

Title: Random quantum satisfiability: statistical mechanics of quantum optimization

Date: Dec 14, 2009 04:00 PM

URL: <http://pirsa.org/09120031>

Abstract: Alongside the effort underway to build quantum computers, it is important to better understand which classes of problems they will find easy and which others even they will find intractable. Inspired by the success of the statistical study of classical constraint optimization problems, we study random ensembles of the QMA₁-complete quantum satisfiability (QSAT) problem introduced by Bravyi. QSAT appropriately generalizes the NP-complete classical satisfiability (SAT) problem. We show that, as the density of clauses/projectors is varied, the ensembles exhibit quantum phase transitions between phases that are product satisfiable, entangled satisfiable and unsatisfiable. Remarkably, almost all instances of QSAT for a fixed interaction graph exhibit the same dimension of the satisfying manifold. This establishes the generic QSAT decision problem as equivalent to a purely graph theoretic property and that the hardest typical instances are likely to be localized in a bounded range of clause density.

Based on papers:

C.R. Laumann, R. Moessner, A. Scardicchio, and S.L. Sondhi. Phase transitions and random quantum satisfiability. QIC 10 (1/2), (2009). arXiv:0903.1904

C.R. Laumann, A.M. Lauchli, R. Moessner, A. Scardicchio, and S.L. Sondhi. On product, generic and random generic quantum satisfiability. arXiv:0910.2058

Random Quantum Satisfiability: Statistical Mechanics of Quantum Optimization

Chris Laumann¹

A. Läuchli² R. Moessner² A. Scardicchio³ S.L. Sondhi¹

¹Department of Physics
Princeton University

²Max Planck Institut für Physik Complexer Systeme
Dresden, Germany

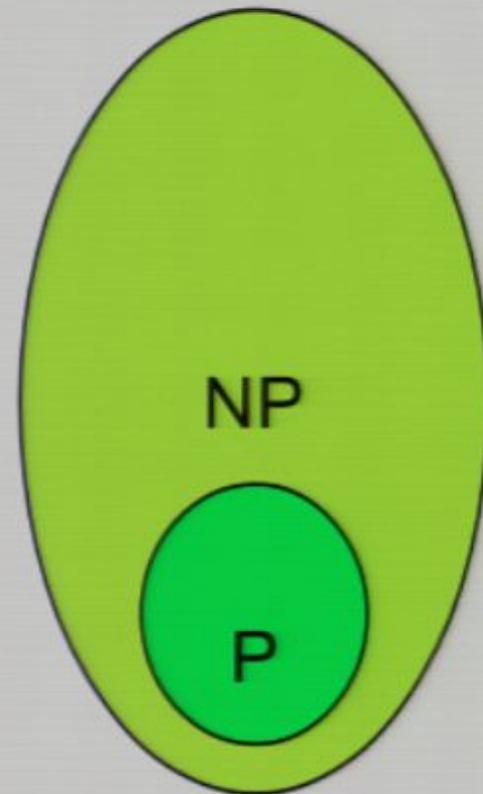
³Abdus Salam International Centre for Theoretical Physics
Trieste, Italy

Perimeter, December 14, 2009

Complexity theory: the lightning intro

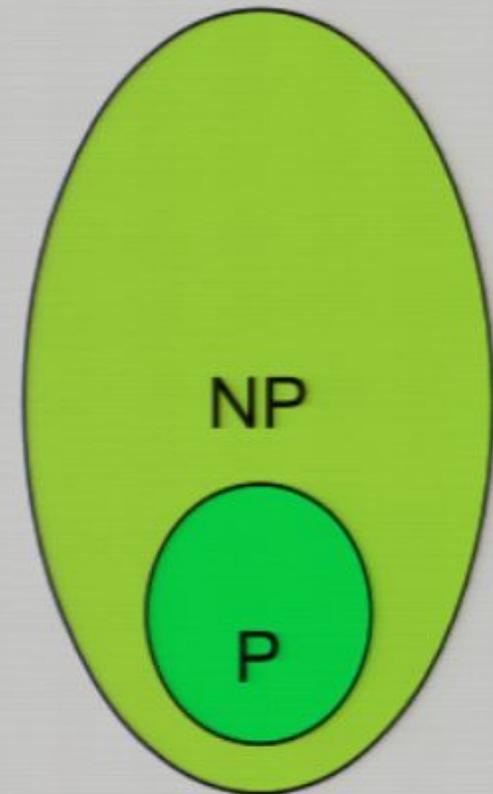
Complexity theory classifies problems according to how *quickly* computers can solve large examples

