Title: Concentration of measure and the mean energy ensemble

Date: Dec 07, 2009 04:00 PM

URL: http://pirsa.org/09120030

Abstract: If a pure quantum state is drawn at random, this state will almost surely be almost maximally entangled. This is a well-known example for the "concentration of measure" phenomenon, which has proved to be tremendously helpful in recent years in quantum information theory. It was also used as a new method to justify some foundational aspects of statistical mechanics.
In this talk, I discuss recent work with David Gross and Jens Eisert on concentration in the set of pure quantum states with fixed mean energy: We show typicality in this manifold of quantum states, and give a method to evaluate expectation values explicitly. This involves some interesting mathematics beyond Levy's Lemma, and suggests potential applications such as finding stronger counterexamples to the additivity conjecture.

Concentration of measure and the mean energy ensemble

Markus Müller

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Joint work with David Gross and Jens Eisert.

Outline of the talk

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1. Foundations of statistical mechanics
   - Problem: How to justify stat. mech.?
   - Possible solution: Concentration of measure

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- Problem: How to justify stat. mech.?
- Possible solution: Concentration of measure

2. The mean energy ensemble

- Going beyond subspaces
- Our result, proof idea and tools
- What does it tell us about physics?

1. Foundations of statistical mechanics

The trouble with statistical physics

Two kinds of missing information:

- Observer's lack of knowledge: knows only volume, temperature, ...
- Physical uncertainty: different cups prepared differently, time evolution, ...

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Statistical physics: makes *objective predictions*, based on *subjective lack of knowledge*.

"Postulate of equal apriori probabilities":

=

=...

Why does it work?

I. Foundations of statistical mechanics

What about ergodicity?

Idea: Time evolution explores all accessible phase space uniformly.

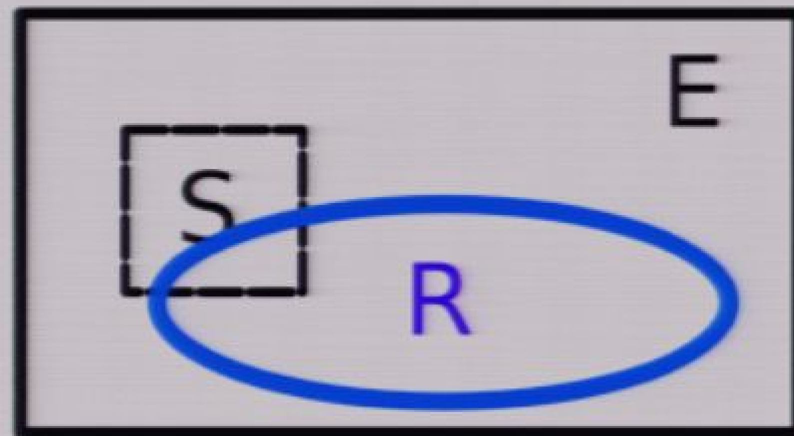
Problems:

- Proven only for some special systems.
- May take very long time.

1. Foundations of statistical mechanics

S. Popescu, A. J. Short, A. Winter, Nature Physics 2(11), 2006

$$\mathcal{H}_R \subset \mathcal{H}_S \otimes \mathcal{H}_E$$



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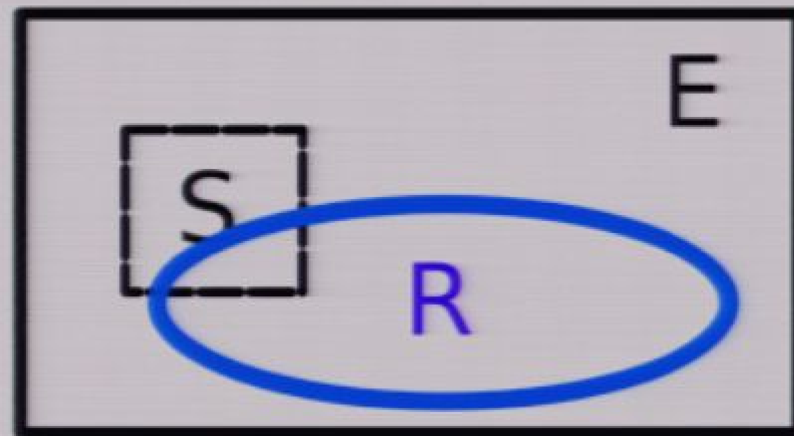

$\mathcal{H}_R$: subspace; restricted set of physically allowed q-states;
$\mathcal{H}_S \otimes \mathcal{H}_E$: the "universe".

Example: S=system, E=bath, R=subspace spanned by global energy eigenstates in $[E - \Delta E, E + \Delta E]$

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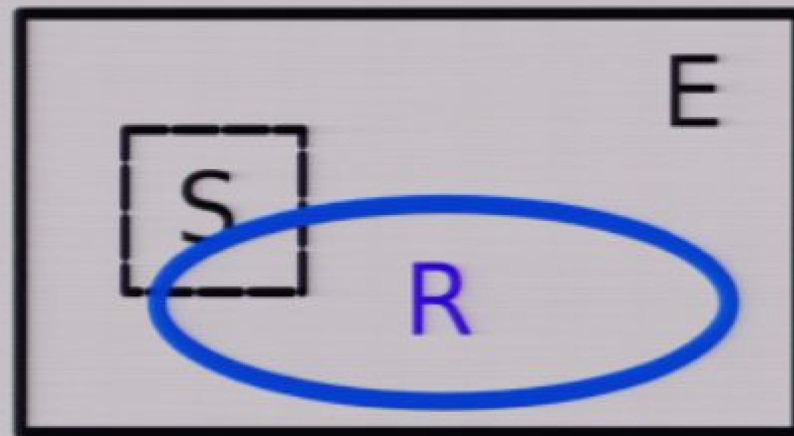

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Statistical mechanics recipe: equidistribution on R gives "microcanonical ensemble" $\Omega_S := \mathrm{Tr}_E \left(\mathbf{1}_R / d_R \right)$.

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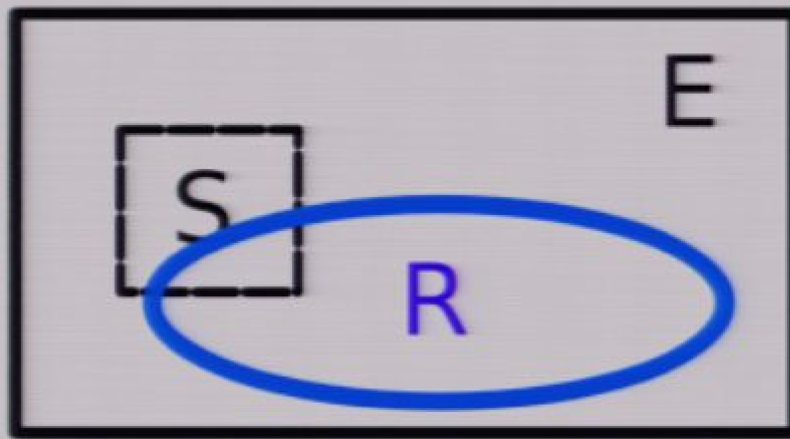

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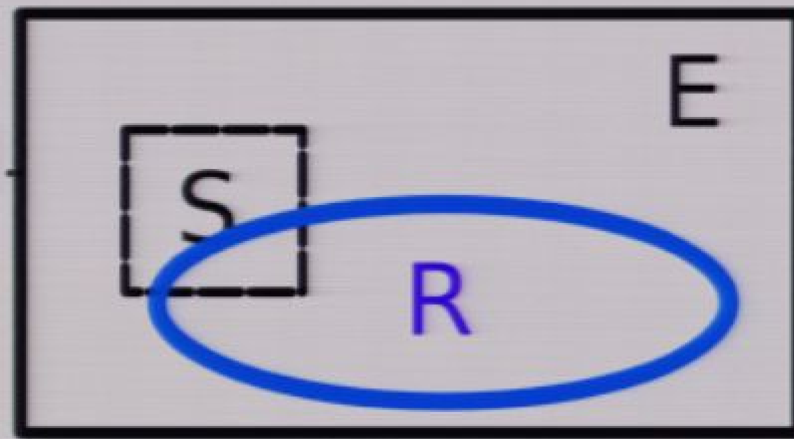

