Title: Mapping Spacetime using Gravitational Wave Detectors

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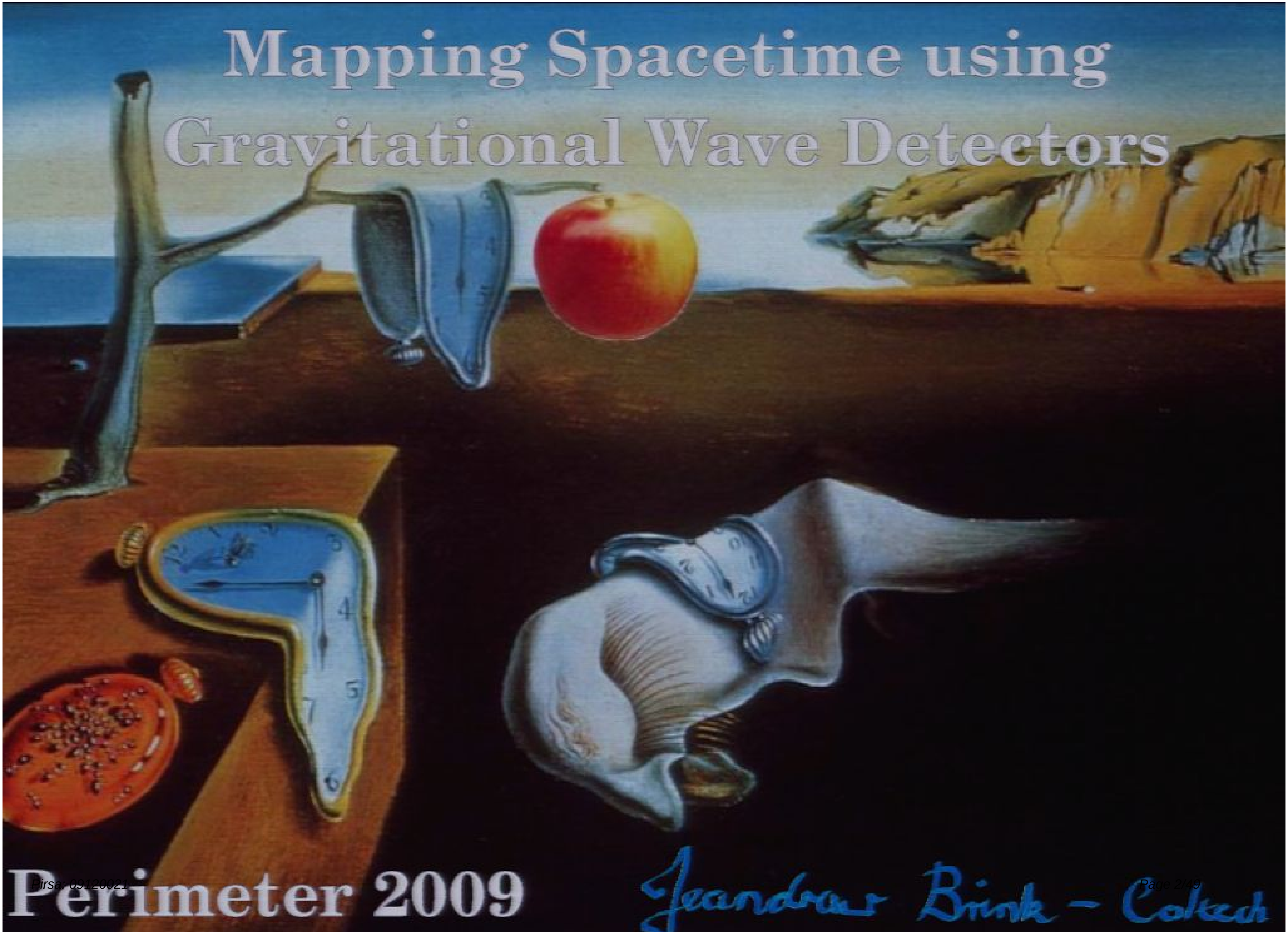
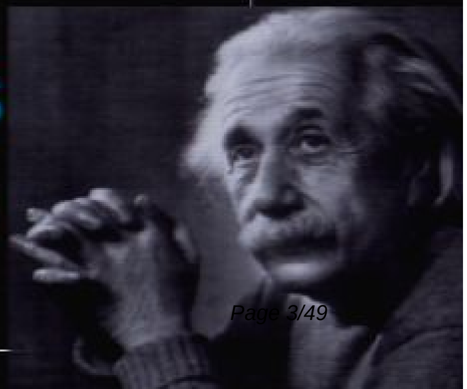
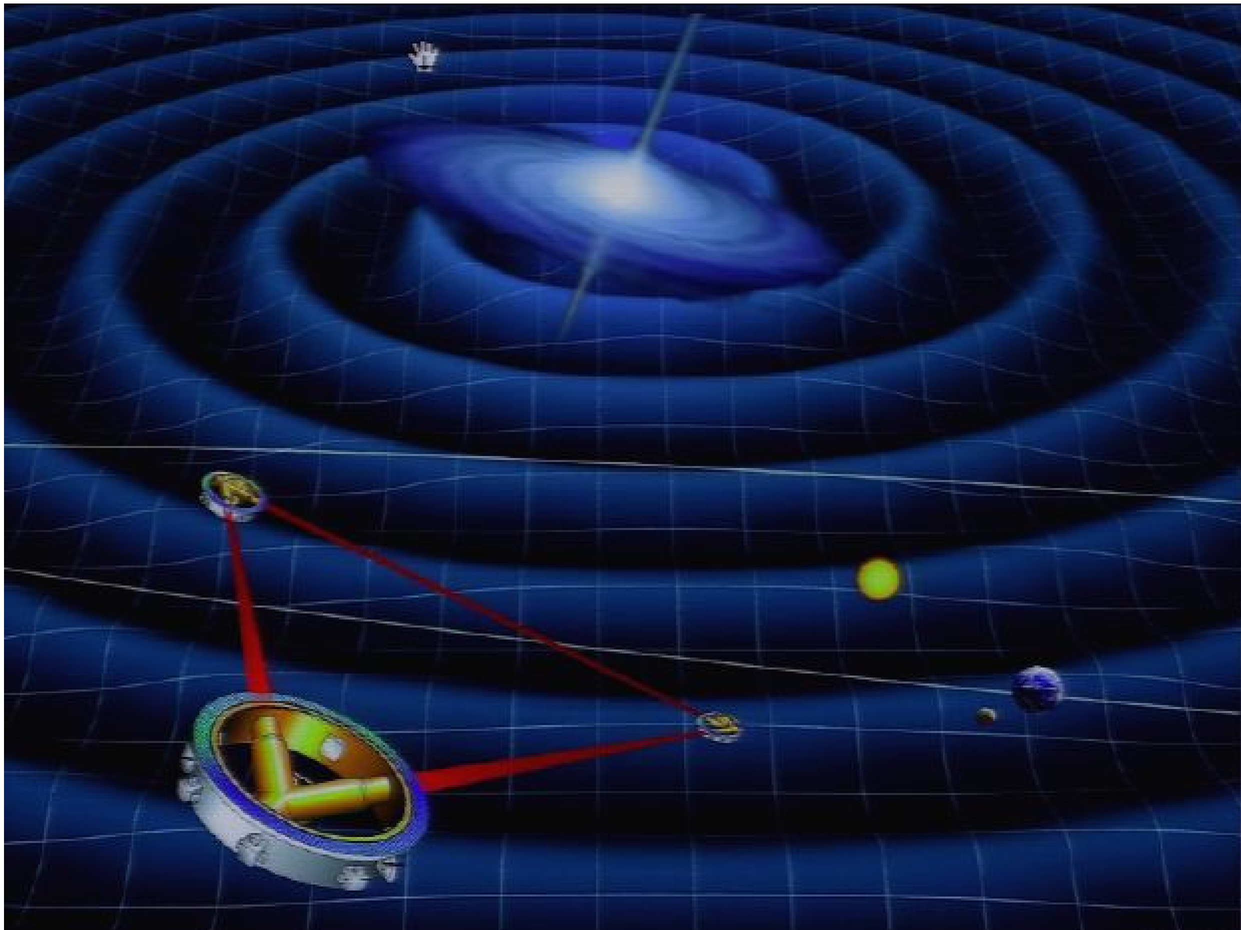
Abstract: One of the main science objectives for the Laser Interferometer Space Antenna (LISA) is to quantitatively map the strong field regions around compact objects using Extreme-Mass-Ratio Inspirals (EMRIs). This idea has been shown to be possible in principle, however in practice only inspirals in a Kerr spacetime have been studied in detail. A spacetime mapping algorithm for an EMRI inspiral into a generic compact object is formulated using ideas from integrable systems. I discuss several aspects of the theoretical development required to make the problem tractable. Some recent results about particle orbits around "Bumpy Black" holes are highlighted.


Mapping Spacetime using
Gravitational Wave Detectors
Perimeter 2009
Jeandrew Brink - Caltech

Outline

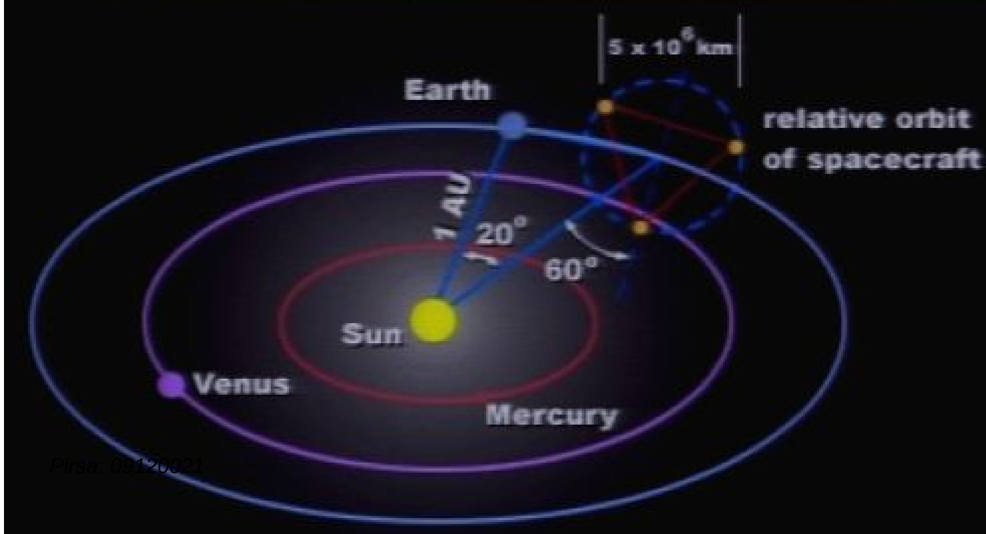
- LISA Science
  - -Experiment, Science Goals
- Historic Tools for Mapping Spacetime
- Problem formulation
- Current Math Machinery for GW detectors
- More General Mathematical Formulation
- Analogies that make it go
- Two manifolds
- Results for Orbits in SAV spacetimes
- Comments on the Spectre of Chaos
- Conclusions and What Next

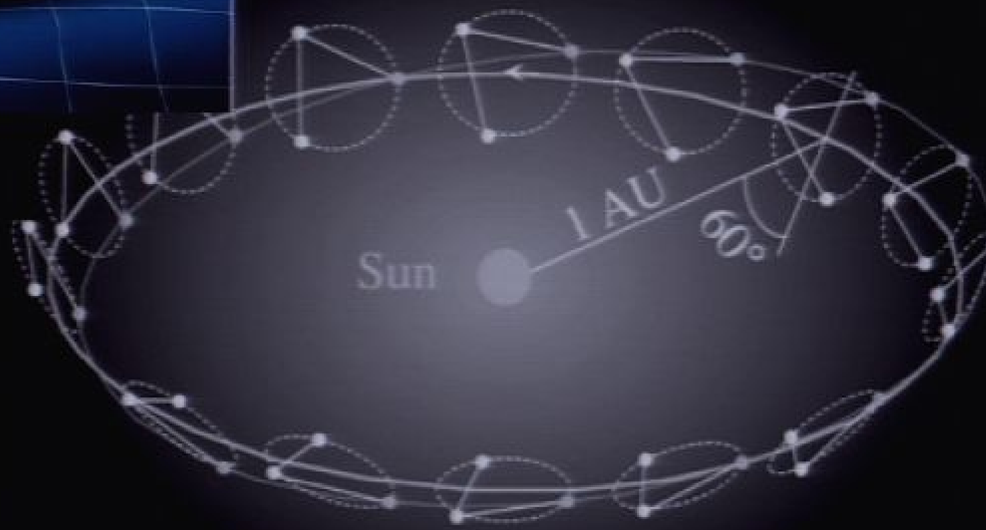




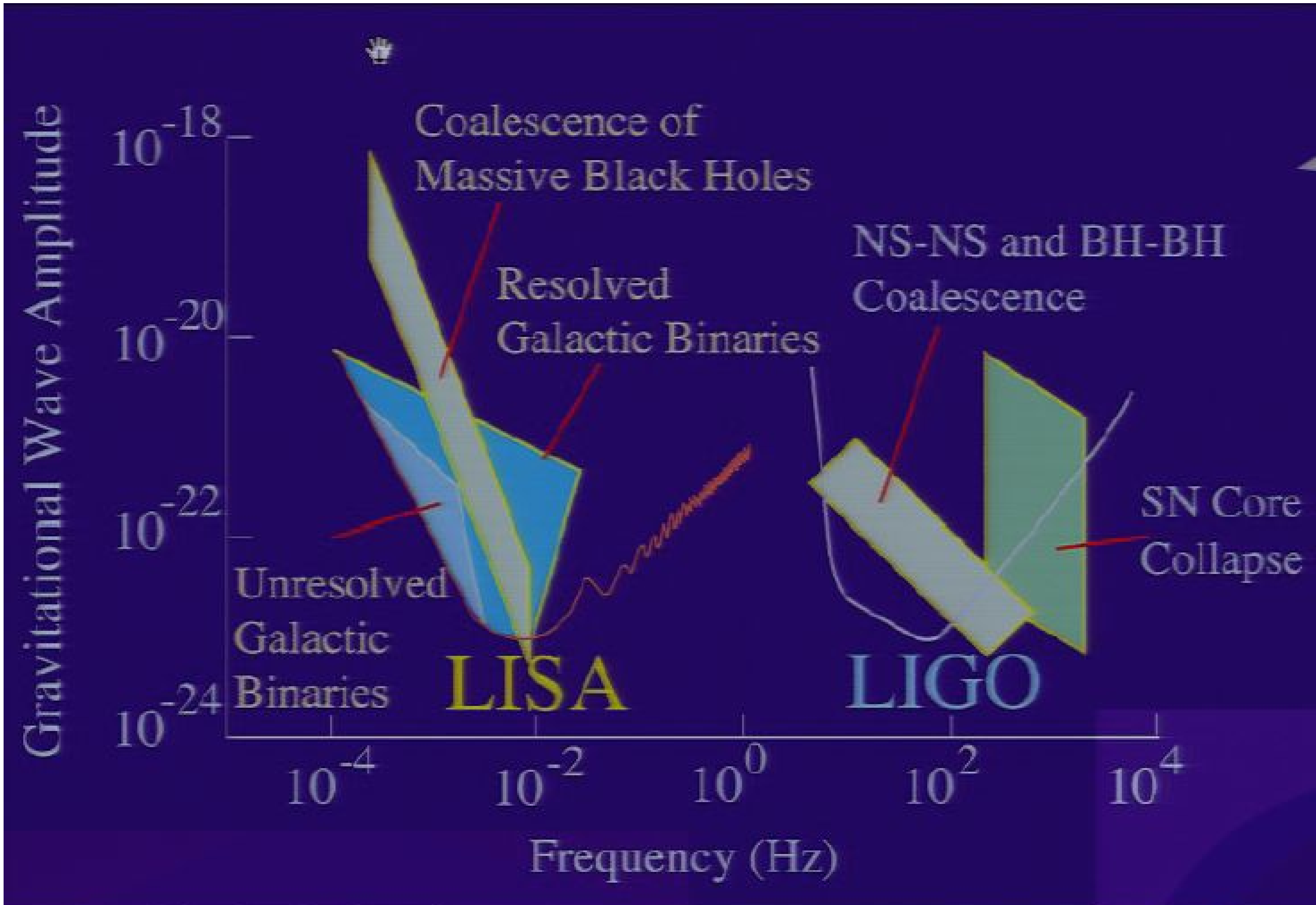
Laser Interferometer Space Antenna

Signal to noise ratios of 50-1000
Positions known to an arcminute, decoded from orbital motion of antennae pattern (see below)
Measure both polarizations of GR waves

