Title: Relativistic Non-Hermitian Quantum Mechanics

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Abstract: The basic structure of quantum mechanics was delineated in the early days of the theory and has not been modified since. Still it is interesting to ask whether that basic structure can be altered or generalized. In the last decade Bender et al have shown that one of the fundamental assumptions of quantum mechanics, that operators are represented by Hermitian matrices, can to an extent be relaxed. In this theory, the parity (P) and time-reversal (T) operators play a role analogous to the Hermitian conjugate. Recently we have extended the realm of `PT quantum mechanics' to include systems that are odd under time-reversal (T^2 = - 1), in the interest of constructing the PT-analogue of the Dirac equation. We find that the fundamental representation of the Dirac equation, which describes relativistic fermions, remains unchanged in the generalization to the non-Hermitian theory. Higher dimensional representations, which ordinarily decouple into pairs of Dirac fermions in Hermitian quantum mechanics, here describe new types of particles with extremely compelling properties. Most notably we have constructed a toy model representing two generations of massless neutrinos that nonetheless undergo flavor oscillations; furthermore this model is Lorentz invariant and unitary in time. The Standard Model requires that the neutrino be massive in order to accomodate the observed flavor oscillations, thus this toy model represents a significant departure from Standard Model physics.

Relativistic Non-Hermitian Quantum Mechanics

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Why Modify Quantum Mechanics?

Why Not?

- **Weinberg**: non-linear quantum mechanics , PRL **62**, 485 (1989)
- **Wilczek**: fractional statistics in 2+1, PRL **49**, 957 (1982)
- **Bender**: non-Hermitian quantum mechanics, PRL **80**, 5243 (1998)

Outline

I. Hermiticity in Quantum Mechanics
Intro to PT Quantum Mechanics

II. Relativistic Non-Hermitian Quantum Mechanics
Hermitian, non-Hermitian Dirac Equation

III. Non-Hermitian Quantum Field Theory
Looks Interesting

Role of Hermiticity in Ordinary QM

I. Eigenvalues.

$A = A^\dagger \Rightarrow$ eigenvalues $\in \mathbb{R}$

II. Inner Product

Hermitian operator defined as $\langle\phi|A\psi\rangle = \langle A\phi|\psi\rangle$

usually, this is $\langle\phi|\psi\rangle = \sum_i \phi_i^* \psi_i$

III. Unitary Time Evolution

$$\frac{\partial}{\partial t}\langle\psi|\psi\rangle = 0$$

$$\langle\psi(t)|\psi(t)\rangle = \langle\psi(0)|U^\dagger U|\psi(0)\rangle = 1$$

$U \rightarrow e^{-iHt}$ $\quad H = H^\dagger \Rightarrow U$ unitary

Intro to PT QM: T-symmetry

- Reverse direction of motion
- Anti-linear, two types:

/ \

Even: $T^2 = \mathbb{1}$ | Odd: $T^2 = -\mathbb{1}$

$\Rightarrow T\Psi = \Psi^*$ | $\Rightarrow T\Psi = Z\Psi^*$ where $Z = \begin{pmatrix} i\sigma_y & 0 \\ 0 & i\sigma_y \end{pmatrix}$

Bosons | Fermions

- Extension to T_{odd} : KJS &Harsh Mathur arXiv:0908.4255

Intro to PT QM: P-symmetry

- Reverse spatial, left/right components
  - Linear, only one type: $P^2=\mathbb{1}$

$$P\Psi = S\,\Psi$$

- Assume [P,T]=0

$$S = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$$

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Intro to PT QM: Inner Product

Adjoint:

If $\langle\phi|A\psi\rangle = \langle A\phi|\psi\rangle$ $\forall$ φ, ψ then A is 'self-adjoint'

New Inner Product: $(\phi, \psi)_{PT} = (PT\phi)^T\psi = \phi^\dagger SZ\psi$

where $Z = \begin{cases} \mathbb{1} & T_{\mathrm{even}} \\ i\sigma_y & T_{\mathrm{odd}} \end{cases}$

- Indefinite sign

Intro to PT QM: P-symmetry

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- Indefinite sign

Observable Operators

- PT inner product-- indefinite norm.