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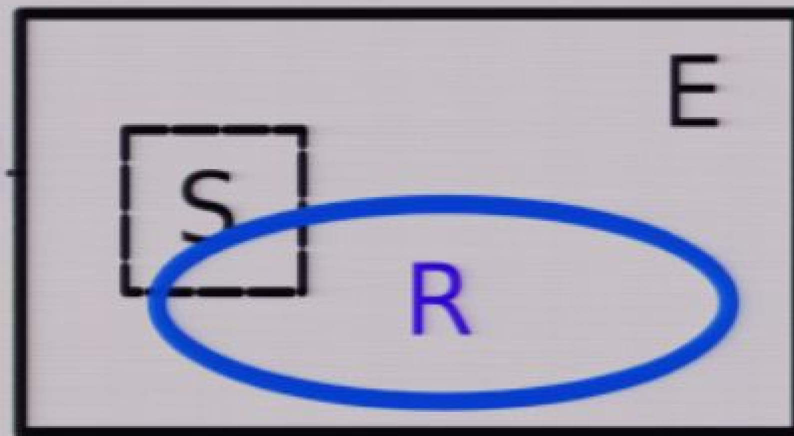

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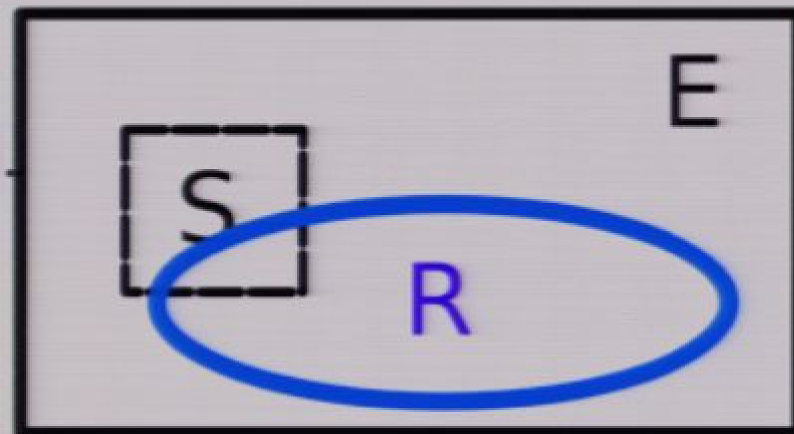

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Theorem (Concentration of measure): Draw $|\psi\rangle \in \mathcal{H}_R$ randomly acc. to unitarily invariant measure. Then,

$$\mathrm{Prob}\left[\|\rho_S - \Omega_S\|_1 \geq \varepsilon + \frac{d_S}{\sqrt{d_R}} \right] \leq 2 \exp\left(-C d_R \varepsilon^2\right),$$

where $C = 1/18\pi^3$, $d_R = \dim \mathcal{H}_R$, $d_S = \dim \mathcal{H}_S$, $\Omega_S = \mathrm{Tr}_E \left(\mathbf{1}_S / d_S\right)$.

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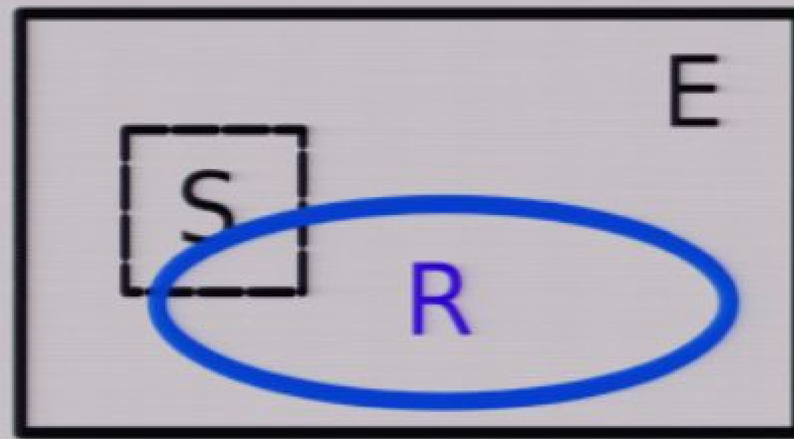

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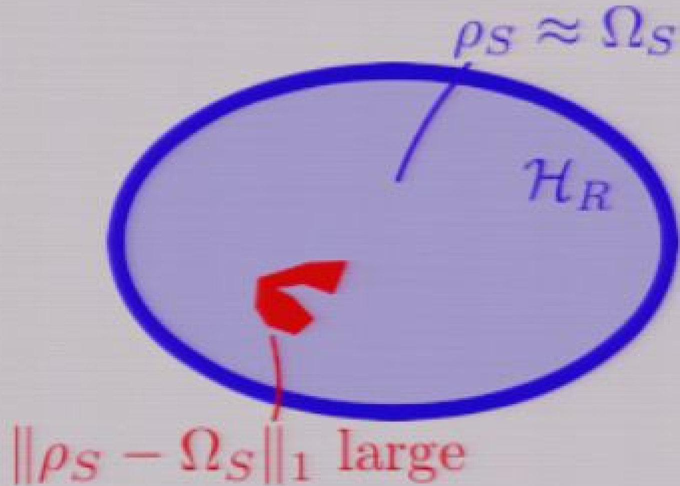

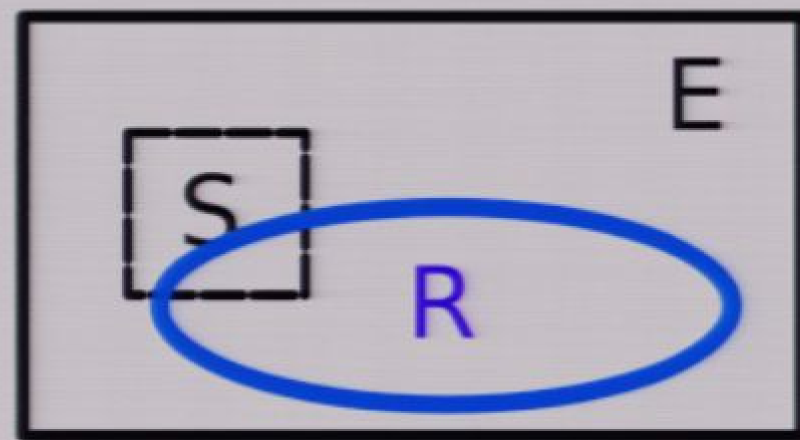

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I. Foundations of statistical mechanics

