

Title: Foundations of Quantum Mech. (PHYS 639) - Lecture 7

Date: Dec 08, 2009 11:00 AM

URL: <http://pirsa.org/09120071>

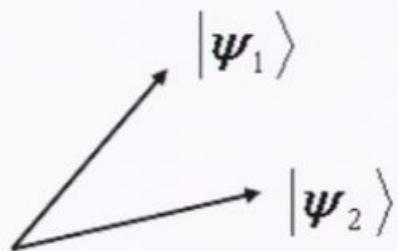
Abstract:

Evidence in favour of ψ -epistemic models

The analogy to Liouville mechanics

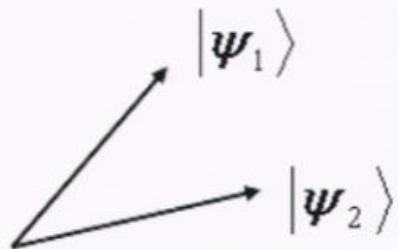
Example 1: The impossibility of discriminating non-orthogonal states

Consider



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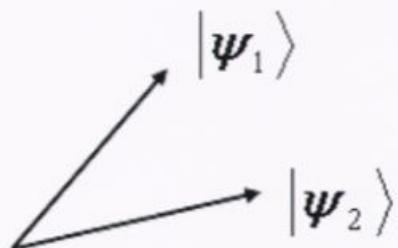
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Newtonian analogy: Mysterious. No analogue of non-orthogonality.

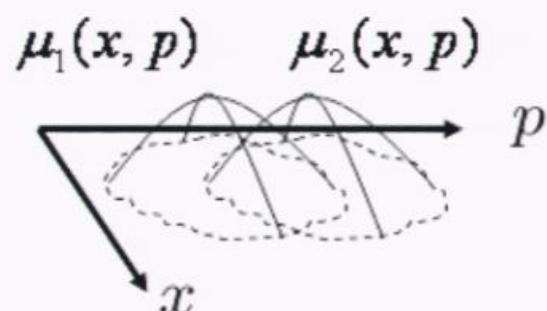
Example 1: The impossibility of discriminating non-orthogonal states

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Liouville analogy: Natural



Example 2: Lack of exponential divergence of states under chaotic evolution

In Newtonian mechanics, exponential divergence of ontic states is the signature of chaos

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In quantum theory,

$$\langle \psi_1(t) | \psi_2(t) \rangle = \langle \psi_1(0) | U^t U | \psi_2(0) \rangle$$

$$= \langle \psi_1(0) | \psi_2(0) \rangle$$

No divergence!

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$$\begin{aligned}\langle \psi_1(t) | \psi_2(t) \rangle &= \langle \psi_1(0) | U^t U | \psi_2(0) \rangle \\ &= \langle \psi_1(0) | \psi_2(0) \rangle\end{aligned}$$

No divergence!

Newtonian analogy: This is puzzling

Liouville analogy: This is natural, due to Liouville's theorem

$$\int dx dp \sqrt{\mu_1(x, p, t)} \sqrt{\mu_2(x, p, t)} = \int dx dp \sqrt{\mu_1(x, p, 0)} \sqrt{\mu_2(x, p, 0)}$$

Example 3: The impossibility of cloning non-orthogonal states

(C. Fuchs, 1996)

Cloning the set $\{|\psi_1\rangle, |\psi_2\rangle\}$ implies $|\psi_s\rangle|\chi\rangle \rightarrow |\psi_s\rangle|\psi_s\rangle$ for $s = 1, 2$

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Cloning the set $\{|\psi_1\rangle, |\psi_2\rangle\}$ implies $|\psi_s\rangle|\chi\rangle \rightarrow |\psi_s\rangle|\psi_s\rangle$ for $s = 1, 2$

By unitarity, the inner product must be constant

$$\text{But } |\langle\psi_1|\langle\chi|(\psi_2\rangle|\chi\rangle)| = |\langle\psi_1|\psi_2\rangle|$$

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Cloning the set $\{\mu_1(z), \mu_2(z)\}$ implies $\mu_s(z)v(y) \rightarrow \mu_s(z)\mu_s(y)$ for $s = 1, 2$

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$$\text{But } \int dz dy \sqrt{\mu_1(z)v(y)} \sqrt{\mu_2(z)v(y)} = \int dz \sqrt{\mu_1(z)} \sqrt{\mu_2(z)}$$

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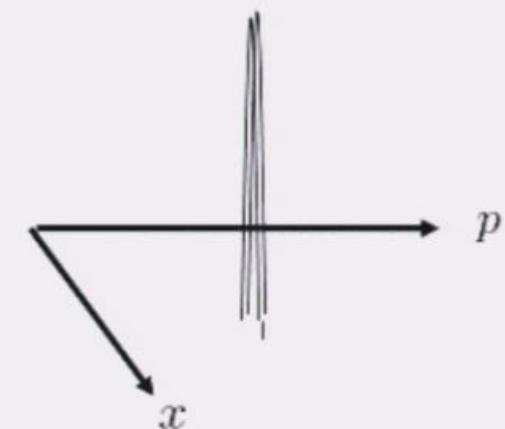
These are equal iff $\int dz \sqrt{\mu_1(z)} \sqrt{\mu_2(z)} = 0 \text{ or } 1$ i.e. disjoint or identical

Where the Liouville analogy fails

Pure states: In Liouville mechanics, they are Dirac-delta functions on phase space

Thus, they have strictly disjoint support
(hence distinguishable, clonable)

State of complete knowledge = Newtonian state



In other words: Quantum states are analogous to states of incomplete knowledge

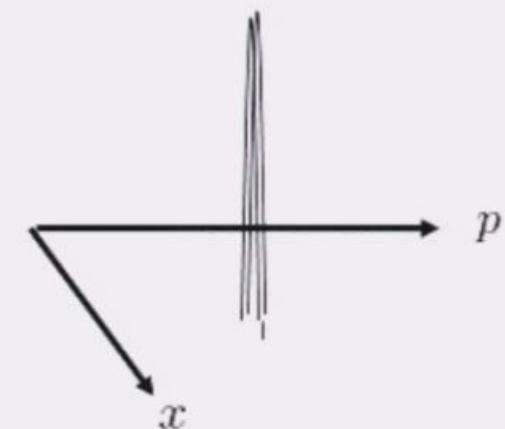
Consider: Liouville mechanics with an epistemic restriction

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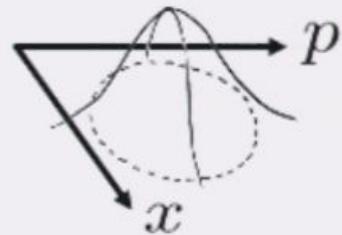
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The analogy to Liouville mechanics with an epistemic restriction

