Title: Foundations of Quantum Mech. (PHYS 639) - Lecture 3

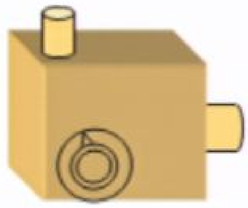
Date: Dec 02, 2009 11:00 AM

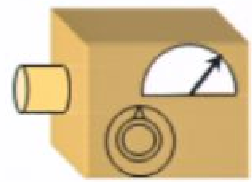
URL: http://pirsa.org/09120067

Abstract:

Towards a purely operational formulation of quantum theory

Operational Quantum Mechanics





Preparation

P

Measurement

M

Vector

$|\psi\rangle$

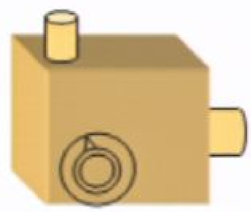
Hermitian operator

A

$A = \sum_k a_k \Pi_k$

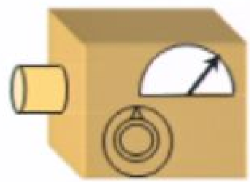
$$Pr(k|\mathsf{P},\mathsf{M}) = \langle\psi|\Pi_k|\psi\rangle$$

Operational Quantum Mechanics



Preparation

P



Measurement

M

Density operator

ρ

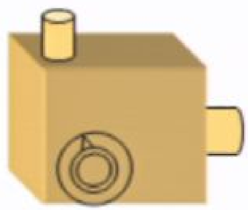
Hermitian operator

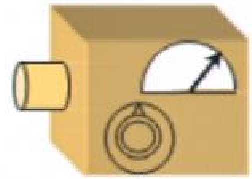
A

$A = \sum_k a_k \Pi_k$

$$Pr(k|\mathsf{P},\mathsf{M}) = \mathrm{Tr}(\rho \Pi_k)$$

Operational Quantum Mechanics

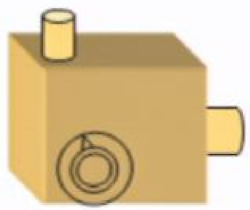




Preparation	Measurement
P	M
Density operator	Projection valued measure (PVM)
ρ	$\{\Pi_k\}$

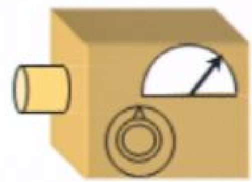
$$Pr(k|\mathsf{P},\mathsf{M}) = \mathrm{Tr}(\rho\Pi_k)$$

Operational Quantum Mechanics



Preparation

P



Measurement

M

Density operator

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Hermitian operator

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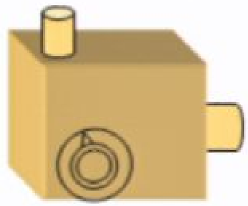
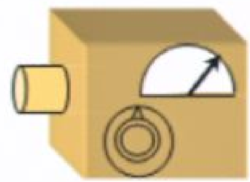
$A = \sum_k a_k \Pi_k$

$$Pr(k|\mathsf{P},\mathsf{M}) = \mathrm{Tr}(\rho \Pi_k)$$

Operational Quantum Mechanics





Preparation

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Measurement

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Density operator

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Projection valued measure (PVM)

$\{\Pi_k\}$

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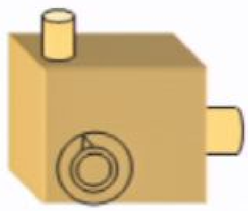
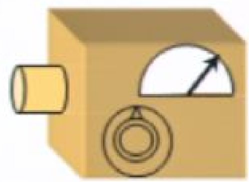
Operational Quantum Mechanics





Preparation

P

Measurement

M

Density operator

ρ

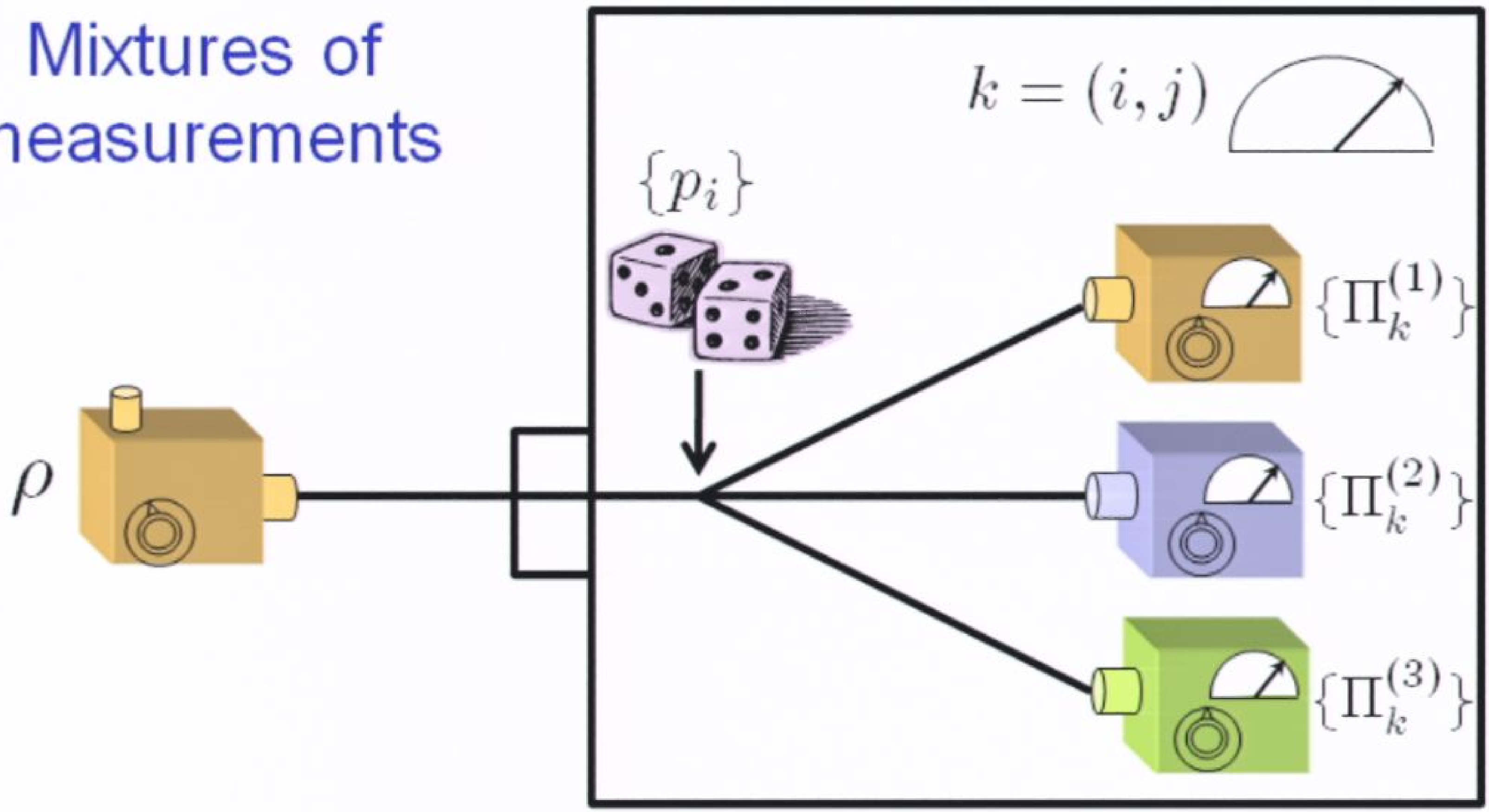
Position operator valued measure (POVM)

$\{E_k\}$

$$Pr(k|\mathsf{P},\mathsf{M}) = \mathrm{Tr}(\rho E_k)$$

Standard Measurements	Generalized Measurements
$\{\Pi_i\}$	$\{E_d\}$
$\langle\psi\|\Pi_i\|\psi\rangle \geq 0\,,\ \forall\|\psi\rangle$	$\langle\psi\|E_d\|\psi\rangle \geq 0\,,\ \forall\|\psi\rangle$
$\sum_i \Pi_i = I$	$\sum_d E_d = I$
$P(i) = \mathrm{tr}(\rho\Pi_i)$	$P(d) = \mathrm{tr}(\rho E_d)$
$\Pi_i\Pi_j = \delta_{ij}\,\Pi_i$	———

Mixtures of measurements



Standard Measurements	**Generalized Measurements**
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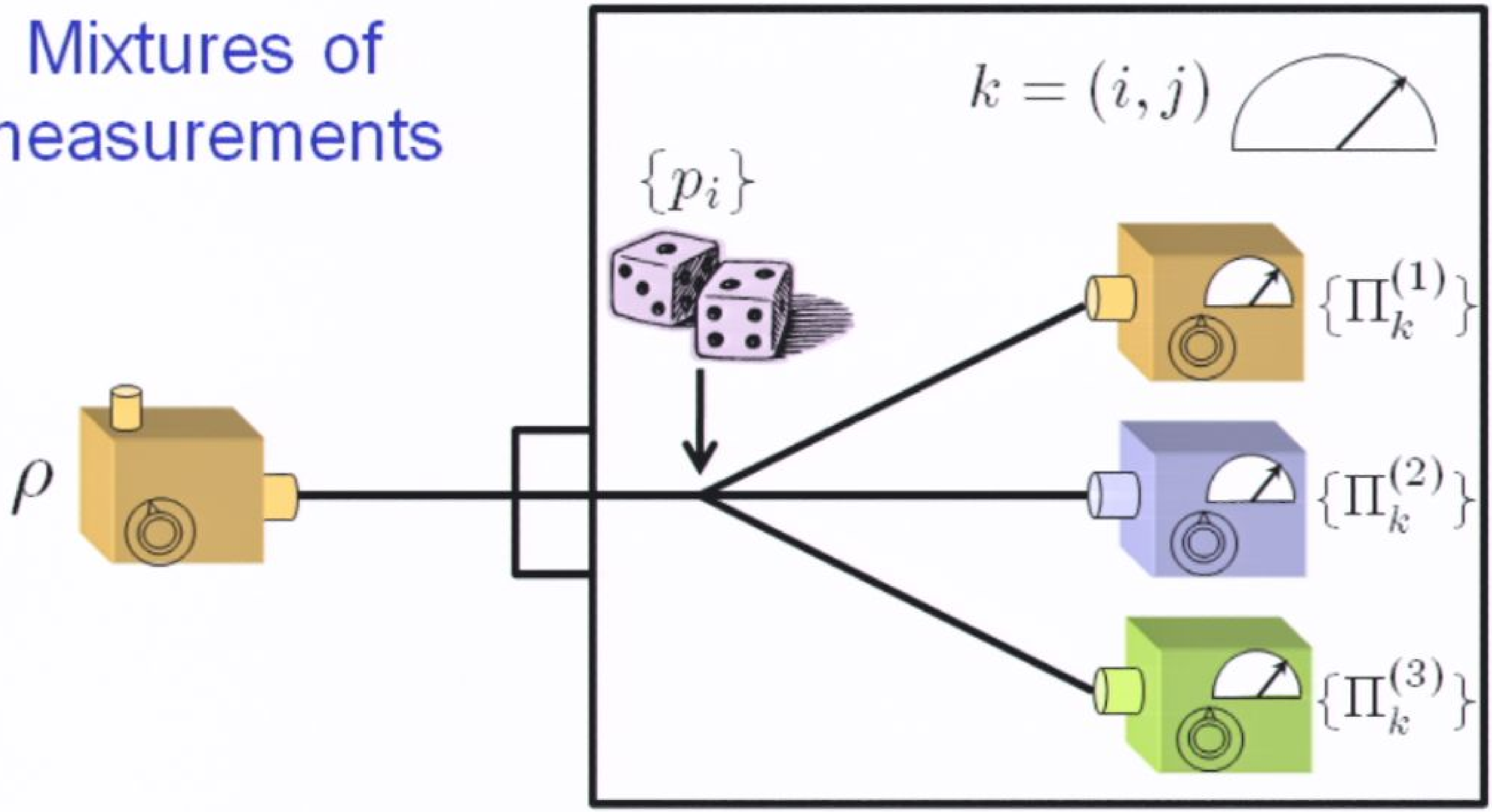
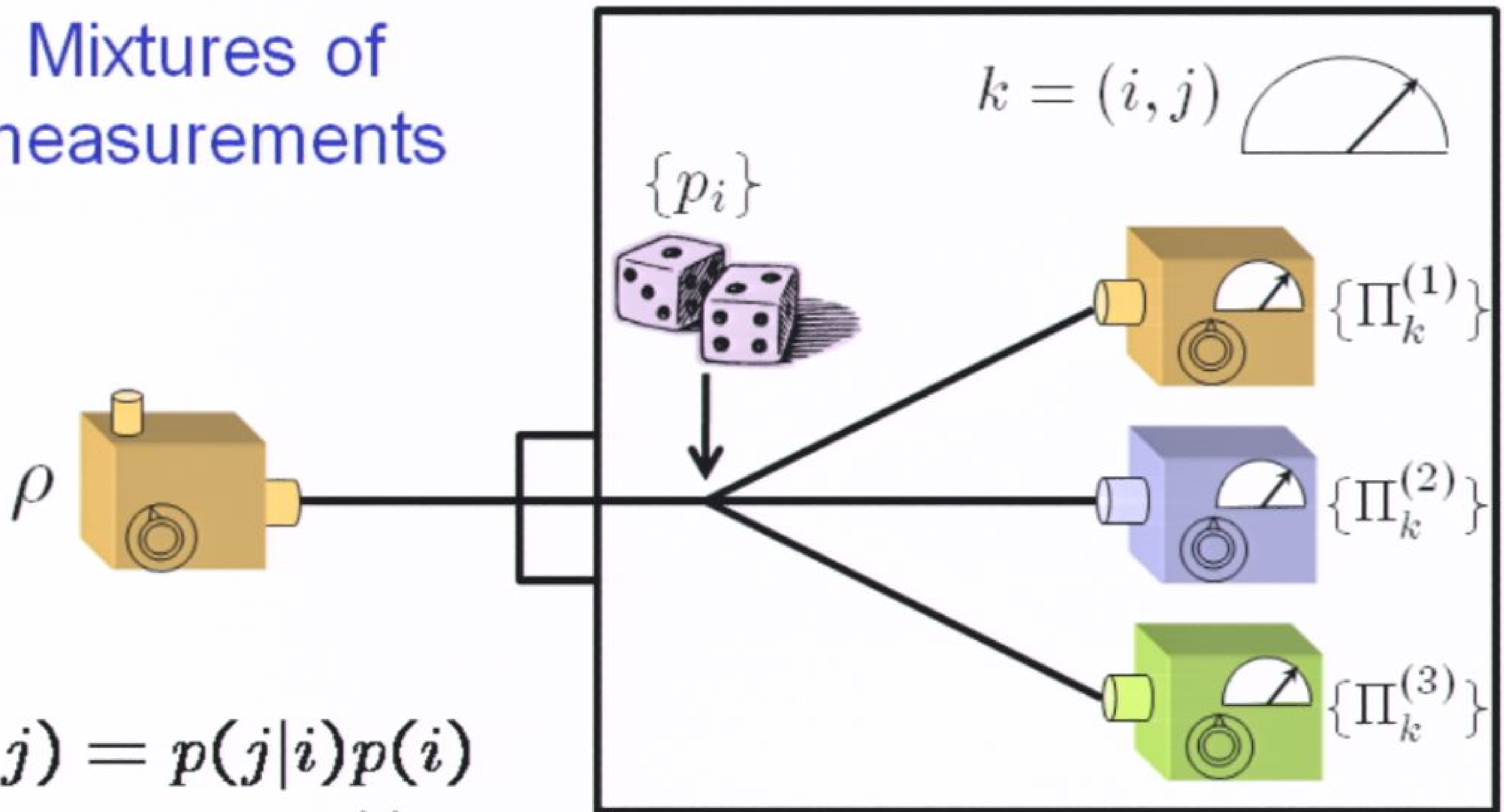
Mixtures of measurements

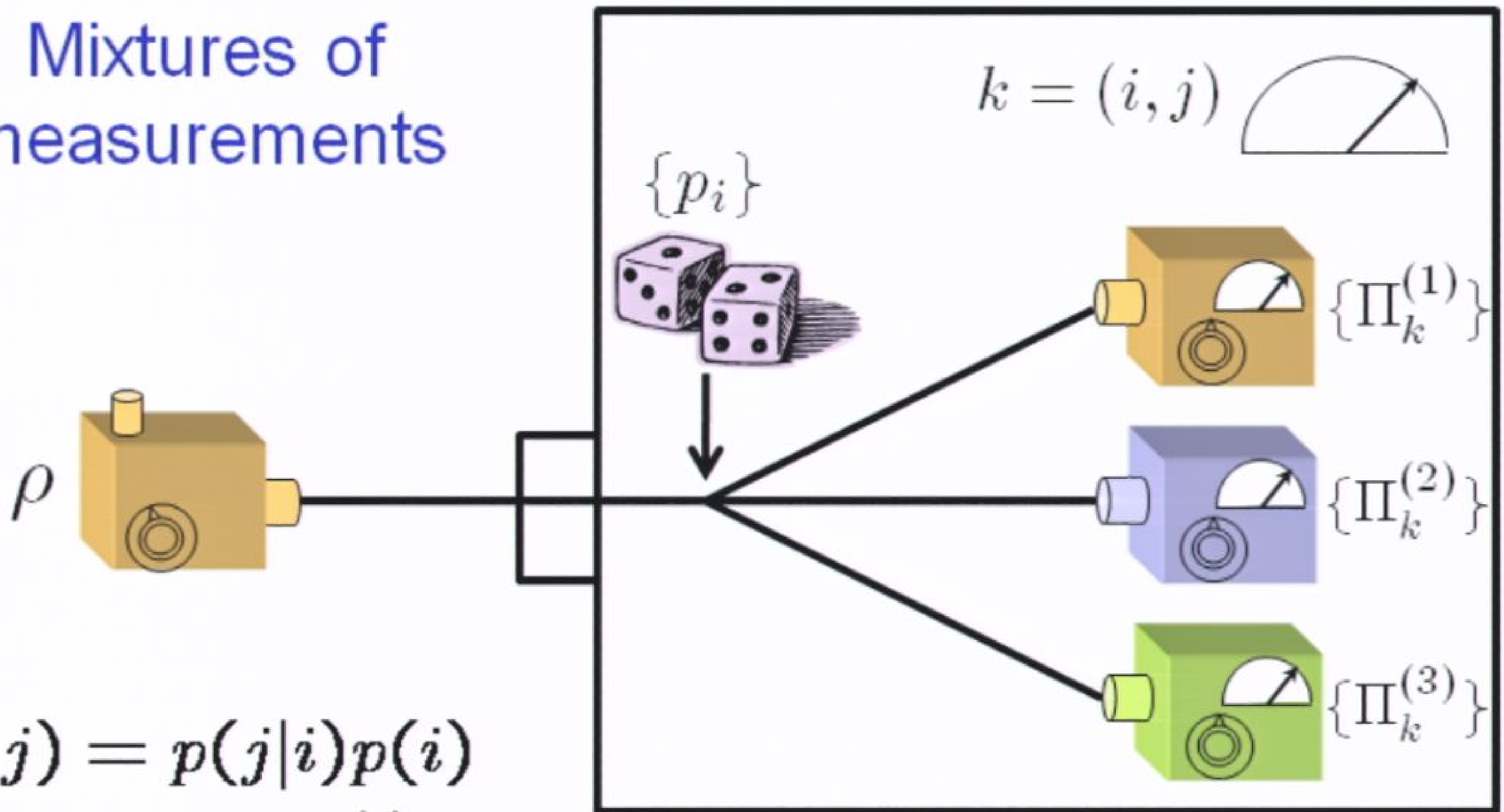


Mixtures of measurements



$$\begin{aligned} p(i,j) &= p(j|i)p(i) \\ &= \mathrm{Tr}(\Pi_j^{(i)}\rho)p_i \\ &= \mathrm{Tr}(p_i\Pi_j^{(i)}\rho) \end{aligned}$$

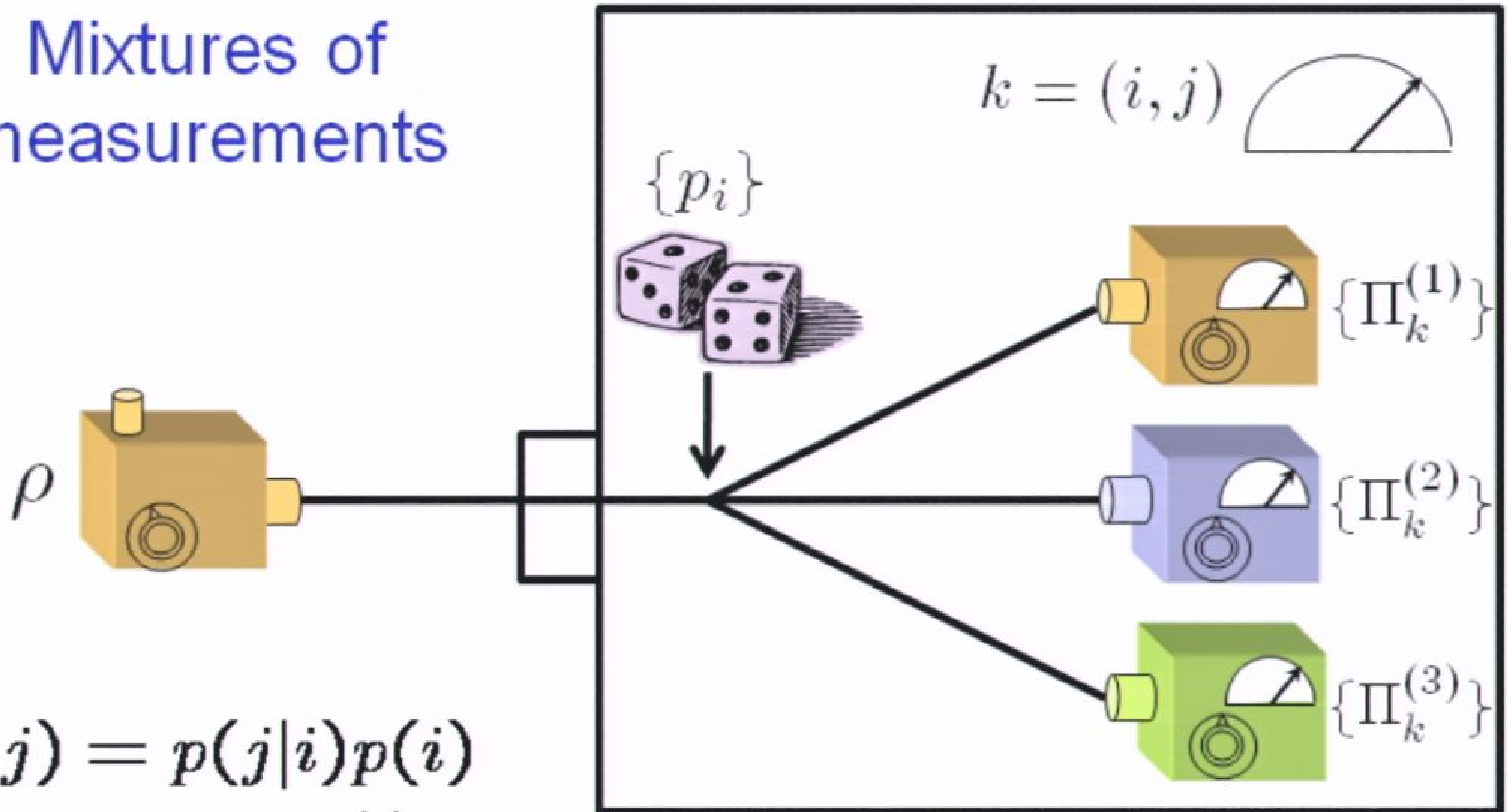
Mixtures of measurements



$$
\begin{aligned}
p(i,j) &= p(j|i)p(i) \\
&= \mathrm{Tr}(\Pi_j^{(i)}\rho)p_i \\
&= \mathrm{Tr}(\underbrace{p_i\Pi_j^{(i)}}_{E_{i,j}}\rho)
\end{aligned}
$$

$$p(k) = \mathrm{Tr}(E_k\rho)$$

Mixtures of measurements



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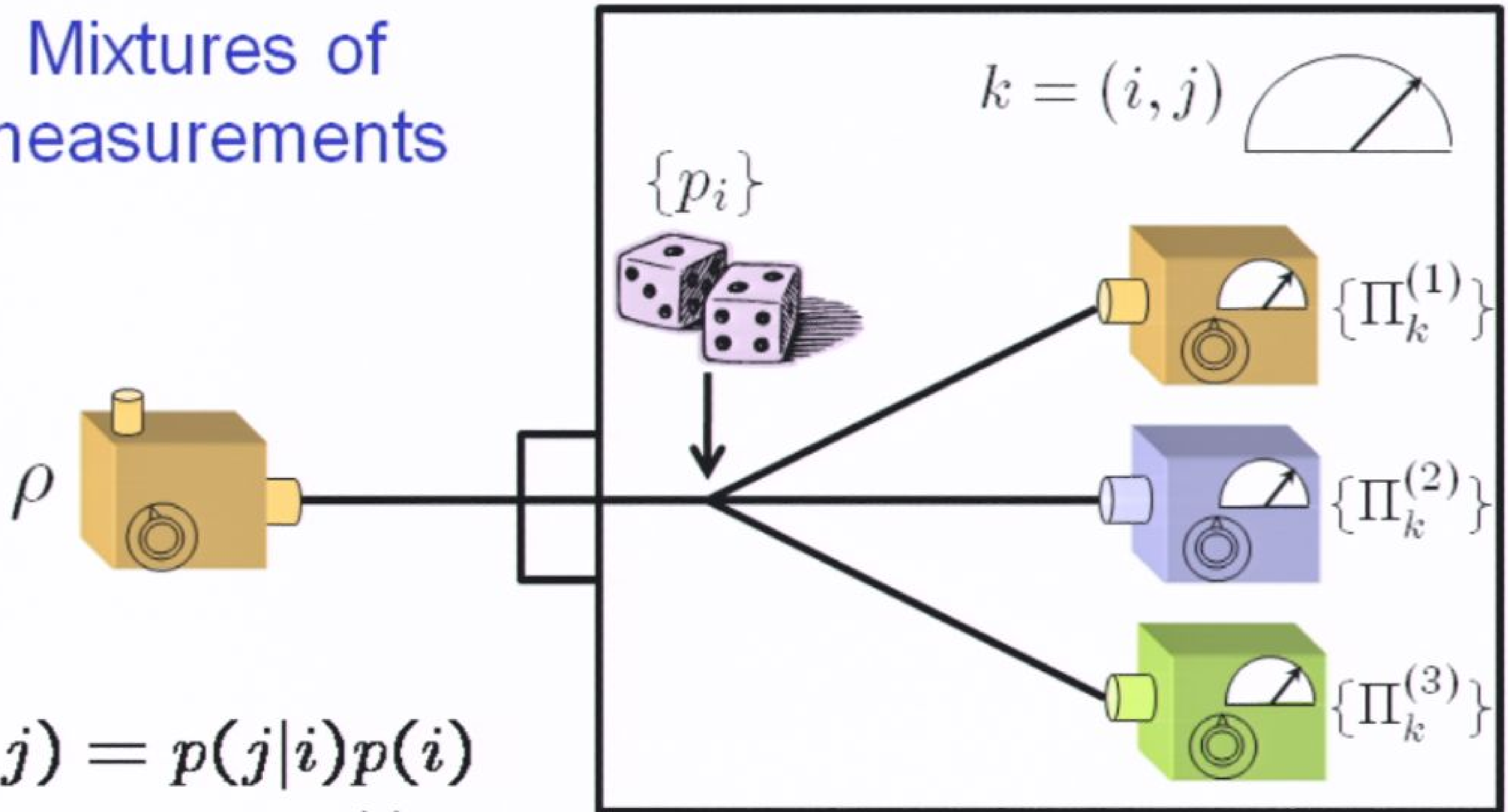
$$p(k) = \mathrm{Tr}(E_k\rho)$$

Positive $\langle\psi|E_k|\psi\rangle \geq 0 \;\; \forall|\psi\rangle \in \mathcal{H}$