The perfect coffee machine

$n = 1$ $n = 2$ $n = 3$

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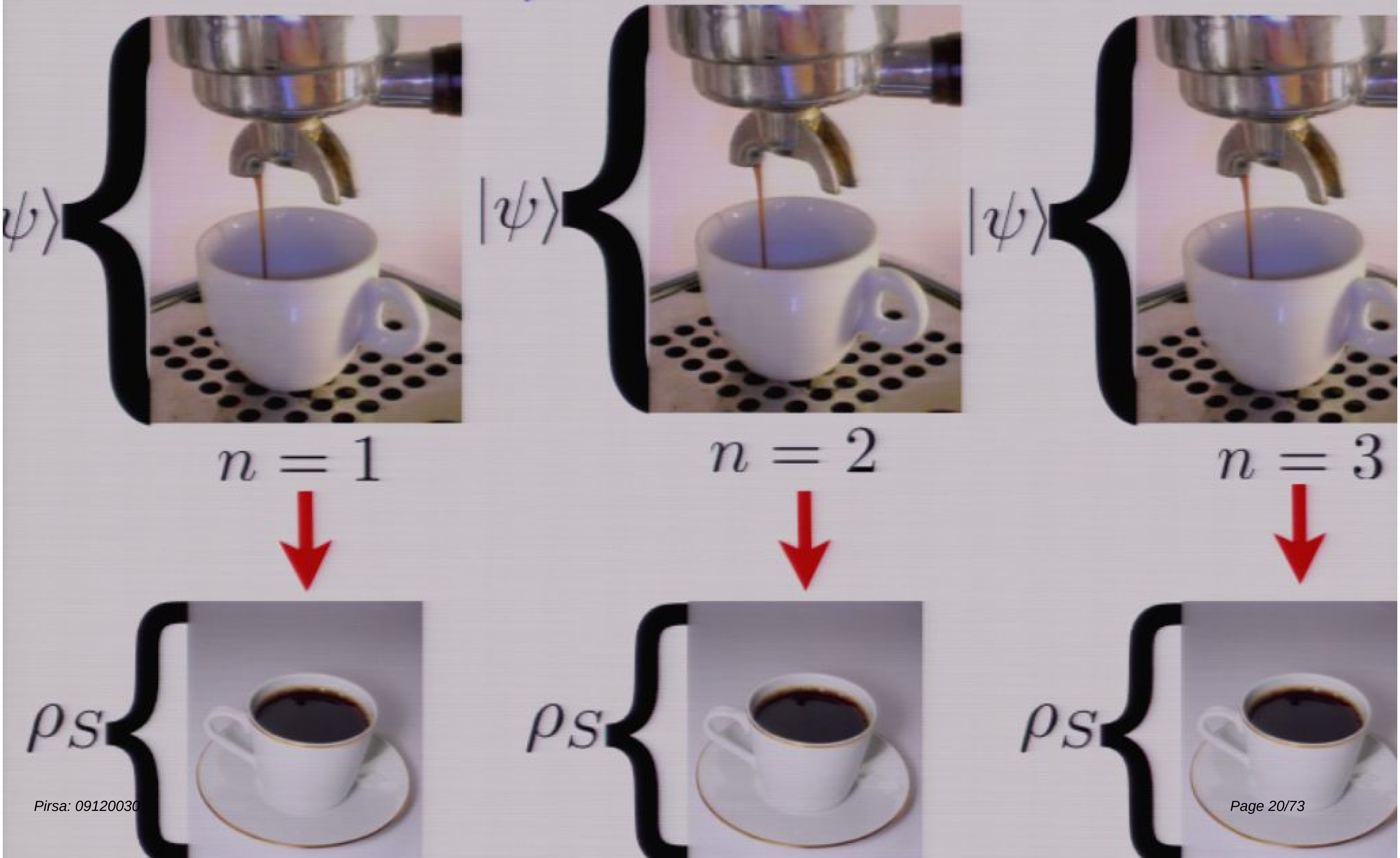

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Reveals ρ_S. But $\rho_S \approx \Omega_S$ (microcanonical ensemble) for "almost all" $|\psi\rangle \in \mathcal{H}_R$.
Hence, almost all coffee machines (compatible with restrictions) prepare the microcanonical ensemble.

↑

measurements ("coffee tomography")

↑ ↑ ↑

ρ_S {  ρ_S {  ρ_S { 

I. Foundations of statistical mechanics

So far: restrictions are subspaces.

- Exact form of Ω_S is not given by Popescu et al. (generality!).
- Goldstein, Lebowitz, Tumulka, Zanghi, PRL **96** (2006): no interaction $H = H_S + H_{env}$, fixed energy E, restriction $\mathcal{H}_R$ spanned by spectral window $[E - \Delta, E + \Delta]$, bath's spectral density exponential around E, then

$$\Omega_S \sim \exp(-\beta H_S).$$

2. The mean energy ensemble

Reasons for going beyond subspaces

- Observers may have knowledge on systems that is different from "*being element of a subspace*".
- Example: given Hamiltonian H, the energy expectation value $\langle\psi|H|\psi\rangle = E$ might be known instead.

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$$M_E = \{|\psi\rangle \in \mathbb{C}^n \mid \langle\psi|H|\psi\rangle = E,\ \ \|\psi\| = 1\}$$

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Goal: prove "concentration of measure" for M_E. Proof tools will be useful also for other nonlinear constraints.

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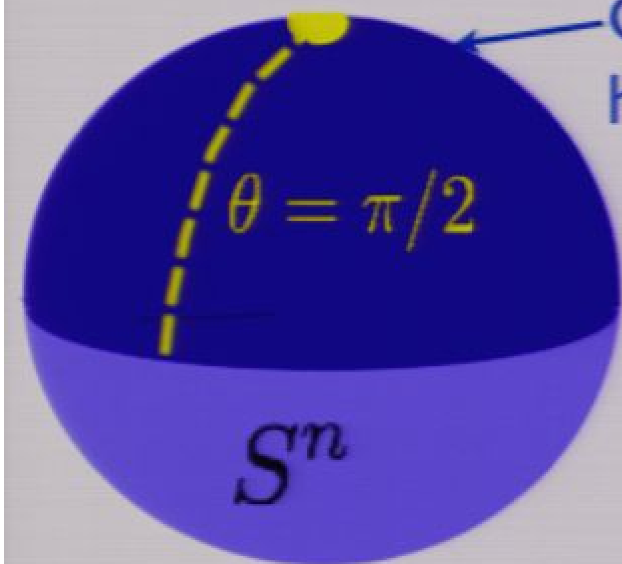
Subspace restrictions (Popescu et al.): Lévy's Lemma

All pure quantum states in $\mathcal{H}_R$: sphere S^n.



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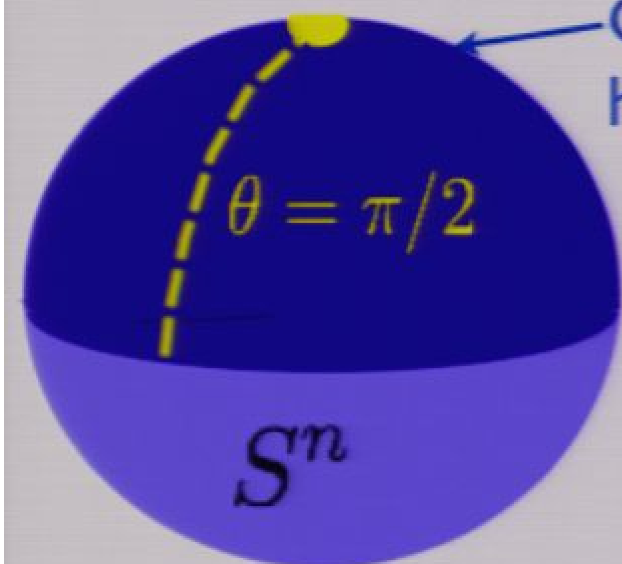

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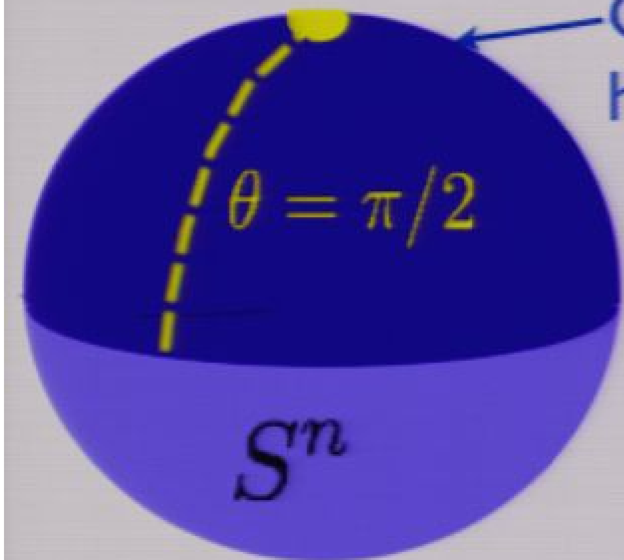

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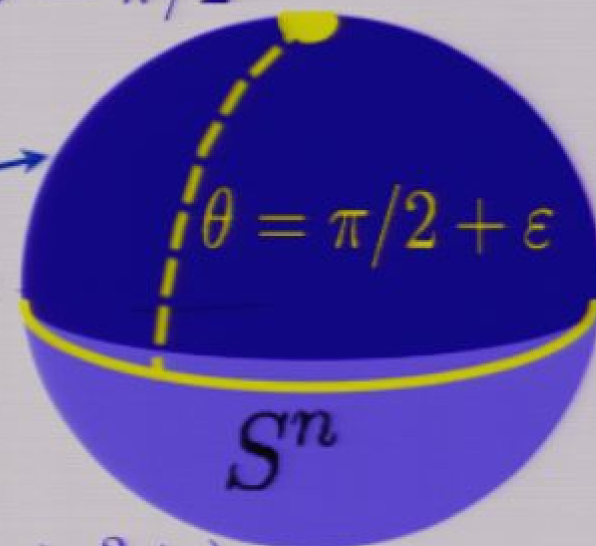
As soon as the angle exceeds $\pi/2 + \varepsilon$, the cap has almost full measure 1.