Based primarily on unpublished work
with Stephen Bartlett and Terry Rudolph

Liouville mechanics

$$\mu(x, p)$$



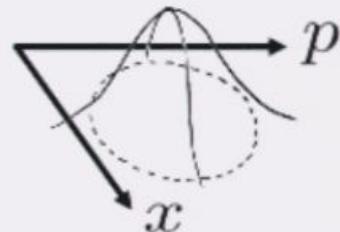
What is a good epistemic restriction to apply?
-- look to quantum mechanics

$$\Delta x \times \Delta p \geq \frac{\hbar}{2}$$

$$C_{x,p} = \left\langle \frac{\hat{x}\hat{p} + \hat{p}\hat{x}}{2} \right\rangle - \langle \hat{x} \rangle \langle \hat{p} \rangle$$

Liouville mechanics

$$\mu(x, p)$$



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Quantum mechanics

Uncertainty principle:

$$\Delta^2 x \Delta^2 p - C_{x,p}^2 \geq (\hbar/2)^2$$

where

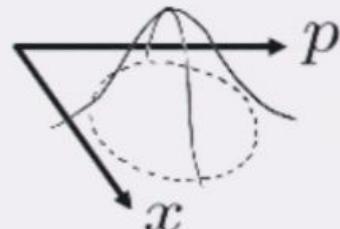
$$\Delta^2 x \equiv \langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2$$

$$C_{x,p} \equiv \frac{1}{2} \langle \hat{x} \hat{p} + \hat{p} \hat{x} \rangle - \langle \hat{x} \rangle \langle \hat{p} \rangle$$

$$\langle \hat{A} \rangle \equiv \text{Tr}(\hat{A} \hat{\rho})$$

Liouville mechanics

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Liouville mechanics with an epistemic restriction

Uncertainty principle:

$$\Delta^2 x \Delta^2 p - C_{x,p}^2 \geq (\hbar/2)^2$$

where

$$\Delta^2 x \equiv \langle x^2 \rangle - \langle x \rangle^2$$

$$C_{x,p} \equiv \langle xp \rangle - \langle x \rangle \langle p \rangle$$

$$\langle f(x, p) \rangle \equiv \int dx dp f(x, p) \mu(x, p)$$

Liouville mechanics with an epistemic restriction

Assume:

The classical uncertainty principle (for a single particle in 1D):

The only Liouville distributions that can be prepared are those that satisfy

$$\Delta^2 x \Delta^2 p - C_{x,p}^2 \geq (\hbar/2)^2$$

and that have maximal entropy for a given set of second-order moments.

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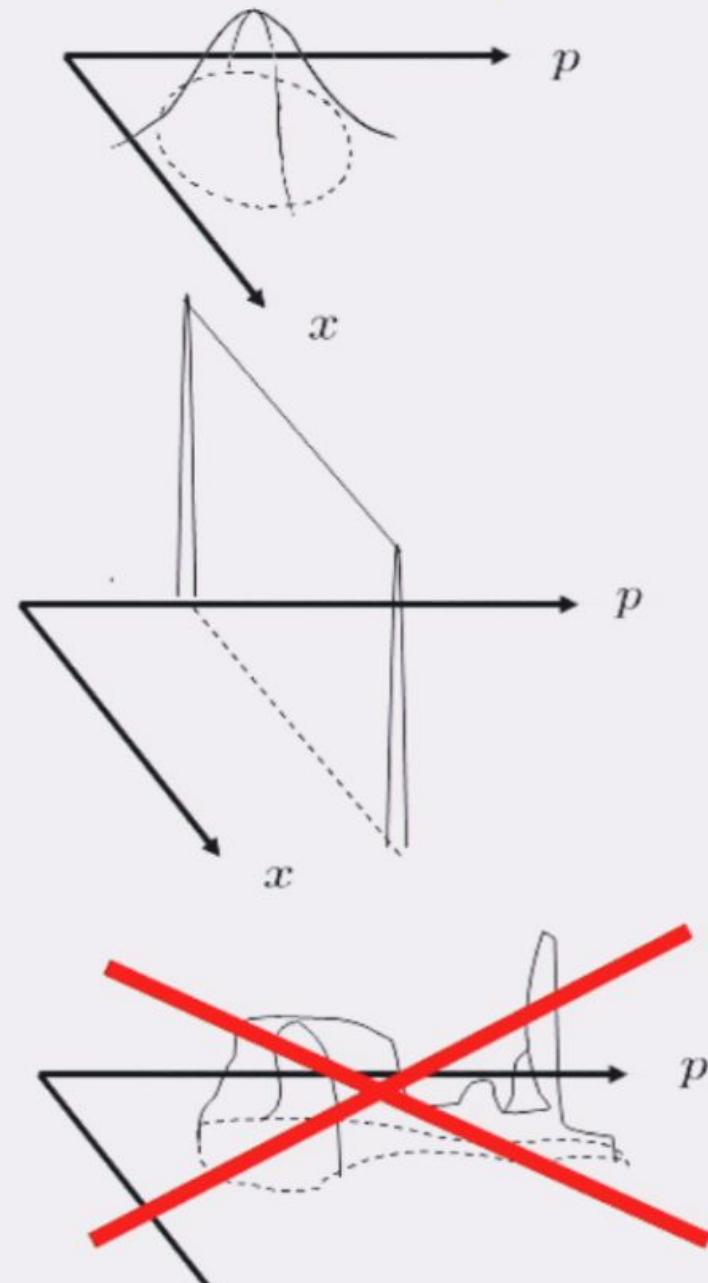
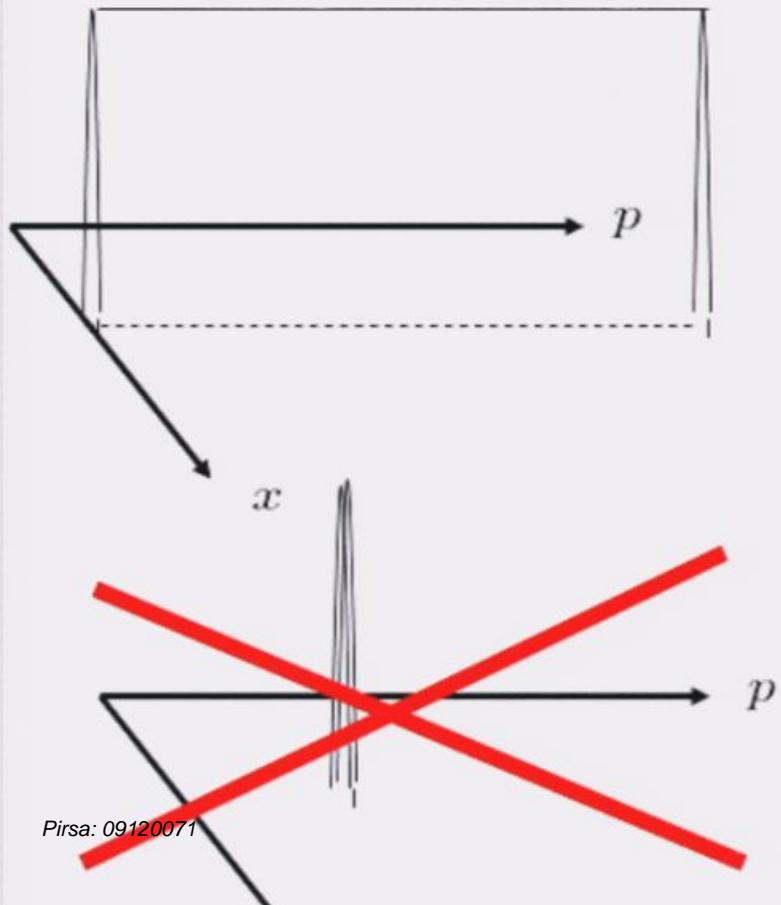
and that have maximal entropy for a given set of second-order moments.