- P -- Efficiently solvable
- NP -- Efficiently checkable



$P \neq NP$

There are hard problems.*

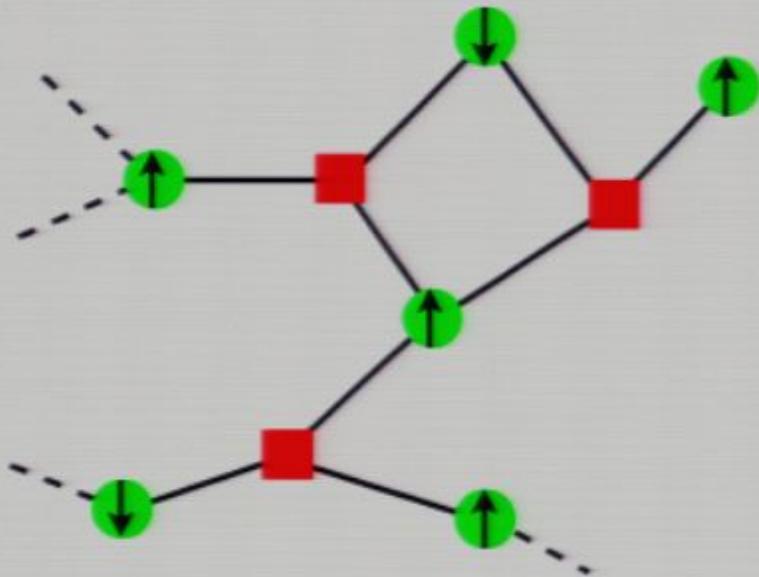


Strong Church-Turing hypothesis

A computer can *efficiently* simulate any physical model of computation.

- All physical models of computation are equivalent
- Any physical object undergoing natural dynamics can be viewed as a computer.
- If $P \neq NP$ there must be glassy physical systems

Classical 3-SAT: An 'Ising' model



N bits

$$\vec{\sigma} \in \{\pm 1\}^N$$

M constraints

$$E^m = \delta_{\sigma_{m_1}, \phi_1^m} \delta_{\sigma_{m_2}, \phi_2^m} \delta_{\sigma_{m_3}, \phi_3^m}$$

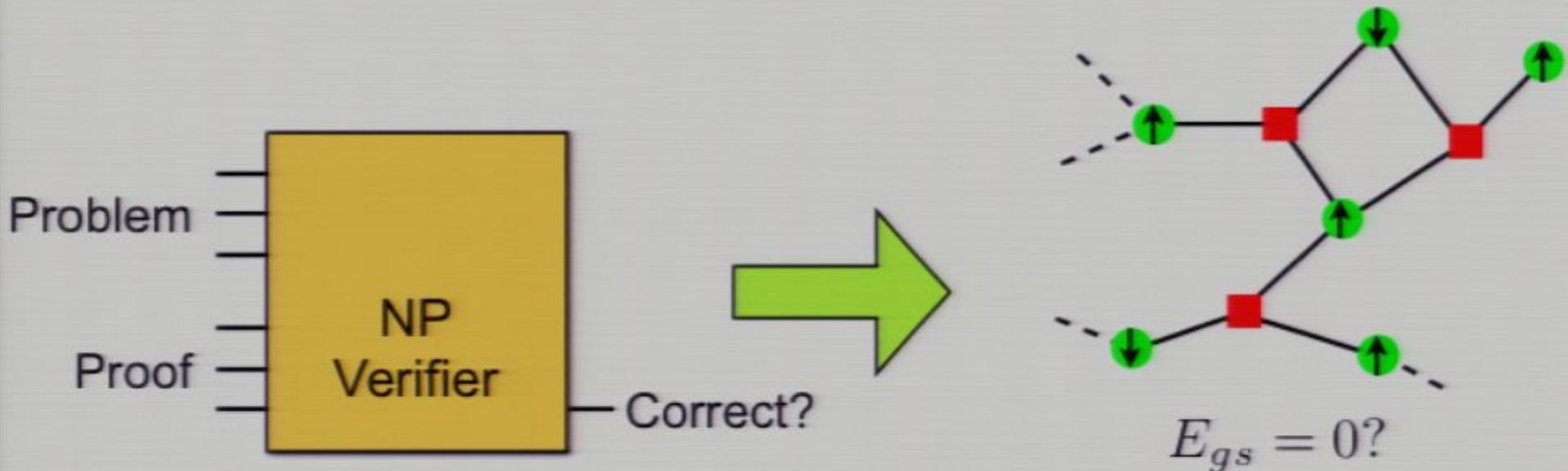
$$H = \sum_{m \in G} E^m(\sigma_{m_1}, \sigma_{m_2}, \sigma_{m_3})$$

E^m is 0 for satisfying states

Is the ground state energy zero?

$$\exists \vec{\sigma} \text{ s.t. } E^m(\sigma_{m_1}, \sigma_{m_2}, \sigma_{m_3}) = 0 \quad \forall m \in G?$$

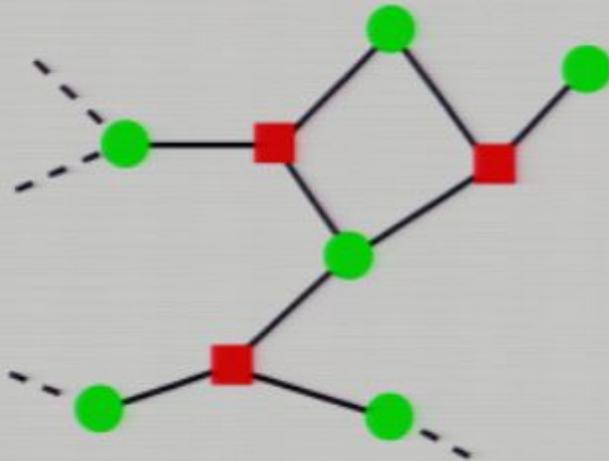
3-SAT: Worst-case complexity



- 3-SAT can encode the operation of the verification circuit.
- 3-SAT is NP-complete: solve 3-SAT efficiently and you could solve all of NP efficiently ($P=NP$)