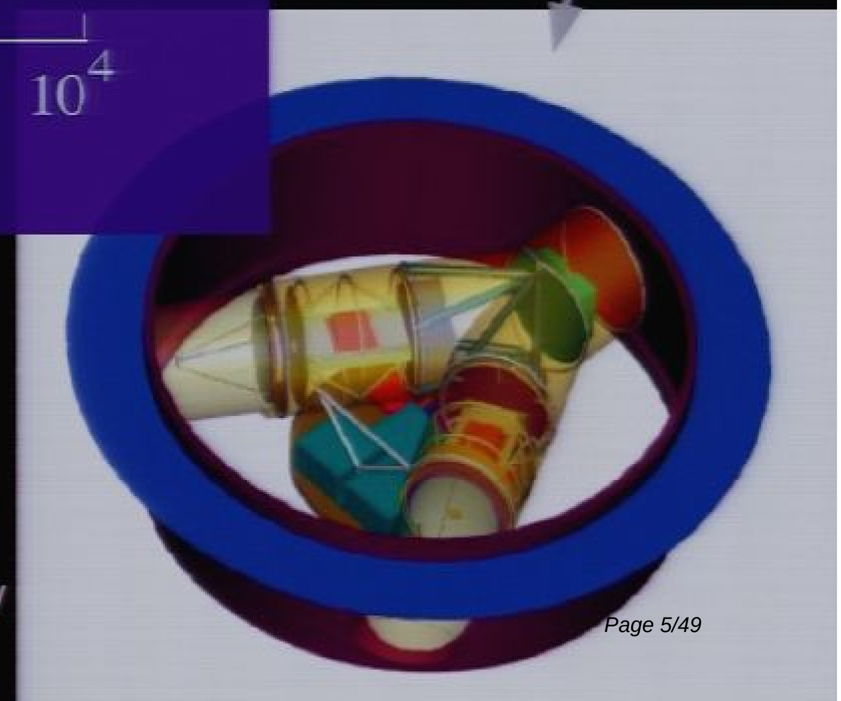




Noise Curve Comparison



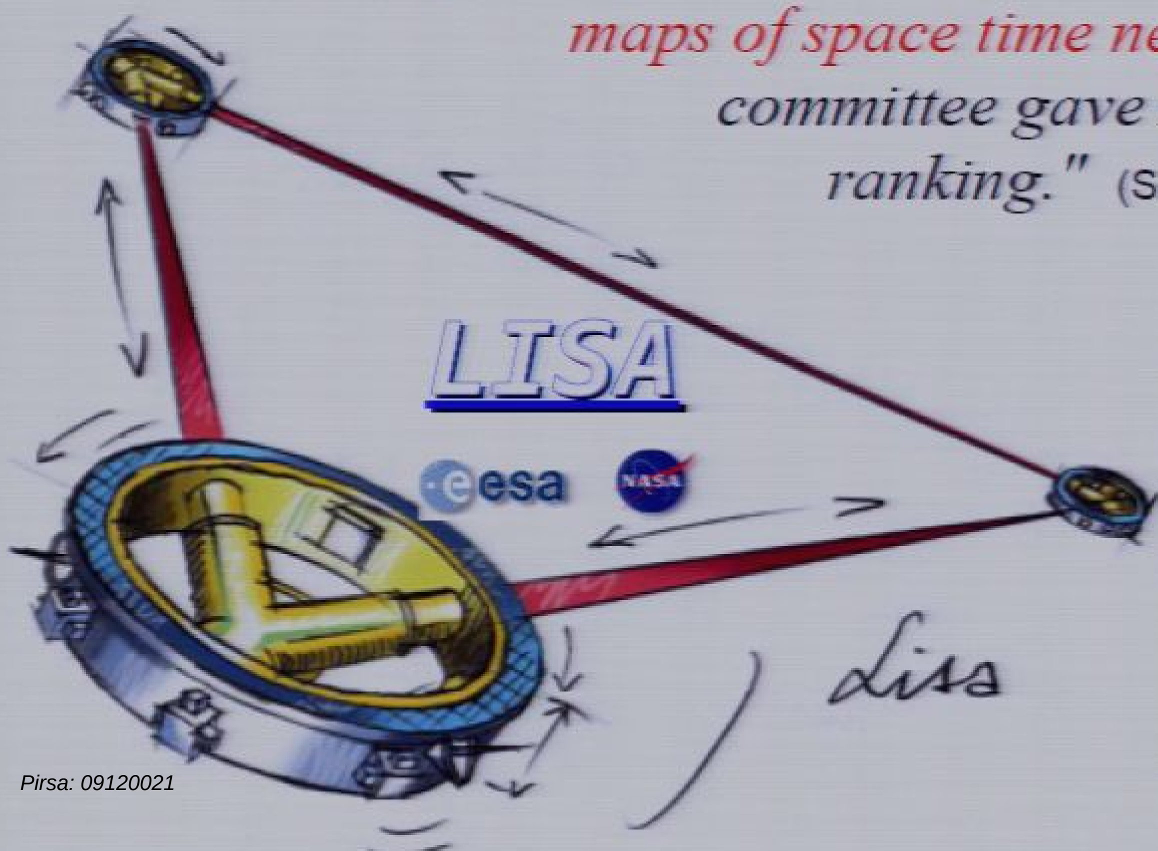
LISA Dual telescope system for each spacecraft in the Constellation



At the "Heart" of each LISA Satellite are the Freely Falling Proof Masses which define the end of each interferometer arm. Their positions are very accurately measured

NASA's Beyond Einstein Program: an architecture for Implementation

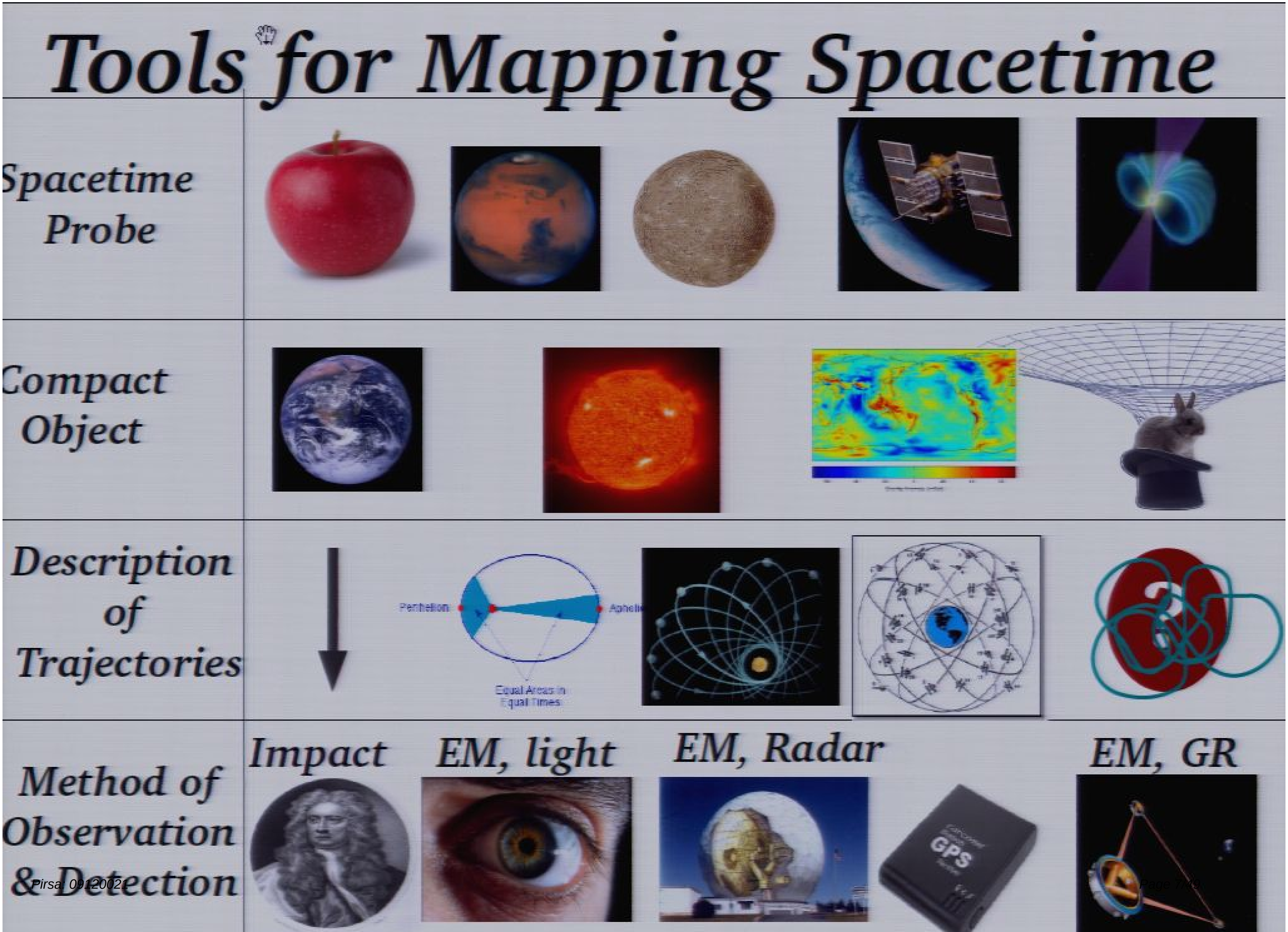
"On purely scientific grounds LISA is the mission that is most promising and least scientifically risky. Even with pessimistic assumptions about event rates, it should provide unambiguous and clean tests of the theory of general relativity in the strong field dynamical regime and be able to make detailed maps of space time near black holes. Thus, the committee gave LISA its highest scientific ranking." (Sep. 2007)



DATES

LISA Pathfinder – Launch 2011
Technology demo (ESA)

LISA Constellation
Launch 2020 (Provisional)
(ESA + NASA)



Tools for Mapping Spacetime
Spacetime Probe
Compact Object
Description of Trajectories
Perihelion
Equal Areas in Equal Times
Method of Observation & Detection
Impact
EM, light
EM, Radar
GPS
EM, GR

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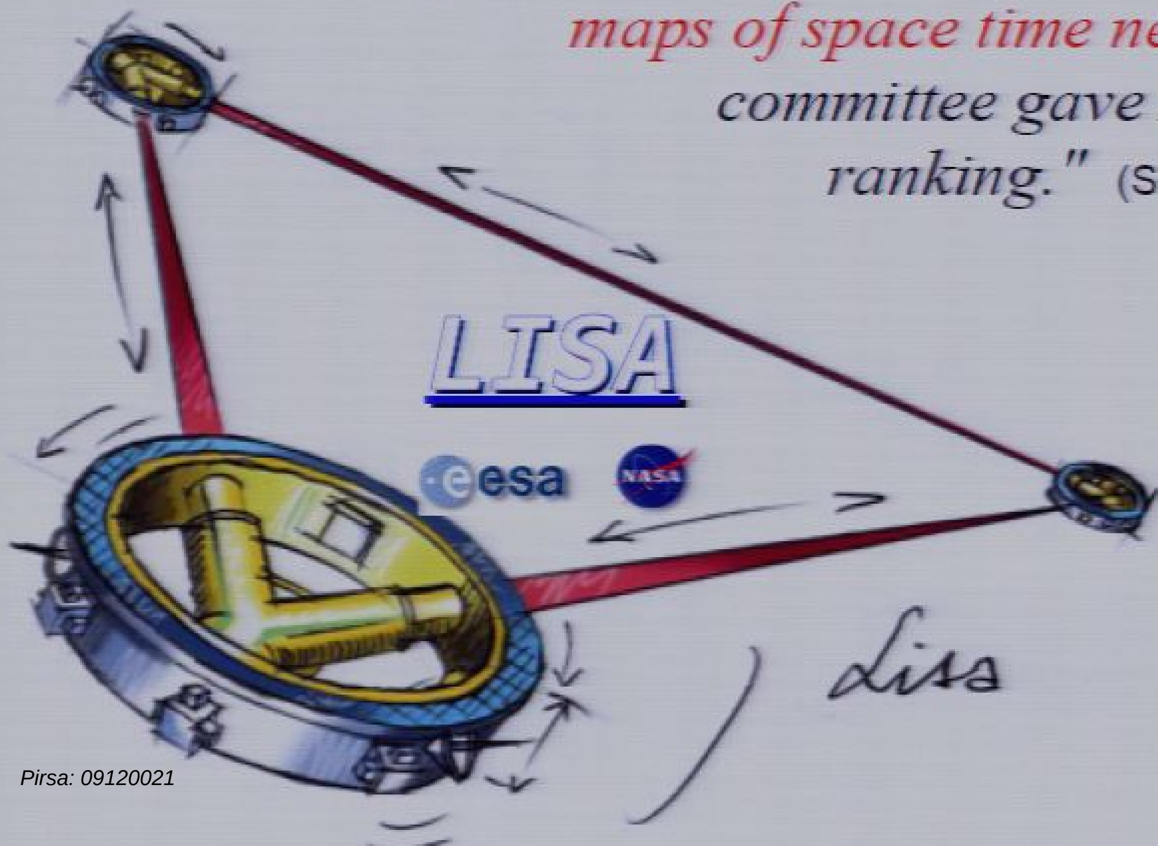

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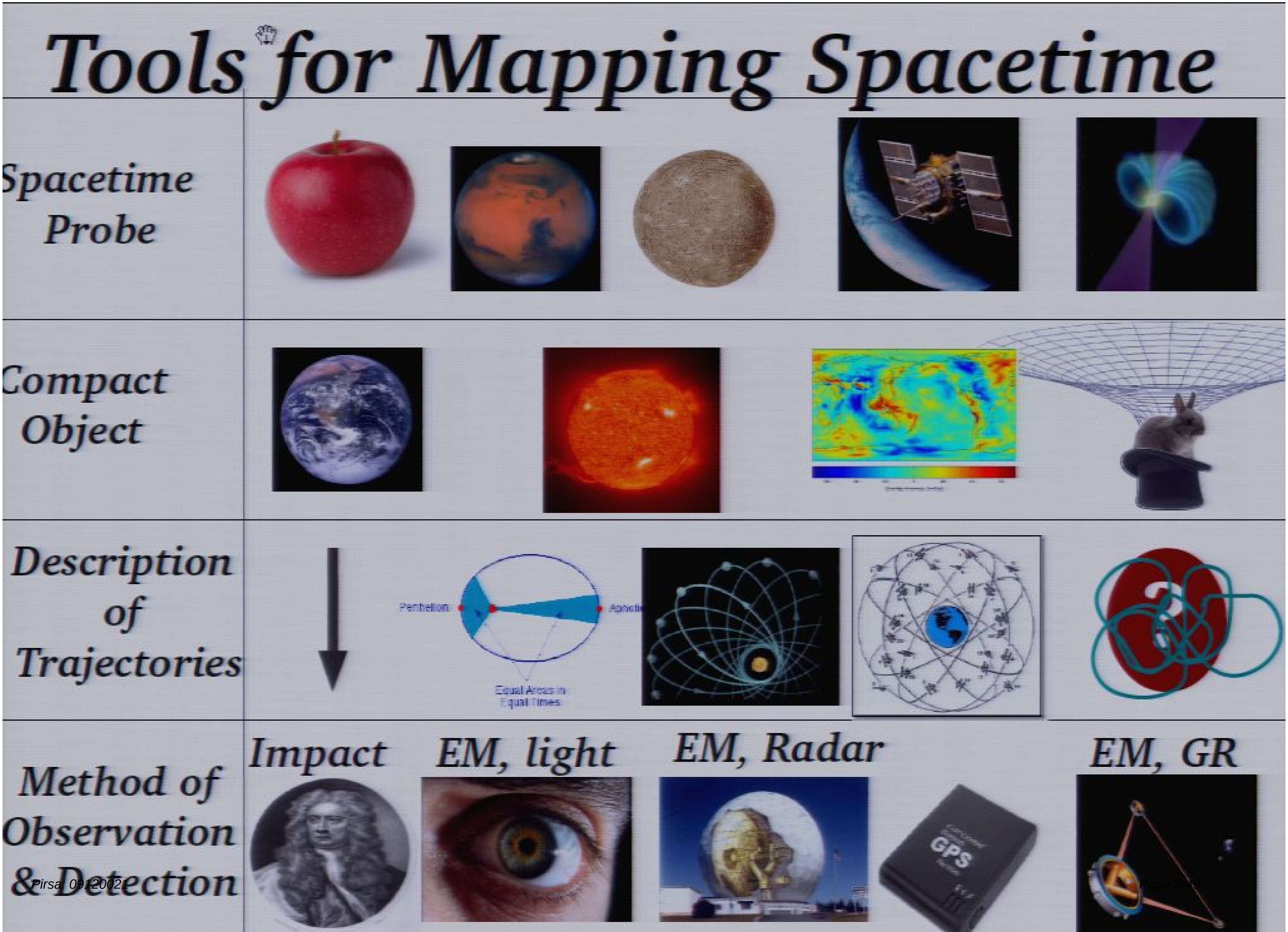
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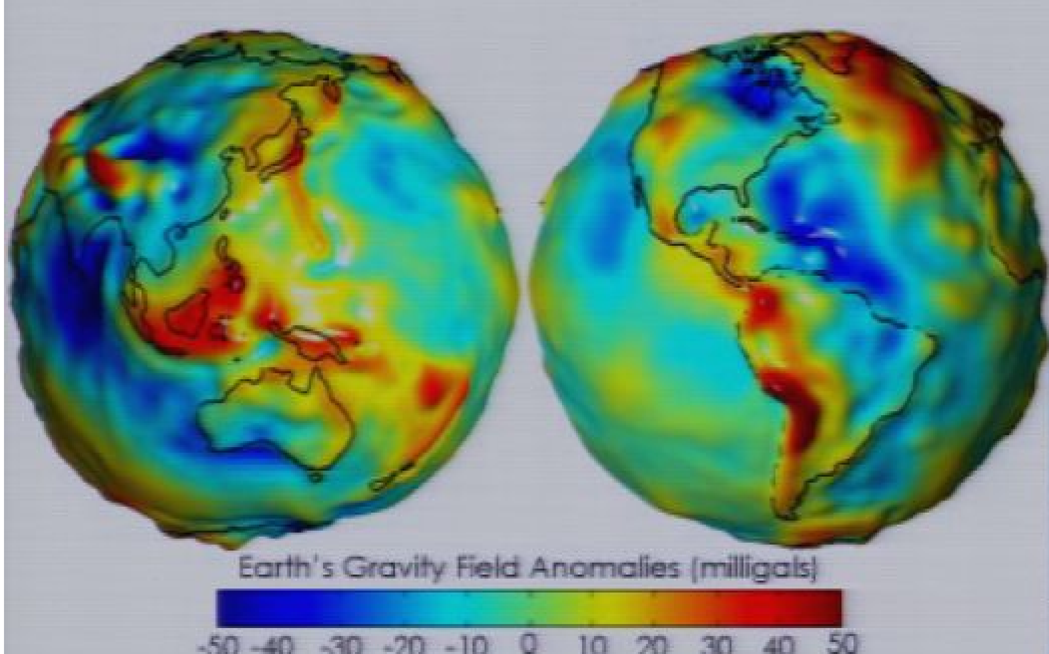
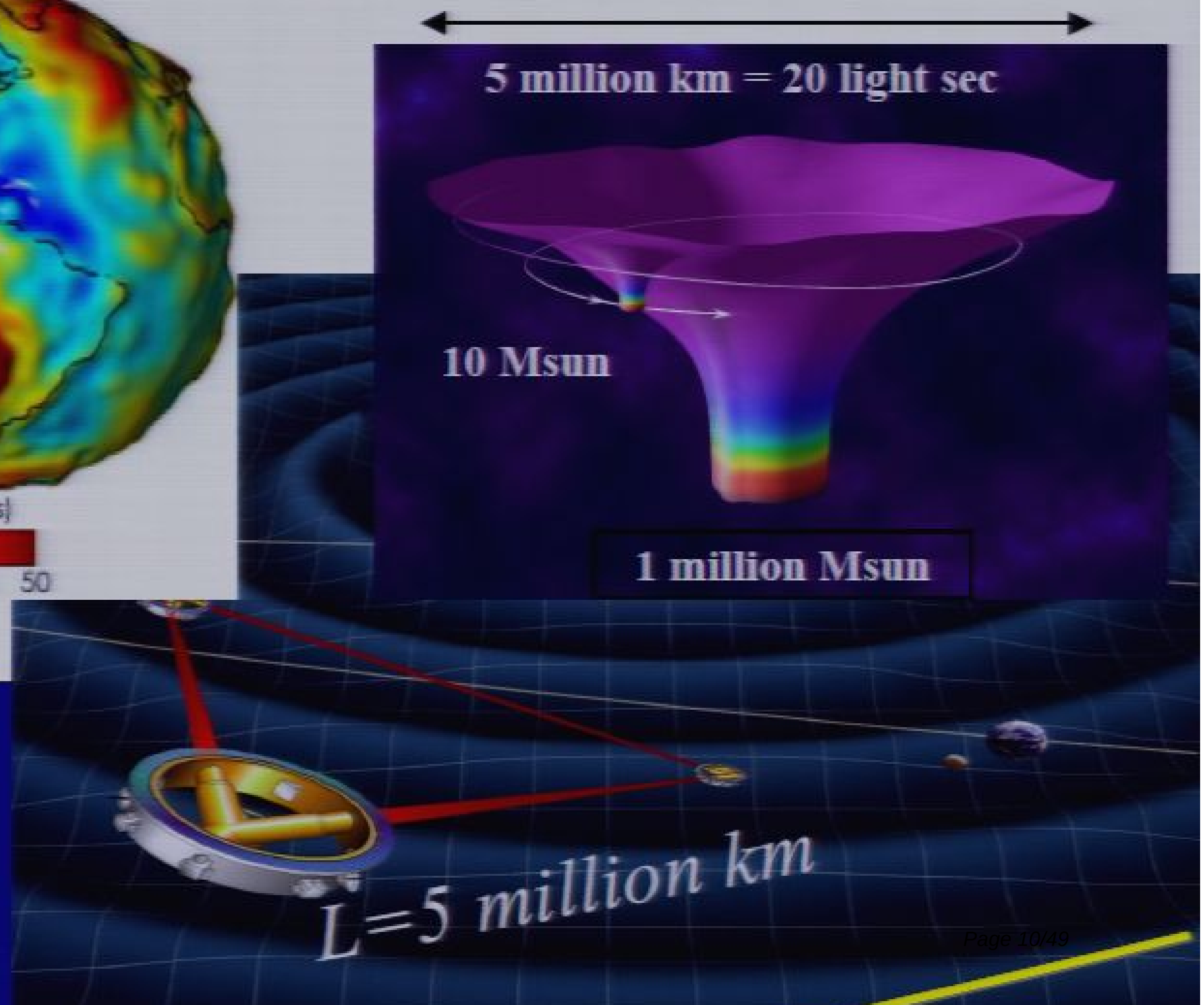
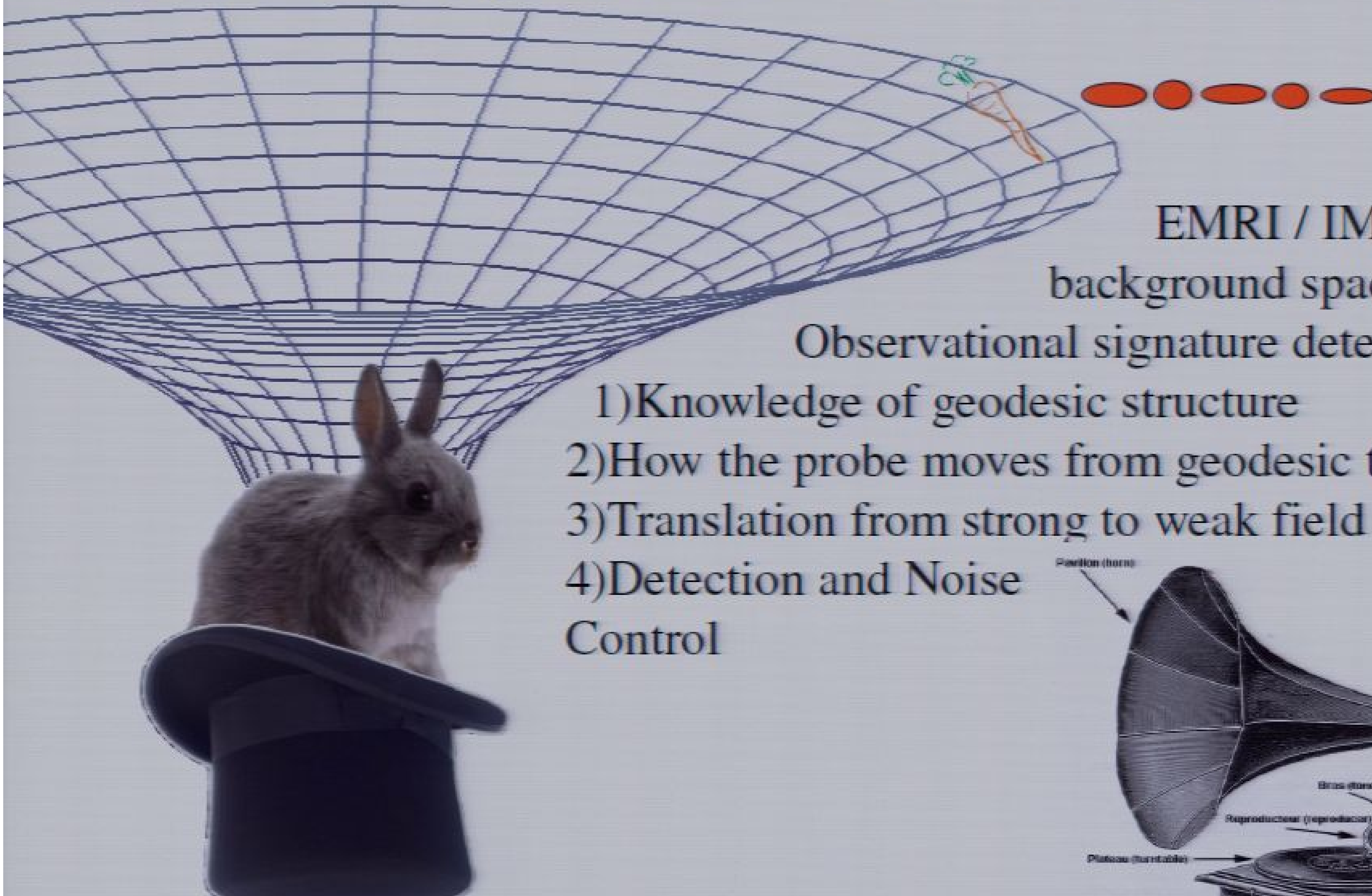
Mapping Spacetime : Done for the earth. Can we do it for Black holes ?