Among $\mu(x,p)$ with a given set of second-order moments, Gaussian distributions maximize the entropy

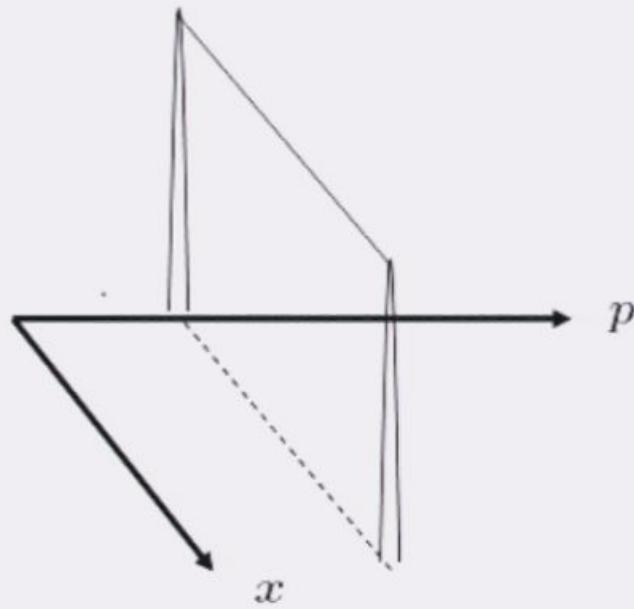
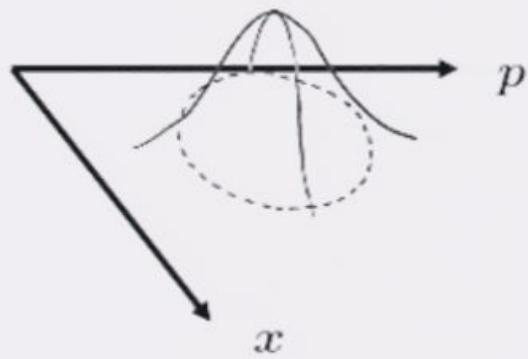
Valid epistemic states for one canonical system

$$\mu(x, p) \geq 0$$

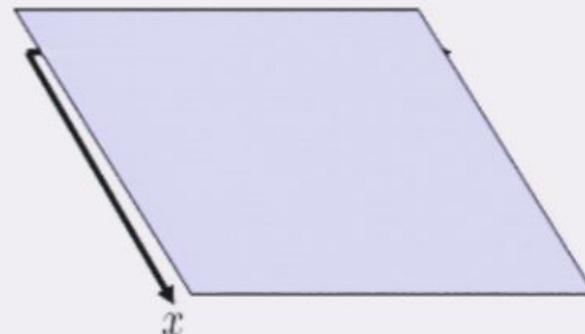
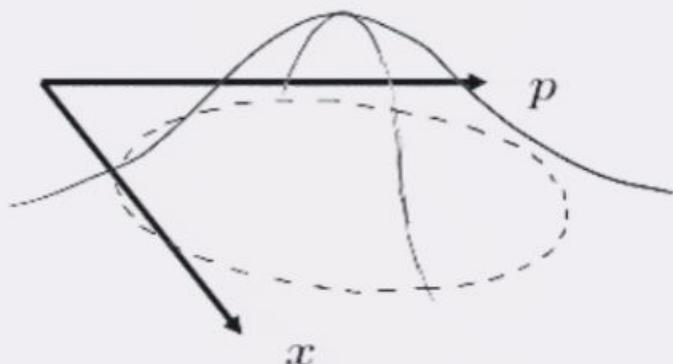
$$\int \mu(x, p) dx dp = 1$$



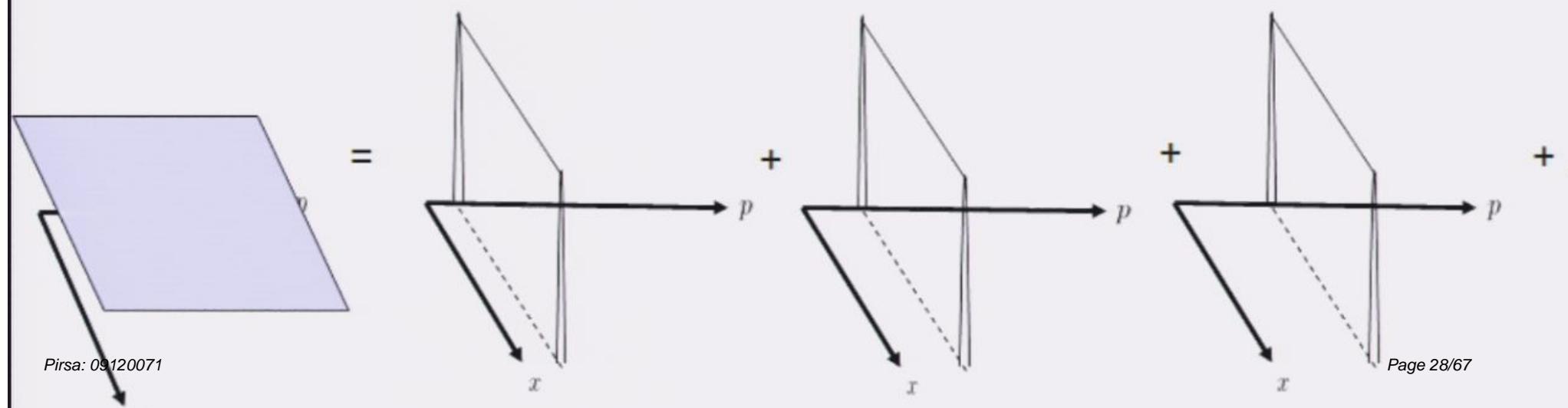
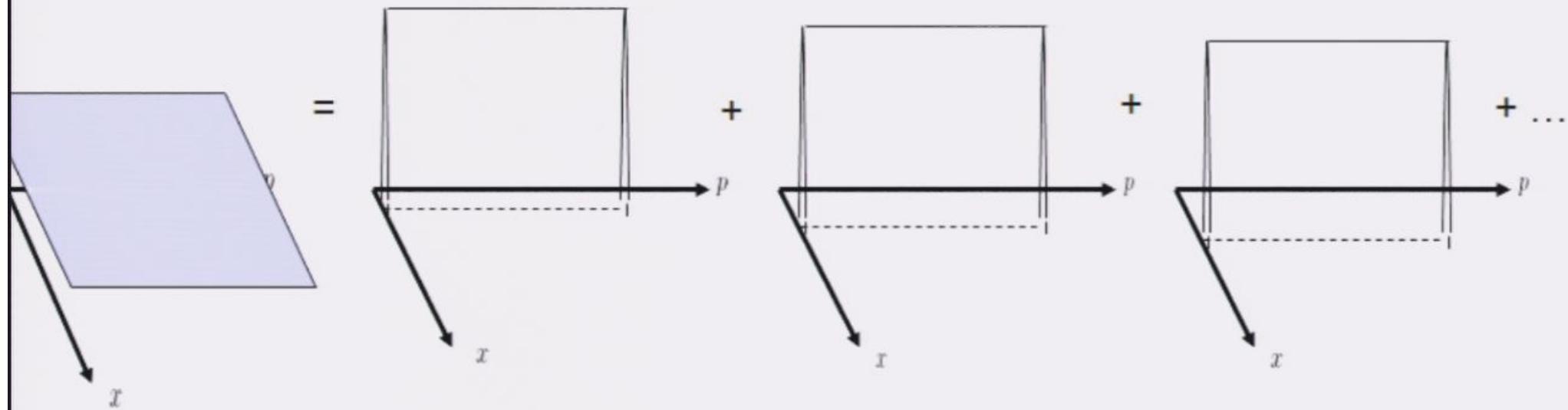
Pure epistemic states