Ensemble of 3-SAT: Average complexity

Random 3-SAT



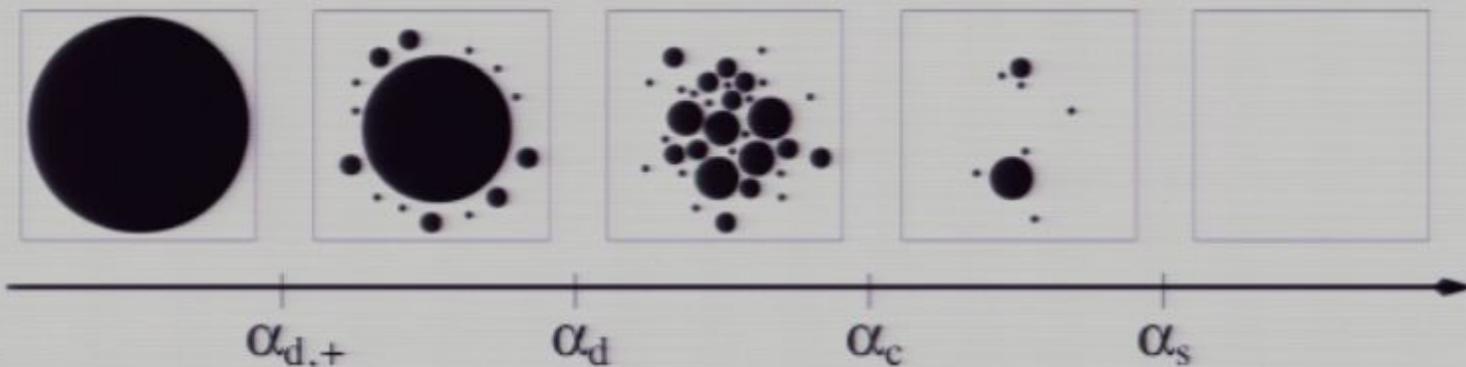
- Which instances are hard?
- Ensembles of 'typical' instances
- Control parameters
- Spin glass physics

Clause density $\alpha = M/N$

Random graph

Disordered couplings

Glass theorist's phase diagram



Krzakala, et al., PNAS 2007

- Qualitative phase diagram of random constraint satisfaction problems
- Phase transitions: clustering of satisfying assignments
- Based on cavity methods
- Quantum cavity methods?

cf. Mezard, et al., Science, 2002

cf. CRL, et al., PRB 2008
Hastings, PRB 2007
Leifer, Poulin, Ann Phys 2008

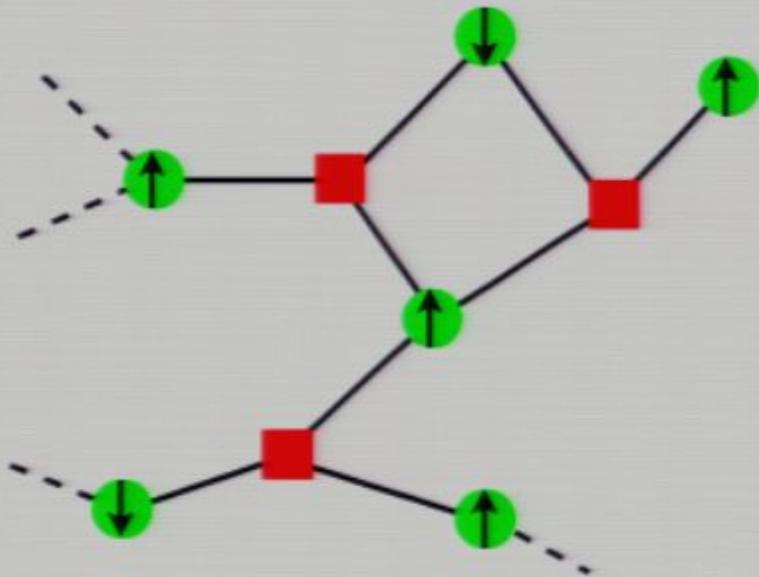
Statistical physics of constraint satisfaction

	Classical
Worst-case complexity	NP-completeness
Statistical physics	Satisfiability transitions Dynamical transitions Clustering transitions
Heuristic algorithms	Simulated annealing Belief propagation Survey propagation

Quantum Satisfiability

- Natural quantum generalization of classical satisfiability (k-SAT)
- Quantum 'hard' worst-case complexity: QMA₁-complete
- Are 'typical' instances hard?
- Motivated by classical story, but has its own features...