Mixtures of measurements



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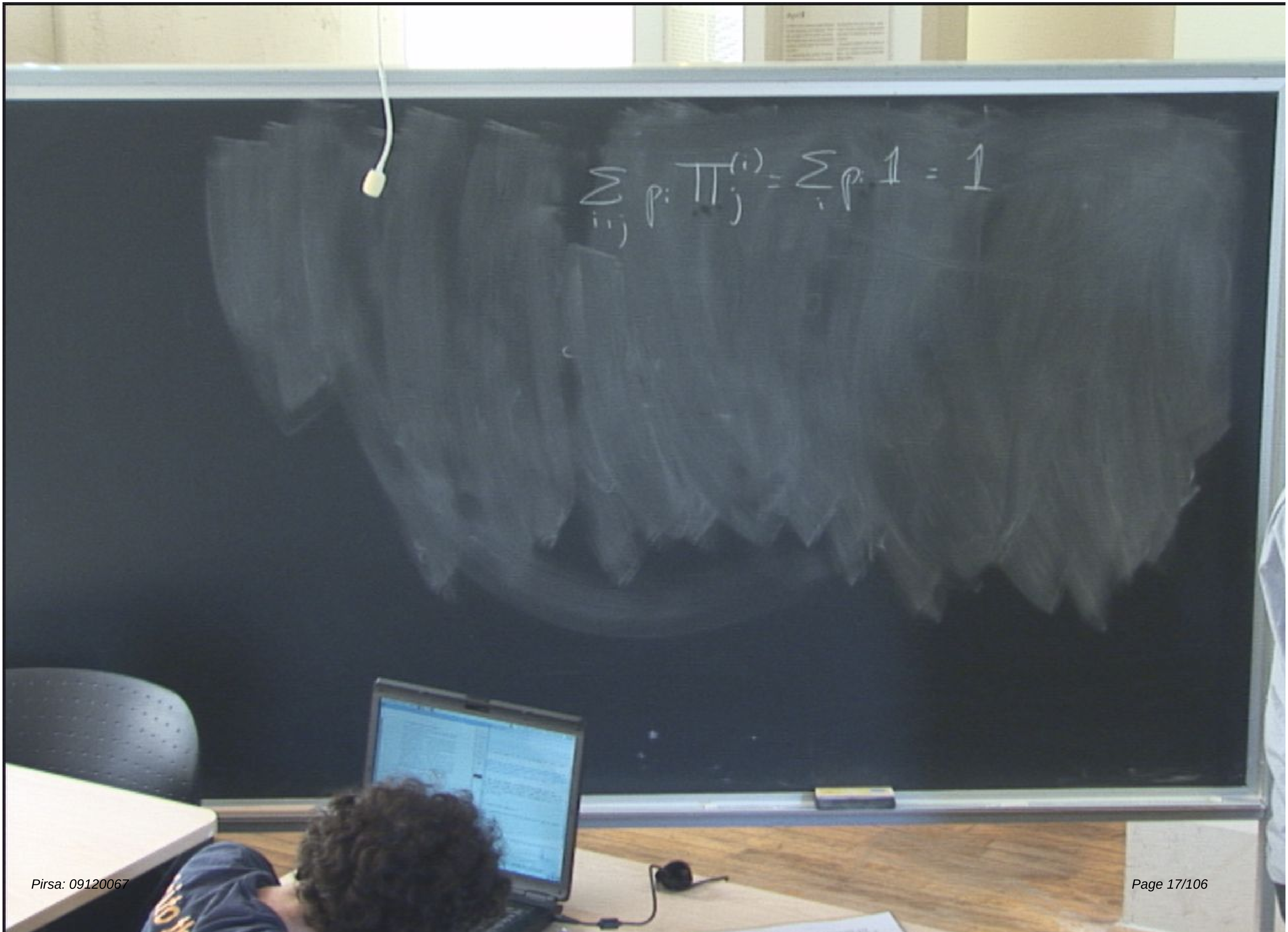
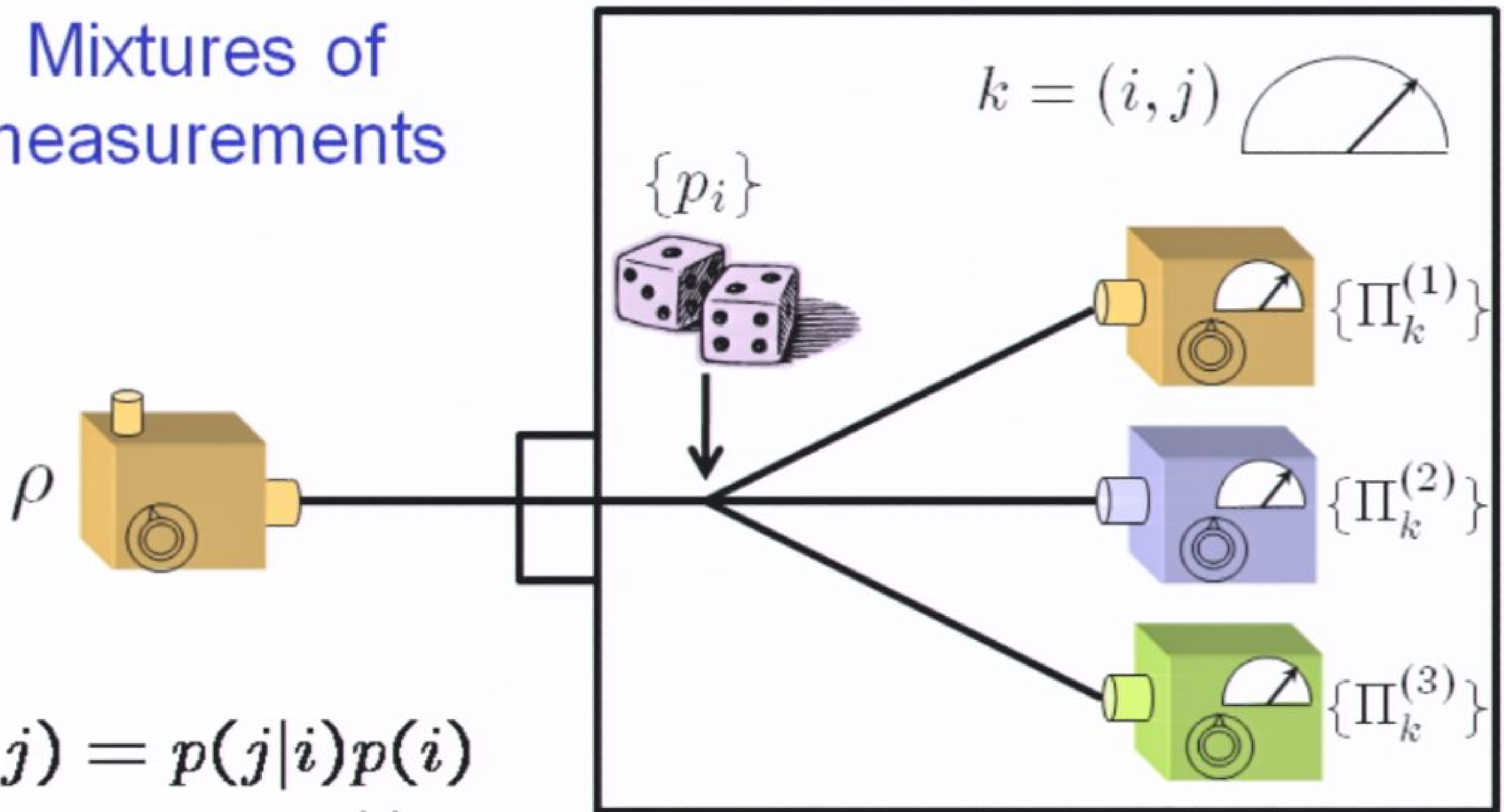
Sum to identity $\sum_k E_k = I$


$\sum_{i,j} p_i \Pi_j^{(i)} = \sum_i p_i \mathbb{1} = \mathbb{1}$

Mixtures of measurements



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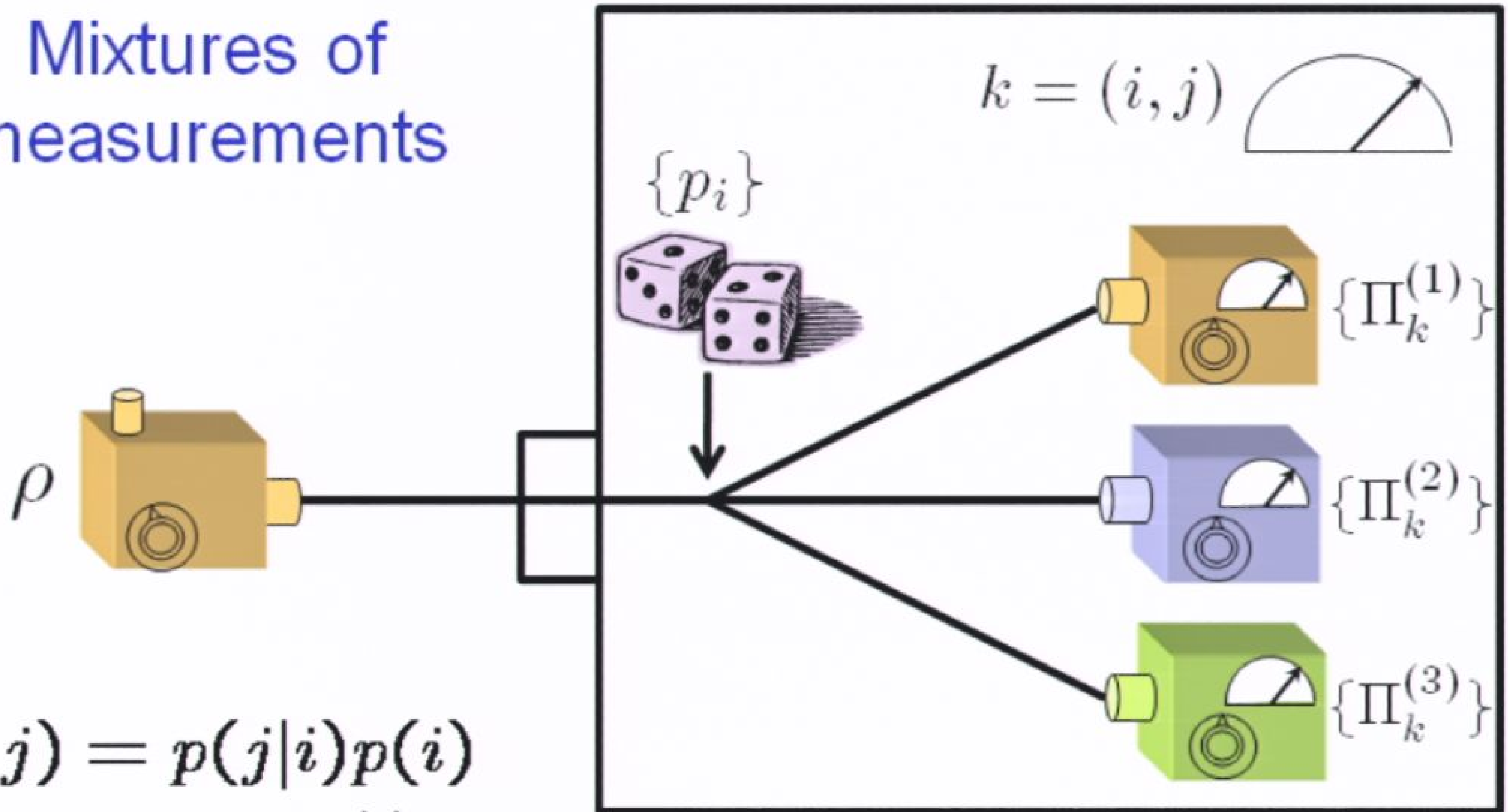
$\{E_k\}$ "Positive operator valued measure (POVM)"

Mixtures of measurements



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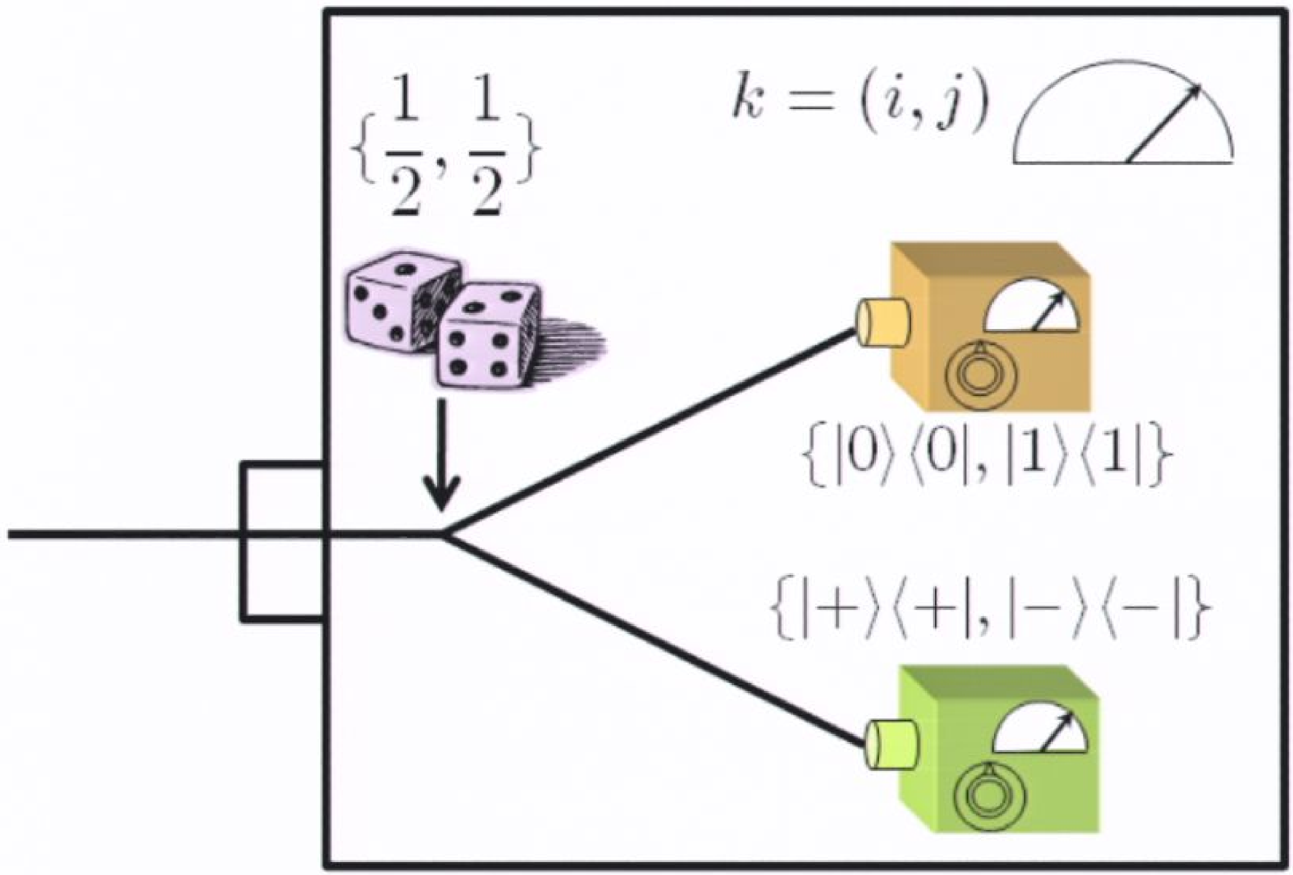
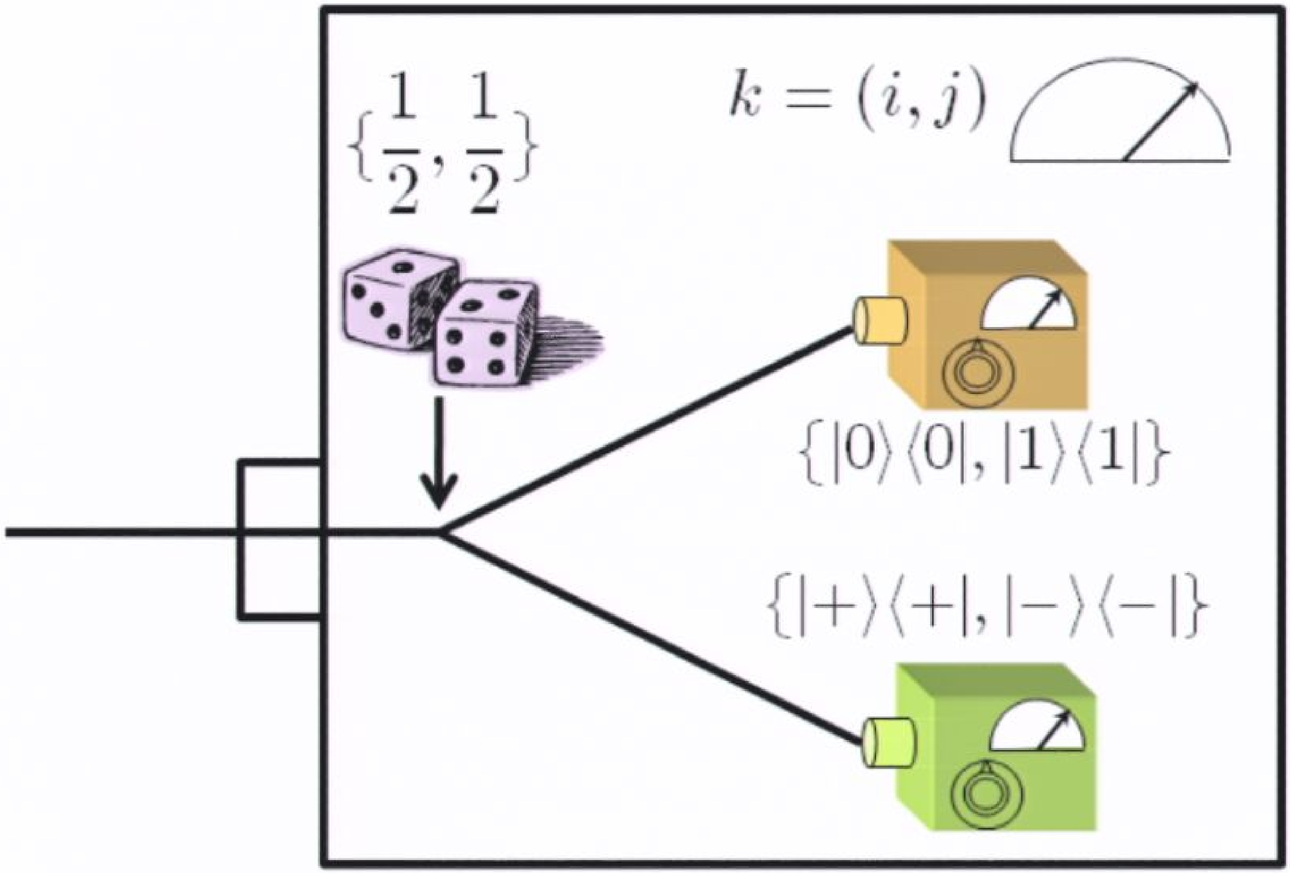
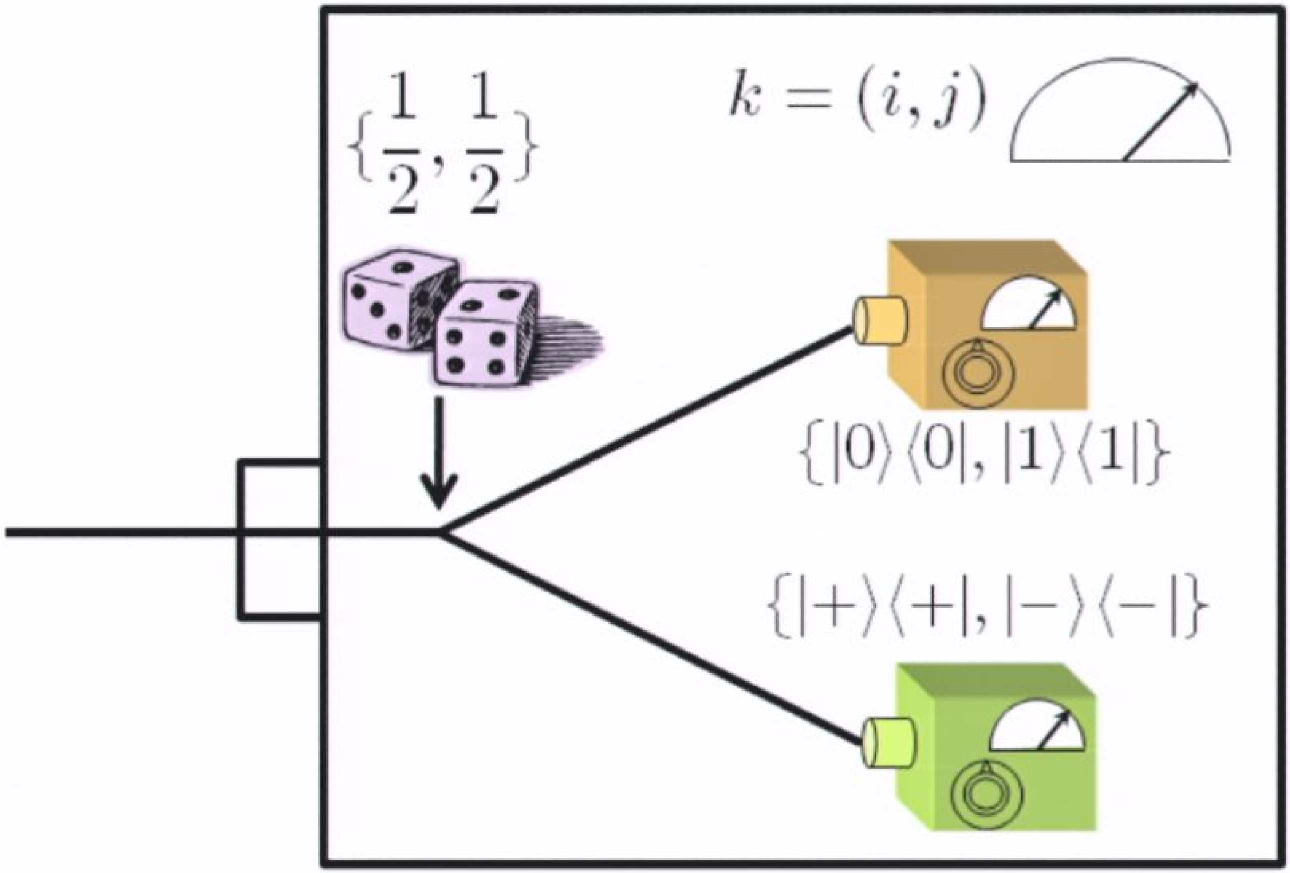


$\{\frac{1}{2}, \frac{1}{2}\}$
$k = (i, j)$
$\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$
$\{|+\rangle\langle +|, |-\rangle\langle -|\}$



$$\{\frac{1}{2}|0\rangle\langle 0| \, , \; \frac{1}{2}|1\rangle\langle 1| \, , \; \frac{1}{2}|+\rangle\langle +| \, , \; \frac{1}{2}|-\rangle\langle -|\}$$

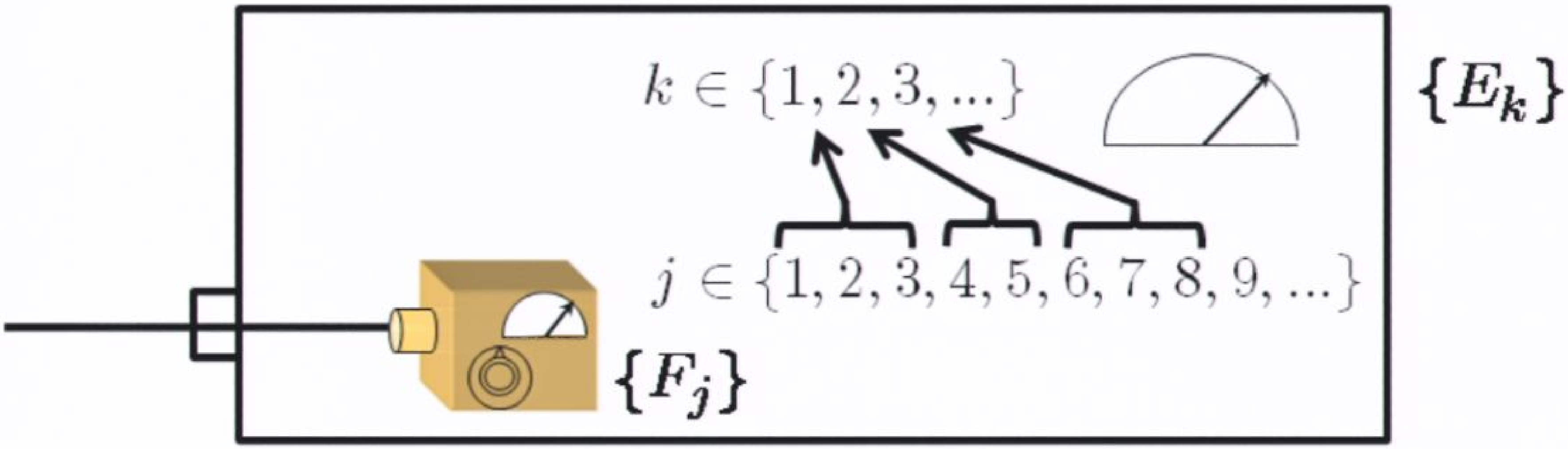


$$\{\frac{1}{2}|0\rangle\langle 0|\ ,\ \frac{1}{2}|1\rangle\langle 1|\ ,\ \frac{1}{2}|+\rangle\langle +|\ ,\ \frac{1}{2}|-\rangle\langle -|\}$$

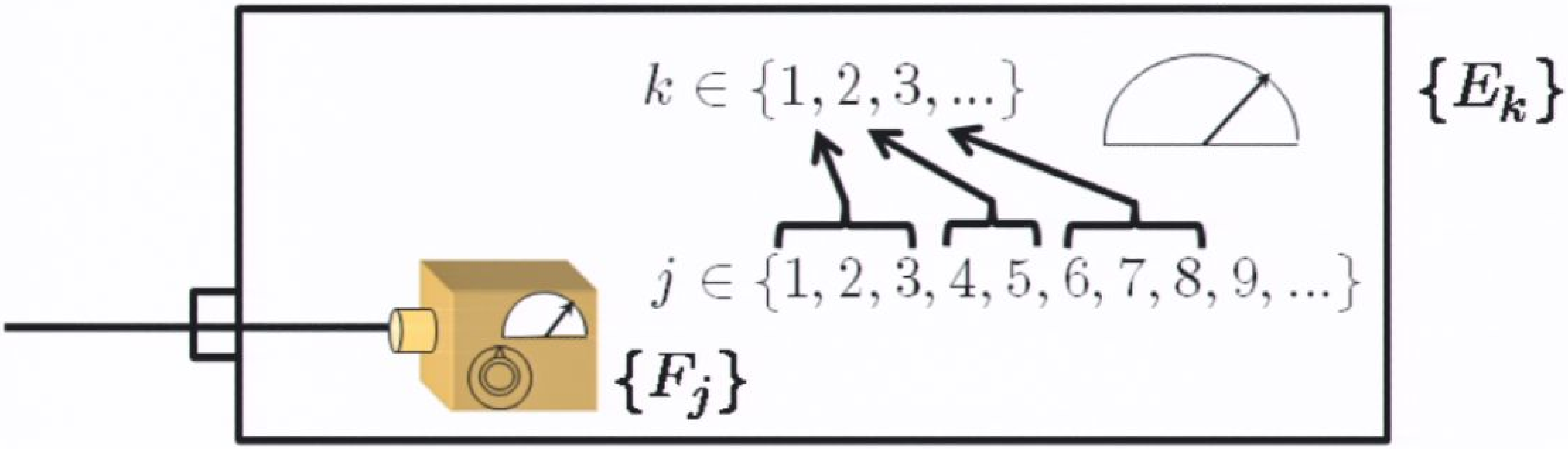
Recall

$$\frac{1}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1| + \frac{1}{4}|+\rangle\langle +| + \frac{1}{4}|-\rangle\langle -| = \frac{1}{2}I$$