Length Scales

A Question: Spacetime Reconstruction Problem



LIGO
LISA

EMRI / IMRI probes background spacetime

Observational signature determined by

1)Knowledge of geodesic structure
2)How the probe moves from geodesic to geodesic
3)Translation from strong to weak field
4)Detection and Noise Control

Anatomie d'un phonographe à disques

Can one draw the black hole bunny out of the hat by watching the gravitational radiation ?

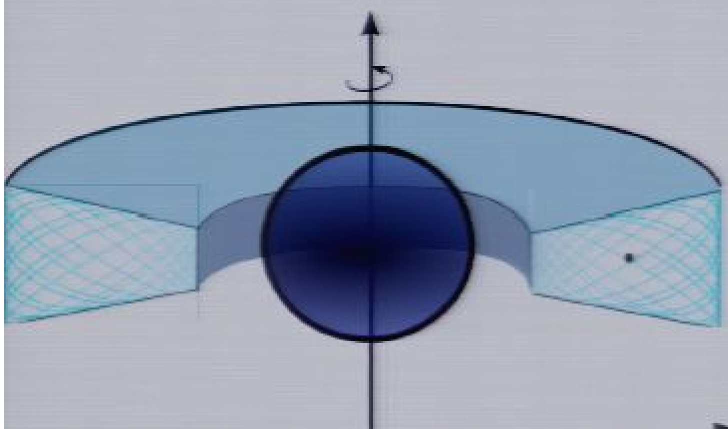
Current machinery and No Hair Theorems

Uniqueness theorem assumptions (Mazure)
- Cosmic Censorship Conjecture "No Naked Singularities"
- Causality "No Closed Timelike Curves"

→ **KERR**

(full set of isolating integrals)

EMRI/IMRI Wave Form Generation Machine



Teukolsky Master EQ.

$$\delta\Psi^{;\alpha}_{;\alpha} + \cdots \delta\Psi = \mathrm{Source}(\delta(\mathrm{orbit}))$$

(Drasco/Hughes)

Particle Motion in strong field region → translates into waves → in the asymptotic region

Action variables $E,\ L_z,\ Q,\ \mu$ **uniquely identify the orbit**
Physical and Phase Space Confinement
Angle variables record where in the orbit your are

Separability of Hamilton Jacobi & Wave Equations (Woodhouse, Carter)

Observables have three fundamental frequencies

More General Mathematical Formulation

The set of current and mass multipole moments of any Axially Symmetric Stationary Asymptotically Flat Vacuum Spacetime uniquely identify the spacetime (Geroch, HKX+C) (WANTED)

Killing vectors ∂_t and ∂_ϕ

$$R_{ij} = 0$$

Ernst Equation $\Re(\mathcal{E})\ \overline{\nabla}^2 \mathcal{E} = \overline{\nabla}\mathcal{E} \cdot \overline{\nabla}\mathcal{E}$

Metric $ds^2 \;= e^{-2\psi}\left[e^{2\gamma}(d\rho^2 + dz^2) + R^2 d\phi^2\right] - e^{2\psi}(dt - \omega d\phi)^2$

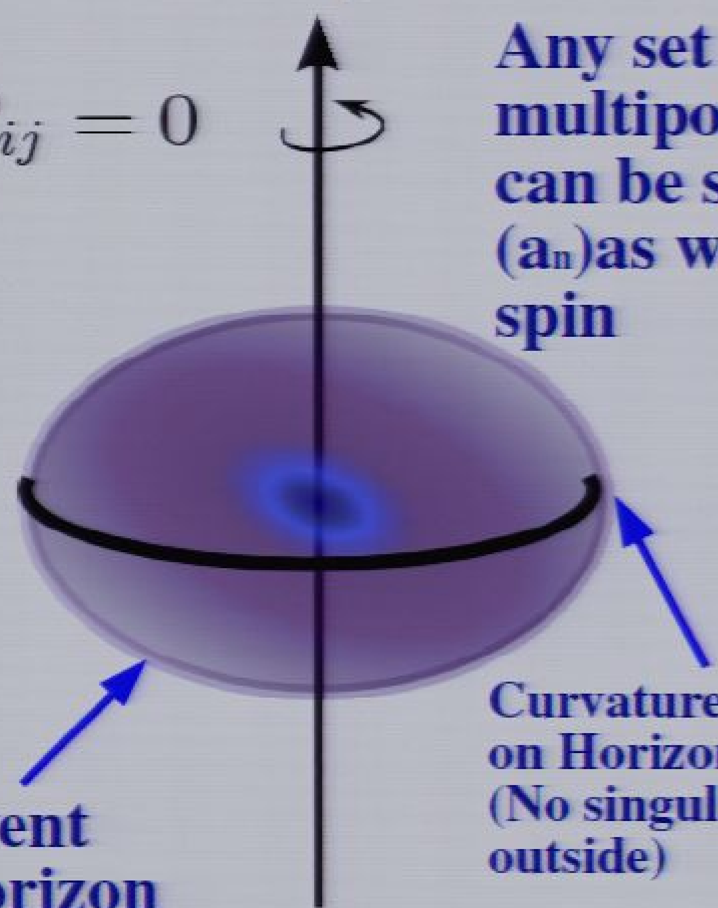
(OBSERVED) Particle orbits are effectively being observed by the detector. Mathematically governed by the study of the Geodesic Equations.

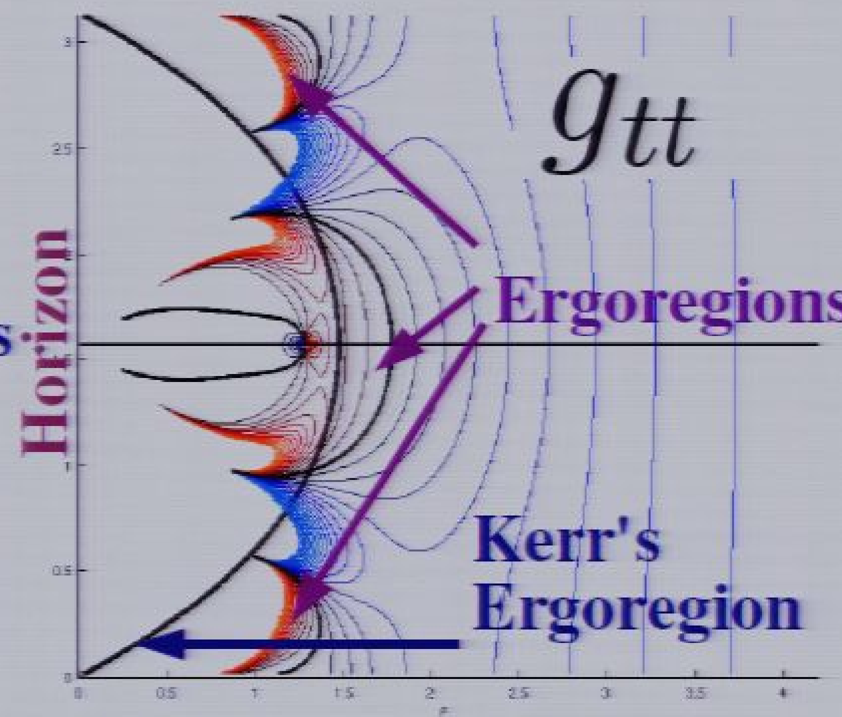
Two degree of freedom problem in dynamical systems

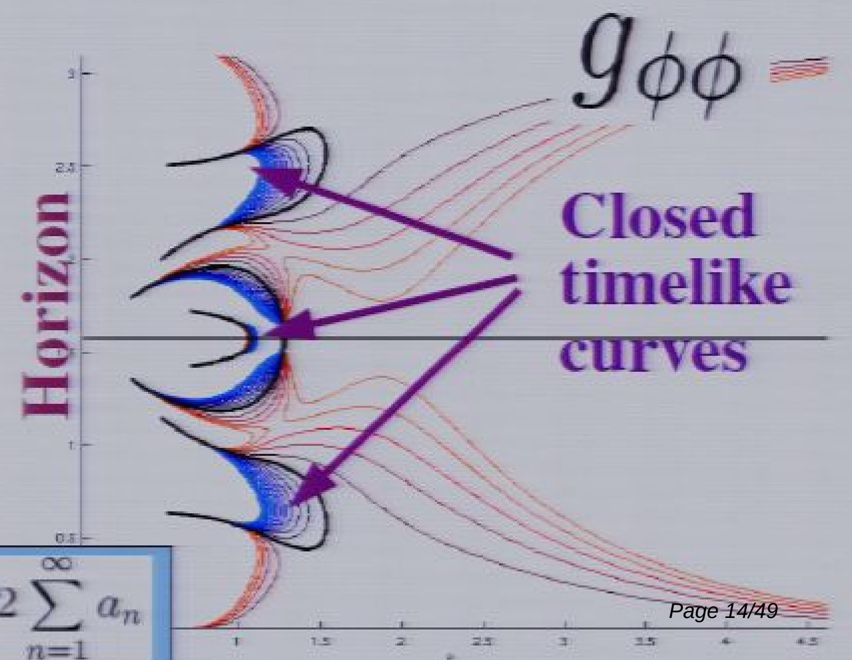
Manko Novikov Spacetime

$R_{ij} = 0$

Any set of mass multipole moments can be specified, (a_n)as wells as the spin



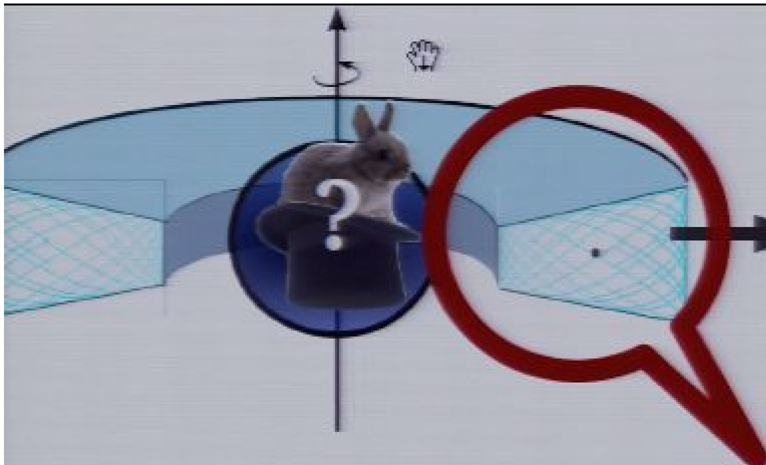




Horizon Area

$$A = 16\pi k^2 \left(e^{-\sigma} + \alpha^2 e^{\sigma}\right) / (1 - \alpha^2)^2$$

$$\sigma = 2 \sum_{n=1}^{\infty} a_n$$



Orbital Description ?