Quantum k-QSAT: A k-local qubit model



N qubits $\mathcal{H} = (\mathbb{C}^2)^{\otimes N}$

M constraints $\Pi^m = |\phi^m\rangle\langle\phi^m|$

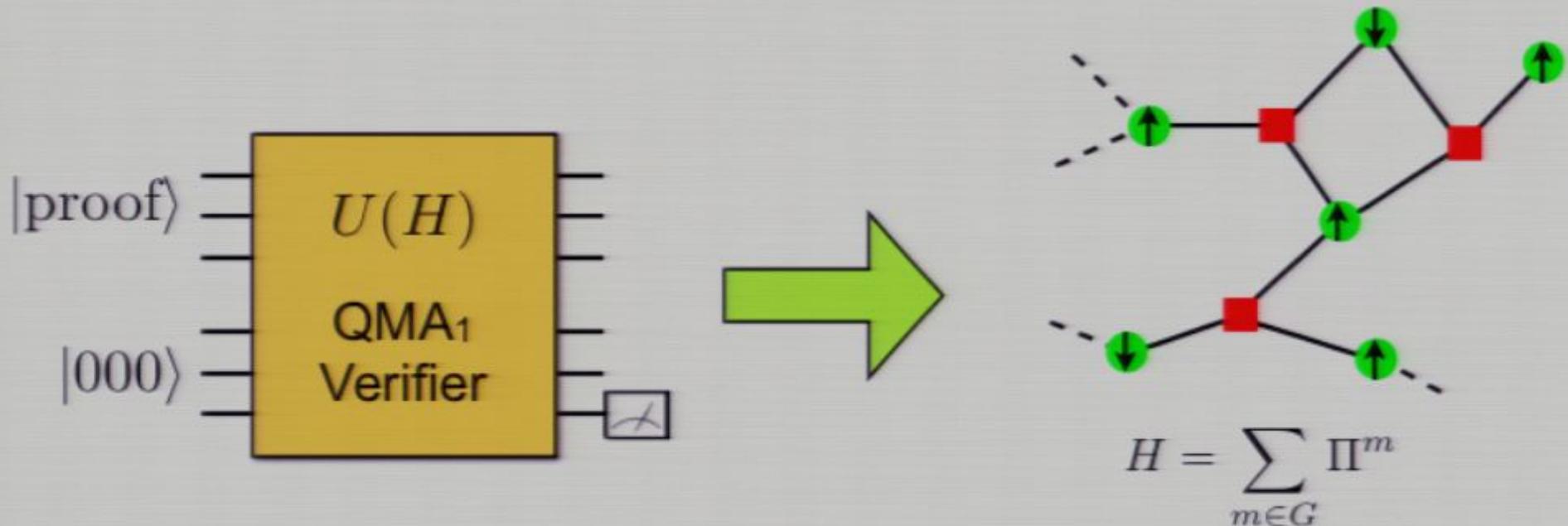
$$H = \sum_{m \in G} \Pi^m$$

Π^m penalizes 1 out of 2^k states

Is the ground state energy zero?

$\exists |\psi\rangle \in \mathcal{H}$ s.t. $\Pi^m |\psi\rangle = 0 \forall m \in G$?

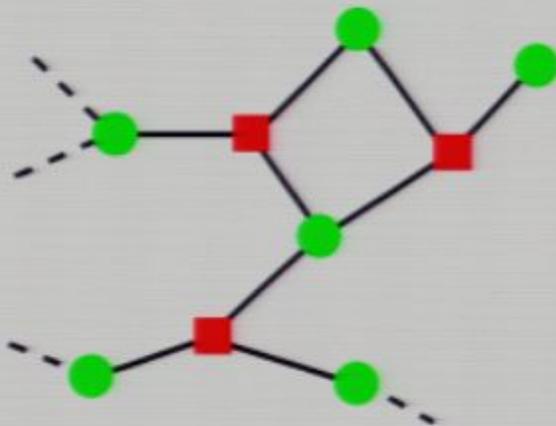
k-QSAT: Worst-case complexity



- 2-QSAT in P -- classical algorithm to solve -- Easy!
- k -QSAT ($k > 3$) is QMA_1 -complete -- Hard!

