

Title: Foundations of Quantum Mech. (PHYS 639) - Lecture 3

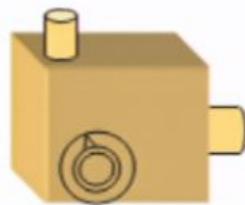
Date: Dec 02, 2009 11:00 AM

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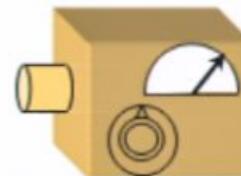
Abstract:

# Towards a purely operational formulation of quantum theory

# Operational Quantum Mechanics



Preparation  
P



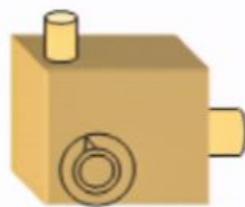
Measurement  
M

Vector  
 $|\psi\rangle$

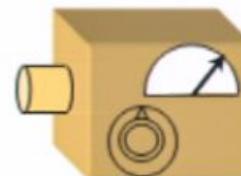
Hermitian operator  
 $A$   
 $A = \sum_k a_k \Pi_k$

$$Pr(k|P, M) = \langle \psi | \Pi_k | \psi \rangle$$

# Operational Quantum Mechanics



Preparation  
P



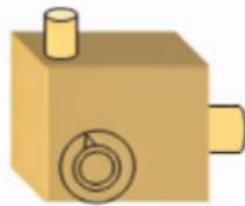
Measurement  
M

Density operator  
 $\rho$

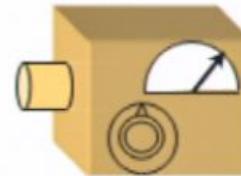
Hermitian operator  
 $A$   
$$A = \sum_k a_k \Pi_k$$

$$Pr(k|P, M) = \text{Tr}(\rho \Pi_k)$$

# Operational Quantum Mechanics



Preparation  
P



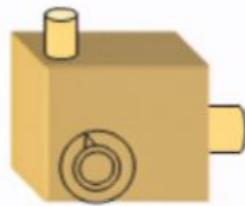
Measurement  
M

Density operator  
 $\rho$

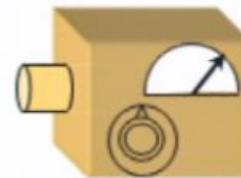
Projection valued  
measure (PVM)  
 $\{\Pi_k\}$

$$Pr(k|P, M) = \text{Tr}(\rho \Pi_k)$$

# Operational Quantum Mechanics



Preparation  
 $P$



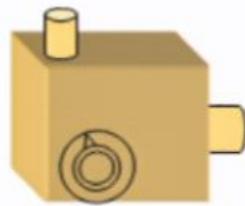
Measurement  
 $M$

Density operator  
 $\rho$

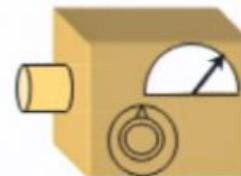
Hermitian operator  
 $A$   
$$A = \sum_k a_k \Pi_k$$

$$Pr(k|P, M) = \text{Tr}(\rho \Pi_k)$$

# Operational Quantum Mechanics



Preparation  
P



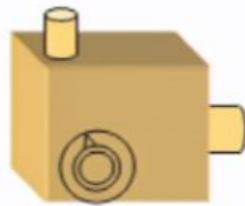
Measurement  
M

Density operator  
 $\rho$

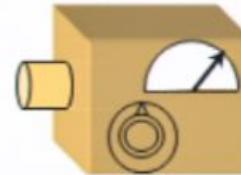
Projection valued  
measure (PVM)  
 $\{\Pi_k\}$

$$Pr(k|P, M) = \text{Tr}(\rho \Pi_k)$$

# Operational Quantum Mechanics



Preparation  
P



Measurement  
M

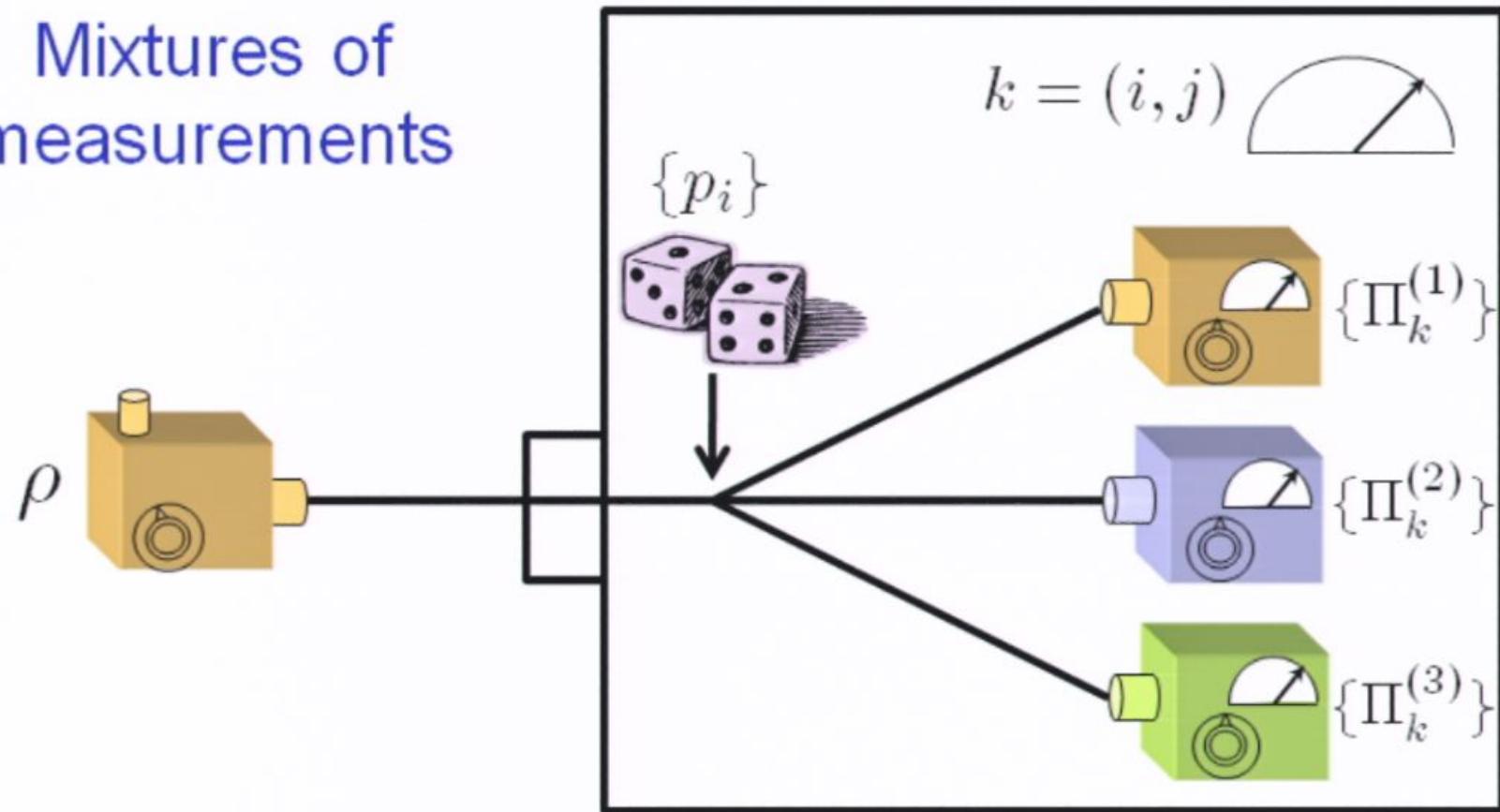
Density operator  
 $\rho$

Position operator valued  
measure (POVM)  
 $\{E_k\}$

$$Pr(k|P, M) = \text{Tr}(\rho E_k)$$

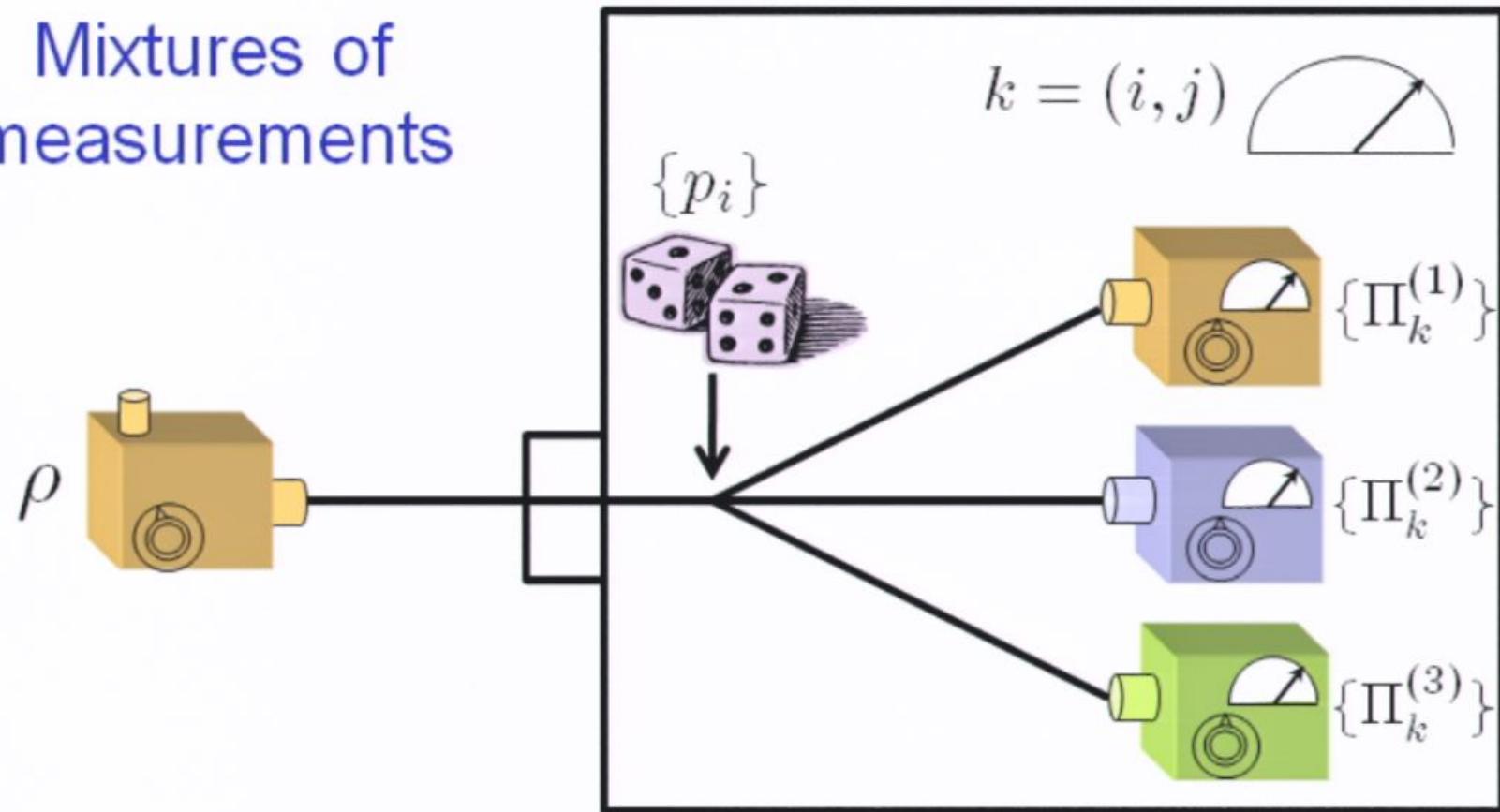
Standard Measurements	Generalized Measurements
$\{\Pi_i\}$	$\{E_d\}$
$\langle \psi   \Pi_i   \psi \rangle \geq 0, \forall  \psi\rangle$	$\langle \psi   E_d   \psi \rangle \geq 0, \forall  \psi\rangle$
$\sum_i \Pi_i = I$	$\sum_d E_d = I$
$P(i) = \text{tr}(\rho \Pi_i)$	$P(d) = \text{tr}(\rho E_d)$
$\Pi_i \Pi_j = \delta_{ij} \Pi_i$	_____

## Mixtures of measurements



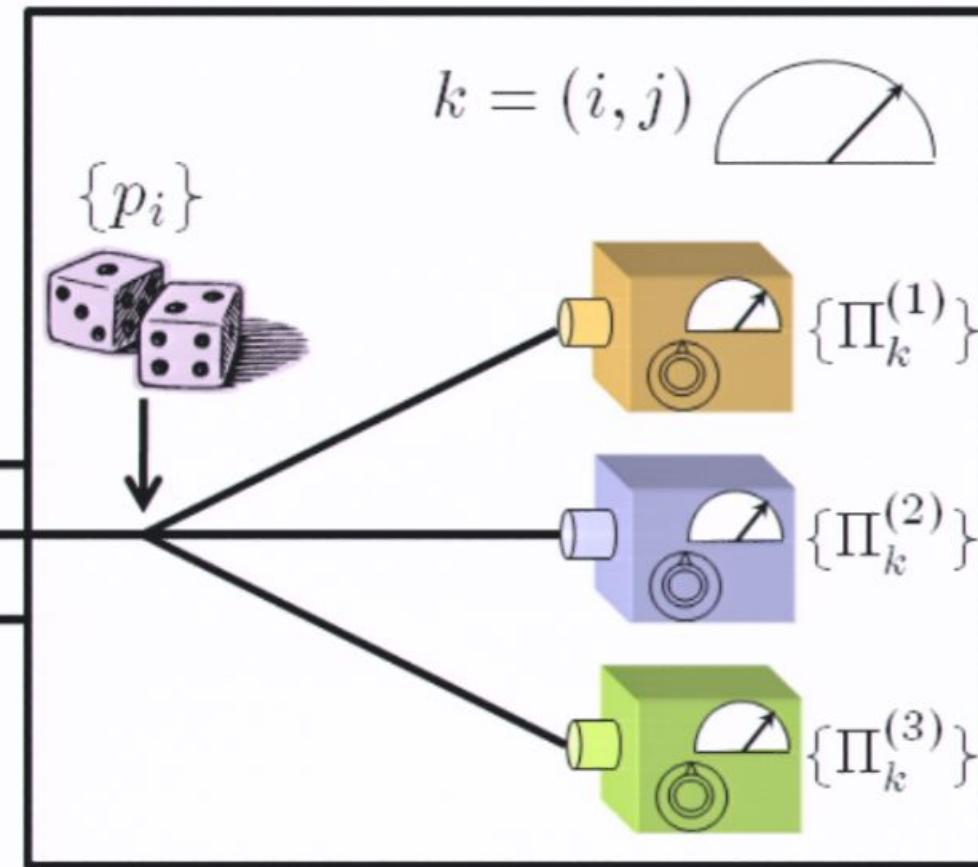
Standard Measurements	Generalized Measurements
$\{\Pi_i\}$	$\{E_d\}$
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$\sum_i \Pi_i = I$	$\sum_d E_d = I$
$P(i) = \text{tr}(\rho \Pi_i)$	$P(d) = \text{tr}(\rho E_d)$
$\Pi_i \Pi_j = \delta_{ij} \Pi_i$	—————

## Mixtures of measurements



## Mixtures of measurements

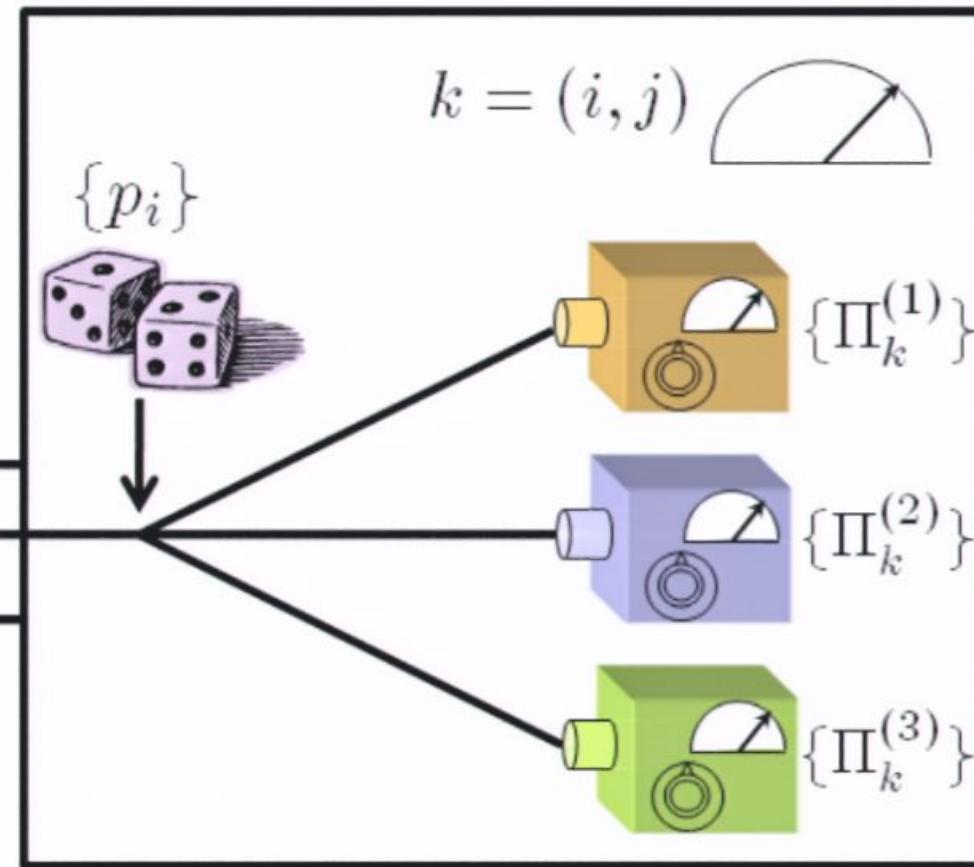
$\rho$



$$\begin{aligned} p(i, j) &= p(j|i)p(i) \\ &= \text{Tr}(\Pi_j^{(i)} \rho) p_i \\ &= \text{Tr}(p_i \Pi_j^{(i)} \rho) \end{aligned}$$

## Mixtures of measurements

$$\rho$$

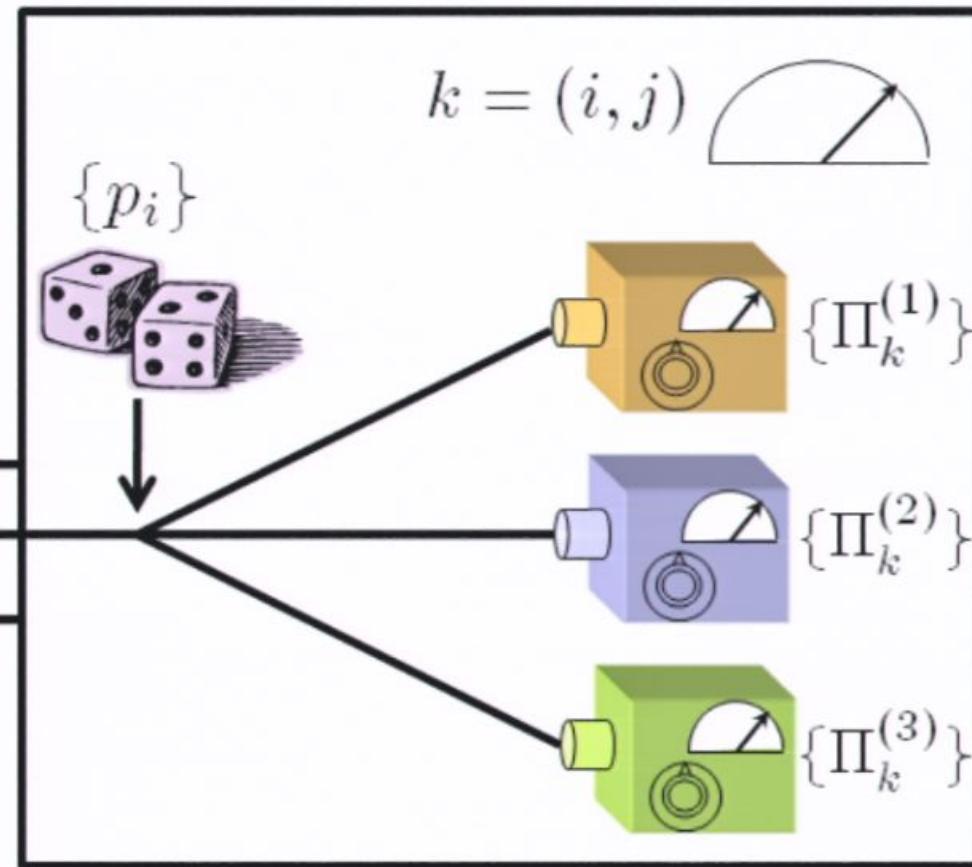


$$\begin{aligned} p(i, j) &= p(j|i)p(i) \\ &= \text{Tr}(\Pi_j^{(i)} \rho) p_i \\ &= \text{Tr}(\underbrace{p_i \Pi_j^{(i)}}_{E_{i,j}} \rho) \end{aligned}$$

$$p(k) = \text{Tr}(E_k \rho)$$

## Mixtures of measurements

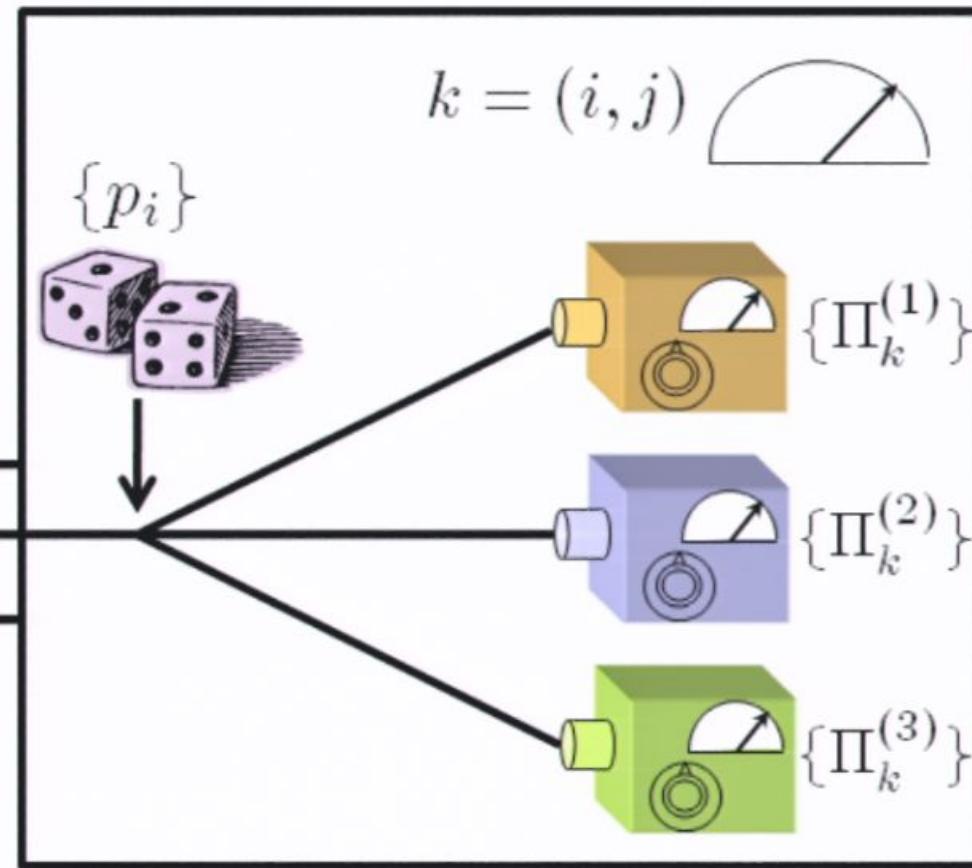
$$\begin{aligned}
 \rho & \\
 & \xrightarrow{\text{dice}} \{p_i\} \\
 & \xrightarrow{\text{dice}} \{E_{i,j}\} \\
 &= p(j|i)p(i) \\
 &= \text{Tr}(\Pi_j^{(i)} \rho) p_i \\
 &= \text{Tr}(\underbrace{p_i \Pi_j^{(i)}}_{E_{i,j}} \rho)
 \end{aligned}$$



$$\begin{aligned}
 p(k) &= \text{Tr}(E_k \rho) \\
 \text{Positive } \langle \psi | E_k | \psi \rangle &\geq 0 \quad \forall |\psi\rangle \in \mathcal{H}
 \end{aligned}$$

## Mixtures of measurements

$$\rho = \text{Tr}(\underbrace{p_i \Pi_j^{(i)}}_{E_{i,j}} \rho)$$



$$p(k) = \text{Tr}(E_k \rho)$$

Positive  $\langle \psi | E_k | \psi \rangle \geq 0 \quad \forall |\psi\rangle \in \mathcal{H}$