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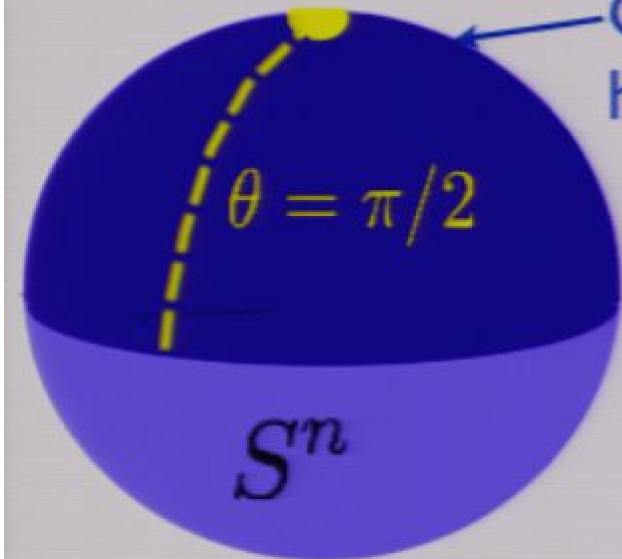

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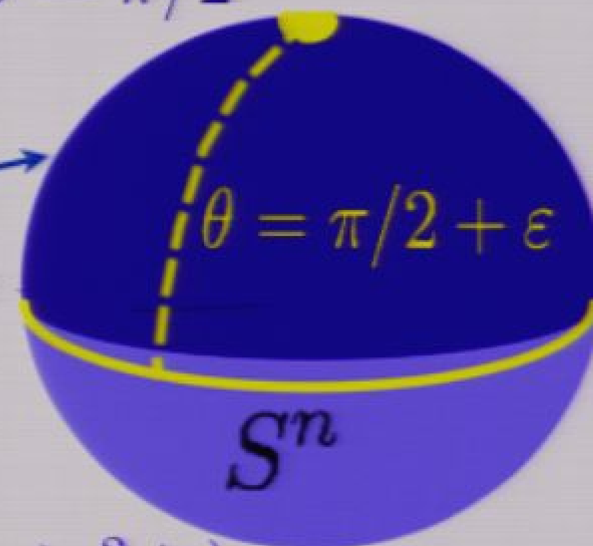

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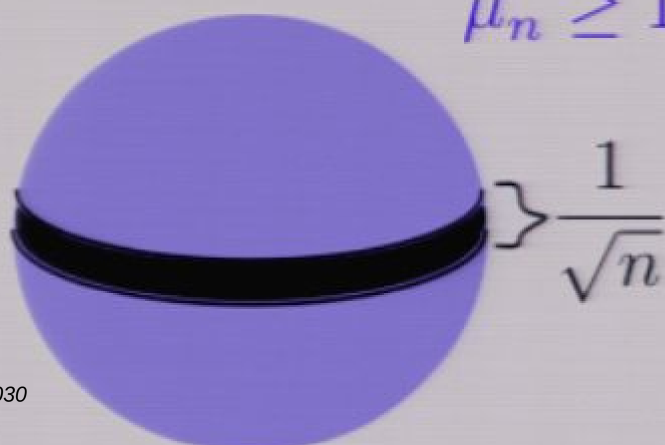

Measure is exponentially concentrated around any equator.

2. The mean energy ensemble

Subspace restrictions (Popescu et al.): Lévy's Lemma

Consequence of geometry: Lévy's Lemma

Let $f : S^n \to \mathbb{R}$ be Lipschitz continuous with constant η, i.e. $|f(x) - f(y)| \leq \eta \cdot \|x - y\|$. Then,

$$\mathrm{Prob}\{|f(x) - \mathbb{E}f| > \varepsilon\} \leq 2\exp\left(-c(n+1)\varepsilon^2/\eta^2\right),$$

where $c = (9\pi^3 \ln 2)^{-1}$.

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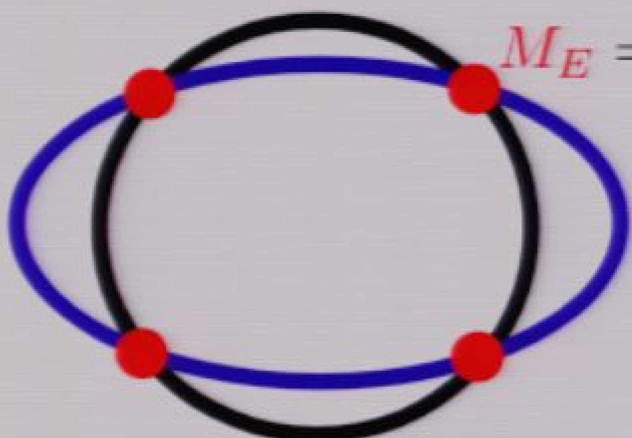
2. The mean energy ensemble
Beyond spheres and subspaces

Setting: We fix some Hamiltonian *H* on $\mathbb{C}^n$, and we draw vector states $|\psi\rangle \in \mathbb{C}^n$ with $\|\psi\| = 1$ randomly under the constraint $\langle\psi|H|\psi\rangle = E.$

$$M_E = \{|\psi\rangle \in \mathbb{C}^n \mid \|\psi\|^2 = 1, \langle\psi|H|\psi\rangle = E\}$$



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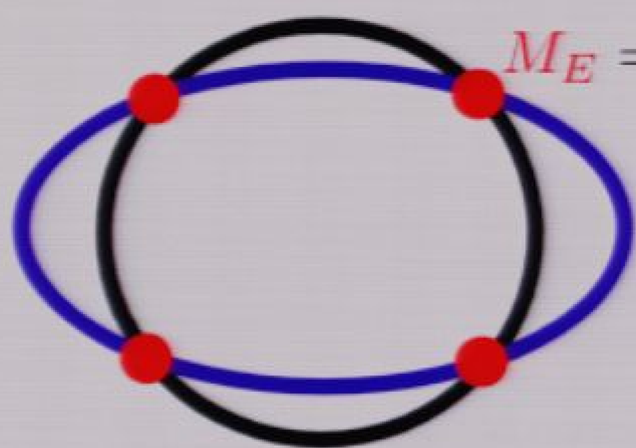

$$M_E = \{|\psi\rangle \in \mathbb{C}^n \mid \|\psi\|^2 = 1, \langle\psi|H|\psi\rangle = E\}$$

- Can we prove concentration of measure on M_E?
- Do typical bipartite states look like Gibbs states?

2. The mean energy ensemble

Main Result, Part I (concentration in general)

Theorem 1 (Main Theorem). *Given* $H = H^\dagger \geq 0$ *on* $\mathbb{C}^n$ *and some energy value* $E > 0$, *draw a state* $|\psi\rangle \in \mathbb{C}^n$ *randomly under the constraint* $\langle\psi|H|\psi\rangle = E$. *Suppose that the following three conditions hold:*

- E *is not equal to any eigenvalue of* H,
- $E \leq \frac{1}{n}\mathrm{Tr}(H) - \mathcal{O}\left((\log n/n)^{1/2}\right)$,
- $E = E_H\,[1 + \mathcal{O}(1/\sqrt{n})]$.

