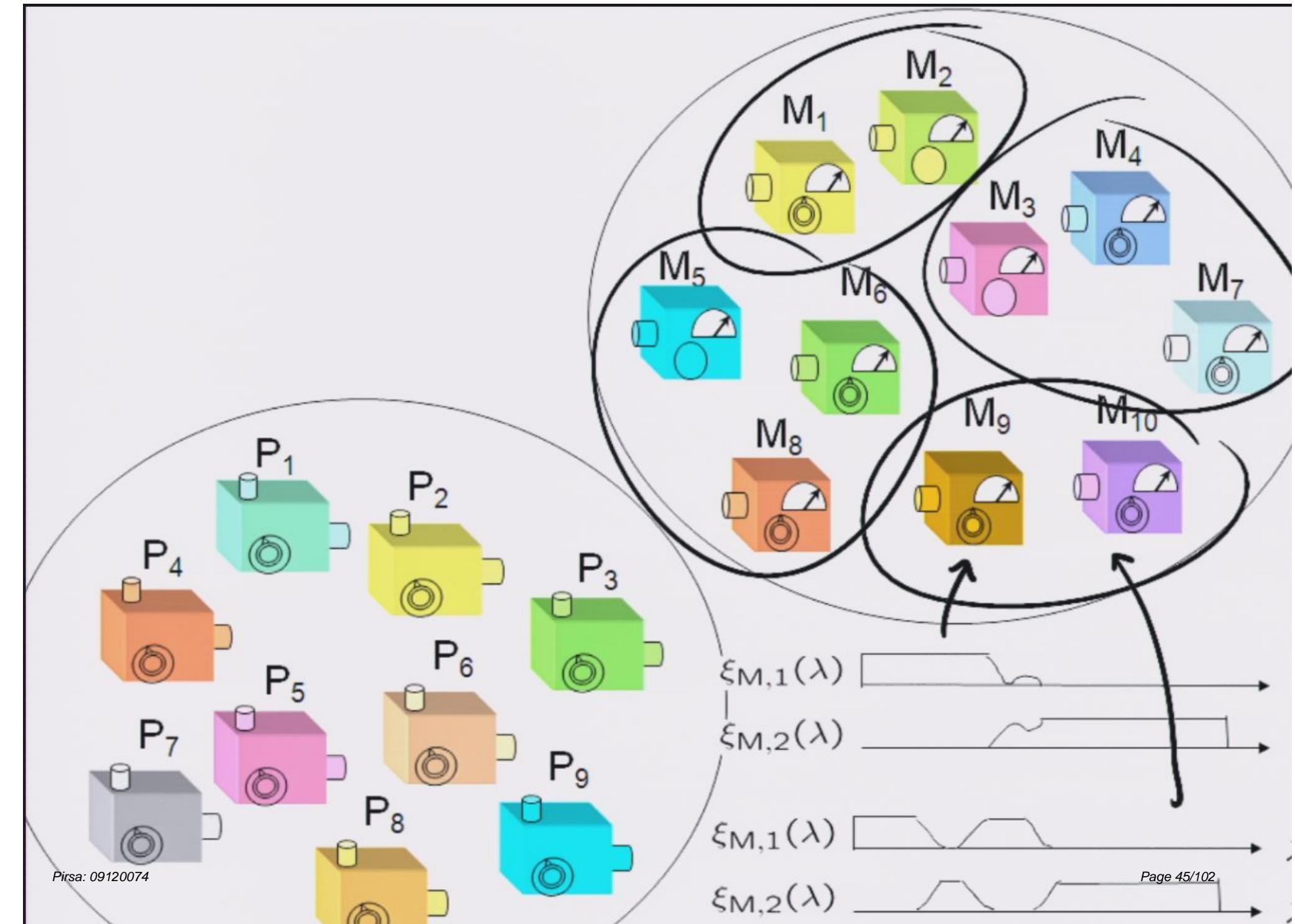


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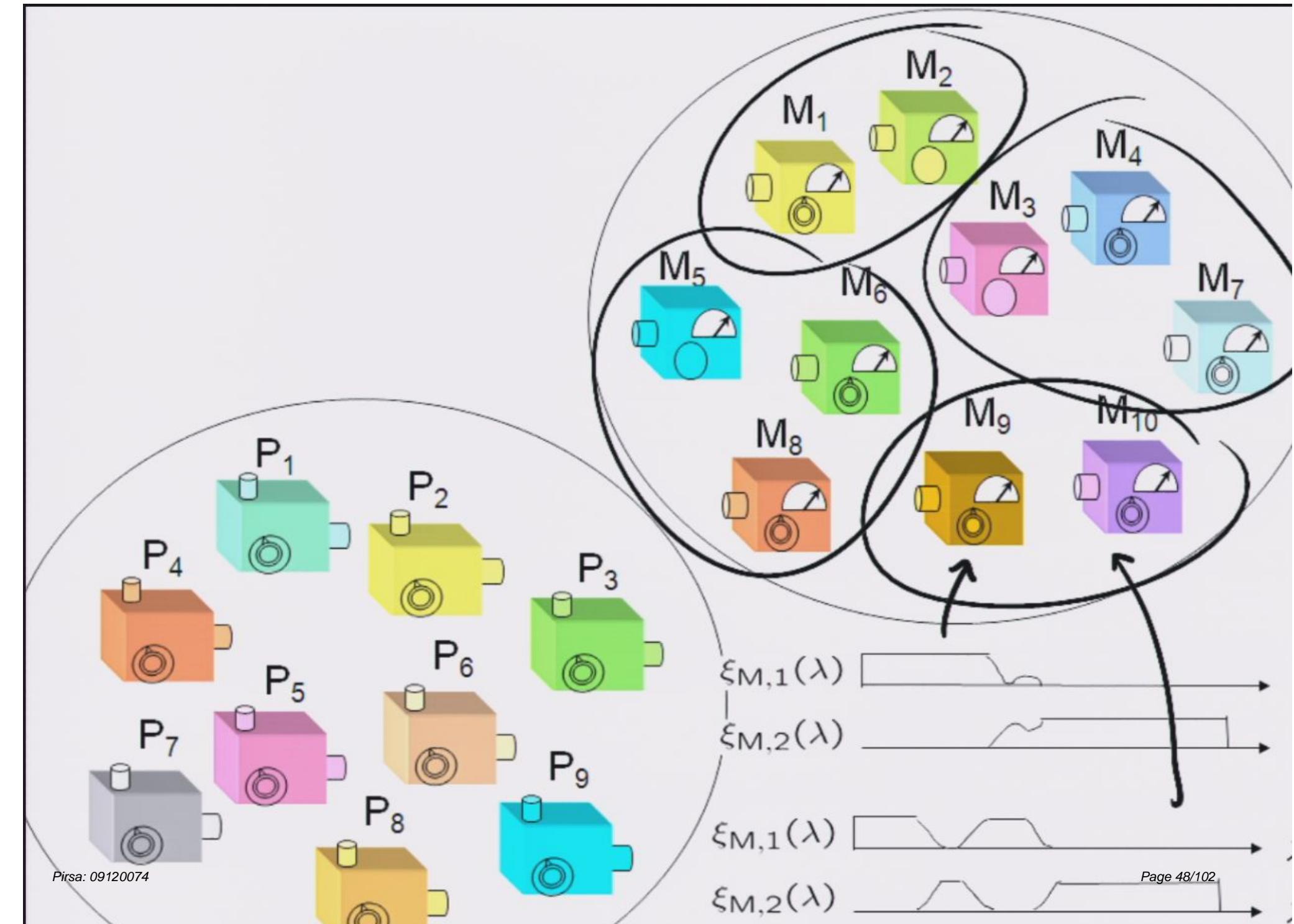