Coarse-graining

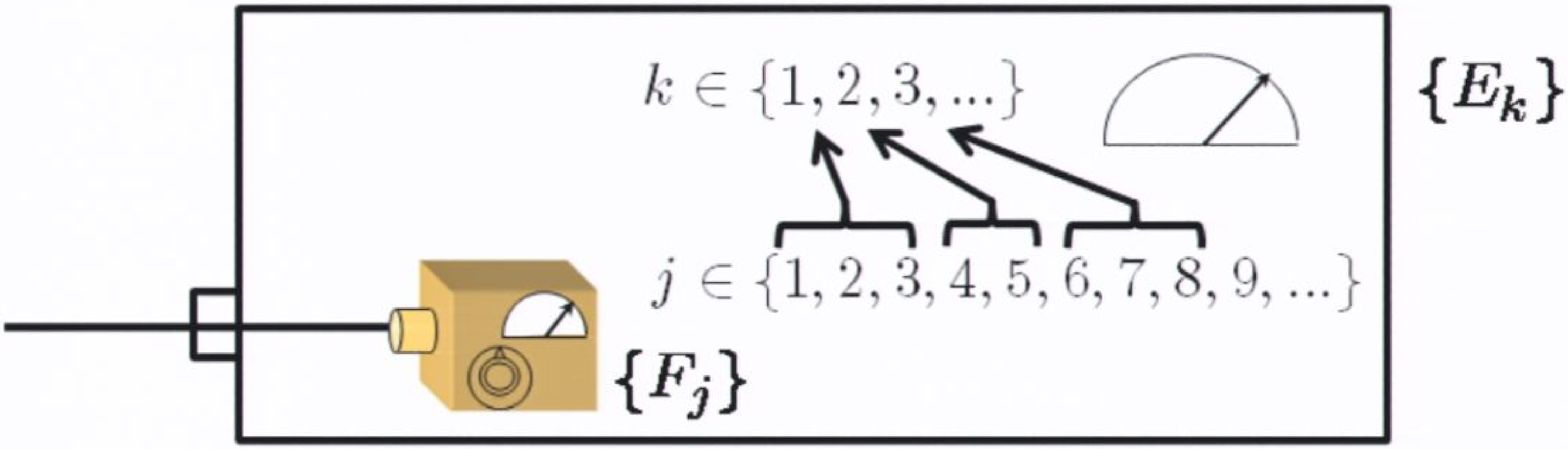


Coarse-graining



$$p(k) = \sum_{j \in S_k} p(j)$$

Coarse-graining

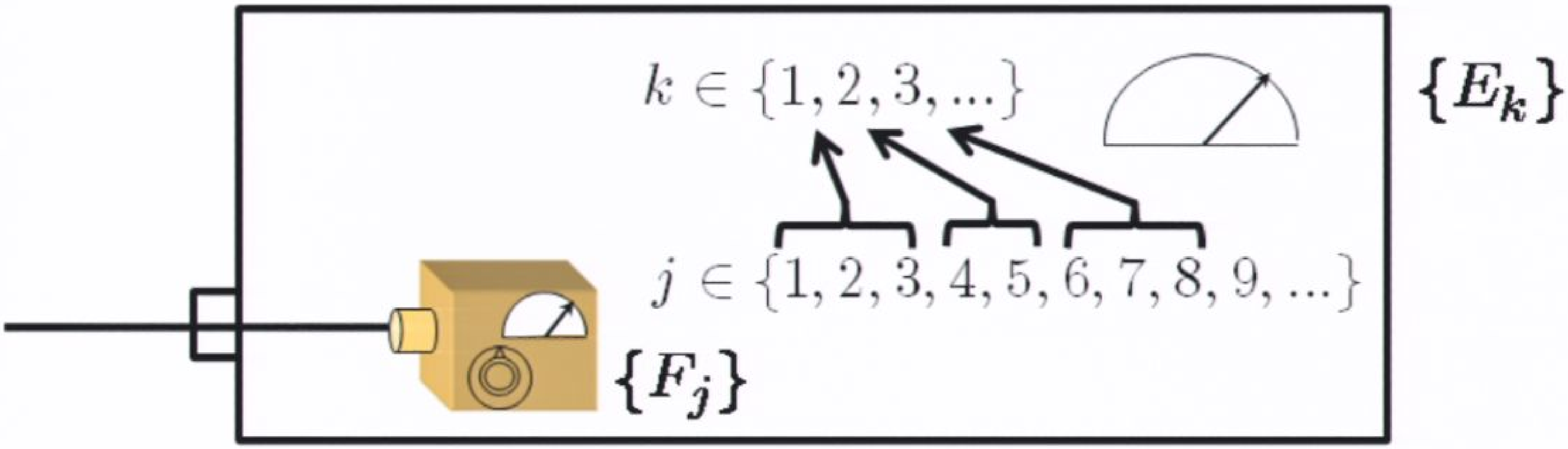


$$p(k) = \sum_{j \in S_k} p(j)$$

$$\mathrm{Tr}(E_k \rho) = \sum_{j \in S_k} \mathrm{Tr}(F_j \rho) \ \forall \rho$$

$$= \mathrm{Tr}[(\sum_{j \in S_k} F_j)\rho] \ \forall \rho$$

Coarse-graining



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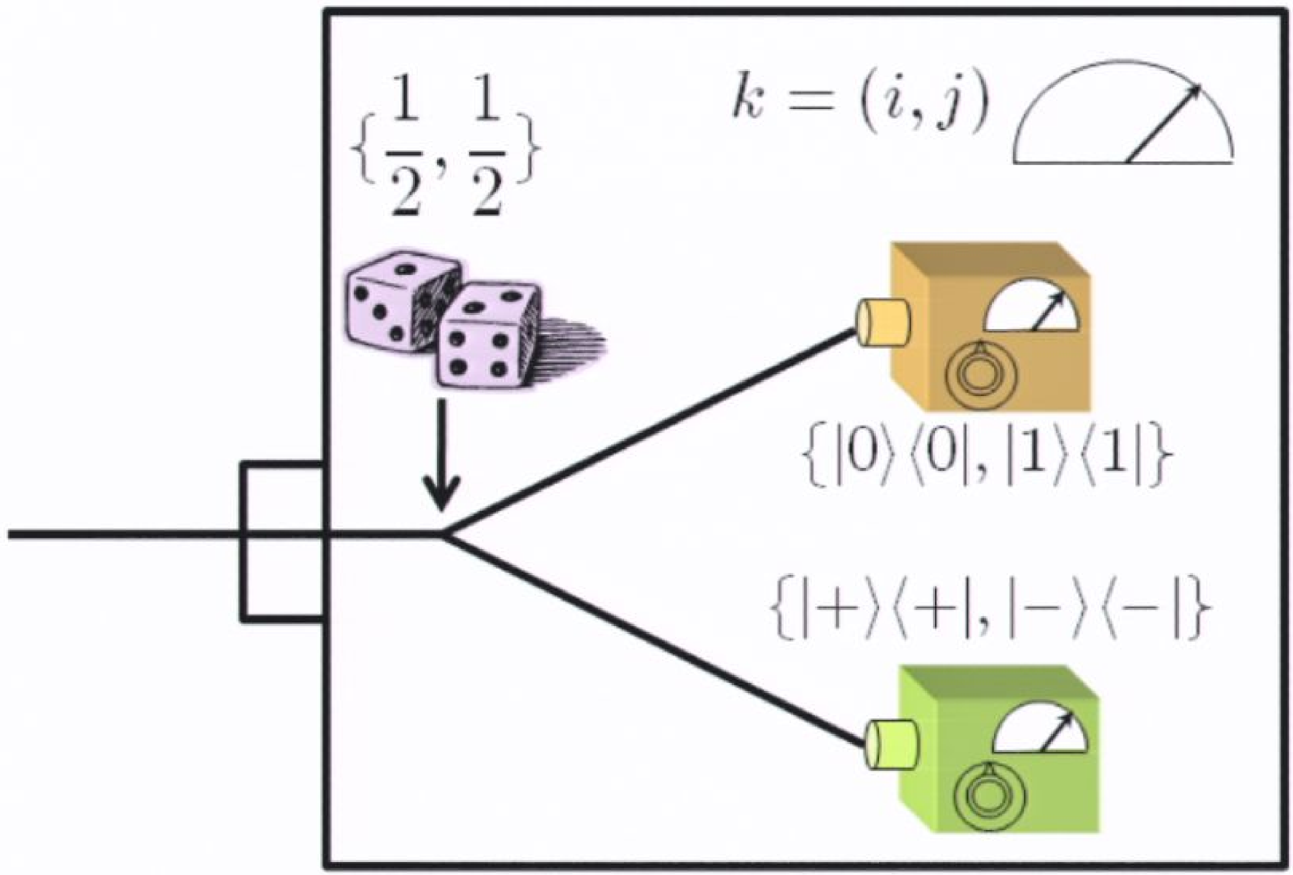
$$E_k = \sum_{j \in S_k} F_j$$

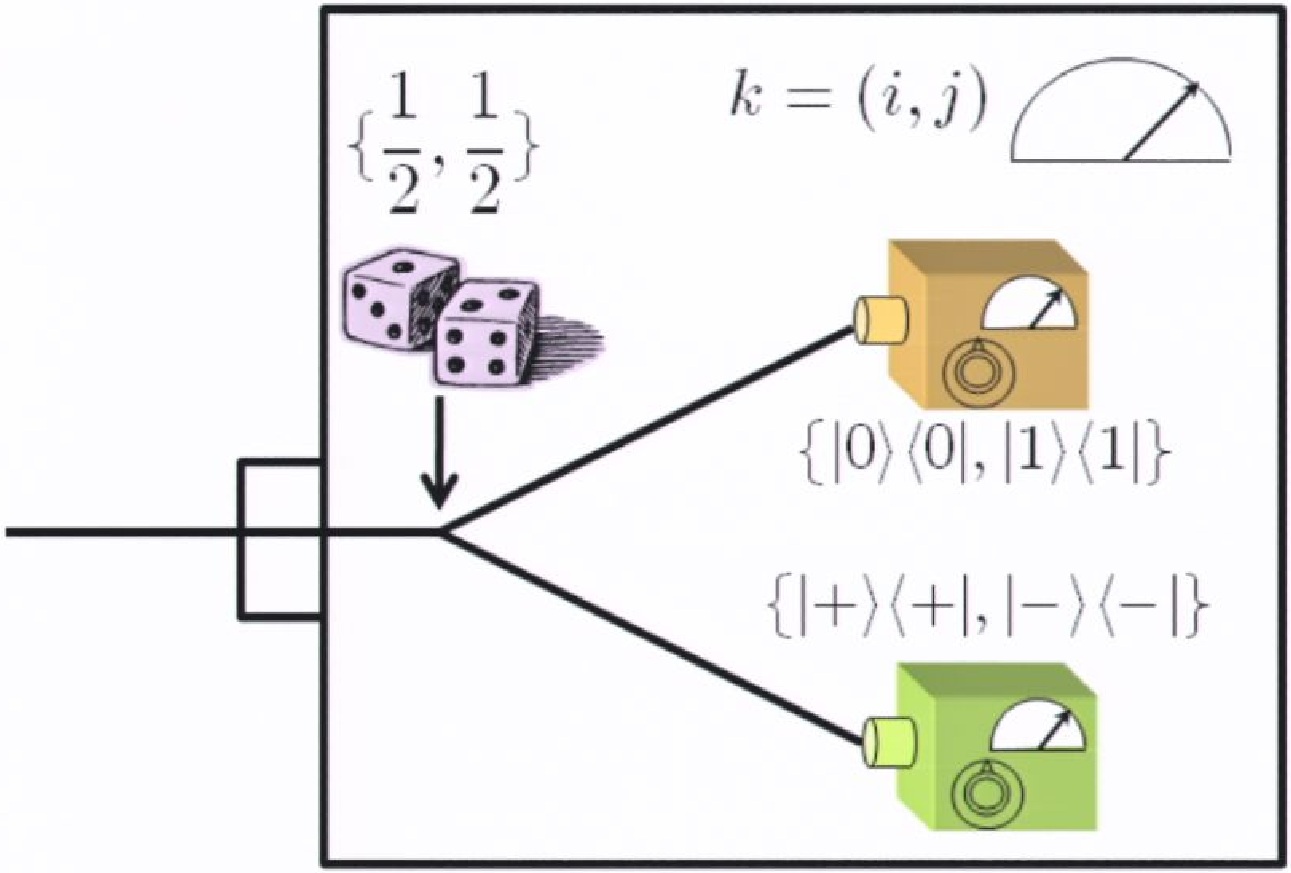
Note: the E_k need not be rank 1

Example



$$\{\frac{1}{2}|0\rangle\langle 0| \,,\ \frac{1}{2}|1\rangle\langle 1| \,,\ \frac{1}{2}|+\rangle\langle +| \,,\ \frac{1}{2}|-\rangle\langle -|\}$$

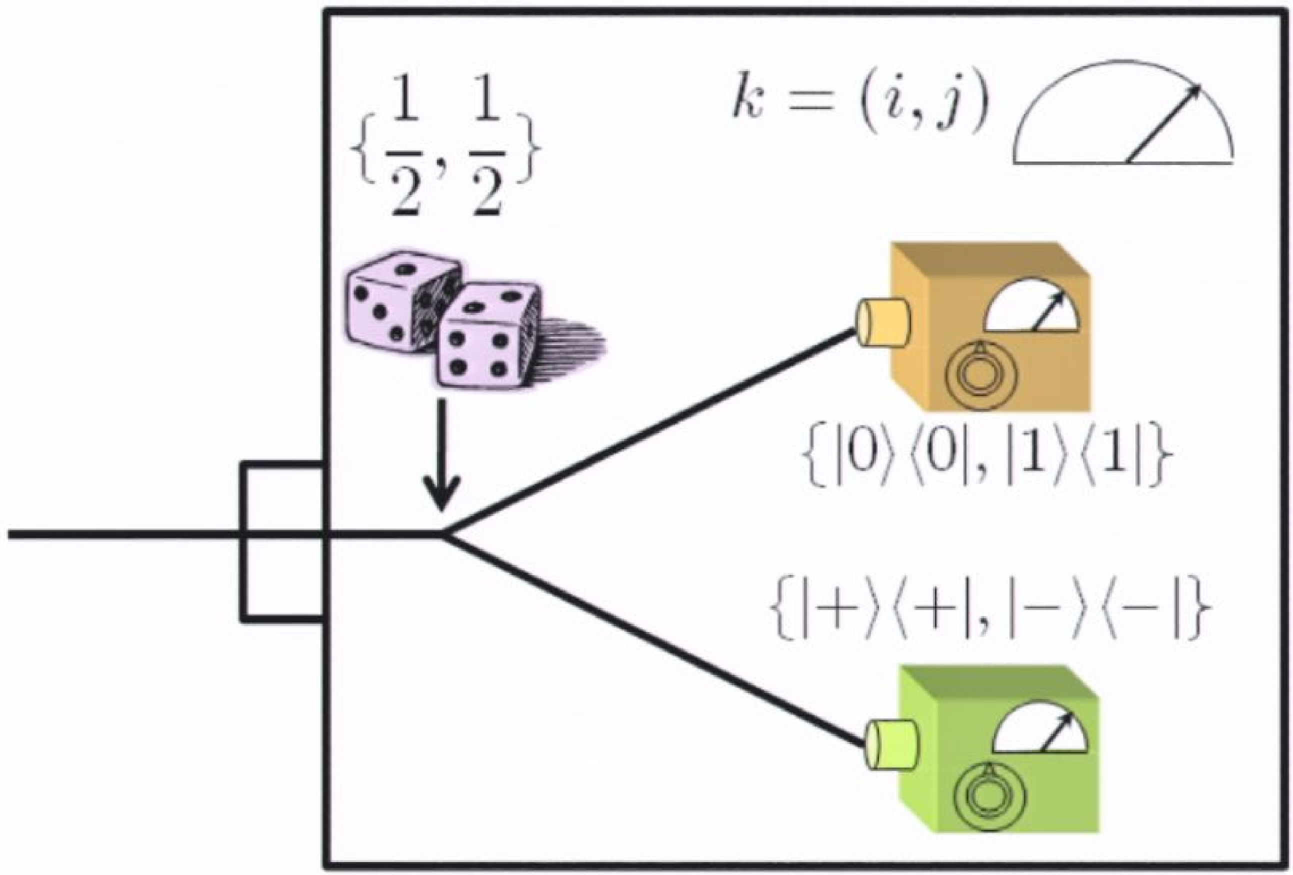
Example



$$\{\frac{1}{2}|0\rangle\langle 0| \,,\; \frac{1}{2}|1\rangle\langle 1| \,,\; \frac{1}{2}|+\rangle\langle +| \,,\; \frac{1}{2}|-\rangle\langle -|\}$$

$$\{\frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| \,,\; \frac{1}{2}|+\rangle\langle +| + \frac{1}{2}|-\rangle\langle -|\}$$

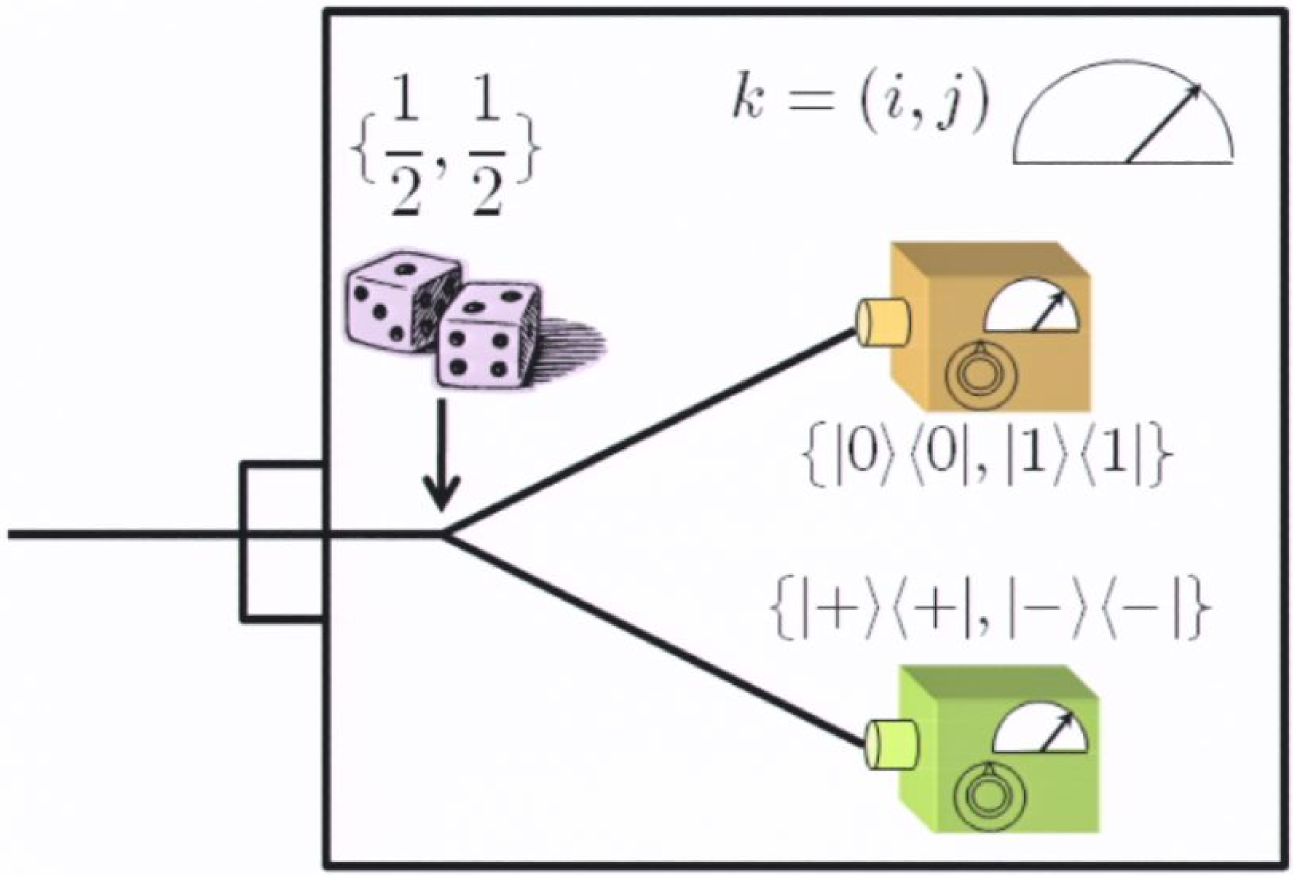
Example



$$\{\frac{1}{2}|0\rangle\langle 0|\ ,\ \frac{1}{2}|1\rangle\langle 1|\ ,\ \frac{1}{2}|+\rangle\langle +|\ ,\ \frac{1}{2}|-\rangle\langle -|\}$$

$$\{\frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|\ ,\ \frac{1}{2}|+\rangle\langle +| + \frac{1}{2}|-\rangle\langle -|\}$$

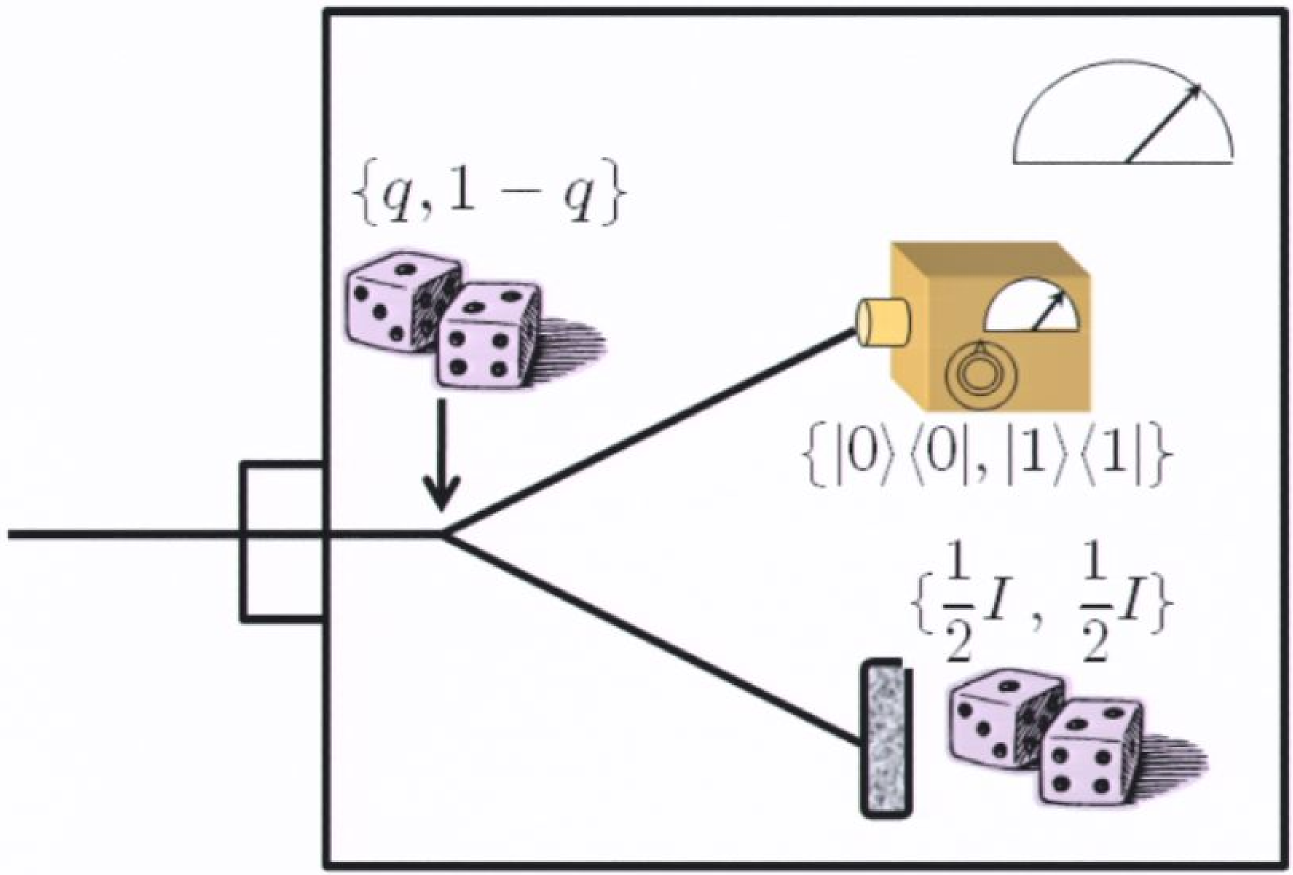
Example



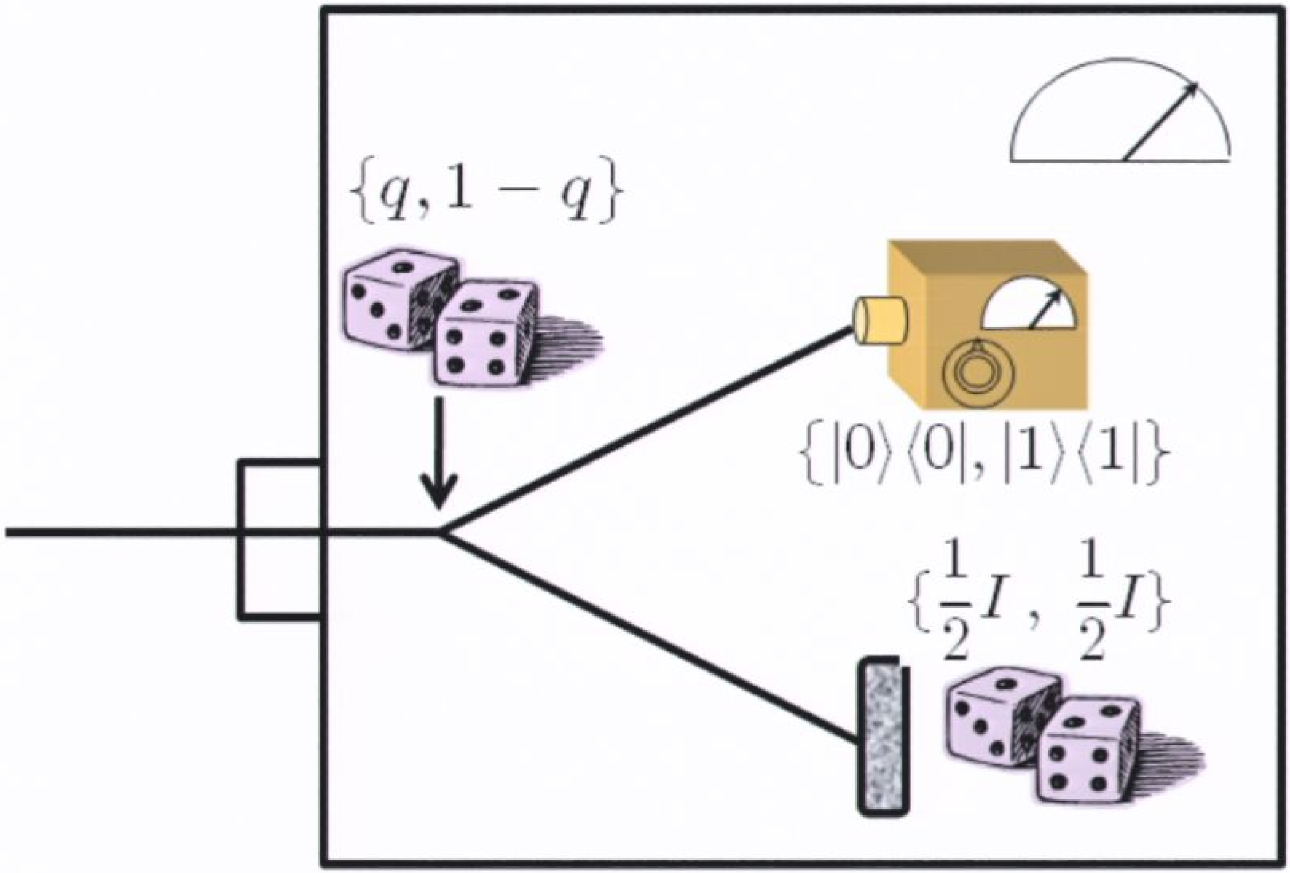
$$\{\frac{1}{2}|0\rangle\langle 0|\ ,\ \frac{1}{2}|1\rangle\langle 1|\ ,\ \frac{1}{2}|+\rangle\langle +|\ ,\ \frac{1}{2}|-\rangle\langle -|\}$$

$$\{\frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|\ ,\ \frac{1}{2}|+\rangle\langle +| + \frac{1}{2}|-\rangle\langle -|\} = \{\frac{1}{2}I\ ,\ \frac{1}{2}I$$

Another example

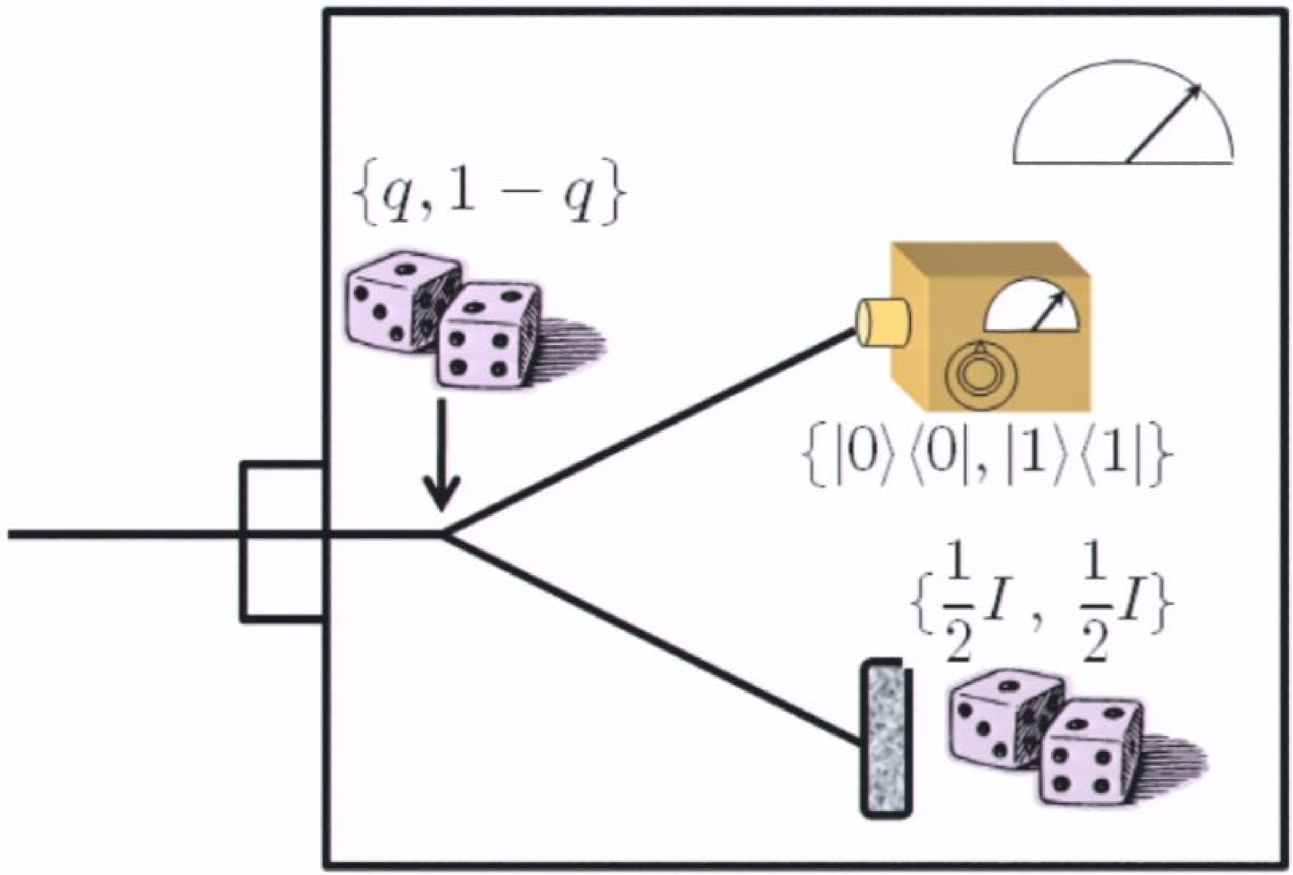


Another example



$$\{q|0\rangle\langle 0|\ ,\ q|1\rangle\langle 1|\ ,\ (1-q)\frac{1}{2}I\ ,\ (1-q)\frac{1}{2}I\}$$