Ensemble of k-QSAT: Average complexity

Random graph



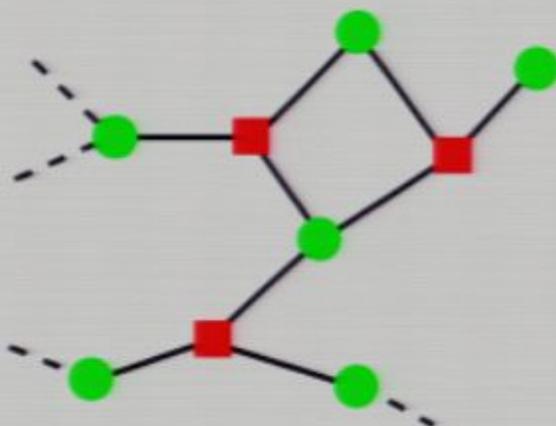
Discrete

Clause density $\alpha = M/N$

Place edges w.p. $p = \alpha / \binom{N}{k}$

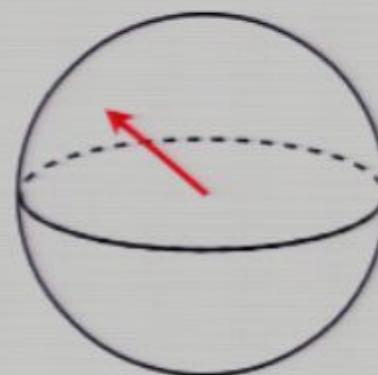
Ensemble of k-QSAT

Random graph



Discrete

Generic projectors



Continuous

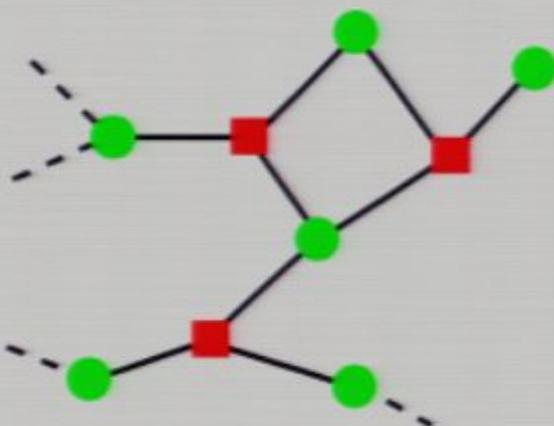
Clause density $\alpha = M/N$

Place edges w.p. $p = \alpha / \binom{N}{k}$

$$\Pi^m \leftarrow \mathbb{C}\mathbb{P}^{2^k - 1}$$

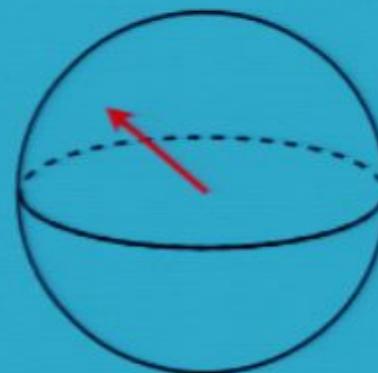
Ensemble of k-QSAT

Random graph



Discrete

Generic projectors



Continuous

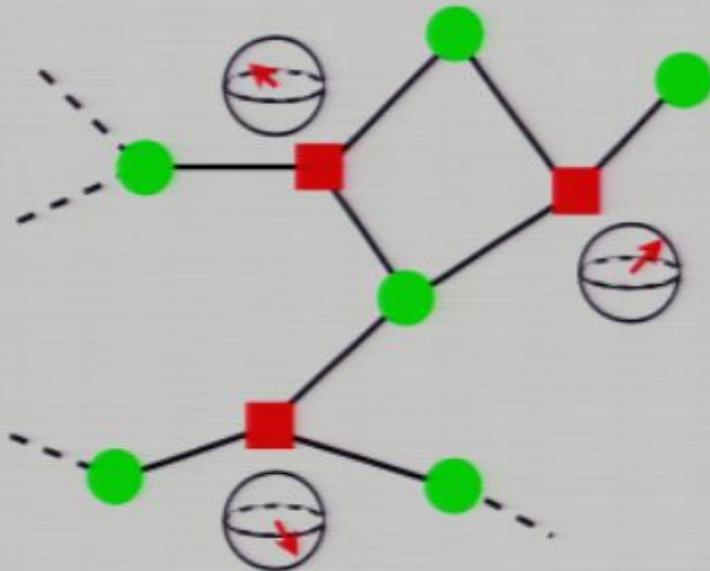
$$\Pi^m \leftarrow \mathbb{C}\mathbb{P}^{2^k - 1}$$

Clause density $\alpha = M/N$

Place edges w.p. $p = \alpha / \binom{N}{k}$

Geometrization

For fixed G , generic choices of Π produce QSAT instances with minimal SAT dimension.

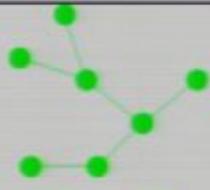
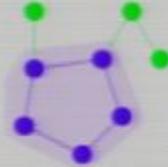
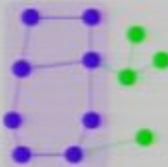