Mixed epistemic states



Multiplicity of convex decompositions of a mixed epistemic state into pure epistemic states



Quantum mechanics

Uncertainty principle:

$$\gamma(\hat{\rho}) + i\hbar\Sigma \geq 0$$

$$\gamma(\hat{\rho}) = 2 \begin{pmatrix} \Delta^2 x_1 & C_{x_1,p_1} & C_{x_1,x_2} & C_{x_1,p_2} & \dots \\ C_{p_1,x_1} & \Delta^2 p_1 & C_{p_1,x_2} & C_{p_1,p_2} & \\ C_{x_2,x_1} & C_{x_2,p_1} & \Delta^2 x_2 & C_{x_2,p_2} & \\ C_{p_2,x_1} & C_{p_2,p_1} & C_{p_2,x_2} & \Delta^2 p_2 & \\ \vdots & & & & \ddots \end{pmatrix}$$

$$\Sigma = \begin{pmatrix} 0 & -1 & & \dots \\ 1 & 0 & & \\ & & 0 & -1 \\ & & 1 & 0 \\ \vdots & & & \ddots \end{pmatrix}$$

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Single particle in 1d:

$$2 \begin{pmatrix} \Delta^2 x & C_{x,p} \\ C_{p,x} & \Delta^2 p \end{pmatrix} + i\hbar \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \geq 0$$

$$2 \begin{pmatrix} \Delta^2 x & C_{x,p} - \frac{1}{2}i\hbar \\ C_{p,x} + \frac{1}{2}i\hbar & \Delta^2 p \end{pmatrix} \geq 0$$

$$\Delta^2 x \Delta^2 p - C_{x,p}^2 \geq (\hbar/2)^2$$

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Liouville mechanics with an epistemic restriction

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$$\mu(x_1, p_1, x_2, p_2, \dots)$$

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$$\Delta^2 x \Delta^2 p - C_{x,p}^2 \geq (\hbar/2)^2$$

$$A \geq 0$$

$$\langle \psi | A | \psi \rangle \geq 0 \quad \forall |\psi\rangle$$

$$A = \sum_n a_n |\phi_n\rangle \langle \phi_n| \quad a_n \geq 0$$

$$\Delta x \times \Delta p \geq \frac{\hbar}{2}$$

$$C_{x,p} = \underbrace{\langle \hat{x} \hat{p} + \hat{p} \hat{x} \rangle}_{z} - \langle \hat{x} \rangle \langle \hat{p} \rangle$$

Liouville mechanics with an epistemic restriction

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and that have maximal entropy for a given set of second-order moments.

Among valid μ with a given γ , multi-variate Gaussians maximize the entropy

$$\mu(\mathbf{z}) = \frac{1}{(2\pi)^{n/2}|\gamma|^{1/2}} \exp\left(-\frac{1}{2}(\mathbf{z} - \langle \mathbf{z} \rangle)^T \gamma^{-1} (\mathbf{z} - \langle \mathbf{z} \rangle)\right)$$

Quantum mechanics

$$\hat{\mathbf{R}} = (\hat{x}_1, \hat{p}_1, \hat{x}_2, \hat{p}_2, \dots)$$

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$$C_{ij} = \frac{1}{\hbar} \langle \hat{x}_i \hat{x}_j + \hat{x}_j \hat{x}_i \rangle - \langle \hat{x}_i \rangle \langle \hat{x}_j \rangle$$

Quantum mechanics

Uncertainty principle:

$$\gamma(\hat{\rho}) + i\hbar\Sigma \geq 0$$

$$\begin{aligned}\hat{\mathbf{R}} &= (\hat{x}_1, \hat{p}_1, \hat{x}_2, \hat{p}_2, \dots) \\ \gamma_{ij} &= 2\left(\frac{1}{2}\langle\{R_i, R_j\}\rangle - \langle R_i \rangle \langle R_j \rangle\right) \\ [R_i, R_j] &= i\hbar\Sigma_{ij}\end{aligned}$$

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$$= 2(\langle R_i R_j \rangle - \langle R_i \rangle \langle R_j \rangle)$$

$$= \langle \{R_i, R_j\} \rangle + \langle [R_i, R_j] \rangle - 2\langle R_i \rangle \langle R_j \rangle$$

$$= \gamma_{ij} + i\hbar\Sigma_{ij}$$

$$(\mathbf{Y}, (\gamma(\hat{\rho}) + i\hbar\Sigma)\mathbf{Y})$$

$$= \sum_{i,j} Y_i^* (\gamma_{ij} + i\hbar\Sigma_{ij}) Y_j$$

$$= 2 \langle \sum_i Y_i^* (R_i - \langle R_i \rangle) \sum_j Y_j (R_j - \langle R_j \rangle) \rangle$$

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$$= 2\langle \sum_i Y_i^* (R_i - \langle R_i \rangle) \sum_j Y_j (R_j - \langle R_j \rangle) \rangle$$

$$= 2\langle A_{\mathbf{Y}}^\dagger A_{\mathbf{Y}} \rangle \geq 0 \quad \forall \mathbf{Y}$$