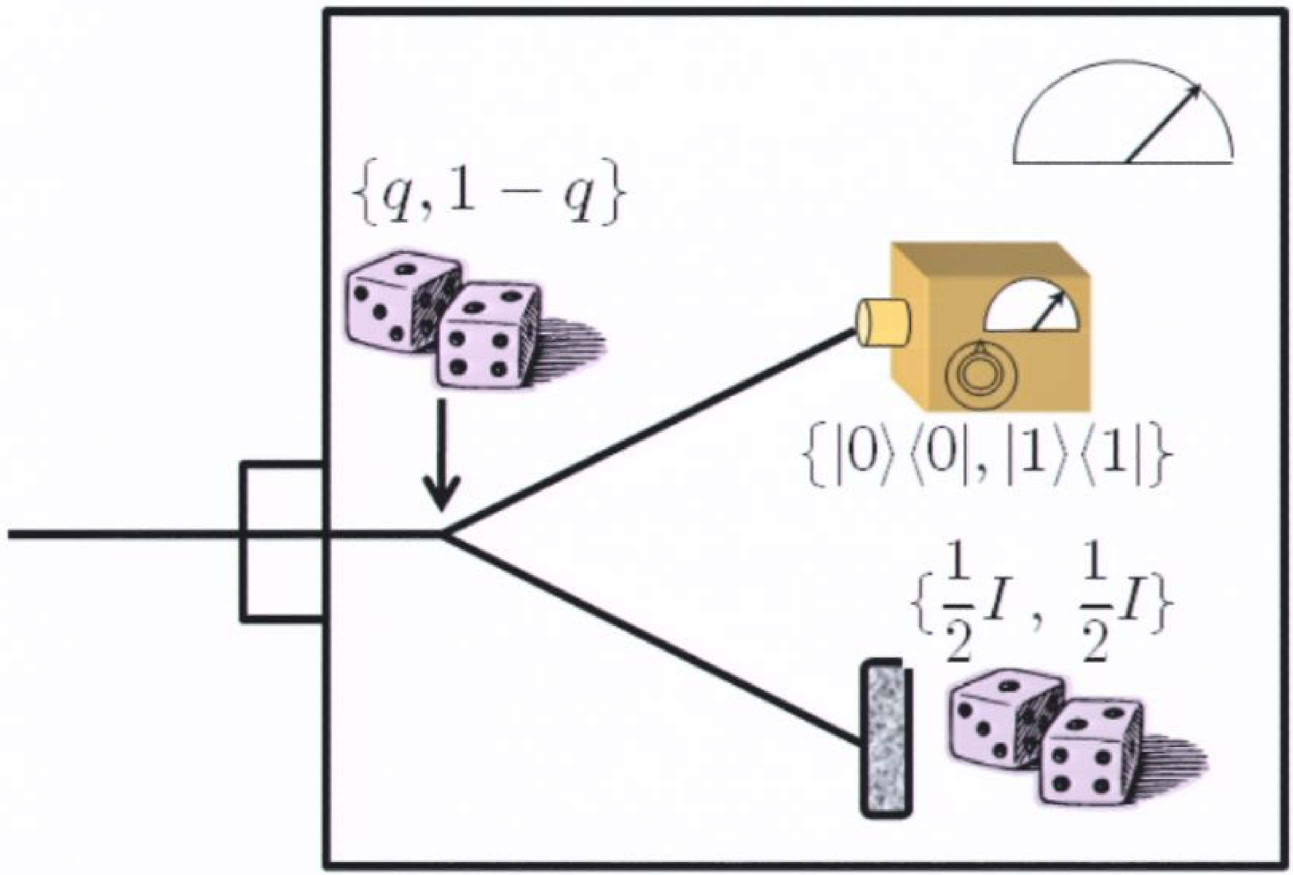
Another example



$$\{q|0\rangle\langle 0|\ ,\ q|1\rangle\langle 1|\ ,\ (1-q)\frac{1}{2}I\ ,\ (1-q)\frac{1}{2}I\}$$

$$\{q|0\rangle\langle 0| + (1-q)\frac{1}{2}I\ \ ,\ \ q|1\rangle\langle 1| + (1-q)\frac{1}{2}I\}$$

Another example

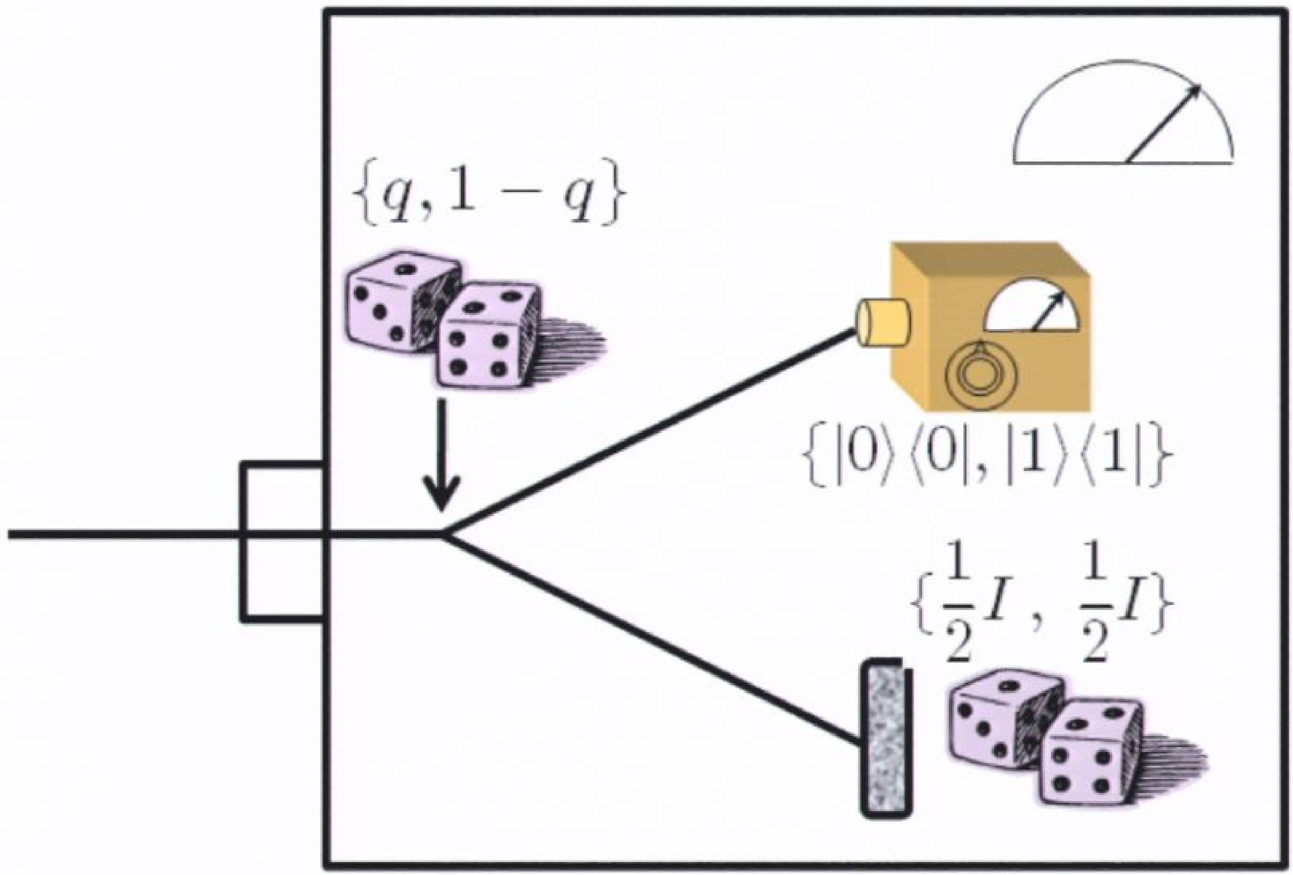


$$\{q|0\rangle\langle 0|\ ,\ q|1\rangle\langle 1|\ ,\ (1-q)\frac{1}{2}I\ ,\ (1-q)\frac{1}{2}I\}$$

$$\{q|0\rangle\langle 0| + (1-q)\frac{1}{2}I\ \ ,\ \ q|1\rangle\langle 1| + (1-q)\frac{1}{2}I\}$$

$$= \{\frac{1+q}{2}|0\rangle\langle 0| + \frac{1-q}{2}|1\rangle\langle 1|\ ,\ \frac{1-q}{2}|0\rangle\langle 0| + \frac{1+q}{2}|1\rangle\langle 1|\}$$

Another example



$$\{q|0\rangle\langle 0|\ ,\ q|1\rangle\langle 1|\ ,\ (1-q)\frac{1}{2}I\ ,\ (1-q)\frac{1}{2}I\}$$

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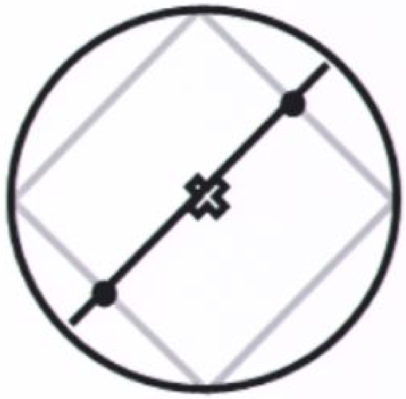
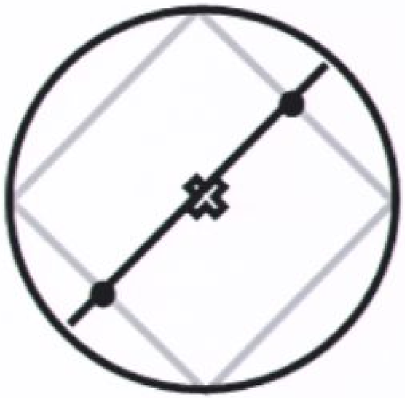
$$\{\frac{1}{2}|0\rangle\langle 0|\ ,\ \frac{1}{2}|1\rangle\langle 1|\ ,\ \frac{1}{2}|+\rangle\langle +|\ ,\ \frac{1}{2}|-\rangle\langle -|\}$$



$$\{\frac{1}{2}|0\rangle\langle 0|\,,\ \frac{1}{2}|1\rangle\langle 1|\,,\ \frac{1}{2}|+\rangle\langle +|\,,\ \frac{1}{2}|-\rangle\langle -|\}$$

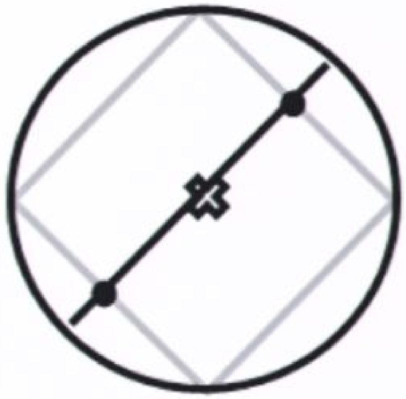
$$\{E_0, E_1\}$$



$$\{\frac{1}{2}|0\rangle\langle 0|\,,\ \frac{1}{2}|1\rangle\langle 1|\,,\ \frac{1}{2}|+\rangle\langle +|\,,\ \frac{1}{2}|-\rangle\langle -|\}$$

$$\{E_0, E_1\}$$

Noisy S· n



$$\{\frac{1}{2}|0\rangle\langle 0|\,,\ \frac{1}{2}|1\rangle\langle 1|\,,\ \frac{1}{2}|+\rangle\langle +|\,,\ \frac{1}{2}|-\rangle\langle -|\}$$

$$\{E_0, E_1\}$$

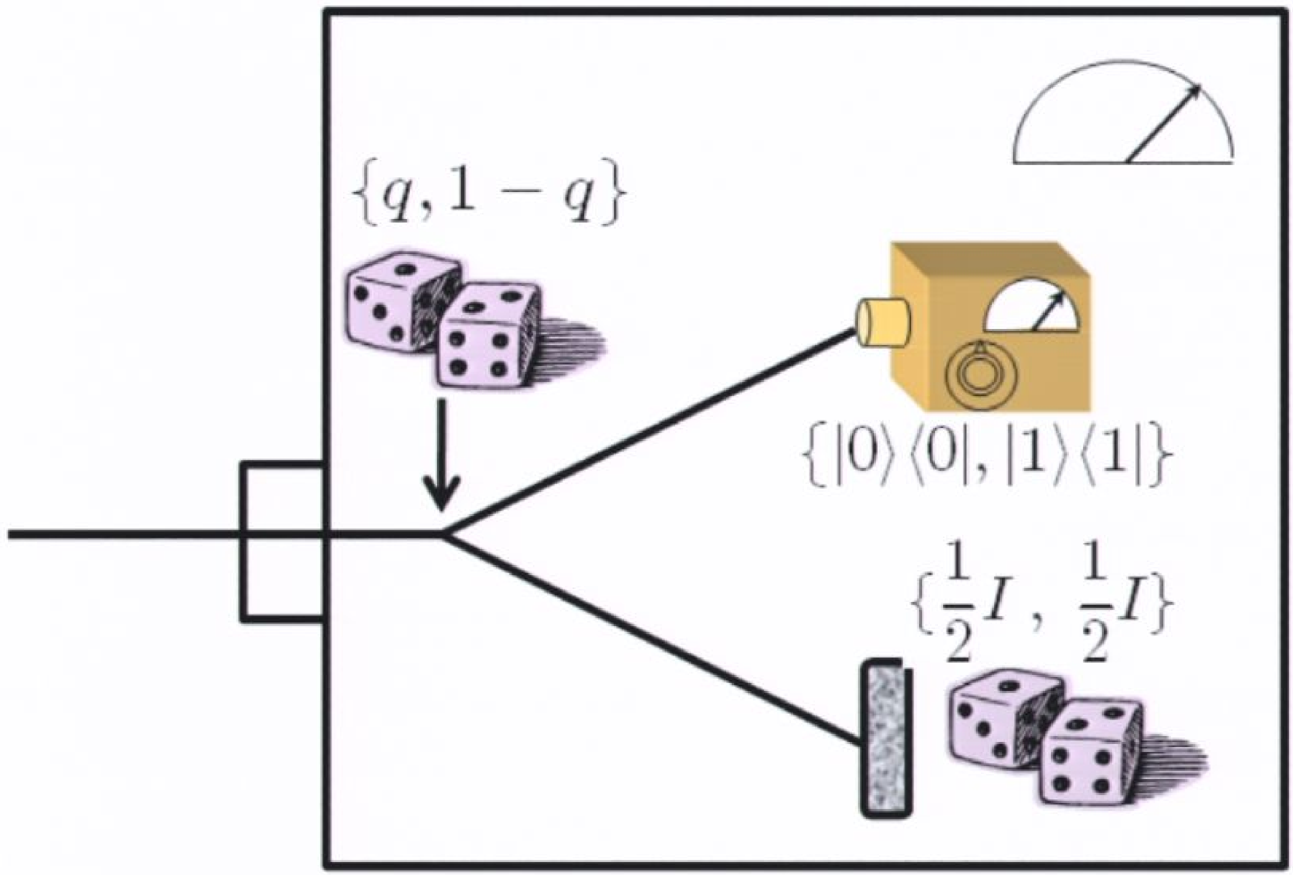
Noisy S· n

$$\{\frac{1}{2}|0\rangle\langle 0|\,,\ \frac{1}{2}|1\rangle\langle 1|\,,\ \frac{1}{2}|+\rangle\langle +|\,,\ \frac{1}{2}|-\rangle\langle -|\}$$

$$\{F_0, F_1\}$$

Noisy S· $n^{\perp}$

Another example

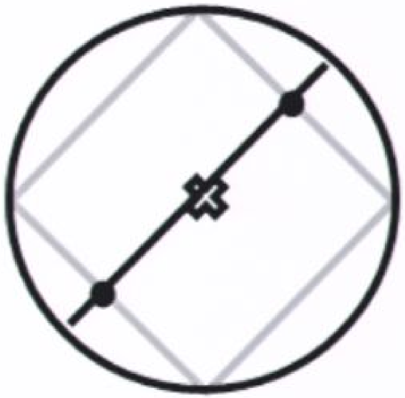


$$\{q|0\rangle\langle 0| \;,\; q|1\rangle\langle 1| \;,\; (1-q)\frac{1}{2}I \;,\; (1-q)\frac{1}{2}I\}$$

$$\{q|0\rangle\langle 0| + (1-q)\frac{1}{2}I \;\;,\;\; q|1\rangle\langle 1| + (1-q)\frac{1}{2}I\}$$

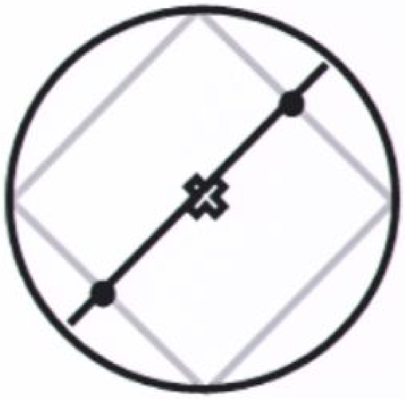
$$= \{\frac{1+q}{2}|0\rangle\langle 0| + \frac{1-q}{2}|1\rangle\langle 1| \;,\; \frac{1-q}{2}|0\rangle\langle 0| + \frac{1+q}{2}|1\rangle\langle 1|\}$$

Noisy S =



$$\{\frac{1}{2}|0\rangle\langle 0|\,,\ \frac{1}{2}|1\rangle\langle 1|\,,\ \frac{1}{2}|+\rangle\langle +|\,,\ \frac{1}{2}|-\rangle\langle -|\}$$

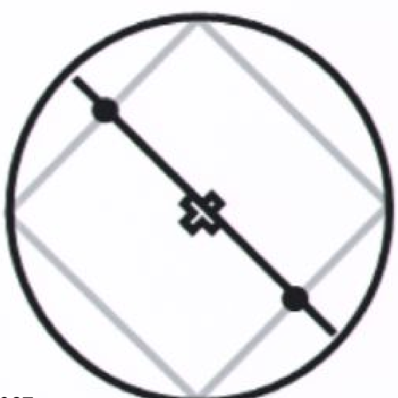
$$\{E_0, E_1\}$$



$$\{\frac{1}{2}|0\rangle\langle 0|\,,\ \frac{1}{2}|1\rangle\langle 1|\,,\ \frac{1}{2}|+\rangle\langle +|\,,\ \frac{1}{2}|-\rangle\langle -|\}$$

$$\{E_0, E_1\}$$

Noisy S· n

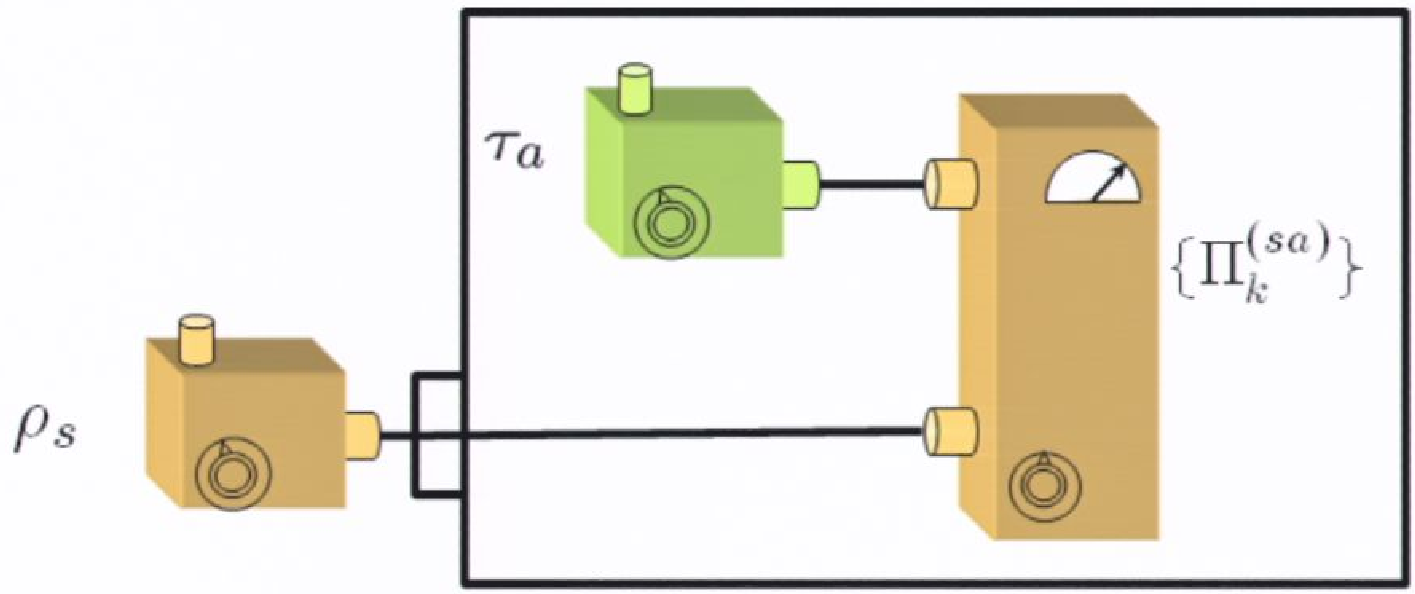


$$\{\frac{1}{2}|0\rangle\langle 0|\,,\ \frac{1}{2}|1\rangle\langle 1|\,,\ \frac{1}{2}|+\rangle\langle +|\,,\ \frac{1}{2}|-\rangle\langle -|\}$$

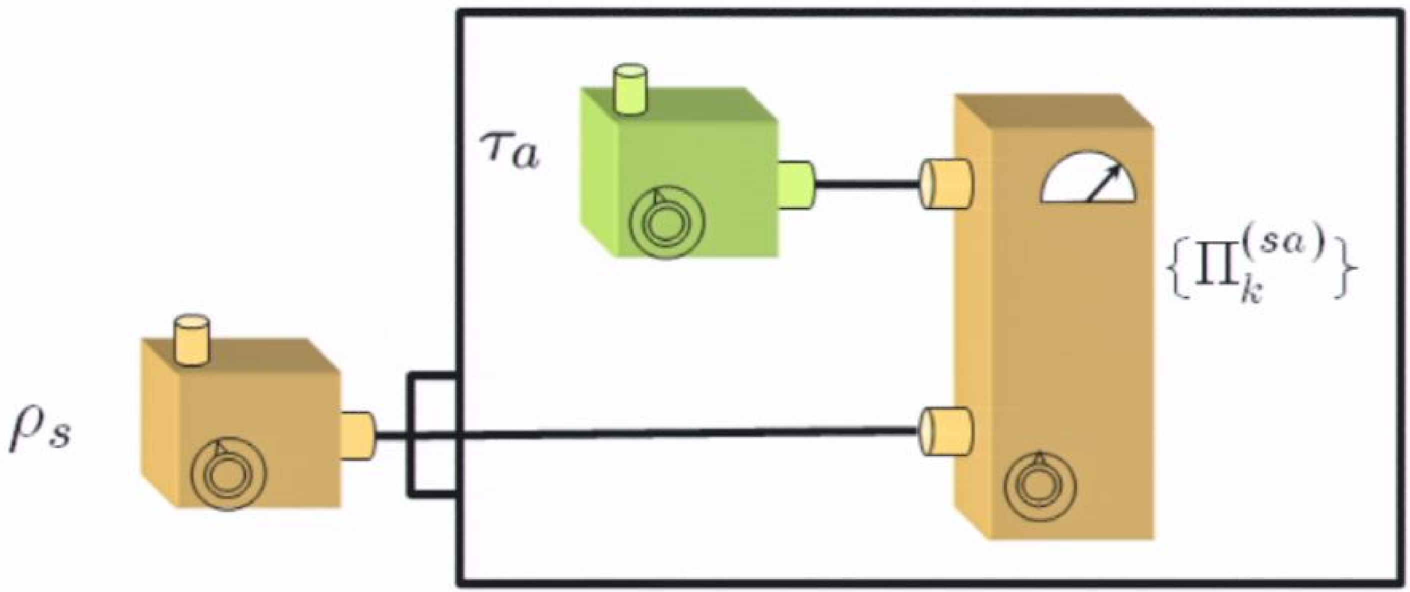
$$\{F_0, F_1\}$$

Noisy S· $n^{\perp}$

Measurement by coupling to an ancilla

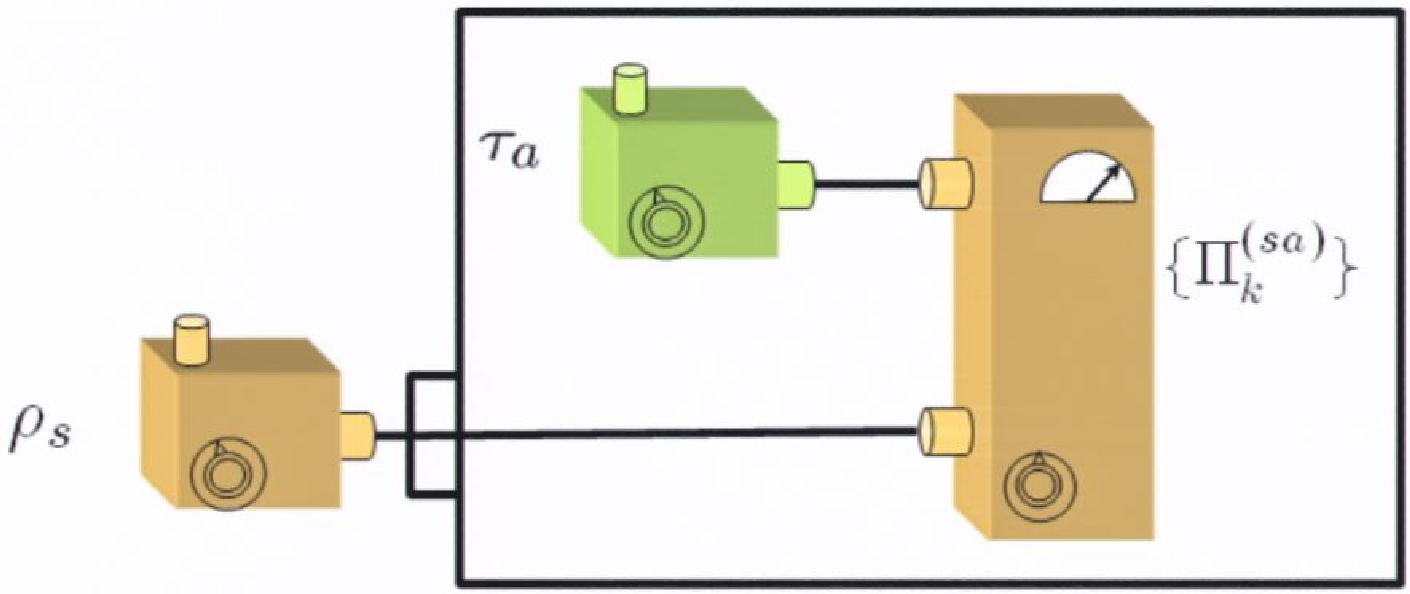


Measurement by coupling to an ancilla



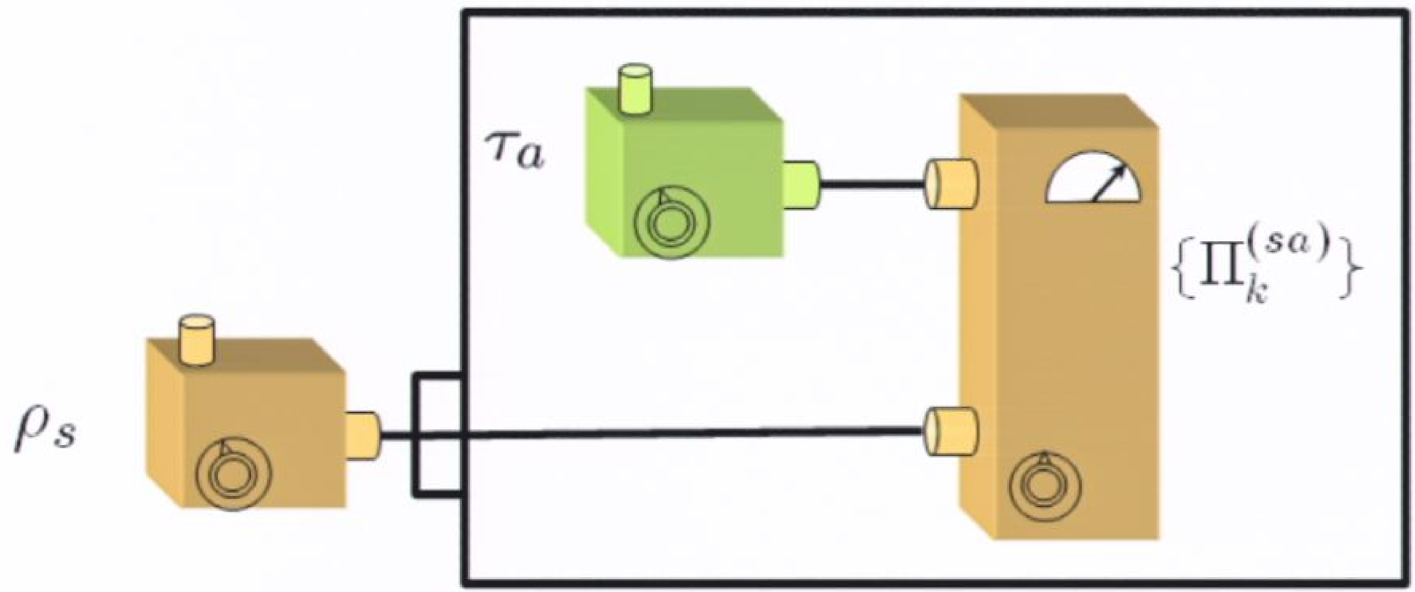
$$p(k) = \mathrm{Tr}_{sa}[\Pi_k^{(sa)}(\rho_s \otimes \tau_a)]$$
$$= \mathrm{Tr}_s[\mathrm{Tr}_a(\Pi_k^{(sa)}\tau_a)\,\rho_s]$$