Hunting Constants of motion

- Hamilton Jacobi Method
- Painleve Check
- Lax Pairs
- **Poincare Maps** (2DoF)
- **Killing Tensors** (4D)
- **Serendipity ~ Guess**

Equatorially Bagged Bunny

Zipoy Voorhees Metric

Timeline of Analytic Results in Vacuum GR

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Spherically symmetric static exact solution
Black hole solution 1916

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Exact solution for an uncharged, *rotating* body, the **Kerr metric**
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Existence of a 2nd order Killing Tensor

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Intimidation Slide

$$T_{1111,3} = 2T_{1111}\gamma_{131} - 4T_{1112}\gamma_{131} - 12T_{1134}\gamma_{131} - 4T_{1111}\gamma_{132} - 12T_{1133}\gamma_{141} - 2T_{1111}\gamma_{232},$$
$$T_{1111,4} = -12T_{1144}\gamma_{131} + 2T_{1111}\gamma_{141} - 4T_{1112}\gamma_{141} - 12T_{1134}\gamma_{141} - 4T_{1111}\gamma_{142} - 2T_{1111}\gamma_{242},$$
$$T_{2222,3} = -2T_{2222}\gamma_{131} - 4T_{2222}\gamma_{132} - 4T_{1222}\gamma_{232} + 2T_{2222}\gamma_{232} - 12T_{2234}\gamma_{232} - 12T_{2233}\gamma_{242},$$
$$T_{2222,4} = -2T_{2222}\gamma_{141} - 4T_{2222}\gamma_{142} - 12T_{2244}\gamma_{232} - 4T_{1222}\gamma_{242} + 2T_{2222}\gamma_{242} - 12T_{2234}\gamma_{242},$$
$$T_{1112,3} = (T_{1112} - 3T_{1122} - 6T_{1234})\gamma_{131} - (4T_{1112} + 6T_{1134})\gamma_{132} - 6T_{1233}\gamma_{141} - 6T_{1133}\gamma_{142} - (T_{1111} + T_{1112})\gamma_{232},$$
$$T_{1112,4} = -6T_{1244}\gamma_{131} - 6T_{1144}\gamma_{132} + (T_{1112} - 3T_{1122} - 6T_{1234})\gamma_{141} - (4T_{1112} + 6T_{1134})\gamma_{142} - (T_{1111} + T_{1112})\gamma_{242},$$
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$$T_{1122,3} = -2(T_{12}[illegible]_{34})\gamma_{131} - 4(T_{1122} + 2T_{1234})\gamma_{132} - 2T_{2233}\gamma_{1}[illegible] - 8T_{1233}\gamma_{1}[illegible] - 2(T_{1112} + T_{1134})\gamma_{232} - 2T_{11}[illegible]$$
$$T_{1122,4} = -2T_{2244}\gamma_{1}[illegible] - 8[illegible] - 4([illegible] + 2[illegible]) [illegible] - 2(T_{1112} + T_{11}[illegible])\gamma_{242}.$$

$$T_{1134,3} = \frac{1}{2}(-T_{11}[illegible] T_{1134}(2\gamma_{132} + \gamma_{232} - \gamma_{131}) + T_{1133}(2\gamma_{142} [illegible] + 2\gamma_{344}))$$
$$- (2T_{1234} + T_{3344})\gamma_{131} - (T_{1233} + T_{3334})\gamma_{141},$$
$$T_{1134,4} = \frac{1}{2}(-T_{1144,3} - 2T_{1134}(-\gamma_{141} + 2\gamma_{142} + \gamma_{242}) + T_{1144}(\gamma_{131} - 2\gamma_{132} - \gamma_{232} + 2\gamma_{343}))$$
$$- (T_{1244} + T_{3444})\gamma_{131} - (2T_{1234} + T_{3344})\gamma_{141},$$
$$T_{2234,3} = \frac{1}{2}(-T_{2233,4} - 2T_{2234}(\gamma_{131} + 2\gamma_{132} - \gamma_{232}) - T_{2233}(\gamma_{141} + 2\gamma_{142} - \gamma_{242} + 2\gamma_{344}))$$
$$- (2T_{1234} + T_{3344})\gamma_{232} - (T_{1233} + T_{3334})\gamma_{242},$$
$$T_{2234,4} = \frac{1}{2}(-T_{2244,3} - 2(+T_{2234}(\gamma_{141} + 2\gamma_{142} - \gamma_{242}) +) - T_{2244}(\gamma_{131} + 2\gamma_{132} - \gamma_{232} - 2\gamma_{343}))$$
$$- (T_{1244} + T_{3444})\gamma_{232} - (2T_{1234} + T_{3344})\gamma_{242},$$
$$T_{1234,3} = \frac{1}{2}(-T_{1233,4} - 2T_{2234}\gamma_{131} - T_{2233}\gamma_{141} - 2T_{1134}\gamma_{232} - T_{1133}\gamma_{242} - 2T_{1233}\gamma_{344})$$
$$- (2T_{1234} + T_{3344})\gamma_{132} - (T_{1233} + T_{3334})\gamma_{142},$$
$$T_{1234,4} = \frac{1}{2}(-T_{1244,}[illegible] T_{2244}\gamma_{131} + 2T_{2234}\gamma_{141} - T_{1144}\gamma_{232} - 3T_{1134}\gamma_{242} + 2T_{1144}\gamma_{343})$$
$$- (T_{1244} [illegible] \gamma_{131} - 2T[illegible] \gamma_{141}$$

Mathematical

$$T_{1133,3} = T_{1133}(\gamma_{131} - 2\gamma_{1}[illegible] \gamma_{131} + T_{3333}\gamma_{1}[illegible]$$
$$T_{2233,3} = -\frac{2}{3}(3T_{1233}\gamma_{232} + [illegible]_{132} - \gamma_{232} + 2\gamma[illegible]$$
$$T_{1233,3} = \frac{1}{3}(-3T_{2233}\gamma_{131} - [illegible] T_{1233}(\gamma_{132} + \gamma[illegible]$$
$$T_{1144,4} = -\frac{2}{3}(T_{4444}\gamma_{131} + [illegible] - \gamma_{242} + 2\gamma_{34}[illegible]$$
$$T_{2244,4} = -\frac{2}{3}((T_{4444}\gamma_{232} + [illegible]_{42} - \gamma_{242} - 2\gamma[illegible]$$
$$T_{1244,4} = \frac{1}{3}(-2T_{4444}\gamma_{132} - [illegible] T_{1144}\gamma_{242} + 6T[illegible]$$

$$T_{3344,4} = -\frac{2}{3}(T_{3444,3} - 2T_{3444}\gamma_{343}),$$
$$T_{3444,4} = -\frac{1}{4}T_{4444,3} + T_{4444}\gamma_{343} + 2T_{3444}\gamma_{344},$$
$$T_{4444,4} = 4T_{4444}\gamma_{344}.$$

$$T_{3344,3} = -\frac{2}{3}(T_{3334,4} + 2T_{3334}\gamma_{344}),$$
$$T_{3334,3} = -\frac{1}{4}T_{3333,4} - T_{3333}\gamma_{344} - 2T_{3334}\gamma_{343},$$
$$T_{3333,3} = -4T_{3333}\gamma_{343}.$$

$$M_1 = \partial_\zeta \gamma, \quad M_2 = \frac{\partial_\zeta R}{R}, \quad M_3 = \frac{\partial_\zeta \mathcal{E}}{\mathcal{E} + \overline{\mathcal{E}}}, \quad M_4 = \frac{\partial_\zeta \overline{\mathcal{E}}}{\mathcal{E} + \overline{\mathcal{E}}},$$
$$M_1^* = \partial_{\bar\zeta} \gamma, \quad M_2^* = \frac{\partial_{\bar\zeta} R}{R}, \quad M_3^* = \frac{\partial_{\bar\zeta} \mathcal{E}}{\mathcal{E} + \overline{\mathcal{E}}}, \quad M_4^* = \frac{\partial_{\bar\zeta} \overline{\mathcal{E}}}{\mathcal{E} + \overline{\mathcal{E}}}.$$

$$\gamma_{123} = \frac{M_3^* - M_4^*}{2\sqrt{2V}}, \qquad \gamma_{124} = \frac{M_4 - M_3}{2\sqrt{2V}},$$
$$\gamma_{131} = \frac{M_2^* - 2M_4^*}{2\sqrt{2V}}, \qquad \gamma_{141} = \frac{M_2 - 2M_3}{2\sqrt{2V}},$$
$$\gamma_{132} = -\frac{M_2^*}{2\sqrt{2V}}, \qquad \gamma_{142} = -\frac{M_2}{2\sqrt{2V}},$$
$$\gamma_{231} = -\frac{M_2^*}{2\sqrt{2V}}, \qquad \gamma_{241} = -\frac{M_2}{2\sqrt{2V}},$$
$$M_2^* - 2M_3^* \qquad M_2 - 2M_4$$

Field EQ

$$M_{1,\zeta} = -\frac{1}{2}(M_3 M_4^* + M_4 M_3^*),$$
$$M_{2,\zeta} = -M_2 M_2^*,$$
$$M_{3,\zeta} = -\left(\frac{1}{2}(M_2 M_3^* + M_3 M_2^*) - M_3 M_3^* + M_3 M_4^*\right),$$
$$M_{4,\zeta} = -\left(\frac{1}{2}(M_2 M_4^* + M_4 M_2^*) + M_4 M_3^* - M_4 M_4^*\right),$$
$$M_{2,\bar\zeta} = -M_2^2 + 2(M_1 M_2 - M_3 M_4).$$

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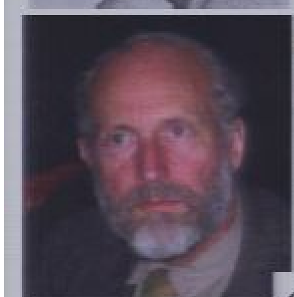

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$$T_{2234,3} = \frac{1}{2}(-T_{2233,4} - 2T_{2234}(\gamma_{131} + 2\gamma_{132} - \gamma_{232}) - T_{2233}(\gamma_{141} + 2\gamma_{142} - \gamma_{242} + 2\gamma_{344}))$$
$$- (2T_{1234} + T_{3344})\gamma_{232} - (T_{1233} + T_{3334})\gamma_{242},$$
$$T_{2234,4} = \frac{1}{2}(-T_{2244,3} - 2(+T_{2234}(\gamma_{141} + 2\gamma_{142} - \gamma_{242}) +) - T_{2244}(\gamma_{131} + 2\gamma_{132} - \gamma_{232} - 2\gamma_{343}))$$
$$- (T_{1244} + T_{3444})\gamma_{232} - (2T_{1234} + T_{3344})\gamma_{242},$$
$$T_{1234,3} = \frac{1}{2}(-T_{1233,4} - 2T_{2234}\gamma_{131} - T_{2233}\gamma_{141} - 2T_{1134}\gamma_{232} - T_{1133}\gamma_{242} - 2T_{1233}\gamma_{344})$$
$$- (2T_{1234} + T_{3344})\gamma_{132} - (T_{1233} + T_{3334})\gamma_{142},$$
$$T_{1234,4} = \frac{1}{2}(-T_{1244,\ldots} \ldots T_{2244}\gamma_{131} + 2T_{2234}\gamma_{141} - T_{1144}\gamma_{232} - \ldots T_{1134}\gamma_{242} + 2T_{1\ldots 44}\gamma_{343})$$
$$- (T_{1244} \ldots)\gamma_{1\ldots} - 2T_{\ldots}\ldots)\gamma_{14\ldots}$$

$$T_{1133,3} = T_{1133}(\gamma_{131} - 2\gamma_{1\ldots} \ldots \gamma_{131} + T_{3333}\gamma_{1\ldots}$$
$$T_{2233,3} = -\frac{2}{3}(3T_{1233}\gamma_{232} + \ldots \gamma_{132} - \gamma_{232} + 2\ldots$$
$$T_{1233,3} = \frac{1}{3}(-3T_{2233}\gamma_{131} - \ldots T_{1233}(\gamma_{132} + \ldots$$
$$T_{1144,4} = -\frac{2}{3}(T_{4444}\gamma_{131} + \ldots - \gamma_{242} + 2\gamma_{34\ldots}$$
$$T_{2244,4} = -\frac{2}{3}((T_{4444}\gamma_{232} + \ldots \gamma_{242} - 2\ldots$$
$$T_{1244,4} = \frac{1}{3}(-2T_{4444}\gamma_{132} - \ldots T_{1144}\gamma_{242} + 6T\ldots$$

$$T_{3344,4} = -\frac{2}{3}(T_{3444,3} - 2T_{3444}\gamma_{343}),$$
$$T_{3444,4} = -\frac{1}{4}T_{4444,3} + T_{4444}\gamma_{343} + 2T_{3444}\gamma_{344},$$
$$T_{4444,4} = 4T_{4444}\gamma_{344}.$$

$$T_{3344,3} = -\frac{2}{3}(T_{3334,4} + 2T_{3334}\gamma_{344}),$$
$$T_{3334,3} = -\frac{1}{4}T_{3333,4} - T_{3333}\gamma_{344} - 2T_{3334}\gamma_{343},$$
$$T_{3333,3} = -4T_{33\ldots}\gamma_{343},$$

$$M_1 = \partial_\zeta \gamma, \quad M_2 = \frac{\partial_\zeta R}{R}, \quad M_3 = \frac{\partial_\zeta \mathcal{E}}{\mathcal{E} + \overline{\mathcal{E}}}, \quad M_4 = \frac{\partial_\zeta \overline{\mathcal{E}}}{\mathcal{E} + \overline{\mathcal{E}}},$$
$$M_1^* = \partial_{\bar\zeta} \gamma, \quad M_2^* = \frac{\partial_{\bar\zeta} R}{R}, \quad M_3^* = \frac{\partial_{\bar\zeta} \mathcal{E}}{\mathcal{E} + \overline{\mathcal{E}}}, \quad M_4^* = \frac{\partial_{\bar\zeta} \overline{\mathcal{E}}}{\mathcal{E} + \overline{\mathcal{E}}},$$

$$\gamma_{123} = \frac{M_3^* - M_4^*}{2\sqrt{2V}}, \qquad \gamma_{124} = \frac{M_4 - M_3}{2\sqrt{2V}},$$
$$\gamma_{131} = \frac{M_2^* - 2M_4^*}{2\sqrt{2V}}, \qquad \gamma_{141} = \frac{M_2 - 2M_3}{2\sqrt{2V}},$$
$$\gamma_{132} = -\frac{M_2^*}{2\sqrt{2V}}, \qquad \gamma_{142} = -\frac{M_2}{2\sqrt{2V}},$$
$$\gamma_{231} = -\frac{M_2^*}{2\sqrt{2V}}, \qquad \gamma_{241} = -\frac{M_2}{2\sqrt{2V}},$$
$$\ldots = \frac{M_2^* - 2M_3^*}{\ldots}, \qquad \ldots = \frac{M_2 - 2M_4}{\ldots}$$

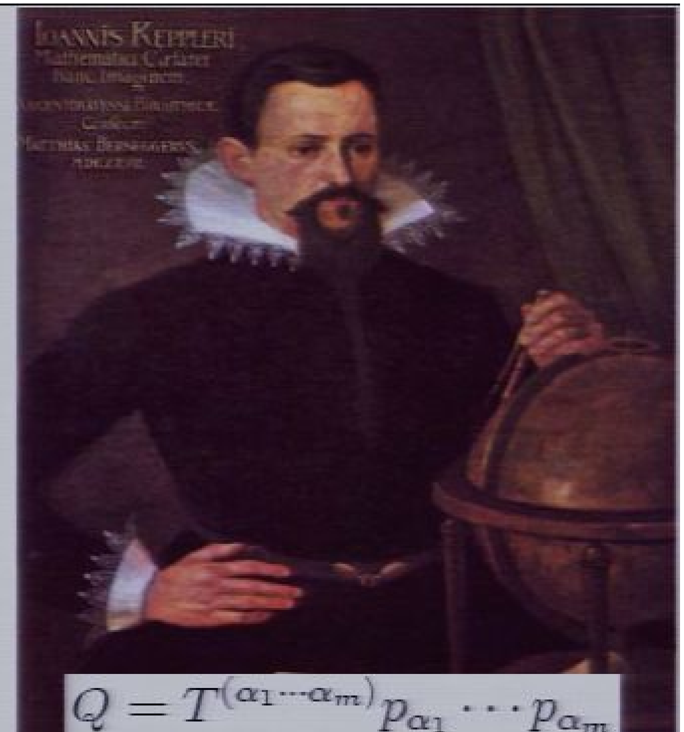
Field EQ

$$M_{1,\zeta} = -\frac{1}{2}(M_3 M_4^* + M_4 M_3^*),$$
$$M_{2,\zeta} = -M_2 M_2^*,$$
$$M_{3,\zeta} = -\left(\frac{1}{2}(M_2 M_3^* + M_3 M_2^*) - M_3 M_3^* + M_3 M_4^*\right),$$
$$M_{4,\zeta} = -\left(\frac{1}{2}(M_2 M_4^* + M_4 M_2^*) + M_4 M_3^* - M_4 M_4^*\right),$$
$$M_{2,\bar\zeta} = -M_2^2 + 2(M_1 M_2 - M_3 M_4).$$

Tycho's Tetrad

Petrov Orientator

The Greats of Planetary Astronomy in a highly twisted spacetime



$$Q = T^{(\alpha_1 \cdots \alpha_m)} p_{\alpha_1} \cdots p_{\alpha_m}$$

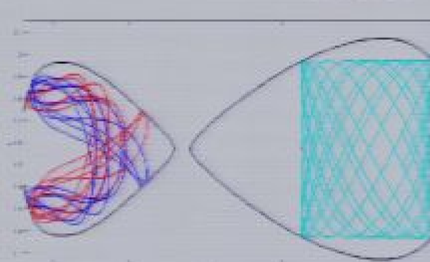
$$k = \frac{1}{\sqrt{2}}(E_1 + E_2) \qquad l = \frac{1}{\sqrt{2}}(E_1 - E_2)$$

$$m = \frac{1}{\sqrt{2}}(E_3 - iE_4)$$

Kepler's Killing Tensors

Newton's

Newman-Penrose



Direct Non-perturbative Algebraic Approach

SAV Field Eqs.