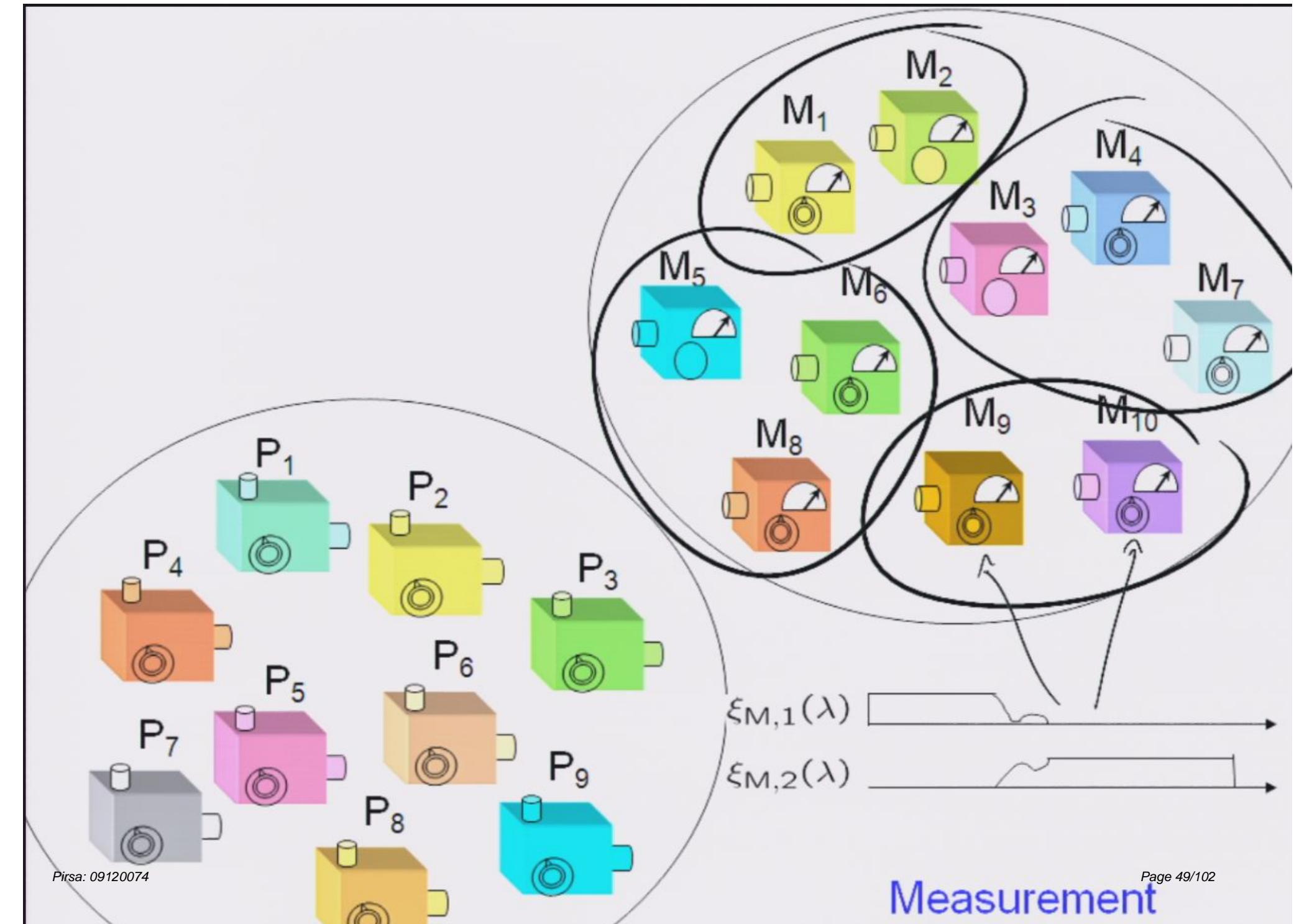
universal noncontextuality

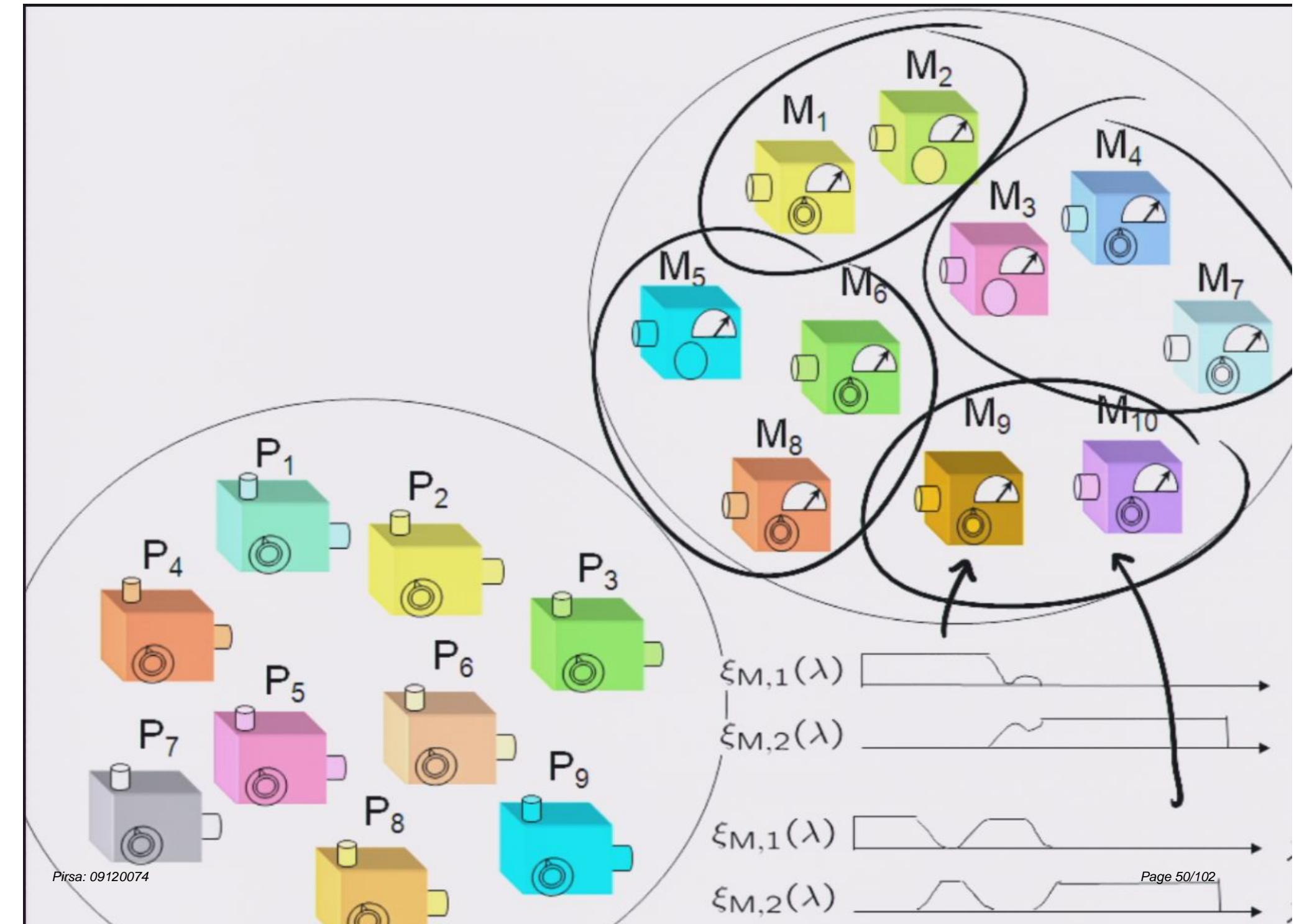
= noncontextuality for preparations *and* measurements

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universal noncontextuality

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Important features of realist models

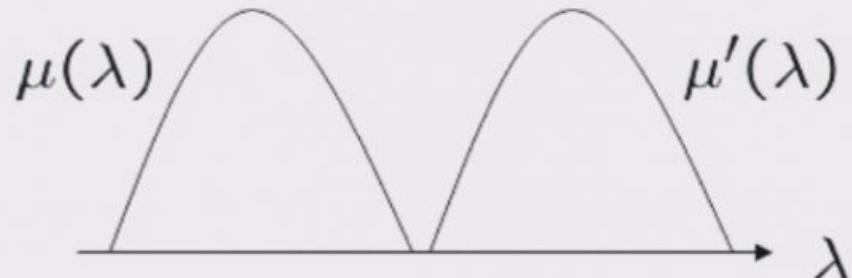
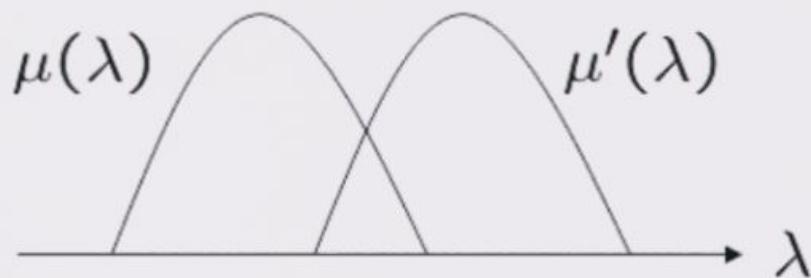
Let $P \leftrightarrow \mu(\lambda)$

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If P and P' are distinguishable with certainty

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Important features of realist models

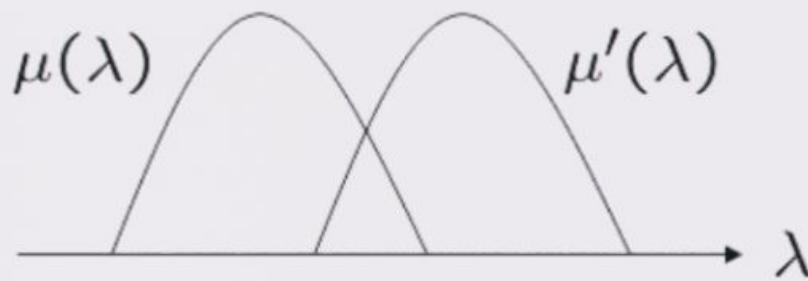
Let $P \leftrightarrow \mu(\lambda)$

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Representing one-shot distinguishability:

If P and P' are distinguishable with certainty

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Representing convex combination:

If $P'' = P$ with prob. p and P' with prob. $1 - p$

Then $\mu''(\lambda) = p \mu(\lambda) + (1 - p) \mu'(\lambda)$

$$P(\lambda|P'') = P(\lambda|P) \underbrace{P(P)}_{\rho} + P(\lambda|P') \underbrace{P(P')}_{1-\rho} =$$

$$\frac{1}{2} \left(\frac{1}{2}, \frac{1}{2}, 0, 0 \right) + \frac{1}{2} \left(0, 0, \frac{1}{2}, \frac{1}{2} \right) =$$

$$\left(\frac{1}{2}, 0, \frac{1}{2}, 0 \right)$$

$$\left(0, \frac{1}{2}, 0, \frac{1}{2} \right)$$

Proof based on finite construction in 2d

$$P_a \leftrightarrow \psi_a = (1, 0)$$

$$P_A \leftrightarrow \psi_A = (0, 1)$$

$$P_b \leftrightarrow \psi_b = (1/2, \sqrt{3}/2)$$

$$P_B \leftrightarrow \psi_B = (\sqrt{3}/2, -1/2)$$

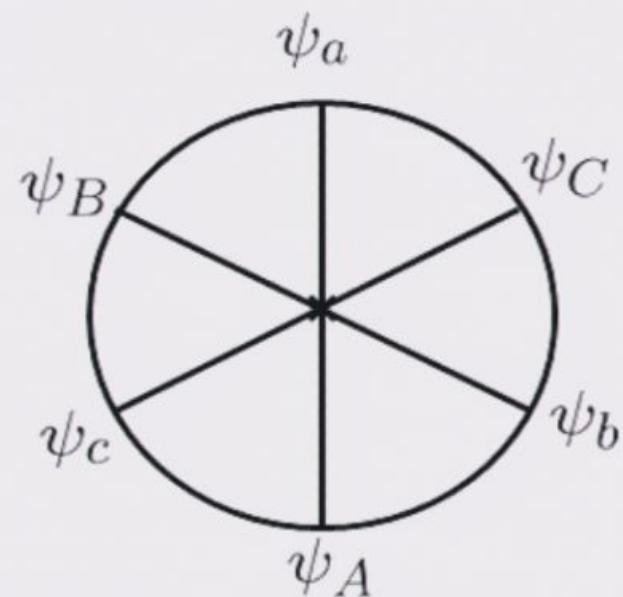
$$P_c \leftrightarrow \psi_c = (1/2, -\sqrt{3}/2)$$

$$P_C \leftrightarrow \psi_C = (\sqrt{3}/2, 1/2)$$

$$\psi_a \perp \psi_A$$

$$\psi_b \perp \psi_B$$

$$\psi_c \perp \psi_C$$



$$\begin{aligned}
 & \rho(\lambda|P) \rho(P) + \rho(\lambda|P'') \rho(P'') \xrightarrow{\dagger} \left(\frac{1}{2}, \frac{1}{2}, 0, 0 \right) \\
 & \qquad \qquad \qquad \dagger - \rho \qquad \qquad \qquad \left(0, 0, \frac{1}{2}, \frac{1}{2} \right) = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4} \right) \\
 & \qquad \qquad \qquad \left(\frac{1}{2}, 0, \frac{1}{2}, 0 \right) \\
 & \qquad \qquad \qquad \left(0, \frac{1}{2}, 0, \frac{1}{2} \right)
 \end{aligned}$$