Valid epistemic states for a pair of canonical systems

Uncorrelated distributions

$$\mu(x_1, p_1, x_2, p_2) = \mu(x_1, p_1) \mu(x_2, p_2)$$

Correlated distributions

e.g. $\mu(x_1, p_1, x_2, p_2) = \frac{1}{N} \delta(x_1 - x_2) \delta(p_1 + p_2)$

This corresponds to the **entangled state** of Einstein, Podolsky and Rosen

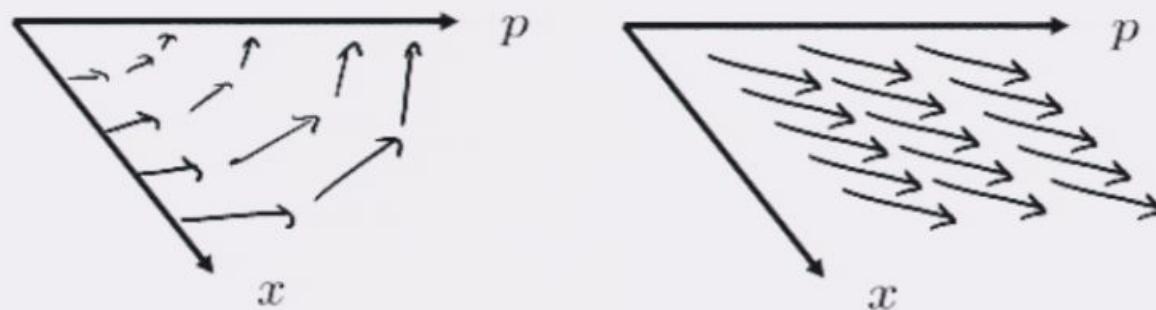
$$\begin{aligned} |\psi\rangle &= \int dx_1 dx_2 \delta(x_1 - x_2) |x_1\rangle |x_2\rangle \\ &= \int dp_1 dp_2 \delta(p_1 + p_2) |p_1\rangle |p_2\rangle \end{aligned}$$

Valid deterministic transformations

The group of canonical transformations with quadratic Hamiltonian

Only canonical transformations preserve the uncertainty principle

Only quadratic Hamiltonians preserve the gaussianity

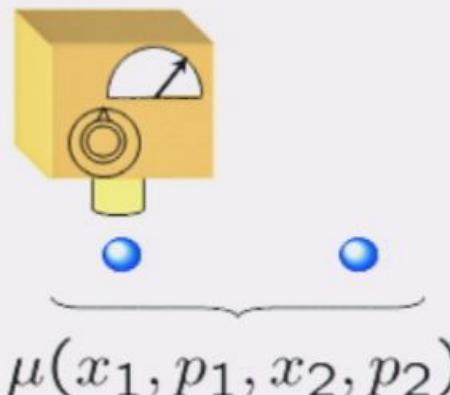


Valid measurements

Sets of indicator functions $\{\xi_k(x, p)\}$

$\xi_k(x, p)$ = probability of k given (x,p)

$$\xi_k(x_1, p_1)$$

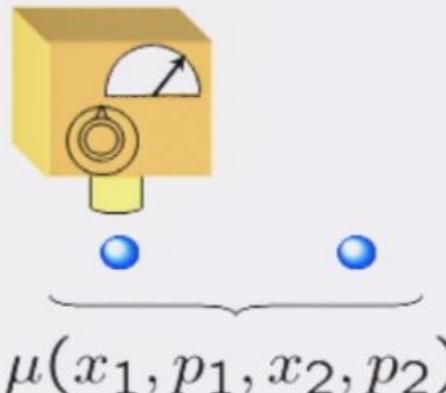


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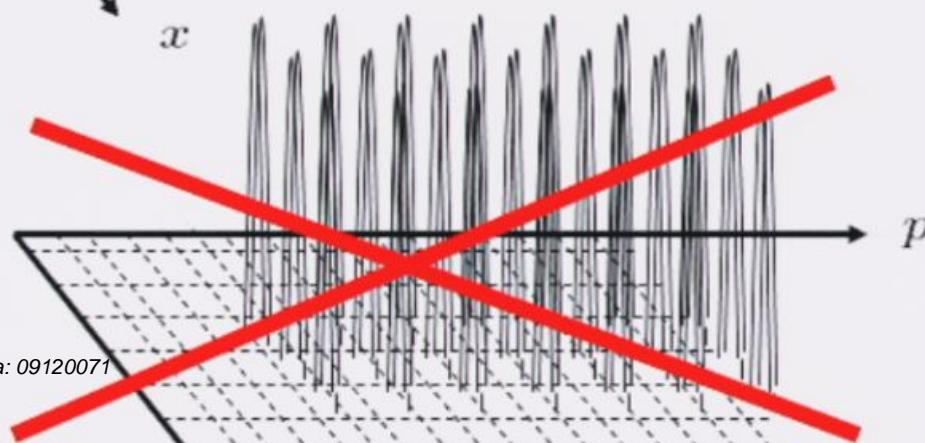
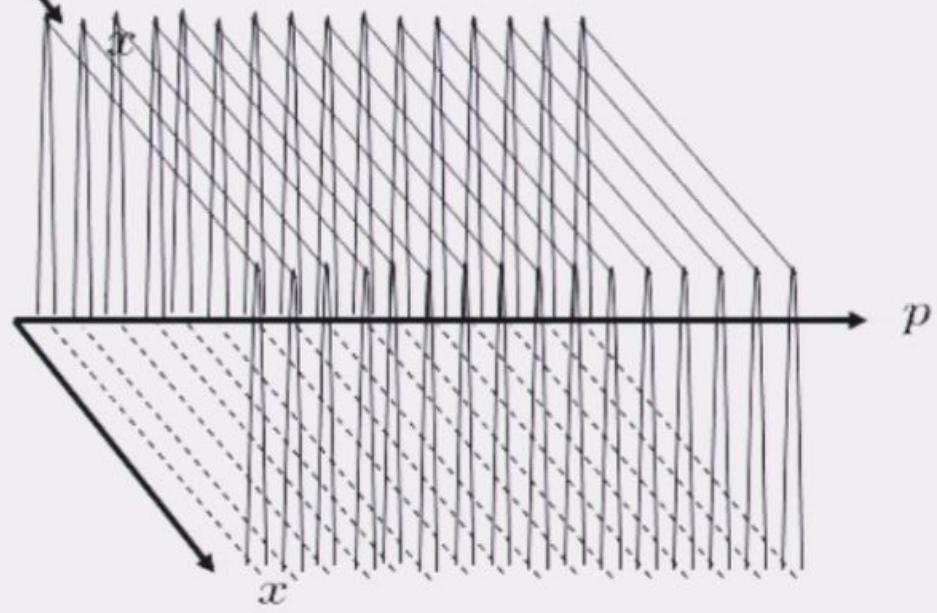
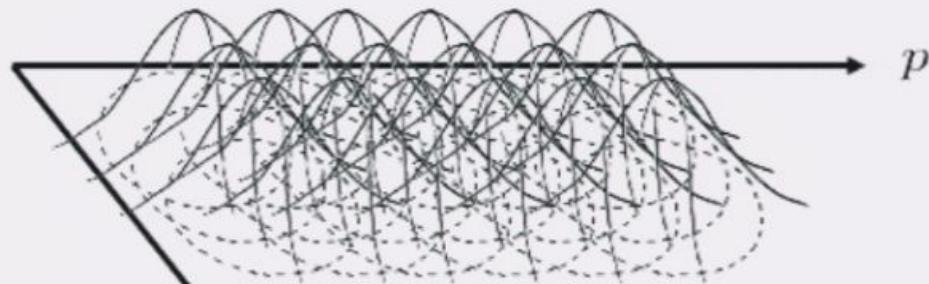
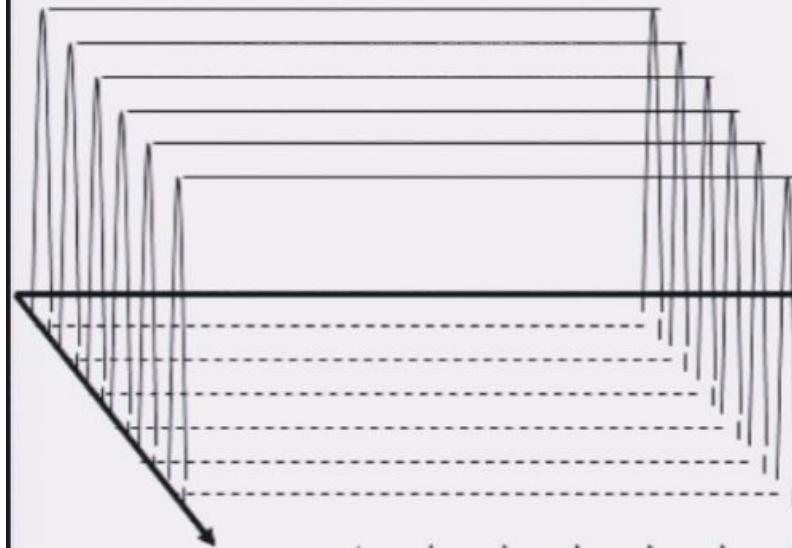
$$\begin{aligned}\mu(x_2, p_2) &\propto \int dx_1 dp_1 \xi(x_1, p_1) \delta(x_1 - x_2) \delta(p_1 + p_2) \\ &= \xi(x_2, -p_2)\end{aligned}$$