$R_{ab}=0$ *Specifies some but not all derivatives of* γ_{abc}

Killing Eqs. (KE)

$T_{(a_1\cdots a_m,b)} = m\,\eta^{cd}\gamma_{c(a_1 a_2}T_{a_3\cdots a_m b)d}$

Specifies some but not all derivatives of T

Integrability conditions

Conditions for existence of all the Killing Tensor comp.
Massive overdetermined Linear System for T and derivatives not fixed by KE (34 Unknowns)
Coefficients polynomial in Field Variables and derivatives

Conditions on Coefs. for solution
=> Conditions on Spacetime

Solution
Explicit Construction of Some Comps. of T

Transverse Tetrad. Solution Generation Tech.. Ordering of eq., choice of unknowns. Newman Penrose.
Pattern recognition. SAV, KE. Retain implicit coordinate freedom. Local manifold test. Ernst Potential

SAV Metrics

$R_{ij} = 0$ Ernst Eq. $\Re(\mathcal{E})\,\overline{\nabla}^2\mathcal{E} = \overline{\nabla}\mathcal{E}\cdot\overline{\nabla}\mathcal{E}$

Killing vectors ∂_t and ∂_ϕ Metric $ds^2 = e^{-2\psi}[e^{2\gamma}(d\rho^2 + dz^2) + R^2 d\phi^2] - e^{2\psi}(dt - \omega d\phi)^2$

Type D (4 Parameters)	Type I (Bi-infinite series of Parameters)		
		Mass moments	Current Moments
Second Order Killing Tensors (Kerr) Expressed in terms of Principle Null Tetrad	EVEN Parity	Schwarzschild Equatorial Symmetry Zipoy- Voorhees	Kerr Manko Novikov Complex Ernst Pot
Some type D Metrics do not admit a 2nd order KT eg C Metric	ODD Parity	Weyl Class Real Ernst Pot. Fewer K. T. and metric components to consider	*BIG TROUBLE*

The 2 manifold checker

Back in the year 1889 ... (Koenigs)
1) Any 2 Manifold with #killing vectors >1 is a space of constant curvature (cc)
2) More than 3 Killing tensors => cc
3) 3 Killing tensors => space of revolution.
4 Explicit types given.
Interpreted the second invariant as the Hamiltonian constant of a related metric
Same picture proposed by Moser

2005
Rediscovered by Kalnins, Kress, Miller
Super-integrable systems + Separation of HJE + Wave Equation
Manifolds with 4 th order Killing tensors have 6 generators. But we can't solve the equations to see what they look like

Why is it so difficult to check whether a 2 manifold is integrable or not ?
One reason is coordinate freedom (Hietarinta)

Direct methods for constructing Q

Hamiltonian Constraint

$$p_\rho^2 + p_z^2 = V$$

Phase Space Angle

$$\begin{aligned} p_\rho &= \sqrt{V}\cos\theta \\ p_z &= \sqrt{V}\sin\theta \end{aligned}$$

2 Metric

$${}_2g_{ij} = 2V(\rho,z)\delta_{ij}$$

Guess at the invariant

$$\mathrm{Q}(\mathrm{p}_\rho, p_z, \rho, z) = \tfrac{1}{2}Q_0 + \textstyle\sum_{n=1}^{N} Q_{C_n}\cos(n\theta) + Q_{S_n}\sin(n\theta)$$

Compute conditions on V, Q_o, Q_c, and Q_s for guess to work

$$\frac{dQ}{d\lambda} = 0$$

Complex functions $Q_n = Q_{C_n} - iQ_{S_n}, \quad \zeta = \tfrac{1}{2}(\rho + iz)$

A little bit of Algebra

if n=N, N-1

$$\partial_\zeta(Q_n V^{-n/2}) = 0$$

Unknown Analytic function t

Generates a coordinate transformation which maintains the structure of 2 metric

if 0<n<N recursion relationships for lower order terms

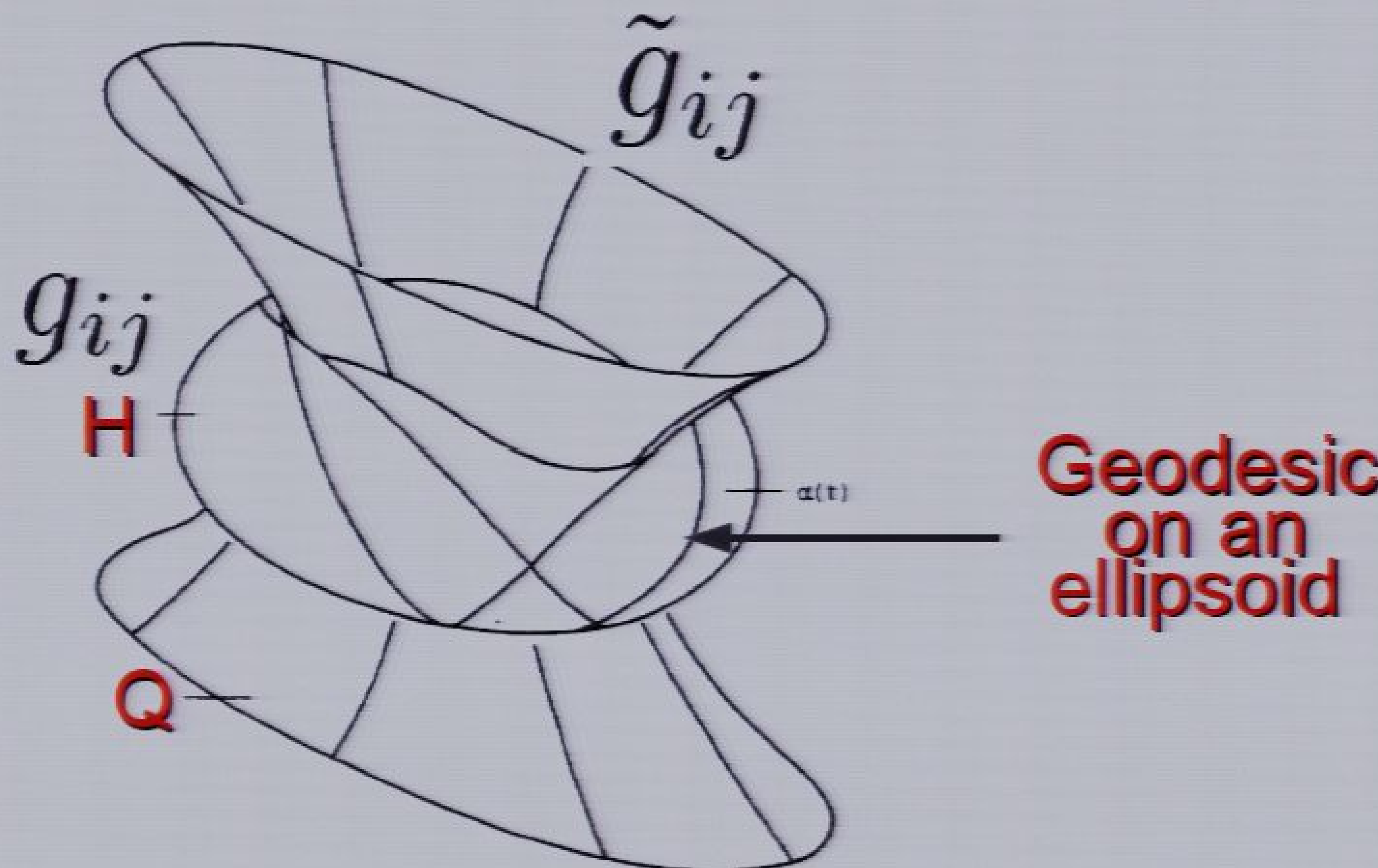
$$\partial_{\bar\zeta}(Q_{n+1}V^{(n+1)/2}) = -V^n\partial_\zeta(Q_{n-1}V^{-(n-1)/2})$$

, Q_o, is real

Even and odd terms

Geometric Interpretation

(Koenigs, Arnold, Knorrer)



(Horst Knorrer)

Make the Following Ansatz

$$T_{3444} = P_{<1:4>} e^{2\gamma - 2\psi}, \qquad T_{3334} = P_{<1:3>} e^{2\gamma - 2\psi},$$

$$T_{3333} = P_{<5:3>} e^{4\gamma - 4\psi}, \qquad T_{4444} = P_{<5:4>} e^{4\gamma - 4\psi},$$

$$T_{[illegible]44} = P_{<-1:0>}.$$

$$T_{1233} = \frac{1}{12} e^{[illegible]} \left(-4e^{2\psi} P_{<1:3>} + e^{4\psi} \left(-3\omega^2 P_{[illegible]} + 6\omega P_{[illegible]} + 6P_{<4:3>}\right) + [illegible] P_{<2:3>}\right),$$

$$T_{1244} = \frac{1}{12} e^{2\gamma - 4\psi} \left(-4e^{2\psi} P_{<1:4>} + e^{4\psi} \left(-3\omega^2 P_{<2:4>} + 6\omega P_{<3:4>} + 6P_{<4:4>}\right) + 3R^2 P_{<2:4>}\right),$$

$$T_{1133} = [illegible] \left(e^{4\psi}\left([illegible] P_{<2:3>} - 2\omega P_{<3:3>} - 2P_{<4:3>}\right) + 2Re^{2\psi} \left(\omega P_{<2:3>} - P_{<3:3>}\right) + R^2 P_{<2:3>}\right),$$

$$T_{1144} = -\frac{1}{4} e^{2\gamma - 4\psi} \left(e^{4\psi} \left(\omega^2 P_{<2:4>} - 2\omega P_{<3:4>} - 2P_{<4:4>}\right) + 2Re^{2\psi} \left(\omega P_{<2:4>} - P_{<3:4>}\right) + R^2 P_{<2:4>}\right),$$

$$T_{2233} = -\frac{1}{4} e^{2\gamma - 4\psi} \left(e^{4\psi} \left(\omega^2 P_{<2:3>} - 2\omega P_{<3:3>} - 2P_{<4:3>}\right) - 2Re^{2\psi} \left(\omega P_{<2:3>} - P_{<3:3>}\right) + R^2 P_{<2:3>}\right),$$

$$T_{2244} = -\frac{1}{4} e^{2\gamma - 4\psi} \left(e^{4\psi} \left(\omega^2 P_{<2:4>} - 2\omega P_{<3:4>} - 2P_{<4:4>}\right) - 2Re^{2\psi} \left(\omega P_{<2:4>} - P_{<3:4>}\right) + R^2 P_{<2:4>}\right),$$

$$T_{1234} = \frac{1}{8} e^{-2\psi} \left(-4e^{2\psi} P_{<-1:0>} + e^{4\psi} \left(\omega \left(2P_{<-3:0>} - \omega P_{[illegible]}\right) + 2P_{<-4:0>}\right) + R^2 P_{<-2:0>}\right),$$

And Then ....

$$T_{1134} = \frac{1}{8} e^{-2\psi} \left(e^{4\psi} \left(\omega \left(2P_{<-3:0>} [illegible]\right) + 2P_{[illegible]}\right) [illegible] - \omega P_{<-2:0>}\right) + R^2 \left(-P_{<-2:0>}\right)\right),$$

$$T_{2234} = \frac{1}{8} e^{-2\psi} \left(e^{4\psi} \left(\omega \left(2P_{<-3:0>} - \omega P_{<-2:0>}\right) + 2P_{<-4:0>}\right) - 2Re^{2\psi} \left(P_{<-3:0>} - \omega P_{<-2:0>}\right) + R^2 \left(-P_{<-2:0>}\right)\right).$$

$$T_{1111} = \frac{9}{64} \left[e^{-4\psi} P_{<1:0>} \left(R + e^{2\psi}\omega\right)^4 + 4e^{-2\psi} P_{<2:0>} \left(R + e^{2\psi}\omega\right)^3 + 8P_{<3:0>} \left(R + e^{2\psi}\omega\right)^2\right] + \frac{9}{16} \left[e^{2\psi} P_{<4:0>} \left(R + e^{2\psi}\omega\right) + e^{4\psi} P_{<5:0>}\right],$$

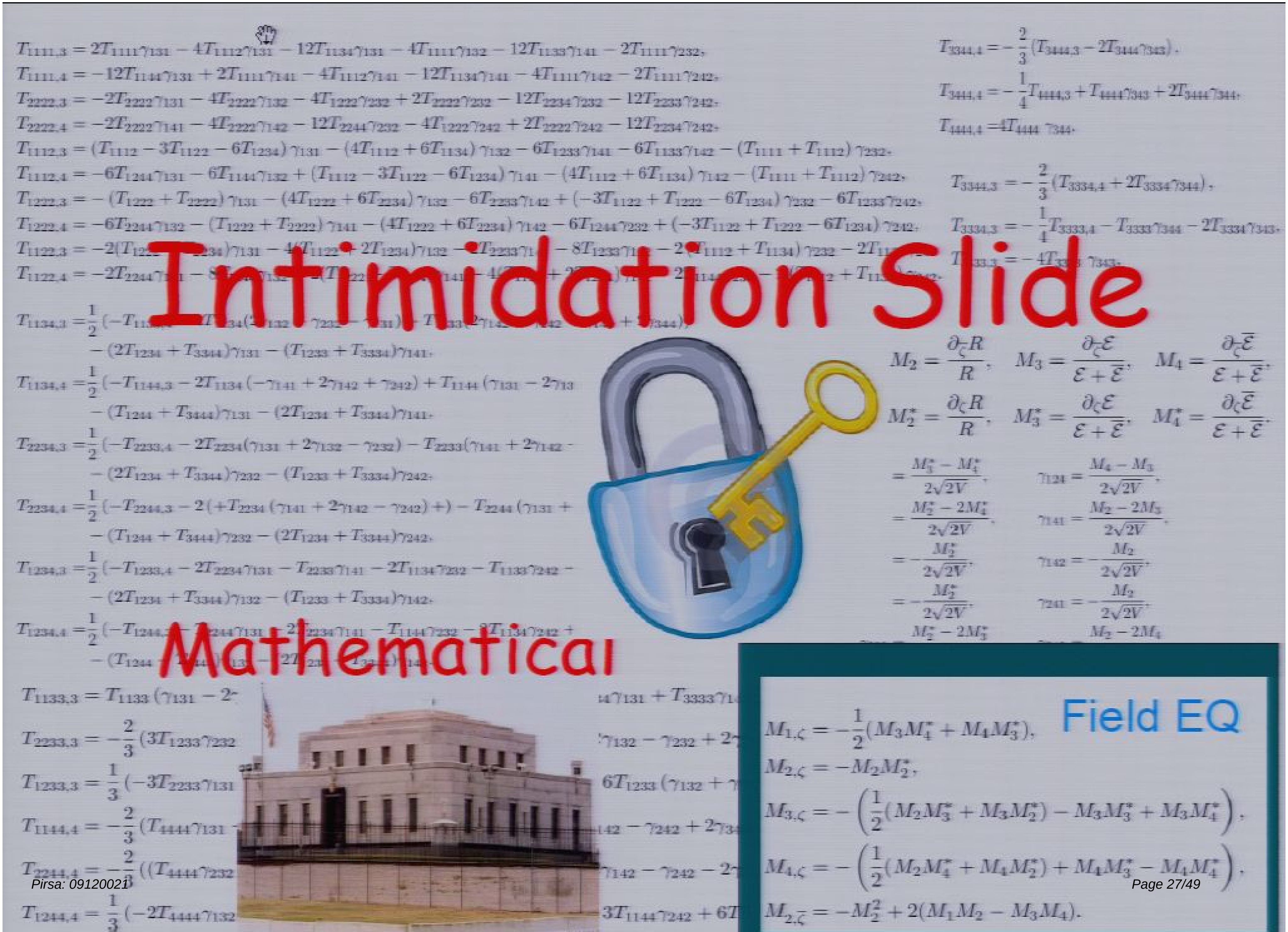
$$T_{2222} = \frac{9}{64} \left[e^{-4\psi} P_{<1:0>} \left(R - e^{2\psi}\omega\right)^4 + 4e^{-2\psi} P_{<2:0>} \left(e^{2\psi}\omega - R\right)^3 + 8P_{<3:0>} \left(R - e^{2\psi}\omega\right)^2\right] + \frac{9}{16} \left[e^{2\psi} P_{<4:0>} \left(e^{2\psi}\omega - R\right) + e^{4\psi} P_{<5:0>}\right],$$

$$T_{1112} = \left[\frac{9}{64} e^{-4\psi} P_{<1:0>} \left(e^{4\psi}\omega^2 - R^2\right) + \frac{3}{8} e^{-2\psi} P_{<-2:0>} - \frac{9}{32} e^{-2\psi} P_{<2:0>} \left(R - 2e^{2\psi}\omega\right)\right] \left(R + e^{2\psi}\omega\right)^2 + \left(\frac{9}{8} e^{2\psi}\omega P_{<3:0>} - \frac{3}{4} P_{<-3:0>}\right) \left(R + e^{2\psi}\omega\right) - \frac{3}{4} e^{2\psi} P_{<-4:0>} + \frac{9}{32} e^{2\psi} P_{<4:0>} \left(R + 2e^{2\psi}\omega\right) + \frac{9}{16} e^{4\psi} P_{<5:0>},$$

Voila !!!

$$T_{1222} = \left[\frac{9}{64} e^{-4\psi} P_{<1:0>} \left(e^{4\psi}\omega^2 - R^2\right) + \frac{3}{8} e^{-2\psi} P_{<-2:0>} + \frac{9}{32} e^{-2\psi} P_{<2:0>} \left(R + 2e^{2\psi}\omega\right)\right] \left(R - e^{2\psi}\omega\right)^2 + \left(\frac{9}{8} e^{2\psi}\omega P_{<3:0>} - \frac{3}{4} P_{<-3:0>}\right) \left(e^{2\psi}\omega - R\right) - \frac{3}{4} e^{2\psi} P_{<-4:0>} + \frac{9}{32} e^{2\psi} P_{<4:0>} \left(2e^{2\psi}\omega - R\right) + \frac{9}{16} e^{4\psi} P_{<5:0>},$$

$$T_{1122} = P_{<-1:0>} + \frac{9}{64} e^{-4\psi} P_{<1:0>} \left(R^2 - e^{4\psi}\omega^2\right)^2 + \left(\frac{1}{2} e^{-2\psi} P_{<-2:0>} + \frac{9}{16} \omega P_{<2:0>}\right) \left(e^{4\psi}\omega^2 - R^2\right) - e^{2\psi}\omega P_{<-3:0>} - \frac{3}{8} P_{<3:0>} \left(R^2 - 3e^{4\psi}\omega^2\right) - e^{2\psi} P_{<-4:0>} + \frac{9}{16} e^{4\psi}\omega P_{<4:0>} + \frac{9}{16} e^{4\psi} P_{<5:0>}.$$


Intimidation Slide
Mathematical
Field EQ
$M_{1,\zeta} = -\frac{1}{2}(M_3 M_4^* + M_4 M_3^*),$
$M_{2,\zeta} = -M_2 M_2^*,$
$M_{3,\zeta} = -\left(\frac{1}{2}(M_2 M_3^* + M_3 M_2^*) - M_3 M_3^* + M_3 M_4^*\right),$
$M_{4,\zeta} = -\left(\frac{1}{2}(M_2 M_4^* + M_4 M_2^*) + M_4 M_3^* - M_4 M_4^*\right),$
$M_{2,\bar{\zeta}} = -M_2^2 + 2(M_1 M_2 - M_3 M_4).$

Equivalent system to Intimidation Slide

$$P_{<i:3>,\zeta} = -f_{i,\bar{\zeta}} \quad P_{<i:4>,\bar{\zeta}} = -f_{i,\zeta} \quad i \in \{1 \cdots 4\}$$

$$P_{<k,0>,\zeta} = \sum_n C_n fp(i_n, j_n), \quad \times 9 + \text{Complex Conj}$$

$$fp(i,j) = -2(P_{<\text{j}:3>} f_{\text{i}})_{,\bar{\zeta}} + f_i P_{<\text{j}:3>,\bar{\zeta}}$$

$$f_1 = e^{2\gamma - 2\psi} = V, \qquad f_2 = \frac{2e^{2\gamma}}{3R^2},$$

$$f_3 = \frac{2e^{2\gamma}\omega}{3R^2}, \qquad f_4 = \frac{e^{2\gamma}\left(R^2 e^{-4\psi} - \omega^2\right)}{3R^2}.$$

n, i_n and j_n (integers) and C_n (rational)

