Title: Symmetric informationally complete measurements: Can we make big ones out of small ones?

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Abstract: For a quantum system with a d-dimensional Hilbert space, a symmetric informationally complete measurement (SIC) can be thought of as a set of d^2 pure states all having the same overlap. Constructions of SICs for composite systems usually do not make use of the composite structure but treat the system as a whole. Indeed for some cases, one can prove that a SIC cannot have the symmetry that one naturally associates with the composite structure.

In this talk I give one example showing how a SIC for three qubits can be constructed from SICs for the individual qubits. I ask whether the strategy used in this example might apply to other composite cases.

Symmetric informationally complete measurements: Can we make big ones out of small ones?

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Outline

I. SICs

II. Measurement for three qubits

III. Other composite systems?

Symmetric Informationally Complete Measurements (SICs)

It is possible to express a quantum state via the probabilities it specifies for a sufficient number of measurement outcomes. (Minimally, d^2 outcomes for a d-dimension system.)

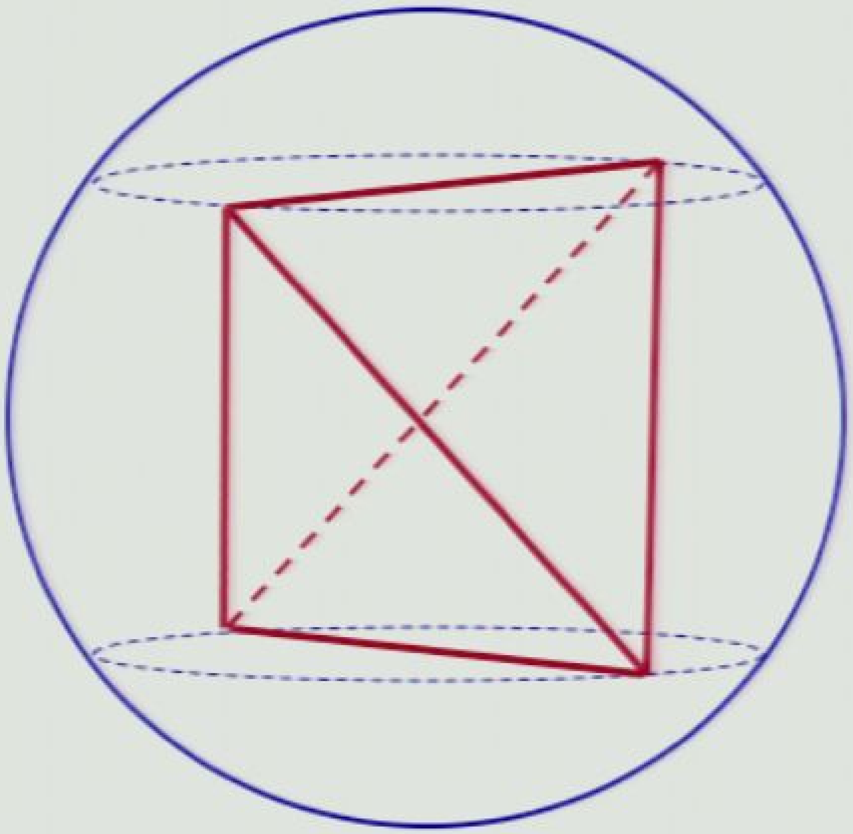
The expression is particularly simple if the probabilities refer to the outcomes of a symmetric measurement or set of measurements.

A complete set of mutually unbiased bases has a lot of symmetry, but such a set probably doesn't exist in every dimension.

A SIC is a single measurement with d^2 outcomes, and it is quite possible that a SIC exists in every dimension.

Symmetric Informationally Complete Measurements (SICs)

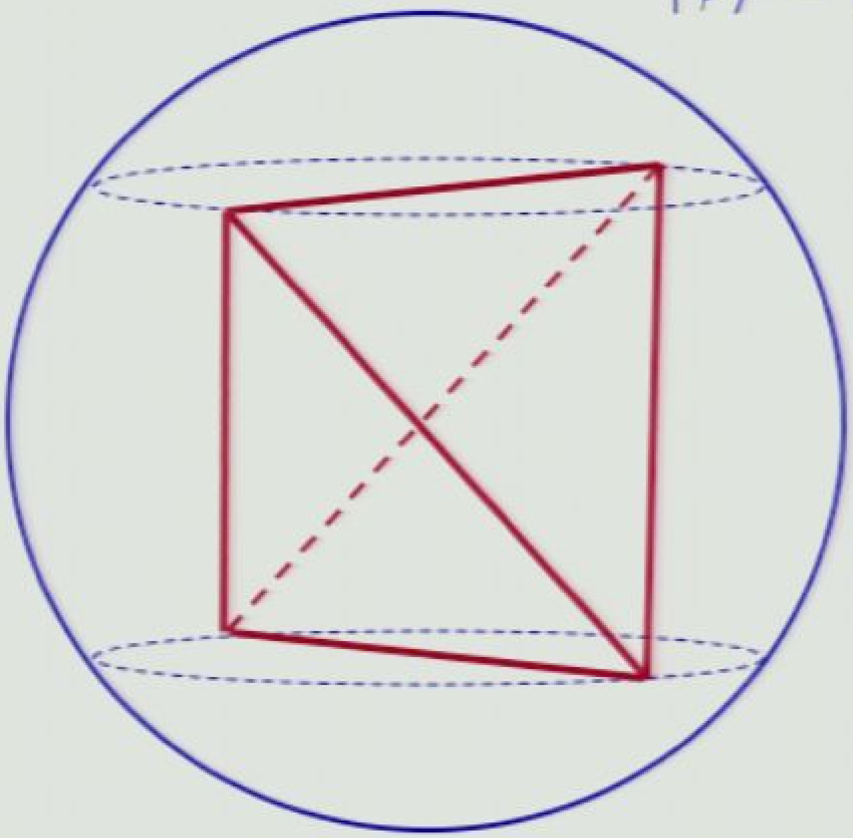
Single qubit



Symmetric Informationally Complete Measurements (SICs)

Single qubit $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \; Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$|\psi\rangle = (+1)\text{-eigenstate of } (X + Y + Z)/\sqrt{3}$



The four operators

$\frac{1}{2}\Pi_0, \; \frac{1}{2}\Pi_1, \; \frac{1}{2}\Pi_2, \; \frac{1}{2}\Pi_3$

where the Π's project onto

$|\psi\rangle, \; X|\psi\rangle, \; Y|\psi\rangle, \; Z|\psi\rangle$

constitute a POVM.