Then, for every function $f : M_E \to \mathbb{R}$ *with* $\max \|\nabla f\| \leq \lambda$, *we have*

$$\mathrm{Prob}\left\{|f(\psi) - \bar{f}| > \varepsilon\right\} \leq c' \cdot n^{3/2} \cdot e^{-cn\left(\frac{\varepsilon}{\lambda} - \frac{1}{4n}\right)^2 + \mathcal{O}(\sqrt{n})}, \qquad (4)$$

where $\bar{f}$ *is the median of* f *on* M_E. *Moreover,* $c = 3E_{min}/64E$, *and* c' *depends on* E_{max}/E *and* $E/\bar{E}$, *where* $E_H^{-1} = \frac{1}{n}\sum_k E_k^{-1}$ *is the harmonic mean energy,* $\bar{E}^{-2} = \frac{1}{n}\sum_k \frac{1}{E_k^2}$, $E_{min} = \min_k E_k$, $E_{max} = \max_k E_k$, *if* $\{E_k\}_{k=1}^n$ *are the eigenvalues of* H.

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- Dependence of constants on spectrum and *n*: can be difficult, due to "energy shift".
- Rule of thumb: if relevant energy ratios grow with dimension n slower than $\approx n^{1/4}$, concentration is strong.

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Main Result, Part I (concentration in general)

- Dependence of constants on spectrum and *n*: can be difficult, due to "energy shift".
- Rule of thumb: if relevant energy ratios grow with dimension n slower than $\approx n^{1/4}$, concentration is strong.
- Simplest example: $H = H_S \otimes \mathbf{1}_{env}^{(n)}$: strong concentration. Spin chain interactions: work in progress.
- How to compute the median $\bar{f}$? We have a formula: Consider the full ellipsoid $N := \{\psi \mid \langle\psi|H|\psi\rangle \leq E(1+1/2n)\}$. Then, $|\bar{f} - \mathbb{E}_N f| \leq \mathcal{O}\left((\log n/\sqrt{n})^{1/2}\right)$.

Explicitly computable via spherical integration.

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Can always be achieved by an energy shift $E \mapsto E + s$, $H \mapsto H + s \cdot \mathbf{1}$.

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2. The mean energy ensemble

Main Result, Part I (concentration in general)

- Dependence of constants on spectrum and *n*: can be difficult, due to "energy shift".
- Rule of thumb: if relevant energy ratios grow with dimension n slower than $\approx n^{1/4}$, concentration is strong.
- Simplest example: $H = H_S \otimes \mathbf{1}_{env}^{(n)}$: strong concentration. Spin chain interactions: work in progress.
- How to compute the median $\bar{f}$? We have a formula: Consider the full ellipsoid $N := \{\psi \mid \langle\psi|H|\psi\rangle \leq E(1 + 1/2n)\}$. Then, $|\bar{f} - \mathbb{E}_N f| \leq \mathcal{O}\left((\log n/\sqrt{n})^{1/2}\right)$.

Explicitly computable via spherical integration.

2. The mean energy ensemble

Main Result, Part II (special case: bipartite system)

Applying the previous result to the functions

$$f(\psi) := (\mathrm{Tr}_{env}|\psi\rangle\langle\psi|)_{i,j}$$

gives the typical reduced density matrix:

Theorem 2 (Typical reduced density matrix). *Let $H = H_A + H_B$ be a Hamiltonian on $\mathbb{C}^A \otimes \mathbb{C}^B$ of dimension $n = AB$ with $A, B \geq 2$ and positive energy eigenvalues $\{E_i^A\}_{i=1}^A$ and $\{E_j^B\}_{j=1}^B$ respectively. If the conditions of Theorem 1 are satisfied, the reduced density matrix $\rho_A := \mathrm{Tr}_B |\psi\rangle\langle\psi|$ for states $|\psi\rangle$ in the mean energy ensemble is typically close to*

$$\rho_c := \frac{E}{n+1} \begin{pmatrix} \sum_{i=1}^B \frac{1}{E_1^A + E_i^B} & 0 & 0 \\ \vdots & \ddots & \vdots \\ 0 & 0 & \sum_{i=1}^B \frac{1}{E_A^A + E_i^B} \end{pmatrix}. \qquad (7)$$

ρ_A is exponentially concentrated on the "canonical state" ρ_c.

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In general, this is not a Gibbs state. Work in progress: find special situations where this is close to Gibbs.

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What does it tell us about physics?

- Small system S, large bath B, Hamiltonian $H = H_S + H_B + H_I$ with small interaction $H_I \approx 0$: usual subspace restriction is a better description of physics (produces Gibbs!). Explanation: projective measurement; observer knows more than expectation value.

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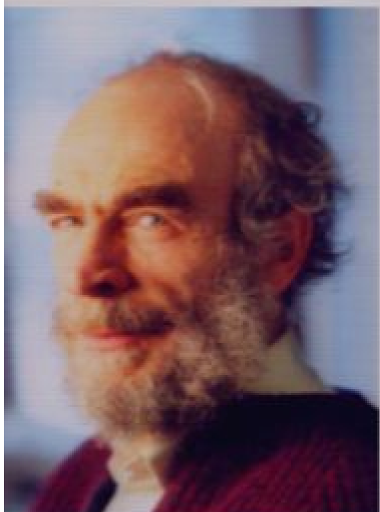
- Work in progress: find a physical system where knowledge of (only!) the mean energy is natural. Possible candidate: $H = H_S + H_B + H_I$, H_I large, and highly entangled eigenstates. Observer measures $H_S + H_B$ and waits some time.

2. The mean energy ensemble

Idea of proof: integral geometry



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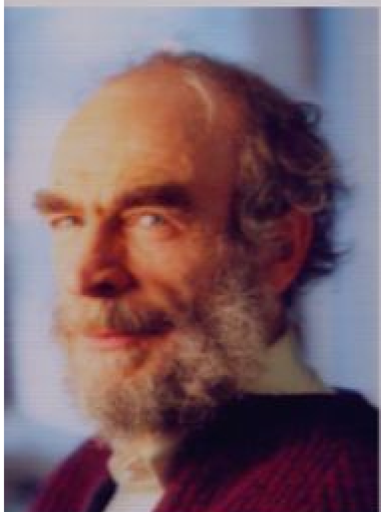

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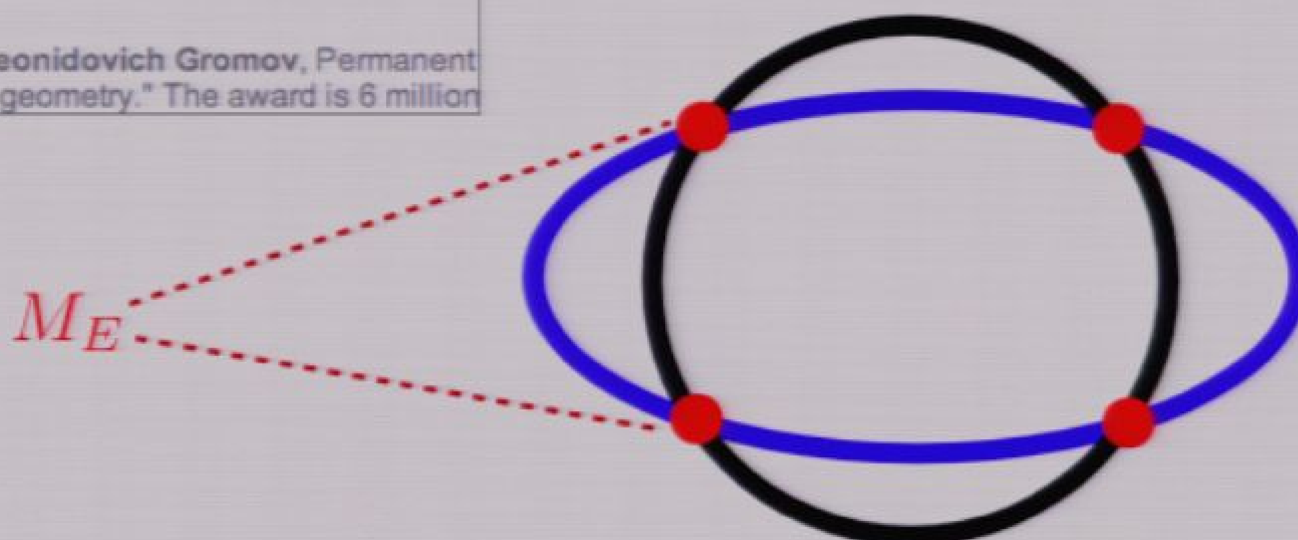