•Define "C" operator which changes negative states to positive:

$C\Psi_i = s_i \Psi_i = K\Psi_i$ (s_i = sign of Ψ_i)

CPT Inner Product:

$$(\phi, \psi)_{CPT} = (CPT\phi)^T \psi = \phi^\dagger K^T S Z \psi$$

$$\text{where } Z = \begin{cases} \mathbb{1} & T_{even} \\ i\sigma_y & T_{odd} \end{cases}$$

Observables are 'CPT self-adjoint' $A = A^\star$

Conditions for PT Hamiltonians

I. Invariant under PT

$$[H, PT]=0$$

II. 'Unbroken' PT

Can find simultaneous eigenvectors of H and PT

III. H is self-adjoint wrt PT inner product

$$H = Z\, H^T Z^T$$

$$\text{where } Z = \begin{cases} \mathbb{1} & T_{even} \\ i\sigma_y & T_{odd} \end{cases}$$

Kramers Degeneracy

Hermitian

- If T_{even}, $\exists$ states φ that are invariant under T: $\varphi = T\varphi$

- If T_{odd}, $\nexists$ states φ that are invariant under T: $\varphi \neq T\varphi$
  but we construct 'Kramers doublets' $(\varphi, T\varphi)$ that are the eigenstates of H if [H,T]=0.

- Thus , these states have a two-fold degeneracy . ('Kramers degeneracy')

Kramers Degeneracy

PT QM

- If T_{even}, $\exists$ states φ that are invariant under PT: $\varphi = PT\,\varphi$.

This is 'unbroken PT' symmetry and guarantees real eigenvalues.

- If T_{odd}, $\nexists$ states φ that are invariant under PT: $\varphi \neq PT\,\varphi$

so we need to generalize definition of unbroken PT in such a way to guarantee real eigenvalues.

- We construct 'PT doublets' (φ, $PT\,\varphi$) ; if these are eigenstates of H then we say PT is unbroken (and these states have a two-fold degeneracy)

Generic Hamiltonians*

*assuming [H,T]=0

Hermitian

PT

T_{even} (Hermitian)

$$H = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}$$

T_{even} (PT)

$$H = \begin{pmatrix} a & ib \\ ib & -a \end{pmatrix}$$

$a, b, \in \mathbb{R}$

T_{odd} (Hermitian)

$$H = \begin{pmatrix} a & b \\ b^\dagger & -a \end{pmatrix}$$

T_{odd} (PT)

$$H = \begin{pmatrix} a & ib \\ ib^\dagger & -a \end{pmatrix}$$

$$b = b_0 + i\mathbf{b} \cdot \sigma$$

$$a = a_0 \sigma_0$$

Summary I

Hermitian QM

- real eigenvalues
- unitary time evolution wrt standard inner product
- T_{even}, T_{odd}
- observables are $A=A^{\dagger}$
- Kramers degeneracy with T_{odd}
- same ol' Hamiltonians

PT QM

- real eigenvalues
- unitary time evolution wrt CPT inner product
- T_{even}, T_{odd}
- observables are $A=A^{\star}$
- Kramers degeneracy with T_{odd}
- new Hamiltonians!

Part II
Relativistic Non-Hermitian Quantum Mechanics

Relativistic Hermitian Quantum Mechanics

Dirac fermion: a pair of Weyl spinors ψ_R , ψ_L
coupled by a mass term m

Familiar Dirac equation:

$$i\sigma^\mu \partial_\mu \psi_L = m\psi_R$$
$$i\bar{\sigma}^\mu \partial_\mu \psi_R = m\psi_L$$

Follows from Lagrangian:

$$\mathcal{L} = \psi_L^\dagger i\sigma^\mu \partial_\mu \psi_L + \psi_R^\dagger i\bar{\sigma}^\mu \partial_\mu \psi_R + m\psi_L^\dagger \psi_R + m\psi_R^\dagger \psi_L$$

Relativistic Hermitian Quantum Mechanics

Multiple fermions: $$\mathcal{L} = \sum_i \psi^\dagger_{L_i} i\sigma^\mu \partial_\mu \psi_{L_i} + \psi^\dagger_{R_i} i\bar{\sigma}^\mu \partial_\mu \psi_{R_i} + \sum_{i,j} m_{ij}\psi^\dagger_{L_i}\psi_{R_j} + m^*_{ij}\psi^\dagger_{R_i}\psi_{L_j}$$

Add interactions, not all mass matrices can be simultaneously diagonalized.