Symmetric Informationally Complete Measurements (SICs)

A qudit (*d*-dimensional object)

A SIC is a set of operators

$$\left\{\frac{1}{d}\Pi_0,\ \ldots,\ \frac{1}{d}\Pi_{d^2-1}\right\}$$

where the Π's are rank-1 projection operators satisfying

$$\mathrm{Tr}(\Pi_j\Pi_k) = \frac{1}{d+1}$$

We can think of a set of pure states separated by equal angles.

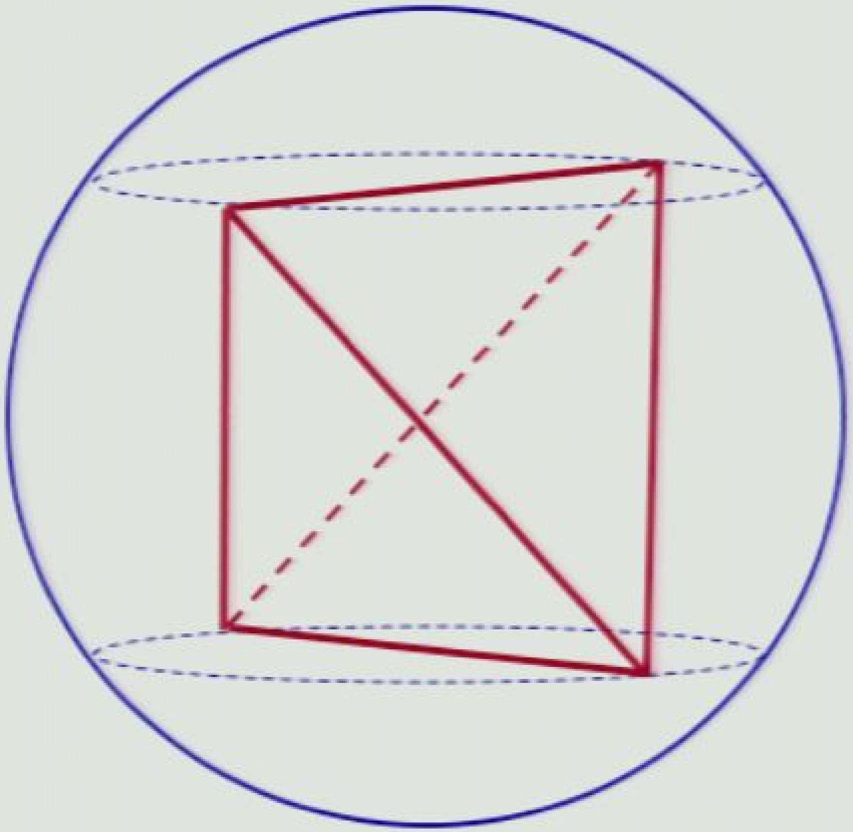
Symmetric Informationally Complete Measurements (SICs)

A state with density matrix ρ can be described by $(p_0, \ldots, p_{d^2-1})$, where $p_j = (1/d)\mathrm{Tr}(\rho\Pi_j)$.

The density matrix can be recovered from the p_j's by the equation

$$\rho = \sum_j \left[(d+1)p_j - \frac{1}{d}\right]\Pi_j$$

Symmetric Informationally Complete Measurements (SICs)



Example:

$|\psi\rangle$ itself is expressed as

$(p_0, p_1, p_2, p_3) = (1/2, 1/6, 1/6, 1/6)$

$$|\psi\rangle = (+1)\text{-eigenstate of } (X + Y + Z)/\sqrt{3}$$

The Weyl-Heisenberg Symmetry

Again, in the qubit case, a SIC can be obtained from

$$|\psi_j\rangle = D_j|\psi\rangle, \;\; j = 0, 1, 2, 3, \;\; \text{where} \;\; D_j \in \{I, X, Y, Z\}$$

In the qudit case, it has been fruitful to look for SICs based on

$$|\psi_j\rangle = D_j|\psi\rangle, \;\; j = 0, 1, \ldots, d^2 - 1,$$

where $D_j \in \{X^a Z^b \,|\, a, b = 0, 1, \ldots, d-1\}$

Here $X|m\rangle = |m + 1 \pmod{d}\rangle$ $\qquad Z|m\rangle = e^{2\pi i m/d}\,|m\rangle$

The starting state $|\psi\rangle$ is called a fiducial vector.