$$|\Psi^+\rangle\langle\Psi^+| = \sigma$$

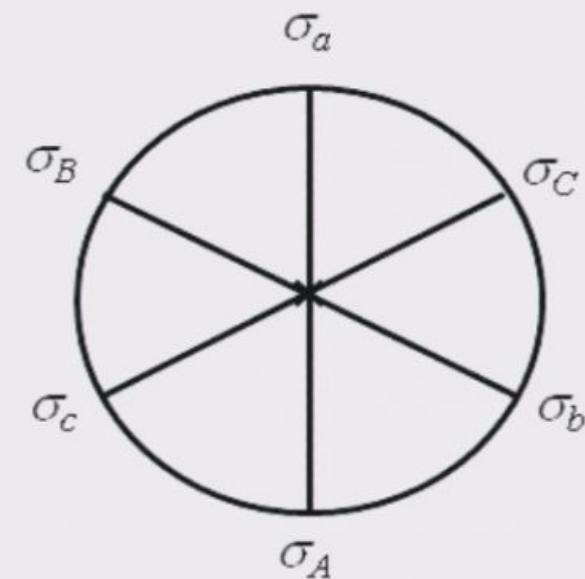
Proof based on finite construction in 2d

$$\begin{array}{ll}
 P_a \leftrightarrow \sigma_a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \sigma_a \sigma_A = 0 \\
 P_A \leftrightarrow \sigma_A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & \sigma_b \sigma_B = 0 \\
 P_b \leftrightarrow \sigma_b = \begin{pmatrix} \frac{1}{4} & \frac{1}{4}\sqrt{3} \\ \frac{1}{4}\sqrt{3} & \frac{3}{4} \end{pmatrix} & \sigma_c \sigma_C = 0 \\
 P_B \leftrightarrow \sigma_B = \begin{pmatrix} \frac{3}{4} & -\frac{1}{4}\sqrt{3} \\ -\frac{1}{4}\sqrt{3} & \frac{1}{4} \end{pmatrix} & \\
 P_c \leftrightarrow \sigma_c = \begin{pmatrix} \frac{1}{4} & -\frac{1}{4}\sqrt{3} \\ -\frac{1}{4}\sqrt{3} & \frac{3}{4} \end{pmatrix} & \\
 P_C \leftrightarrow \sigma_C = \begin{pmatrix} \frac{3}{4} & \frac{1}{4}\sqrt{3} \\ \frac{1}{4}\sqrt{3} & \frac{1}{4} \end{pmatrix} &
 \end{array}$$

P_a and P_A are distinguishable with certainty

P_b and P_B are distinguishable with certainty

P_c and P_C are distinguishable with certainty



$$\mu_a(\lambda) \mu_A(\lambda) = 0$$

$$\rightarrow \mu_b(\lambda) \mu_B(\lambda) = 0$$

$P_{aA} \equiv P_a$ and P_A with prob. 1/2 each

$P_{bB} \equiv P_b$ and P_B with prob. 1/2 each

$P_{cC} \equiv P_c$ and P_C with prob. 1/2 each

$P_{abc} \equiv P_a, P_b$ and P_c with prob. 1/3 each

$P_{ABC} \equiv P_A, P_B$ and P_C with prob. 1/3 each

$P_{aA} \equiv P_a$ and P_A with prob. 1/2 each

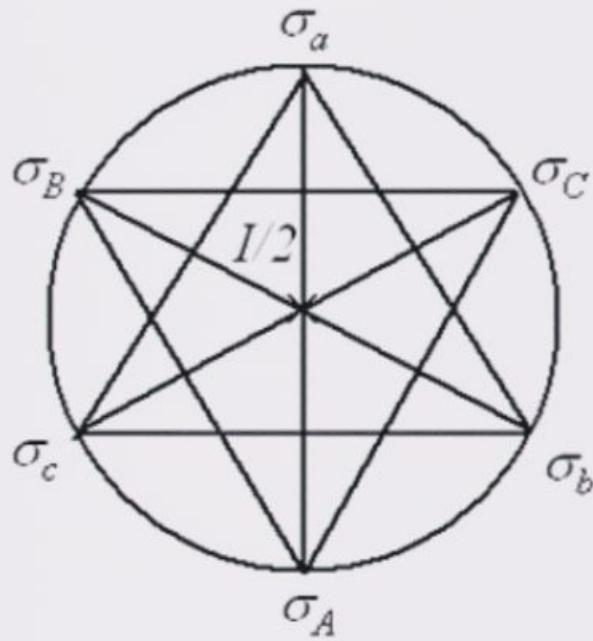
$P_{bB} \equiv P_b$ and P_B with prob. 1/2 each

$P_{cC} \equiv P_c$ and P_C with prob. 1/2 each

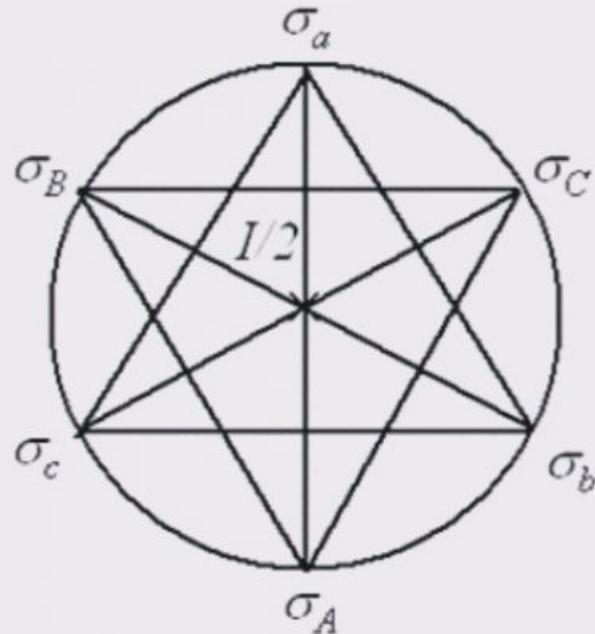
$P_{abc} \equiv P_a, P_b$ and P_c with prob. 1/3 each

$P_{ABC} \equiv P_A, P_B$ and P_C with prob. 1/3 each

$$\begin{aligned}\mu_{aA}(\lambda) &= \frac{1}{2}\mu_a(\lambda) + \frac{1}{2}\mu_A(\lambda) \\ \mu_{bB}(\lambda) &= \frac{1}{2}\mu_b(\lambda) + \frac{1}{2}\mu_B(\lambda) \\ \mu_{cC}(\lambda) &= \frac{1}{2}\mu_c(\lambda) + \frac{1}{2}\mu_C(\lambda) \\ \mu_{abc}(\lambda) &= \frac{1}{3}\mu_a(\lambda) + \frac{1}{3}\mu_b(\lambda) + \frac{1}{3}\mu_c(\lambda) \\ \mu_{ABC}(\lambda) &= \frac{1}{3}\mu_A(\lambda) + \frac{1}{3}\mu_B(\lambda) + \frac{1}{3}\mu_C(\lambda)\end{aligned}$$

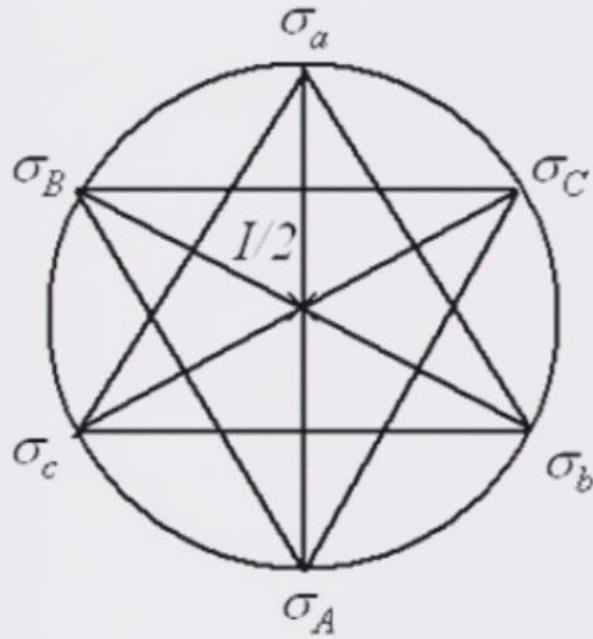


$$\begin{aligned}
 I/2 &= \frac{1}{2}\sigma_a + \frac{1}{2}\sigma_A \\
 &= \frac{1}{2}\sigma_b + \frac{1}{2}\sigma_B \\
 &= \frac{1}{2}\sigma_c + \frac{1}{2}\sigma_C \\
 &= \frac{1}{3}\sigma_a + \frac{1}{3}\sigma_b + \frac{1}{3}\sigma_c \\
 &= \frac{1}{3}\sigma_A + \frac{1}{3}\sigma_B + \frac{1}{3}\sigma_C.
 \end{aligned}$$



$$\begin{aligned}
 I/2 &= \frac{1}{2}\sigma_a + \frac{1}{2}\sigma_A \\
 &= \frac{1}{2}\sigma_b + \frac{1}{2}\sigma_B \\
 &= \frac{1}{2}\sigma_c + \frac{1}{2}\sigma_C \\
 &= \frac{1}{3}\sigma_a + \frac{1}{3}\sigma_b + \frac{1}{3}\sigma_c \\
 &= \frac{1}{3}\sigma_A + \frac{1}{3}\sigma_B + \frac{1}{3}\sigma_C.
 \end{aligned}$$

$$\begin{aligned}
 \mathsf{P}_{aA} &\simeq \mathsf{P}_{bB} \simeq \mathsf{P}_{cC} \\
 &\simeq \mathsf{P}_{abc} \simeq \mathsf{P}_{ABC}
 \end{aligned}$$



$$\begin{aligned}
 I/2 &= \frac{1}{2}\sigma_a + \frac{1}{2}\sigma_A \\
 &= \frac{1}{2}\sigma_b + \frac{1}{2}\sigma_B \\
 &= \frac{1}{2}\sigma_c + \frac{1}{2}\sigma_C \\
 &= \frac{1}{3}\sigma_a + \frac{1}{3}\sigma_b + \frac{1}{3}\sigma_c \\
 &= \frac{1}{3}\sigma_A + \frac{1}{3}\sigma_B + \frac{1}{3}\sigma_C.
 \end{aligned}$$

$$\begin{aligned}
 P_{aA} &\simeq P_{bB} \simeq P_{cC} \\
 &\simeq P_{abc} \simeq P_{ABC}
 \end{aligned}$$

By preparation noncontextuality

$$\begin{aligned}
 \mu_{aA}(\lambda) &= \mu_{bB}(\lambda) = \mu_{cC}(\lambda) \\
 &= \mu_{abc}(\lambda) = \mu_{ABC}(\lambda) \\
 &\equiv \nu(\lambda)
 \end{aligned}$$