Non-diagonal mass matrices ~ ‘oscillations’

SM Neutrino Oscillations

Pre 1999: Neutrinos massless:

$$\mathcal{L} = \sum_{i,j} e^\dagger_{L_i} i\sigma^\mu \partial_\mu e_{L_i} + e^\dagger_{R_i} i\bar{\sigma}^\mu \partial_\mu e_{R_i}$$

$$+ m_{ij} e^\dagger_{L_i} e_{R_j} + m^*_{ij} e^\dagger_{R_i} e_{L_j}$$

$$+ \nu^\dagger_{L_i} i\sigma^\mu \partial_\mu \nu_{L_i} \quad (+\text{interactions})$$

$|\psi_{electron}\rangle \rightarrow \begin{pmatrix} e_L \\ e_R \end{pmatrix}$

$|\psi_{neutrino}\rangle \rightarrow \nu_L$

Post 1999: Neutrinos oscillate:

$|\psi_{electron}\rangle \rightarrow \begin{pmatrix} e_L \\ e_R \end{pmatrix}$

$|\psi_{neutrino}\rangle \rightarrow \begin{pmatrix} \nu_L \\ \nu_R \end{pmatrix}$

$$\mathcal{L} = \sum_{i,j} e^\dagger_{L_i} i\sigma^\mu \partial_\mu e_{L_i} + e^\dagger_{R_i} i\bar{\sigma}^\mu \partial_\mu e_{R_i} + m_{ij} e^\dagger_{L_i} e_{R_j} + m^*_{ij} e^\dagger_{R_i} e_{L_j}$$

$$+ \nu^\dagger_{L_i} i\sigma^\mu \partial_\mu \nu_{L_i} + \nu^\dagger_{R_i} i\bar{\sigma}^\mu \partial_\mu \nu_{R_i} + \mu_{ij} \nu^\dagger_{L_i} \nu_{R_j} + \mu^*_{ij} \nu^\dagger_{R_j} \nu_{L_i}$$

(+interactions)

Hermitian Dirac Equation

What Hamiltonian has eigenvalues $E = \pm\sqrt{p^2 + m^2}$?

$$H_D = -i\alpha \cdot \nabla + \beta$$

assuming

$$\begin{aligned} \{\alpha_i, \alpha_j\} &= 2\delta_{ij} \\ \{\alpha, \beta\} &= 0 \end{aligned}$$

(Dirac algebra)

Also enforces
Lorentz algebra :

$$\begin{aligned} J_x &= \frac{-i}{2}\alpha_y\alpha_z \\ K_x &= \frac{i}{2}\alpha_x \end{aligned}$$

Hermitian Dirac Equation: Solutions

$$H_D = -i\alpha \cdot \nabla + \beta \qquad \begin{aligned} \{\alpha_i, \alpha_j\} &= 2\delta_{ij} \\ \{\alpha, \beta\} &= 0 \end{aligned}$$

2×2 : Weyl fermions

$$\begin{array}{llll} \alpha_i & \rightarrow & \sigma_i & \text{(left-handed)} \\ \alpha_i & \rightarrow & -\sigma_i & \text{(right-handed)} \end{array}, \quad \beta = 0 \quad ; \quad E = \pm p$$

Hermitian Dirac Equation: Solutions

4×4 : Fundamental representation (Dirac fermions)

$$\alpha_i \to \begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix} \quad , \; \beta \to \begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix} \quad ; \quad E = \pm\sqrt{p^2 + m^2}$$

m real number

a pair of Weyl spinors coupled by mass m

$$H_D = -i\alpha \cdot \nabla + \beta \qquad H\psi = i\frac{\partial \psi}{\partial t}$$

$$-i\begin{pmatrix} \sigma \cdot \nabla & 0 \\ 0 & -\sigma \cdot \nabla \end{pmatrix}\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} + \begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix}\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = i\frac{\partial}{\partial t}\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

$$\begin{pmatrix} -i(\sigma \cdot \nabla)\psi_L \\ -i(-\sigma \cdot \nabla)\psi_R \end{pmatrix} + \begin{pmatrix} m\psi_R \\ m\psi_L \end{pmatrix} = i\frac{\partial}{\partial t}\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

$$i(\sigma \cdot \nabla)\psi_L + i\frac{\partial \psi_L}{\partial t} = m\psi_L \to i\sigma^\mu \partial_\mu \psi_L = m\psi_R$$

$$i(-\sigma \cdot \nabla)\psi_R + i\frac{\partial \psi_R}{\partial t} = m\psi_R \to i\bar{\sigma}^\mu \partial_\mu \psi_R = m\psi_L$$

Hermitian Dirac Equation: Solutions

$$8 \times 8 \quad : \text{Dirac 'quartet'}$$

note: dagger

$$\alpha_i \rightarrow \begin{pmatrix} \sigma_i & 0 & 0 & 0 \\ 0 & \sigma_i & 0 & 0 \\ 0 & 0 & -\sigma_i & 0 \\ 0 & 0 & 0 & -\sigma_i \end{pmatrix}$$

$$\beta \rightarrow \begin{pmatrix} 0 & M \\ M^\dagger & 0 \end{pmatrix}$$

$$M = \begin{pmatrix} m_1\sigma_0 & m_2\sigma_0 \\ m_3\sigma_0 & m_4\sigma_0 \end{pmatrix}$$

$\sigma_0 = 1_{2x2}$

$m_i \in \mathbb{C}$

2 lefts and 2 rights coupled by more complicated mass terms--
what kind of beast is this?