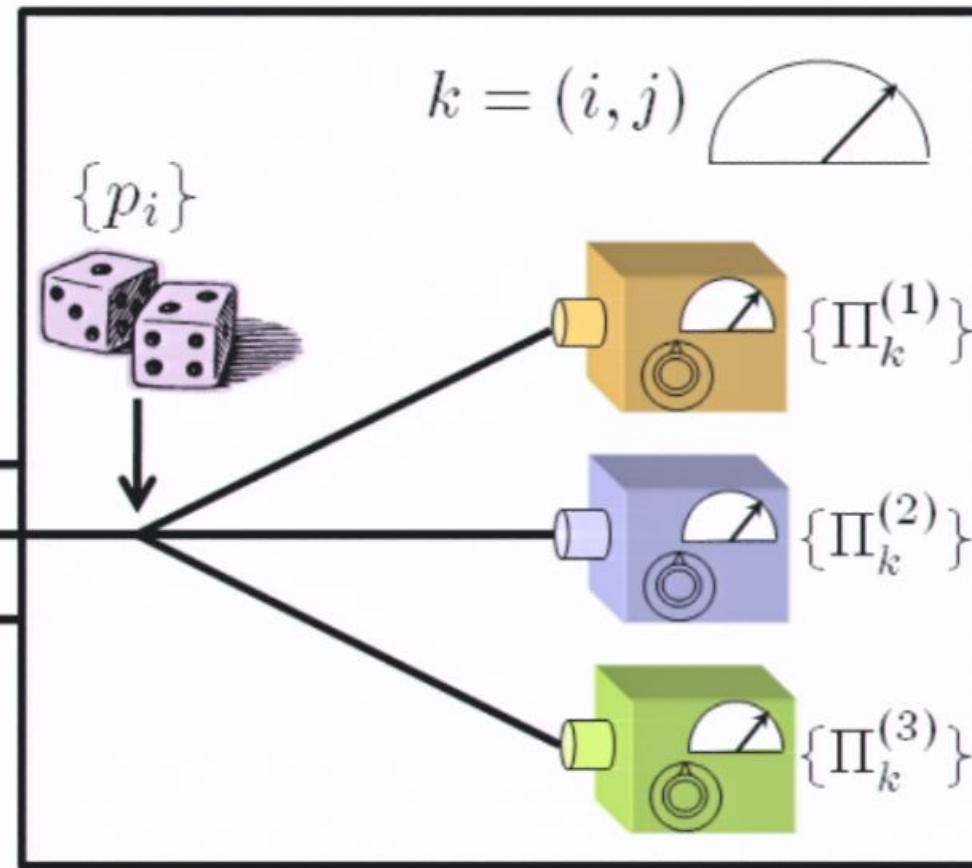
Sum to identity  $\sum_k E_k = I$

$$\sum_{i,j} p_i \Pi_j^{(i)} = \sum_i p_i \mathbb{1} = 1$$



## Mixtures of measurements

$\rho$



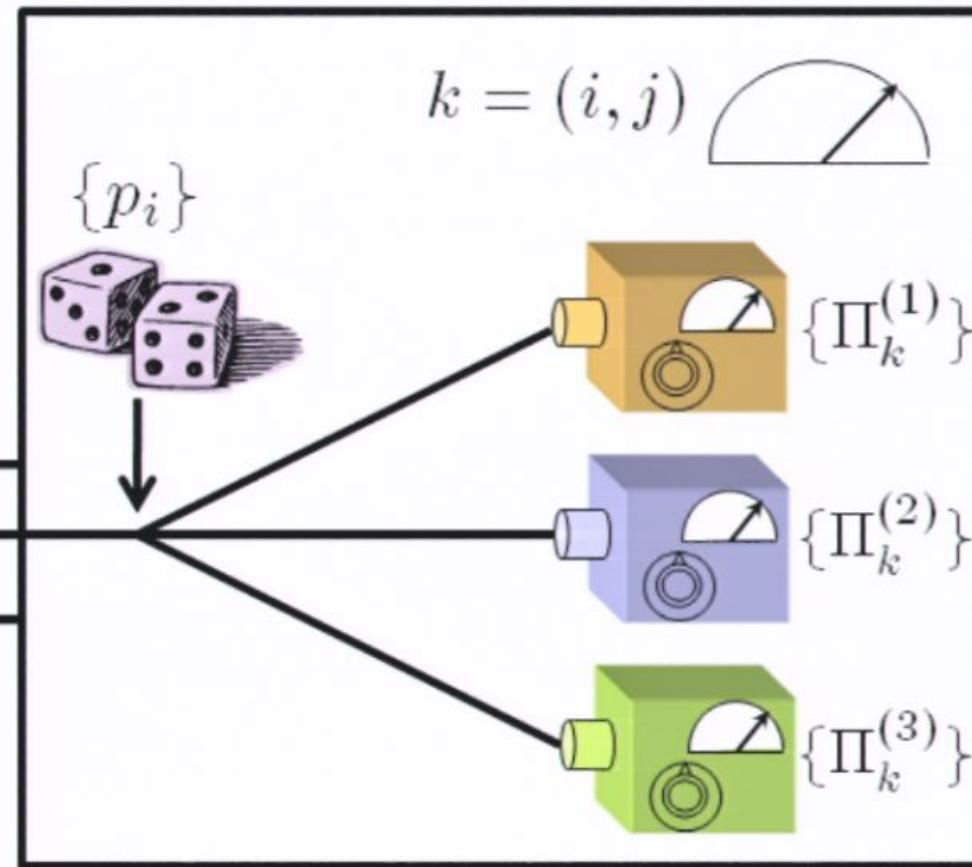
$$\begin{aligned}
 p(i, j) &= p(j|i)p(i) \\
 &= \text{Tr}(\Pi_j^{(i)} \rho) p_i \\
 &= \underbrace{\text{Tr}(p_i \Pi_j^{(i)})}_{E_{i,j}} \rho
 \end{aligned}$$

$$p(k) = \text{Tr}(E_k \rho)$$

$$\begin{aligned}
 &\text{Positive } \langle \psi | E_k | \psi \rangle \geq 0 \quad \forall |\psi\rangle \in \mathcal{H} \\
 &\text{Sum to identity } \sum_k E_k = I
 \end{aligned}$$

## Mixtures of measurements

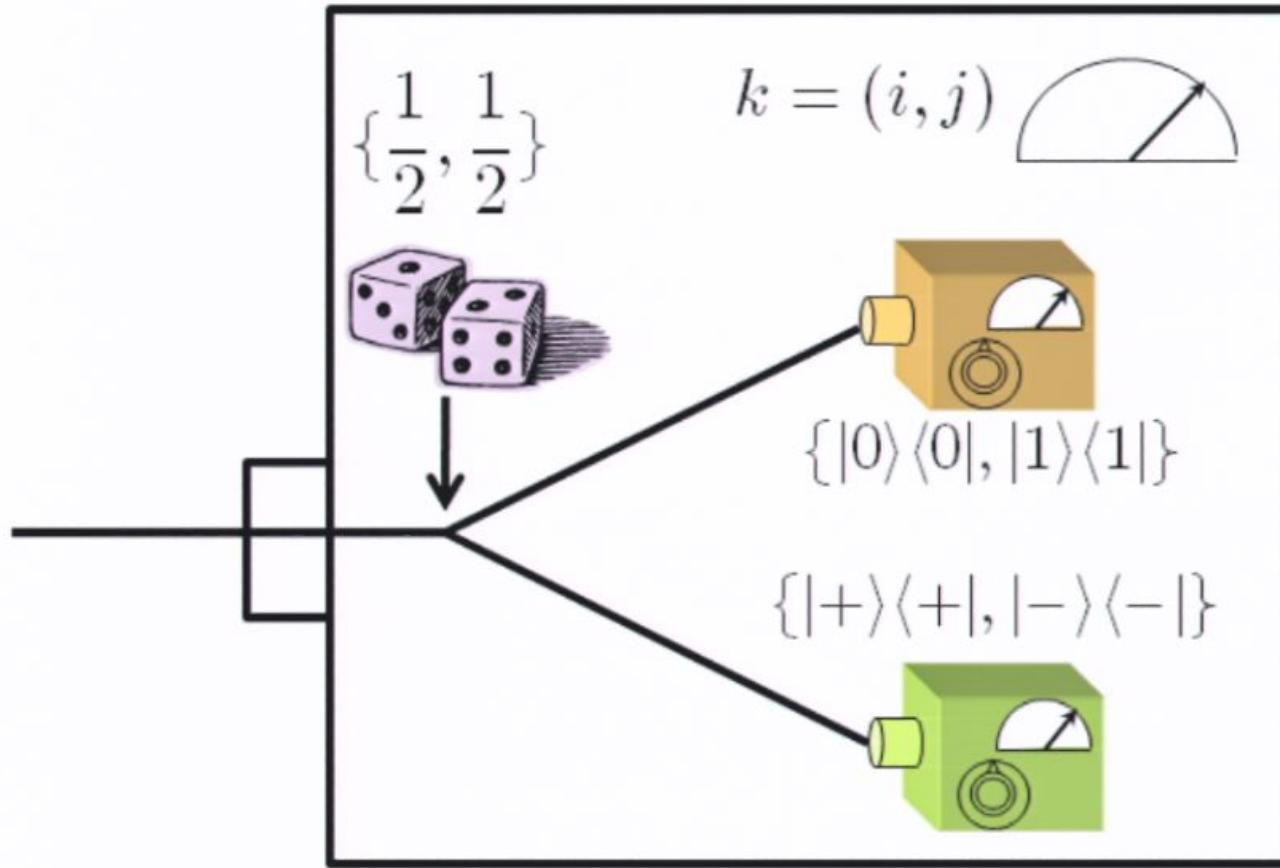
$$\begin{aligned}
 \rho & \xrightarrow{\text{Box}} \\
 \circ (i, j) &= p(j|i)p(i) \\
 &= \text{Tr}(\Pi_j^{(i)} \rho) p_i \\
 &= \text{Tr}(\underbrace{p_i \Pi_j^{(i)}}_{E_{i,j}} \rho)
 \end{aligned}$$

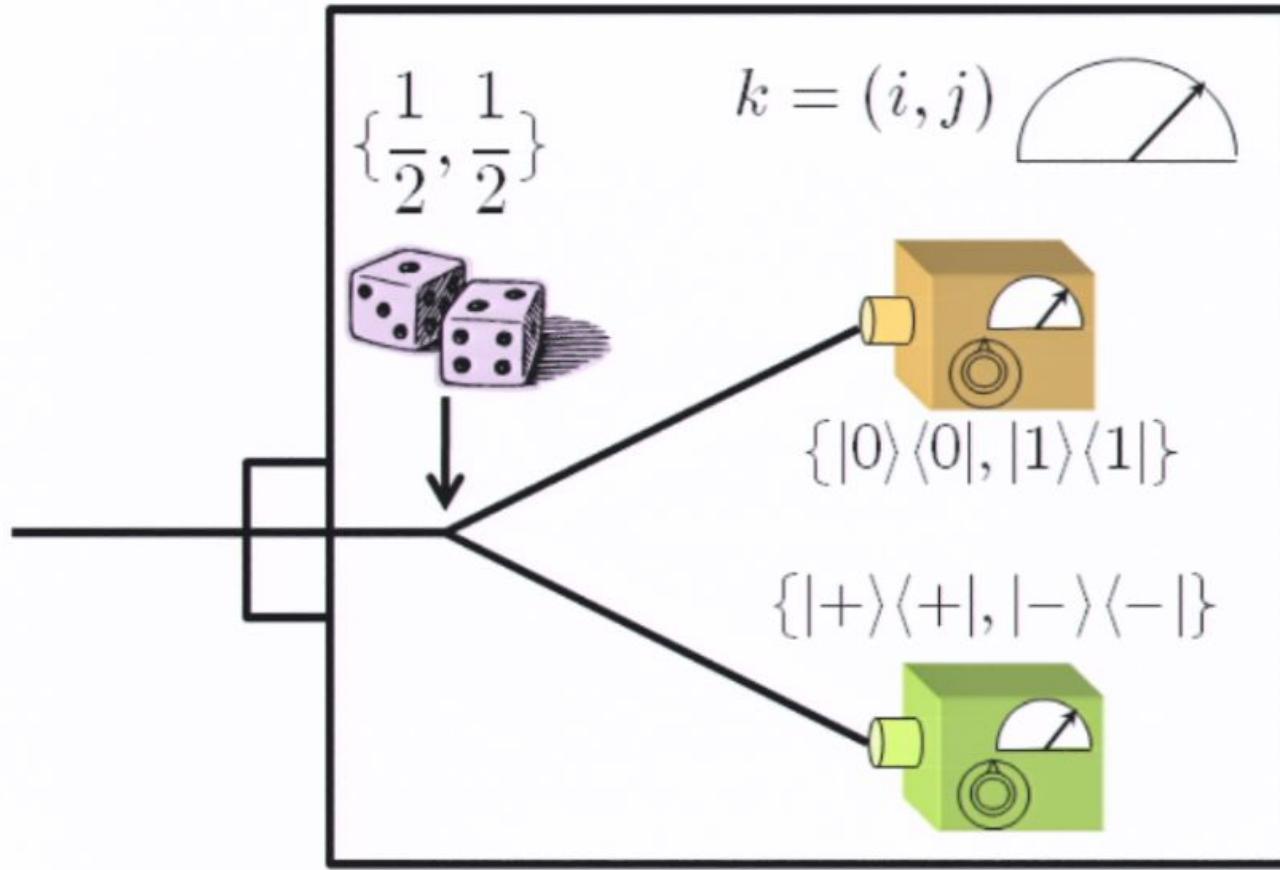


$$p(k) = \text{Tr}(E_k \rho)$$

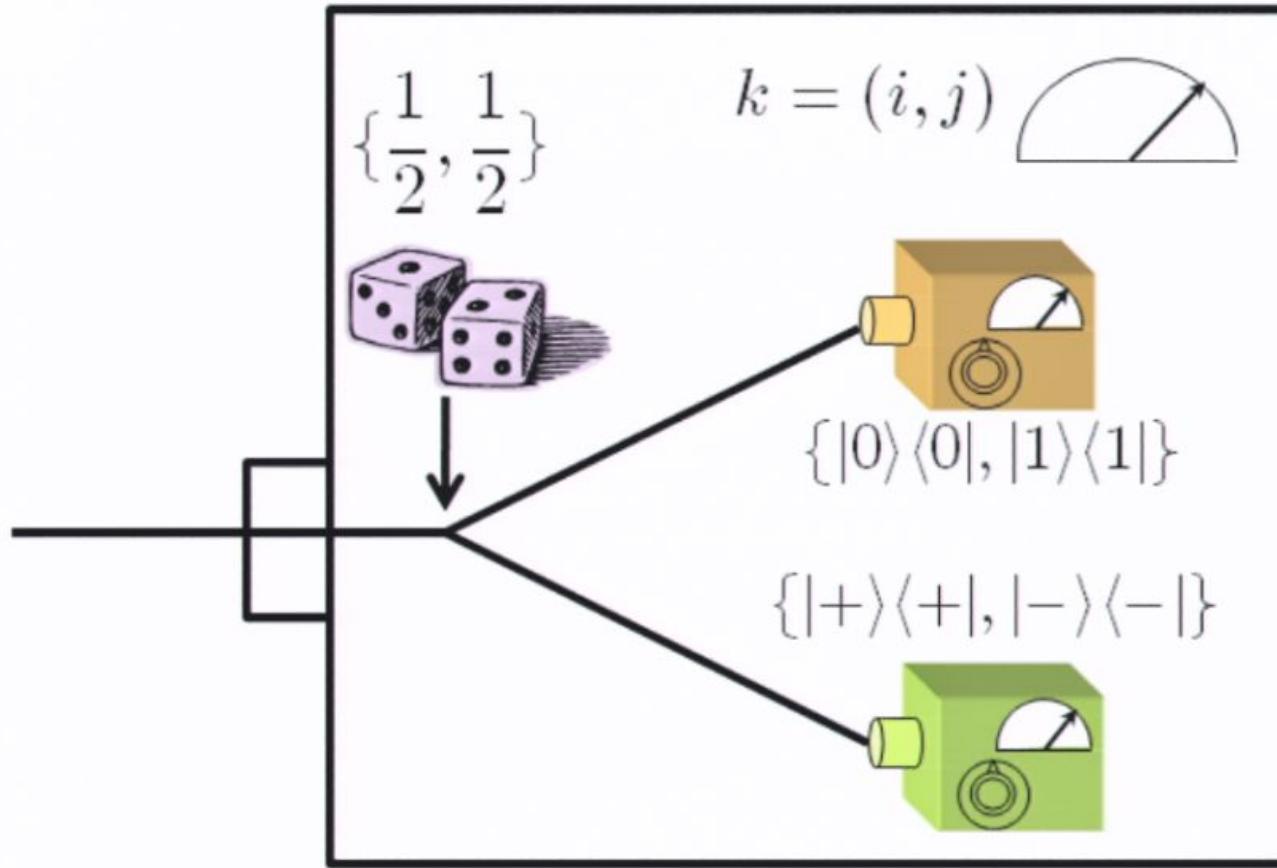
Positive  $\langle \psi | E_k | \psi \rangle \geq 0 \quad \forall |\psi\rangle \in \mathcal{H}$

Sum to identity  $\sum_k E_k = I$





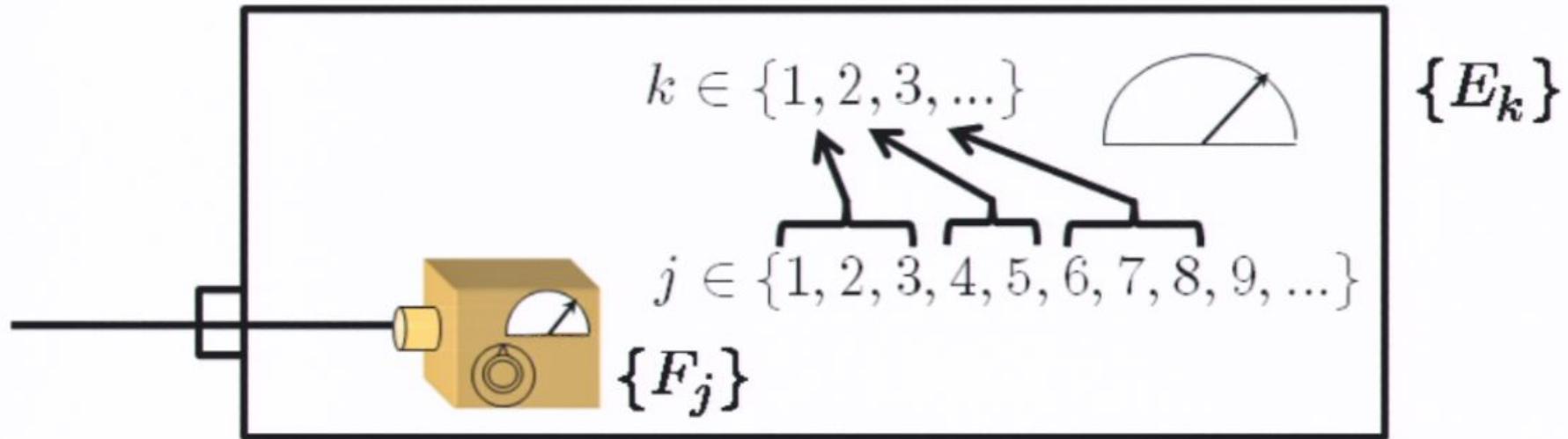
$$\left\{ \frac{1}{2}|0\rangle\langle 0|, \frac{1}{2}|1\rangle\langle 1|, \frac{1}{2}|+\rangle\langle +|, \frac{1}{2}|->\langle -|\right\}$$



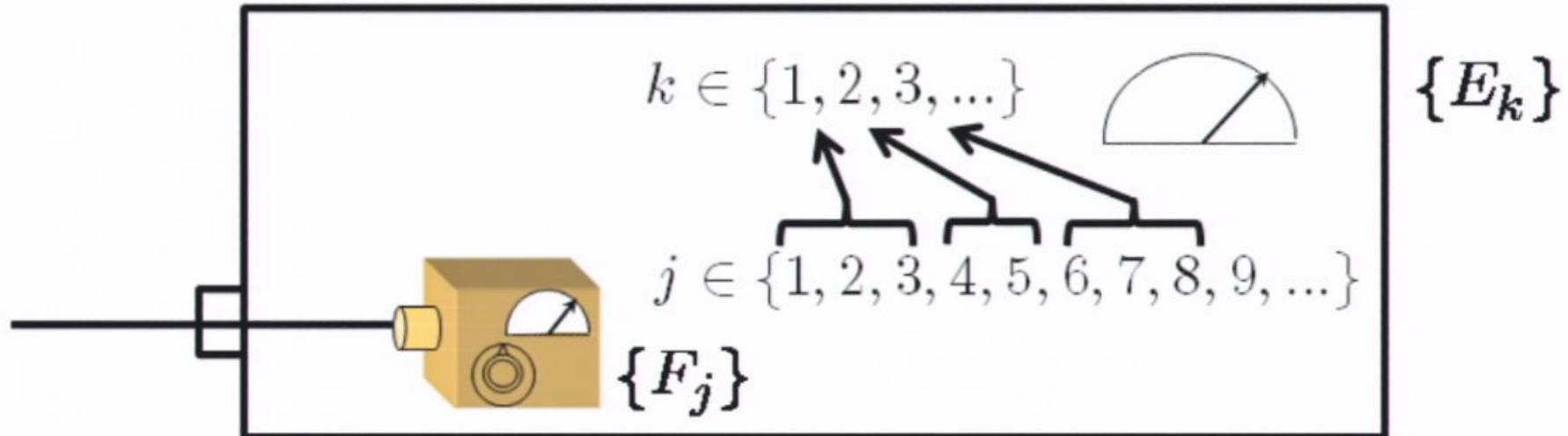
Recall

$$\frac{1}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1| + \frac{1}{4}|+\rangle\langle +| + \frac{1}{4}|-\rangle\langle -| = \frac{1}{2}I$$

## Coarse-graining

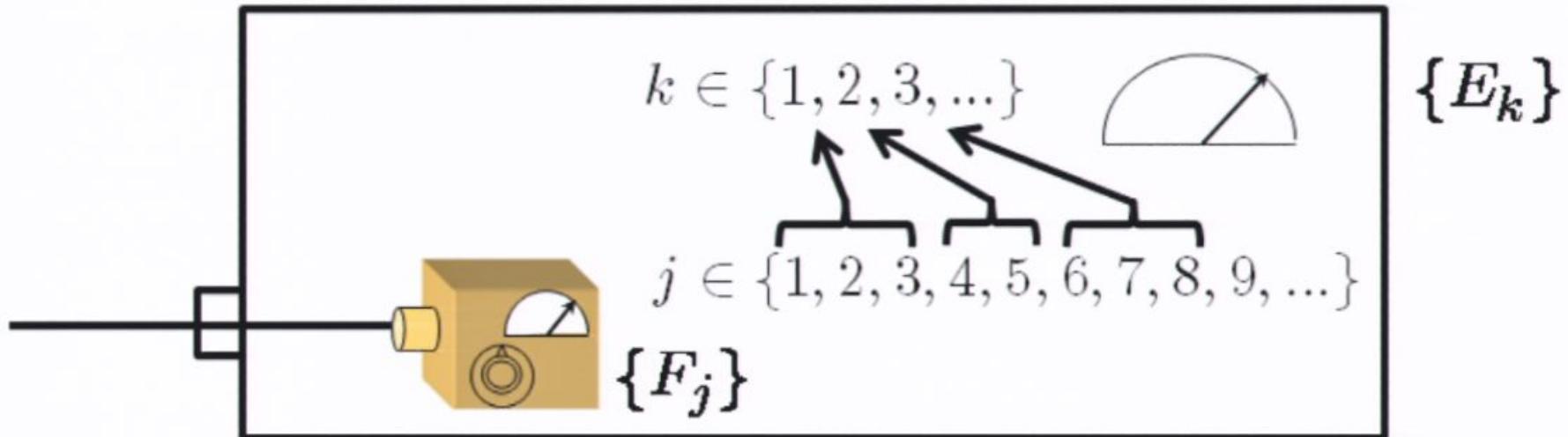


## Coarse-graining



$$p(k) = \sum_{j \in S_k} p(j)$$

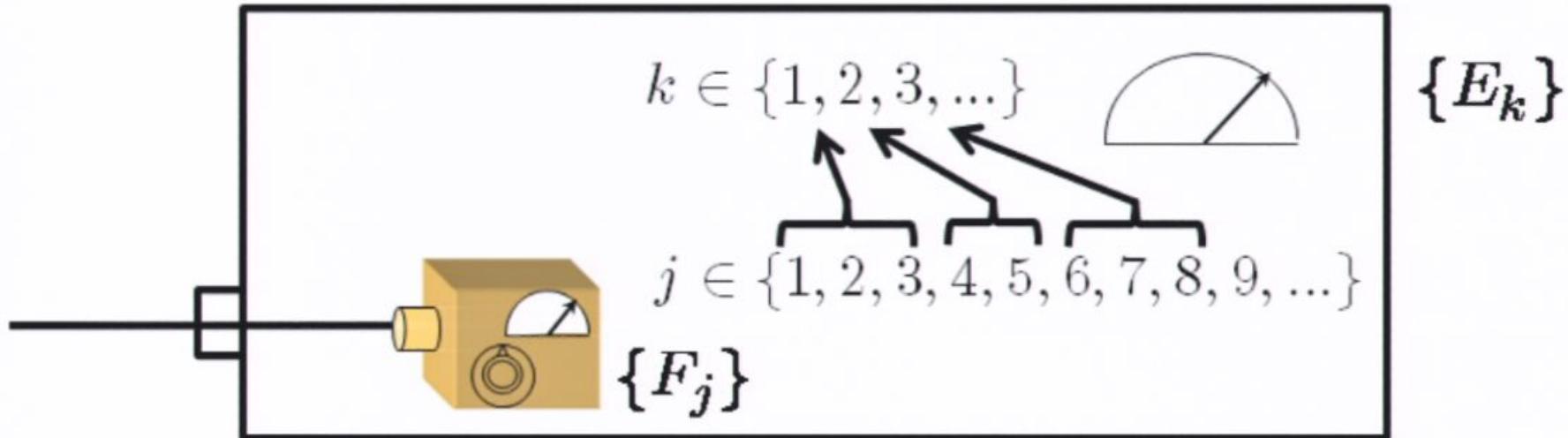
## Coarse-graining



$$p(k) = \sum_{j \in S_k} p(j)$$

$$\begin{aligned}\text{Tr}(E_k \rho) &= \sum_{j \in S_k} \text{Tr}(F_j \rho) \quad \forall \rho \\ &= \text{Tr}\left[\left(\sum_{j \in S_k} F_j\right) \rho\right] \quad \forall \rho\end{aligned}$$