Measurement by coupling to an ancilla



$$p(k) = \mathrm{Tr}_{sa}[\Pi_k^{(sa)}(\rho_s \otimes \tau_a)]$$
$$= \mathrm{Tr}_s[\underbrace{\mathrm{Tr}_a(\Pi_k^{(sa)}\tau_a)}_{E_k^{(s)}}\ \rho_s]$$

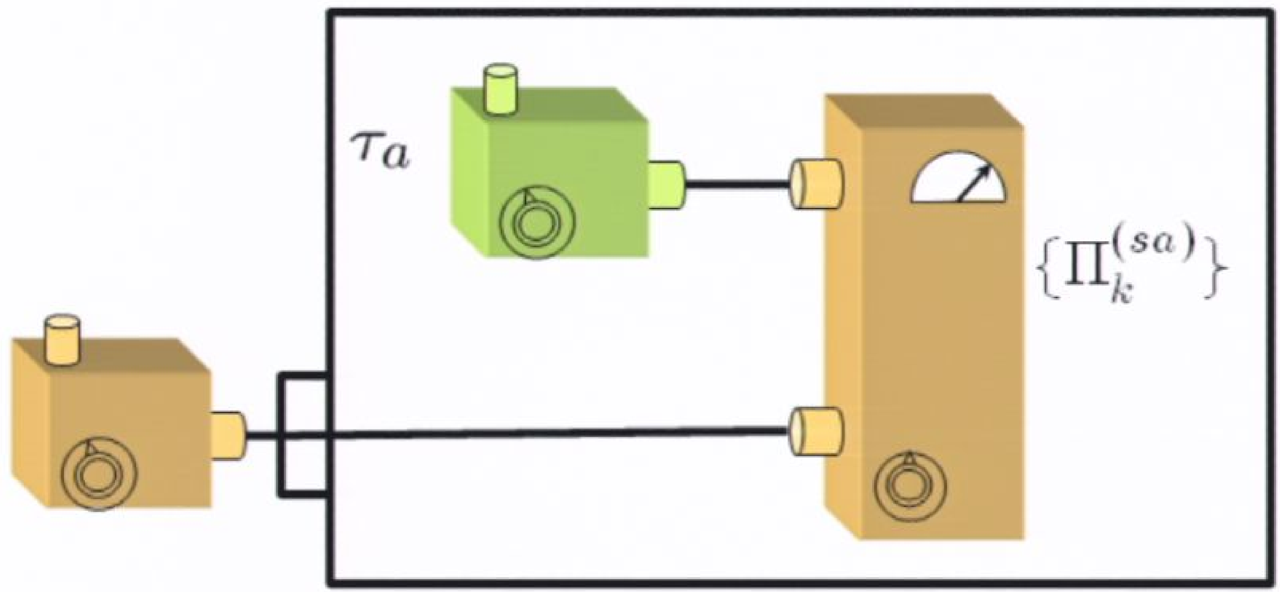
Measurement by coupling to an ancilla



$$p(k) = \mathrm{Tr}_{sa}[\Pi_k^{(sa)}(\rho_s \otimes \tau_a)]$$
$$= \mathrm{Tr}_s[\underbrace{\mathrm{Tr}_a(\Pi_k^{(sa)}\tau_a)}_{E_k^{(s)}}\,\rho_s]$$

$$p(k) = \mathrm{Tr}_s(E_k^{(s)}\rho_s)$$

Measurement by coupling to an ancilla

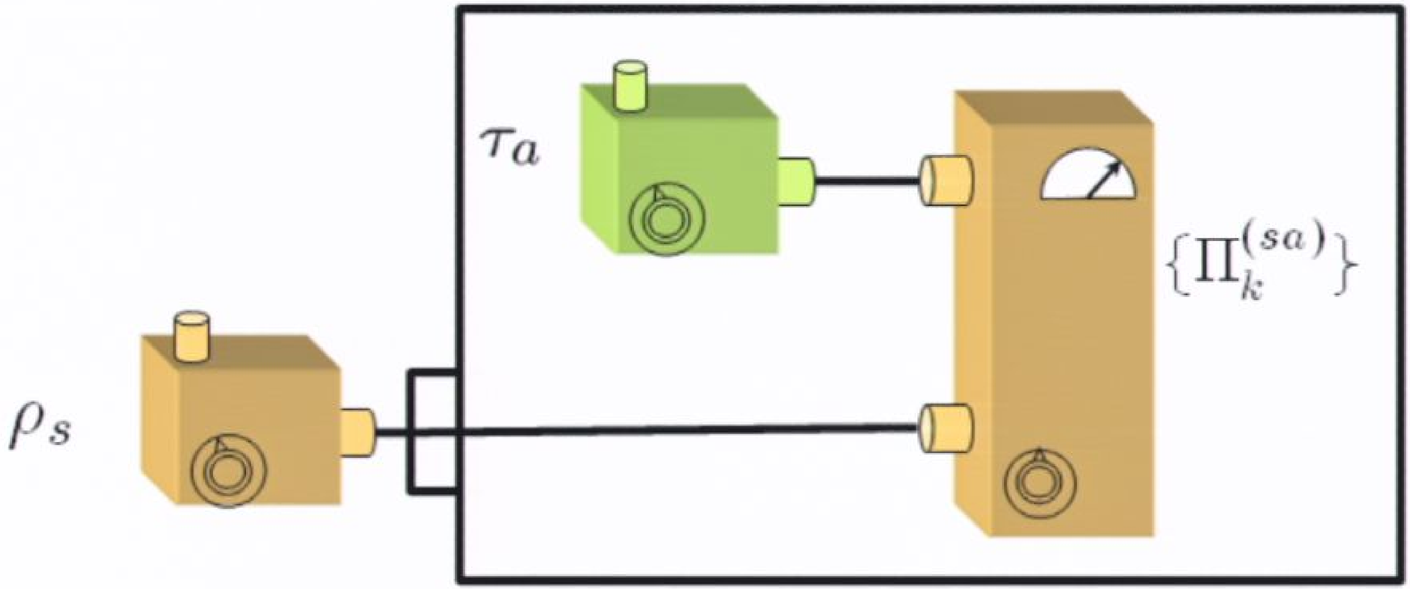


$$p(k) = \mathrm{Tr}_{sa}[\Pi_k^{(sa)}(\rho_s \otimes \tau_a)]$$
$$= \mathrm{Tr}_s[\underbrace{\mathrm{Tr}_a(\Pi_k^{(sa)}\tau_a)}_{E_k^{(s)}}\,\rho_s]$$

$$p(k) = \mathrm{Tr}_s(E_k^{(s)}\rho_s)$$

Positive $\langle\psi|E_k^{(s)}|\psi\rangle \geq 0 \quad \forall|\psi\rangle \in \mathcal{H}$

Measurement by coupling to an ancilla



$$p(k) = \mathrm{Tr}_{sa}[\Pi_k^{(sa)}(\rho_s \otimes \tau_a)]$$
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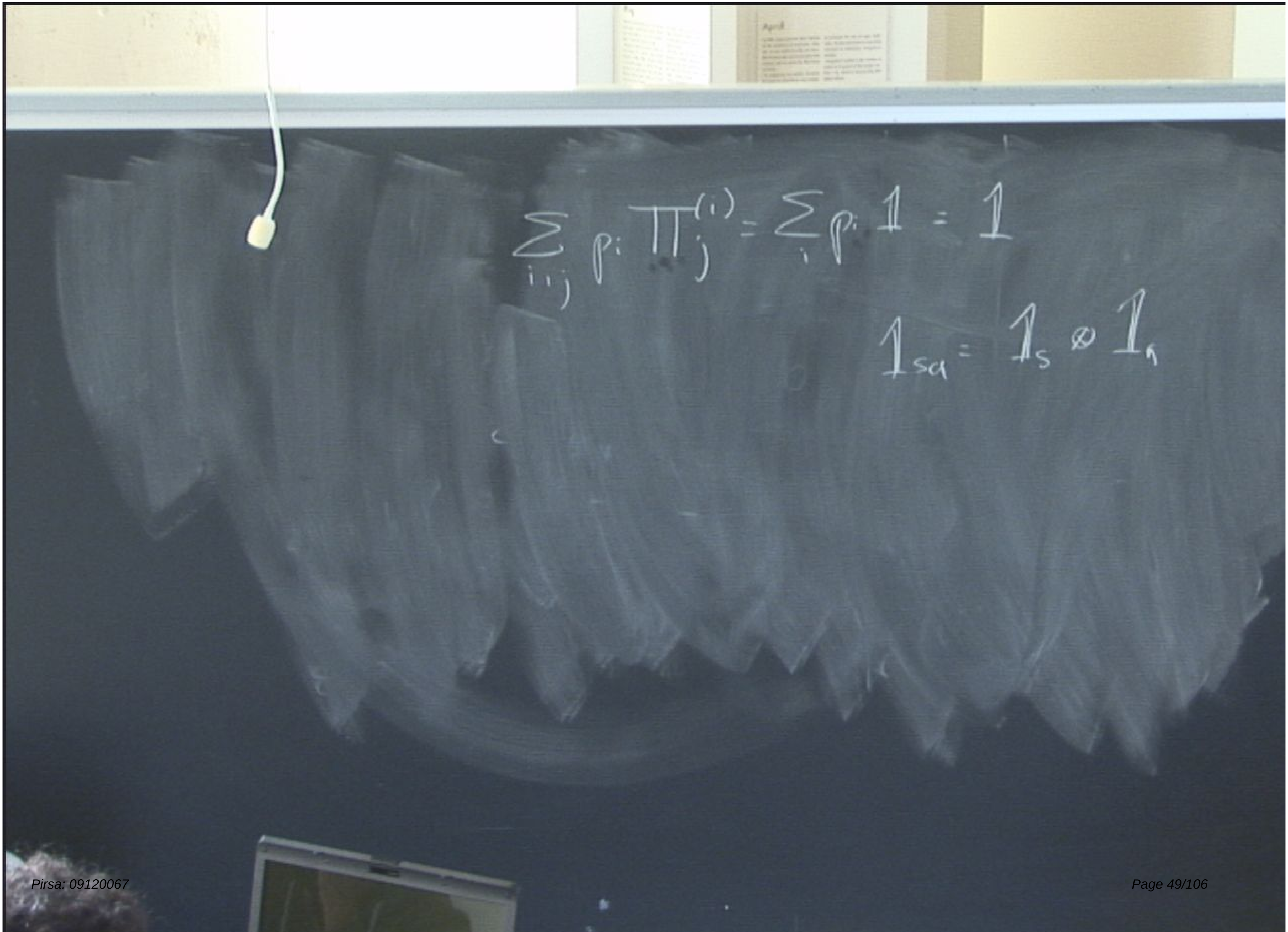
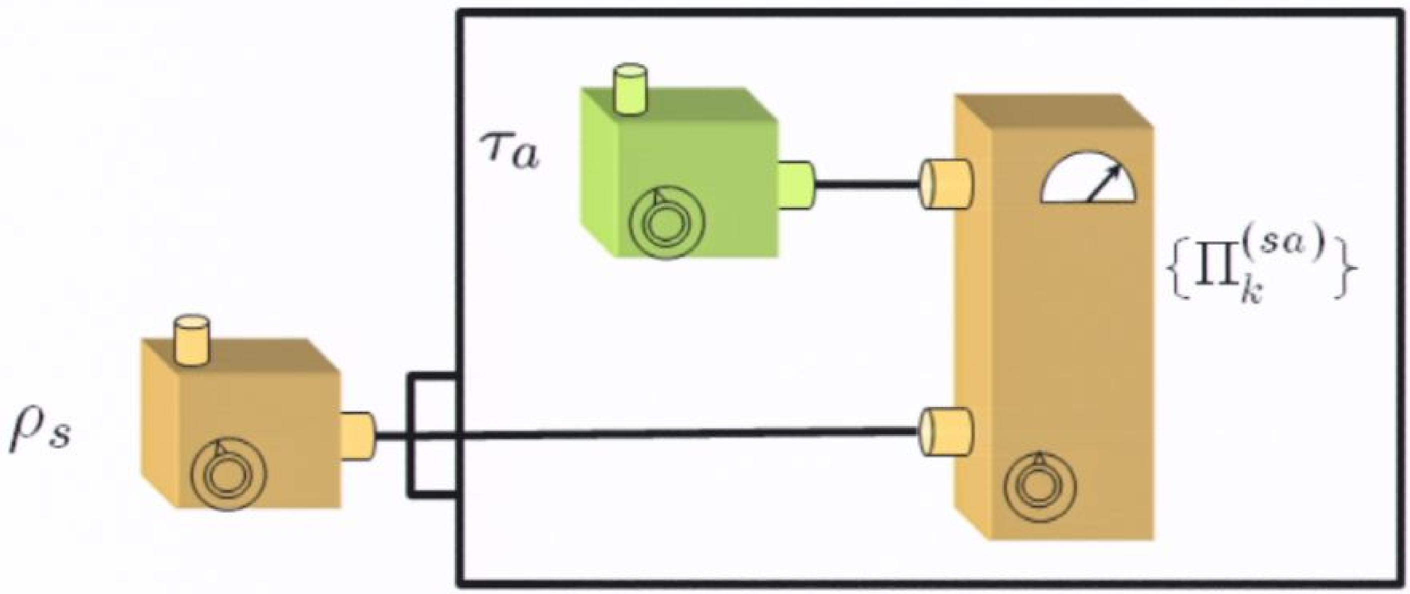
Positive $\langle\psi|E_k^{(s)}|\psi\rangle \geq 0 \ \ \forall|\psi\rangle \in \mathcal{H}$

Sum to identity $\sum_k E_k^{(s)} = I_s$


$\sum_{i,j} p_i \Pi_j^{(i)} = \sum_i p_i \mathbb{1} = \mathbb{1}$
$\mathbb{1}_{sa} = \mathbb{1}_s \otimes \mathbb{1}_a$

Measurement by coupling to an ancilla



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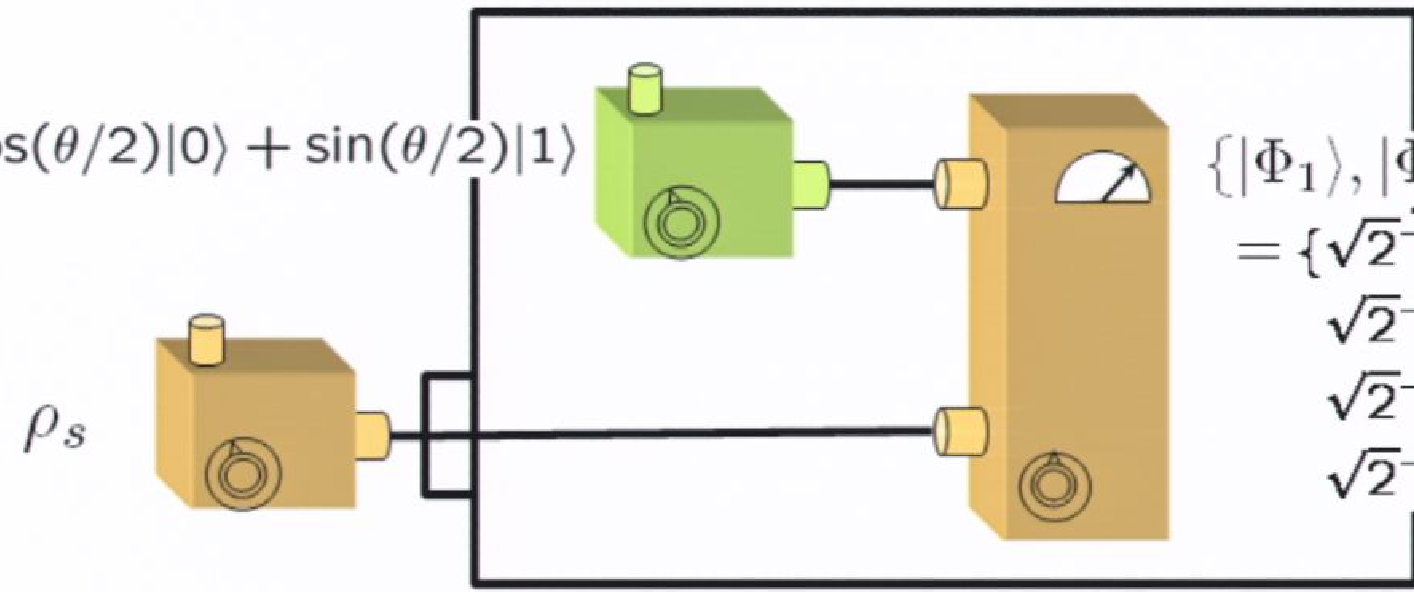
Sum to identity $\sum_k E_k^{(s)} = I_s$

Example

$\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)|1\rangle$



$$\{|\Phi_1\rangle, |\Phi_2\rangle, |\Phi_3\rangle, |\Phi_4\rangle\}$$
$$= \{\sqrt{2}^{-1}(|0\rangle|0\rangle + |1\rangle|1\rangle),$$
$$\sqrt{2}^{-1}(|0\rangle|0\rangle - 1\rangle|1\rangle),$$
$$\sqrt{2}^{-1}(|0\rangle|1\rangle + |1\rangle|0\rangle),$$
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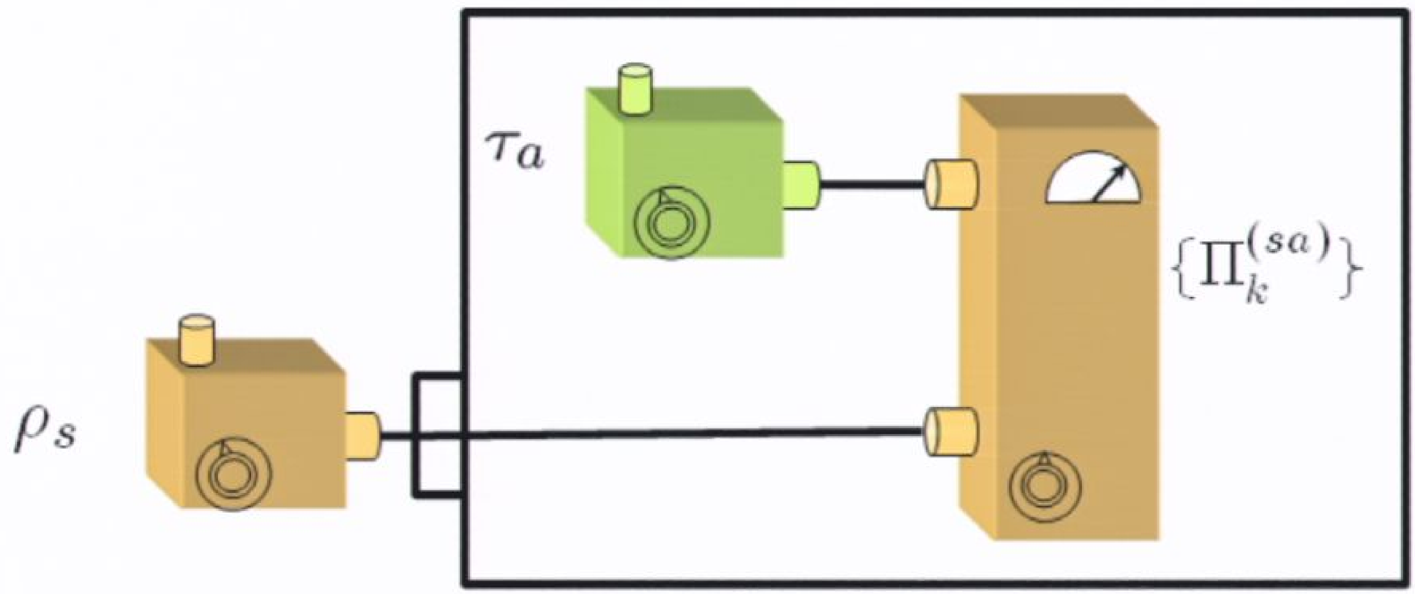
Measurement by coupling to an ancilla



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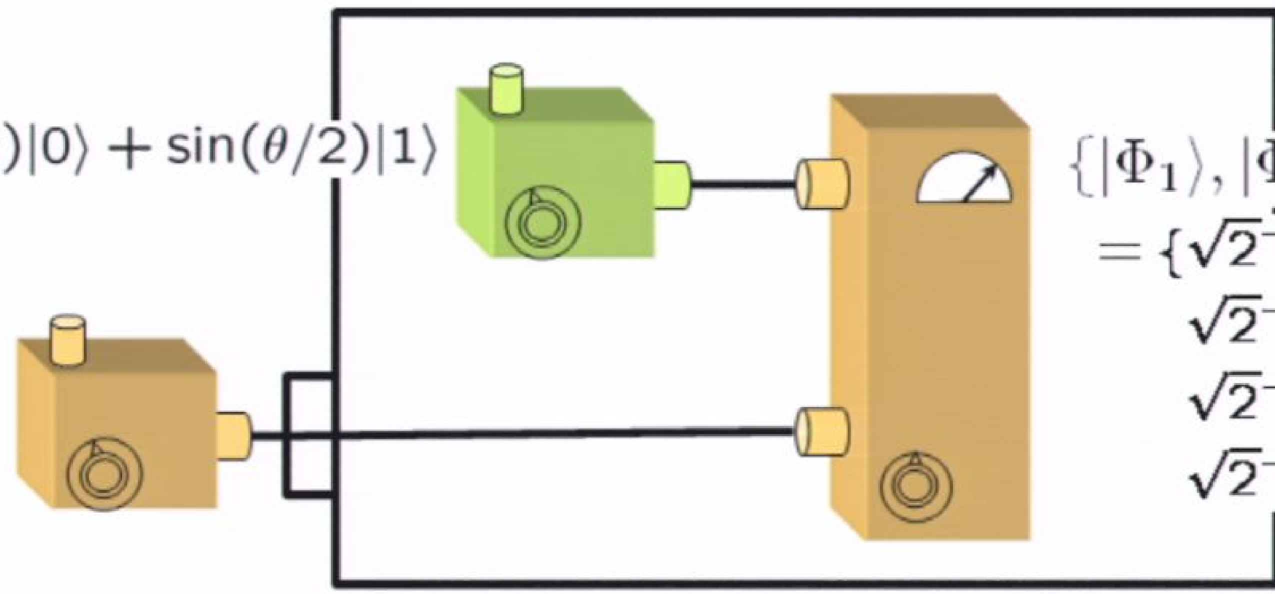
Example

$\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)|1\rangle$

ρ_s



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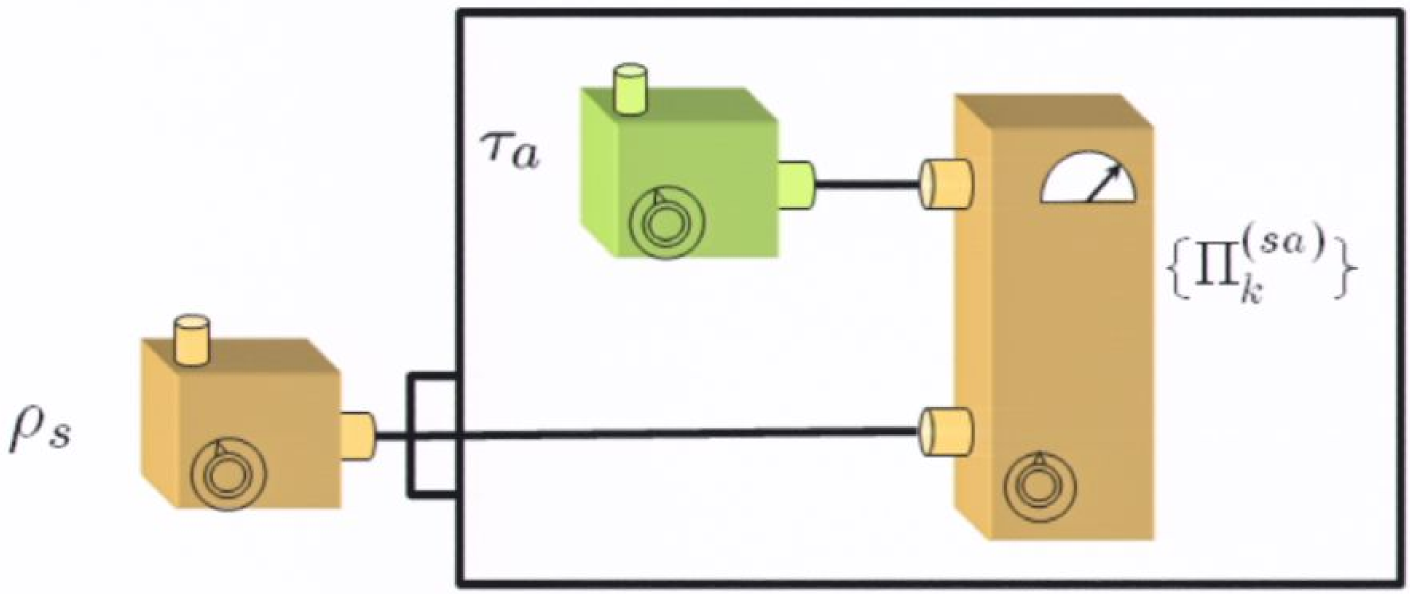
Measurement by coupling to an ancilla



$$p(k) = \mathrm{Tr}_{sa}[\Pi_k^{(sa)}(\rho_s \otimes \tau_a)]$$
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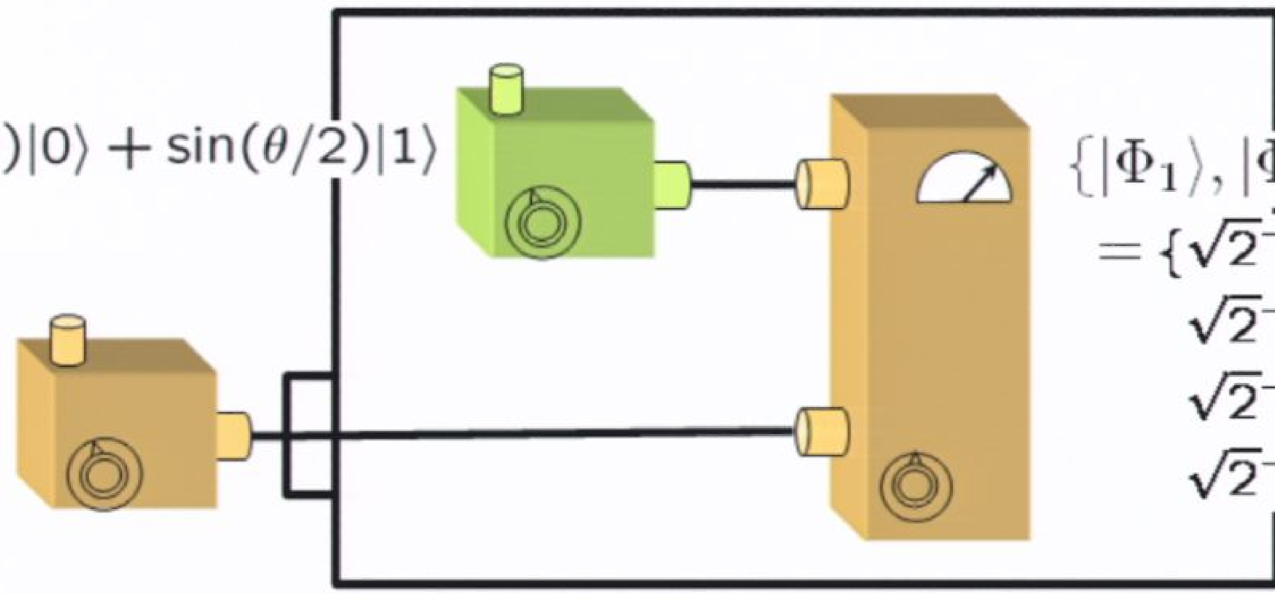
Example