Analytic solution

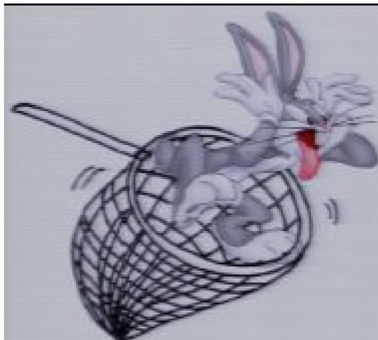
$$P_{<\text{i}:3>}(\rho, z) = \frac{1}{\pi} \int_{-\infty}^{\infty} \int_0^{\pi} f_i(\hat{\rho}, \hat{z}) G(\rho, z, \hat{\rho}, \hat{z})\, d\hat{\rho}\, d\hat{z},$$

$$P_{<\text{k}:0>}(\rho, z) = \sum_n C_n \left(\frac{1}{\pi} \int_{-\infty}^{\infty} \int_0^{\pi} d\tilde{\rho}\, d\tilde{z}\right) \left(\frac{1}{\pi} \int_{-\infty}^{\infty} \int_0^{\pi} d\hat{\rho}\, d\hat{z}\right) f_{i_n}(\tilde{\rho}, \tilde{z}) f_{j_n}(\hat{\rho}, \hat{z}) K(\rho, z, \tilde{\rho}, \tilde{z}, \hat{\rho}, \hat{z})$$

Greens Functions

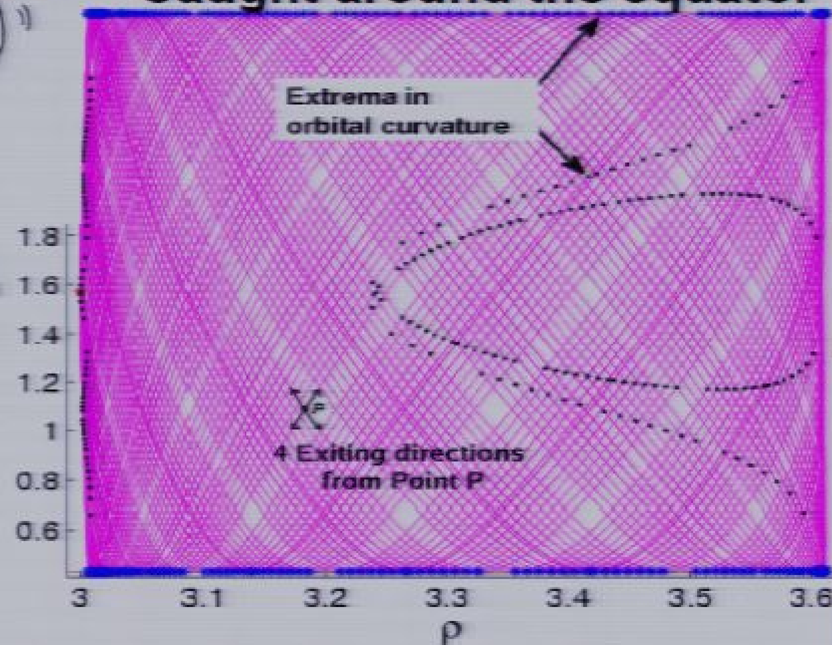
$$G(\rho, z, \hat{\rho}, \hat{z}) = -\pi \delta(z - \hat{z}) \delta(\rho - \hat{\rho}) + \left(\frac{1}{\sinh(2\hat{\bar{\zeta}} - 2\bar{\zeta})}\right)^2.$$

$$K(\rho, z, \tilde{\rho}, \tilde{z}, \hat{\rho}, \hat{z}) = 2G(\tilde{\rho}, \tilde{z}, \hat{\rho}, \hat{z}) G(\rho, z, \tilde{\rho}, \tilde{z}) + \frac{\partial}{\partial \tilde{\bar{\zeta}}} (G(\tilde{\rho}, \tilde{z}, \hat{\rho}, \hat{z})) \left(\frac{1}{1 - e^{4(\tilde{\bar{\zeta}} - \bar{\zeta})}}\right)$$



Zipoy-Voorhees

Caught around the equator



Equatorial Symmetry about y=0

$$ds^2 = e^{-2\psi}\left[e^{2\gamma}(d\rho^2+dz^2)+R^2d\phi^2\right]-e^{2\psi}dt^2$$

$$e^{2\psi}=\left(\frac{x-1}{x+1}\right)^{\delta} \qquad e^{2\gamma}=\frac{(x^2-1)^{\delta^2}}{(x^2-y^2)^{\delta^2-1}}$$

$$R=\sqrt{(x^2-1)(1-y^2)}$$

$$x=\cosh\rho \qquad y=\cos z$$

Naked Singularity
with positive mass
Asymptotically Flat

Guess solution to 4th rank killing equations On equator

$$\{1,2\}\in\{\partial_t,\partial_\phi\}$$
$$\{3,4\}\in\{\partial_z,\partial_\rho\}$$

$$T_{4444}=T_{3333}=T_{3344}=-T_{3444}=-T_{3334}=f_1(x)$$

$$T_{1133}=T_{1144}=T_{2233}=T_{2244}=T_{1234}=-f_2(x)$$

$$T_{1233}=T_{1244}=T_{1134}=T_{2234}=f_2(x)$$

$$T_{1111}=T_{2222}=T_{1122}=-T_{1112}=-T_{1222}=f_3(x)$$

Asymptotic behavior

Point Mass Schwarzschild	(D)	$\delta=1$
Prolate (Field of rod)	(I)	$0<\delta<1$
Oblate (Field of disk)	(I)	$\delta>1$
Flat	(D)	$\delta=0$

Equivalent system to Intimidation Slide

$$P_{<i:3>,\zeta} = -f_{i,\bar{\zeta}} \quad P_{<i:4>,\bar{\zeta}} = -f_{i,\zeta} \quad i \in \{1 \cdots 4\}$$

$$P_{<k,0>,\zeta} = \sum_n C_n fp(i_n, j_n), \quad \times 9 + \text{Complex Conj}$$

$$fp(i,j) = -2(P_{<\mathrm{j:3}>} f_{\mathrm{i}})_{,\bar{\zeta}} + f_i P_{<\mathrm{j:3}>,\bar{\zeta}}$$

$$f_1 = e^{2\gamma - 2\psi} = V, \qquad f_2 = \frac{2e^{2\gamma}}{3R^2},$$

$$f_3 = \frac{2e^{2\gamma}\omega}{3R^2}, \qquad f_4 = \frac{e^{2\gamma}\left(R^2 e^{-4\psi} - \omega^2\right)}{3R^2}.$$

n, i_n and j_n (integers) and C_n (rational)

Analytic solution

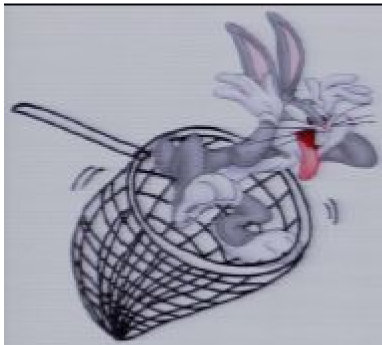
$$P_{<\mathrm{i:3}>}(\rho, z) = \frac{1}{\pi}\int_{-\infty}^{\infty}\int_0^{\pi} f_i(\hat{\rho}, \hat{z}) G(\rho, z, \hat{\rho}, \hat{z})\, d\hat{\rho}\, d\hat{z},$$

$$P_{<\mathrm{k:0}>}(\rho, z) = \sum_n C_n \left(\frac{1}{\pi}\int_{-\infty}^{\infty}\int_0^{\pi} d\tilde{\rho}\, d\tilde{z}\right)\left(\frac{1}{\pi}\int_{-\infty}^{\infty}\int_0^{\pi} d\hat{\rho}\, d\hat{z}\right) f_{i_n}(\tilde{\rho}, \tilde{z}) f_{j_n}(\hat{\rho}, \hat{z}) K(\rho, z, \tilde{\rho}, \tilde{z}, \hat{\rho}, \hat{z})$$

Greens Functions

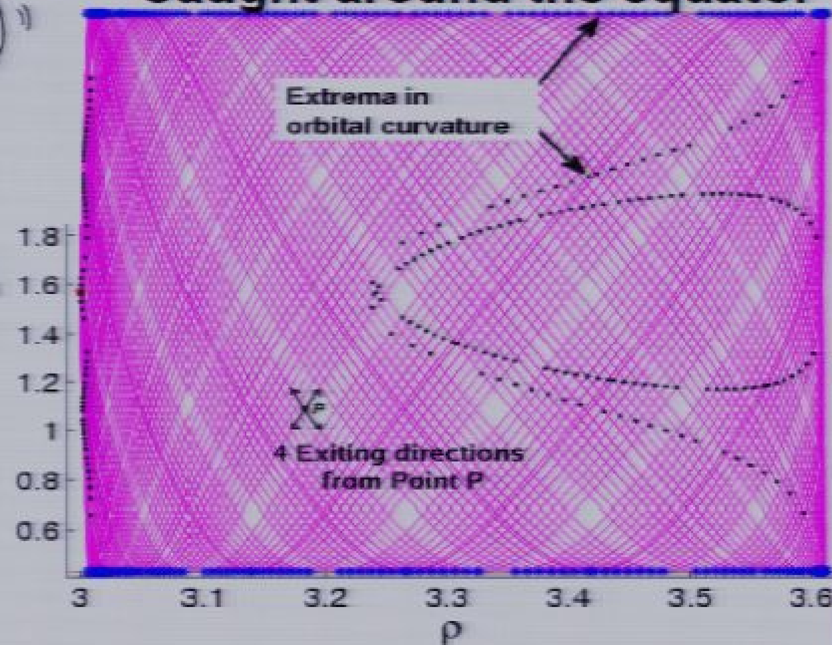
$$G(\rho, z, \hat{\rho}, \hat{z}) = -\pi\delta(z - \hat{z})\delta(\rho - \hat{\rho}) + \left(\frac{1}{\sinh(2\hat{\bar{\zeta}} - 2\bar{\zeta})}\right)^2.$$

$$K(\rho, z, \tilde{\rho}, \tilde{z}, \hat{\rho}, \hat{z}) = 2G(\tilde{\rho}, \tilde{z}, \hat{\rho}, \hat{z}) G(\rho, z, \tilde{\rho}, \tilde{z}) + \frac{\partial}{\partial \bar{\bar{\zeta}}}\left(G(\tilde{\rho}, \tilde{z}, \hat{\rho}, \hat{z})\right)\left(\frac{1}{1 - e^{4(\bar{\bar{\zeta}} - \bar{\zeta})}}\right)$$



Zipoy-Voorhees

Caught around the equator



Equatorial Symmetry about y=0

$$ds^2 = e^{-2\psi}\left[e^{2\gamma}(d\rho^2 + dz^2) + R^2 d\phi^2\right] - e^{2\psi}dt^2$$

$$e^{2\psi} = \left(\frac{x-1}{x+1}\right)^{\delta} \qquad e^{2\gamma} = \frac{(x^2-1)^{\delta^2}}{(x^2-y^2)^{\delta^2-1}}$$

$$R = \sqrt{(x^2-1)(1-y^2)}$$

$$x = \cosh\rho \qquad y = \cos z$$

Naked Singularity with positive mass Asymptotically Flat

Guess solution to 4th rank killing equations On equator

$$\{1,2\} \in \{\partial_t, \partial_\phi\}$$
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Asymptotic behavior

Point Mass Schwarzschild	(D)	$\delta = 1$
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Equivalent system to Intimidation Slide

$$P_{<i:3>,\zeta} = -f_{i,\bar{\zeta}} \quad P_{<i:4>,\bar{\zeta}} = -f_{i,\zeta} \quad i \in \{1 \cdots 4\}$$

$$P_{<k,0>,\zeta} = \sum_n C_n fp(i_n, j_n), \qquad \times 9 + \text{Complex Conj}$$

$$fp(i,j) = -2(P_{<\mathrm{j}:3>} f_{\mathrm{i}})_{,\bar{\zeta}} + f_i P_{<\mathrm{j}:3>,\bar{\zeta}}$$

$$f_1 = e^{2\gamma - 2\psi} = V, \qquad f_2 = \frac{2e^{2\gamma}}{3R^2},$$

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n, i_n and j_n (integers) and C_n (rational)

Analytic solution

$$P_{<\mathrm{i}:3>}(\rho, z) = \frac{1}{\pi}\int_{-\infty}^{\infty}\int_0^{\pi} f_i(\hat{\rho}, \hat{z}) G(\rho, z, \hat{\rho}, \hat{z})\, d\hat{\rho}\, d\hat{z},$$

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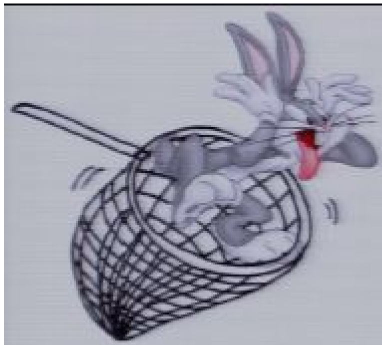
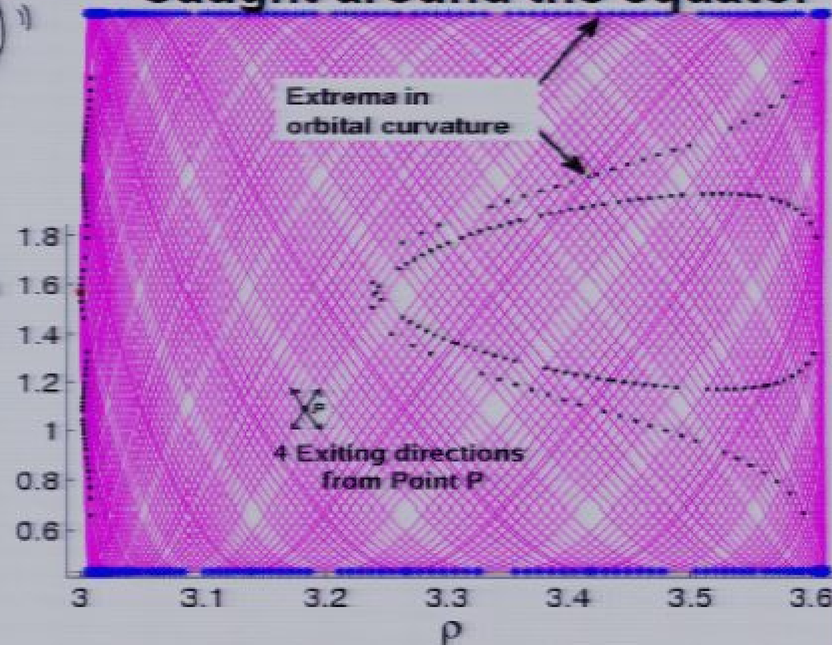
Zipoy-Voorhees



Caught around the equator



Equatorial
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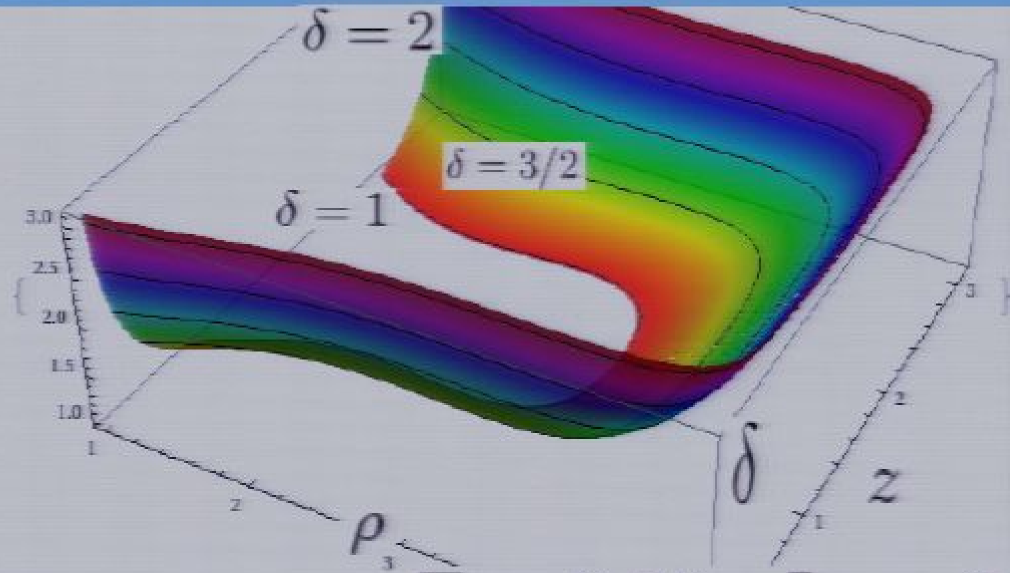

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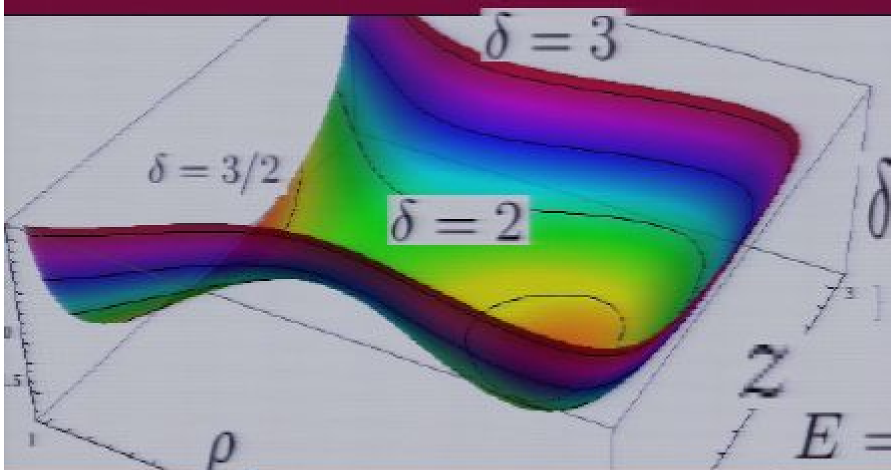