$P_{aA} \equiv P_a$ and P_A with prob. 1/2 each

$P_{bB} \equiv P_b$ and P_B with prob. 1/2 each

$P_{cC} \equiv P_c$ and P_C with prob. 1/2 each

$P_{abc} \equiv P_a, P_b$ and P_c with prob. 1/3 each

$P_{ABC} \equiv P_A, P_B$ and P_C with prob. 1/3 each

$$\begin{aligned}\mu_{aA}(\lambda) &= \frac{1}{2}\mu_a(\lambda) + \frac{1}{2}\mu_A(\lambda) \\ \mu_{bB}(\lambda) &= \frac{1}{2}\mu_b(\lambda) + \frac{1}{2}\mu_B(\lambda) \\ \mu_{cC}(\lambda) &= \frac{1}{2}\mu_c(\lambda) + \frac{1}{2}\mu_C(\lambda) \\ \mu_{abc}(\lambda) &= \frac{1}{3}\mu_a(\lambda) + \frac{1}{3}\mu_b(\lambda) + \frac{1}{3}\mu_c(\lambda) \\ \mu_{ABC}(\lambda) &= \frac{1}{3}\mu_A(\lambda) + \frac{1}{3}\mu_B(\lambda) + \frac{1}{3}\mu_C(\lambda)\end{aligned}$$

Our task is to find

$$\mu_a(\lambda), \mu_A(\lambda), \mu_b(\lambda),$$

$$\mu_B(\lambda), \mu_c(\lambda), \mu_C(\lambda),$$

and $\nu(\lambda)$ such that

$$\mu_a(\lambda) \mu_A(\lambda) = 0$$

$$\mu_b(\lambda) \mu_B(\lambda) = 0$$

$$\mu_c(\lambda) \mu_C(\lambda) = 0$$

$$\begin{aligned}\nu(\lambda) &= \frac{1}{2}\mu_a(\lambda) + \frac{1}{2}\mu_A(\lambda) \\ &= \frac{1}{2}\mu_b(\lambda) + \frac{1}{2}\mu_B(\lambda) \\ &= \frac{1}{2}\mu_c(\lambda) + \frac{1}{2}\mu_C(\lambda) \\ &= \frac{1}{3}\mu_a(\lambda) + \frac{1}{3}\mu_b(\lambda) + \frac{1}{3}\mu_c(\lambda) \\ &= \frac{1}{3}\mu_A(\lambda) + \frac{1}{3}\mu_B(\lambda) + \frac{1}{3}\mu_C(\lambda).\end{aligned}$$

Our task is to find

$\mu_a(\lambda), \mu_A(\lambda), \mu_b(\lambda),$
 $\mu_B(\lambda), \mu_c(\lambda), \mu_C(\lambda),$
 and $\nu(\lambda)$ such that

$$\mu_a(\lambda) \mu_A(\lambda) = 0$$

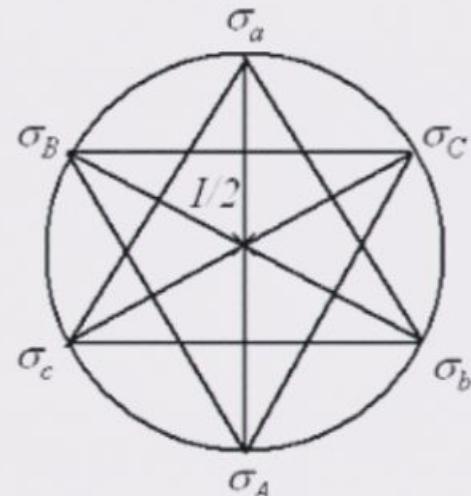
$$\mu_b(\lambda) \mu_B(\lambda) = 0$$

$$\mu_c(\lambda) \mu_C(\lambda) = 0$$

$$\begin{aligned}\nu(\lambda) &= \frac{1}{2}\mu_a(\lambda) + \frac{1}{2}\mu_A(\lambda) \\&= \frac{1}{2}\mu_b(\lambda) + \frac{1}{2}\mu_B(\lambda) \\&= \frac{1}{2}\mu_c(\lambda) + \frac{1}{2}\mu_C(\lambda) \\&= \frac{1}{3}\mu_a(\lambda) + \frac{1}{3}\mu_b(\lambda) + \frac{1}{3}\mu_c(\lambda) \\&= \frac{1}{3}\mu_A(\lambda) + \frac{1}{3}\mu_B(\lambda) + \frac{1}{3}\mu_C(\lambda).\end{aligned}$$

i.e., paralleling the quantum structure:

$$\begin{aligned}\sigma_a \sigma_A &= 0 \\ \sigma_b \sigma_B &= 0 \\ \sigma_c \sigma_C &= 0\end{aligned}$$



$$\begin{aligned}I/2 &= \frac{1}{2}\sigma_a + \frac{1}{2}\sigma_A \\&= \frac{1}{2}\sigma_b + \frac{1}{2}\sigma_B \\&= \frac{1}{2}\sigma_c + \frac{1}{2}\sigma_C \\&= \frac{1}{3}\sigma_a + \frac{1}{3}\sigma_b + \frac{1}{3}\sigma_c \\&= \frac{1}{3}\sigma_A + \frac{1}{3}\sigma_B + \frac{1}{3}\sigma_C.\end{aligned}$$

Our task is to find
 $\mu_a(\lambda)$, $\mu_A(\lambda)$, $\mu_b(\lambda)$,
 $\mu_B(\lambda)$, $\mu_c(\lambda)$, $\mu_C(\lambda)$,
and $\nu(\lambda)$ such that

$$\mu_a(\lambda) \mu_A(\lambda) = 0$$

$$\mu_b(\lambda) \mu_B(\lambda) = 0$$

$$\mu_c(\lambda) \mu_C(\lambda) = 0$$

From decompositions (1)-(3), for $\lambda = \lambda'$

$$\mu_a(\lambda') = 0 \text{ or } 2\nu(\lambda')$$

$$\mu_b(\lambda') = 0 \text{ or } 2\nu(\lambda')$$

$$\mu_c(\lambda') = 0 \text{ or } 2\nu(\lambda')$$

$$\begin{aligned}\nu(\lambda) &= \frac{1}{2}\mu_a(\lambda) + \frac{1}{2}\mu_A(\lambda) \\ &= \frac{1}{2}\mu_b(\lambda) + \frac{1}{2}\mu_B(\lambda) \\ &= \frac{1}{2}\mu_c(\lambda) + \frac{1}{2}\mu_C(\lambda) \\ &= \frac{1}{3}\mu_a(\lambda) + \frac{1}{3}\mu_b(\lambda) + \frac{1}{3}\mu_c(\lambda) \\ &= \frac{1}{3}\mu_A(\lambda) + \frac{1}{3}\mu_B(\lambda) + \frac{1}{3}\mu_C(\lambda)\end{aligned}$$

Our task is to find
 $\mu_a(\lambda)$, $\mu_A(\lambda)$, $\mu_b(\lambda)$,
 $\mu_B(\lambda)$, $\mu_c(\lambda)$, $\mu_C(\lambda)$,
and $\nu(\lambda)$ such that

$$\mu_a(\lambda) \mu_A(\lambda) = 0$$

$$\mu_b(\lambda) \mu_B(\lambda) = 0$$

$$\mu_c(\lambda) \mu_C(\lambda) = 0$$

$$\nu(\lambda) = \frac{1}{2}\mu_a(\lambda) + \frac{1}{2}\mu_A(\lambda)$$

$$= \frac{1}{2}\mu_b(\lambda) + \frac{1}{2}\mu_B(\lambda)$$

$$= \frac{1}{2}\mu_c(\lambda) + \frac{1}{2}\mu_C(\lambda)$$

$$= \frac{1}{3}\mu_a(\lambda) + \frac{1}{3}\mu_b(\lambda) + \frac{1}{3}\mu_c(\lambda)$$

$$= \frac{1}{3}\mu_A(\lambda) + \frac{1}{3}\mu_B(\lambda) + \frac{1}{3}\mu_C(\lambda)$$

From decompositions (1)-(3), for $\lambda = \lambda'$

$$\mu_a(\lambda') = 0 \text{ or } 2\nu(\lambda')$$

$$\mu_b(\lambda') = 0 \text{ or } 2\nu(\lambda')$$

$$\mu_c(\lambda') = 0 \text{ or } 2\nu(\lambda')$$

But then the RHS of decomposition (4) is

$$0, \frac{2}{3}\nu(\lambda'), \frac{4}{3}\nu(\lambda'), 2\nu(\lambda') \\ \neq \nu(\lambda')$$