$\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)|1\rangle$

ρ_s



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Example

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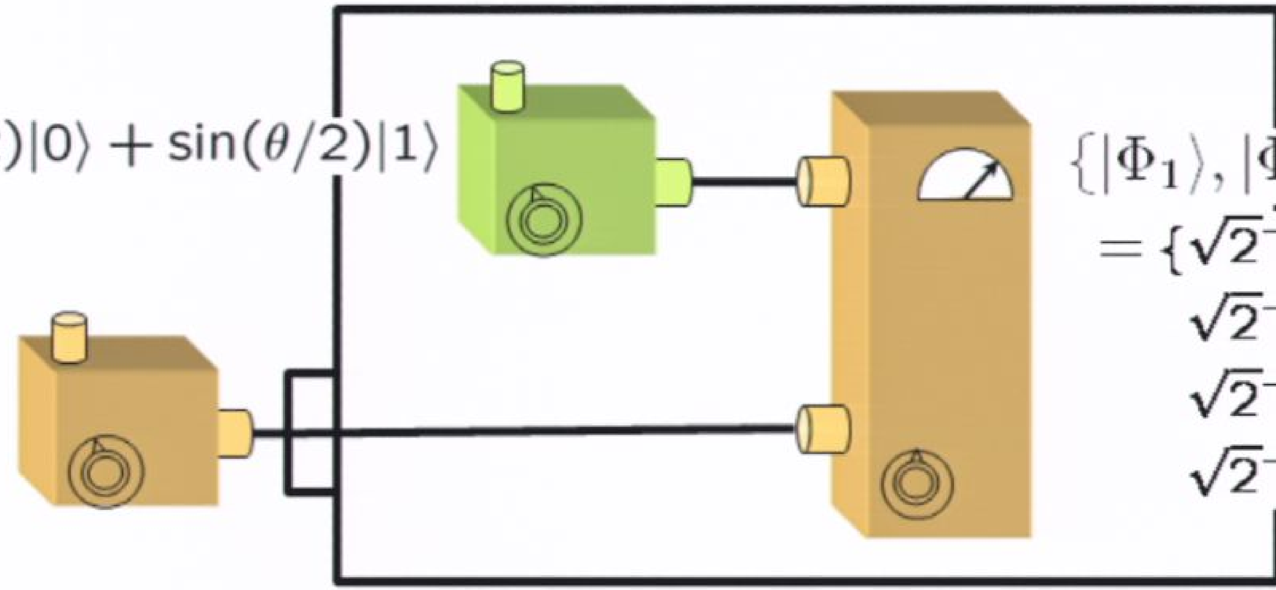
ρ_s



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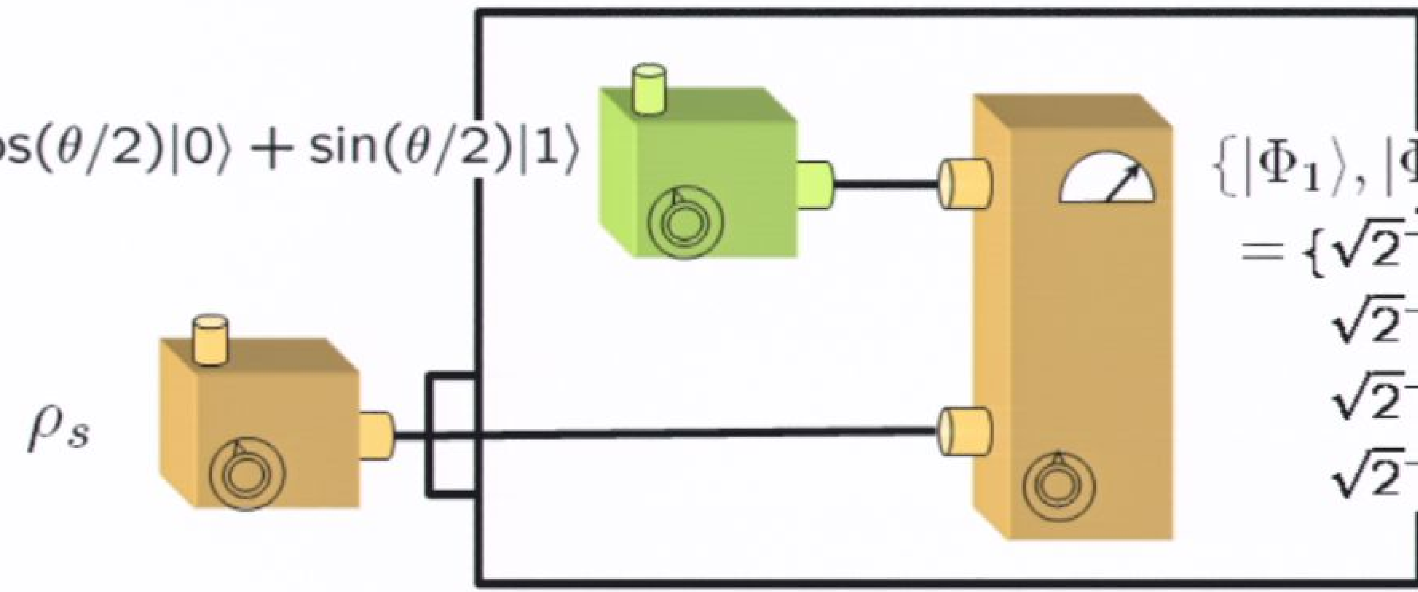
$$\sqrt{2}^{-1}(|0\rangle|1\rangle - |1\rangle|0\rangle)\}$$

$$E_k^{(s)} = \mathrm{Tr}_a(\Pi_k^{(sa)}\tau_a)$$

Example

$$\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)|1\rangle$$



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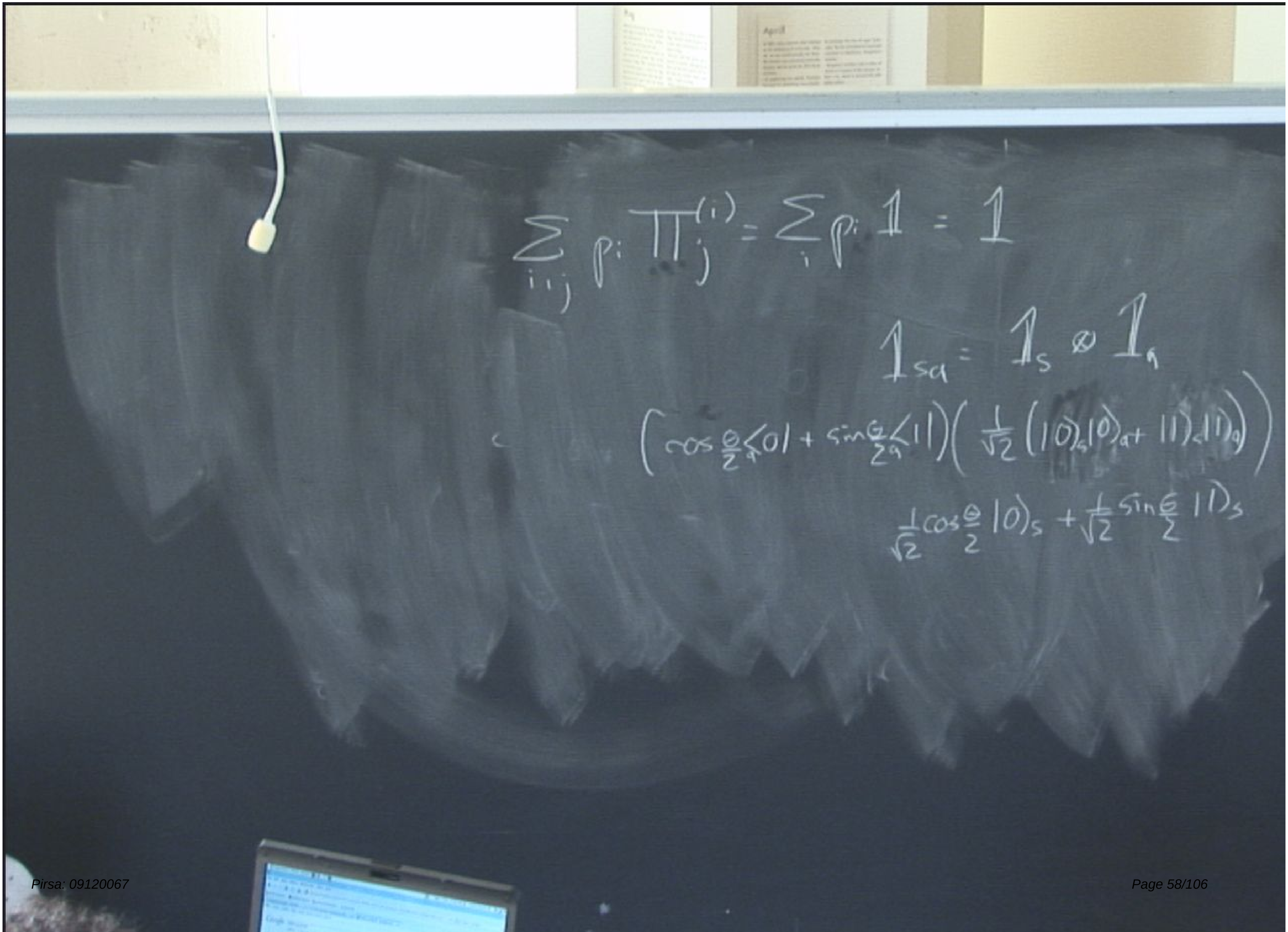
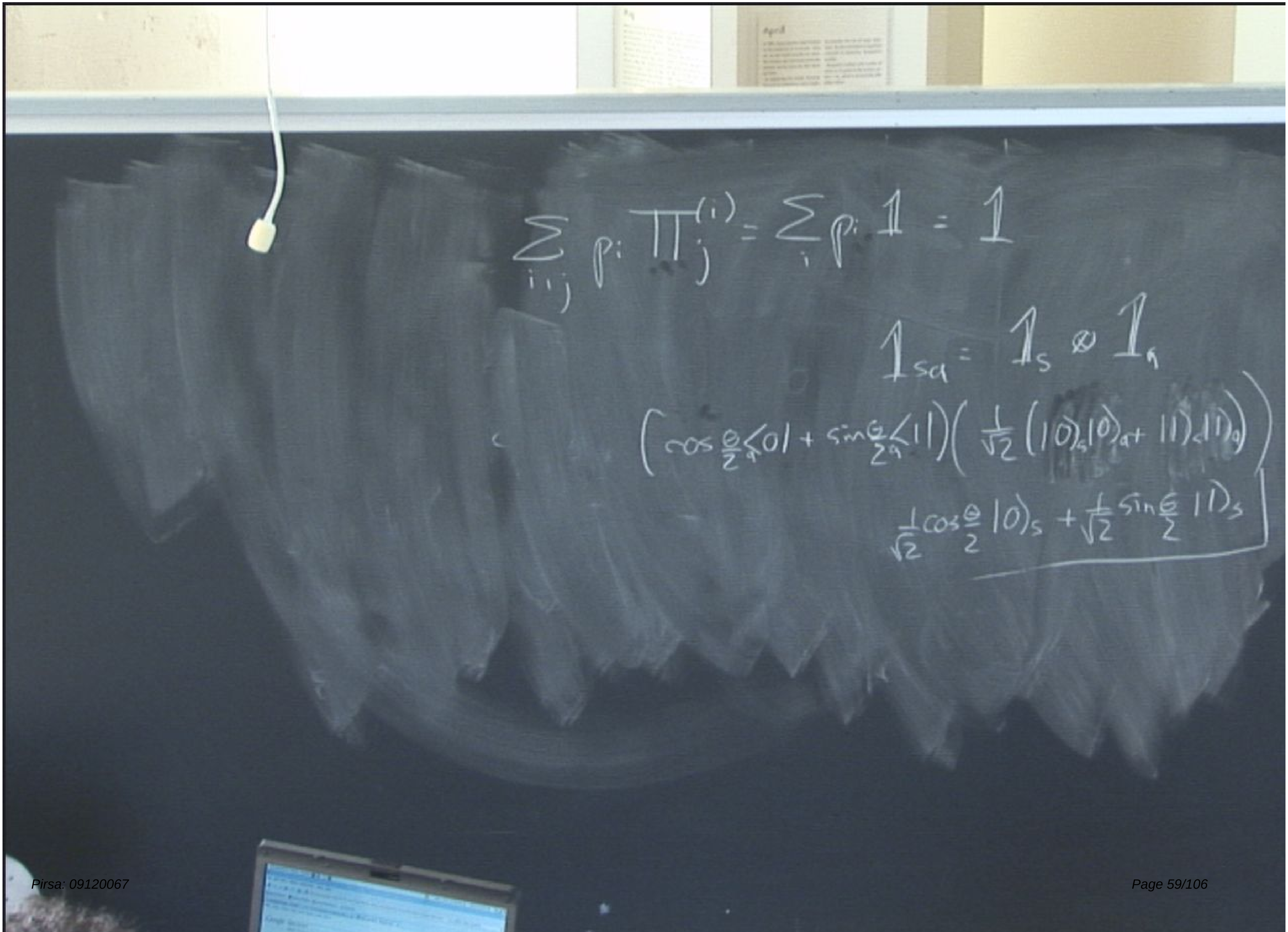
$$= \langle\theta|_a|\Phi_k\rangle_{sa}\langle\Phi_k|_{sa}|\theta\rangle_a$$


$\sum_{i,j} p_i \Pi_j^{(i)} = \sum_i p_i \mathbb{1} = \mathbb{1}$
$\mathbb{1}_{sa} = \mathbb{1}_s \otimes \mathbb{1}_a$
$\left(\cos\frac{\theta}{2}\,{}_a\langle 0| + \sin\frac{\theta}{2}\,{}_a\langle 1|\right)\left(\frac{1}{\sqrt{2}}\left(|0\rangle_s|0\rangle_a + |1\rangle_s|1\rangle_a\right)\right)$
$\frac{1}{\sqrt{2}}\cos\frac{\theta}{2}|0\rangle_s + \frac{1}{\sqrt{2}}\sin\frac{\theta}{2}|1\rangle_s$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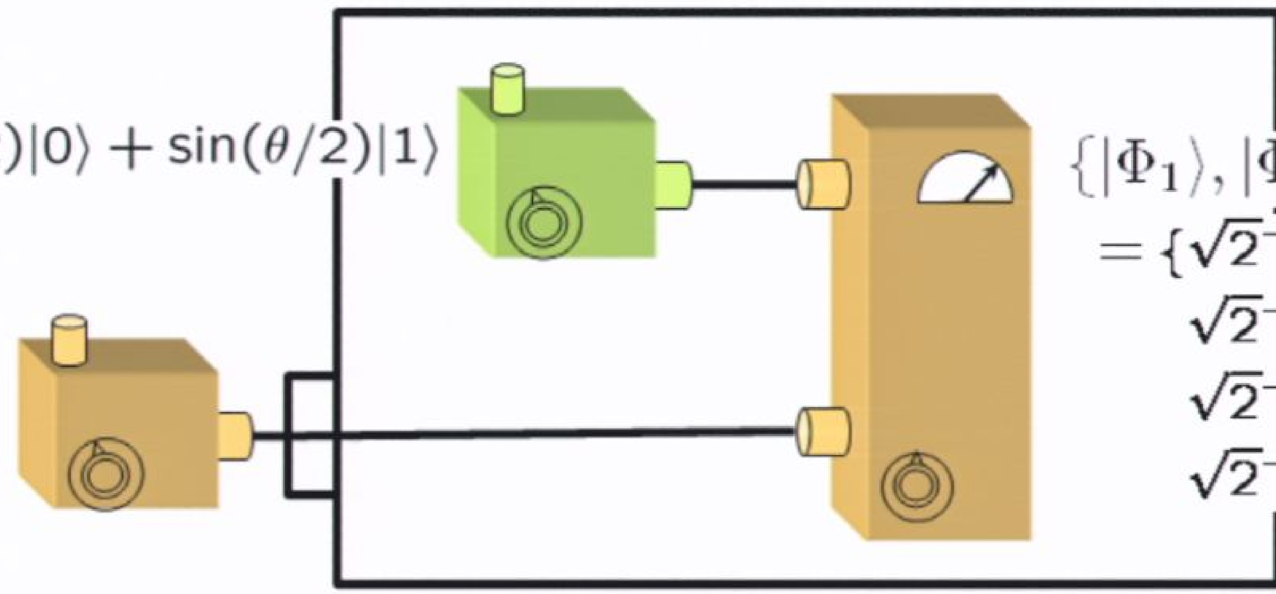
Example

$\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)|1\rangle$

ρ_s



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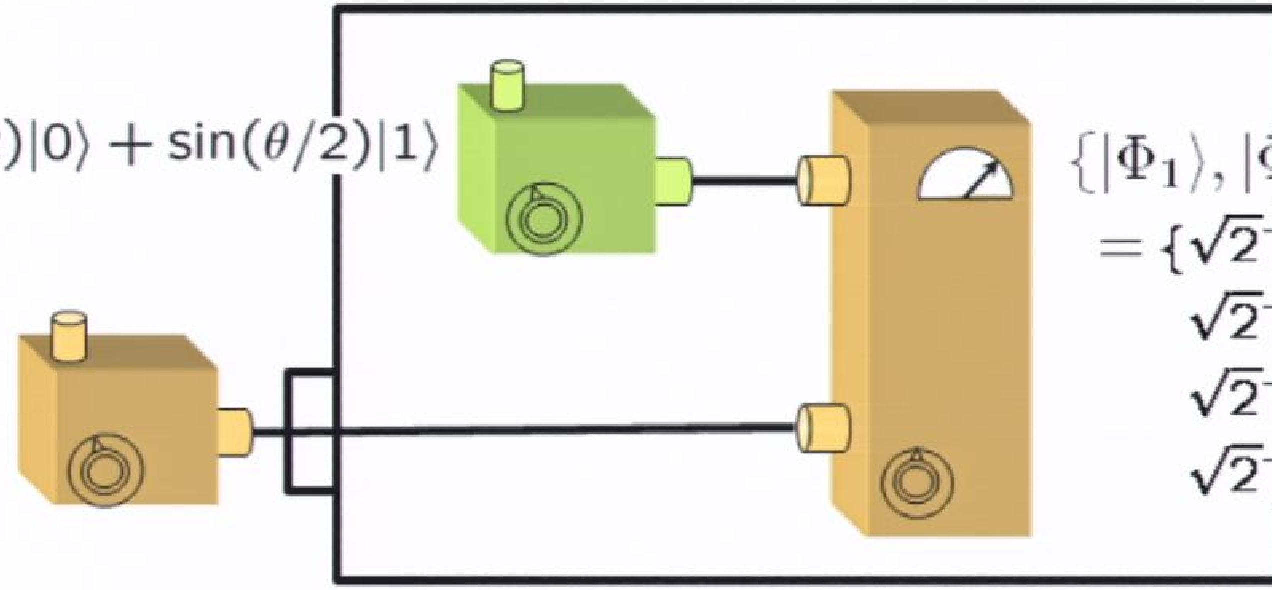
$$= \langle\theta|_a|\Phi_k\rangle_{sa}\langle\Phi_k|_{sa}|\theta\rangle_a$$
$$\langle\theta|_a|\Phi_{1(2)}\rangle_{sa} = \sqrt{2}^{-1}[\cos(\theta/2)|0\rangle_s \pm \sin(\theta/2)|1\rangle_s] = \sqrt{2}^{-1}|\pm\theta\rangle_s$$

Example

$\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)|1\rangle$

ρ_s



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Example

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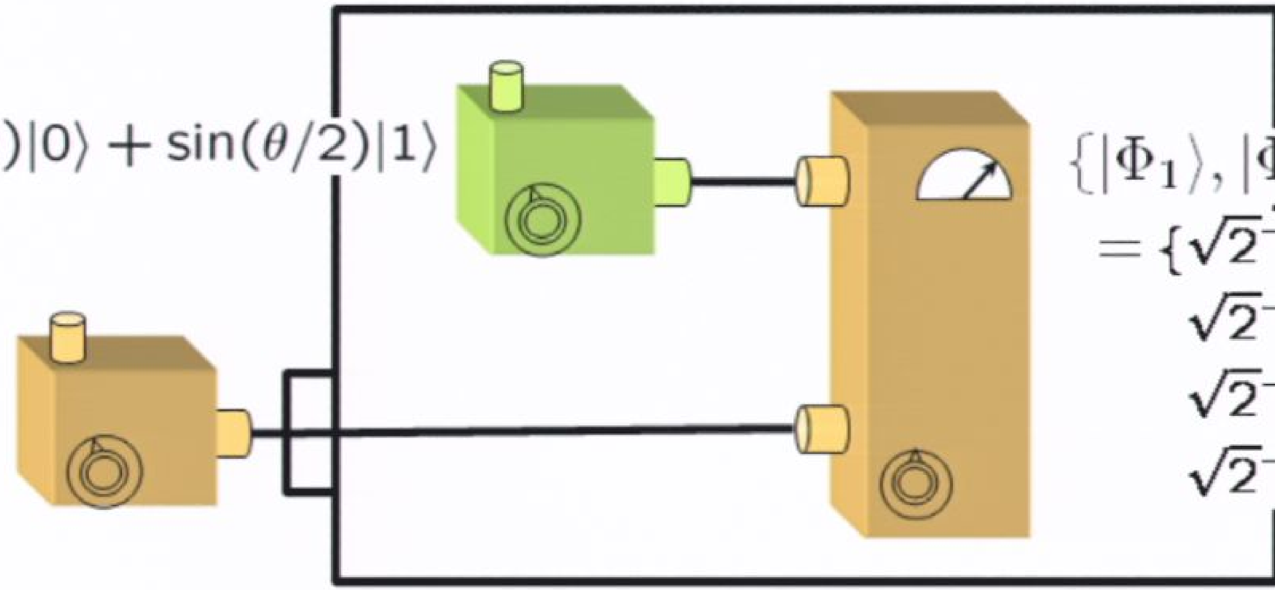
ρ_s



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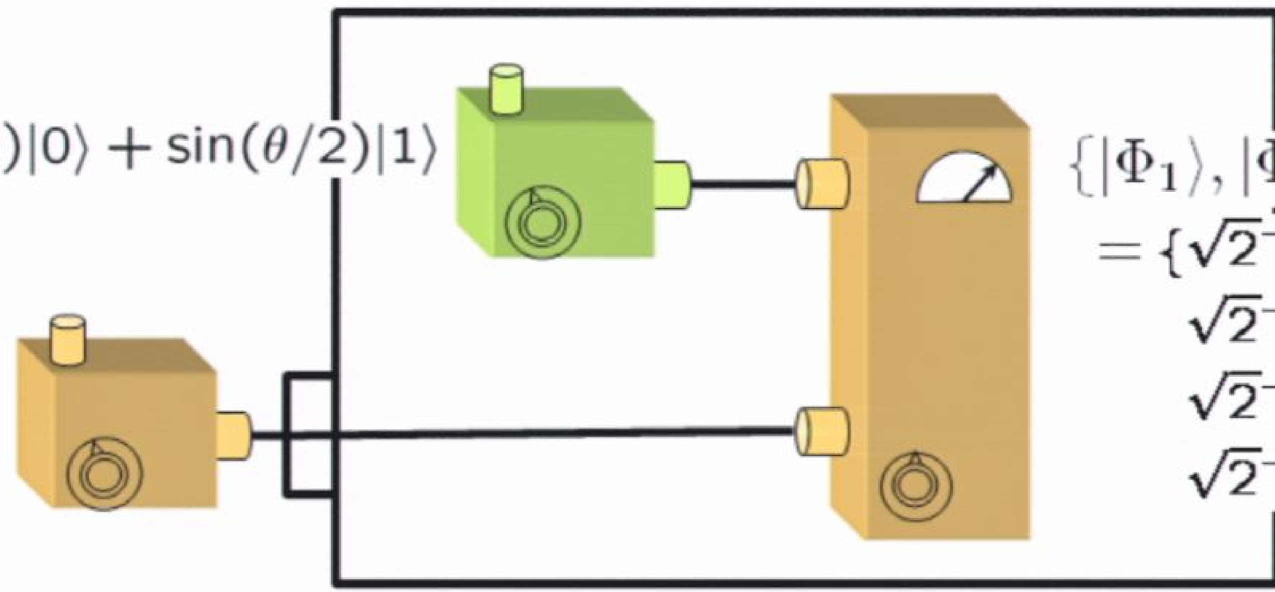
$$\langle\theta|_a|\Phi_{3(4)}\rangle_{sa} = \sqrt{2}^{-1}[\sin(\theta/2)|0\rangle_s \pm \cos(\theta/2)|1\rangle_s] = \sqrt{2}^{-1}|\pi\mp\theta\rangle_s$$

Example

$$\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)|1\rangle$$



ρ_s

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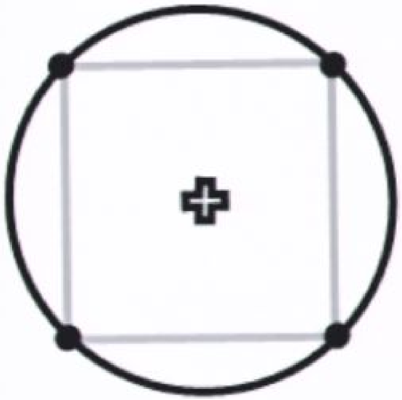
$$\langle\theta|_a|\Phi_{1(2)}\rangle_{sa} = \sqrt{2}^{-1}[\cos(\theta/2)|0\rangle_s \pm \sin(\theta/2)|1\rangle_s] = \sqrt{2}^{-1}|\pm\theta\rangle_s$$

$$\langle\theta|_a|\Phi_{3(4)}\rangle_{sa} = \sqrt{2}^{-1}[\sin(\theta/2)|0\rangle_s \pm \cos(\theta/2)|1\rangle_s] = \sqrt{2}^{-1}|\pi\mp\theta\rangle_s$$

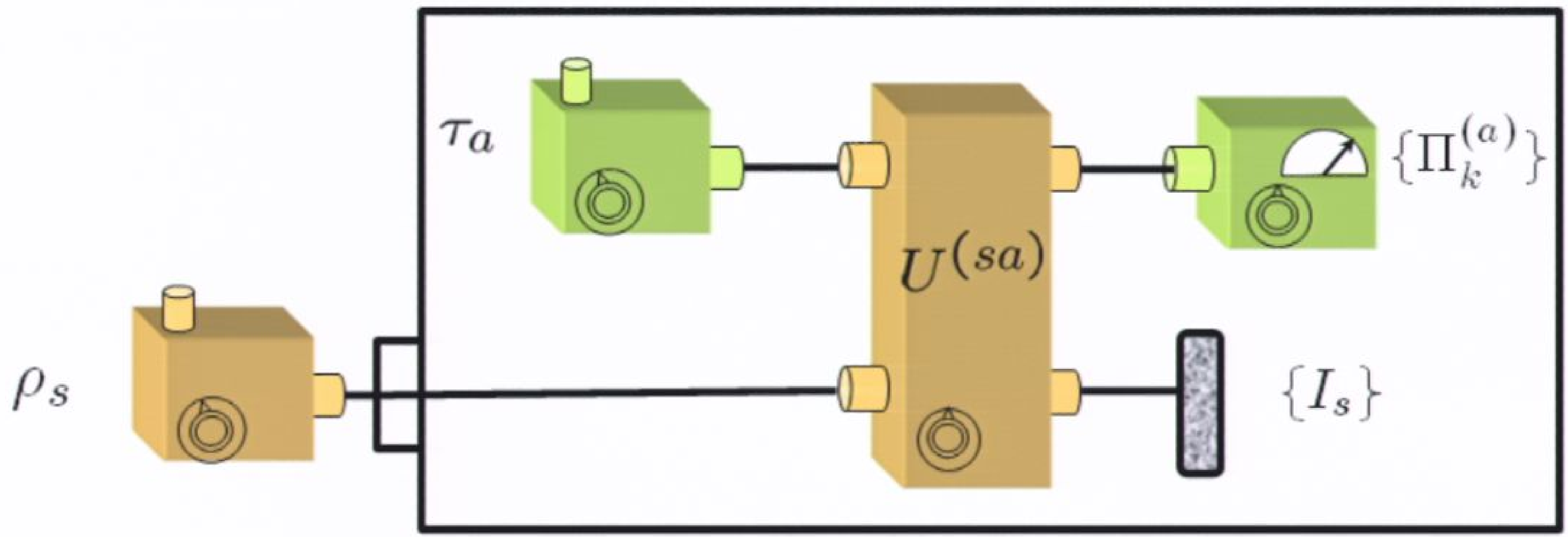
$$\{E_k\} = \{\frac{1}{2}|\theta\rangle\langle\theta|\,,\ \frac{1}{2}|-\theta\rangle\langle-\theta|\,,\ \frac{1}{2}|\pi-\theta\rangle\langle\pi-\theta|\,,\ \frac{1}{2}|\pi+\theta\rangle\langle\pi+\theta|\}$$