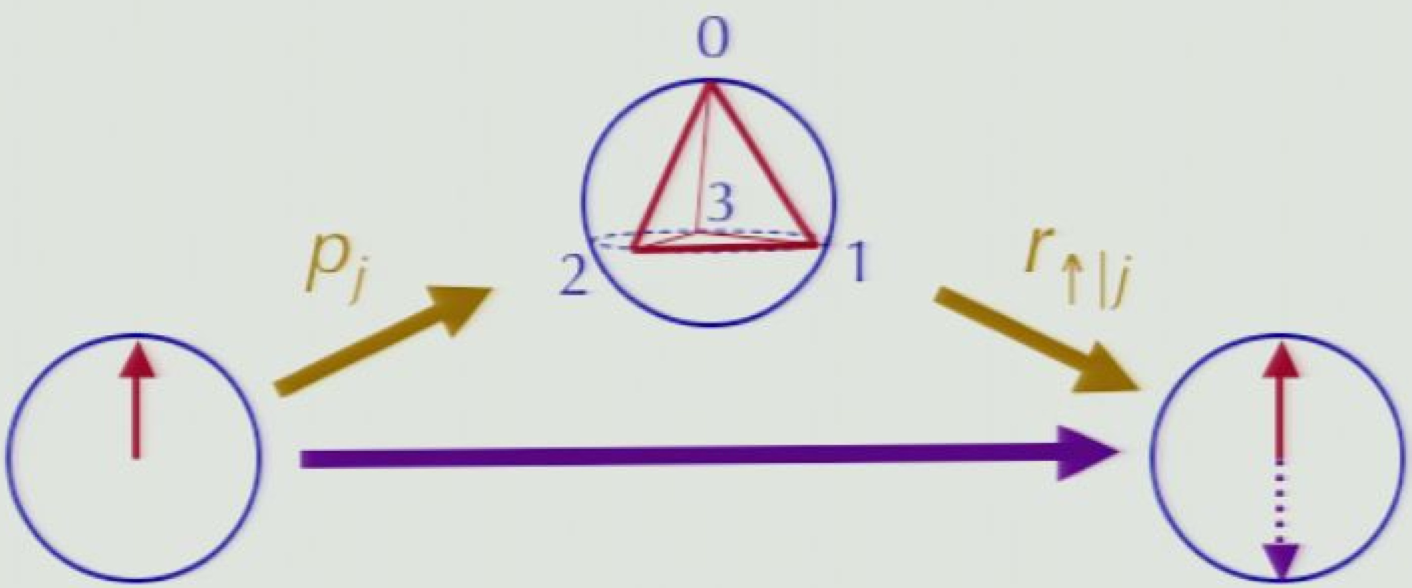
Examples of fiducial vectors (Renes, Blume-Kohout, Scott, Caves)

d=3: $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}$

d=4: $\begin{pmatrix}(1-1/\sqrt{5})\Big/\left(2\sqrt{2-\sqrt{2}}\right)\\ \frac{i}{2}\sqrt{1+1/\sqrt{5}+\sqrt{1/5+1/\sqrt{5}}}\\ i(1-1/\sqrt{5})\left(\sqrt{\sqrt{2}-1}\right)/(2\sqrt{2})\\ \frac{i}{2}\sqrt{1+1/\sqrt{5}-\sqrt{1/5+1/\sqrt{5}}}\end{pmatrix}$

Such vectors have been found analytically for d = 1-15, 19, 24, and numerically for $d \leq 67$.

The Born rule in terms of a SIC (Fuchs and Schack, 2009)



upper path

$$P_\uparrow = p_0\, r_{\uparrow|0} + p_1\, r_{\uparrow|1} + p_2\, r_{\uparrow|2} + p_3\, r_{\uparrow|3}$$
$$= (1/2)(1) + (1/6)(1/3) + (1/6)(1/3) + (1/6)(1/3) = 2/3$$

lower path

$$P_\uparrow = \left[3\left(\tfrac{1}{2}\right) - \left(\tfrac{1}{2}\right)\right](1) + \left[3\left(\tfrac{1}{6}\right) - \left(\tfrac{1}{2}\right)\right]\left(\tfrac{1}{3}\right) + \left[3\left(\tfrac{1}{6}\right) - \left(\tfrac{1}{2}\right)\right]\left(\tfrac{1}{3}\right) + \left[3\left(\tfrac{1}{6}\right) - \left(\tfrac{1}{2}\right)\right]\left(\tfrac{1}{3}\right)$$
$$= 1(1) + 0(1/3) + 0(1/3) + 0(1/3) = 1$$

The Born rule in terms of a SIC (Fuchs and Schack, 2009)

In general, the standard formula

$$P_\alpha = \sum_j p_j \, r_{\alpha|j}$$

is replaced, when the SIC is counterfactual, with

$$P_\alpha = \sum_j \left[(d+1)p_j - \frac{1}{d} \right] r_{\alpha|j}$$

Probabilities larger than average (i.e., larger than $1/d^2$) are replaced by larger numbers.

Probabilities smaller than average are replaced by smaller numbers.

Composite Systems---Example: Three Qubits

For three qubits, we could define a POVM using

$$|\psi_{jk\ell}\rangle = |\psi_j\rangle \otimes |\psi_k\rangle \otimes |\psi_\ell\rangle \qquad |\psi_j\rangle = D_j|\psi\rangle$$

but it would not be a SIC.

Can we find a SIC having the following symmetry?

$$|\psi_{jk\ell}\rangle = D_j \otimes D_k \otimes D_\ell \, |\psi\rangle, \quad j,k,\ell = 0,1,2,3.$$

If so, the p_{jkl} will presumably reflect the three-qubit structure.

The good news: For three qubits, yes!

And it even uses the properties of single-qubit SICs.