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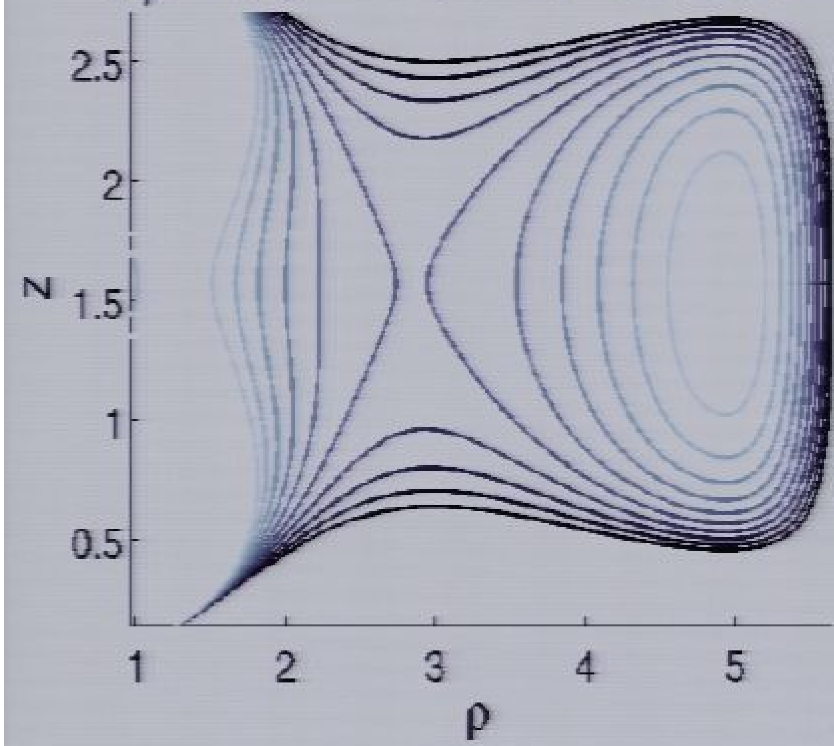
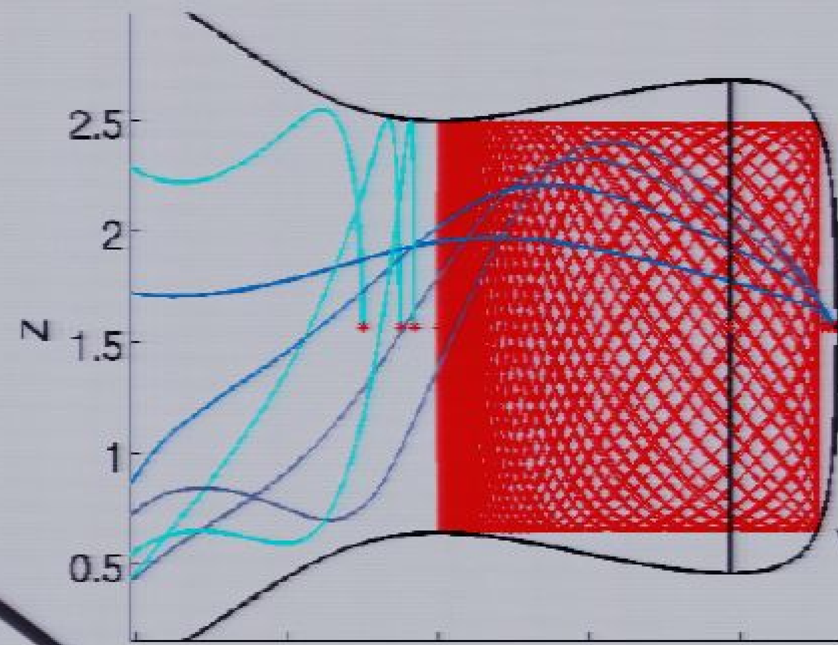
Hamiltonian Constraint sets allowed region for orbit

$$P_\rho^4 \frac{4}{x^4} C3 \left(x^2\right)^{2\delta^2} \left(x^2-1\right)^{2-2\delta^2} - P_\rho^2 \frac{8}{x^2}\left(\frac{x-1}{x+1}\right)^{-2\delta}\left(x^2-1\right)^{-\delta^2}\left(\begin{array}{c} 3C_2 Lz^2 x^2 \left(\frac{x-1}{x+1}\right)^{2\delta}\left(x^2-1\right)^{\delta^2} \\ + C_3\left(x^2-1\right)\left(x^2\right)^{\delta^2}\left(E^2\left(x^2-1\right) - \left(\frac{x-1}{x+1}\right)^{\delta}\left(Lz^2\left(\frac{x-1}{x+1}\right)^{\delta} + \left(x^2-1\right)\mu^2\right)\right)\end{array}\right)$$

$$Q = \frac{4\left(x^2\right)^{-2\delta^2}}{\left(x^2-1\right)^2}\left(\frac{x-1}{x+1}\right)^{-4\delta}\left(\begin{array}{c} L_z^4 x^4 \left(\frac{x-1}{x+1}\right)^{4\delta}\left(x^2-1\right)^{2\delta^2} \\ + L_z^2\left(\frac{x-1}{x+1}\right)^{2\delta}\left(6\, C_2\left(x^2\right)^{\delta^2+1}\left(x^2-1\right)^{\delta^2+1}\left(E^2\left(x^2-1\right) - \left(\frac{x-1}{x+1}\right)^{\delta}\left(L_z^2\left(\frac{x-1}{x+1}\right)^{\delta} + \left(x^2-1\right)\mu^2\right)\right)\right) \\ + C_3\left(x^2-1\right)^2\left(x^2\right)^{2\delta^2}\left(e^2\left(x^2-1\right) - \left(\frac{x-1}{x+1}\right)^{\delta}\left(Lz^2\left(\frac{x-1}{x+1}\right)^{\delta} + \left(x^2-1\right)\mu^2\right)\right)^2 \end{array}\right)$$

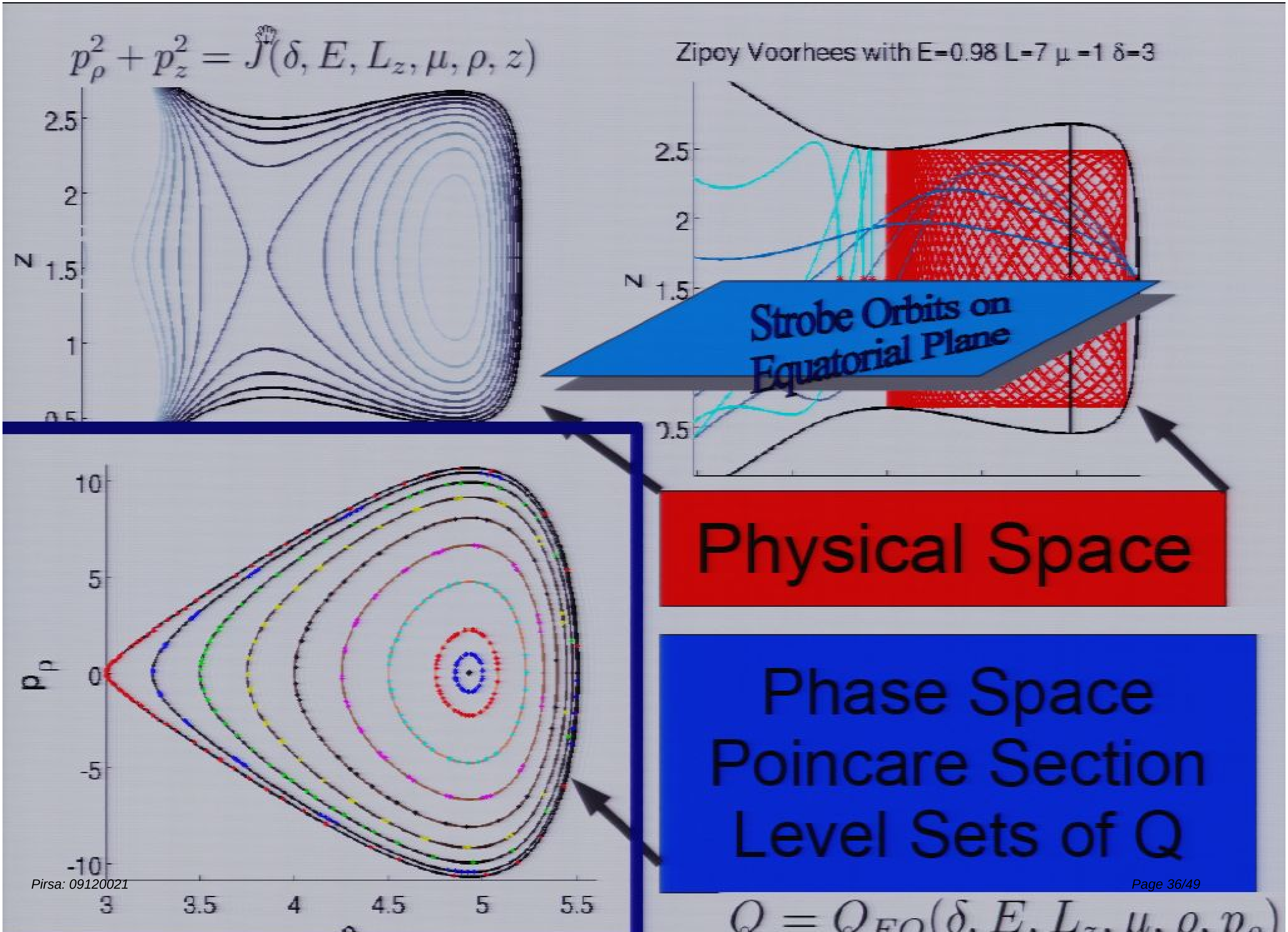
$$p_\rho^2 + p_z^2 = J(\delta, E, L_z, \mu, \rho, z)$$





Physical Space

$$E = 0.98,\ L_z = 7 \quad \delta = 3$$



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Zipoy Voorhees with E=0.98 L=7 μ =1 δ=3
Strobe Orbits on Equatorial Plane
Physical Space
Phase Space
Poincare Section
Level Sets of Q
$Q = Q_{EQ}(\delta, E, L_z, \mu, \rho, p_\rho)$

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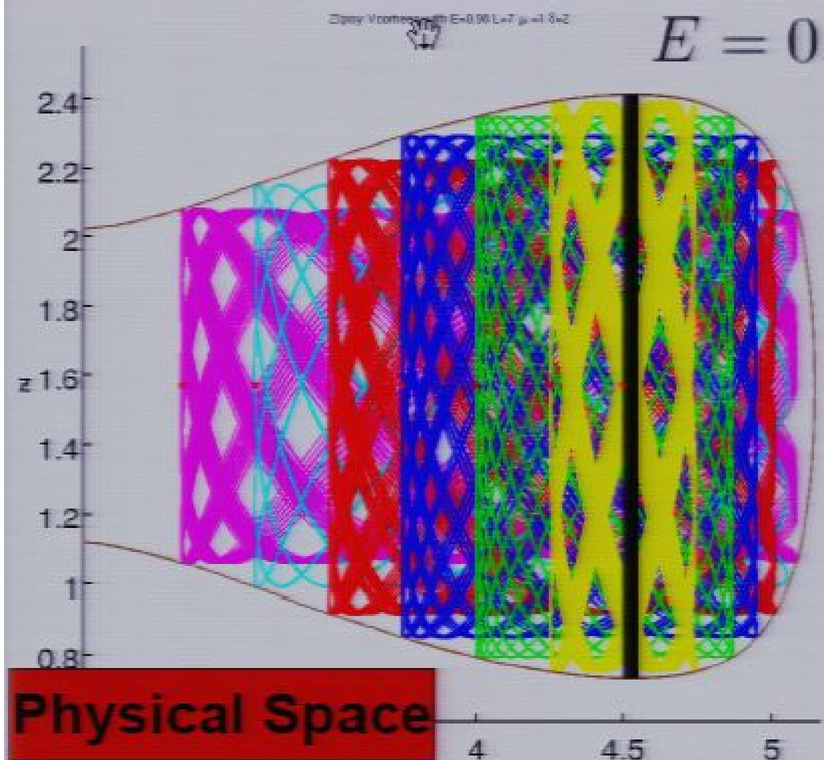

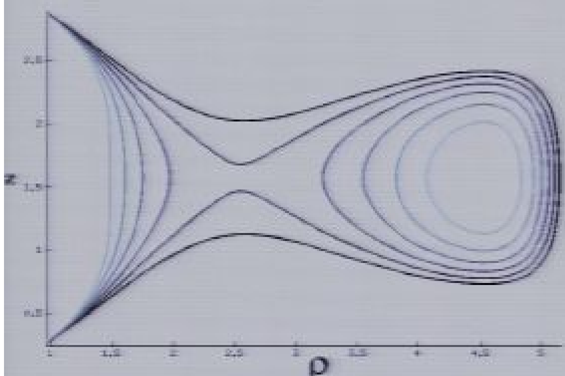

Knowledge of Q on the equatorial plane, allows the radial period of geodesic motion to be determined analytically for any orbit

For a given Q associated with the orbit use :

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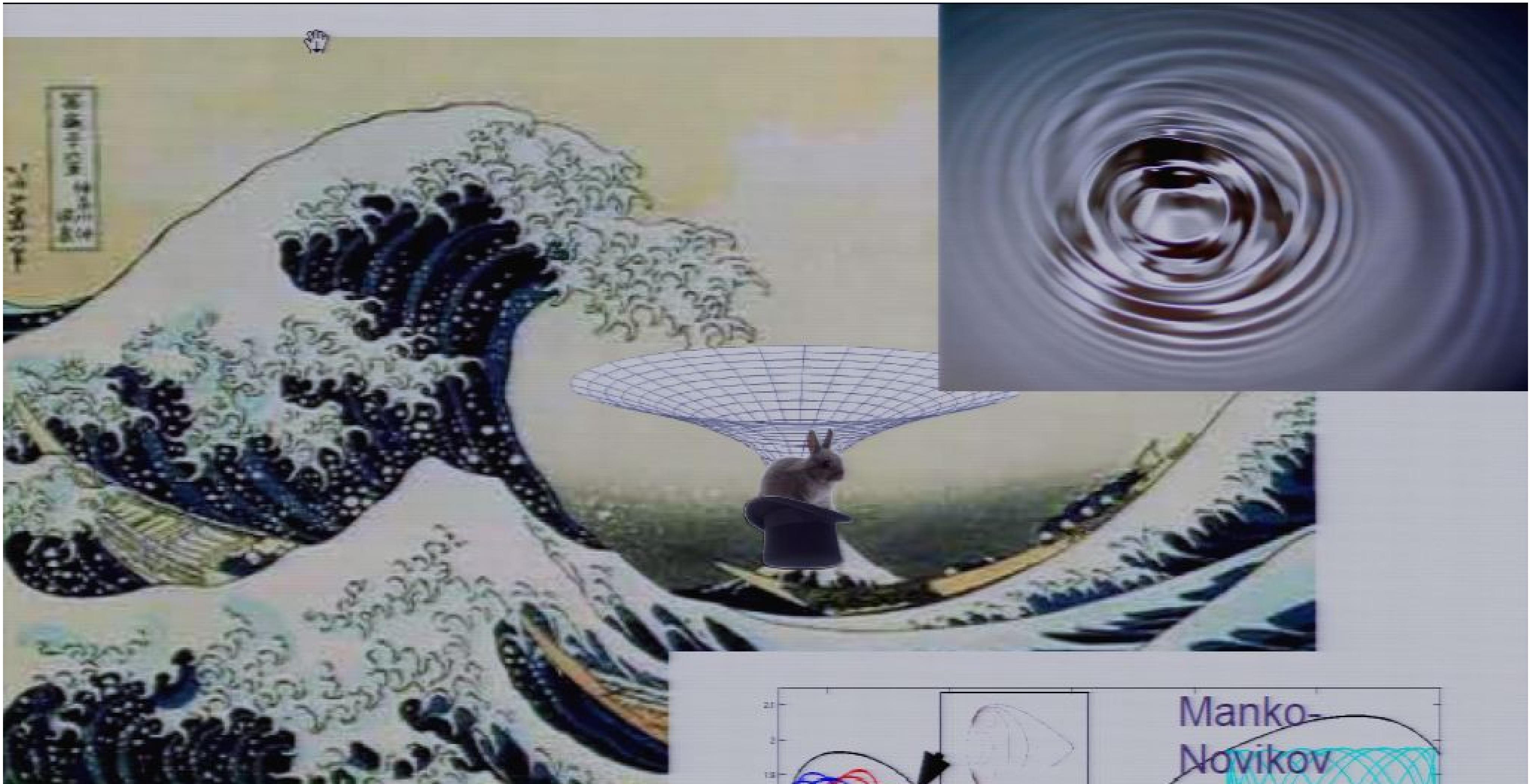
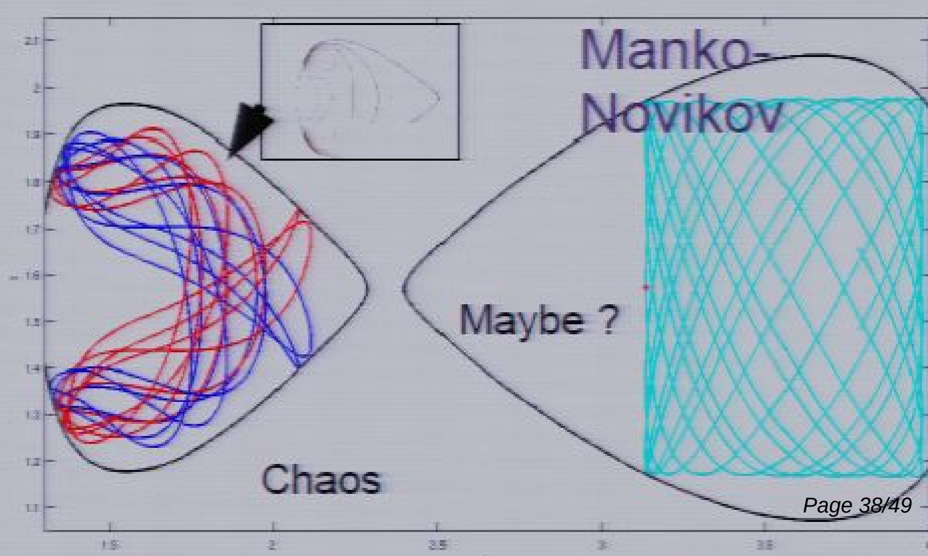
To solve for the momentum p_ρ in terms of ρ