## Coarse-graining



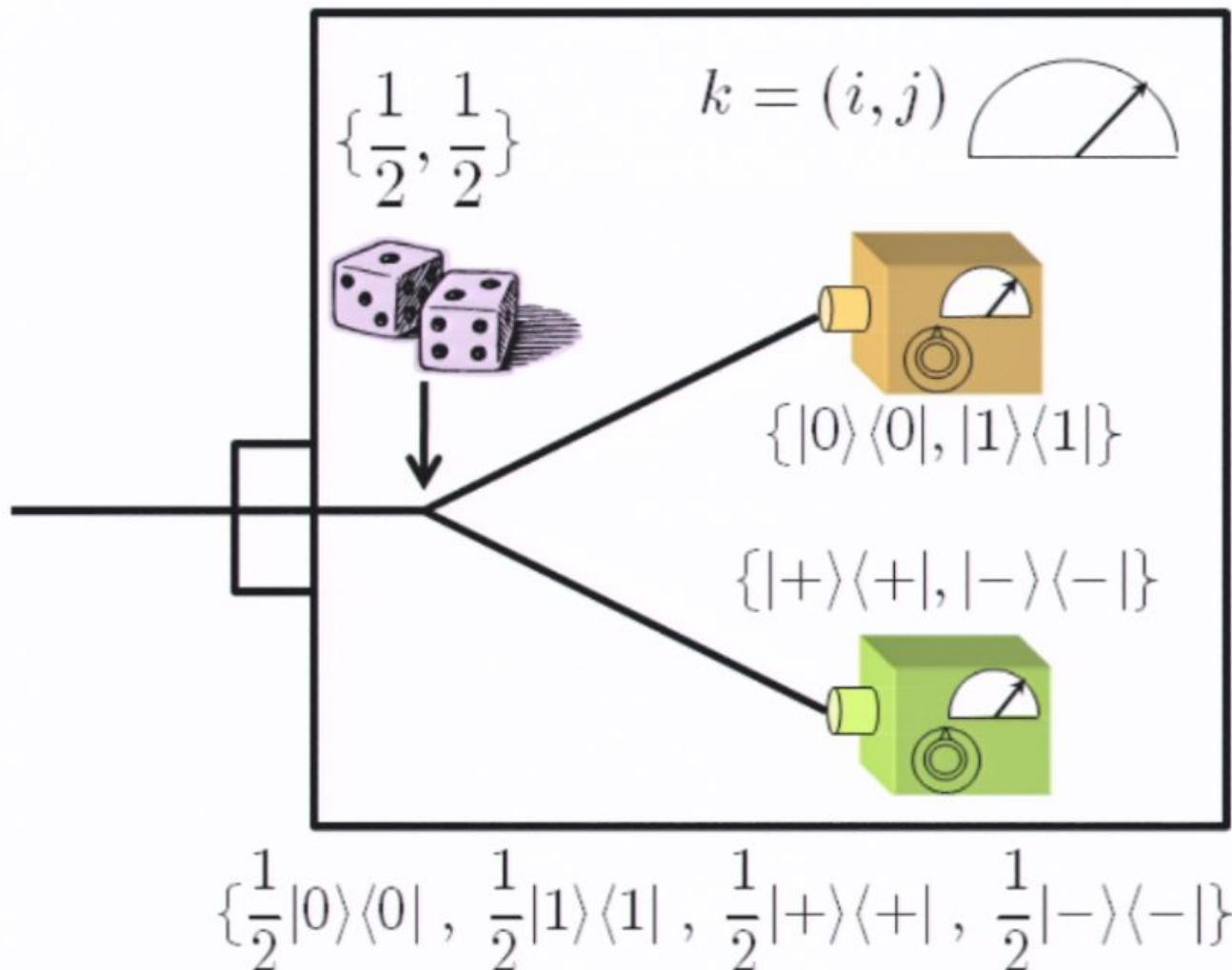
$$p(k) = \sum_{j \in S_k} p(j)$$

$$\begin{aligned}\text{Tr}(E_k \rho) &= \sum_{j \in S_k} \text{Tr}(F_j \rho) \quad \forall \rho \\ &= \text{Tr}\left[\left(\sum_{j \in S_k} F_j\right) \rho\right] \quad \forall \rho\end{aligned}$$

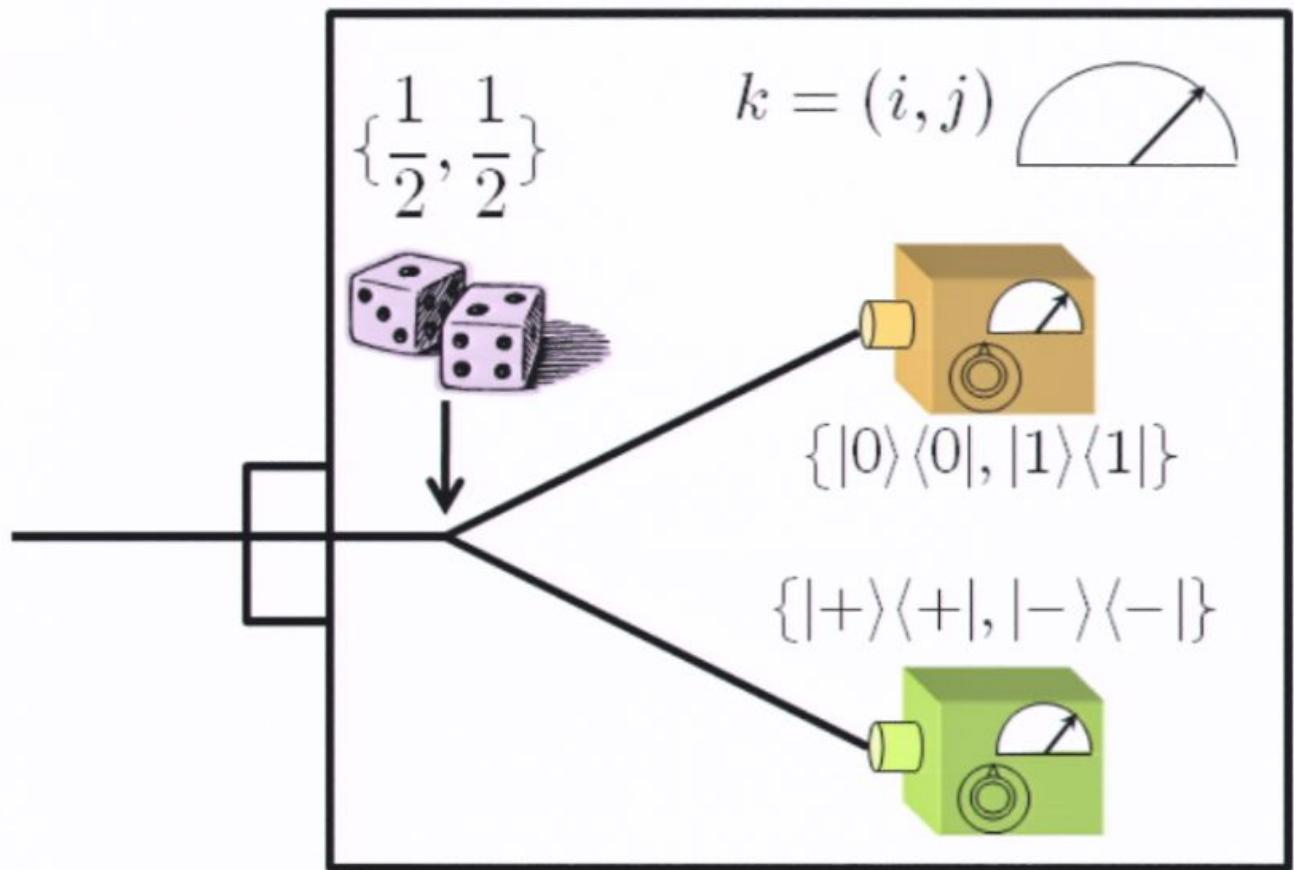
$$E_k = \sum_{j \in S_k} F_j$$

Note: the  $E_k$  need not be rank 1

## Example



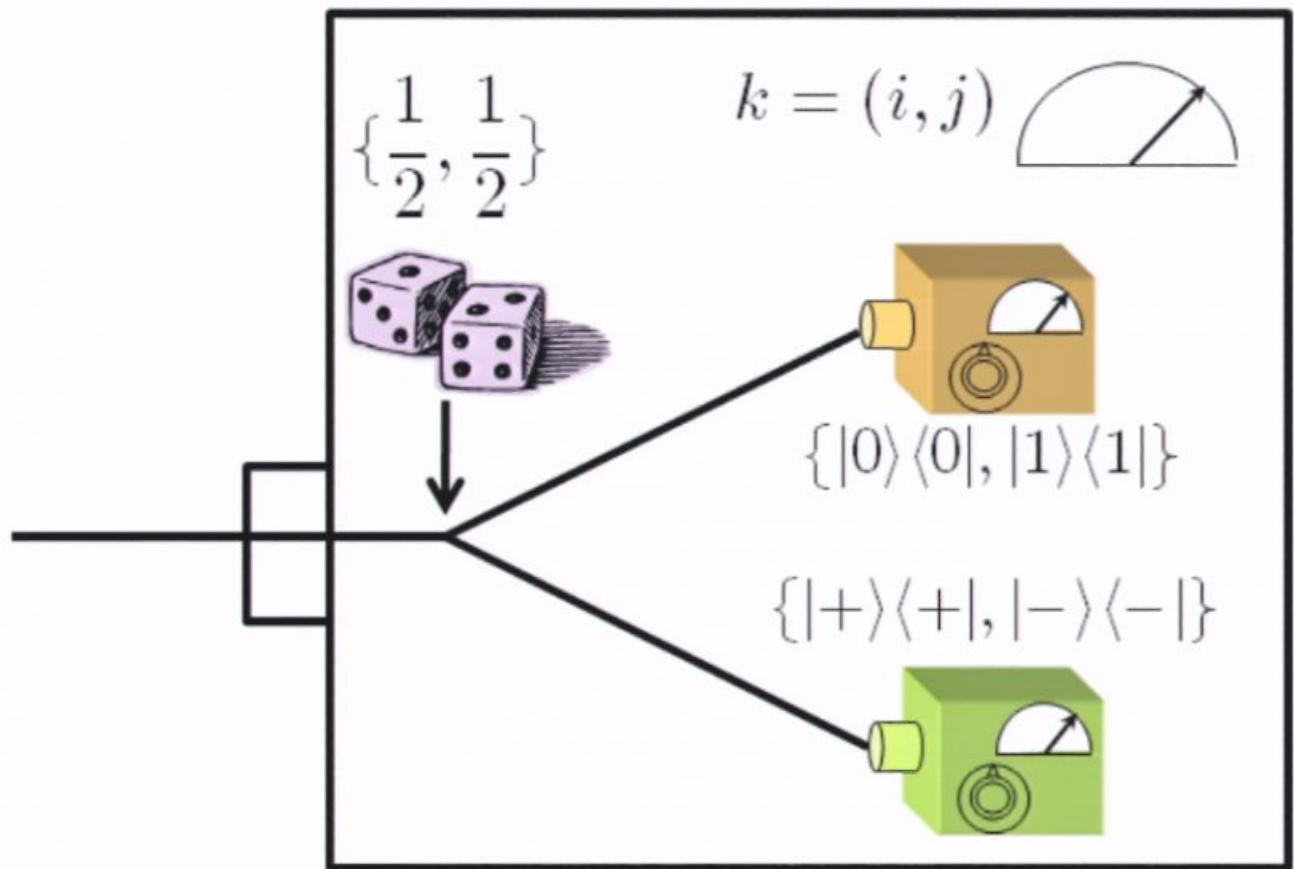
## Example



$$\left\{ \frac{1}{2}|0\rangle\langle 0|, \frac{1}{2}|1\rangle\langle 1|, \frac{1}{2}|+\rangle\langle +|, \frac{1}{2}|-\rangle\langle -| \right\}$$

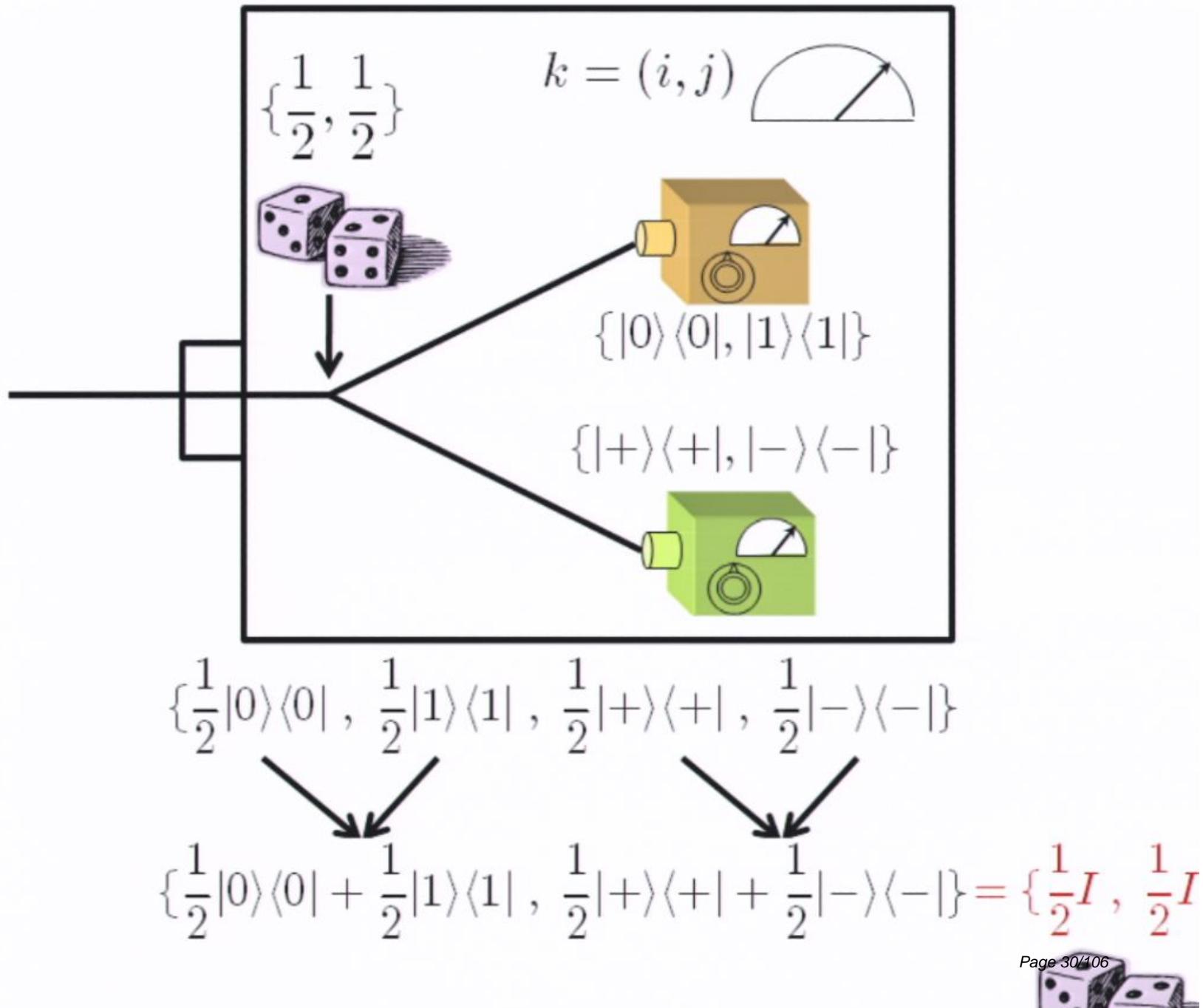
$$\left\{ \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|, \frac{1}{2}|+\rangle\langle +| + \frac{1}{2}|-\rangle\langle -| \right\}$$

## Example

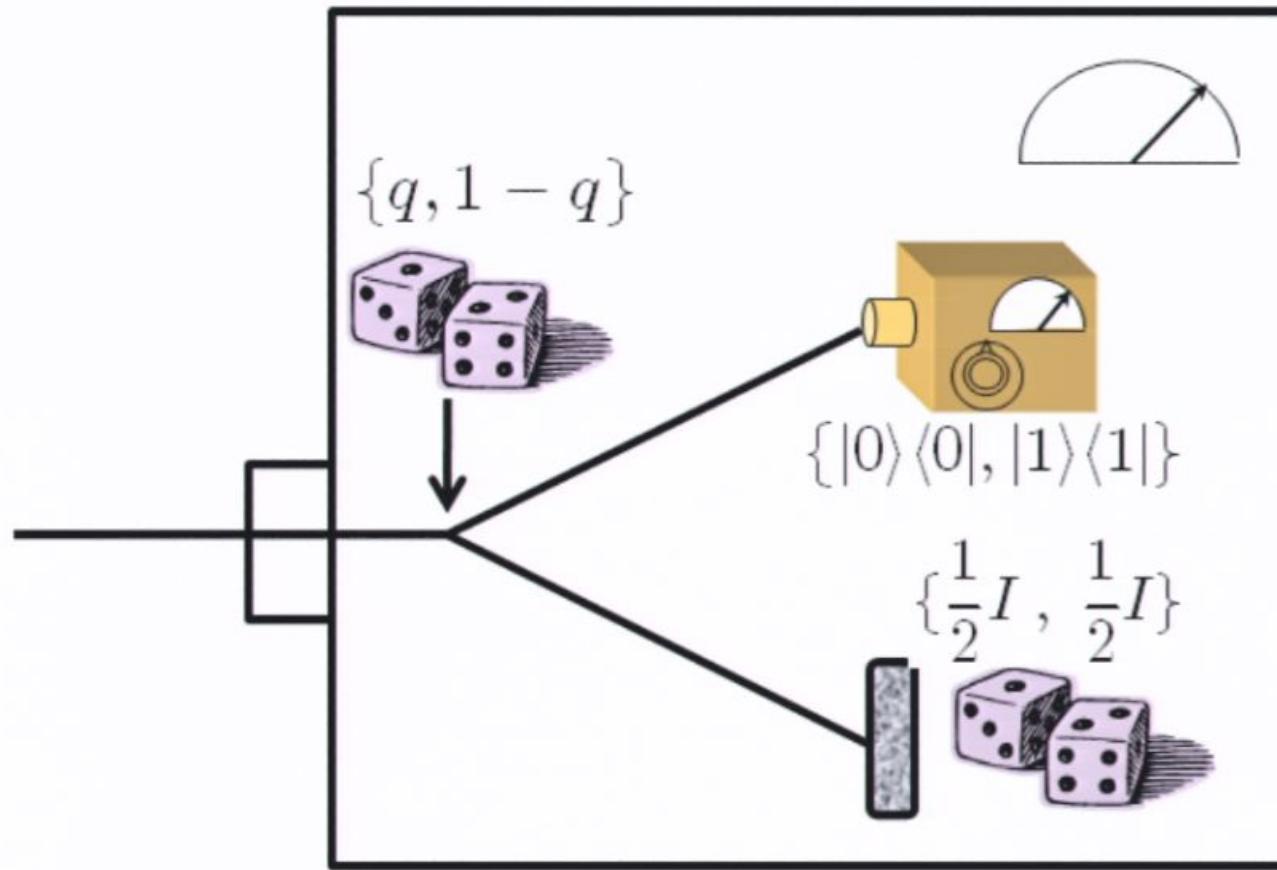


$$\left\{ \frac{1}{2}|0\rangle\langle 0|, \frac{1}{2}|1\rangle\langle 1|, \frac{1}{2}|+\rangle\langle +|, \frac{1}{2}|-\rangle\langle -| \right\}$$
$$\downarrow \qquad \qquad \downarrow$$
$$\left\{ \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|, \frac{1}{2}|+\rangle\langle +| + \frac{1}{2}|-\rangle\langle -| \right\}$$