First, a few more facts about single-qubit SICs:

$$|\phi_+\rangle = (+1)\text{-eigenstate of } (X+Y+Z)/\sqrt{3}$$

$$|\phi_-\rangle = (-1)\text{-eigenstate of } (X+Y+Z)/\sqrt{3}$$

Those two vectors define two distinct SICs (dual tetrahedra).
Inner products:

$$\langle\phi_+|D_j|\phi_+\rangle = 1/\sqrt{3}, \qquad j = 1, 2, 3.$$

$$\langle\phi_-|D_j|\phi_-\rangle = -1/\sqrt{3}, \qquad j = 1, 2, 3.$$

$$\langle\phi_+|D_j|\phi_-\rangle = -\sqrt{2/3}\, e^{2\pi i j/3}, \qquad j = 1, 2, 3.$$

A simple fiducial vector for three qubits

$$|\psi\rangle = a\,|\phi_+,\,\phi_+,\,\phi_+\rangle + b\,|\phi_-,\,\phi_-,\,\phi_-\rangle$$

$$a^2 + b^2 = 1 \qquad a^2 - b^2 = 1/\sqrt{3}$$

Claim:

$$\big|\langle\psi|D_j \otimes D_k \otimes D_\ell|\psi\rangle\big| = \frac{1}{3} \quad \text{if } (j,k,l) \neq (0,0,0).$$

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If all three D's are not the identity,

$$\langle\psi|D_j \otimes D_k \otimes D_\ell|\psi\rangle = \overbrace{(a^2 - b^2)\langle\phi_+|D_j|\phi_+\rangle^3}^{1/9}$$

$$\underbrace{+2ab\,\mathrm{Re}\,\langle\phi_+|D_j|\phi_-\rangle\langle\phi_+|D_k|\phi_-\rangle\langle\phi_+|D_\ell|\phi_-\rangle}_{2/9 \text{ or } -4/9}$$

The OK news: A very similar SIC had been found before.

(Hoggar, 1998)

Our fiducial vector in the standard basis:

$$|\psi\rangle = \frac{1}{\sqrt{6}}\left[\sqrt{2}\,|000\rangle + q\,(|100\rangle + |010\rangle + |001\rangle) + q^3|111\rangle\right]$$

$$q = \frac{1+i}{\sqrt{2}}$$

Hoggar's fiducial vector (thanks to Åsa Ericsson):

$$|\eta\rangle = \frac{1}{\sqrt{6}}\left[\sqrt{2}\,|011\rangle + q\,(|100\rangle + |110\rangle + |101\rangle) + q^3|111\rangle\right]$$

How they are related (not *locally* equivalent):

$$|\psi\rangle = \mathrm{CNOT}_{21}\,\mathrm{CNOT}_{31}\,\mathrm{CNOT}_{12}\,\mathrm{CNOT}_{13}\,\mathrm{X}_2\,\mathrm{X}_3\,|\eta\rangle$$

The bad news: Three qubits is a special case

Theorem (Godsil and Roy, 2008):

For n qubits, if there is a SIC with states of the form

$$|\psi_{j_1,\ldots,j_n}\rangle = D_{j_1} \otimes \cdots \otimes D_{j_n} |\psi\rangle$$

then n must be either 1 or 3.

This leaves open the possibility of other "composite" SICS, e.g., in 3x3, or 2x2x3, or 2x2x4, or 2x19.

But the above construction for d=8 is special in several ways.

- There are two orthogonal fiducial vectors for d=2. (Also true for d=3.)

- $\frac{\langle\phi_+|D_j|\phi_+\rangle}{\langle\phi_-|D_j|\phi_-\rangle}$ is independent of j Not true for d=3.

- $(2+1)^2 = 2^3 + 1$

  $(d_1+1)(d_2+1) \neq d_1 d_2 d_3 + 1$ unless each $d_j = 2$.

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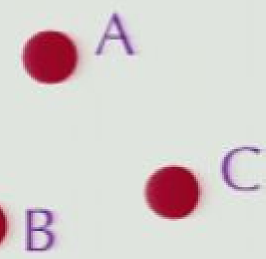
- $(2+1)^2 = 2^3 + 1$

  $(d_1 + 1)(d_2 + 1) \neq d_1 d_2 d_3 + 1$ unless each $d_j = 2$.

Suppose one *does* have three qubits.



State description: $p_{jk\ell} = \frac{1}{8}\langle \psi_{jk\ell} | \rho | \psi_{jk\ell} \rangle$

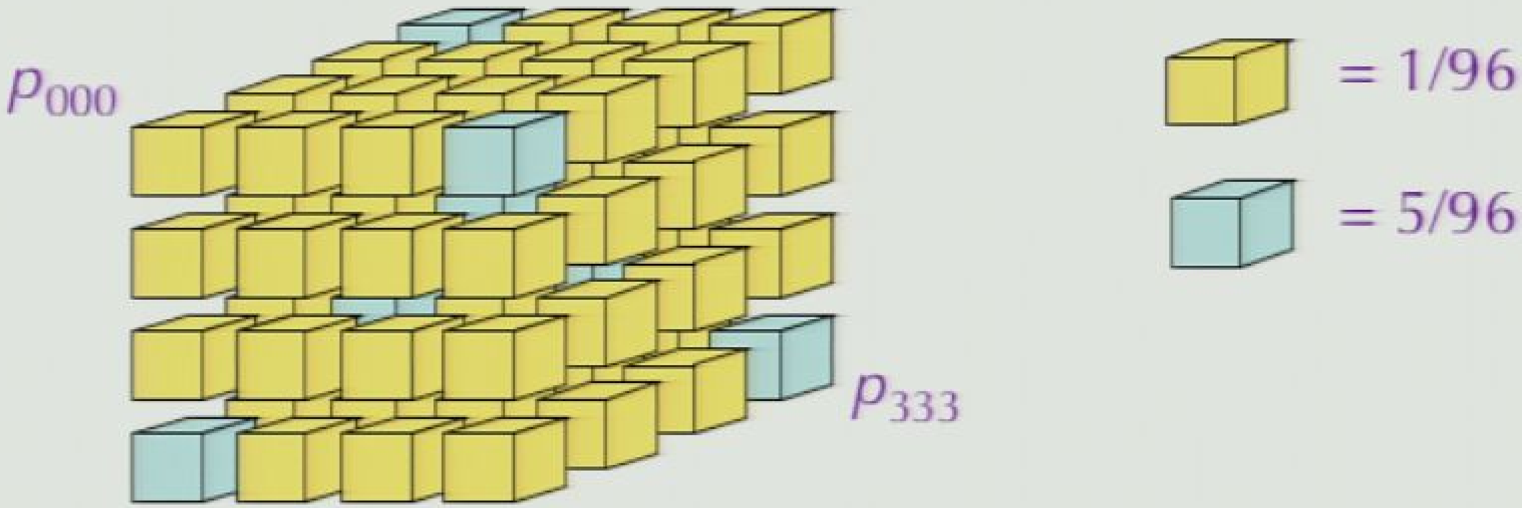
One can easily get certain properties of the components, e.g.,