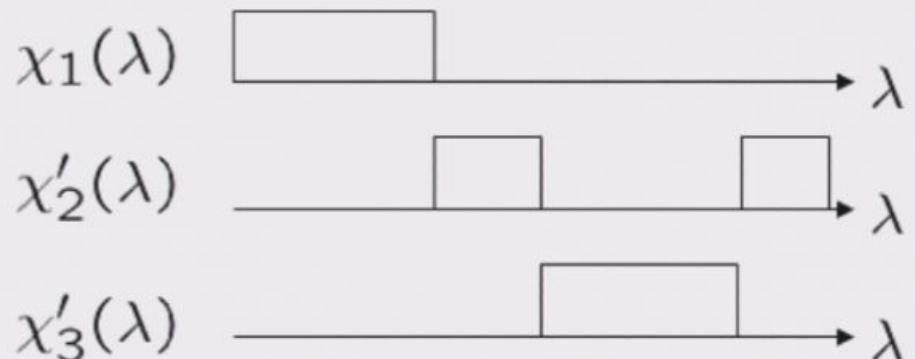
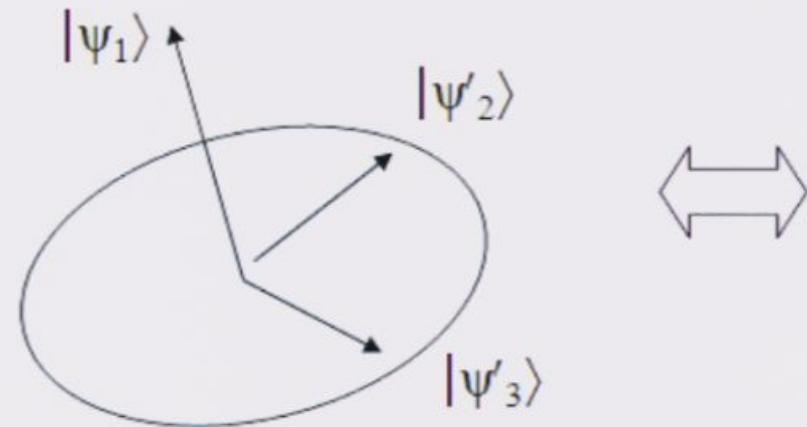
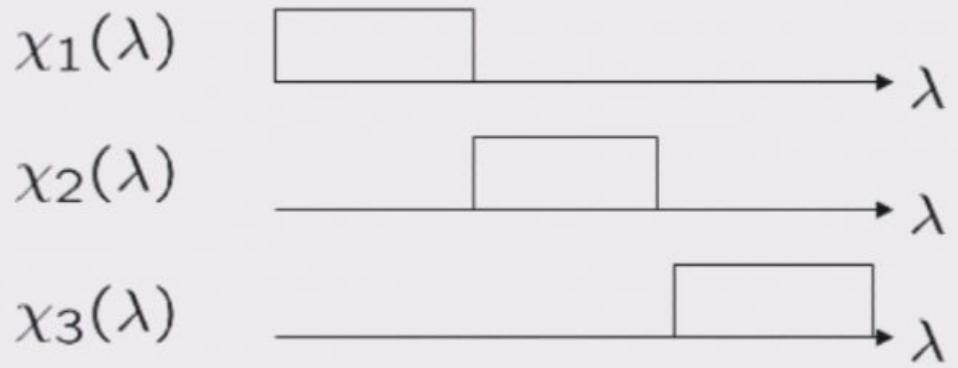
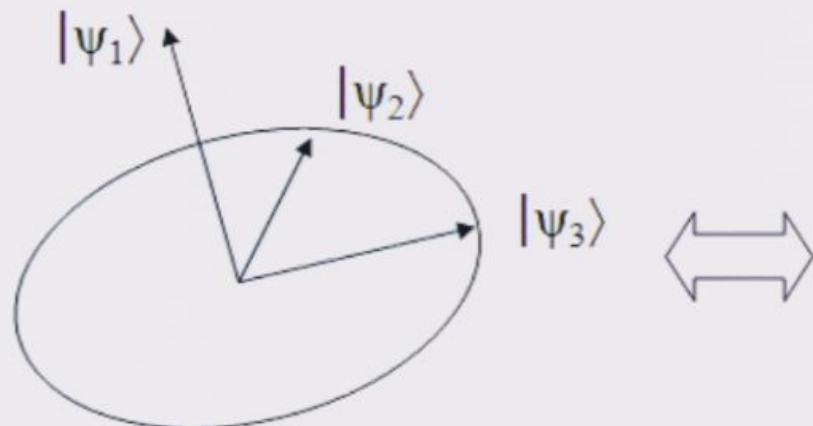
for λ' such that $\nu(\lambda') \neq 0$

CONTRADICTION

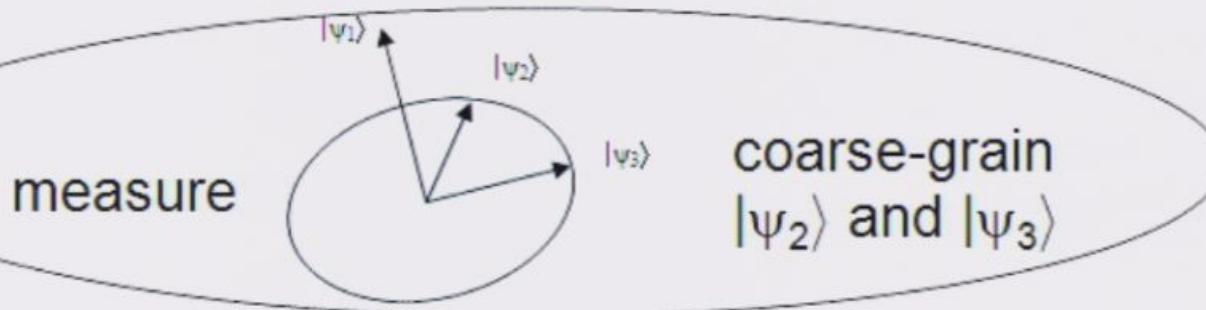
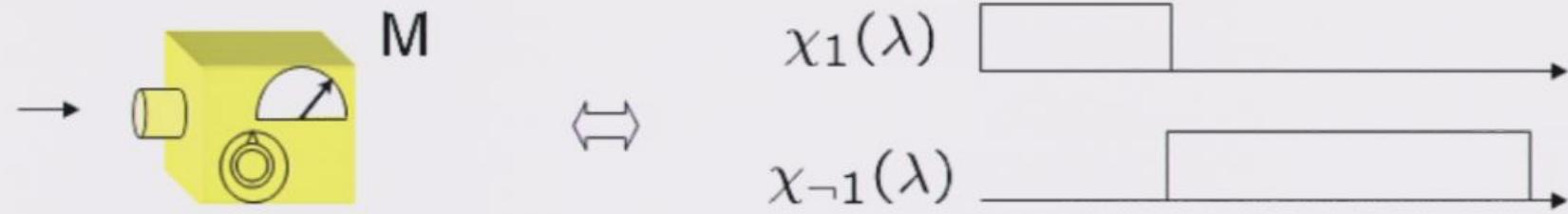
Measurement contextuality

New definition versus traditional definition

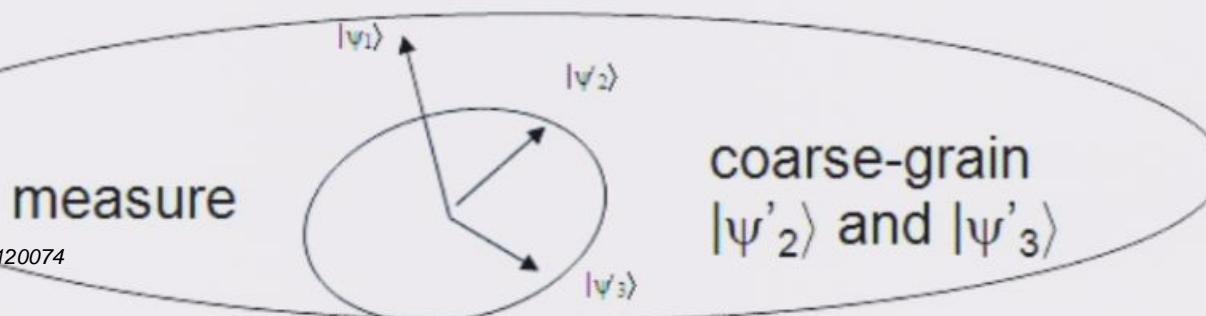
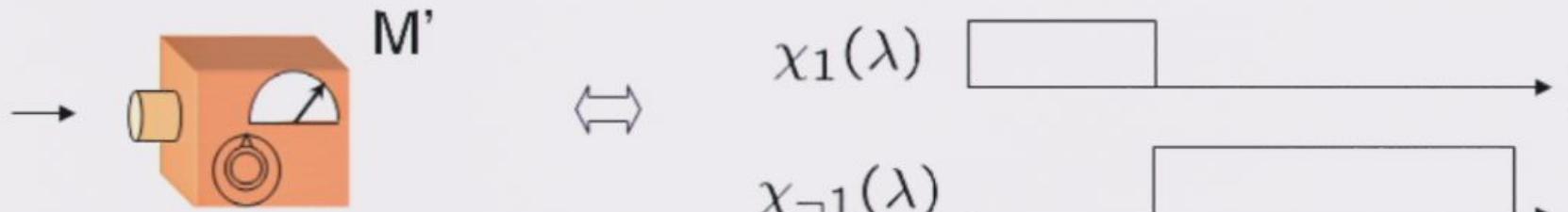
How to formulate the traditional notion of noncontextuality:



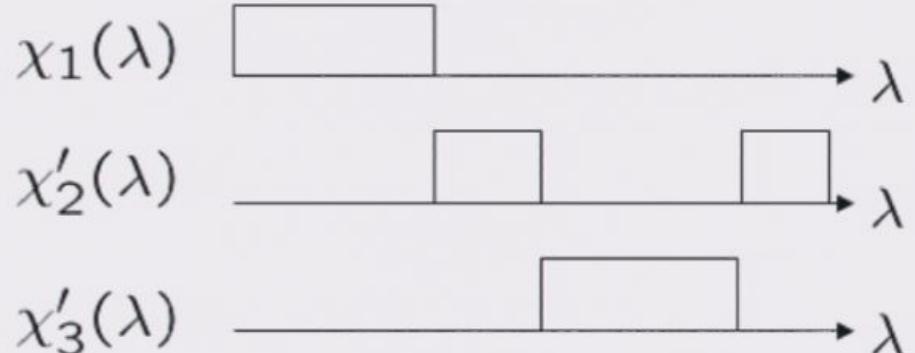
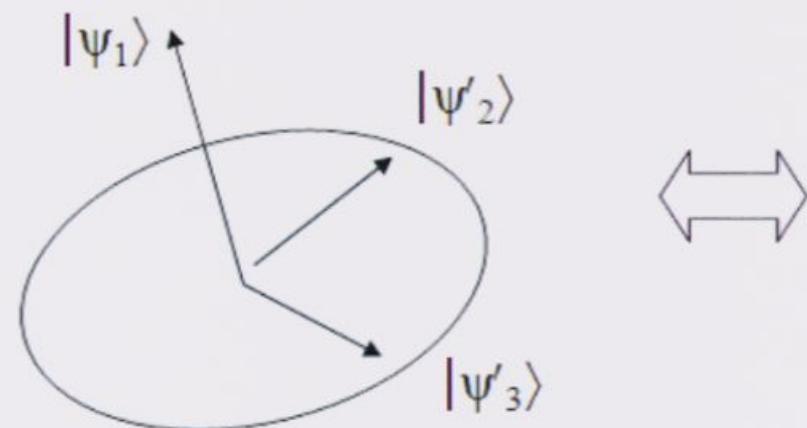
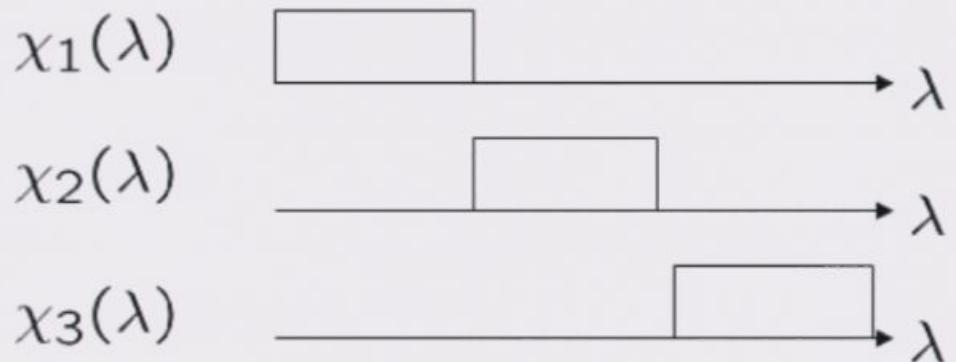
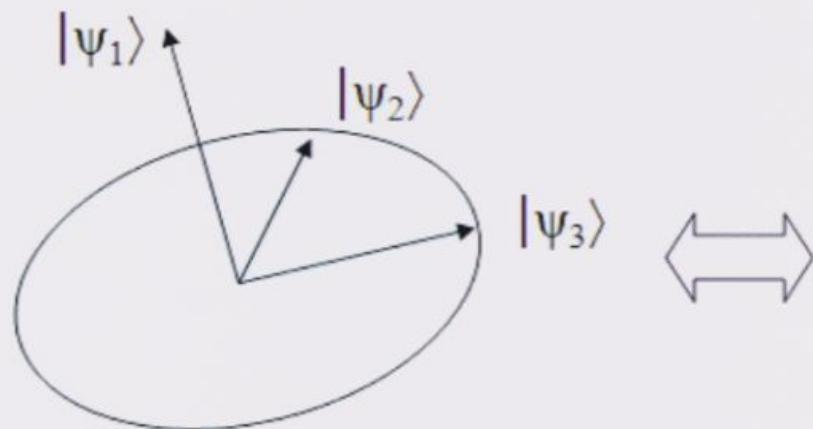
This is equivalent to assuming:



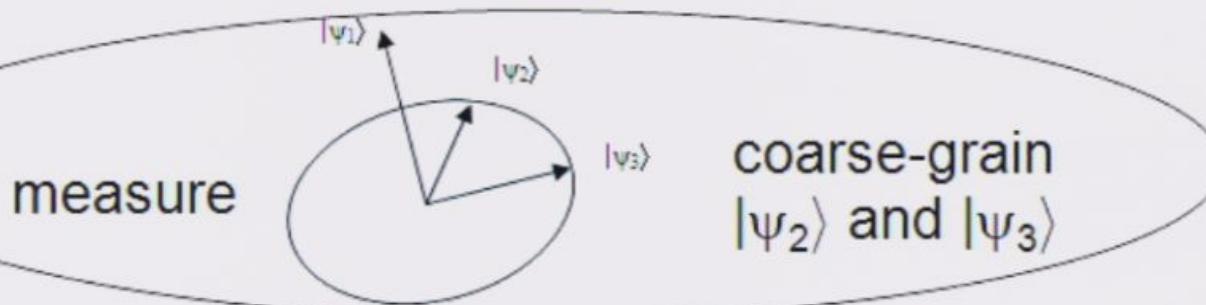
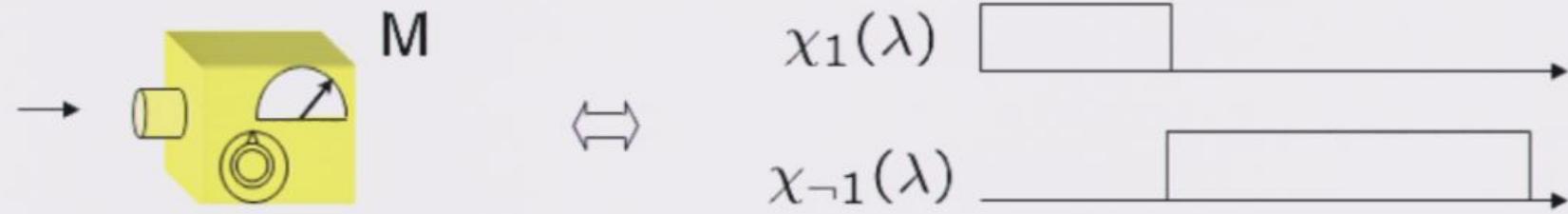
$$\{|\psi_1\rangle\langle\psi_1|, I - |\psi_1\rangle\langle\psi_1|\}$$



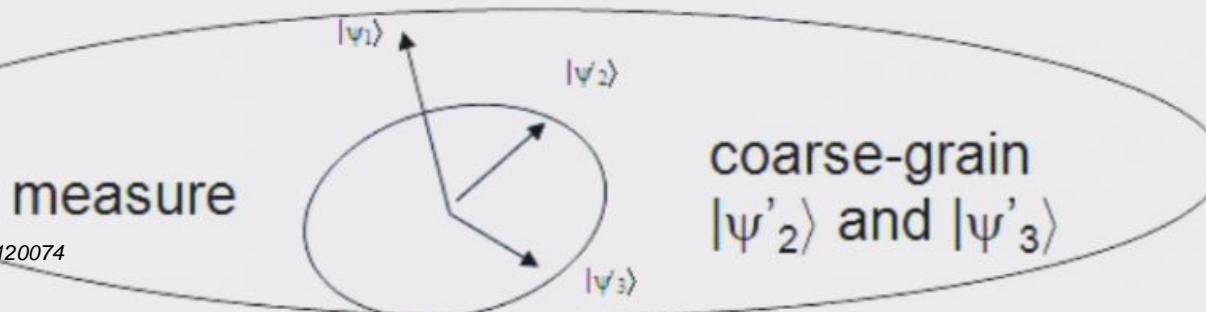
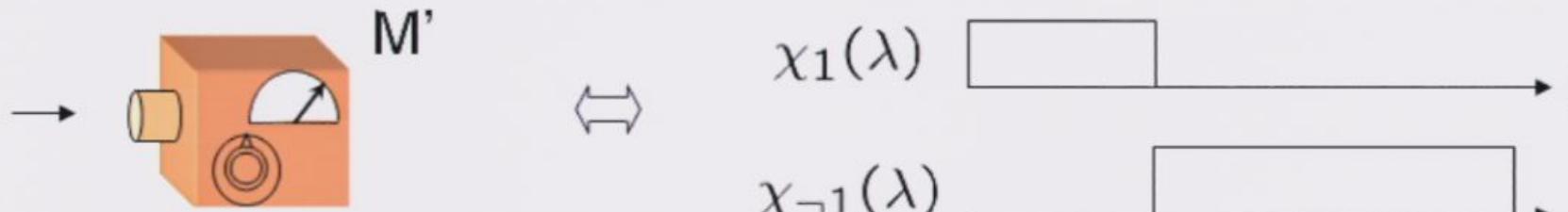
How to formulate the traditional notion of noncontextuality:



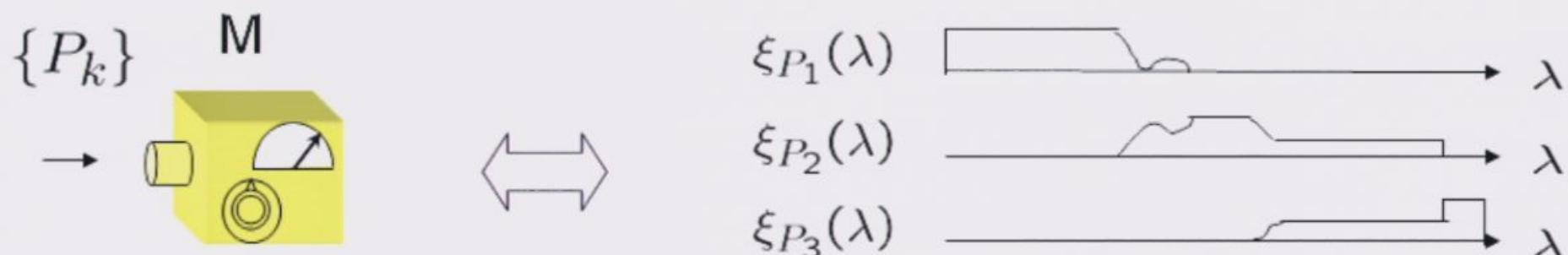
This is equivalent to assuming:



$$\{|\psi_1\rangle\langle\psi_1|, I - |\psi_1\rangle\langle\psi_1|\}$$



But recall that the most general representation was



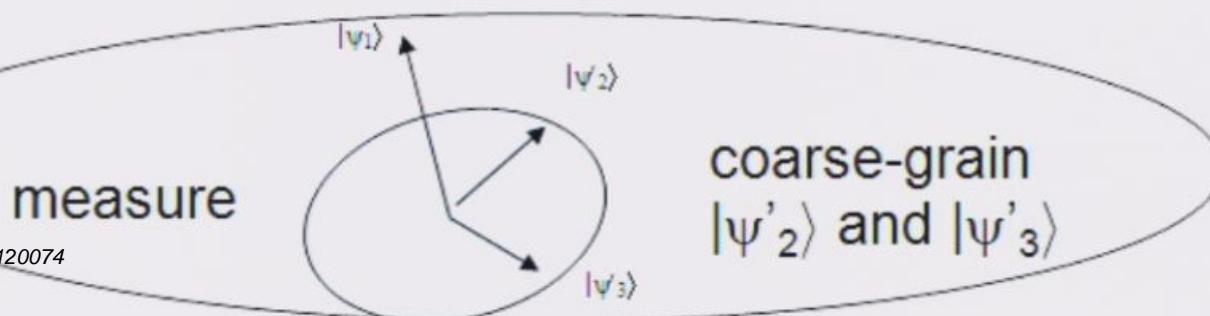
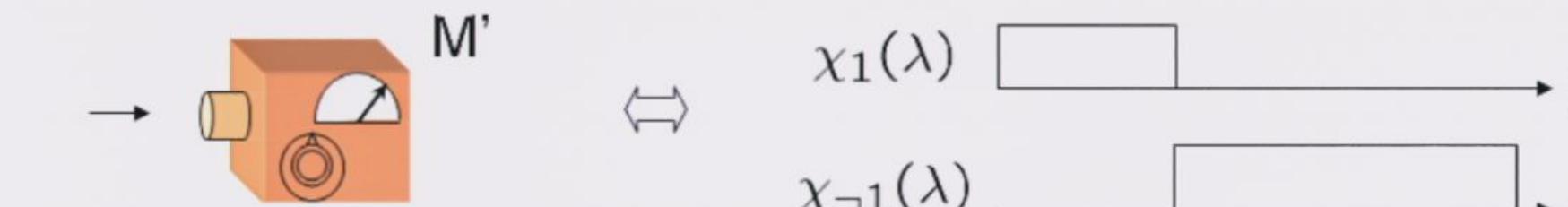
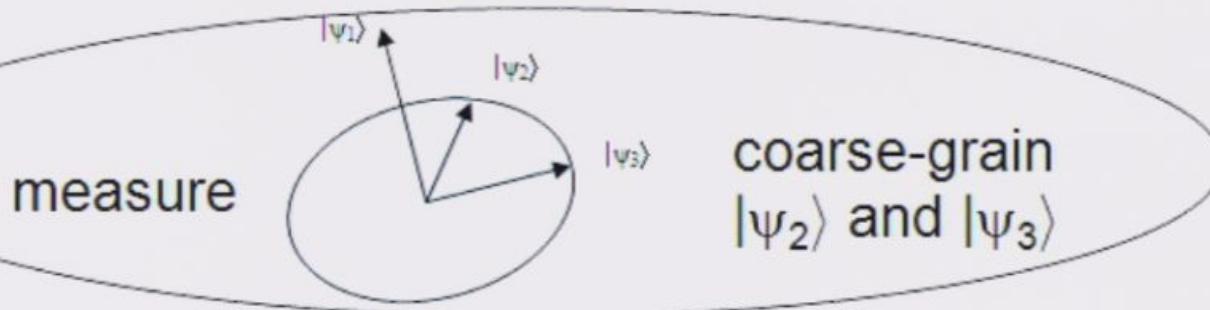
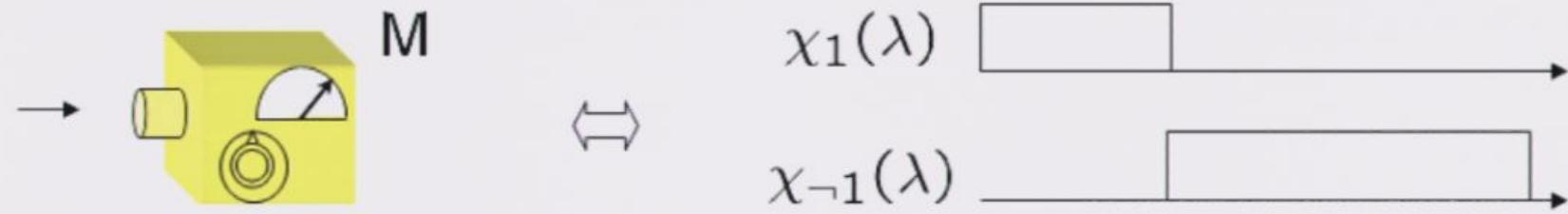
Therefore:

traditional notion of
noncontextuality =

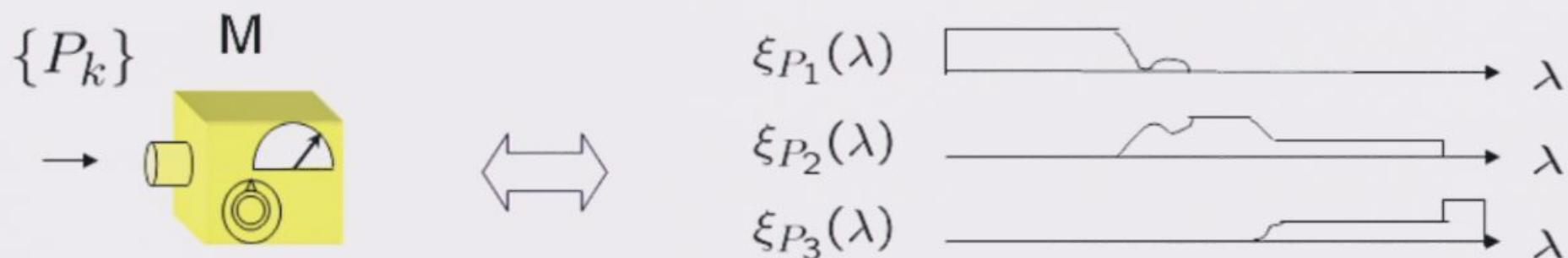
revised notion of
noncontextuality for sharp
measurements

and
outcome determinism for
sharp measurements

This is equivalent to assuming:



But recall that the most general representation was



Therefore:

traditional notion of
noncontextuality =

revised notion of
noncontextuality for sharp
measurements

and
outcome determinism for
sharp measurements

So, the new definition of noncontextuality is **not simply a generalization** of the traditional notion

For sharp measurements, it is a **revision** of the traditional notion

Local determinism:

We ask: Does **the outcome** depend on space-like separated events
(in addition to local settings and λ)?

Bell's local causality:

We ask: Does **the probability of the outcome** depend on space-like separated events (in addition to local settings and λ)?

Local determinism:

We ask: Does **the outcome** depend on space-like separated events
(in addition to local settings and λ)?

Bell's local causality:

We ask: Does **the probability of the outcome** depend on space-like separated events (in addition to local settings and λ)?

Traditional notion of measurement noncontextuality:

We ask: Does **the outcome** depend on the measurement context
(in addition to the observable and λ)?

The revised notion of measurement noncontextuality:

We ask: Does **the probability of the outcome** depend on the measurement context (in addition to the observable and λ)?

Local determinism:

We ask: Does **the outcome** depend on space-like separated events
(in addition to local settings and λ)?

Bell's local causality:

We ask: Does **the probability of the outcome** depend on space-like separated events (in addition to local settings and λ)?

Traditional notion of measurement noncontextuality:

We ask: Does **the outcome** depend on the measurement context
(in addition to the observable and λ)?

The revised notion of measurement noncontextuality:

We ask: Does **the probability of the outcome** depend on the measurement context (in addition to the observable and λ)?

traditional notion of noncontextuality = revised notion of noncontextuality for sharp measurements
and
outcome determinism for sharp measurements

No-go theorems for previous notion are not necessarily no-go theorems for the new notion!

In face of contradiction, could give up ODSM

However, one can prove that

preparation
noncontextuality → outcome determinism for
sharp measurements

Therefore:

measurement
noncontextuality → measurement
noncontextuality
and
preparation
noncontextuality → and
outcome determinism for
sharp measurements

However, one can prove that

preparation
noncontextuality → outcome determinism for
sharp measurements

Therefore:

measurement
noncontextuality → measurement
noncontextuality
and
preparation
noncontextuality → and
outcome determinism for
sharp measurements

However, one can prove that

preparation
noncontextuality → outcome determinism for
sharp measurements

Therefore:

measurement
noncontextuality
and
preparation
noncontextuality → Traditional notion of
noncontextuality

However, one can prove that

preparation
noncontextuality → outcome determinism for
sharp measurements

Therefore:

measurement
noncontextuality → Traditional notion of
noncontextuality
and
preparation
noncontextuality

no-go theorems for the traditional notion of noncontextuality can
be salvaged as no-go theorems for the generalized notion

Measurement-based proof of contextuality

(i.e. of the impossibility of a noncontextual
realist model of quantum theory)