$$R(G, \Pi) = \dim(\ker(H_{G, \Pi}))$$

$$R(G, \Pi) = R_g(G) \text{ w.p. } 1$$

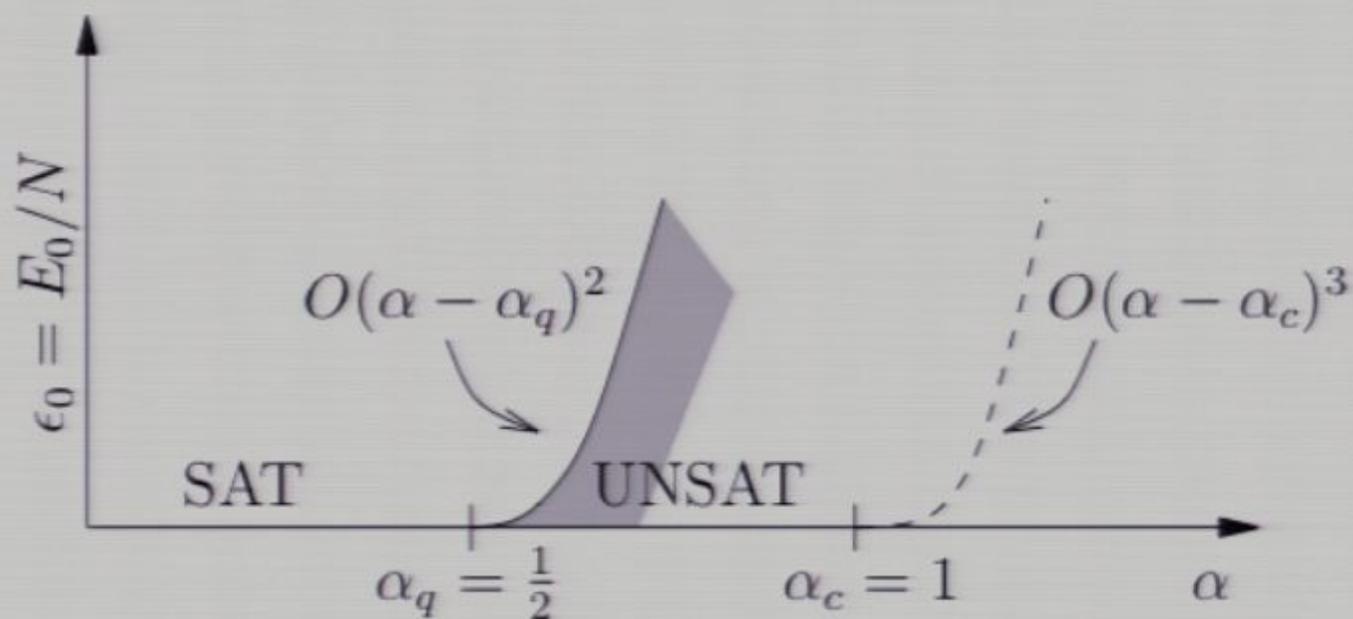
$$R_g(G) \leq R(G, \Pi) \forall \Pi$$

Example: Generic 2-QSAT

Cluster G		$R_g(G)$
Trees (N sites)		$N+1$
One loop		2
More loops		0

- QSAT factors over disconnected clusters
- Proof using nonorthogonal 'transfer basis'

Example: Random 2-QSAT Phase Diagram



- Generic 2-QSAT on Poissonian random graphs
- SAT-UNSAT phase transition at emergence of giant component

Existence of SAT-UNSAT phases at higher k



Construct SAT states:

Transfer matrix states

Dimer covering states

Quantum Lovasz states

Upper bound $R_g(G)$

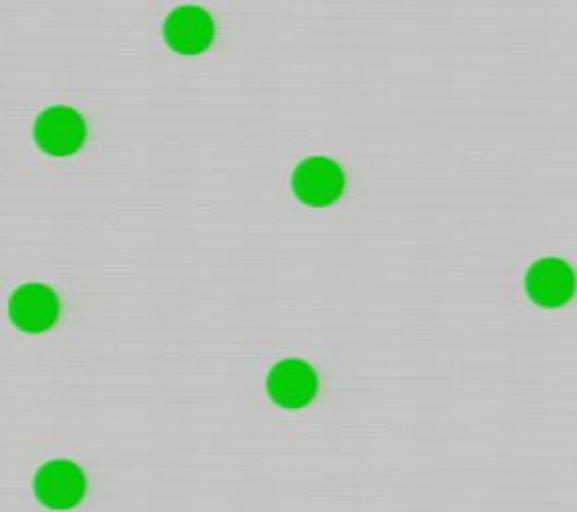
'First moment bound'

Classical bound

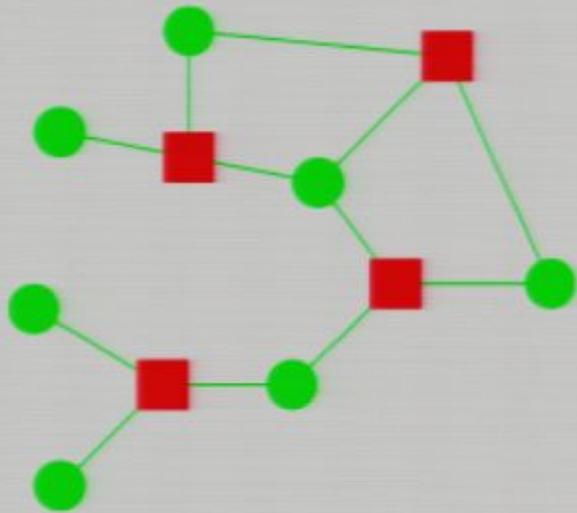
Sunflower bound

k-QSAT: 'First moment' bound

$$R_g^0 = 2^N$$



k-QSAT: 'First moment' bound



$$R_g^0 = 2^N$$

$$R_g^1 = R_g^0(1 - 1/2^k)$$

$$R_g^2 = R_g^1(1 - 1/2^k)$$

$$R_g^3 \leq R_g^2(1 - 1/2^k) \quad \text{w.p. 1}$$

...