$$\mathrm{Tr}\,\rho^2_{ABC} = 72 \sum_{jk\ell} p^2_{jk\ell} - 1$$

$$\mathrm{Tr}\,\rho^2_{AB} = 36 \sum_{jk} p^2_{jk} - 2 \qquad p_{jk} = \sum_{\ell} p_{jk\ell}$$

$$\mathrm{Tr}\,\rho^2_{A} = 18 \sum_{j} p^2_{j} - 4 \qquad p_{j} = \sum_{k} p_{jk}$$

Example: The GHZ state $(|000\rangle+|111\rangle)/\sqrt{2}$



$$p_{jk} = (1/24) \times \begin{bmatrix} 2 & 1 & 1 & 2 \\ 1 & 2 & 2 & 1 \\ 1 & 2 & 2 & 1 \\ 2 & 1 & 1 & 2 \end{bmatrix}$$

$$p_j = (1/4) \times \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$$

But it seems that such "composite" SICs may be rare.

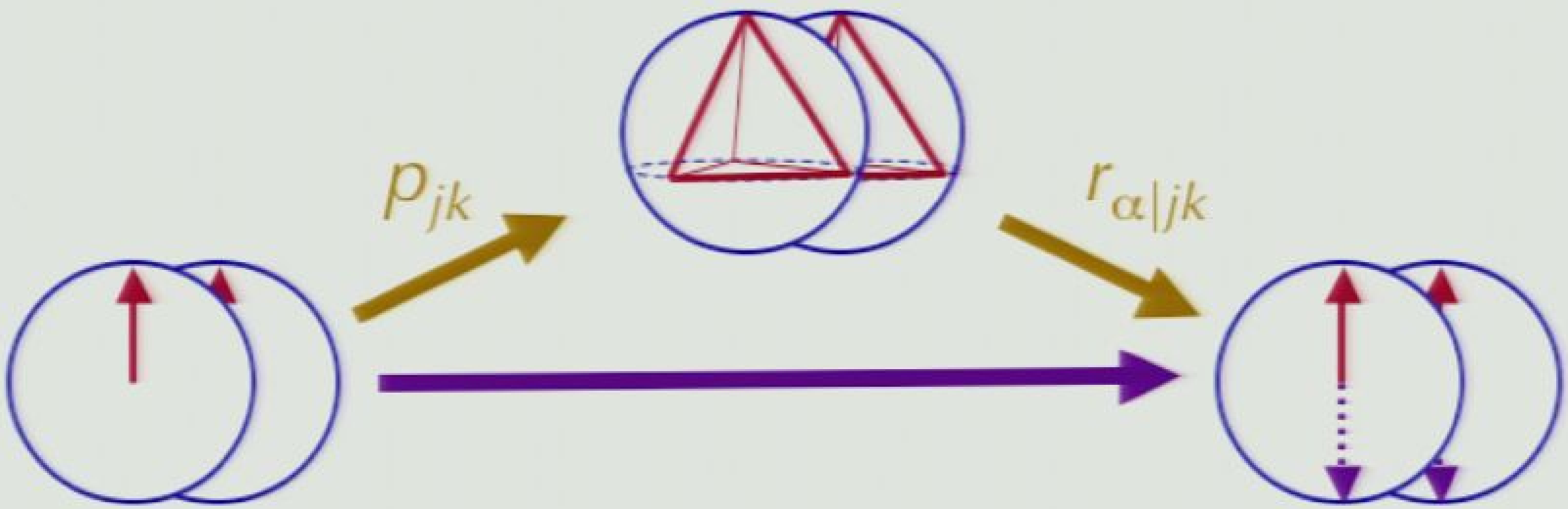
So consider again a POVM made from a product of two SICs:

$$|\psi_{jk}\rangle = |\phi_j^A\rangle \otimes |\phi_k^B\rangle, \; j = 0, \ldots, d_A^2 - 1; \; k = 0, \ldots, d_B^2 - 1.$$

The description of a state:

$$p_{jk} = \frac{1}{d_A d_B} \langle\psi_{jk}|\rho|\psi_{jk}\rangle$$

The Born rule in terms of this POVM



$$P_\alpha = \sum_{jk}\left[(d_A+1)(d_B+1)p_{jk} - \left(\frac{d_A+1}{d_B}\right)p_j^A - \left(\frac{d_B+1}{d_A}\right)p_j^B + \frac{1}{d_A d_B}\right] r_{\alpha|jk}$$

How to get this rule by replacing probabilities

Start with $P_\alpha = \sum_{jk} p_{jk}\, r_{\alpha|jk} = \sum_{jk} \left[\left(p_{jk} - p_j^A p_k^B \right) + p_j^A p_k^B \right] r_{\alpha|jk}$

$$= \sum_{jk} \left[C_{jk} + p_j^A p_k^B \right] r_{\alpha|jk}$$

Make these substitutions: $p_j^A \rightarrow (d_A + 1) p_j^A - \frac{1}{d_A}$

$$p_j^B \rightarrow (d_B + 1) p_j^B - \frac{1}{d_B}$$

$$C_{jk} \rightarrow (d_A + 1)(d_B + 1) C_{jk}$$

Get $P_\alpha = \sum_{jk} \left[(d_A + 1)(d_B + 1) p_{jk} - \left(\frac{d_A + 1}{d_B} \right) p_j^A - \left(\frac{d_B + 1}{d_A} \right) p_j^B + \frac{1}{d_A d_B} \right] r_{\alpha|jk}$