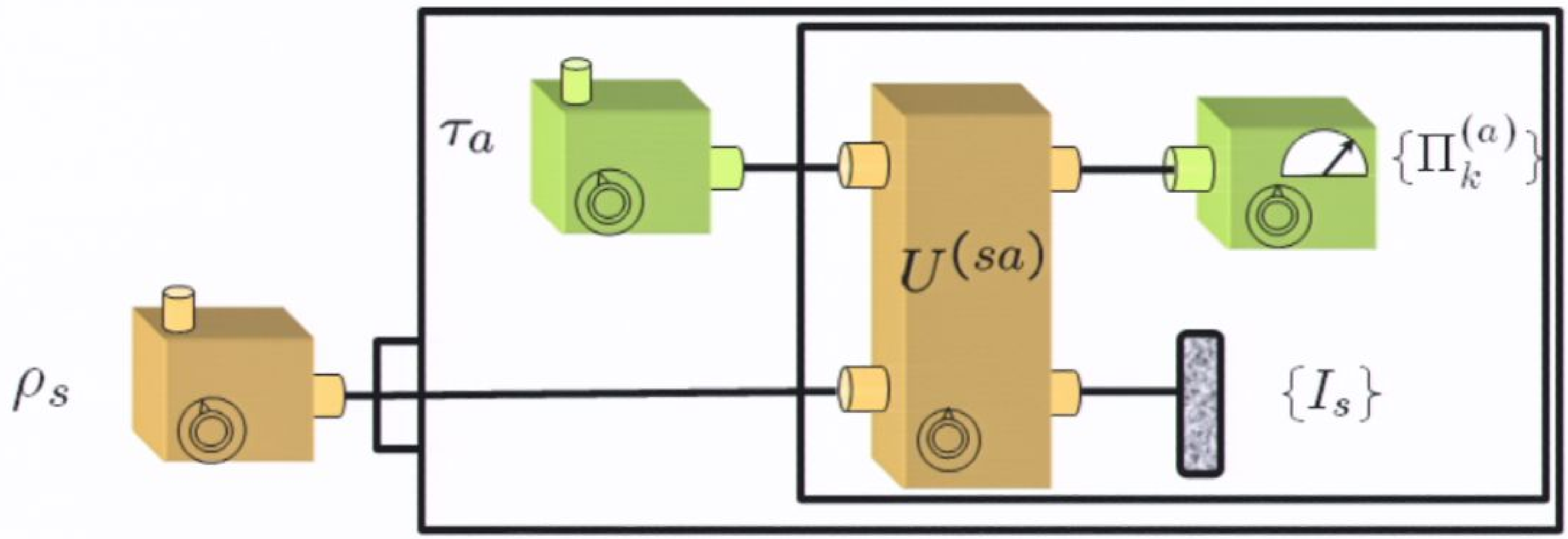
$\theta = \pi/4$

$$\{\frac{1}{2}|\theta\rangle\langle\theta| \;,\; \frac{1}{2}|-\theta\rangle\langle-\theta| \;,\; \frac{1}{2}|\pi-\theta\rangle\langle\pi-\theta| \;,\; \frac{1}{2}|\pi+\theta\rangle\langle\pi+\theta|\}$$

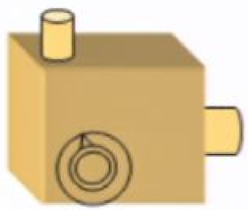
Internalizing the probe system (a la von Neumann)



Internalizing the probe system (a la von Neumann)

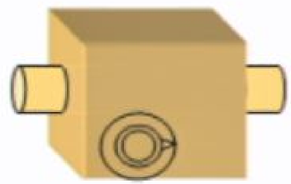


Operational Quantum Mechanics



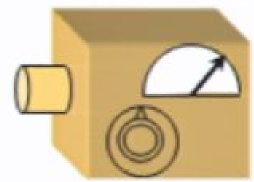
Preparation

P



Transformation

T



Measurement

M

Density operator

ρ

Unitary

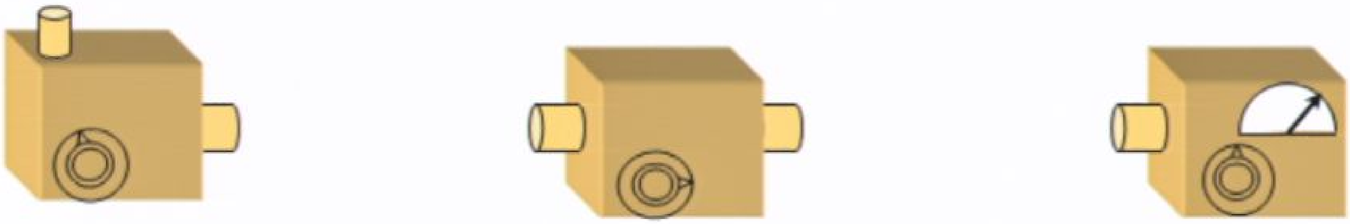
U

Positive operator-valued measure (POVM)

$\{E_k\}$

$$Pr(k|\mathsf{P},\mathsf{T},\mathsf{M}) = \mathrm{Tr}[E_k U \rho U^\dagger]$$

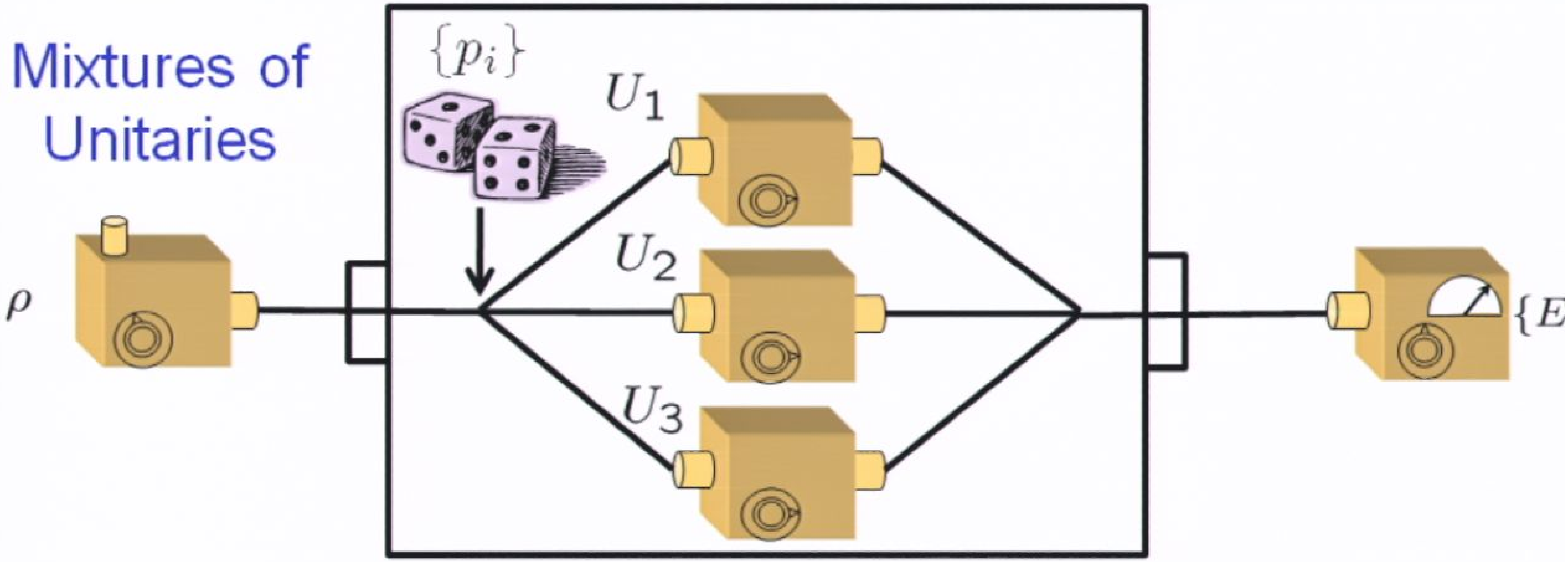
Operational Quantum Mechanics



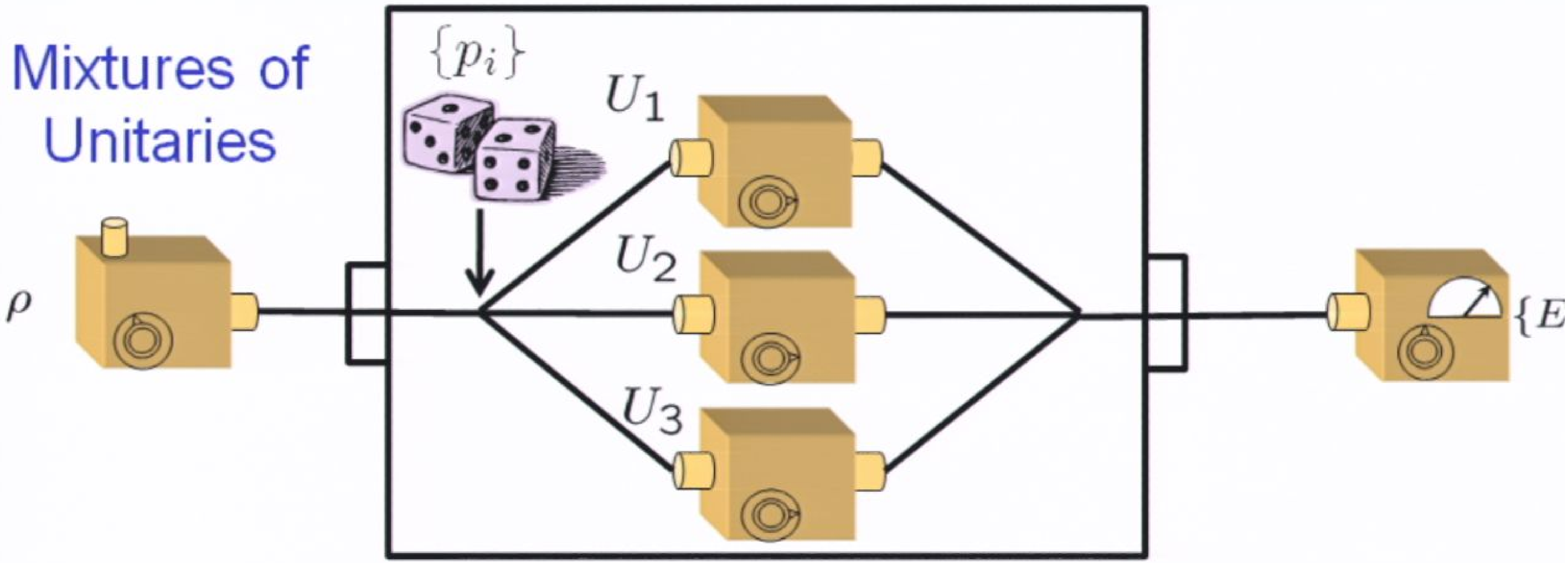
Preparation	Transformation	Measurement
P	T	M
Density operator	Trace-preserving completely positive linear map (CP map)	Positive operator-valued measure (POVM)
ρ	$\mathcal{T}$	$\{E_k\}$

$$Pr(k|\mathsf{P},\mathsf{T},\mathsf{M}) = \mathrm{Tr}[E_k \mathcal{T}(\rho)]$$

Mixtures of Unitaries



Mixtures of Unitaries

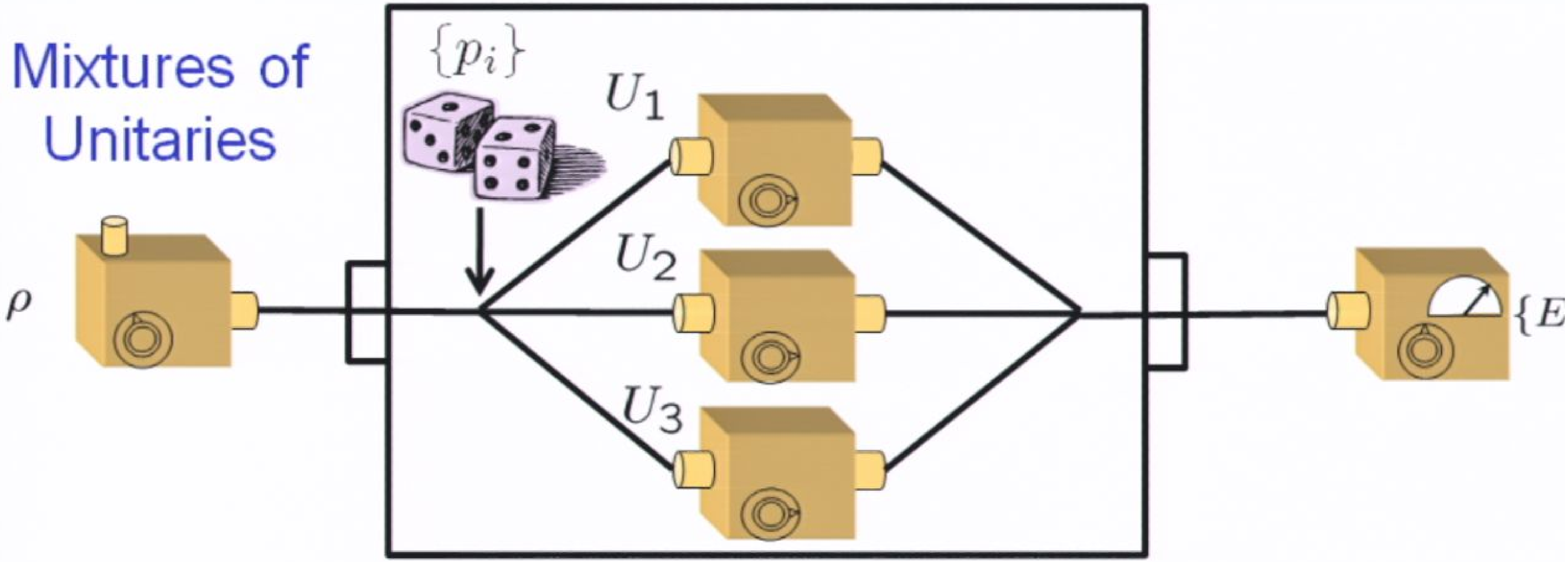


$$p(k) = \sum_i p(k|i)p(i)$$

$$= \sum_i \mathrm{Tr}[E_k U_i \rho U_i^\dagger] p_i$$

$$= \sum_i \mathrm{Tr}[E_k \sum_i p_i U_i \rho U_i^\dagger]$$

Mixtures of Unitaries



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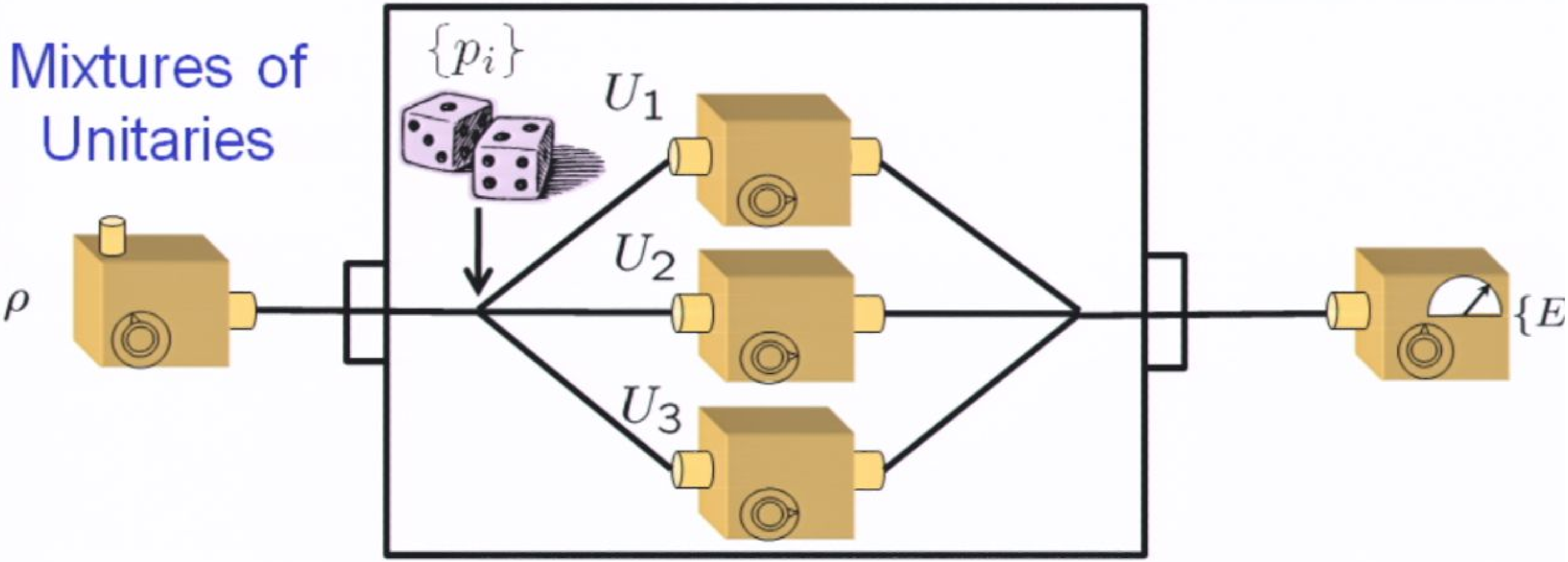

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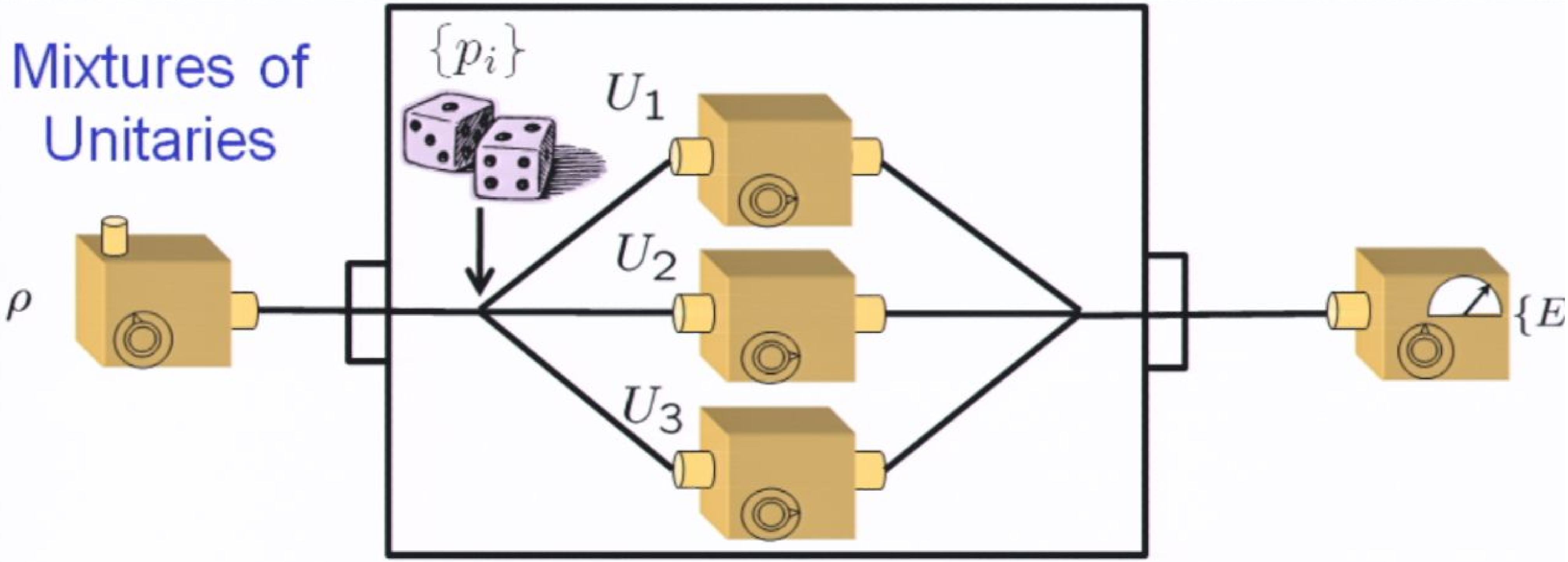

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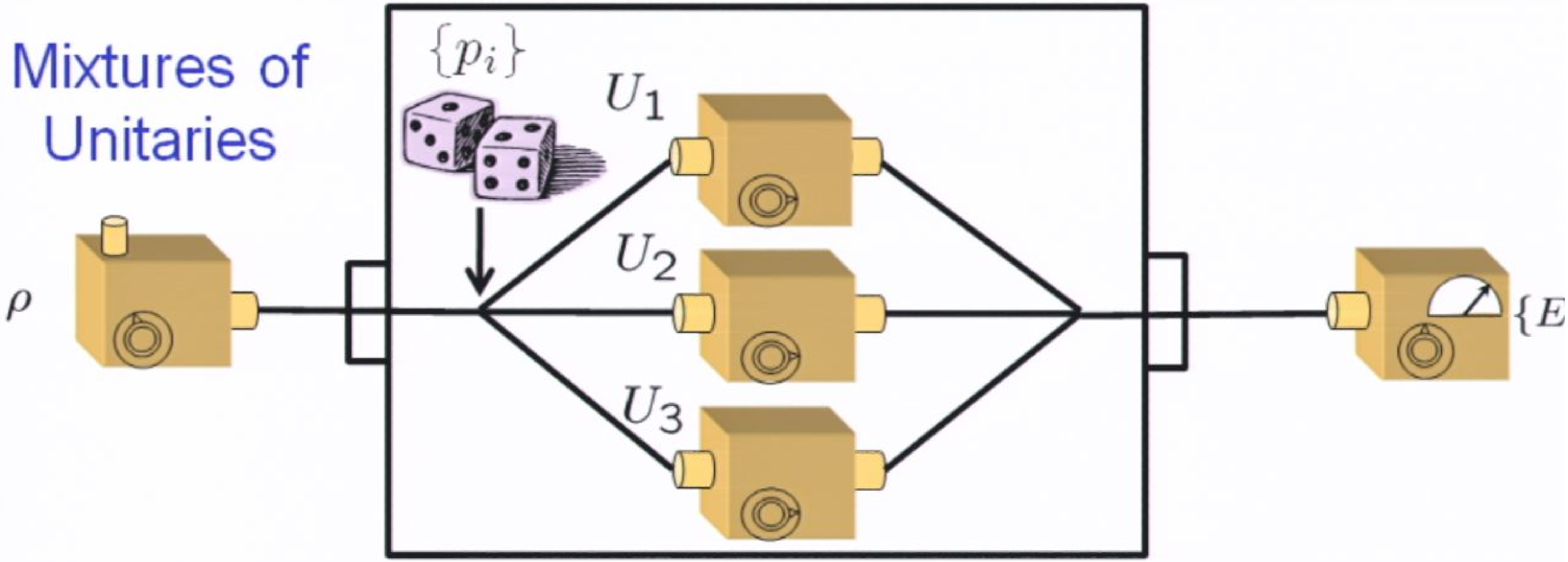

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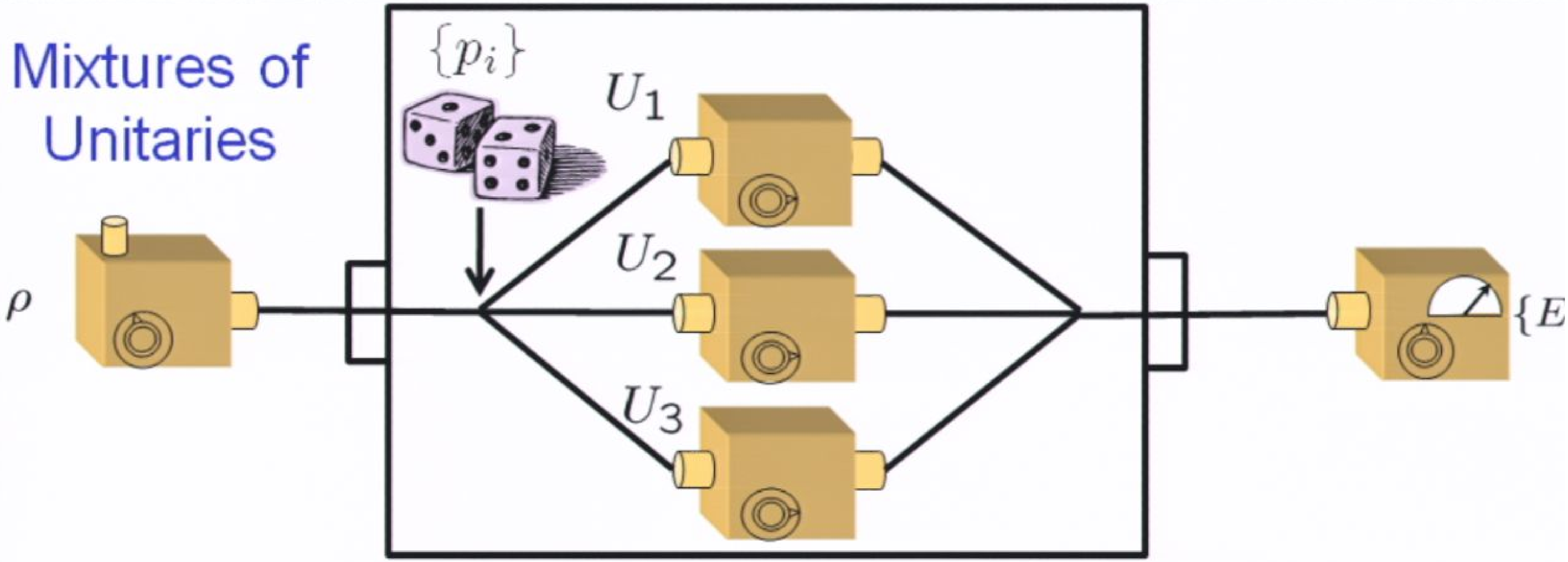

$$
\begin{aligned}
p(k) &= \sum_i p(k|i)p(i) \\
&= \sum_i \mathrm{Tr}[E_k U_i \rho U_i^\dagger] p_i \\
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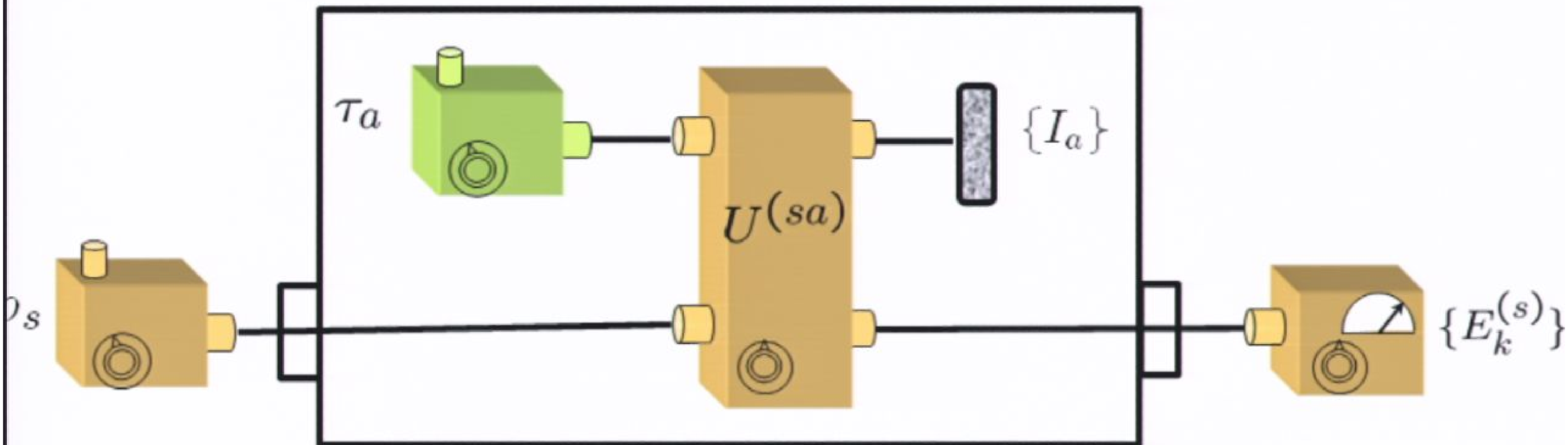
$\mathcal{T}$ "Completely positive trace-preserving linear map"