$$R_g^M \leq 2^N (1 - 1/2^k)^M \quad \text{w.p. 1}$$

$$\alpha_{wb} = -1/\log(1 - 1/2^k) \sim 2^k$$



Sunflowers and nosegays

- Best current upper bound on satisfiability transition in random k -QSAT

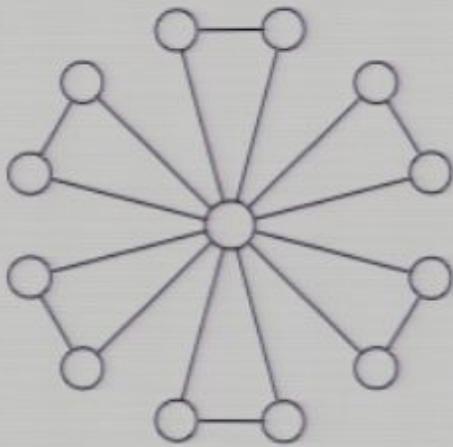


Figure 1: The (6, 3)-sunflower.

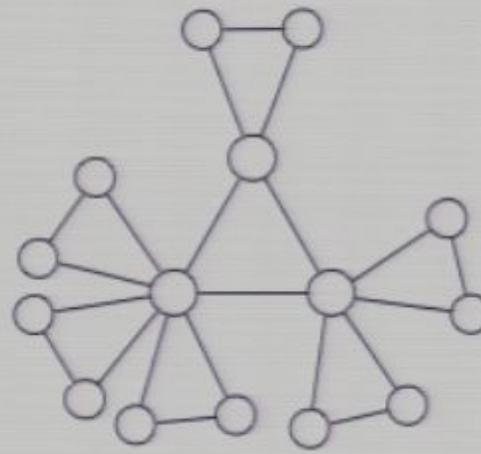
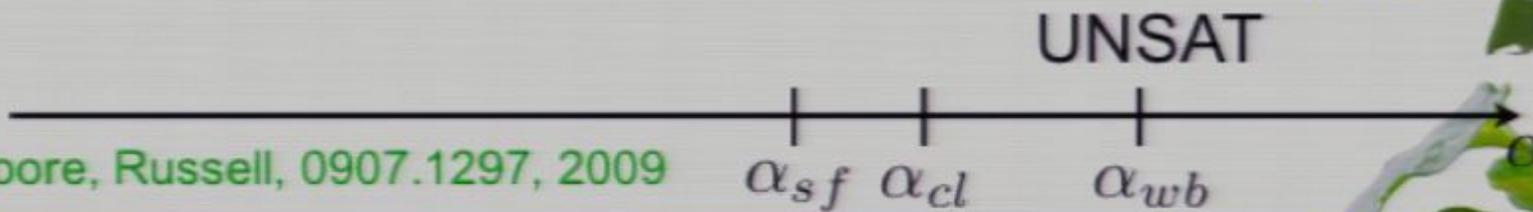


Figure 2: The (1, 2, 3)-nosegay.

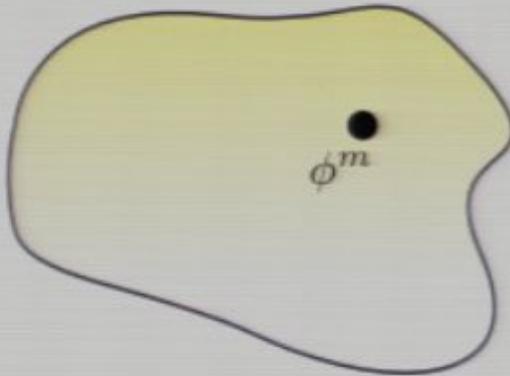
$$k = 3 \quad \alpha_{sf} \approx 3.594 < \alpha_{\text{classical}}$$

$$k \rightarrow \infty \quad \alpha_{sf} \approx 2^k \frac{\ln 2}{2} < \alpha_{\text{classical}}$$