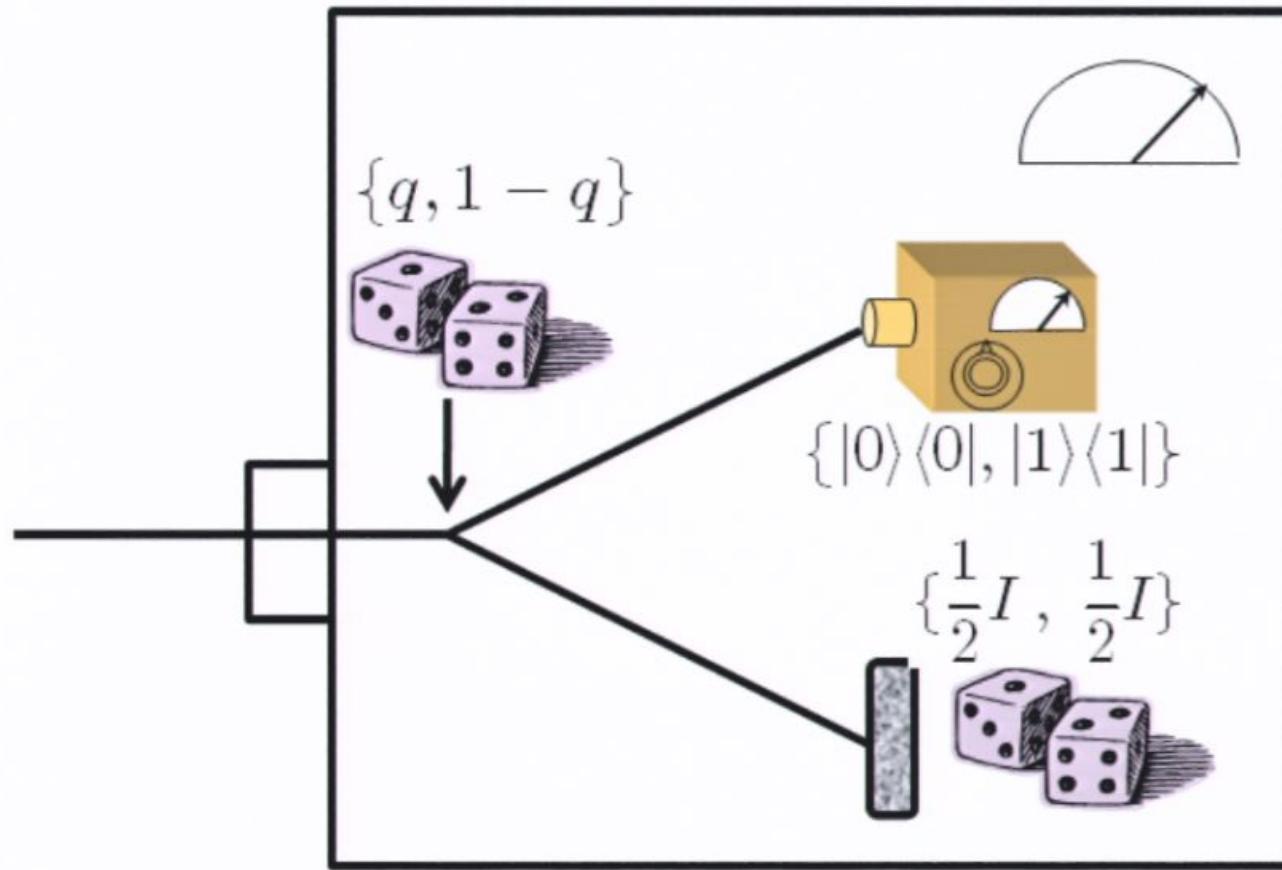
## Example



## Another example

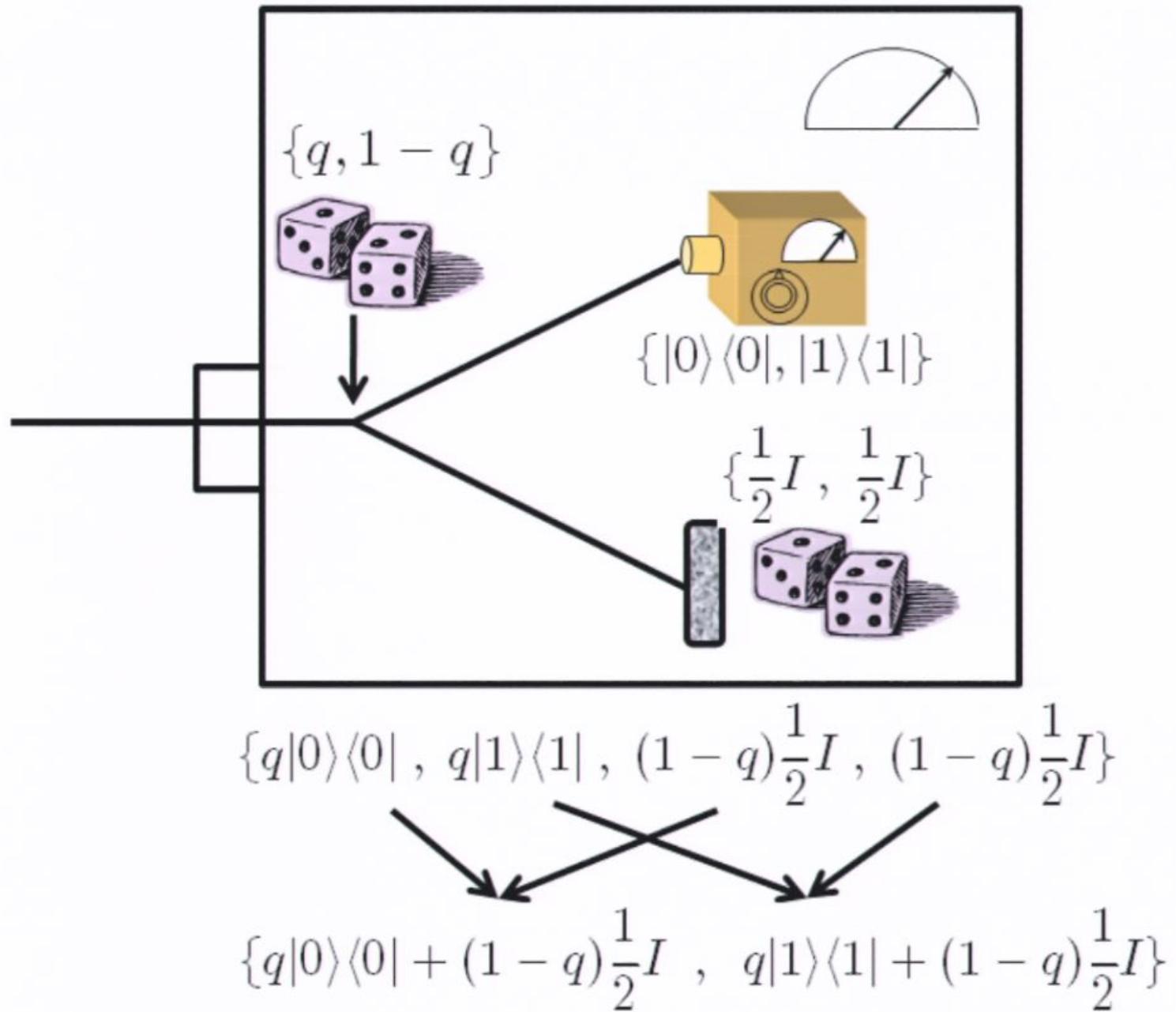


## Another example

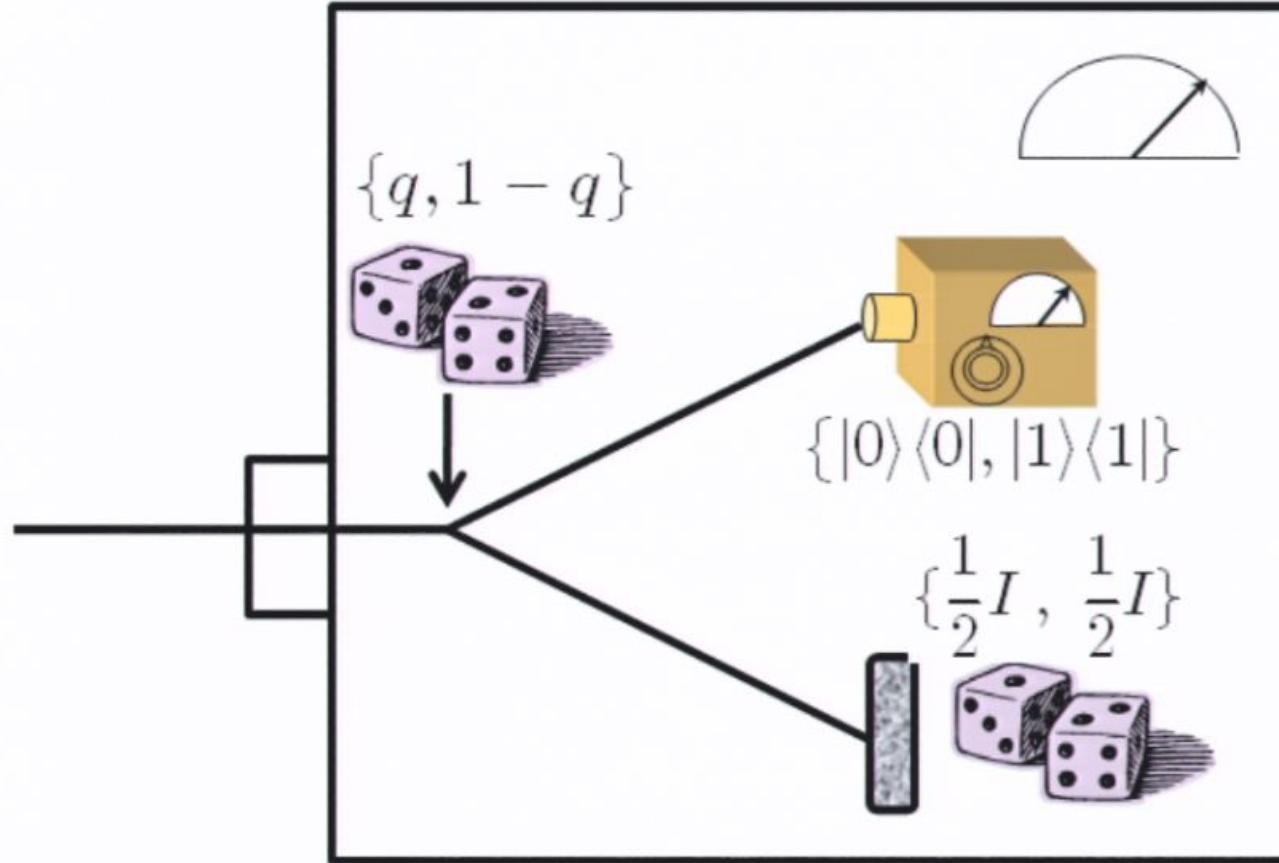


$$\{q|0\rangle\langle 0|, q|1\rangle\langle 1|, (1-q)\frac{1}{2}I, (1-q)\frac{1}{2}I\}$$

## Another example



## Another example

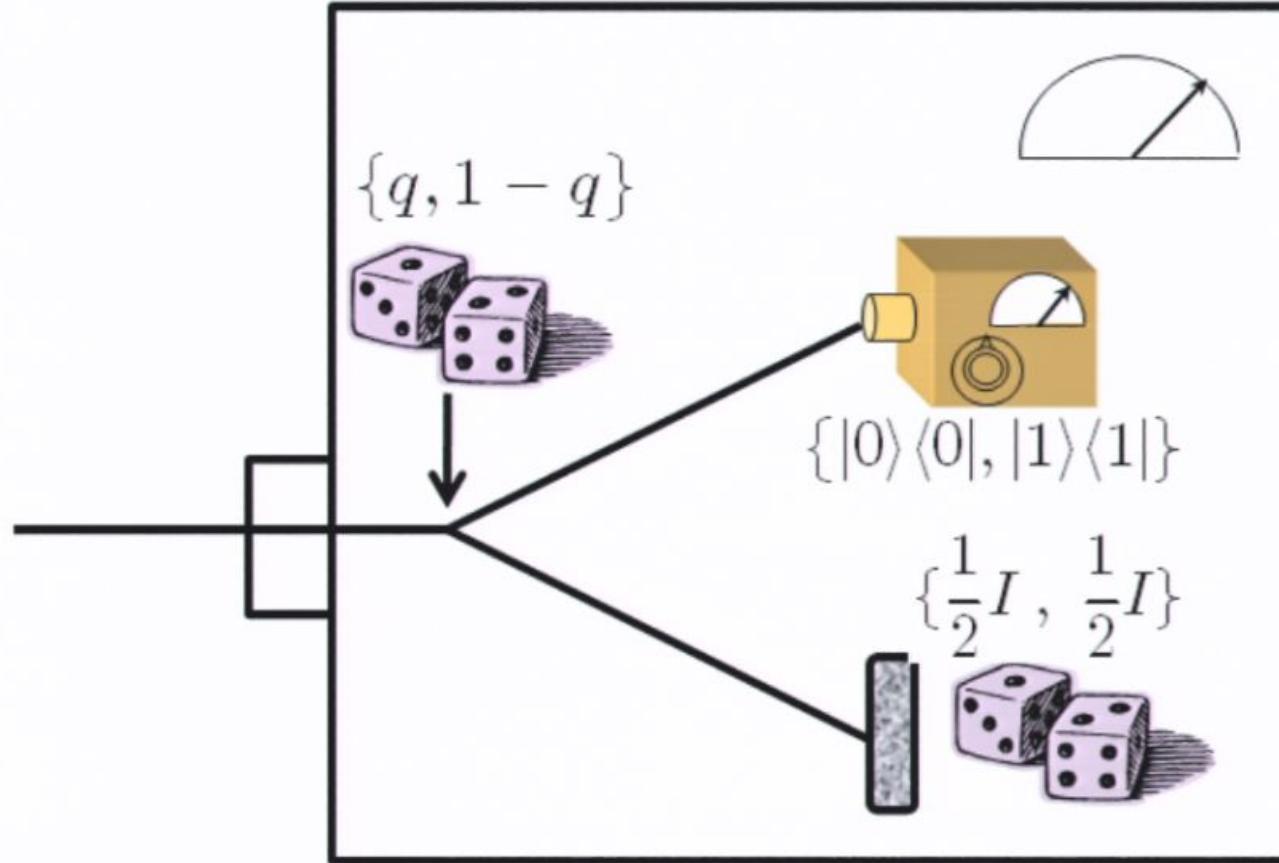


$$\{q|0\rangle\langle 0|, q|1\rangle\langle 1|, (1-q)\frac{1}{2}I, (1-q)\frac{1}{2}I\}$$

$$\{q|0\rangle\langle 0| + (1-q)\frac{1}{2}I, q|1\rangle\langle 1| + (1-q)\frac{1}{2}I\}$$

$$= \left\{ \frac{1+q}{2}|0\rangle\langle 0| + \frac{1-q}{2}|1\rangle\langle 1|, \frac{1-q}{2}|0\rangle\langle 0| + \frac{1+q}{2}|1\rangle\langle 1| \right\}$$

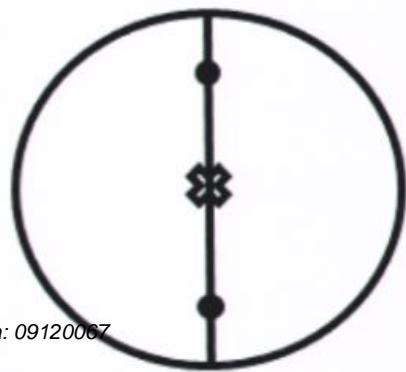
## Another example



$$\{q|\psi\rangle\langle\psi|, q|\phi\rangle\langle\phi|, (1-q)\frac{1}{2}I, (1-q)\frac{1}{2}I\}$$

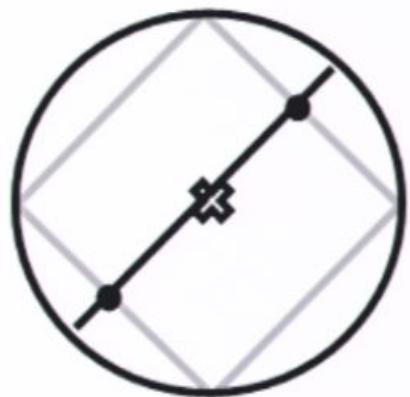
$$\begin{aligned} & \xrightarrow{\quad} \cancel{\quad} \xrightarrow{\quad} \\ & \{q|\psi\rangle\langle\psi| + (1-q)\frac{1}{2}I, q|\phi\rangle\langle\phi| + (1-q)\frac{1}{2}I\} \end{aligned}$$

$$= \left\{ \frac{1+q}{2}|\psi\rangle\langle\psi| + \frac{1-q}{2}|\phi\rangle\langle\phi|, \frac{1-q}{2}|\psi\rangle\langle\psi| + \frac{1+q}{2}|\phi\rangle\langle\phi| \right\}$$





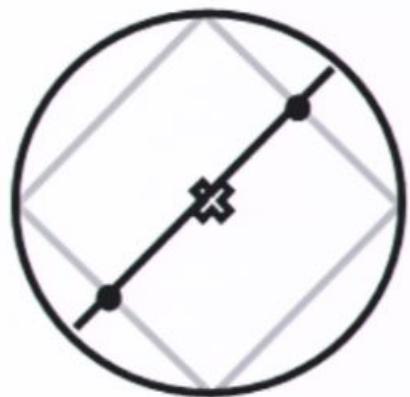
$$\left\{ \frac{1}{2}|0\rangle\langle 0|, \frac{1}{2}|1\rangle\langle 1|, \frac{1}{2}|+\rangle\langle +|, \frac{1}{2}|-\rangle\langle -| \right\}$$



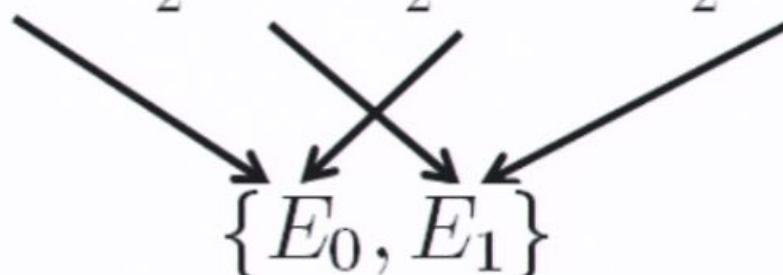
$$\left\{ \frac{1}{2}|0\rangle\langle 0|, \frac{1}{2}|1\rangle\langle 1|, \frac{1}{2}|+\rangle\langle +|, \frac{1}{2}|-\rangle\langle -| \right\}$$

$\searrow$        $\nearrow$        $\swarrow$        $\nwarrow$

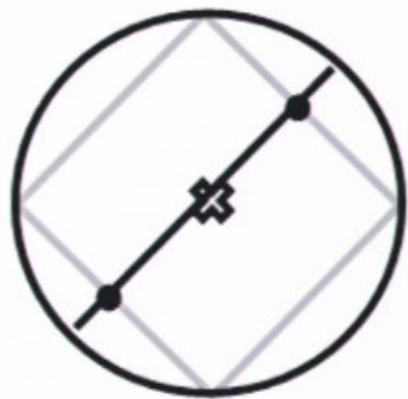
$$\{E_0, E_1\}$$



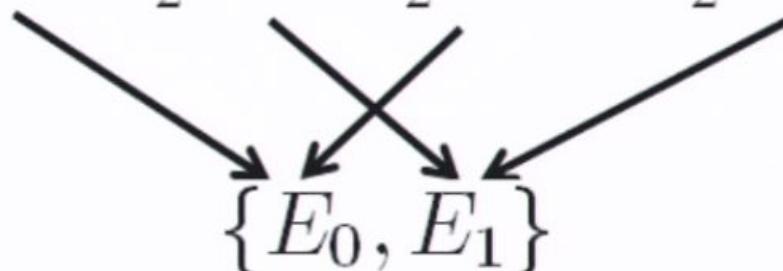
$$\left\{ \frac{1}{2}|0\rangle\langle 0|, \frac{1}{2}|1\rangle\langle 1|, \frac{1}{2}|+\rangle\langle +|, \frac{1}{2}|-\rangle\langle -| \right\}$$



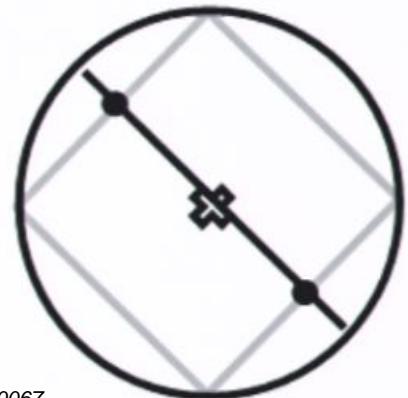
Noisy S·n



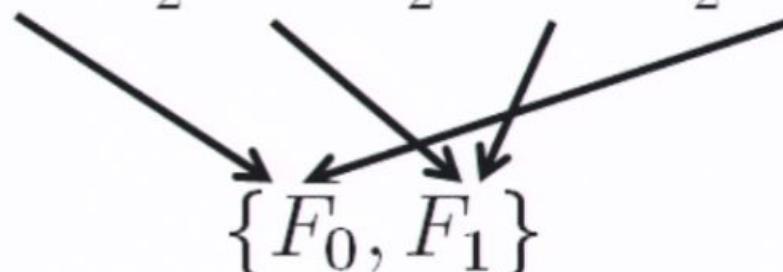
$$\left\{ \frac{1}{2}|0\rangle\langle 0|, \frac{1}{2}|1\rangle\langle 1|, \frac{1}{2}|+\rangle\langle +|, \frac{1}{2}|-\rangle\langle -| \right\}$$



Noisy S·n

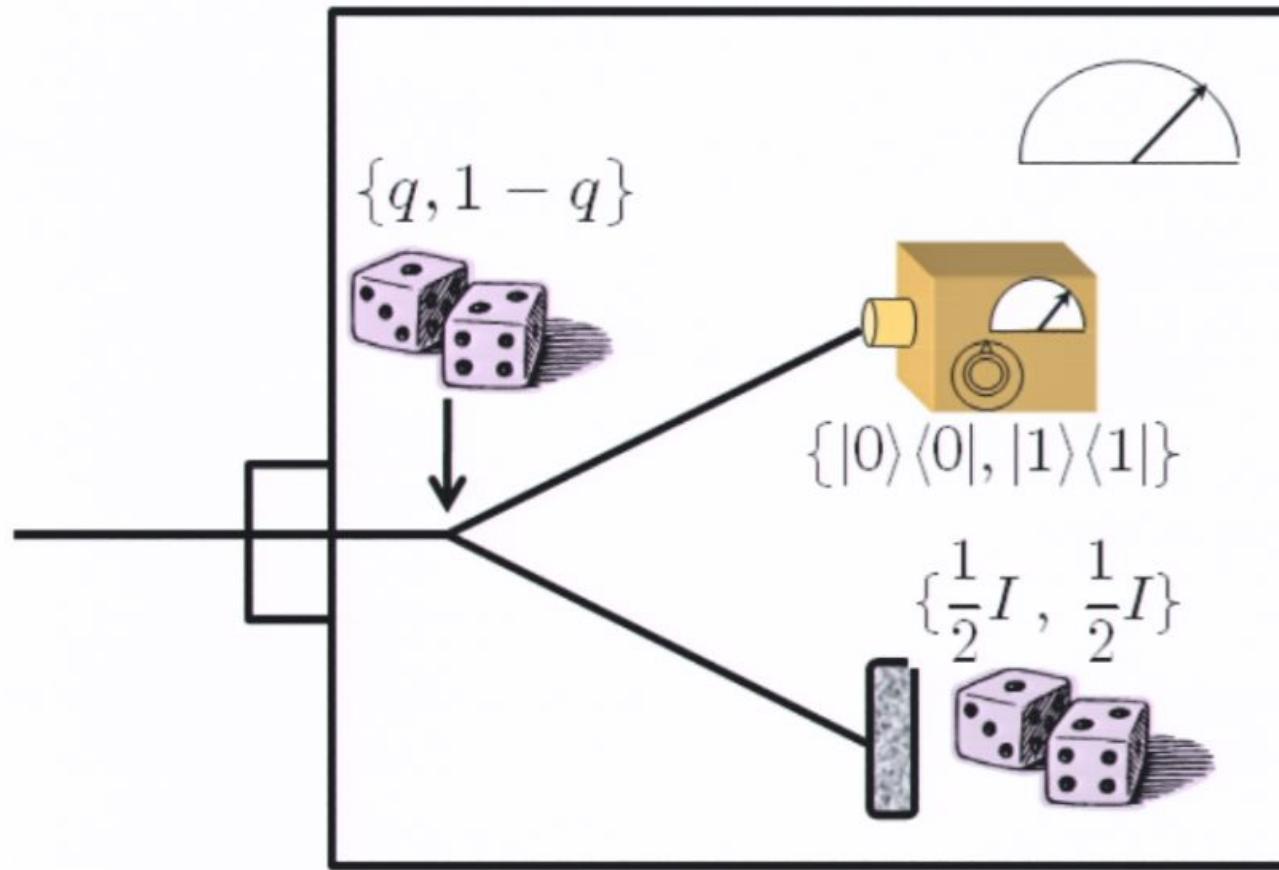


$$\left\{ \frac{1}{2}|0\rangle\langle 0|, \frac{1}{2}|1\rangle\langle 1|, \frac{1}{2}|+\rangle\langle +|, \frac{1}{2}|-\rangle\langle -| \right\}$$



Noisy S·n<sup>⊥</sup>

## Another example

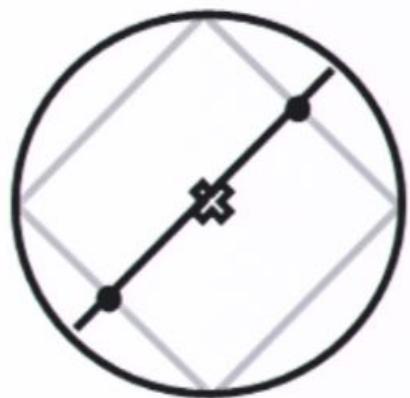


$$\{q|0\rangle\langle 0|, q|1\rangle\langle 1|, (1-q)\frac{1}{2}I, (1-q)\frac{1}{2}I\}$$

$$\{q|0\rangle\langle 0| + (1-q)\frac{1}{2}I, q|1\rangle\langle 1| + (1-q)\frac{1}{2}I\}$$

$$= \left\{ \frac{1+q}{2}|0\rangle\langle 0| + \frac{1-q}{2}|1\rangle\langle 1|, \frac{1-q}{2}|0\rangle\langle 0| + \frac{1+q}{2}|1\rangle\langle 1| \right\}$$

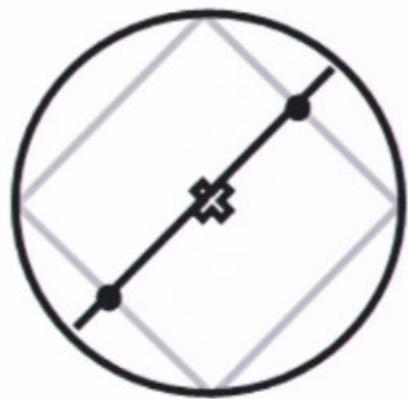




$$\left\{ \frac{1}{2}|0\rangle\langle 0|, \frac{1}{2}|1\rangle\langle 1|, \frac{1}{2}|+\rangle\langle +|, \frac{1}{2}|-\rangle\langle -| \right\}$$

$\swarrow$        $\nearrow$        $\searrow$        $\nwarrow$

$$\{E_0, E_1\}$$

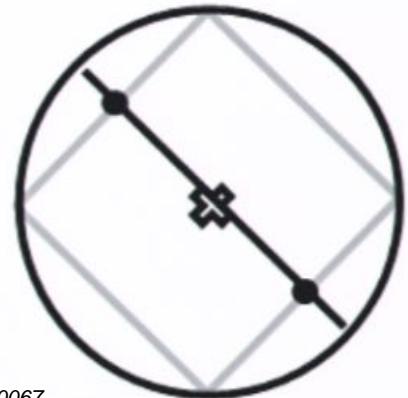


$\{\frac{1}{2}|0\rangle\langle 0|, \frac{1}{2}|1\rangle\langle 1|, \frac{1}{2}|+\rangle\langle +|, \frac{1}{2}|-\rangle\langle -|\}$

↓  
↓  
↓

$\{E_0, E_1\}$

Noisy S·n



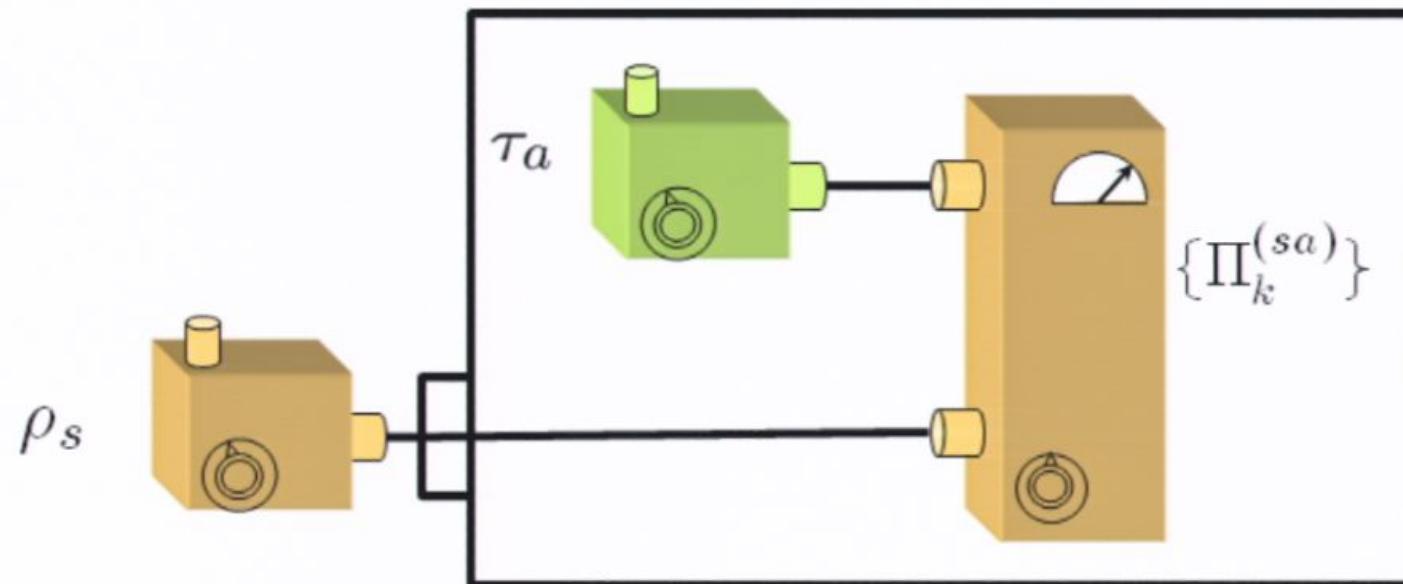
$\{\frac{1}{2}|0\rangle\langle 0|, \frac{1}{2}|1\rangle\langle 1|, \frac{1}{2}|+\rangle\langle +|, \frac{1}{2}|-\rangle\langle -|\}$