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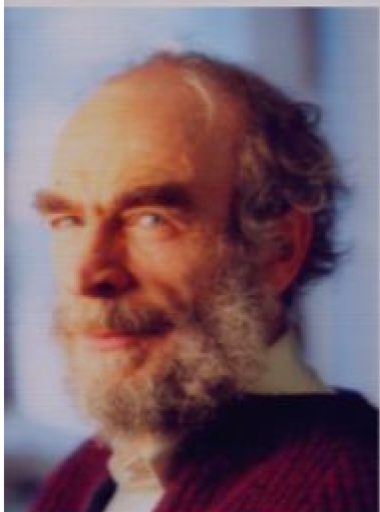

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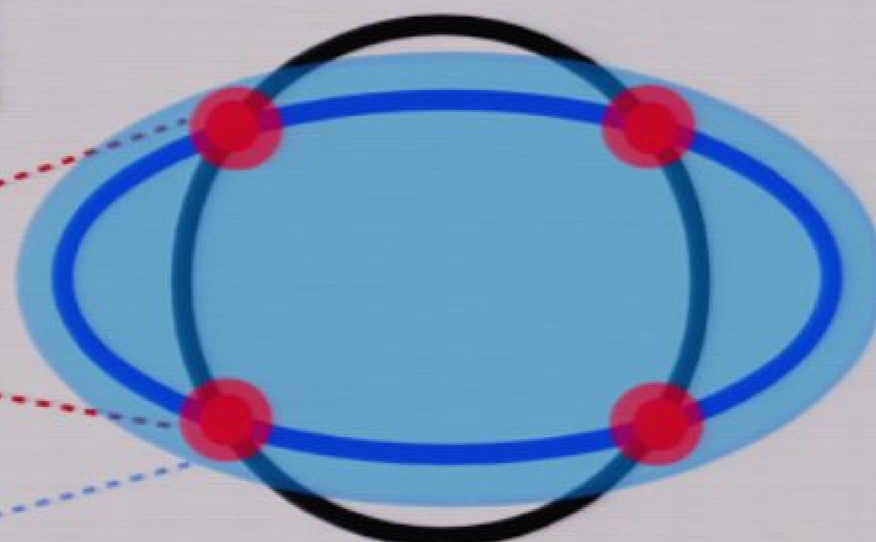

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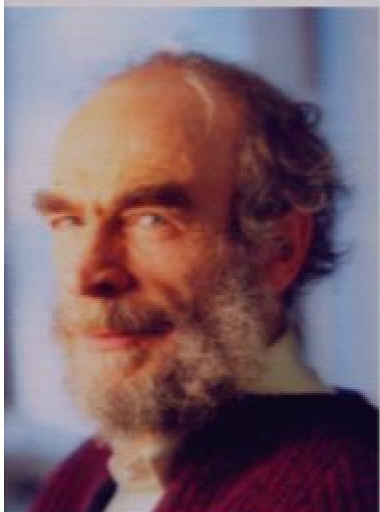

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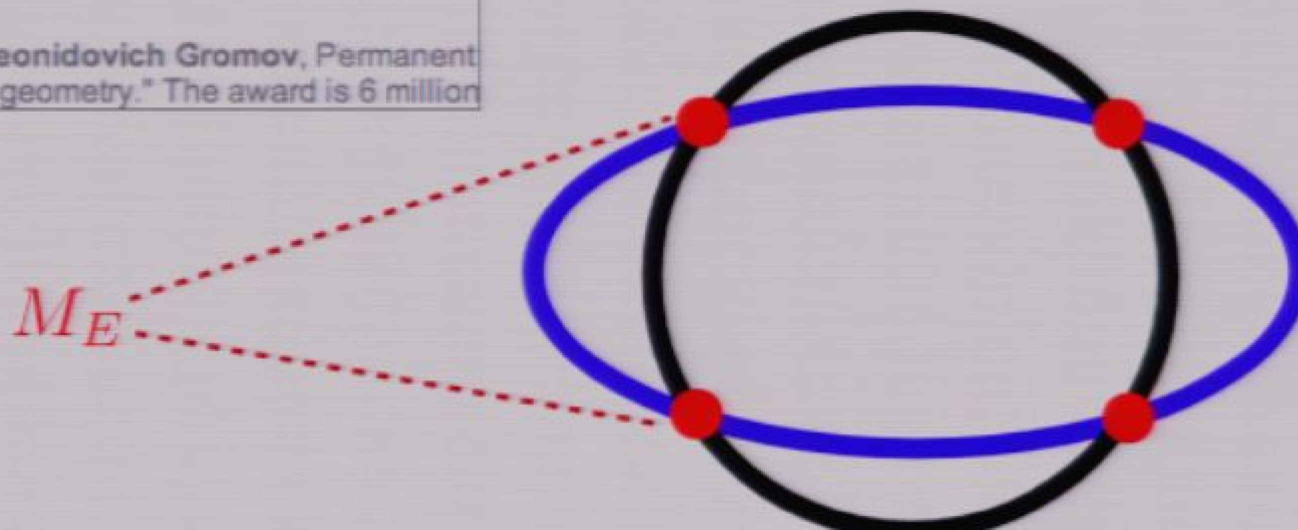

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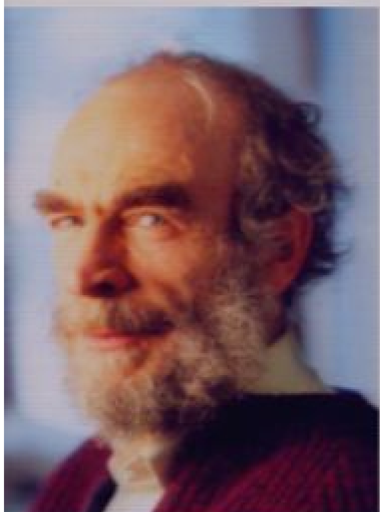

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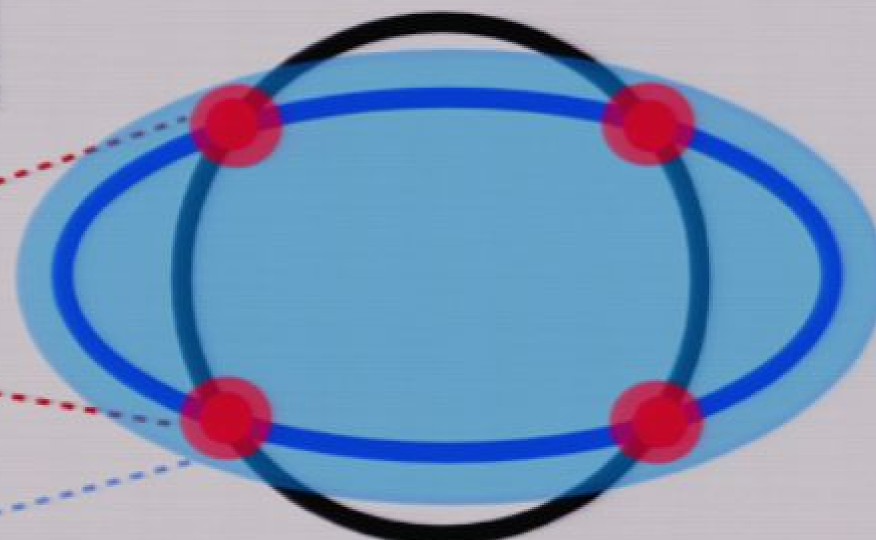

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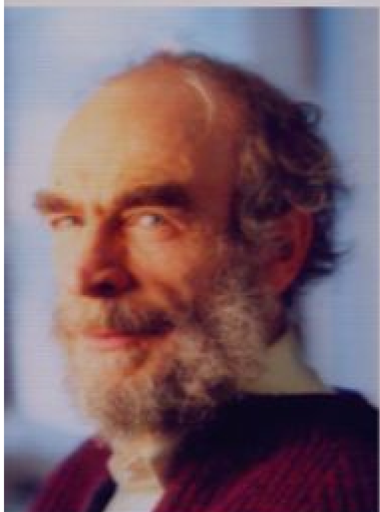

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Mean energy manifold inherits concentration of measure (and expectation values) from surrounding ellipsoid.