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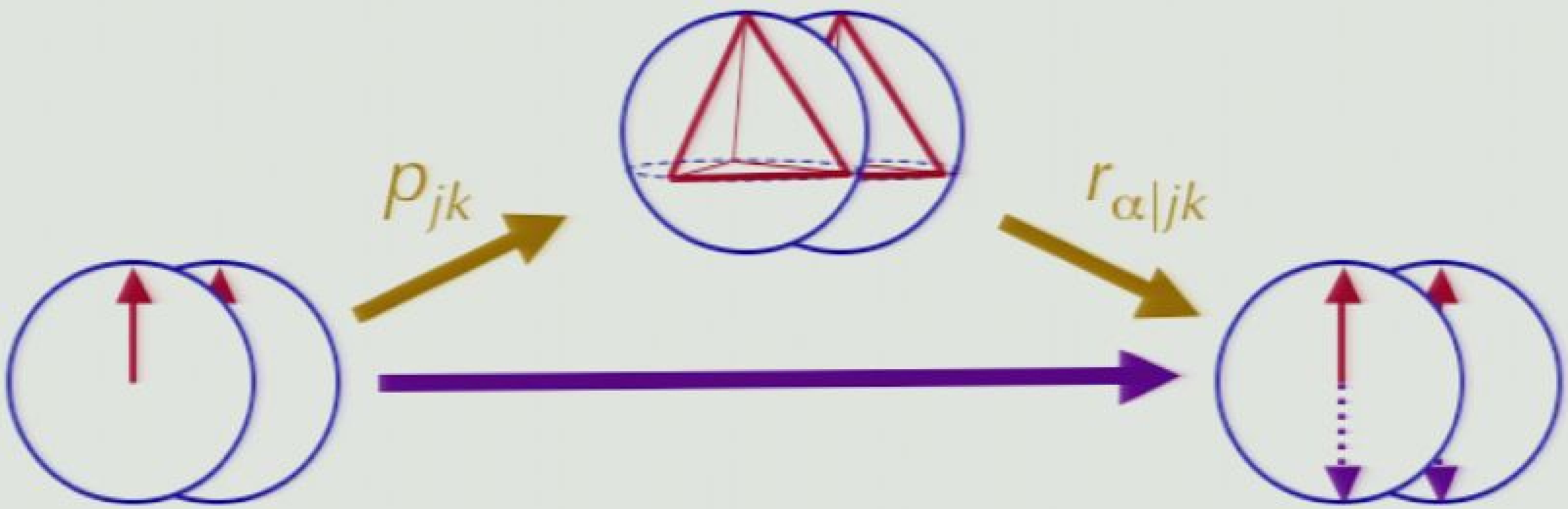

$$P_\alpha = \sum_{jk} \left[(d_A+1)(d_B+1)p_{jk} - \left(\frac{d_A+1}{d_B}\right) p_j^A - \left(\frac{d_B+1}{d_A}\right) p_j^B + \frac{1}{d_A d_B} \right] r_{\alpha|jk}$$

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For each particle, large probabilities are replaced by larger numbers, and small probabilities are replaced by smaller numbers.

Correlations get larger.

Make these substitutions:

$$p_j^A \rightarrow (d_A + 1)p_j^A - \frac{1}{d_A}$$

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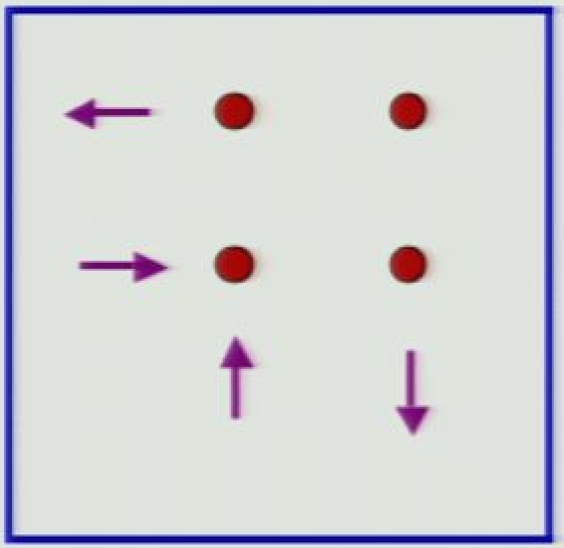
Get $$P_\alpha = \sum_{jk} \left[(d_A + 1)(d_B + 1)p_{jk} - \left(\frac{d_A + 1}{d_B} \right) p_j^A - \left(\frac{d_B + 1}{d_A} \right) p_j^B + \frac{1}{d_A d_B} \right] r_{\alpha|jk}$$

Conclusions

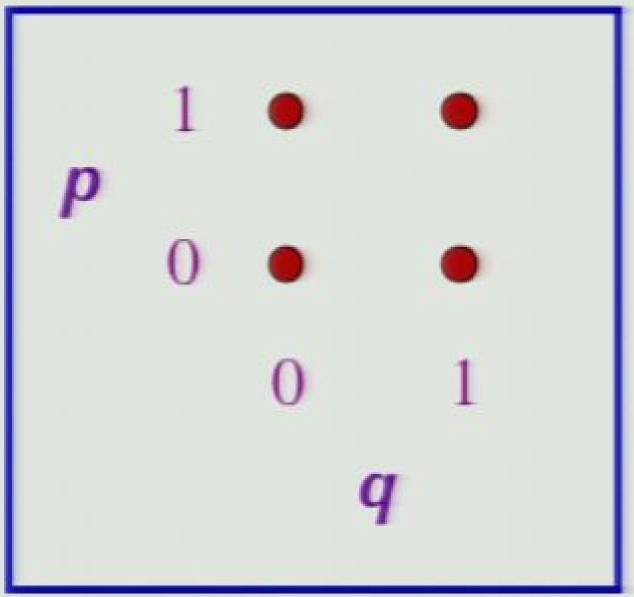
- There exists a SIC for three qubits with the Pauli-Pauli-Pauli symmetry, whose structure can be seen as arising from the properties of SICs for a single qubit.
- It seems unlikely that the same strategy will work for many other composite systems. (And we know it won't work for other numbers of qubits.)
- But *products* of SICs may be a good way of describing systems in situations where the composition is important.