↓  
↓  
↓

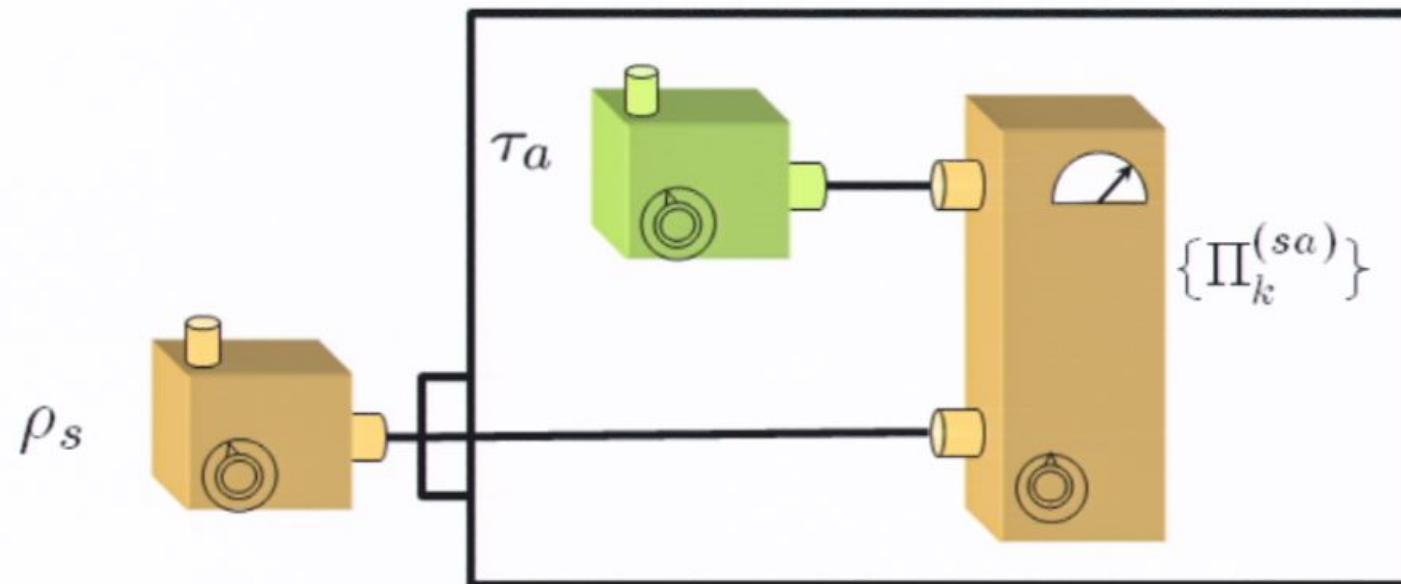
$\{F_0, F_1\}$

Noisy S·n $^\perp$

## Measurement by coupling to an ancilla

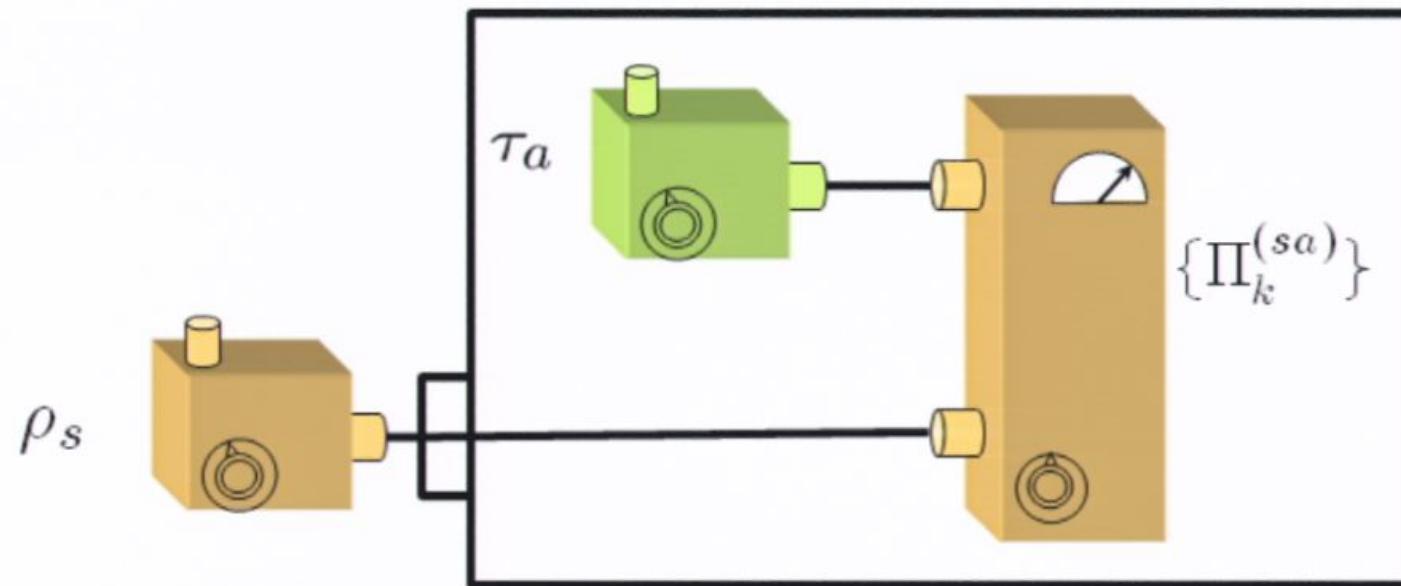


## Measurement by coupling to an ancilla



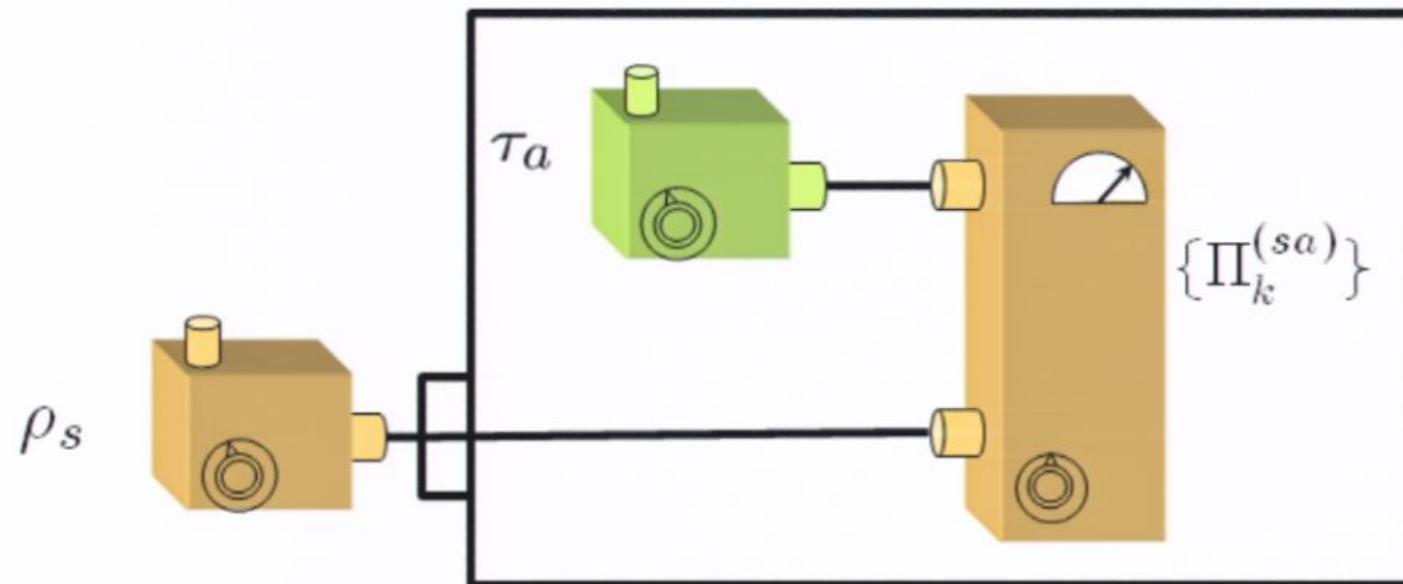
$$\begin{aligned} p(k) &= \text{Tr}_{sa}[\Pi_k^{(sa)}(\rho_s \otimes \tau_a)] \\ &= \text{Tr}_s[\text{Tr}_a(\Pi_k^{(sa)} \tau_a) \rho_s] \end{aligned}$$

## Measurement by coupling to an ancilla



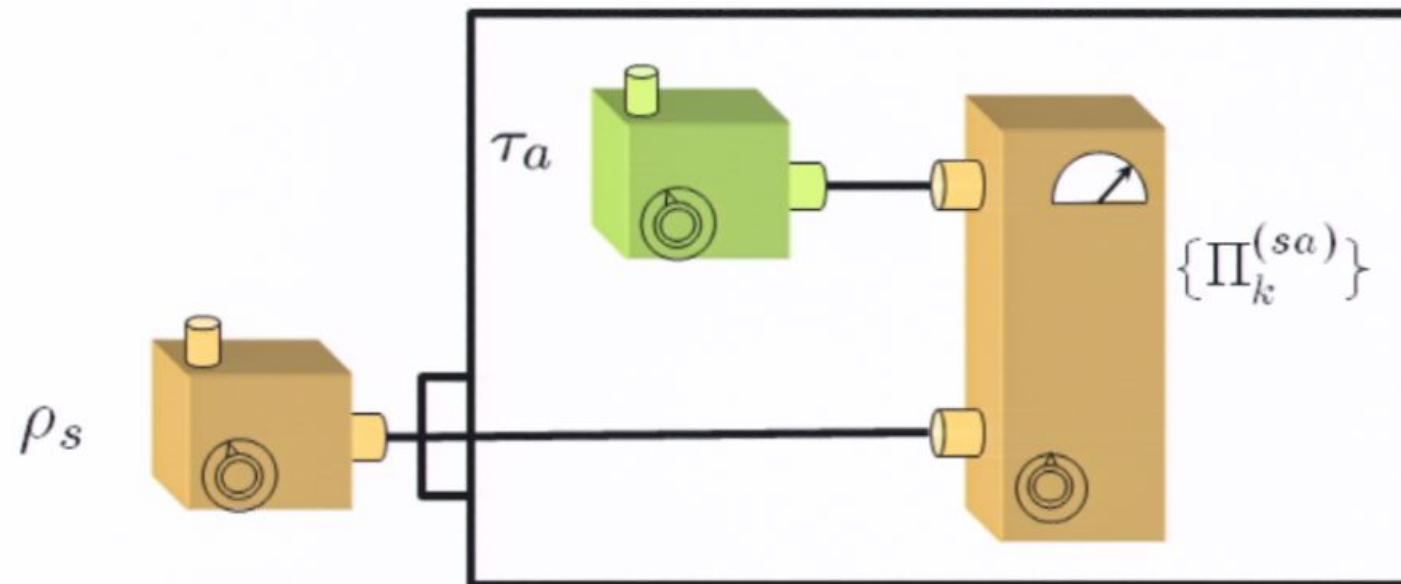
$$\begin{aligned} p(k) &= \text{Tr}_{sa}[\Pi_k^{(sa)}(\rho_s \otimes \tau_a)] \\ &= \text{Tr}_s[\underbrace{\text{Tr}_a(\Pi_k^{(sa)} \tau_a)}_{E_k^{(s)}} \rho_s] \end{aligned}$$

## Measurement by coupling to an ancilla



$$\begin{aligned} p(k) &= \text{Tr}_{sa}[\Pi_k^{(sa)}(\rho_s \otimes \tau_a)] \\ &= \underbrace{\text{Tr}_s[\text{Tr}_a(\Pi_k^{(sa)} \tau_a) \rho_s]}_{E_k^{(s)}} \quad p(k) = \text{Tr}_s(E_k^{(s)} \rho_s) \end{aligned}$$

## Measurement by coupling to an ancilla



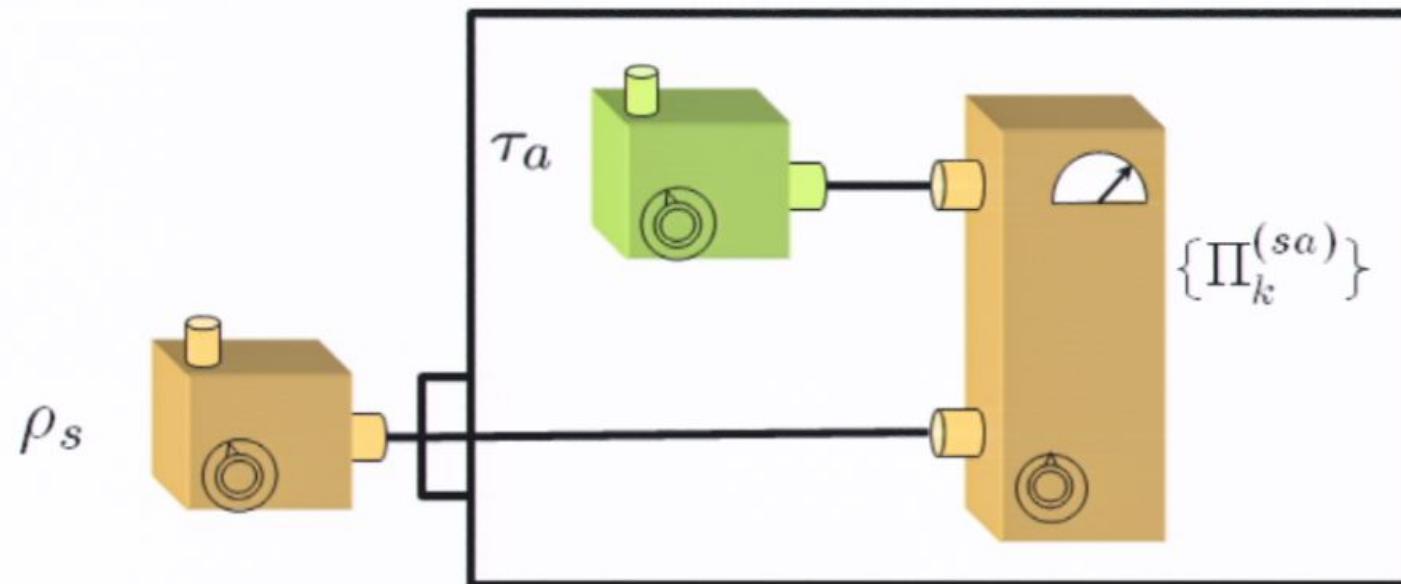
$$p(k) = \text{Tr}_{sa}[\Pi_k^{(sa)}(\rho_s \otimes \tau_a)]$$

$$= \text{Tr}_s[\underbrace{\text{Tr}_a(\Pi_k^{(sa)} \tau_a)}_{E_k^{(s)}} \rho_s]$$

$$p(k) = \text{Tr}_s(E_k^{(s)} \rho_s)$$

Positive  $\langle \psi | E_k^{(s)} | \psi \rangle \geq 0 \quad \forall |\psi\rangle \in \mathcal{V}$

## Measurement by coupling to an ancilla



$$p(k) = \text{Tr}_{sa}[\Pi_k^{(sa)}(\rho_s \otimes \tau_a)]$$

$$= \text{Tr}_s[\underbrace{\text{Tr}_a(\Pi_k^{(sa)} \tau_a)}_{E_k^{(s)}} \rho_s]$$

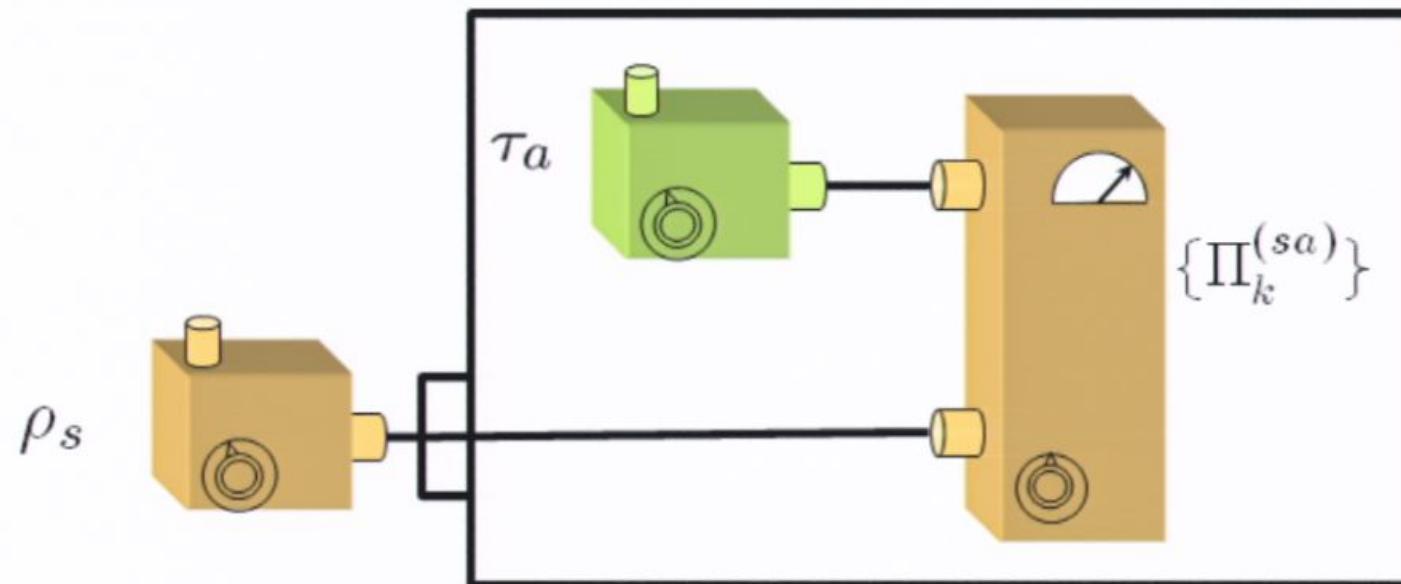
$$p(k) = \text{Tr}_s(E_k^{(s)} \rho_s)$$

Positive  $\langle \psi | E_k^{(s)} | \psi \rangle \geq 0 \quad \forall |\psi\rangle \in \mathcal{V}$   
 Sum to identity  $\sum_k E_k^{(s)} = I_s$

$$\sum_{i \neq j} p_i \pi_j^{(i)} = \sum_i p_i 1 = 1$$

$$1_{S\alpha} = 1_S \otimes 1_\alpha$$

## Measurement by coupling to an ancilla



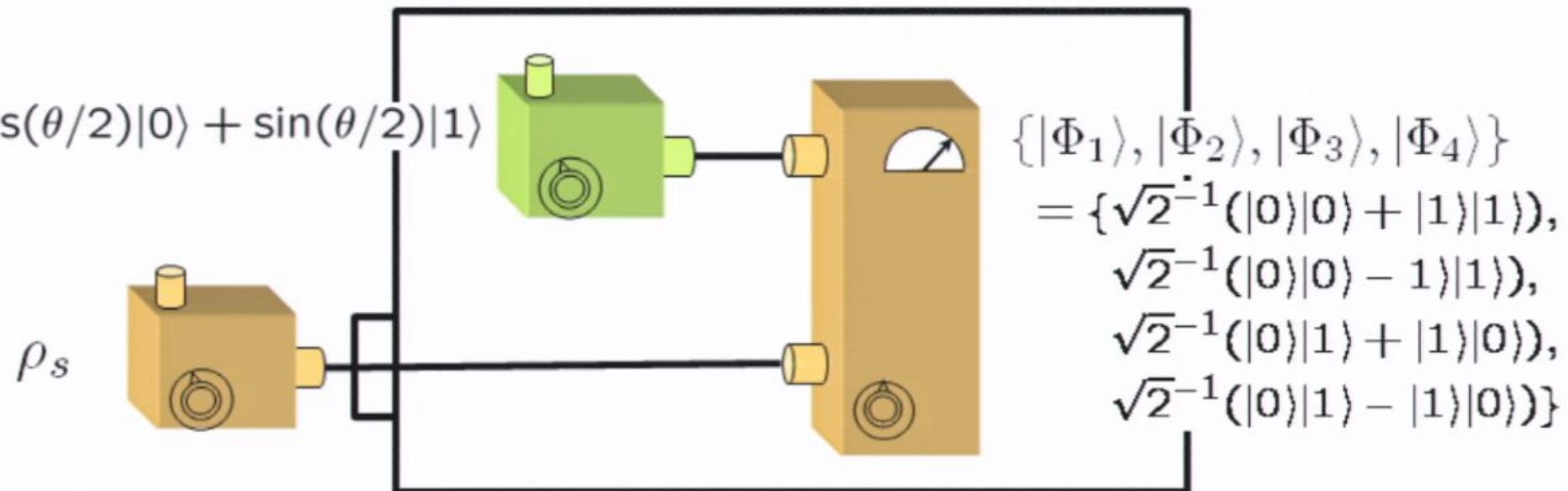
$$p(k) = \text{Tr}_{sa}[\Pi_k^{(sa)}(\rho_s \otimes \tau_a)]$$

$$= \text{Tr}_s[\underbrace{\text{Tr}_a(\Pi_k^{(sa)} \tau_a)}_{E_k^{(s)}} \rho_s]$$

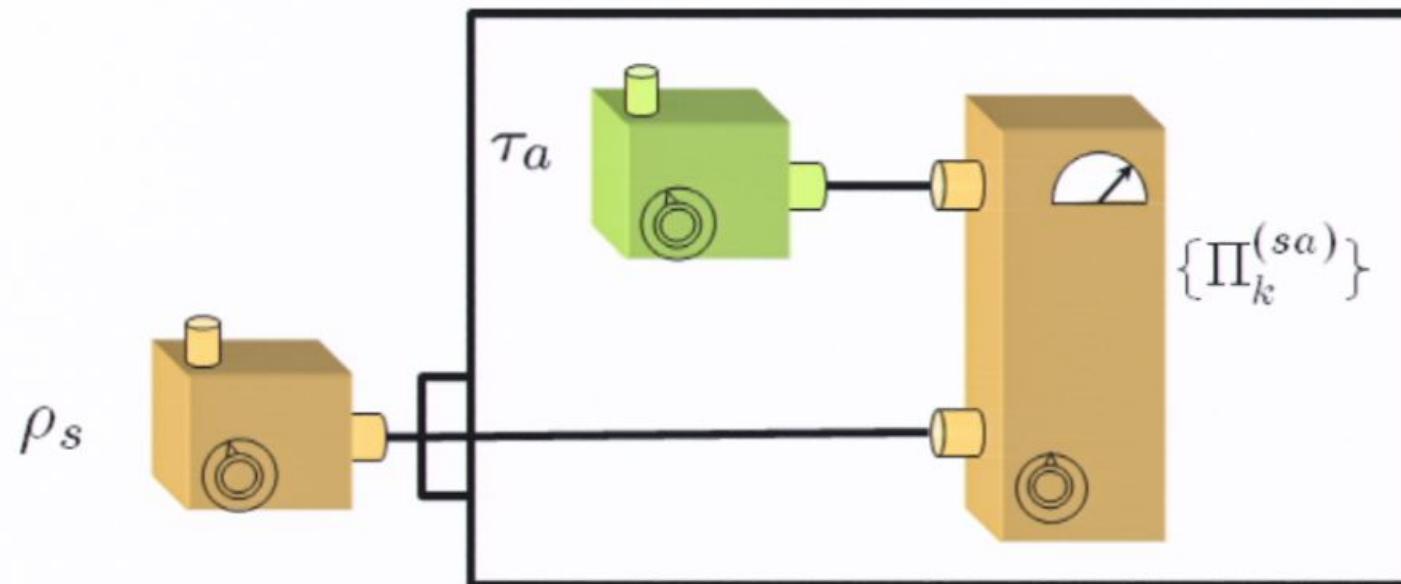
$$p(k) = \text{Tr}_s(E_k^{(s)} \rho_s)$$

Positive  $\langle \psi | E_k^{(s)} | \psi \rangle \geq 0 \quad \forall |\psi\rangle \in \mathcal{V}$   
Sum to identity  $\sum_k E_k^{(s)} = I_s$

## Example



## Measurement by coupling to an ancilla



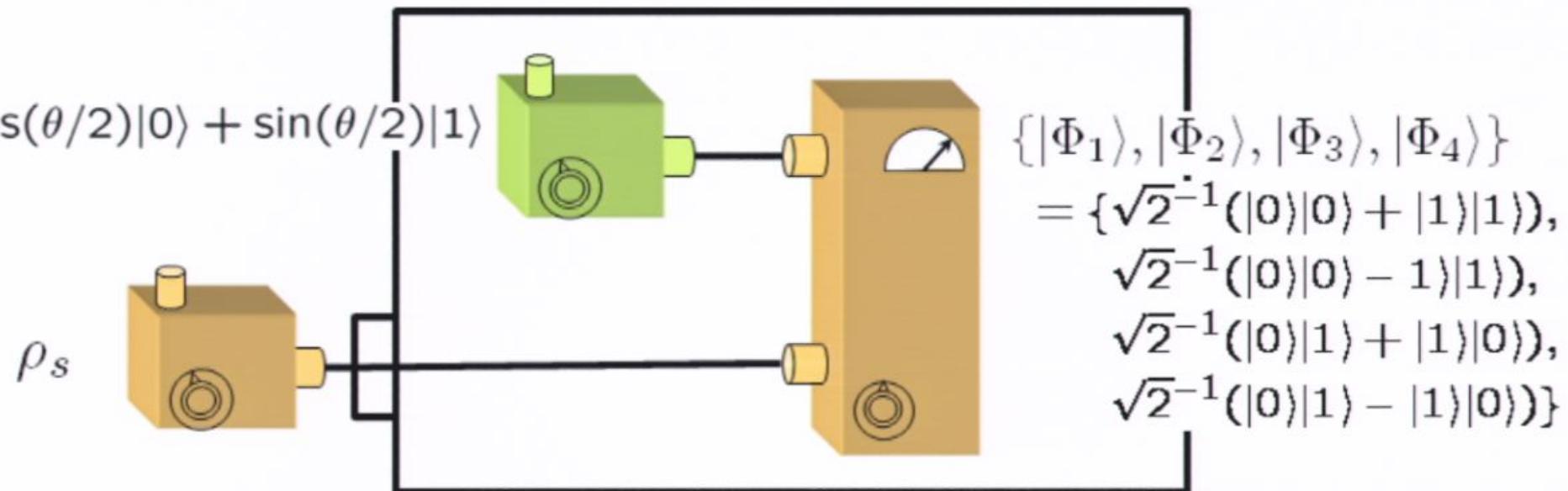
$$p(k) = \text{Tr}_{sa}[\Pi_k^{(sa)}(\rho_s \otimes \tau_a)]$$

$$= \text{Tr}_s[\underbrace{\text{Tr}_a(\Pi_k^{(sa)}\tau_a)}_{E_k^{(s)}} \rho_s]$$