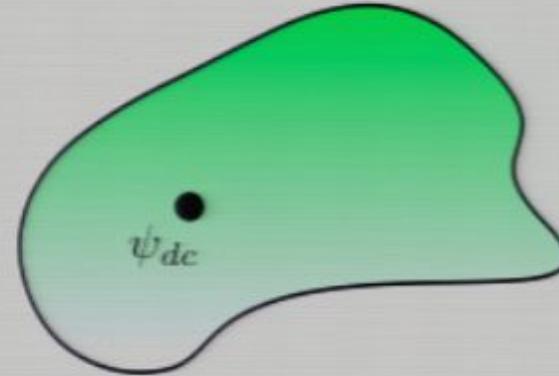
Product state perturbation theory

Projectors



$$(\mathbb{CP}^{2^k - 1})^M$$

Product States

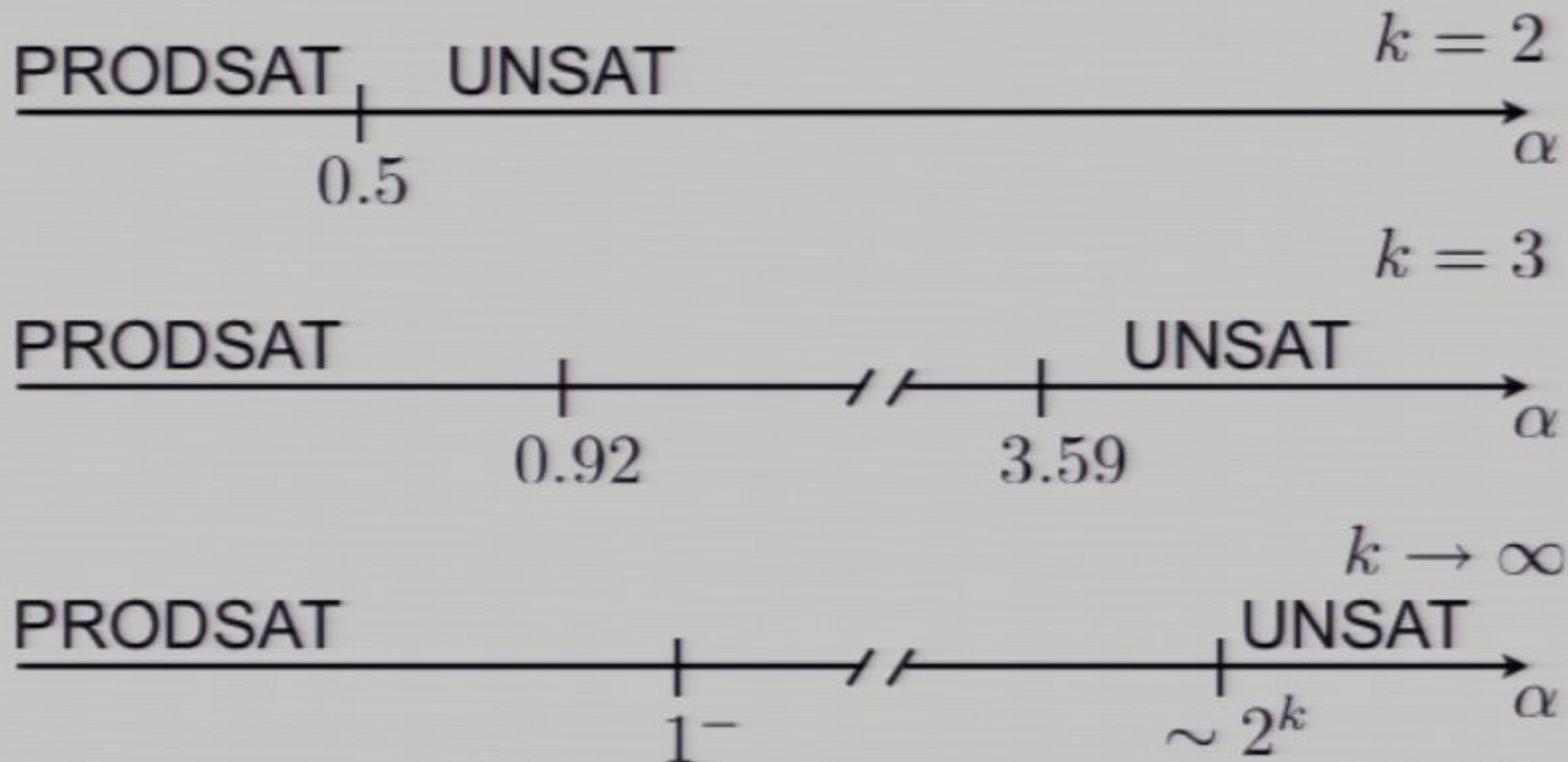


$$(\mathbb{CP}^1)^N$$

$$\phi_{i_1 i_2 \dots i_k}^m z_{m_1}^{i_1} z_{m_2}^{i_2} \dots z_{m_k}^{i_k} = 0$$

- Linearization of product state satisfiability condition
- Generically solvable if and only if G has dimer covers

Entanglement transition



- $k=2$ has direct PRODSAT-UNSAT transition
- $k>2$ bounds are well separated: an entangled SAT phase?
- Numerics for $k=3$ (small sizes): $\alpha_c \approx 1 \pm 0.06$

Conclusion

- Quantum satisfiability is a quantum hard optimization problem. It can encode any problem in the class QMA_1 .
- Generic QSAT is as frustrated as possible and is a graph property.
- Random QSAT provides insight into what makes QSAT hard.
- 2-QSAT: know the property $R_g(G)$; get the SAT-UNSAT phase diagram
- Higher k : indirect arguments provide SAT-UNSAT phase
 - Product satisfiability characterized by dimer coverings.
 - Dimensional upper bounds produce UNSAT phase.
- Entangled SAT phase at higher k . Connections to Quantum Lovász.

Open questions

- Where's the SAT-UNSAT phase transition?
- What is the complexity of computing $R_g(G)$? Does generic QSAT have a classical test?
- *Glass physics*: Quantum analogs of clustering/dynamical phase transitions from classical glass problems?
- *Stat mech*: Cavity methods for QSAT problem?
- *Quantum algorithms*: Quantum generalizations of belief propagation, survey propagation? How does the adiabatic algorithm fare?

Thanks for listening!