Transformation by coupling to an ancilla



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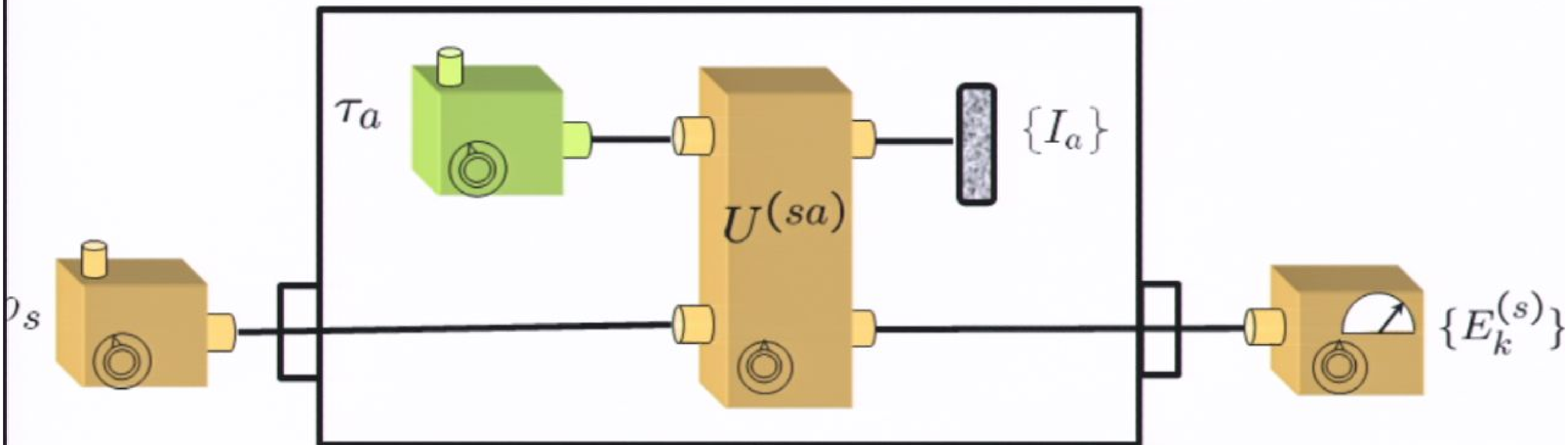

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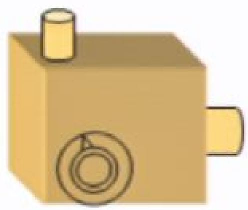
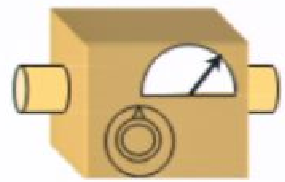
Operational Quantum Mechanics





Preparation   Measurement

P   M

Effective preparation

P_k

General transformations due to measurements

Linear map: $\mathcal{T} : \mathcal{L}(\mathbb{C}_d) \rightarrow \mathcal{L}(\mathbb{C}_d)$

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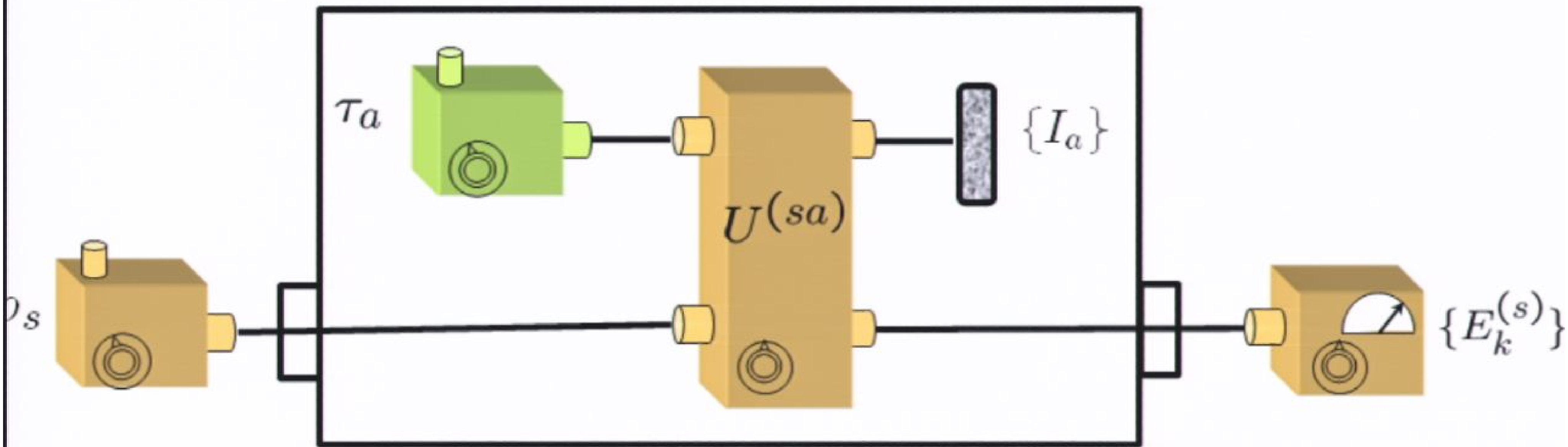

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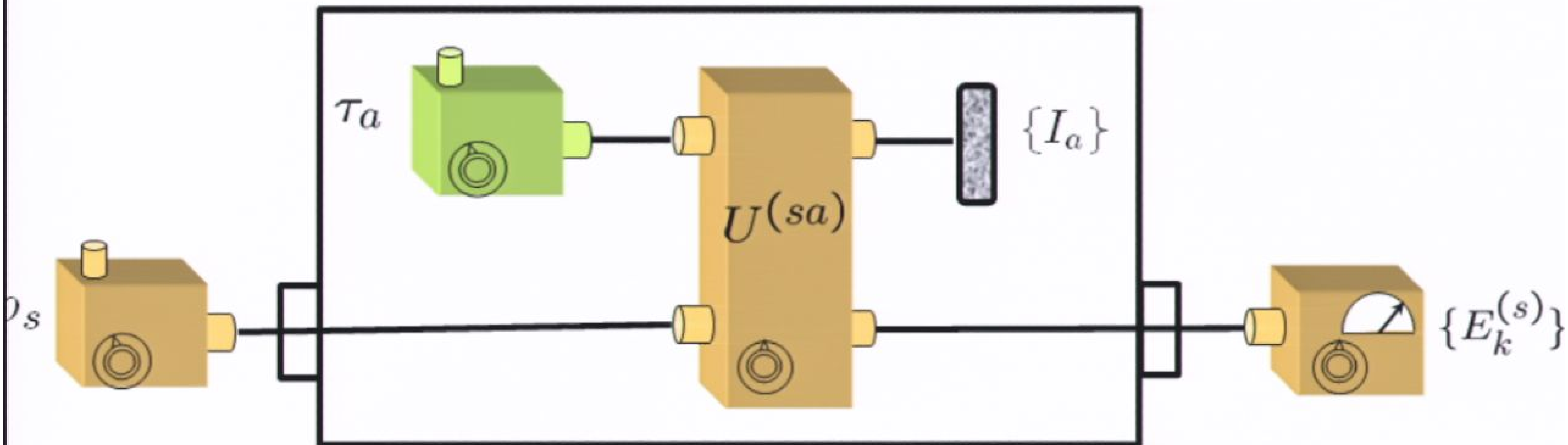

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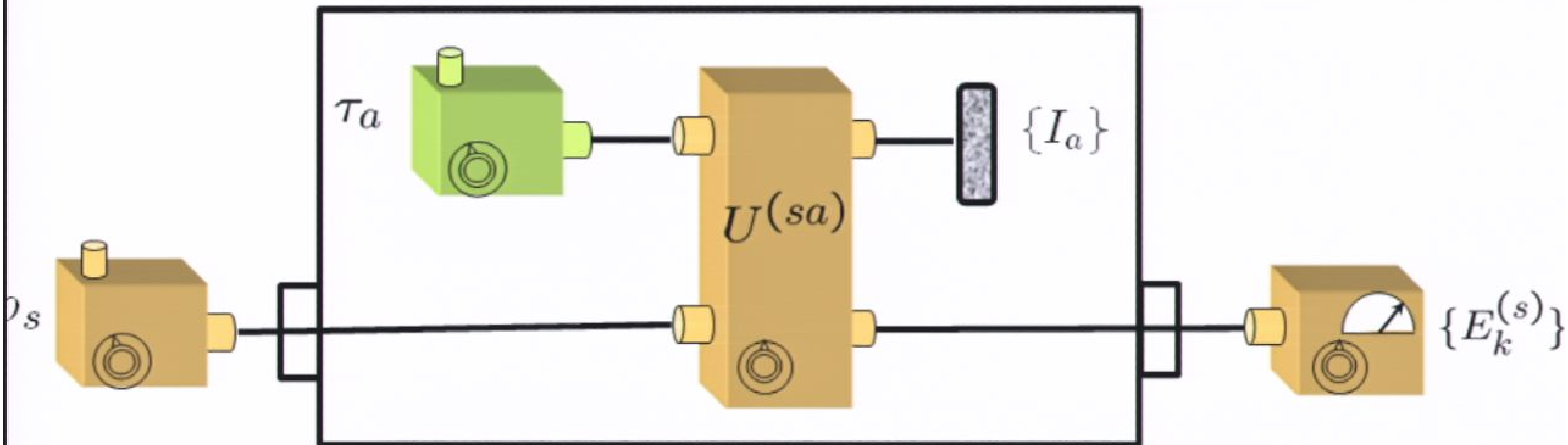

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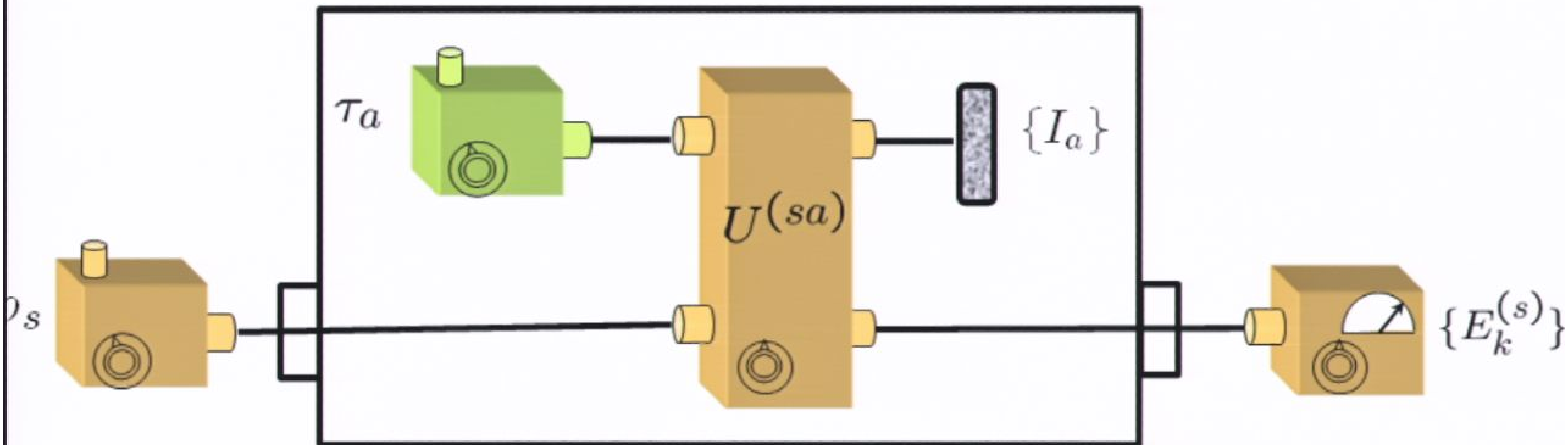

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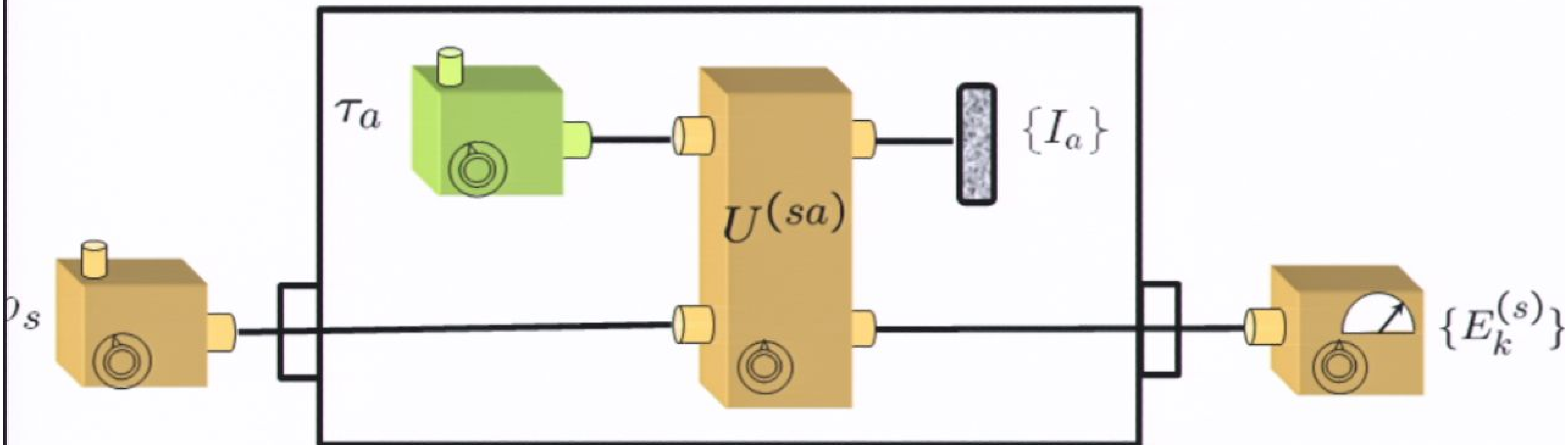

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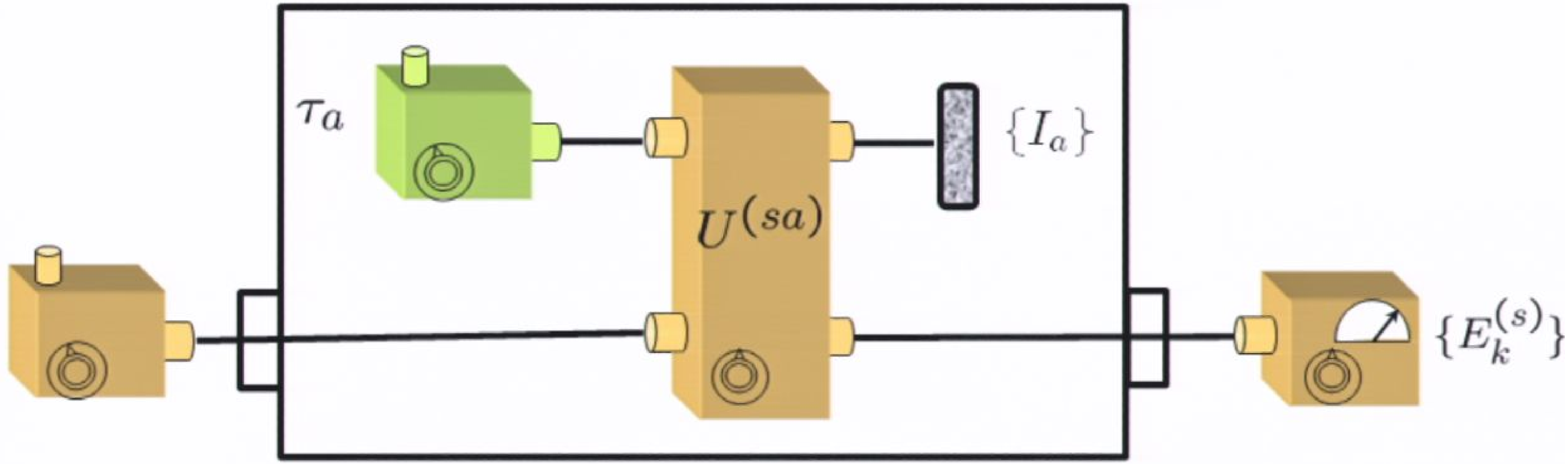

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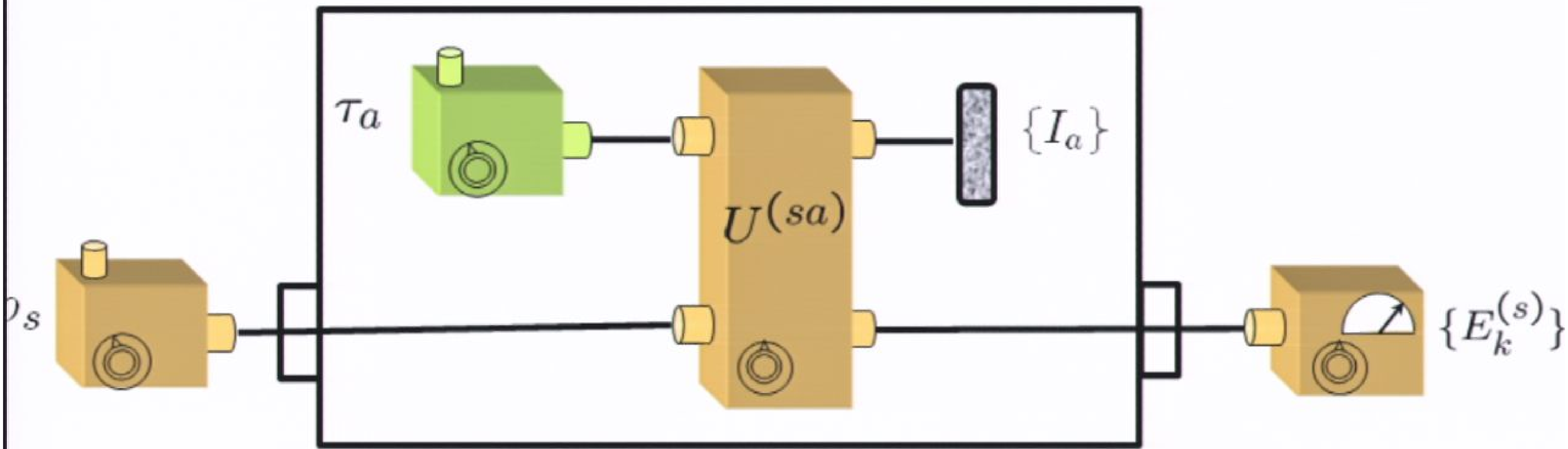

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$\mathcal{T}(\rho) = \sum_k \Pi_k \rho \Pi_k$ Nonselective projective measurement

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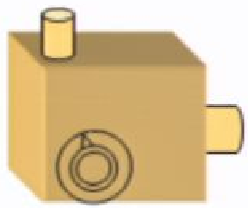
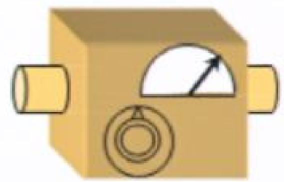
$\mathcal{T}(\rho) = U \rho U^\dagger$ $\leftrightarrow$ Reversible transformation

$\mathcal{T}(\rho) \neq U \rho U^\dagger$ $\leftrightarrow$ Irreversible transformation

Operational Quantum Mechanics





Preparation — Measurement

P — M

Effective preparation

P_k

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$\mathcal{T}(\rho) = \Pi_k \rho \Pi_k$ Selective projective measurement

$\mathcal{T}(\rho) = \Psi(r) \rho \Psi^\dagger(r)$ Photon detected at position r

The non-independence of the structure of preparations and the structure of measurements

A version of Gleason's theorem

Consider a function on density operators $\rho \mapsto f(\rho)$, satisfying:

1) $0 \leq f(\rho) \leq 1$ for all ρ

2) $f(w\rho + (1-w)\rho') = wf(\rho) + (1-w)f(\rho')$ where $0 \leq w \leq 1$.

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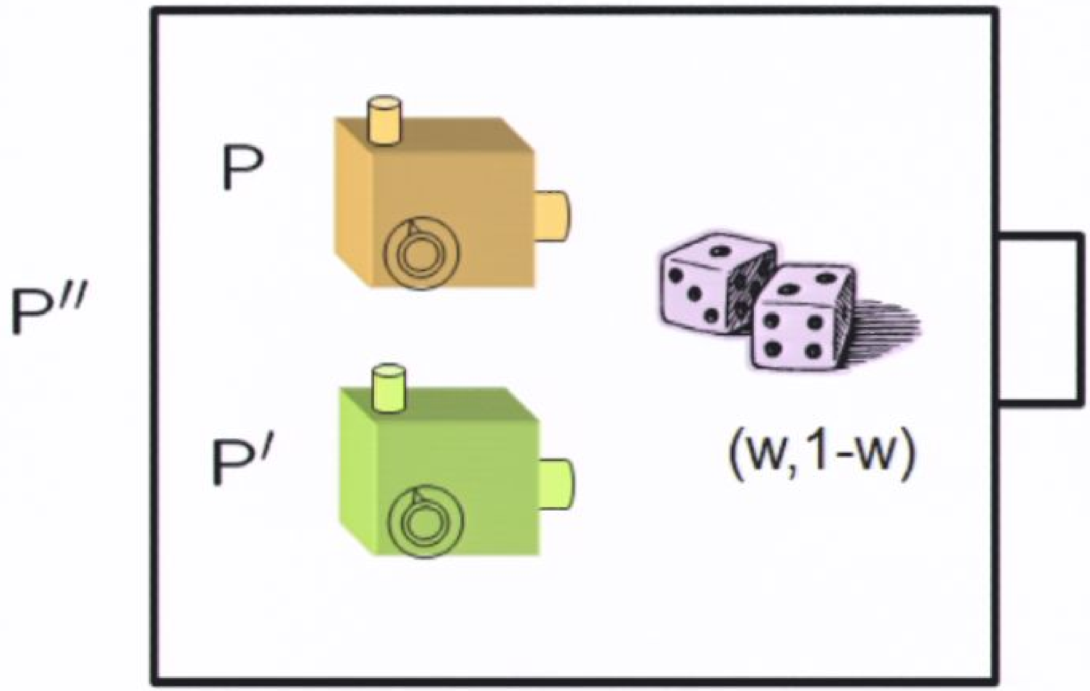
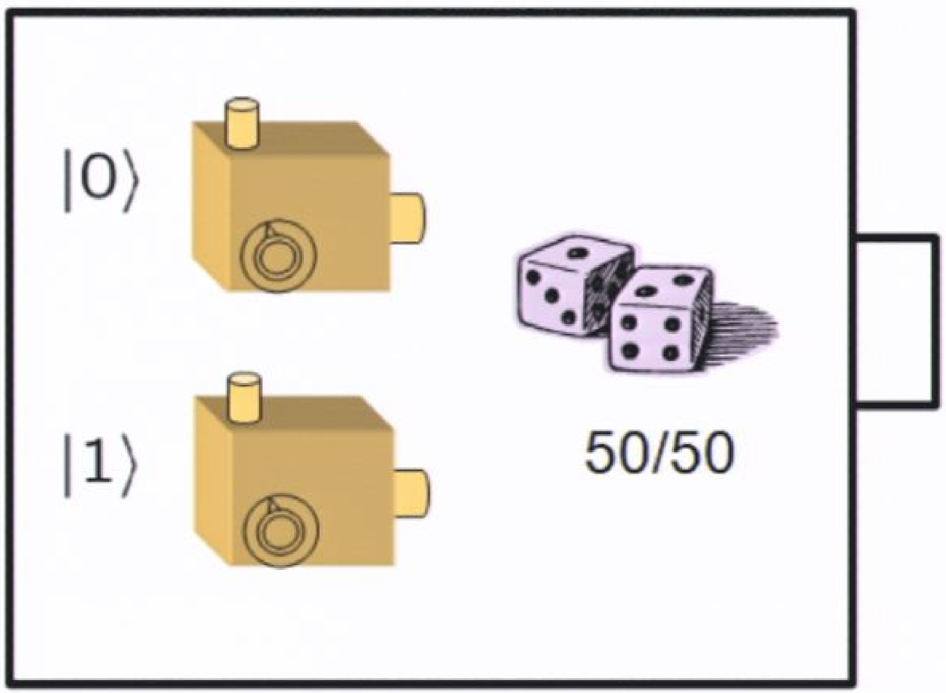
Representing mixtures of preparations



If $\mathsf{P}'' = \mathsf{P}$ with prob. w and P' with prob. $1-w$

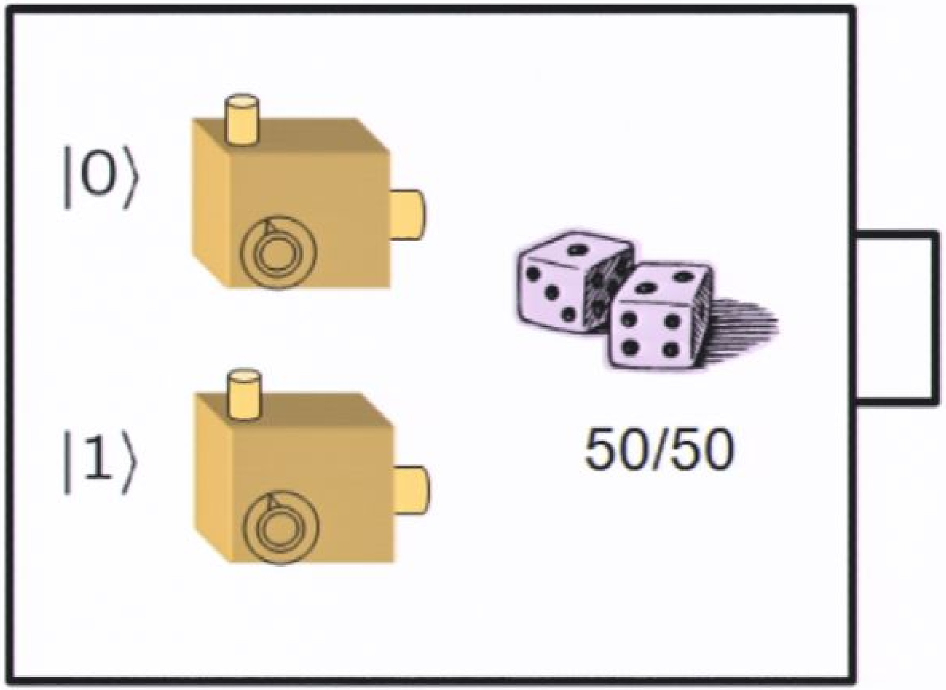
Then $p(k|\mathsf{P}'') = w\, p(k|\mathsf{P}) + (1-w)\, p(k|\mathsf{P}')$

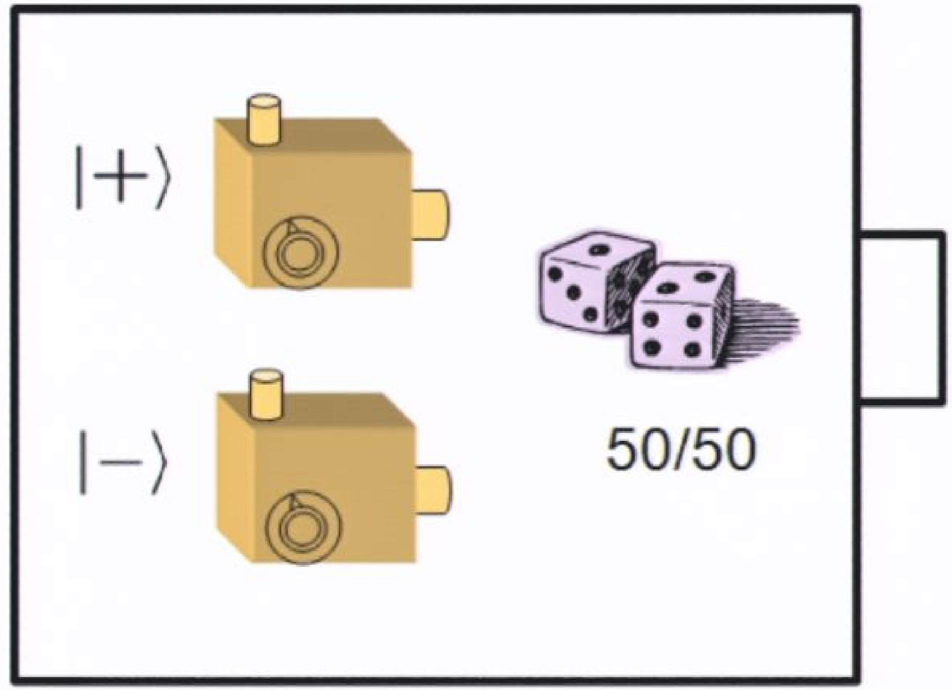
$$f(\mathsf{P}'') = w\, f(\mathsf{P}) + (1-w)\, f(\mathsf{P}')$$



$$\frac{1}{2}I \;=\; \frac{1}{2}\left|0\right\rangle\left\langle 0\right| + \frac{1}{2}\left|1\right\rangle\left\langle 1\right|$$

$$f(\tfrac{1}{2}I) = \tfrac{1}{2}\, f(|0\rangle\langle 0|) + \tfrac{1}{2}\, f(|1\rangle\langle 1|)$$





$$\frac{1}{2}I = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| \qquad \frac{1}{2}I = \frac{1}{2}|+\rangle\langle +| + \frac{1}{2}|-\rangle\langle -|$$

$$f(\tfrac{1}{2}I) = \tfrac{1}{2}\, f(|0\rangle\langle 0|) + \tfrac{1}{2}\, f(|1\rangle\langle 1|) \quad f(\tfrac{1}{2}I) = \tfrac{1}{2}\, f(|+\rangle\langle +|) + \tfrac{1}{2}\, f(|-\rangle\langle -$$

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Consider a function on density operators $\rho \mapsto f(\rho)$, satisfying:

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the indistinguishability of mixtures that are indistinguishable in quantum mechanics

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The theorem:

$$f(\rho) = Tr\,(E\rho)$$

for some effect E (i.e. $0 \leq E \leq I$).