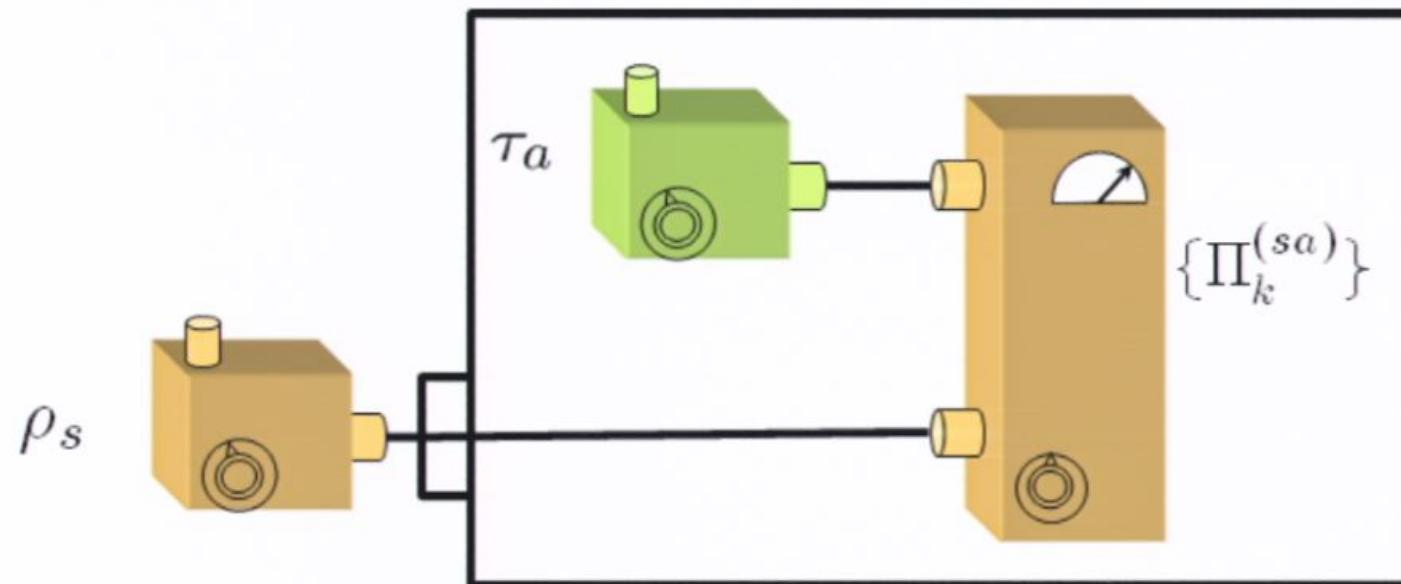
$$p(k) = \text{Tr}_s(E_k^{(s)}\rho_s)$$

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Sum to identity  $\sum_k E_k^{(s)} = I_s$

## Example



## Measurement by coupling to an ancilla

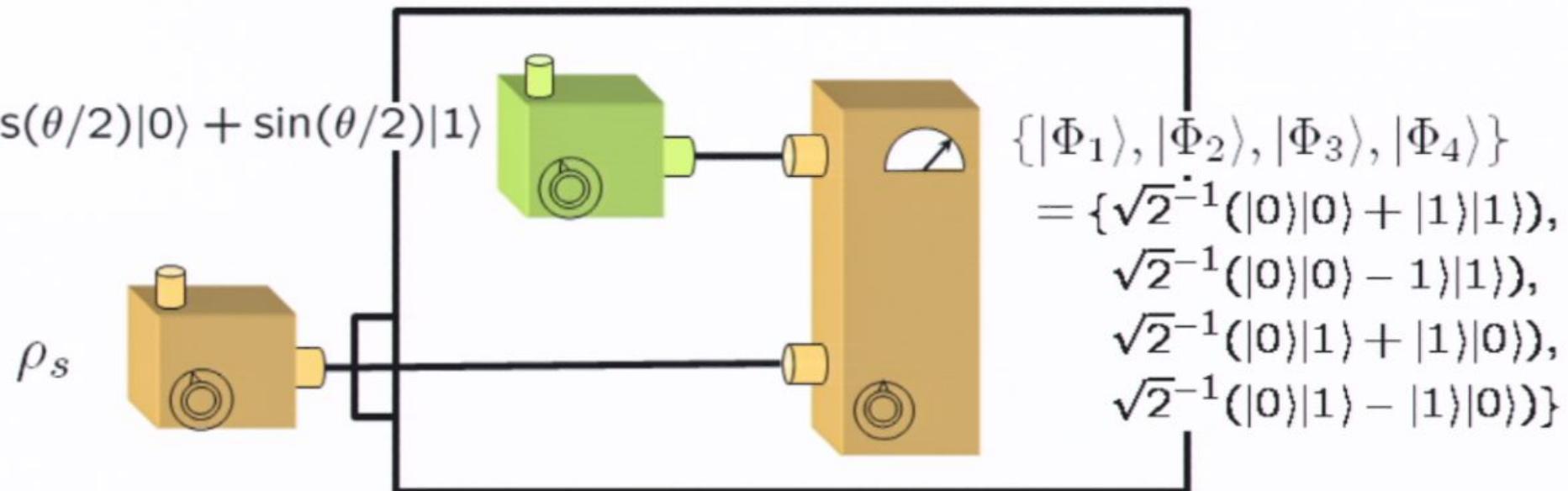


$$\begin{aligned} p(k) &= \text{Tr}_{sa}[\Pi_k^{(sa)}(\rho_s \otimes \tau_a)] \\ &= \underbrace{\text{Tr}_s[\text{Tr}_a(\Pi_k^{(sa)} \tau_a) \rho_s]}_{E_k^{(s)}} \end{aligned}$$

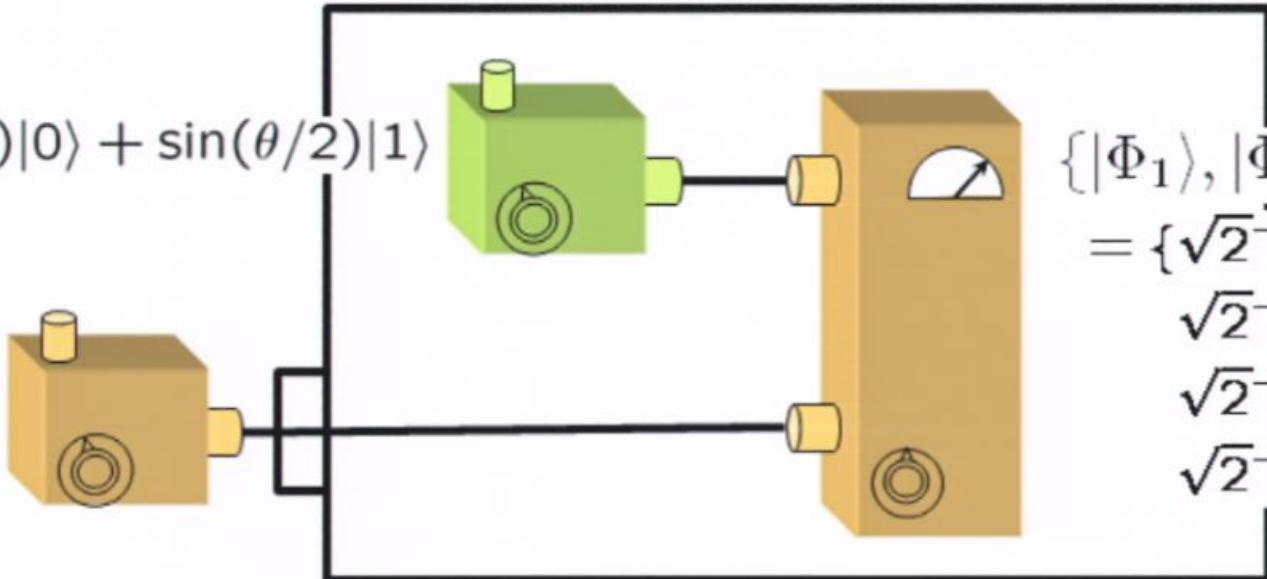
$$p(k) = \text{Tr}_s(E_k^{(s)} \rho_s)$$

Positive  $\langle \psi | E_k^{(s)} | \psi \rangle \geq 0 \quad \forall |\psi\rangle \in \mathcal{V}$   
 Sum to identity  $\sum_k E_k^{(s)} = I_s$

## Example



## Example

$$\left| \psi \right\rangle = \cos(\theta/2) |0\rangle + \sin(\theta/2) |1\rangle$$

$$\{\left| \Phi_1 \right\rangle, \left| \Phi_2 \right\rangle, \left| \Phi_3 \right\rangle, \left| \Phi_4 \right\rangle\}$$
$$= \{\sqrt{2}^{-1}(|0\rangle|0\rangle + |1\rangle|1\rangle),$$
$$\sqrt{2}^{-1}(|0\rangle|0\rangle - |1\rangle|1\rangle),$$
$$\sqrt{2}^{-1}(|0\rangle|1\rangle + |1\rangle|0\rangle),$$
$$\sqrt{2}^{-1}(|0\rangle|1\rangle - |1\rangle|0\rangle)\}$$

$$E_k^{(s)} = \text{Tr}_a(\Pi_k^{(sa)} \tau_a)$$

## Example

$$\left| \psi \right\rangle = \cos(\theta/2) |0\rangle + \sin(\theta/2) |1\rangle$$
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$$\sqrt{2}^{-1}(|0\rangle|1\rangle + |1\rangle|0\rangle),$$
$$\sqrt{2}^{-1}(|0\rangle|1\rangle - |1\rangle|0\rangle)\}$$

$$E_k^{(s)} = \text{Tr}_a(\Pi_k^{(sa)} \tau_a)$$
$$= \langle \theta |_a | \Phi_k \rangle_{sa} \langle \Phi_k |_{sa} | \theta \rangle_a$$

$$\sum_{i,j} p_i \Pi_j^{(i)} = \sum_i p_i \mathbb{1} = 1$$

$$1_{sa} = 1_s \otimes 1_a$$

$$\left( \cos \frac{\theta}{2} |0\rangle + \sin \frac{\theta}{2} |1\rangle \right) \left( \frac{1}{\sqrt{2}} (|0\rangle_s |0\rangle_a + |1\rangle_s |1\rangle_a) \right)$$

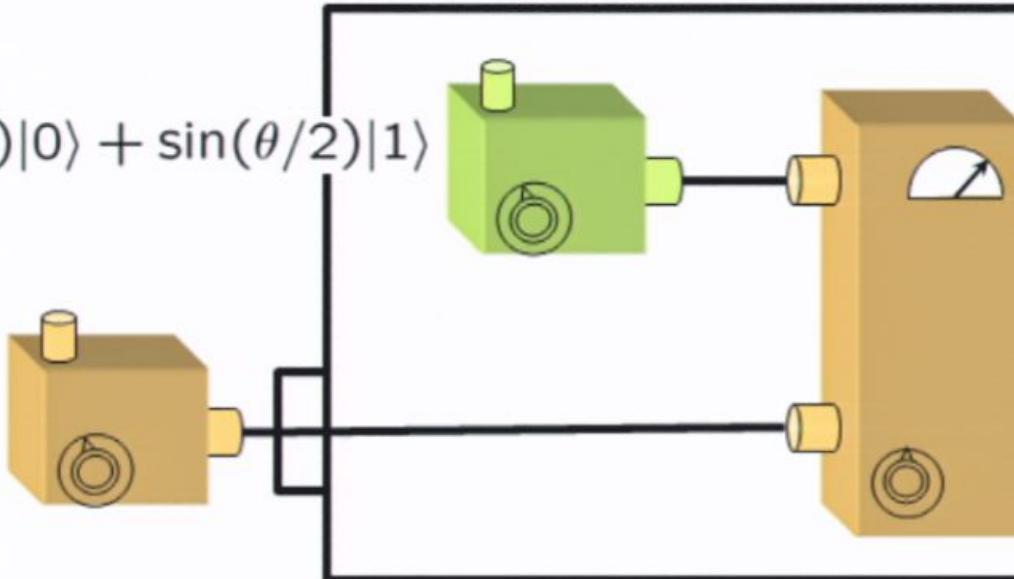
$$\frac{1}{\sqrt{2}} \cos \frac{\theta}{2} |0\rangle_s + \frac{1}{\sqrt{2}} \sin \frac{\theta}{2} |1\rangle_s$$

$$\sum_{i,j} p_i \pi_j^{(i)} = \sum_i p_i \mathbb{1} = 1$$

$$1_{\text{sa}} = 1_s \otimes 1_r$$

$$\left( \cos \frac{\theta}{2} |0\rangle + \sin \frac{\theta}{2} |1\rangle \right) \left( \frac{1}{\sqrt{2}} (|0\rangle_s |0\rangle_r + |1\rangle_s |1\rangle_r) \right)$$
$$\underline{\frac{1}{\sqrt{2}} \cos \frac{\theta}{2} |0\rangle_s + \frac{1}{\sqrt{2}} \sin \frac{\theta}{2} |1\rangle_s}$$

## Example

$$|\psi\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)|1\rangle$$


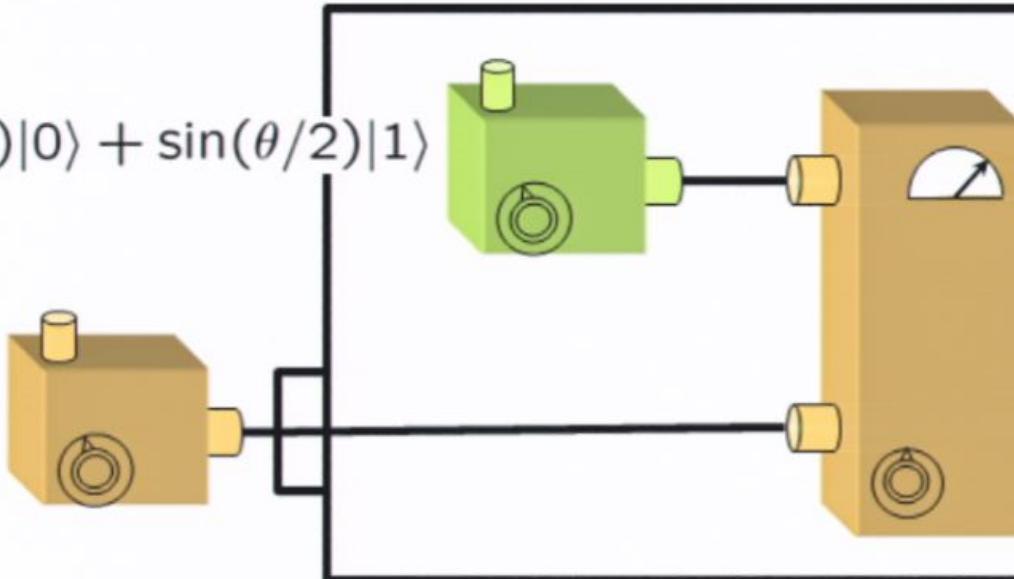
A quantum circuit diagram. On the left, a yellow box labeled  $\rho_s$  contains a yellow circle with a smaller circle inside, representing a density matrix. A wire from this box splits into two paths. The top path leads to a green rectangular block with a yellow circle containing a smaller circle, representing a unitary operator. Above this block is a small yellow cylinder. The bottom path leads to a yellow rectangular block with a yellow circle containing a smaller circle, representing another unitary operator. Above this block is a small yellow cylinder. Both paths converge at a meter-like device with a dial and a needle, representing a measurement.

$$\{|\Phi_1\rangle, |\Phi_2\rangle, |\Phi_3\rangle, |\Phi_4\rangle\}$$
$$= \{\sqrt{2}^{-1}(|0\rangle|0\rangle + |1\rangle|1\rangle),$$
$$\sqrt{2}^{-1}(|0\rangle|0\rangle - |1\rangle|1\rangle),$$
$$\sqrt{2}^{-1}(|0\rangle|1\rangle + |1\rangle|0\rangle),$$
$$\sqrt{2}^{-1}(|0\rangle|1\rangle - |1\rangle|0\rangle)\}$$

$$E_k^{(s)} = \text{Tr}_a(\Pi_k^{(sa)} \tau_a)$$
$$= \langle \theta|_a |\Phi_k\rangle_{sa} \langle \Phi_k|_{sa} |\theta\rangle_a$$

$$\langle \theta|_a |\Phi_{1(2)}\rangle_{sa} = \sqrt{2}^{-1} [\cos(\theta/2)|0\rangle_s \pm \sin(\theta/2)|1\rangle_s] = \sqrt{2}^{-1} |\pm \theta\rangle_s$$

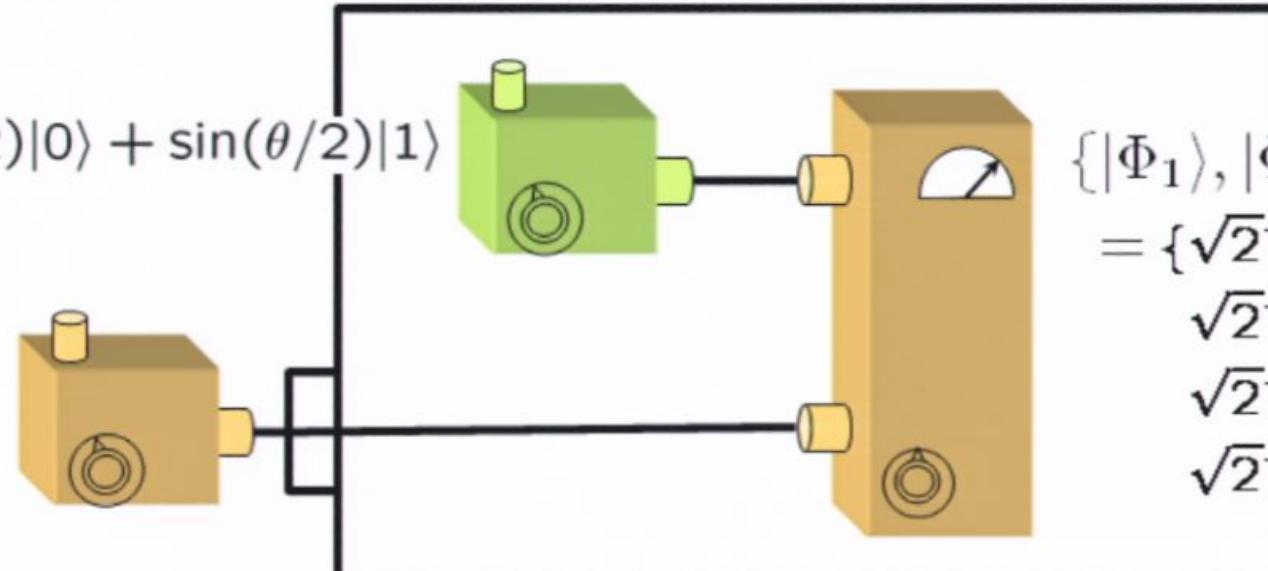
## Example

$$|\psi\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)|1\rangle$$

$$\{\Phi_1, \Phi_2, \Phi_3, \Phi_4\}$$
$$= \{\sqrt{2}^{-1}(|0\rangle|0\rangle + |1\rangle|1\rangle),$$
$$\sqrt{2}^{-1}(|0\rangle|0\rangle - |1\rangle|1\rangle),$$
$$\sqrt{2}^{-1}(|0\rangle|1\rangle + |1\rangle|0\rangle),$$
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## Example

$$\left| \theta \right\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)|1\rangle$$


$$\begin{aligned} &= \{\left| \Phi_1 \right\rangle, \left| \Phi_2 \right\rangle, \left| \Phi_3 \right\rangle, \left| \Phi_4 \right\rangle\} \\ &= \{\sqrt{2}^{-1}(|0\rangle|0\rangle + |1\rangle|1\rangle), \\ &\quad \sqrt{2}^{-1}(|0\rangle|0\rangle - |1\rangle|1\rangle), \\ &\quad \sqrt{2}^{-1}(|0\rangle|1\rangle + |1\rangle|0\rangle), \\ &\quad \sqrt{2}^{-1}(|0\rangle|1\rangle - |1\rangle|0\rangle)\} \end{aligned}$$

$$\begin{aligned} E_k^{(s)} &= \text{Tr}_a(\Pi_k^{(sa)} \tau_a) \\ &= \langle \theta |_a | \Phi_k \rangle_{sa} \langle \Phi_k |_{sa} | \theta \rangle_a \end{aligned}$$

$$\langle \theta |_a | \Phi_{1(2)} \rangle_{sa} = \sqrt{2}^{-1} [\cos(\theta/2)|0\rangle_s \pm \sin(\theta/2)|1\rangle_s] = \sqrt{2}^{-1} |\pm \theta\rangle_s$$

$$\langle \theta |_a | \Phi_{3(4)} \rangle_{sa} = \sqrt{2}^{-1} [\sin(\theta/2)|0\rangle_s \pm \cos(\theta/2)|1\rangle_s] = \sqrt{2}^{-1} |\pi \mp \theta\rangle_s$$

## Example

$$|\psi\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)|1\rangle$$

$$\{|\Phi_1\rangle, |\Phi_2\rangle, |\Phi_3\rangle, |\Phi_4\rangle\}$$

$$= \{\sqrt{2}^{-1}(|0\rangle|0\rangle + |1\rangle|1\rangle),$$

$$\sqrt{2}^{-1}(|0\rangle|0\rangle - |1\rangle|1\rangle),$$

$$\sqrt{2}^{-1}(|0\rangle|1\rangle + |1\rangle|0\rangle),$$

$$\sqrt{2}^{-1}(|0\rangle|1\rangle - |1\rangle|0\rangle)\}$$

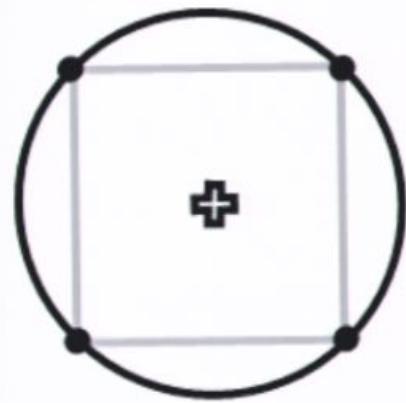
$$E_k^{(s)} = \text{Tr}_a(\Pi_k^{(sa)} \tau_a)$$

$$= \langle \theta|_a |\Phi_k\rangle_{sa} \langle \Phi_k|_{sa} |\theta\rangle_a$$

$$\langle \theta|_a |\Phi_{1(2)}\rangle_{sa} = \sqrt{2}^{-1} [\cos(\theta/2)|0\rangle_s \pm \sin(\theta/2)|1\rangle_s] = \sqrt{2}^{-1} |\pm \theta\rangle_s$$

$$\langle \theta|_a |\Phi_{3(4)}\rangle_{sa} = \sqrt{2}^{-1} [\sin(\theta/2)|0\rangle_s \pm \cos(\theta/2)|1\rangle_s] = \sqrt{2}^{-1} |\pi \mp \theta\rangle_s$$

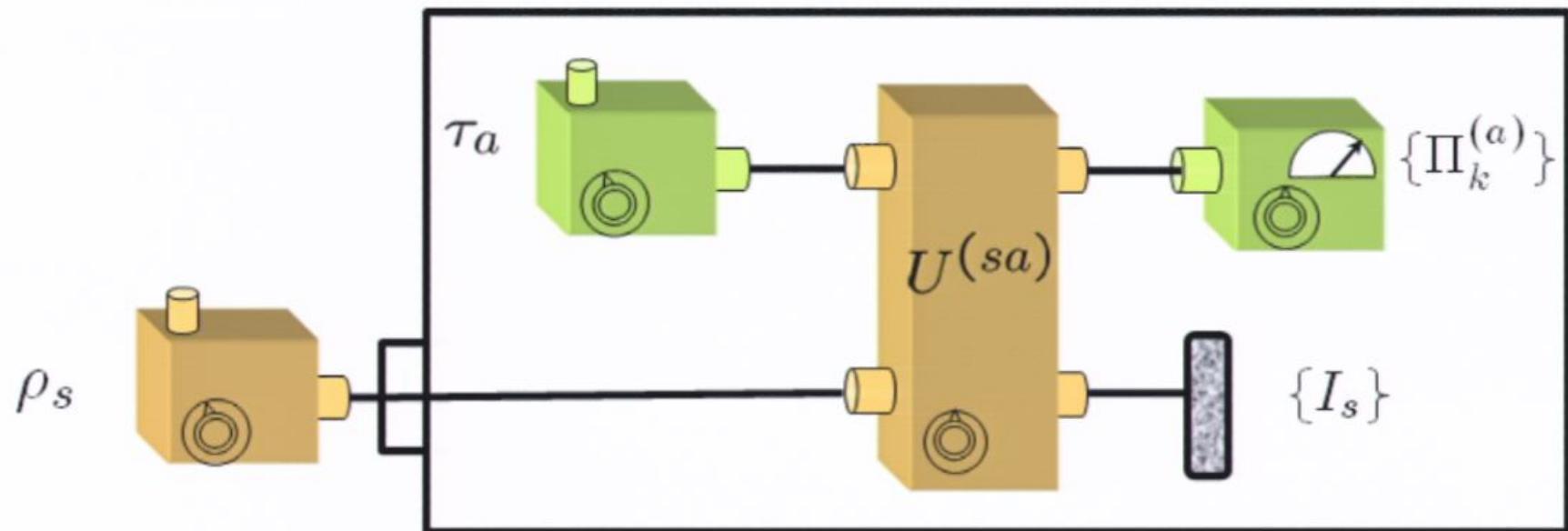
$$\{E_k\} = \left\{ \frac{1}{2}|\theta\rangle\langle\theta|, \frac{1}{2}|-\theta\rangle\langle-\theta|, \frac{1}{2}|\pi-\theta\rangle\langle\pi-\theta|, \frac{1}{2}|\pi+\theta\rangle\langle\pi+\theta| \right\}$$