Suitable decomposition shows 2 independent Dirac fermions:

$$E = \pm\sqrt{p^2 + \mu_1^2}\,;\ \pm\sqrt{p^2 + \mu_2^2}$$

μ_1, μ_2 singular values of M

Non-Hermitian Dirac Equation: Solutions

What happens if we allow α, β to be non-Hermitian?

$$H_D = -i\alpha \cdot \nabla + \beta \qquad \begin{aligned} \{\alpha_i, \alpha_j\} &= 2\delta_{ij} \\ \{\alpha, \beta\} &= 0 \end{aligned}$$

Plan: naively construct
PT Dirac theory.

Compare/contrast.

Non-Hermitian Dirac Equation: Solutions

'Model 4'

$$\alpha_i \to \begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix} \quad , \; \beta \to \begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix} \; ; \quad E = \pm\sqrt{p^2 + m^2}$$

a pair of Weyl spinors coupled by mass *m*

m real number

Model 4 is exactly the same as Dirac fermions. The PT Dirac equation is exactly the same as the Dirac equation. This is a nice result. What is the underlying symmetry of the Dirac equation, Hermiticity, or PT? One cannot tell yet (surprisingly).

Non-Hermitian Dirac Equation: Solutions

'Model 8'

note: star, not dagger

$$\alpha_i \rightarrow \begin{pmatrix} \sigma_i & 0 & 0 & 0 \\ 0 & \sigma_i & 0 & 0 \\ 0 & 0 & -\sigma_i & 0 \\ 0 & 0 & 0 & -\sigma_i \end{pmatrix}$$

$$\beta \rightarrow \begin{pmatrix} 0 & M \\ M^* & 0 \end{pmatrix}$$

$$M = \begin{pmatrix} (m_0 + m_3)\sigma_0 & (m_1 - im_2)\sigma_0 \\ (m_1 + im_2)\sigma_0 & (m_0 - m_3)\sigma_0 \end{pmatrix}$$

$\sigma_0 = 1_{2x2}$

still 2 lefts and 2 rights -- but mass matrix non-hermitian:

$$\beta = \begin{pmatrix} 0 & 0 & m_0 + m_3 & m_1 - im_2 \\ 0 & 0 & m_1 + im_2 & m_0 + m_3 \\ m_0 + m_3 & m_1 + im_2 & 0 & 0 \\ m_1 - im_2 & m_0 + m_3 & 0 & 0 \end{pmatrix}$$

Model 8

Take simplified case: $m_1 = m_3 = 0$

$$\beta = \begin{pmatrix} 0 & 0 & m_0 & -im_2 \\ 0 & 0 & im_2 & m_0 \\ m_0 & im_2 & 0 & 0 \\ -im_2 & m_0 & 0 & 0 \end{pmatrix}$$

$$E = \pm\sqrt{p^2 + m_{eff}^2} \qquad m_{eff} = \sqrt{m_0^2 - m_2^2}$$

For $m_0 = m_2$, $m_{eff} = 0$ but mass matrix non-zero-- Model 8 a toy model of two generations of a massless neutrino

Non-Hermitian Dirac Equation: Solutions

'Model 8'

note: star, not dagger

$$\alpha_i \rightarrow \begin{pmatrix} \sigma_i & 0 & 0 & 0 \\ 0 & \sigma_i & 0 & 0 \\ 0 & 0 & -\sigma_i & 0 \\ 0 & 0 & 0 & -\sigma_i \end{pmatrix}$$

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$$M = \begin{pmatrix} (m_0 + m_3)\sigma_0 & (m_1 - im_2)\sigma_0 \\ (m_1 + im_2)\sigma_0 & (m_0 - m_3)\sigma_0 \end{pmatrix}$$

$\sigma_0 = 1_{2x2}$

still 2 lefts and 2 rights -- but mass matrix non-hermitian:

$$\beta = \begin{pmatrix} 0 & 0 & m_0 + m_3 & m_1 - im_2 \\ 0 & 0 & m_1 + im_2 & m_0 + m_3 \\ m_0 + m_3 & m_1 + im_2 & 0 & 0 \\ m_1 - im_2 & m_0 + m_3 & 0 & 0 \end{pmatrix}$$

Model 8

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$$\beta = \begin{pmatrix} 0 & 0 & m_0 & -im_2 \\ 0 & 0 & im_2 & m_0 \\ m_0 & im_2 & 0 & 0 \\ -im_2 & m_0 & 0 & 0 \end{pmatrix}$$

$$E = \pm\sqrt{p^2 + m_{eff}^2} \qquad m_{eff} = \sqrt{m_0^2 - m_2^2}$$

For $m_0 = m_2$, $m_{eff} = 0$ but mass matrix non-zero--
Model 8 a toy model of two generations of a massless neutrino

Sheep in Wolf's Clothing?