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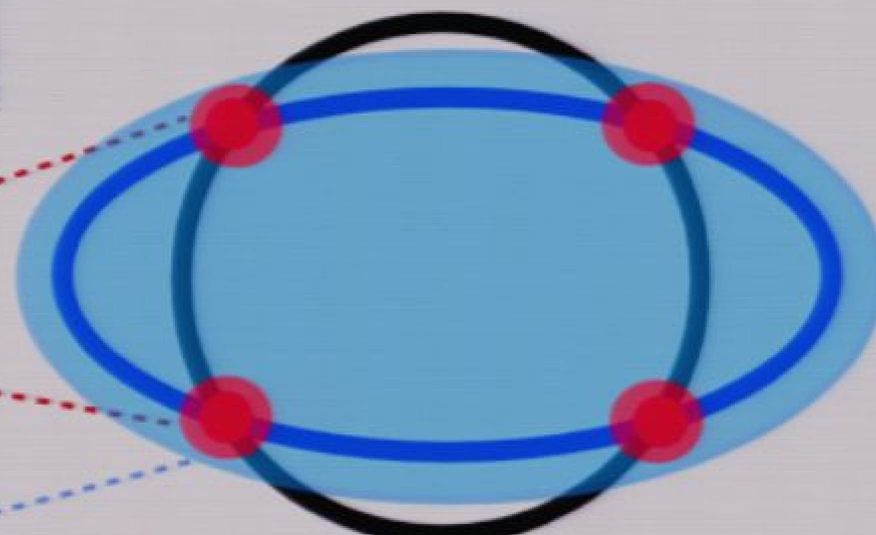

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covers a large part of *N* if

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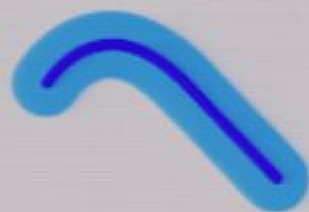
Standard result: measure concentration in ellipsoid N

2. The mean energy ensemble

How to estimate the neighborhood volume

Intuition:



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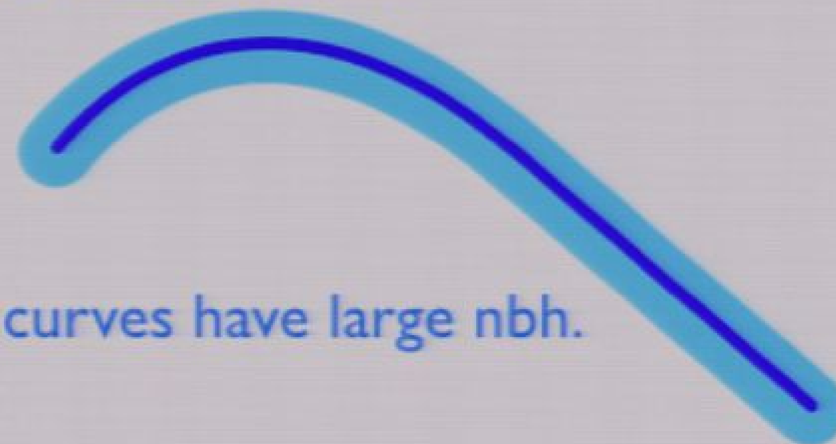

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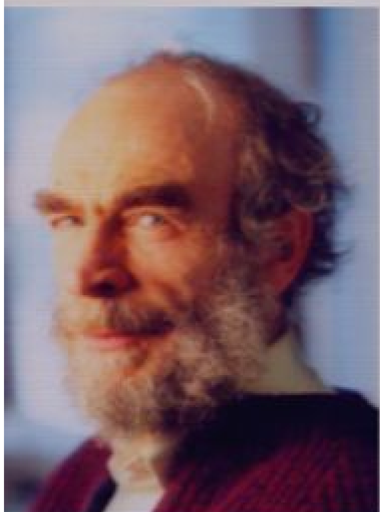

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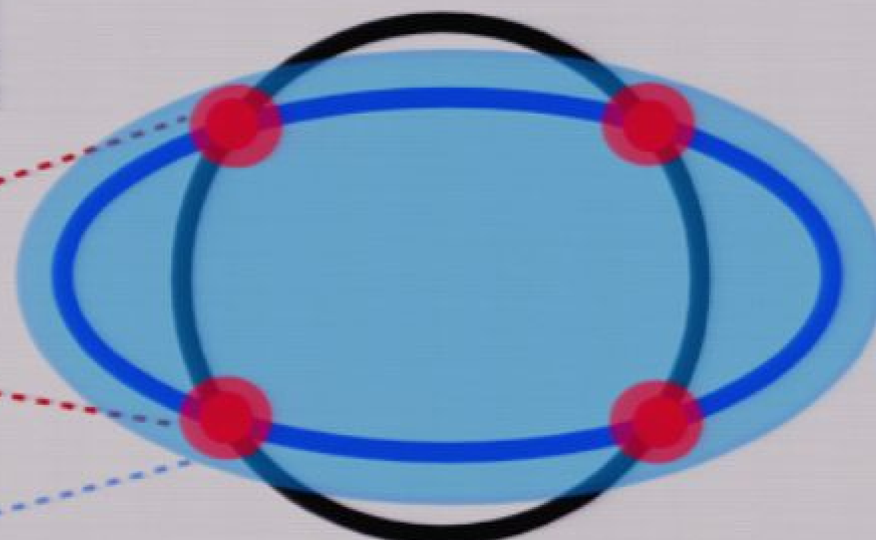

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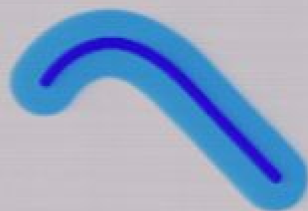

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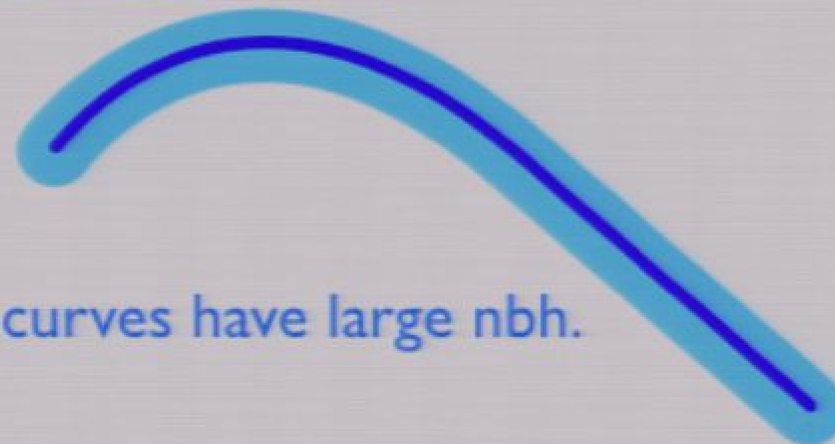