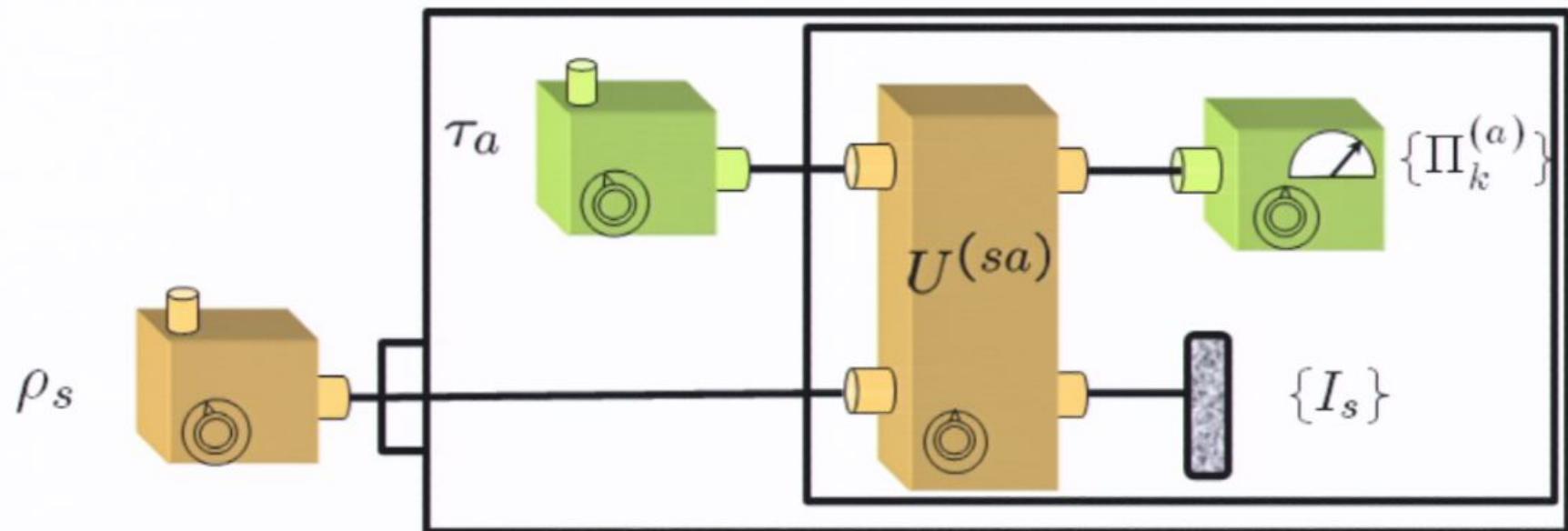
$$\theta = \pi/4$$

$$\left\{ \frac{1}{2}|\theta\rangle\langle\theta|, \frac{1}{2}|-\theta\rangle\langle-\theta|, \frac{1}{2}|\pi-\theta\rangle\langle\pi-\theta|, \frac{1}{2}|\pi+\theta\rangle\langle\pi+\theta| \right\}$$

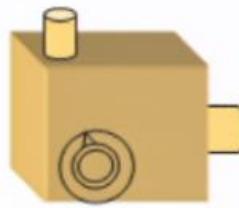
## Internalizing the probe system (a la von Neumann)



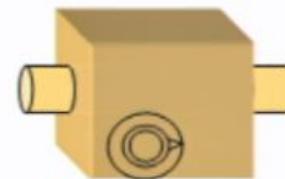
## Internalizing the probe system (a la von Neumann)



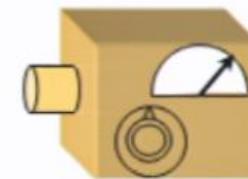
# Operational Quantum Mechanics



Preparation  
P



Transformation  
T



Measurement  
M

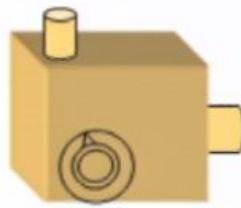
Density operator  
 $\rho$

Unitary  
 $U$

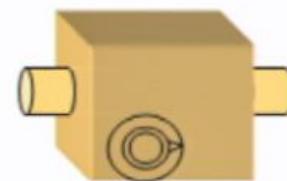
Positive operator-valued  
measure (POVM)  
 $\{E_k\}$

$$Pr(k|P, T, M) = \text{Tr}[E_k U \rho U^\dagger]$$

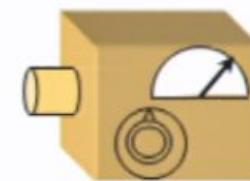
# Operational Quantum Mechanics



Preparation  
P



Transformation  
T



Measurement  
M

Density operator  
 $\rho$

Trace-preserving  
completely positive  
linear map (CP map)

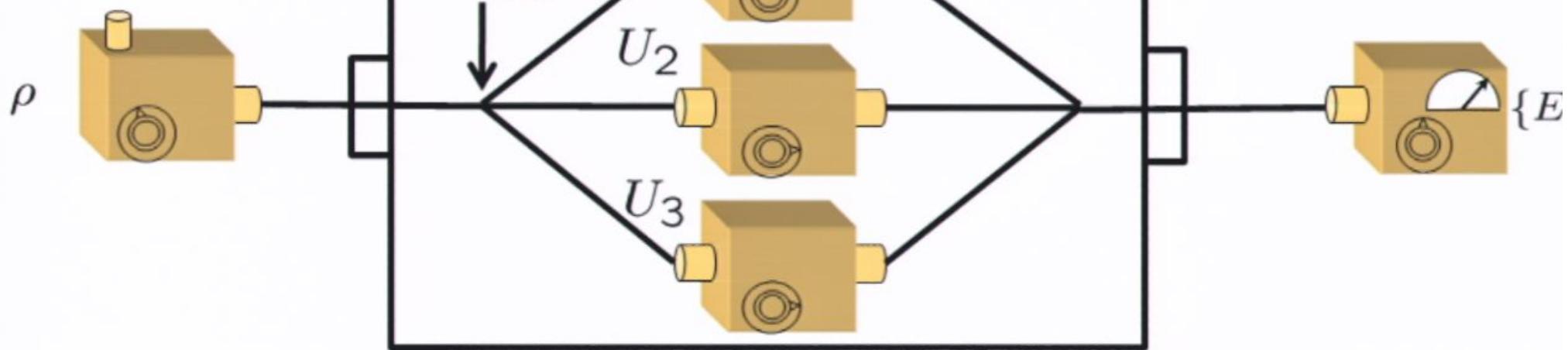
T

Positive operator-valued  
measure (POVM)

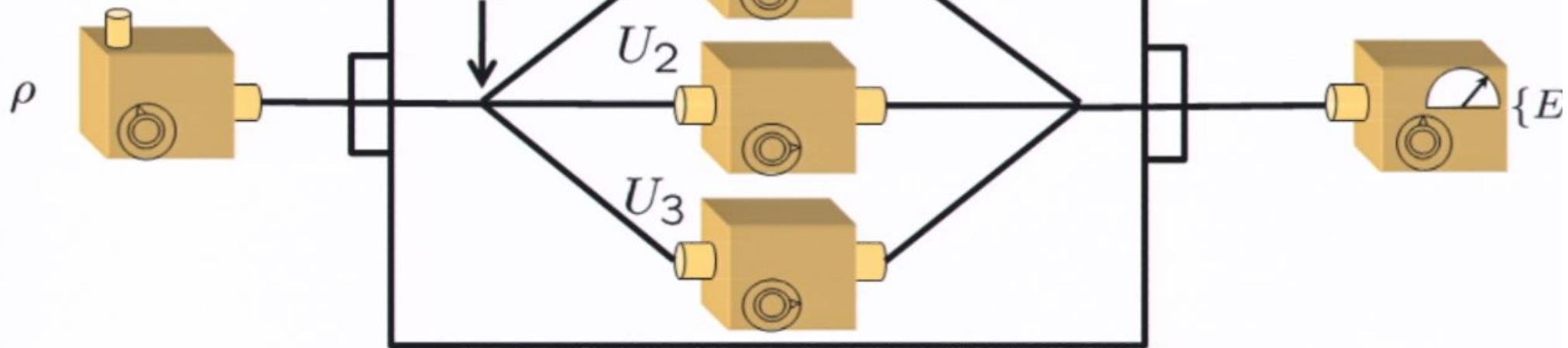
$\{E_k\}$

$$Pr(k|P, T, M) = \text{Tr}[E_k T(\rho)]$$

## Mixtures of Unitaries



## Mixtures of Unitaries



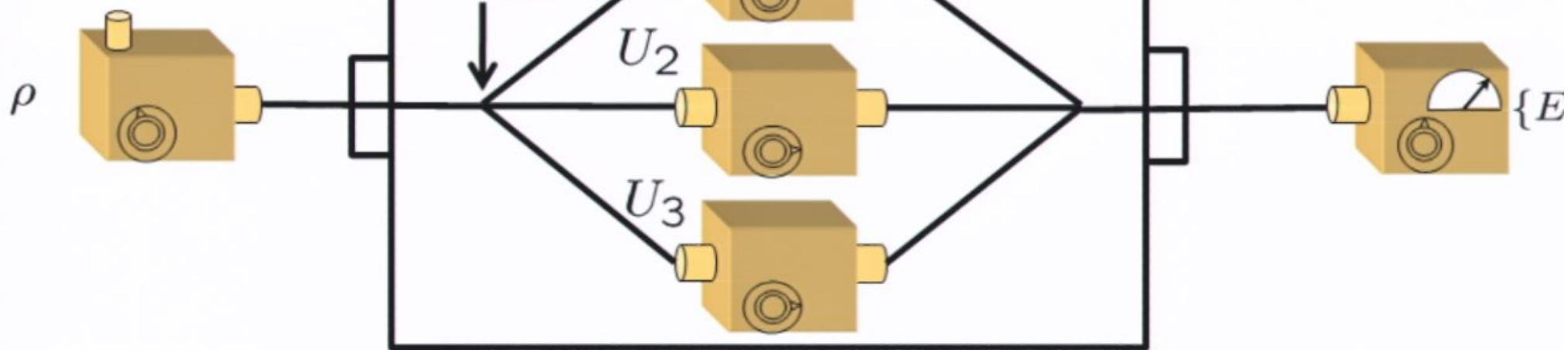
$$\begin{aligned} p(k) &= \sum_i p(k|i)p(i) \\ &= \sum_i \text{Tr}[E_k U_i \rho U_i^\dagger] p_i \\ &= \sum_i \text{Tr}[E_k \sum_i p_i U_i \rho U_i^\dagger] \end{aligned}$$

## Mixtures of Unitaries

A quantum circuit diagram illustrating a mixture of unitaries. It starts with a density operator  $\rho$  (represented by a yellow cube with a dial) entering a beam splitter. The circuit then branches into three parallel paths, each passing through a unitary  $U_i$  (represented by a yellow cube with a dial). The first path has a probability distribution  $\{p_i\}$  (represented by two purple dice) above it, indicating the probability of selecting unitary  $U_1$ . The outputs of the three paths converge at another beam splitter, which then leads to a measurement device  $\{E\}$  (represented by a yellow cube with a meter). The entire process is labeled  $T(\rho)$ .

$$\begin{aligned} p(k) &= \sum_i p(k|i)p(i) \\ &= \sum_i \text{Tr}[E_k U_i \rho U_i^\dagger] p_i \\ &= \sum_i \text{Tr}[E_k \underbrace{\sum_i p_i U_i \rho U_i^\dagger}_T] \end{aligned}$$
$$T(\rho)$$

## Mixtures of Unitaries

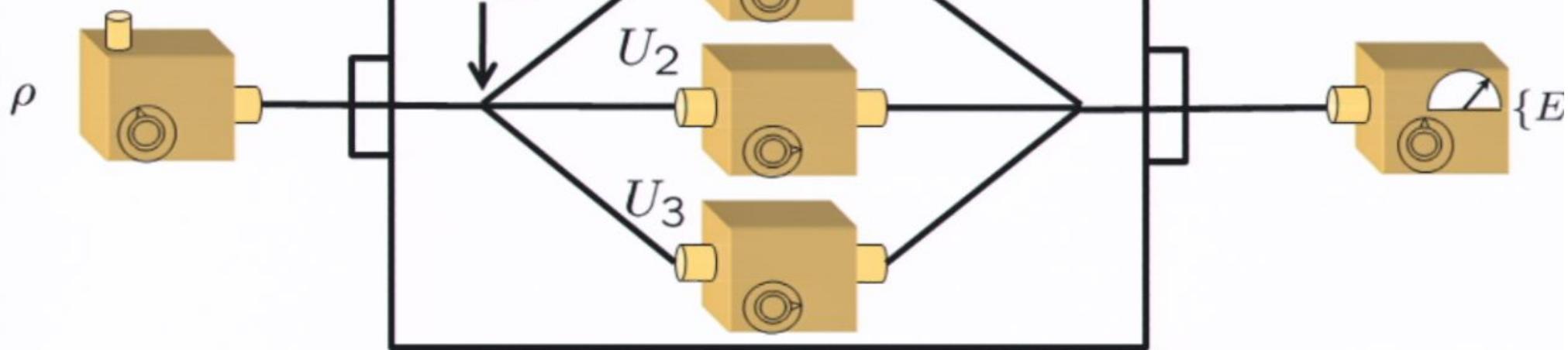


$$\begin{aligned}
 p(k) &= \sum_i p(k|i)p(i) \\
 &= \sum_i \text{Tr}[E_k U_i \rho U_i^\dagger] p_i \\
 &= \sum_i \text{Tr}[E_k \underbrace{\sum_i p_i U_i \rho U_i^\dagger}_T]
 \end{aligned}$$

$T(\rho)$

Positive:  $\mathcal{T}(\rho) > 0$  if  $\rho > 0$

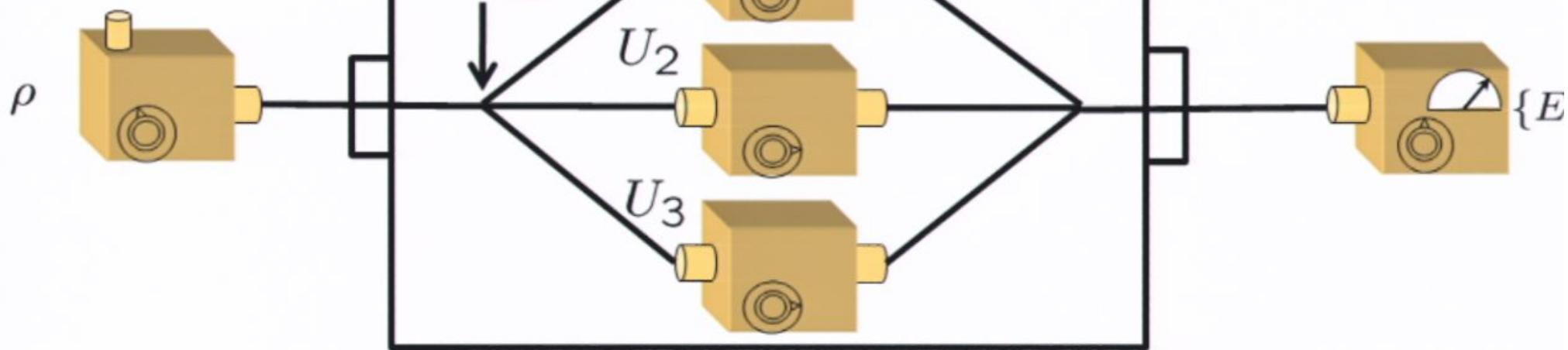
## Mixtures of Unitaries



$$\begin{aligned}
 p(k) &= \sum_i p(k|i)p(i) \\
 &= \sum_i \text{Tr}[E_k U_i \rho U_i^\dagger] p_i \\
 &= \sum_i \text{Tr}[E_k \underbrace{\sum_i p_i U_i \rho U_i^\dagger}_\mathcal{T(\rho)}]
 \end{aligned}$$

Positive:  $\mathcal{T}(\rho) > 0$  if  $\rho > 0$   
 Completely positive:  
 $\mathcal{T}_s \otimes \mathcal{I}_a(\rho_{sa}) > 0$  if  $\rho_{sa} > 0$

## Mixtures of Unitaries

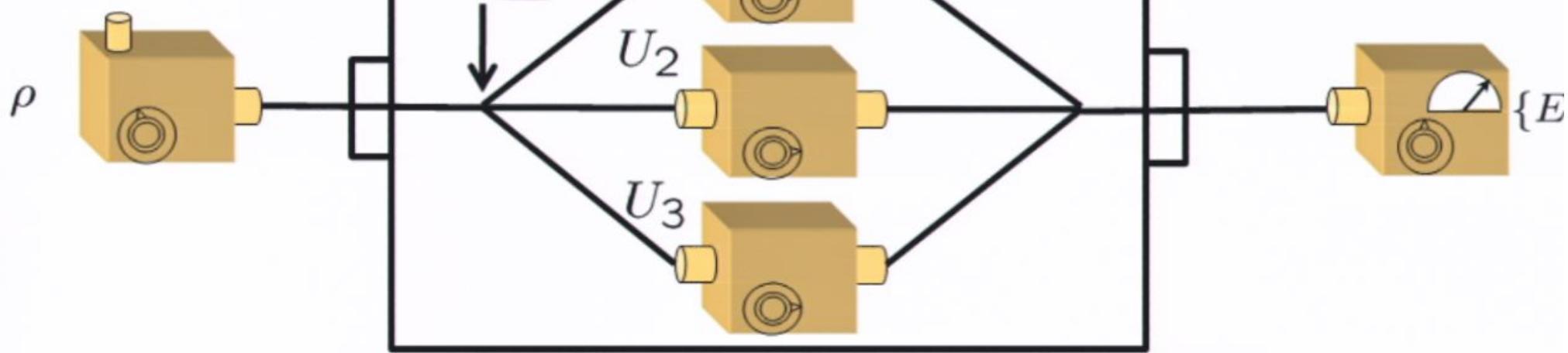


$$\begin{aligned}
 p(k) &= \sum_i p(k|i)p(i) \\
 &= \sum_i \text{Tr}[E_k U_i \rho U_i^\dagger] p_i \\
 &= \sum_i \text{Tr}[E_k \underbrace{\sum_i p_i U_i \rho U_i^\dagger}_T]
 \end{aligned}$$

$T(\rho)$

Positive:  $T(\rho) > 0$  if  $\rho > 0$   
 Completely positive:  
 $T_s \otimes I_a(\rho_{sa}) > 0$  if  $\rho_{sa} > 0$   
 Trace-preserving:  $\text{Tr}(T(\rho)) = \text{Tr}(\rho)$

## Mixtures of Unitaries

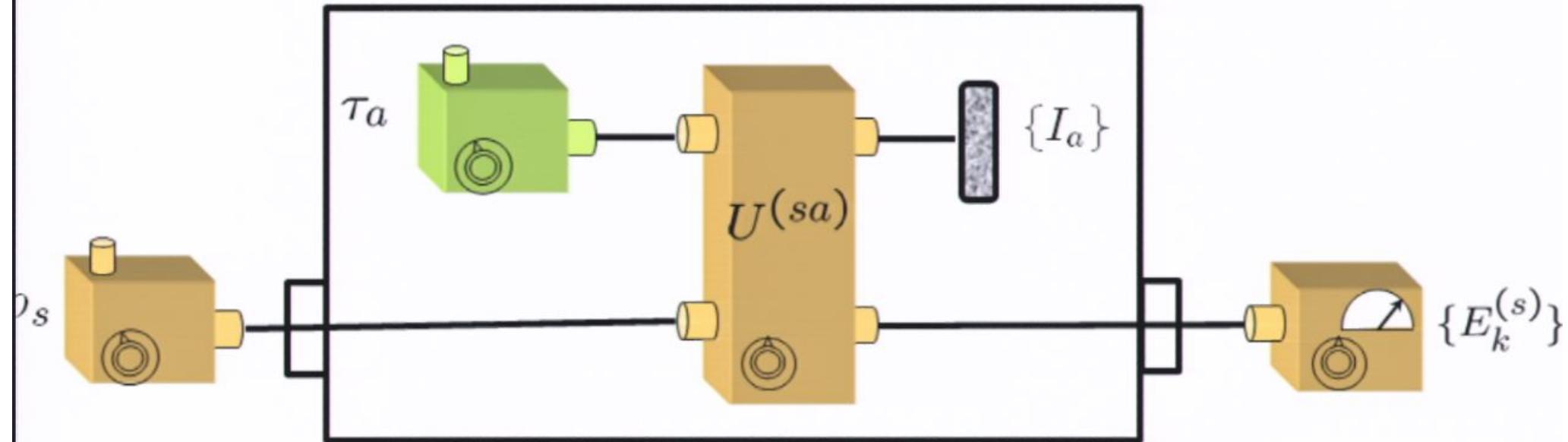


$$\begin{aligned}
 p(k) &= \sum_i p(k|i)p(i) \\
 &= \sum_i \text{Tr}[E_k U_i \rho U_i^\dagger] p_i \\
 &= \sum_i \text{Tr}[E_k \underbrace{\sum_i p_i U_i \rho U_i^\dagger}_T]
 \end{aligned}$$

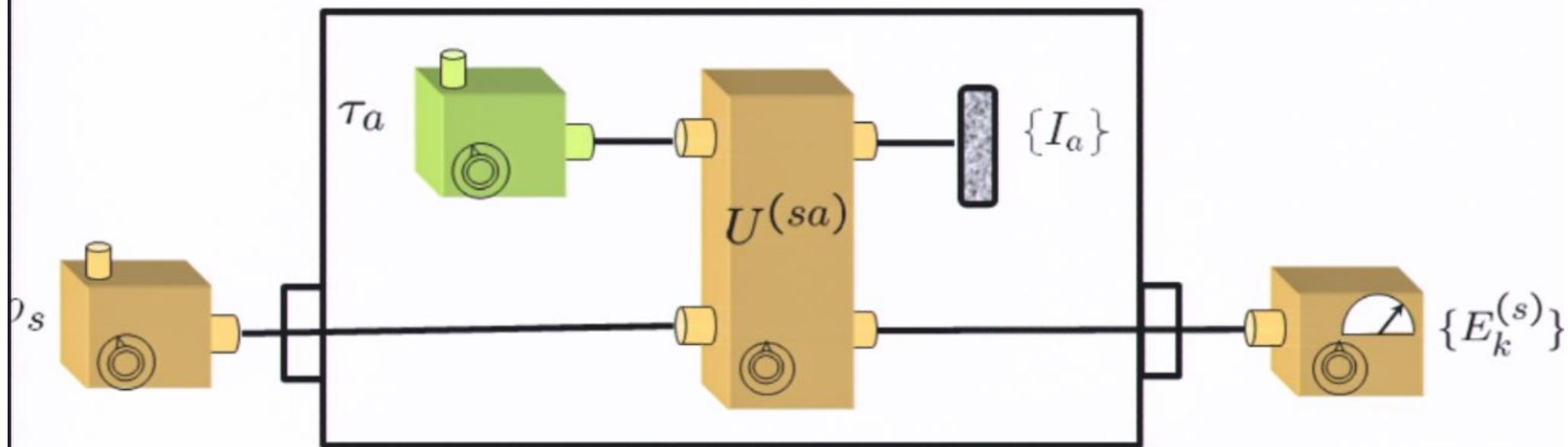
$T(\rho)$

Positive:  $\mathcal{T}(\rho) > 0$  if  $\rho > 0$   
 Completely positive:  
 $\mathcal{T}_s \otimes \mathcal{I}_a(\rho_{sa}) > 0$  if  $\rho_{sa} > 0$   
 Trace-preserving:  $\text{Tr}(\mathcal{T}(\rho)) = \text{Tr}(\rho)$

## Transformation by coupling to an ancilla



## Transformation by coupling to an ancilla



$$\begin{aligned} p(k) &= \text{Tr}_{sa}[(E_k^{(s)} \otimes I_a) U_{sa}(\rho_s \otimes \tau_a) U_{sa}^\dagger] \\ &= \text{Tr}_s[E_k^{(s)} \mathcal{T}(\rho_s)] \end{aligned}$$

## General transformations

Linear map:  $\mathcal{T} : \mathcal{L}(\mathbb{C}_d) \rightarrow \mathcal{L}(\mathbb{C}_d)$

Completely positive:  $\mathcal{T}_s \otimes \mathcal{I}_a(\rho_{sa}) > 0$  if  $\rho_{sa} > 0$

Trace-preserving:  $\text{Tr}(\mathcal{T}(\rho)) = \text{Tr}(\rho)$

Kraus decomposition

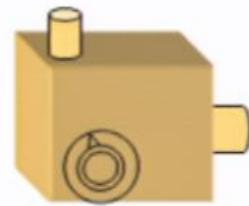
$$\mathcal{T}(\rho) = \sum_{\mu} K_{\mu} \rho K_{\mu}^{\dagger}$$

$K_{\mu}$  linear operators

$$\sum_{\mu} K_{\mu}^{\dagger} K_{\mu} = I$$

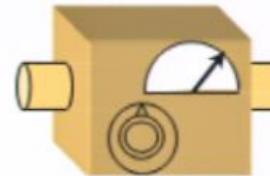
$$\mathcal{T}(\rho) = \sum_i (\sqrt{p_i} U_i) \rho (U_i^{\dagger} \sqrt{p_i}) \quad \text{Mixture of unitaries}$$