Proof of contextuality for unsharp measurements in 2d

$$M_a \leftrightarrow \{\Pi_a, \Pi_A\}$$

$$M_b \leftrightarrow \{\Pi_b, \Pi_B\}$$

$$M_c \leftrightarrow \{\Pi_c, \Pi_C\}$$

Π_x projects onto ψ_x

$$\Pi_a + \Pi_A = I$$

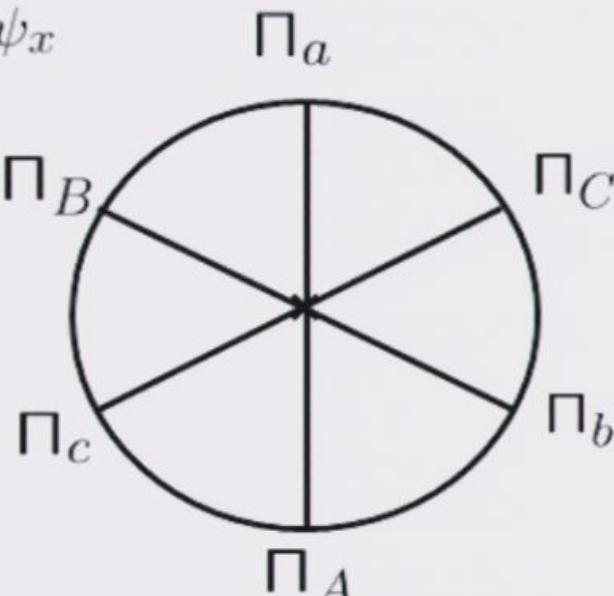
$$\Pi_b + \Pi_B = I$$

$$\Pi_c + \Pi_C = I$$

$$\Pi_a \Pi_A = 0$$

$$\Pi_b \Pi_B = 0$$

$$\Pi_c \Pi_C = 0$$



Proof of contextuality for unsharp measurements in 2d

$$M_a \leftrightarrow \{\Pi_a, \Pi_A\}$$

$$M_b \leftrightarrow \{\Pi_b, \Pi_B\}$$

$$M_c \leftrightarrow \{\Pi_c, \Pi_C\}$$

Π_x projects onto ψ_x

$$\Pi_a + \Pi_A = I$$

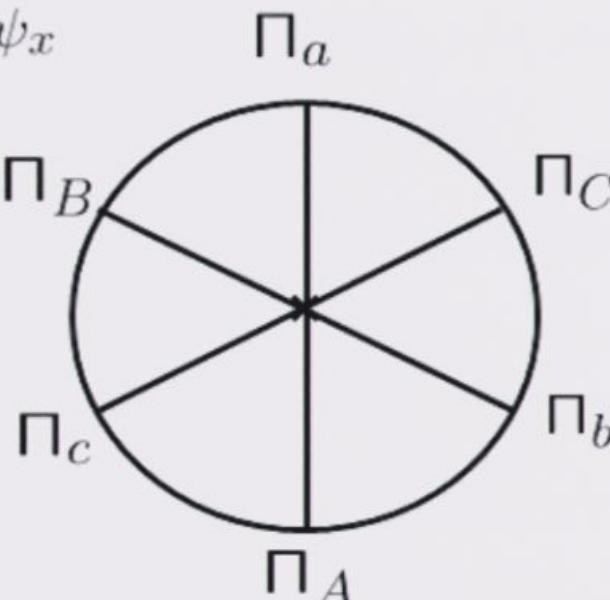
$$\Pi_b + \Pi_B = I$$

$$\Pi_c + \Pi_C = I$$

$$\Pi_a \Pi_A = 0$$

$$\Pi_b \Pi_B = 0$$

$$\Pi_c \Pi_C = 0$$



$$M_a \leftrightarrow \{\chi_a(\lambda), \chi_A(\lambda)\}$$

$$M_b \leftrightarrow \{\chi_b(\lambda), \chi_B(\lambda)\}$$

$$M_c \leftrightarrow \{\chi_c(\lambda), \chi_C(\lambda)\}$$

By definition

$$\chi_a(\lambda) + \chi_A(\lambda) = 1$$

$$\chi_b(\lambda) + \chi_B(\lambda) = 1$$

$$\chi_c(\lambda) + \chi_C(\lambda) = 1$$

Proof of contextuality for unsharp measurements in 2d

$$M_a \leftrightarrow \{\Pi_a, \Pi_A\}$$

$$M_b \leftrightarrow \{\Pi_b, \Pi_B\}$$

$$M_c \leftrightarrow \{\Pi_c, \Pi_C\}$$

Π_x projects onto ψ_x

$$\Pi_a + \Pi_A = I$$

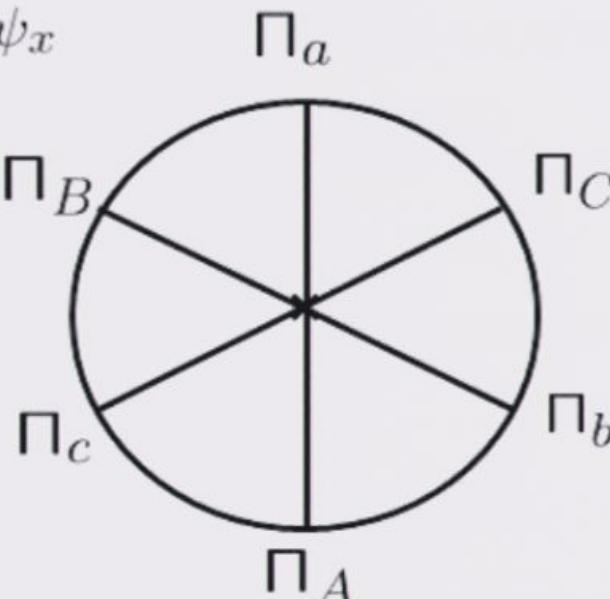
$$\Pi_b + \Pi_B = I$$

$$\Pi_c + \Pi_C = I$$

$$\Pi_a \Pi_A = 0$$

$$\Pi_b \Pi_B = 0$$

$$\Pi_c \Pi_C = 0$$



$$M_a \leftrightarrow \{\chi_a(\lambda), \chi_A(\lambda)\}$$

$$M_b \leftrightarrow \{\chi_b(\lambda), \chi_B(\lambda)\}$$

$$M_c \leftrightarrow \{\chi_c(\lambda), \chi_C(\lambda)\}$$

By definition

$$\chi_a(\lambda) + \chi_A(\lambda) = 1$$

$$\chi_b(\lambda) + \chi_B(\lambda) = 1$$

$$\chi_c(\lambda) + \chi_C(\lambda) = 1$$

Proof of contextuality for unsharp measurements in 2d

$$M_a \leftrightarrow \{\Pi_a, \Pi_A\}$$

$$M_b \leftrightarrow \{\Pi_b, \Pi_B\}$$

$$M_c \leftrightarrow \{\Pi_c, \Pi_C\}$$

Π_x projects onto ψ_x

$$\Pi_a + \Pi_A = I$$

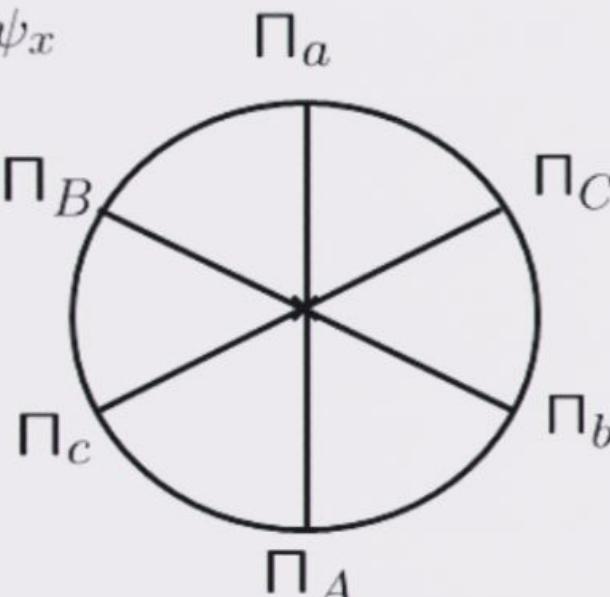
$$\Pi_b + \Pi_B = I$$

$$\Pi_c + \Pi_C = I$$

$$\Pi_a \Pi_A = 0$$

$$\Pi_b \Pi_B = 0$$

$$\Pi_c \Pi_C = 0$$



$$M_a \leftrightarrow \{\chi_a(\lambda), \chi_A(\lambda)\}$$

$$M_b \leftrightarrow \{\chi_b(\lambda), \chi_B(\lambda)\}$$

$$M_c \leftrightarrow \{\chi_c(\lambda), \chi_C(\lambda)\}$$

By definition

$$\chi_a(\lambda) + \chi_A(\lambda) = 1$$

$$\chi_b(\lambda) + \chi_B(\lambda) = 1$$

$$\chi_c(\lambda) + \chi_C(\lambda) = 1$$

By outcome determinism for sharp measurements

$$\chi_a(\lambda)\chi_A(\lambda) = 0$$

$$\chi_b(\lambda)\chi_B(\lambda) = 0$$

$$\chi_c(\lambda)\chi_C(\lambda) = 0$$

$M \equiv$ implement one of M_a , M_b and M_c with prob. $1/3$ each, register only whether first or second outcome occurred

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$$M \leftrightarrow \{\frac{1}{3}\chi_a(\lambda) + \frac{1}{3}\chi_b(\lambda) + \frac{1}{3}\chi_c(\lambda), \frac{1}{3}\chi_A(\lambda) + \frac{1}{3}\chi_B(\lambda) + \frac{1}{3}\chi_C(\lambda)\}$$

Proof of contextuality for unsharp measurements in 2d

$$M_a \leftrightarrow \{\Pi_a, \Pi_A\}$$

$$M_b \leftrightarrow \{\Pi_b, \Pi_B\}$$

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Π_x projects onto ψ_x

$$\Pi_a + \Pi_A = I$$

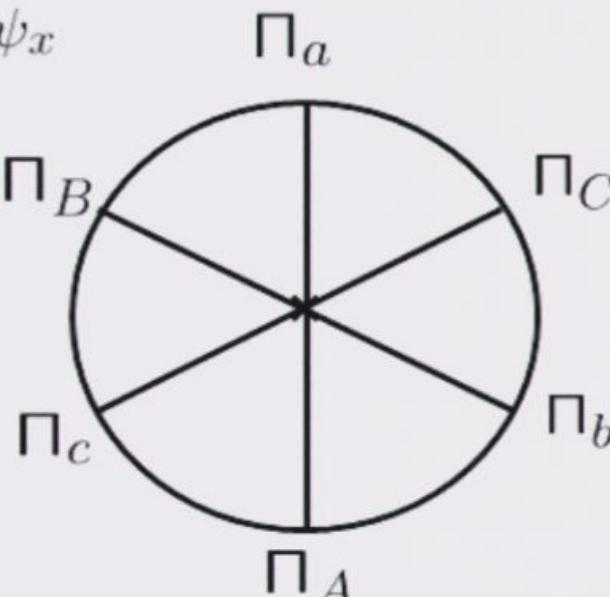
$$\Pi_b + \Pi_B = I$$

$$\Pi_c + \Pi_C = I$$

$$\Pi_a \Pi_A = 0$$

$$\Pi_b \Pi_B = 0$$

$$\Pi_c \Pi_C = 0$$



$$M_a \leftrightarrow \{\chi_a(\lambda), \chi_A(\lambda)\}$$

$$M_b \leftrightarrow \{\chi_b(\lambda), \chi_B(\lambda)\}$$

$$M_c \leftrightarrow \{\chi_c(\lambda), \chi_C(\lambda)\}$$

By definition

$$\chi_a(\lambda) + \chi_A(\lambda) = 1$$

$$\chi_b(\lambda) + \chi_B(\lambda) = 1$$

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$M \equiv$ implement one of M_a , M_b and M_c with prob. $1/3$ each, register only whether first or second outcome occurred

$$M \leftrightarrow \left\{ \frac{1}{3}\Pi_a + \frac{1}{3}\Pi_b + \frac{1}{3}\Pi_c, \frac{1}{3}\Pi_A + \frac{1}{3}\Pi_B + \frac{1}{3}\Pi_C \right\}$$

$$M \leftrightarrow \left\{ \frac{1}{3}\chi_a(\lambda) + \frac{1}{3}\chi_b(\lambda) + \frac{1}{3}\chi_c(\lambda), \frac{1}{3}\chi_A(\lambda) + \frac{1}{3}\chi_B(\lambda) + \frac{1}{3}\chi_C(\lambda) \right\}$$

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$\tilde{M} \equiv$ ignore the system, flip a fair coin

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By the assumption of **measurement noncontextuality**

$$M \simeq \tilde{M} \longrightarrow \left\{ \frac{1}{3}\chi_a + \frac{1}{3}\chi_b + \frac{1}{3}\chi_c, \frac{1}{3}\chi_A + \frac{1}{3}\chi_B + \frac{1}{3}\chi_C \right\} = \left\{ \frac{1}{2}, \frac{1}{2} \right\}$$

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$$\text{But } \{0, 1\}, \{\frac{1}{3}, \frac{2}{3}\}, \{1, 0\}, \{\frac{2}{3}, \frac{1}{3}\} \neq \left\{ \frac{1}{2}, \frac{1}{2} \right\}$$

Proof of contextuality for unsharp measurements in 2d

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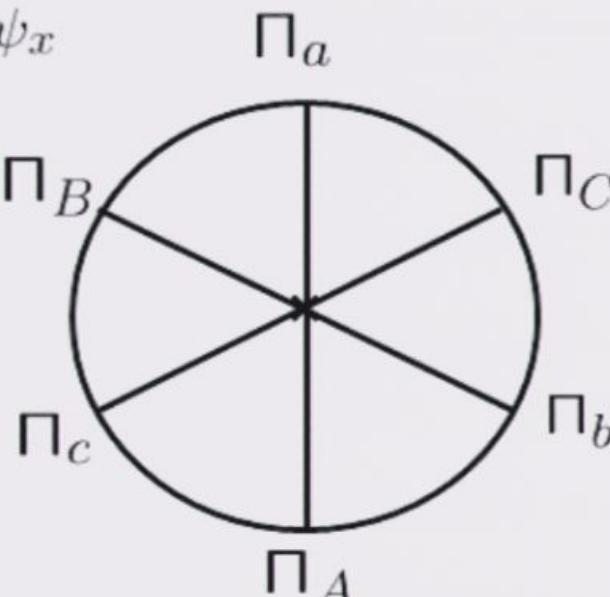
$$\Pi_b + \Pi_B = I$$

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The mystery of contextuality

There is a tension between

1) the dependence of representation on certain details of the experimental procedure

and

2) the independence of outcome statistics on those details of the experimental procedure