$$\mu(x_2, p_2) \propto \int dx_1 dp_1 \xi(x_1, p_1) \mu(x_1, p_1, x_2, p_2)$$

Valid measurements for one canonical system

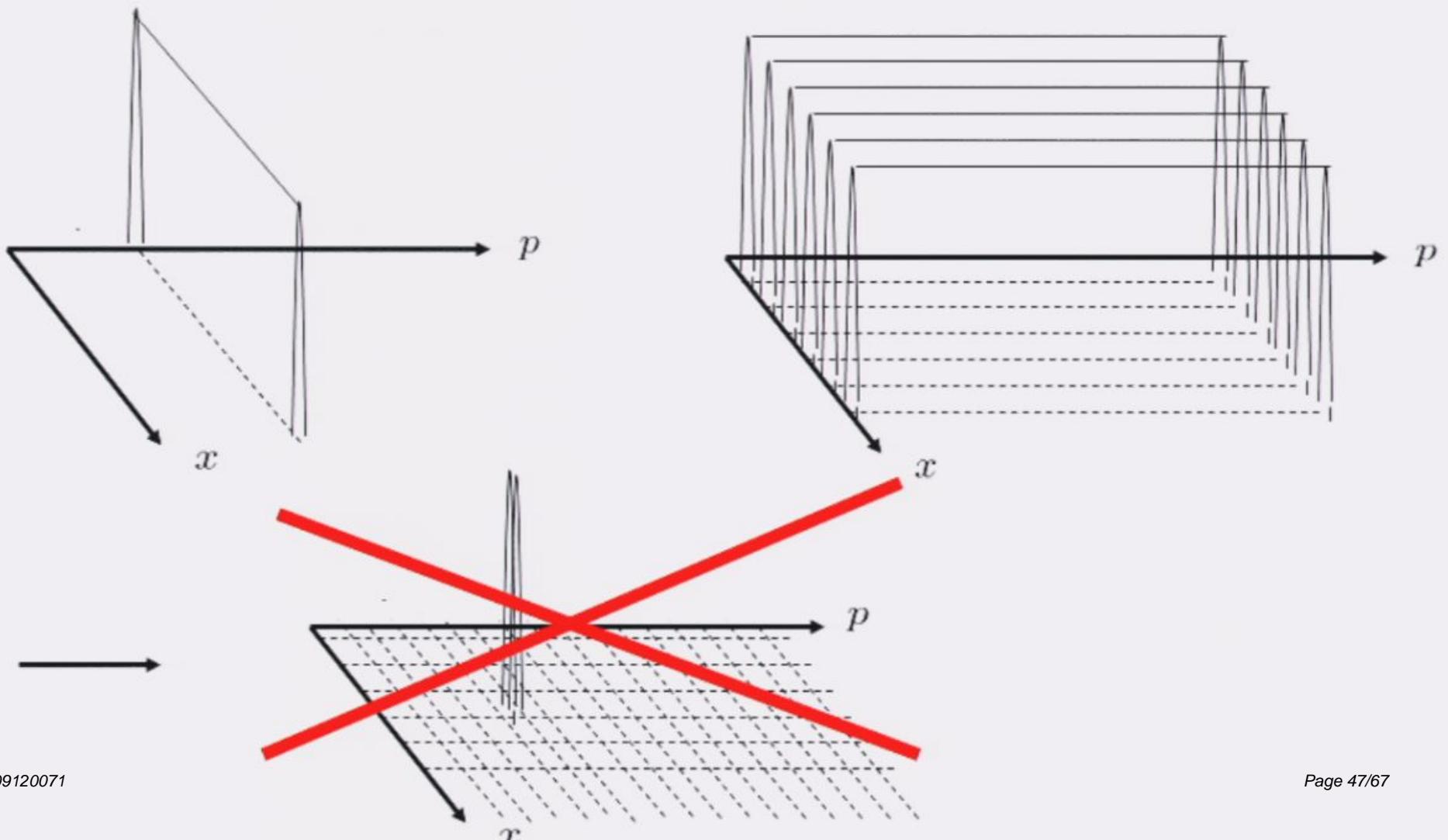
$$\xi_k(x, p) \geq 0$$

$$\sum_k \xi_k(x, p) = 1 \quad \forall x \forall p$$



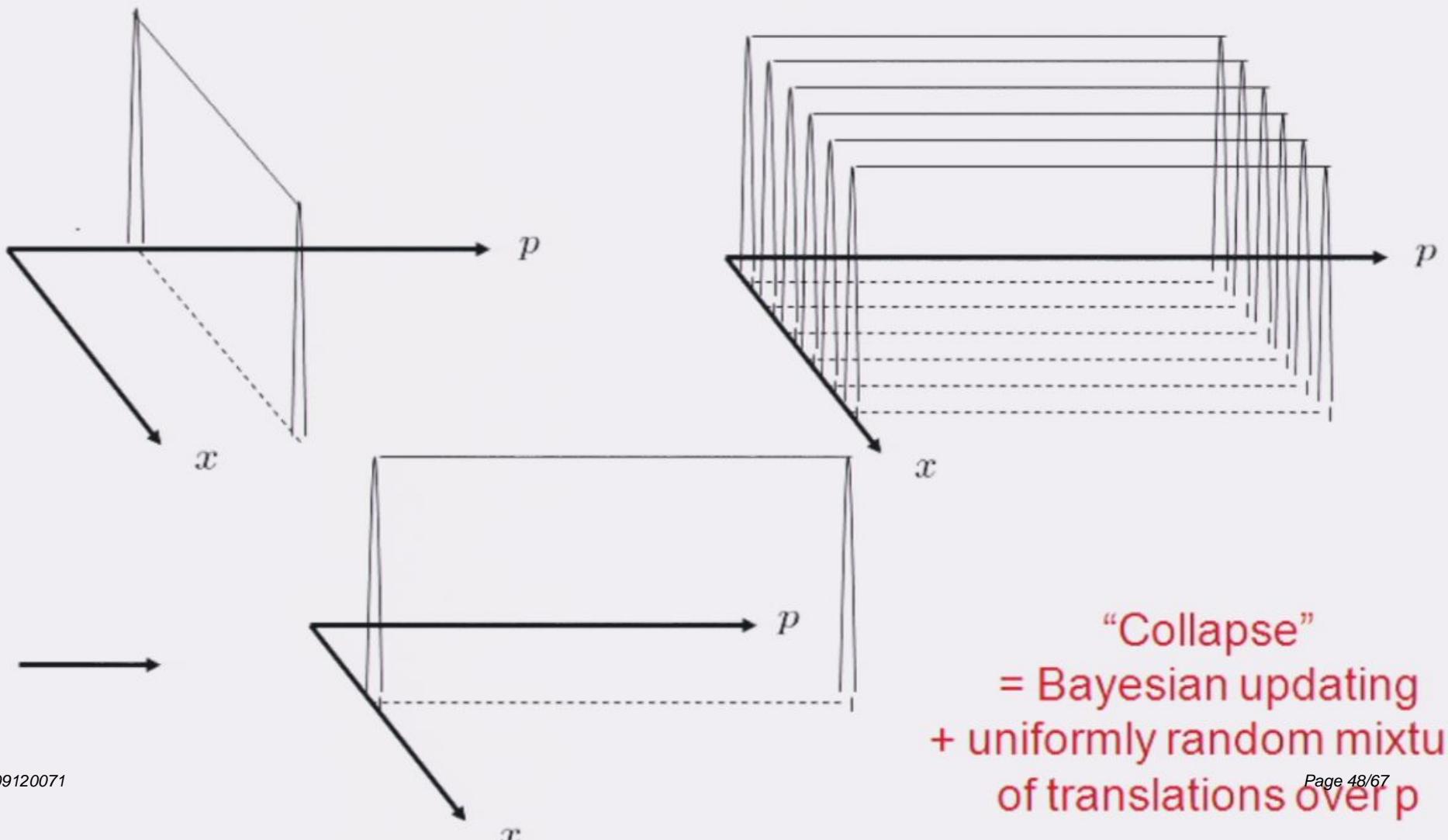
Measurement-induced transformations

Measure x in a
reproducible way



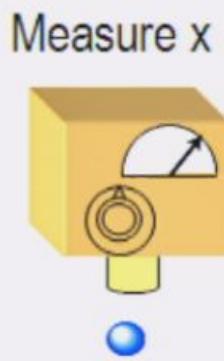
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Note: the evolution is *deterministic* if the apparatus is treated internally

External apparatus



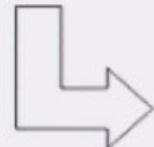