# Operational Quantum Mechanics



Preparation

P



Measurement

M

Effective preparation

$P_k$

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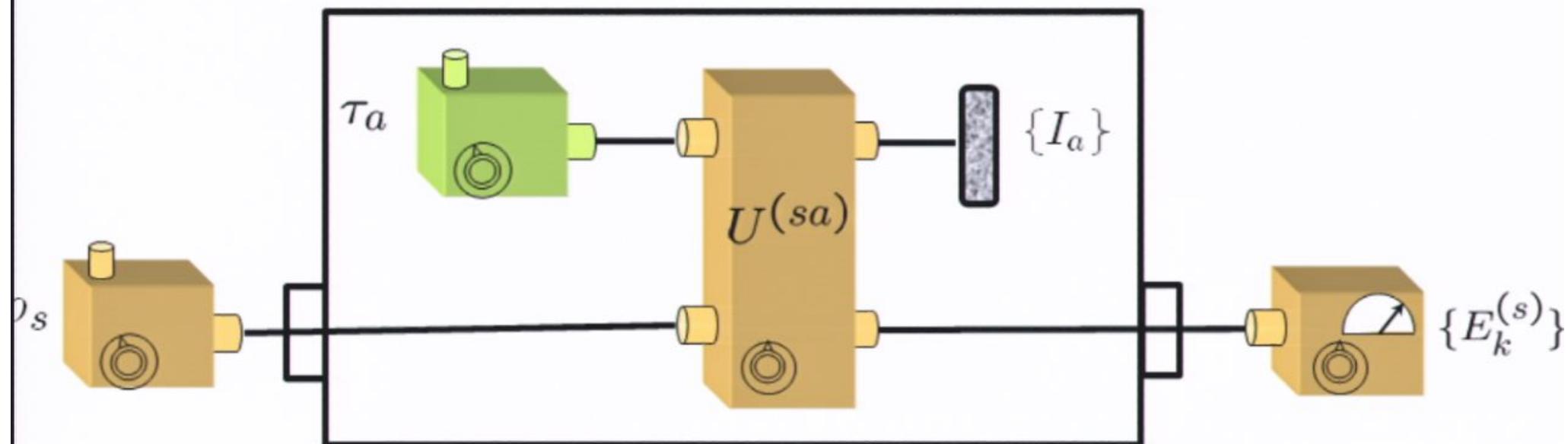
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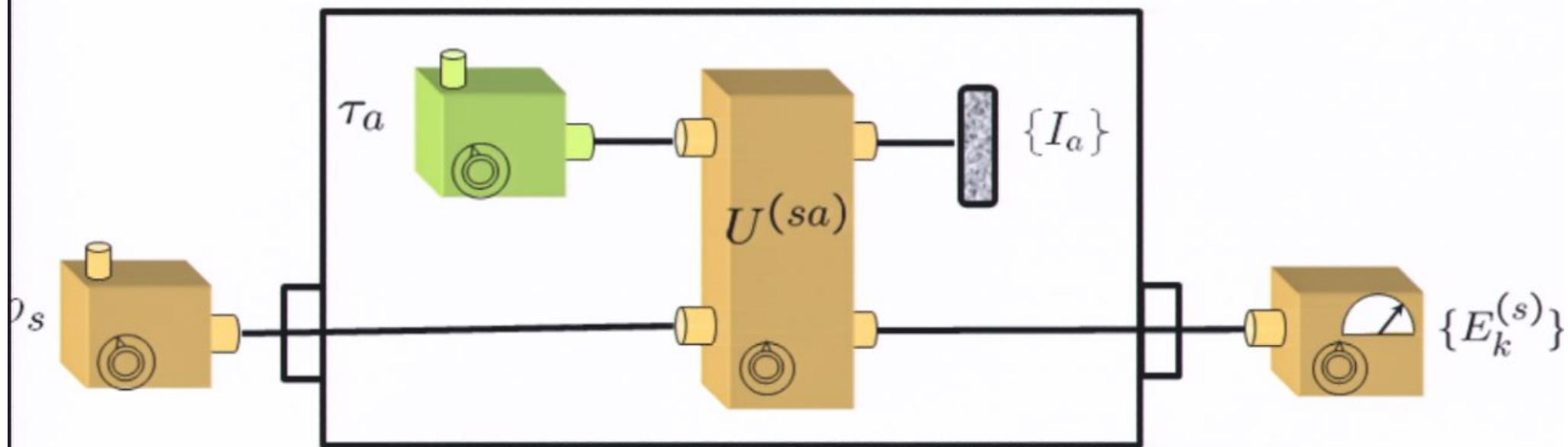
Trace-preserving:

$$\text{where } \tau_a = \sum_i w_i |i\rangle_a \langle i|$$

$$\text{Tr}(\mathcal{T}(\rho)) = \text{Tr}(\rho)$$

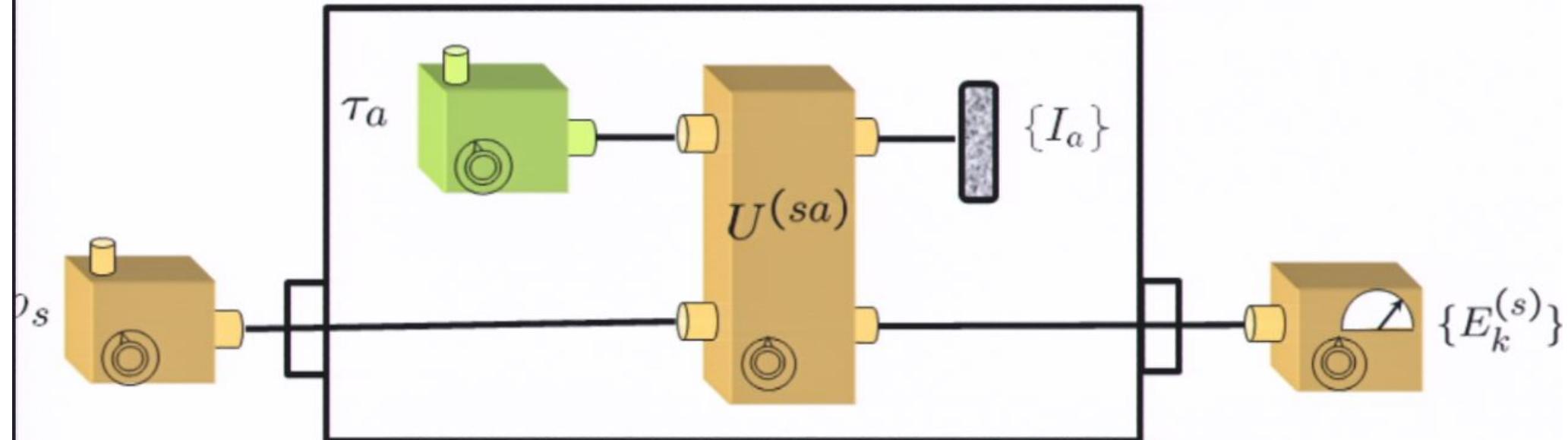
$$\text{because } \sum_\mu K_\mu^{(s)\dagger} K_\mu^{(s)} = I_s$$

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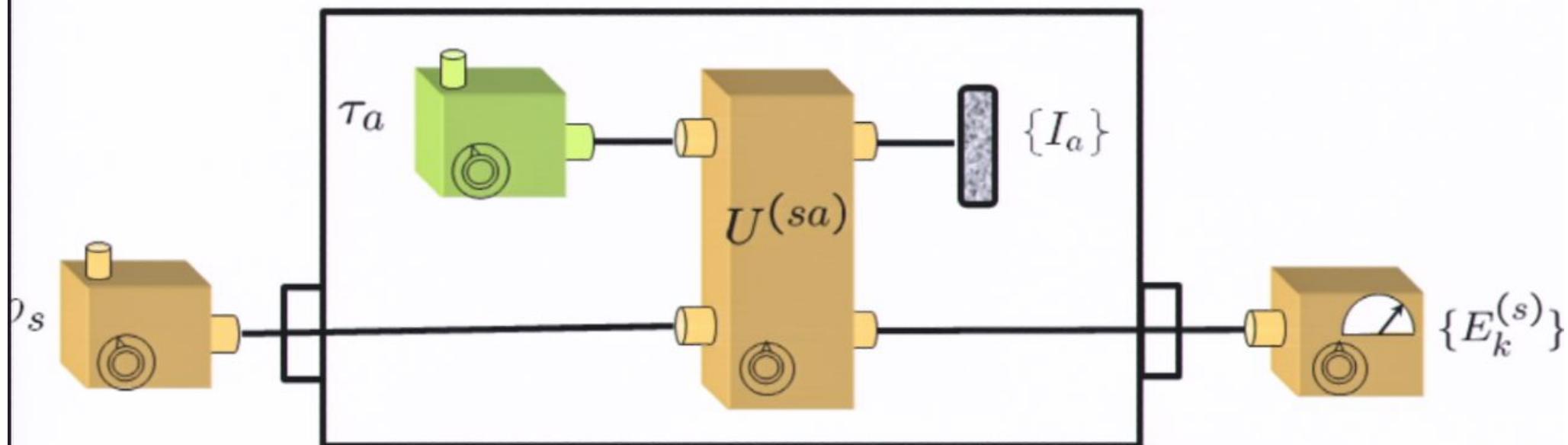


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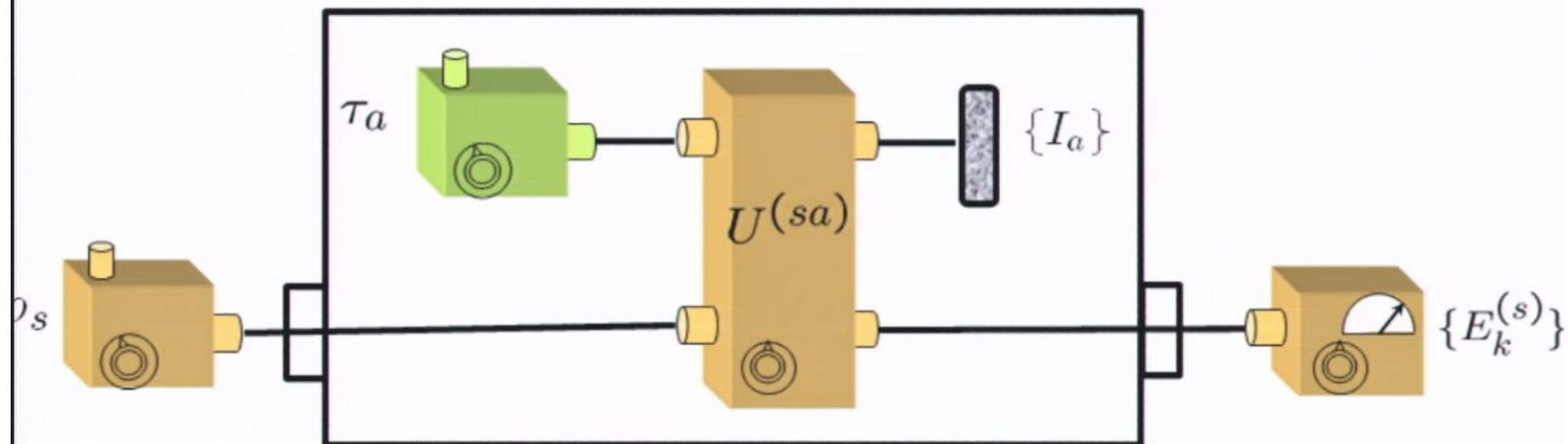


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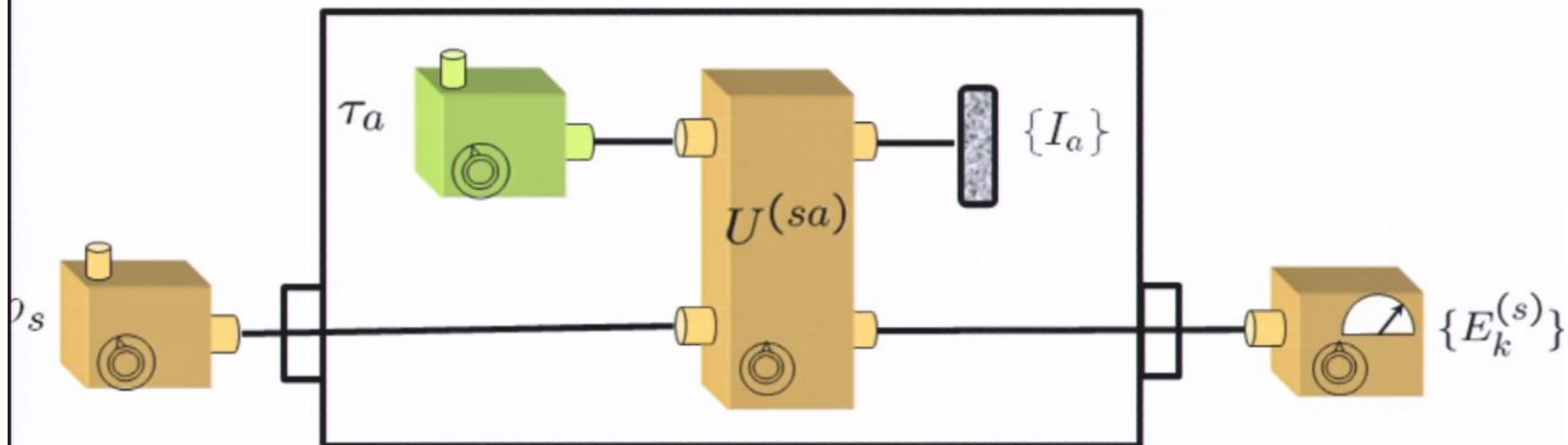


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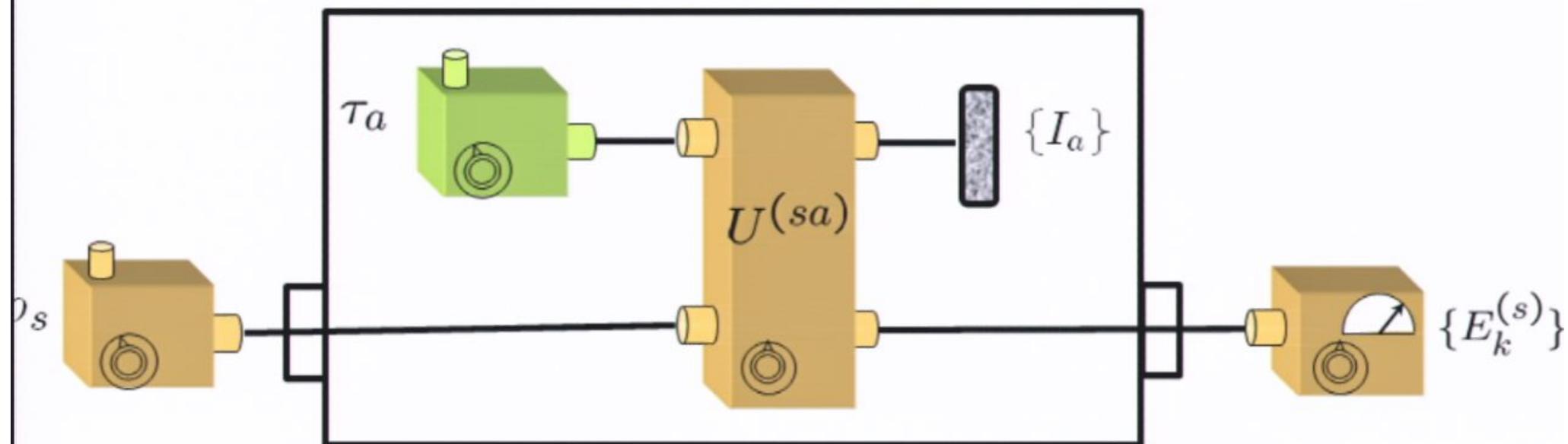


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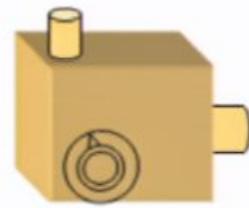
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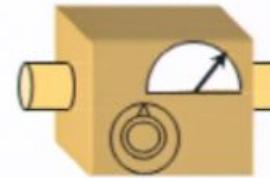
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$\mathcal{T}(\rho) = \prod_k \rho \Pi_k$  Selective projective measurement

$\mathcal{T}(\rho) = \Psi(r) \rho \Psi^{\dagger}(r)$  Photon detected at position r

# The non-independence of the structure of preparations and the structure of measurements

## A version of Gleason's theorem

Consider a function on density operators  
 $\rho \mapsto f(\rho)$ , satisfying:

- 1)  $0 \leq f(\rho) \leq 1$  for all  $\rho$
- 2)  $f(w\rho + (1 - w)\rho') = wf(\rho) + (1 - w)f(\rho')$   
where  $0 \leq w \leq 1$ .

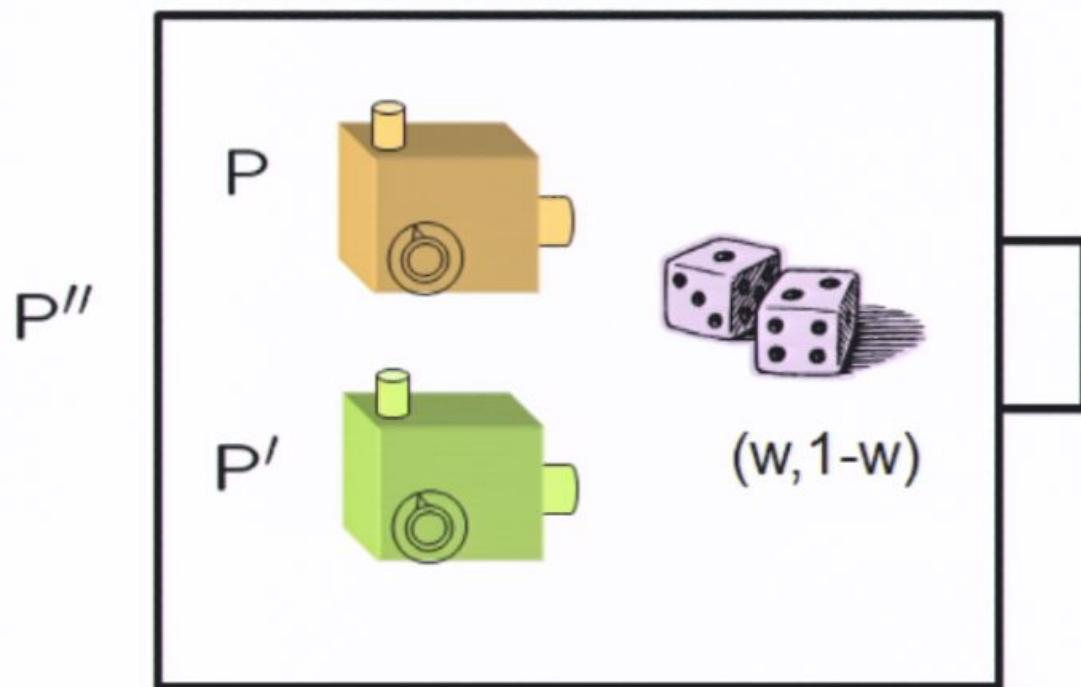
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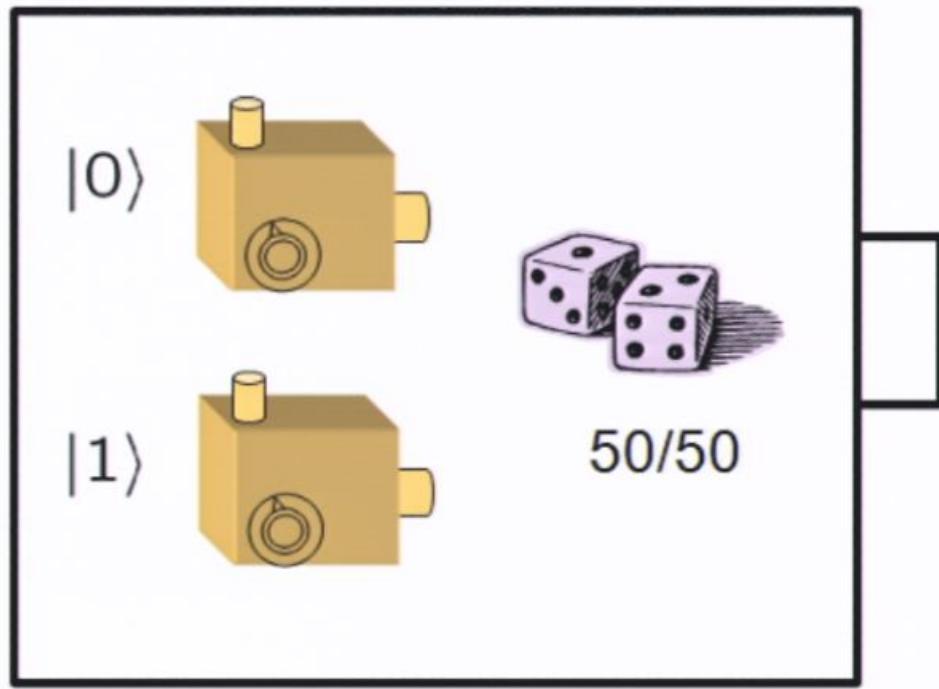
## Representing mixtures of preparations



If  $P'' = P$  with prob.  $w$  and  $P'$  with prob.  $1 - w$

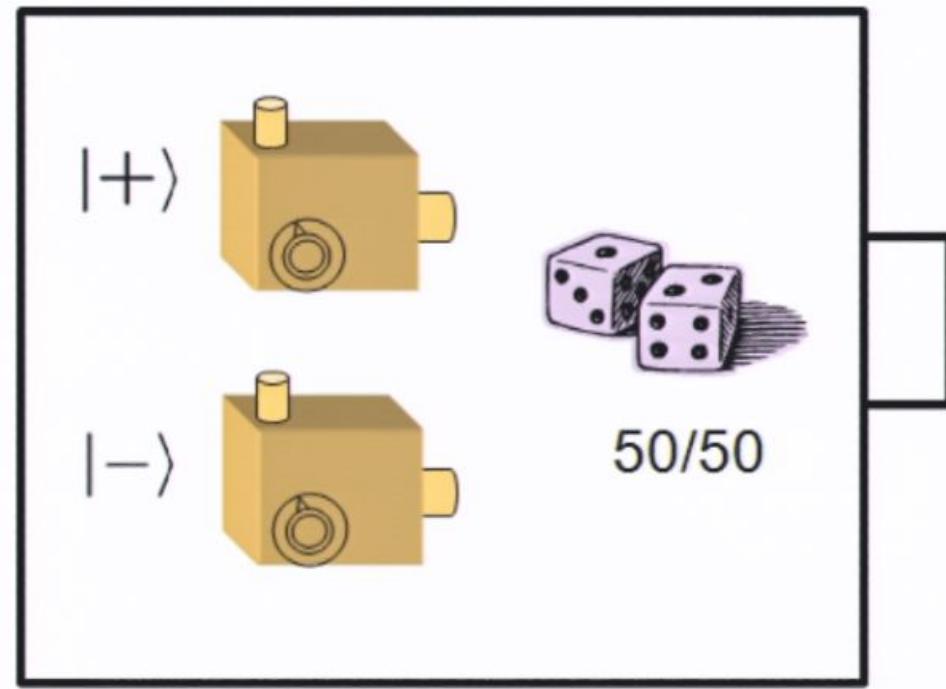
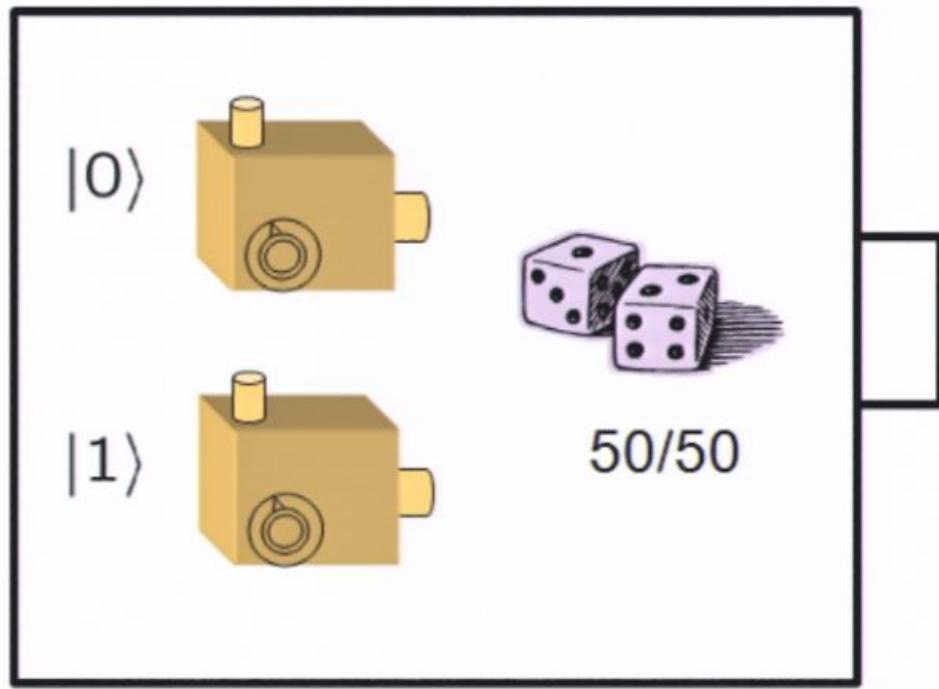
Then  $p(k|P'') = w p(k|P) + (1 - w) p(k|P')$

$$f(P'') = w f(P) + (1 - w) f(P')$$



$$\frac{1}{2}I = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$$

$$f\left(\frac{1}{2}I\right) = \frac{1}{2}f(|0\rangle\langle 0|) + \frac{1}{2}f(|1\rangle\langle 1|)$$



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**The theorem:**

$$f(\rho) = \text{Tr}(E\rho)$$

for some effect  $E$  (i.e.  $0 \leq E \leq I$ ).