Another way in which three qubits seem to be special: Rotationally invariant states in discrete phase space.

Phase Space for a Single Qubit



OR



q and p take values in $\{0,1\}$, with arithmetic mod 2.
Each point can be associated with a Pauli operator $X^q Z^p$.

A Wigner Function on this Phase Space

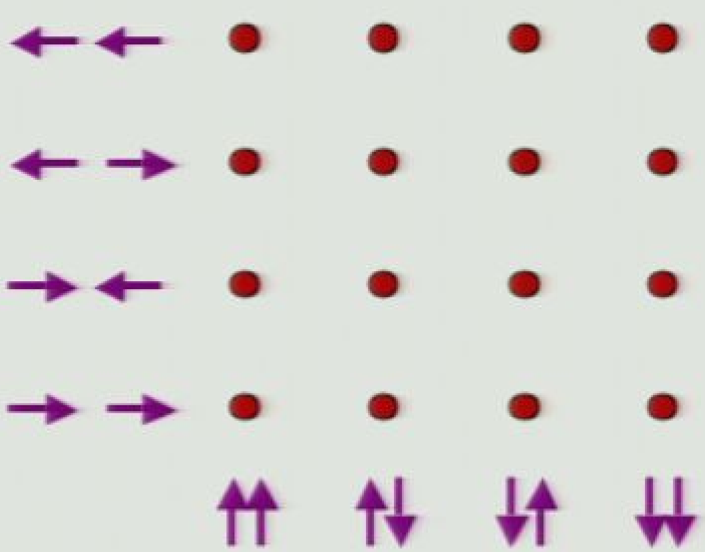
The up state:

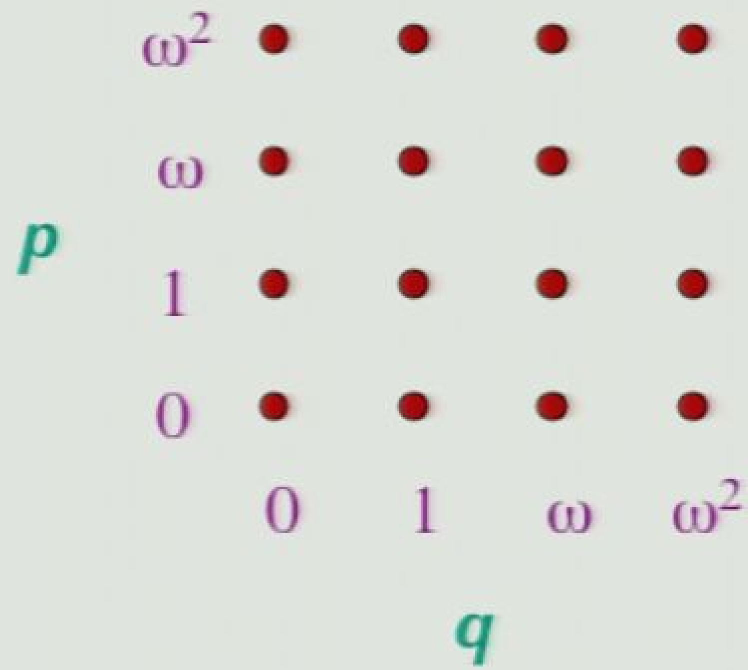
	↑	↓
←	1/2	0
→	1/2	0

An eigenstate of $X + Y + Z$:

	↑	↓
←	0.394	0.394
→	−0.183	0.394

Phase Space for Two Qubits





Each point is associated with an operator $X^{q_1}Z^{p_1} \otimes X^{q_2}Z^{p_2}$

$$1+1 = \omega+\omega = \omega^2+\omega^2 = 0$$
$$1+\omega = \omega^2$$

A Wigner Function for Two Qubits

↑↑

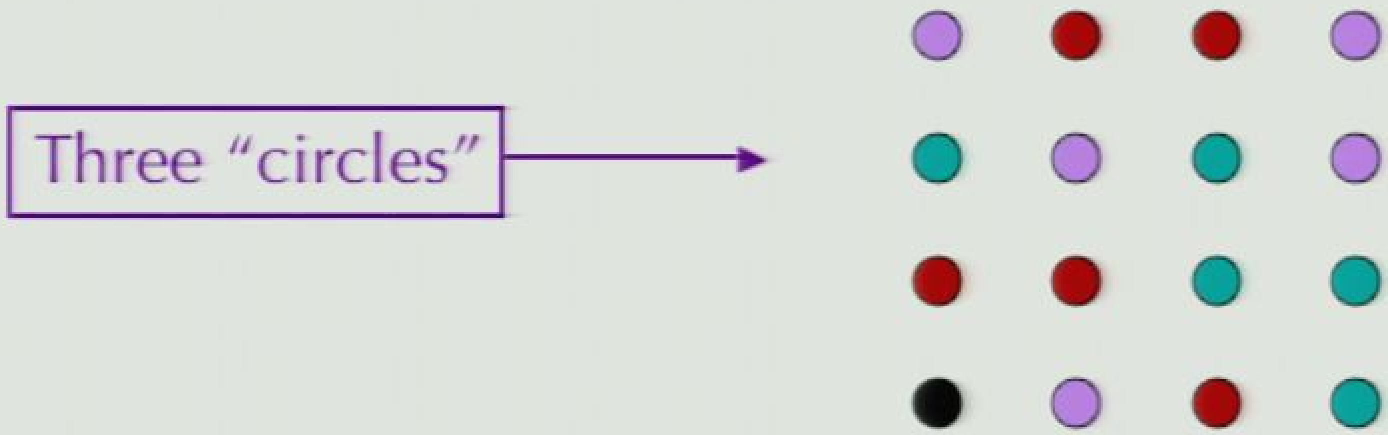
	↑↑	↑↓	↓↑	↓↓
←←	1/4	0	0	0
←→	1/4	0	0	0
→←	1/4	0	0	0
→→	1/4	0	0	0