Unknown disturbance to p

Internal apparatus

Measure x



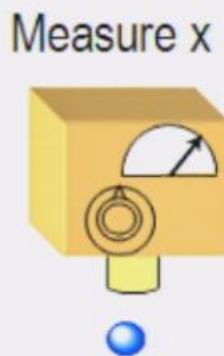
Interact by $H_{int} = x_{sys} p_{app}$



Final x of apparatus reflects initial x of system
Final p of system reflects initial p of apparatus

Note: the evolution is *deterministic* if the apparatus is treated internally

External apparatus



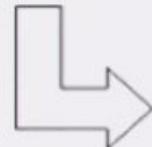
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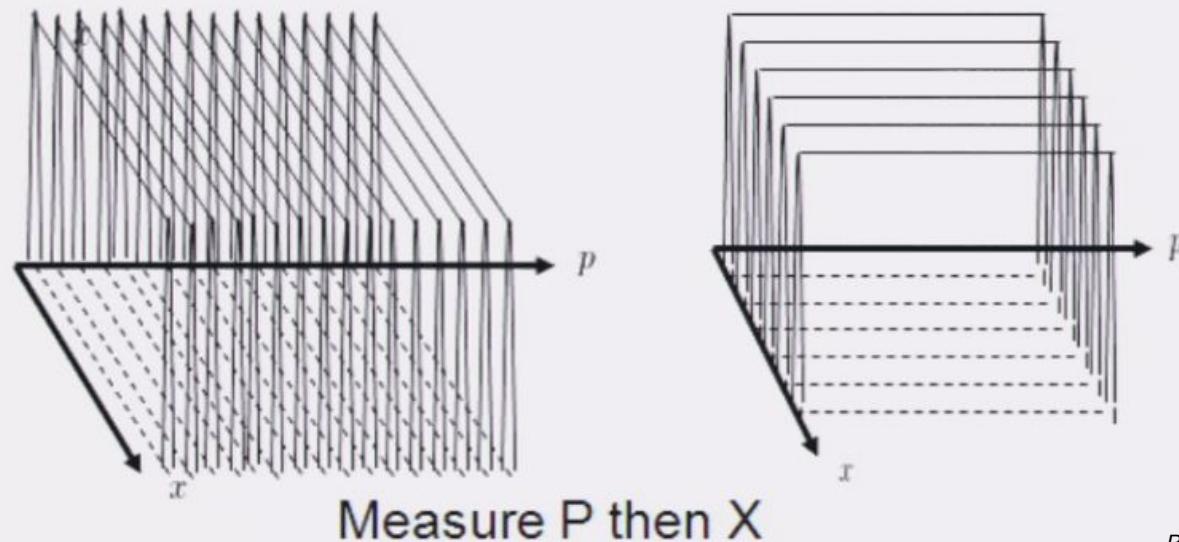
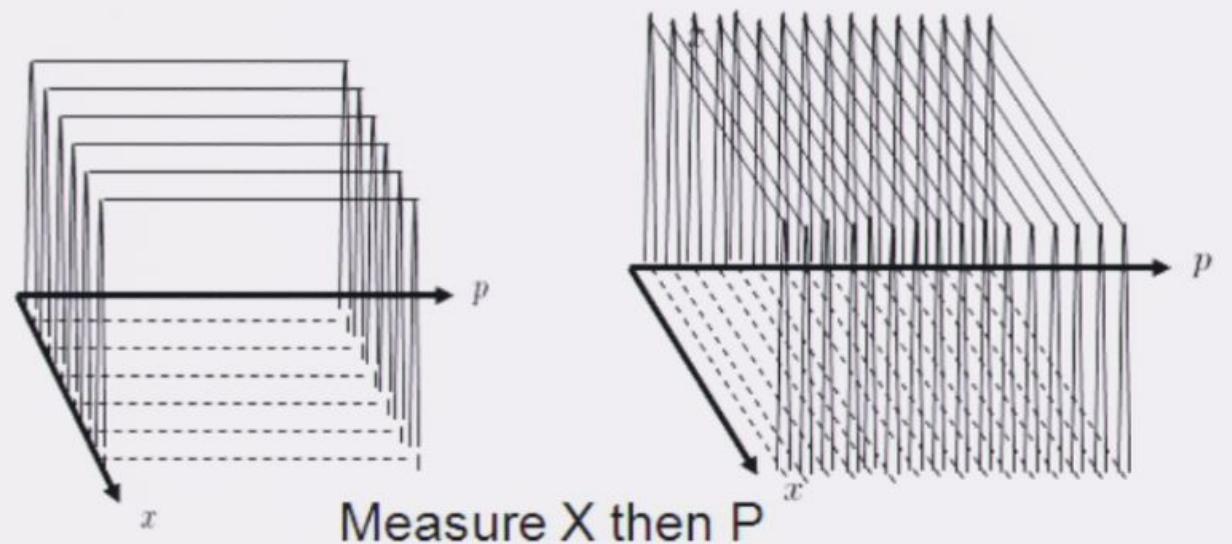
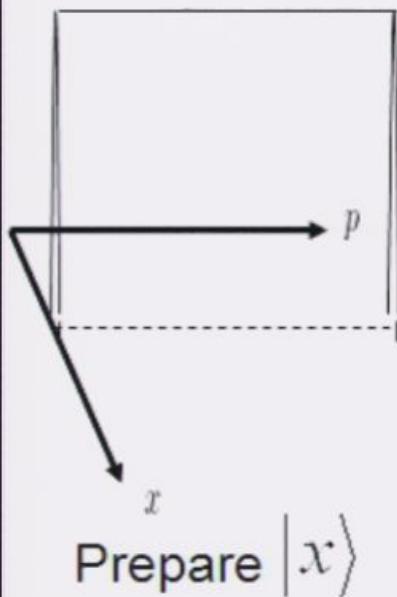