How do we know Model 8 is not Dirac quartet in disguise?

- Construct mapping between theories:

$$\psi_{\rm Dirac}(\mathbf{r}) = \int d\mathbf{r}'\, L(\mathbf{r}-\mathbf{r}')\psi_8(\mathbf{r}').$$

- Eigenfunctions $\psi_{\rm Dirac}$ map onto eigenfunctions ψ_8 via non-local coupling L , where range of L set by non-Hermiticity parameter m_2

- Model 8 P and T violating

- Model 4 checks out (is local)

Summary II

Hermitian Dirac :

- The fermions we know and love-- a pair of Weyl spinors coupled by mass
- Direct sums of these decouple into independent particles
- Hermitian mass matrices:
  'massless neutrinos' = trivial mass matrix → no mass differences

PT Dirac :

- The fermions we know and love-- a pair of Weyl spinors coupled by mass
- Direct sum maintains physics of single particle
- Non-Hermitian mass matrices:
  'massless neutrinos' = effective mass of zero from a non-zero mass matrix

Part III
Non-Hermitian Quantum Field Theory

Hermitian Quantum Field Theory

Basic Entities

- Hamiltonian H_D, and its eigenfunctions u_i and v_i

$$H_D = -i\alpha \cdot \nabla + \beta$$

- Fermionic creation/annihilation operators

$$\{c_{\mathbf{p}s}, c^{\dagger}_{\mathbf{k}s'}\} = \delta(\mathbf{p}-\mathbf{k})\delta_{ss'}$$
$$\{d_{\mathbf{p}s}, d^{\dagger}_{\mathbf{k}s'}\} = \delta(\mathbf{p}-\mathbf{k})\delta_{ss'}$$

- Field operators

$$\psi(\mathbf{r}) = \int \frac{dp}{2\pi^3}\left(u_{ps}e^{i\mathbf{p}\cdot\mathbf{r}}c_{ps} + v_{ps}e^{-i\mathbf{p}\cdot\mathbf{r}}d^{\dagger}_{ps}\right)$$
$$\psi^{\dagger}(\mathbf{r}) = \int \frac{dp}{2\pi^3}\left(u^{\dagger}_{ps}e^{-i\mathbf{p}\cdot\mathbf{r}}c^{\dagger}_{ps} + v^{\dagger}_{ps}e^{i\mathbf{p}\cdot\mathbf{r}}d_{ps}\right)$$

$$\{\psi(\mathbf{r}), \psi^{\dagger}(\mathbf{r}')\} = \delta(\mathbf{r}-\mathbf{r}')$$

Non-Hermitian Quantum Field Theory
Basic Entities

- Hamiltonian H_D, its eigenfunctions u and v; its conjugate $H_D^\dagger$, and its eigenfunctions $\tilde{u}, \tilde{v}$!

- Fermionic creation/annihilation operators

$$\begin{aligned} \{c_{\mathbf{p}s}, c^\star_{\mathbf{k}s'}\} &= \delta(\mathbf{p}-\mathbf{k})\delta_{ss'} \\ \{d_{\mathbf{p}s}, d^\star_{\mathbf{k}s'}\} &= \delta(\mathbf{p}-\mathbf{k})\delta_{ss'} \end{aligned}$$

- Field operators

$$\begin{aligned} \psi(\mathbf{r}) &= \int \frac{dp}{2\pi^3}\left(u_{ps}e^{i\mathbf{p}\cdot\mathbf{r}}c_{ps} + v_{ps}e^{-i\mathbf{p}\cdot\mathbf{r}}d^\star_{ps}\right) \\ \psi^\star(\mathbf{r}) &= \int \frac{dp}{2\pi^3}\left(\tilde{u}^\dagger_{ps}e^{-i\mathbf{p}\cdot\mathbf{r}}c^\dagger_{ps} + \tilde{v}^\dagger_{ps}e^{i\mathbf{p}\cdot\mathbf{r}}d_{ps}\right) \end{aligned}$$

$$\{\psi(\mathbf{r}), \psi^\star(\mathbf{r}')\} = \delta(\mathbf{r}-\mathbf{r}')$$

Is Nature Hermitian?