↑→

	↑↑	↑↓	↓↑	↓↓
←←	0	0	0	0
←→	1/4	1/4	0	0
→←	0	0	0	0
→→	1/4	1/4	0	0

Rotations and Rotationally Invariant States

Choose a good quadratic polynomial, e.g., $q^2 + qp + \omega p^2$.



A rotation (around the origin) is a linear transformation R that preserves these circles.

One can find a unitary U associated with this rotation.

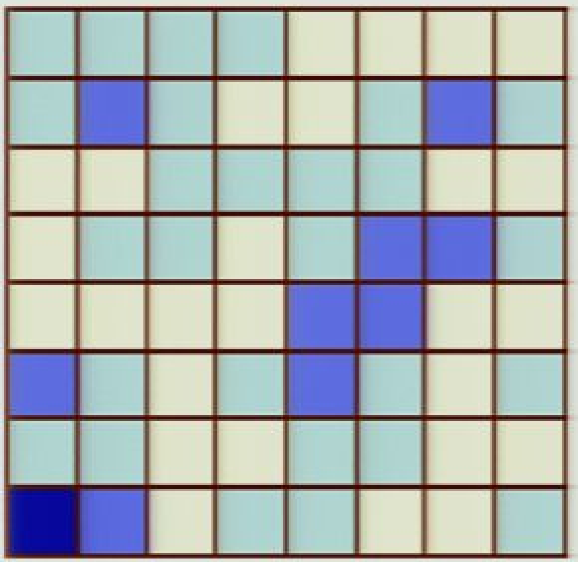
The eigenstates of this U are “rotationally invariant states.”

Do there exist rotationally invariant states with a positive Wigner function?

Like a coherent state in the continuous case.

Daniel Sussman and I found such a state (rotationally invariant and positive) only for one qubit and for three qubits.

Rotationally invariant three-qubit state with positive Wigner function:



$$|\psi\rangle = a\,|\phi_+,\,\phi_+,\,\phi_+\rangle + b\,|\phi_-,\,\phi_-,\,\phi_-\rangle$$

$$a^2 + b^2 = 1 \qquad a^2 - b^2 = 1/3$$

Not *quite* the same as our fiducial state!

How to get this rule by replacing probabilities

Start with $P_\alpha = \sum_{jk} p_{jk}\, r_{\alpha|jk} = \sum_{jk} \left[\left(p_{jk} - p_j^A p_k^B \right) + p_j^A p_k^B \right] r_{\alpha|jk}$

$$= \sum_{jk} \left[C_{jk} + p_j^A p_k^B \right] r_{\alpha|jk}$$

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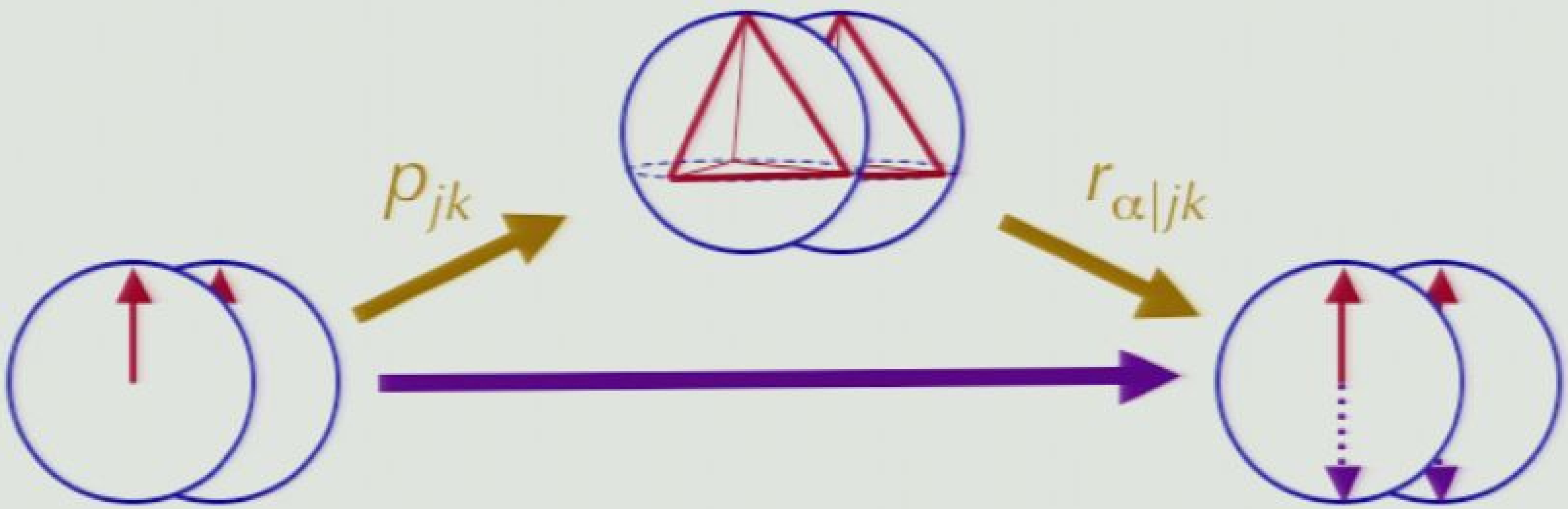
The Born rule in terms of this POVM



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