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Final x of apparatus reflects initial x of system
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Non-commutativity of measurements



Note: the evolution is *deterministic* if the apparatus is treated internally

External apparatus

Measure x

$$\begin{aligned} |\psi\rangle &= \int dx_1 dx_2 \delta(x_1 - x_2) |x_1\rangle |x_2\rangle \\ &= \int dp_1 dp_2 \delta(p_1 + p_2) |p_1\rangle |p_2\rangle \end{aligned}$$

Internal apparatus

Measure x



Prepare x



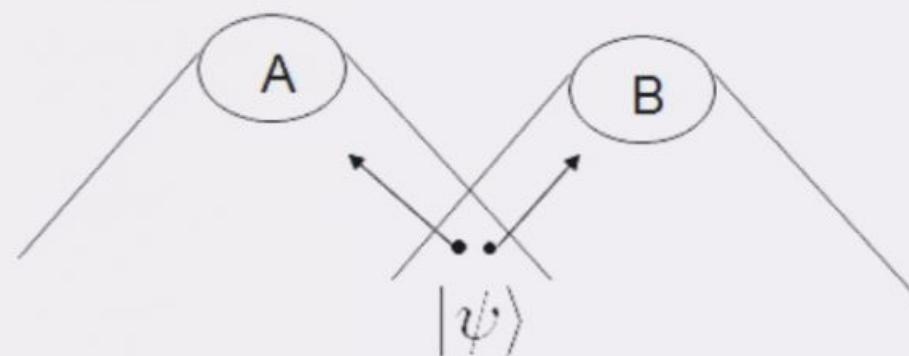
On particle 1, measure either X or P

Outcomes for measurements of X or P on particle 2 become certain

$\langle \psi | \hat{x}_1 = \hat{p}_1 \rangle \psi = \langle \psi | \hat{x}_2 \rangle \psi$

The position and momentum of particle 2 are determined

The EPR experiment



$$\begin{aligned} |\psi\rangle &= \int dx_1 \, dx_2 \, \delta(x_1 - x_2) |x_1\rangle |x_2\rangle \\ &= \int dp_1 \, dp_2 \, \delta(p_1 + p_2) |p_1\rangle |p_2\rangle \end{aligned}$$

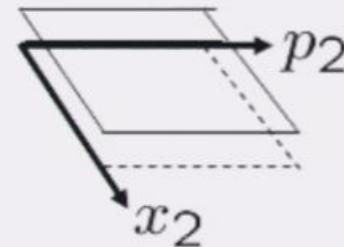
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$$\mu(x_1, p_1, x_2, p_2) = \frac{1}{N} \delta(x_1 - x_2) \delta(p_1 + p_2)$$

$$\mu(x_2, p_2) = \frac{1}{N}$$

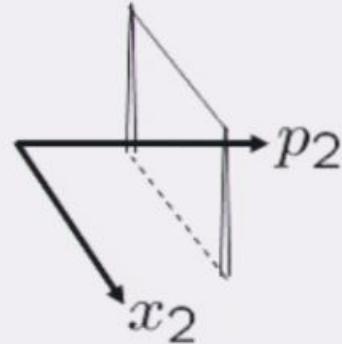
Initially A is completely ignorant of 2



If A measures x on 1, she infers x of 2



If A measures p on 1, she infers p of 2



A's decision does not affect the reality at 2,
the x and p were already elements of reality

The Wigner representation

Weyl operators $\hat{w}(u, v) = \exp(-iv\hat{x} - iu\hat{p})$

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This can be generalized

$W_{\hat{\rho}}(x_1, p_1, x_2, p_2) = \text{Tr}[\hat{\rho}\hat{A}(x_1, p_1) \otimes \hat{A}(x_2, p_2)]$

Gaussian quantum mechanics

Gaussian state ρ : one that has a Gaussian Wigner rep'n

$$W_\rho(\mathbf{z}) = \frac{1}{(2\pi)^{n/2}|\gamma|^{1/2}} \exp\left(-\frac{1}{2}(\mathbf{z} - \langle \mathbf{z} \rangle)^T \gamma^{-1} (\mathbf{z} - \langle \mathbf{z} \rangle)\right)$$

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Therefore, the Wigner rep'n satisfies the classical uncertainty principle

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Gaussian measurements and transformations: preserve Gaussianity

One can prove

Theorem: Liouville mechanics with an epistemic restriction is empirically equivalent to Gaussian quantum mechanics

Categorizing quantum phenomena

Those arising in a restricted statistical classical theory

Those not arising in a restricted statistical classical theory

Categorizing quantum phenomena

Those arising in a restricted statistical classical theory

Those not arising in a restricted statistical classical theory

Wave-particle duality

noncommutativity

entanglement

Quantized spectra

Key distribution

Improvements in metrology

Bell inequality violations

Computational speed-up

Teleportation

Coherent superposition

No cloning

Quantum eraser

Bell-Kochen-Specker theorem

Pre and post-selection
“paradoxes”

Particle statistics

Categorizing quantum phenomena

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