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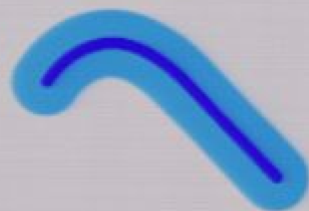

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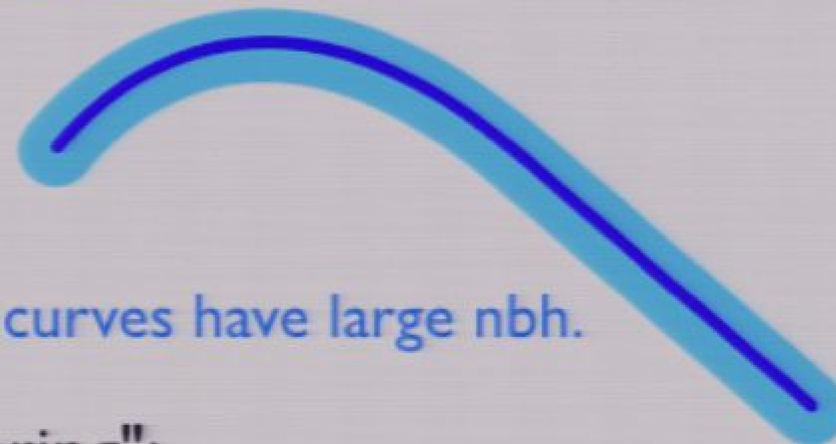

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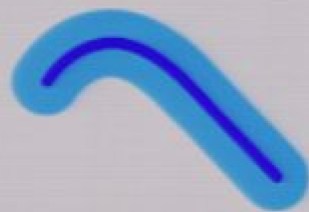

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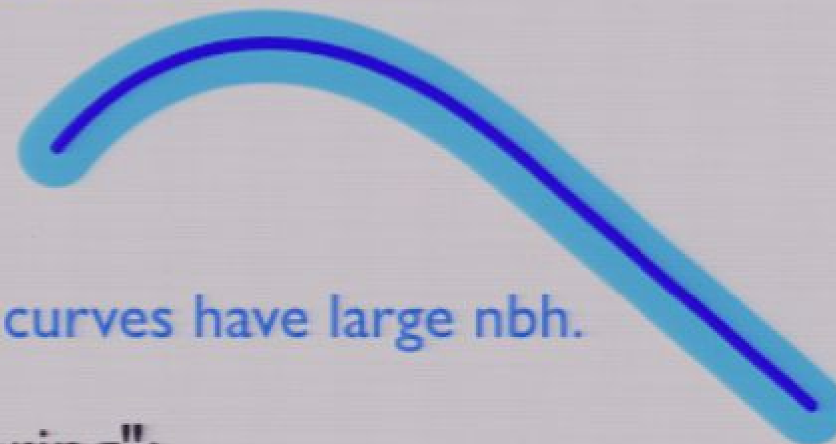

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C: curve, D: domain (e.g. $D = U_\varepsilon(C)$)

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and use that $L \cap C \neq 0 \Rightarrow \text{length}(L \cap U_\varepsilon(C)) \geq 2\varepsilon$

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Apply "concentration under constraints" in general

- Minimum output entropy of a quantum channel $\mathcal{M}$:
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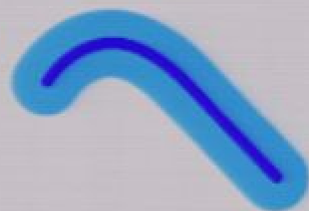

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• Tools from "integral geometry" turned out to be useful (probably also in more general settings).
• Possible applications to the additivity problem?
• Measure concentration can help to understand the foundations of statistical physics, ...
• ... but the mean energy ensemble does not reproduce Gibbs states in standard situations. Yet, it might describe physics correctly in more exotic situations.

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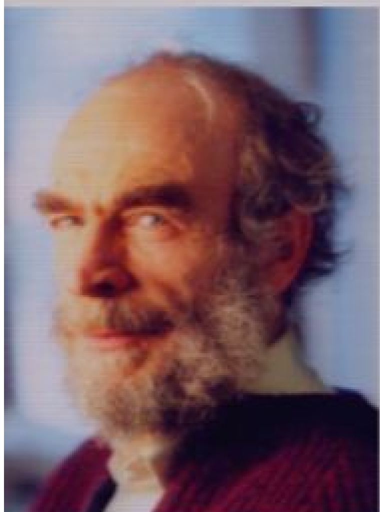

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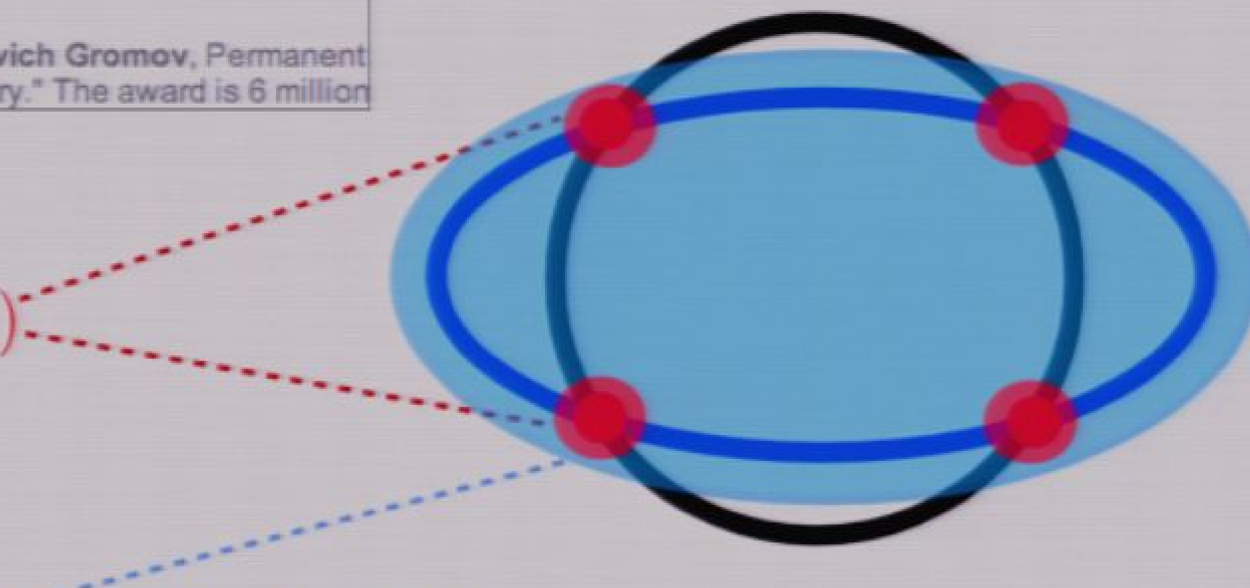

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$$f(\psi) := (\mathrm{Tr}_{env}|\psi\rangle\langle\psi|)_{i,j}$$

gives the typical reduced density matrix:

Theorem 2 (Typical reduced density matrix). *Let $H = H_A + H_B$ be a Hamiltonian on $\mathbb{C}^A \otimes \mathbb{C}^B$ of dimension $n = AB$ with $A, B \geq 2$ and positive energy eigenvalues $\{E_i^A\}_{i=1}^A$ and $\{E_j^B\}_{j=1}^B$ respectively. If the conditions of Theorem 1 are satisfied, the reduced density matrix $\rho_A := \mathrm{Tr}_B |\psi\rangle\langle\psi|$ for states $|\psi\rangle$ in the mean energy ensemble is typically close to*

$$\rho_c := \frac{E}{n+1} \begin{pmatrix} \sum_{i=1}^B \frac{1}{E_1^A + E_i^B} & 0 & 0 \\ \vdots & \ddots & \vdots \\ 0 & 0 & \sum_{i=1}^B \frac{1}{E_A^A + E_i^B} \end{pmatrix}. \quad (7)$$

ρ_A is exponentially concentrated on the "canonical state" ρ_c.