$$\frac{d\rho}{d\tau} = e^{2\psi - 2\gamma} p_\rho$$

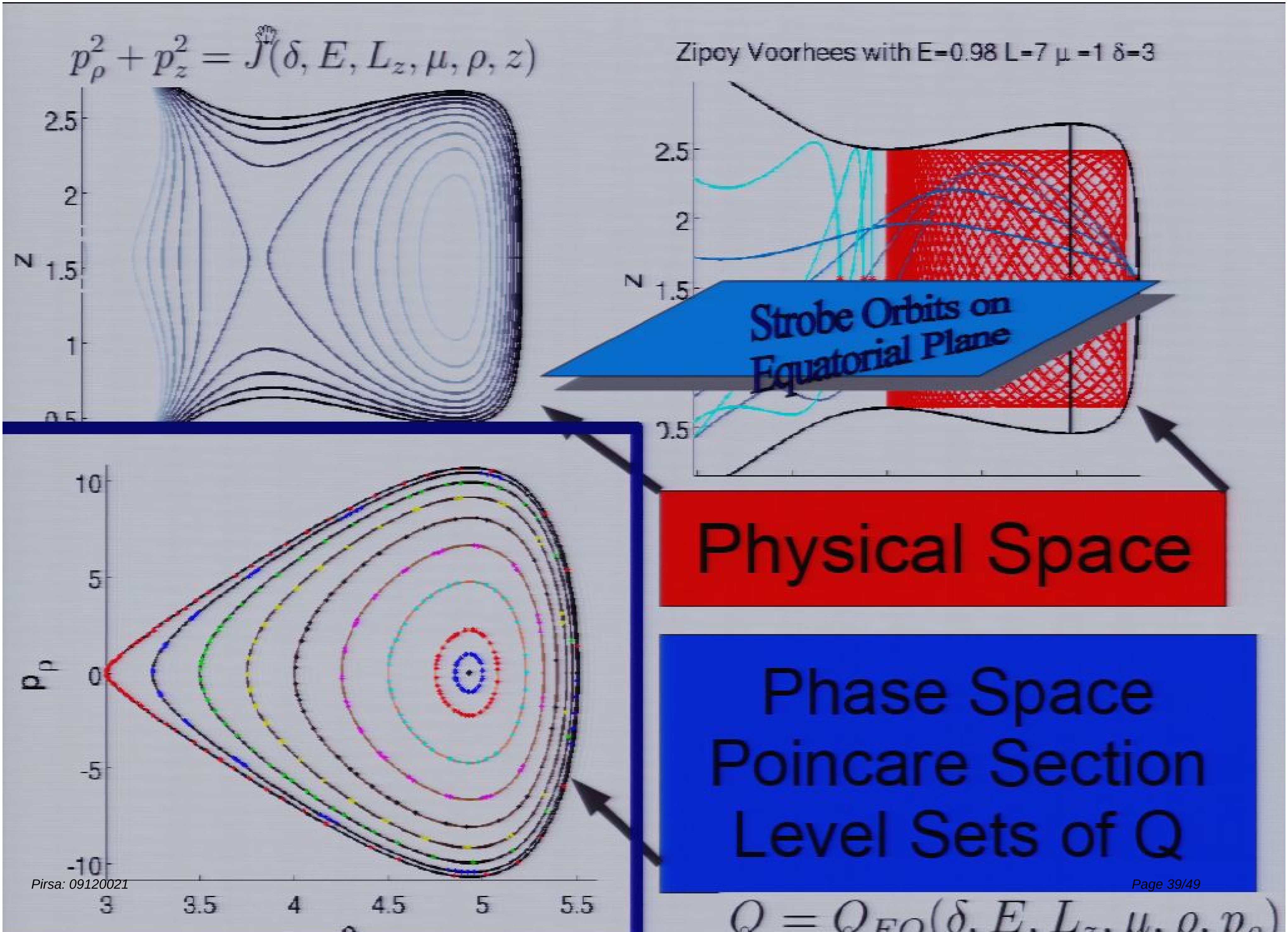
Substitute into one of Hamilton's equations





Spectre Of Chaos Could Swamp Mapping Project

Numerically Observed "chaos"
Gueron and Letelier
Gair, Mandel.



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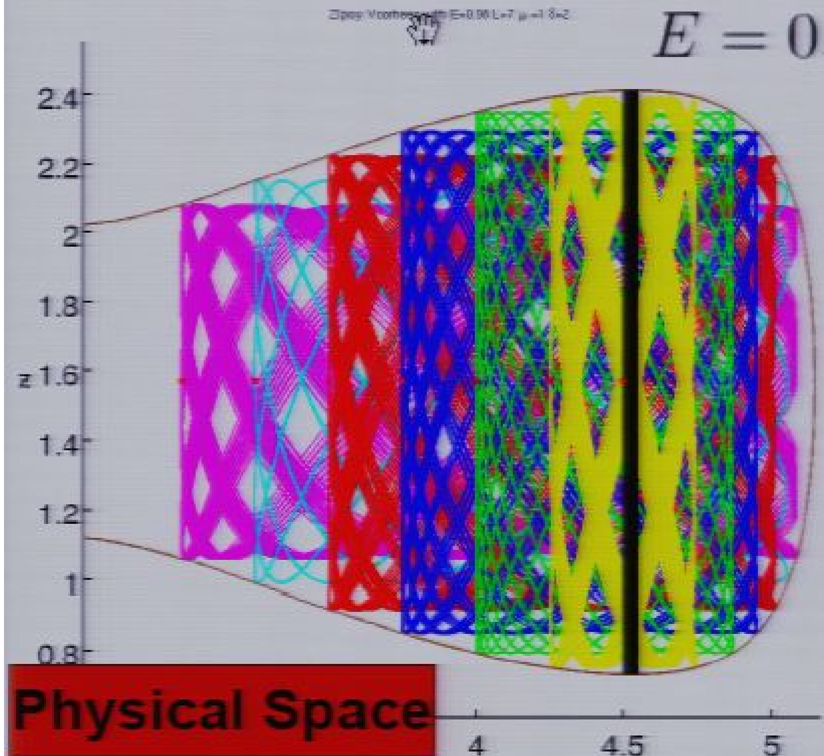

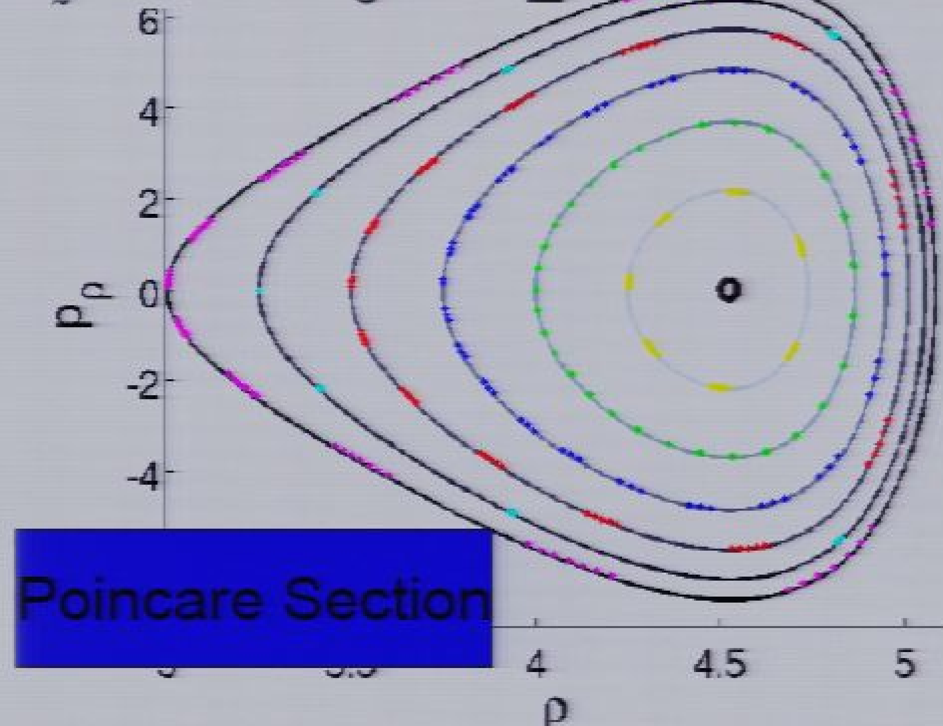

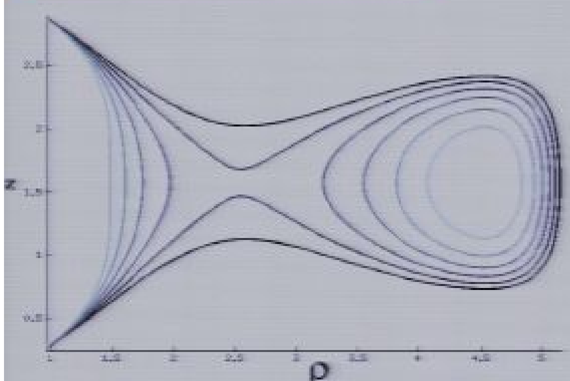

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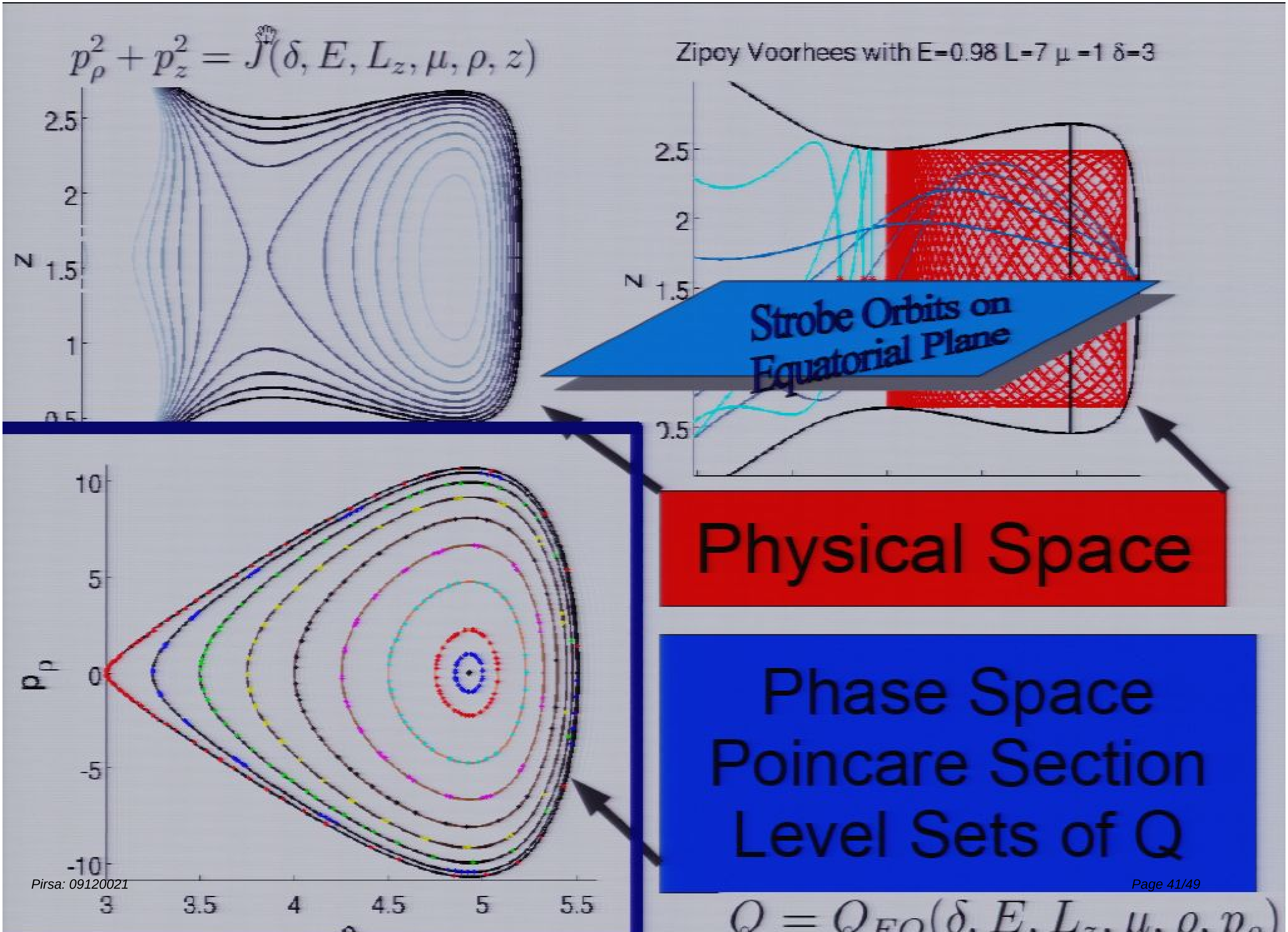
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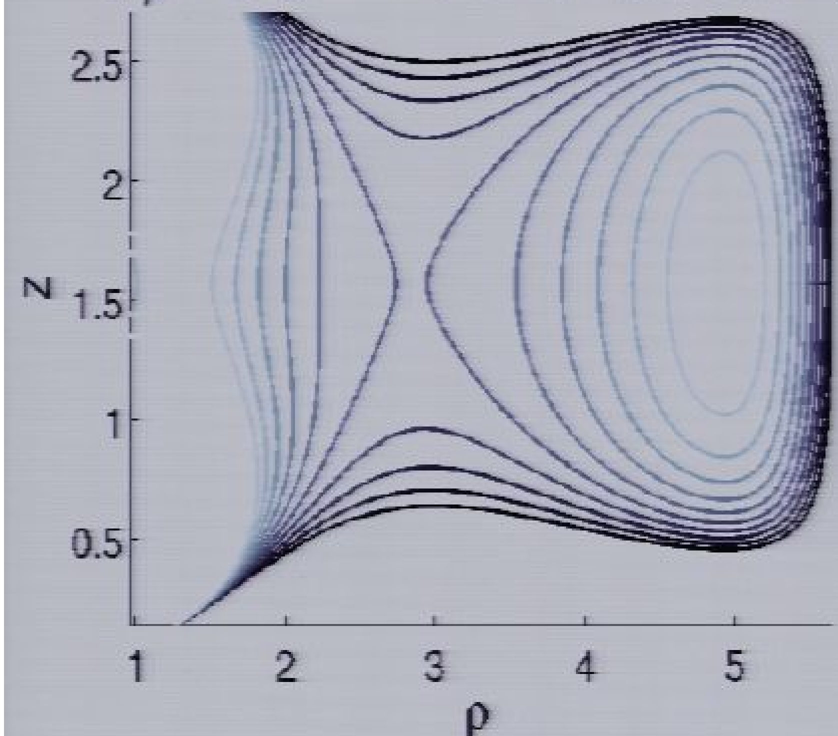

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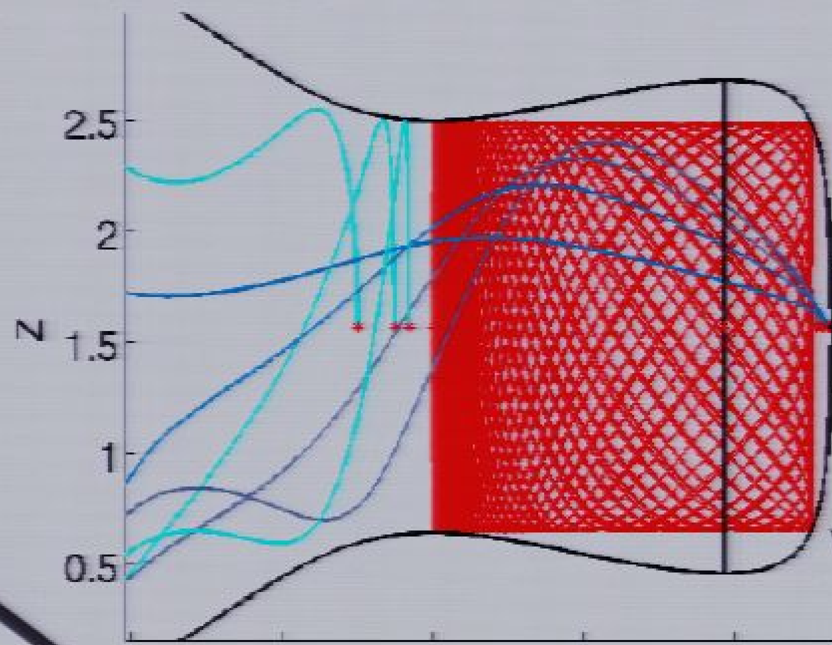

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Full Sets of Constants of motion on equator

Killing Vectors

$p_t = -E$ and $p_\phi - = L_z$

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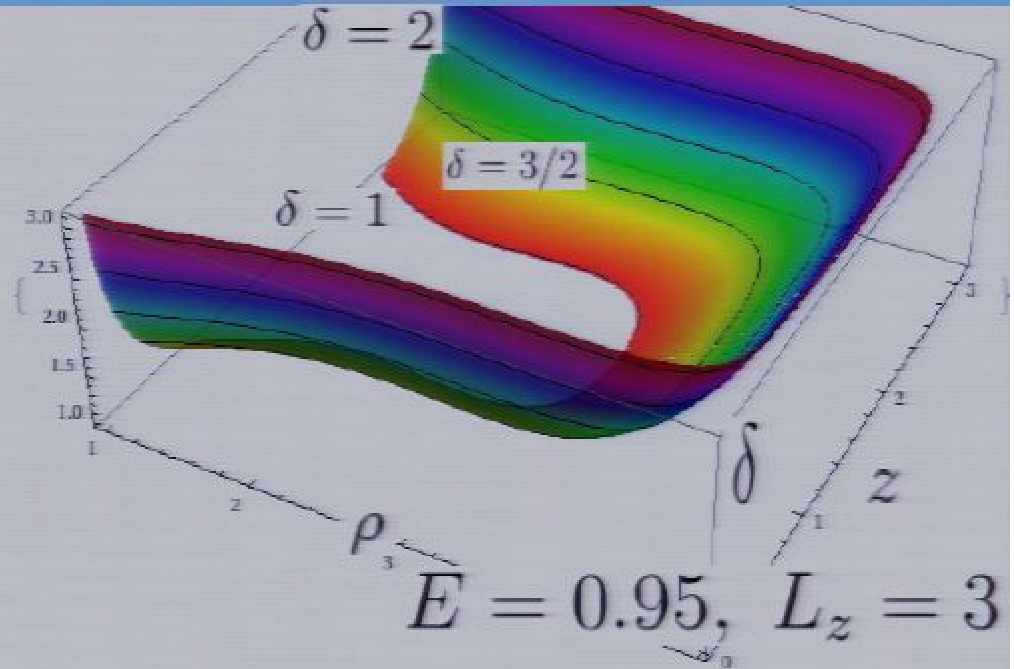

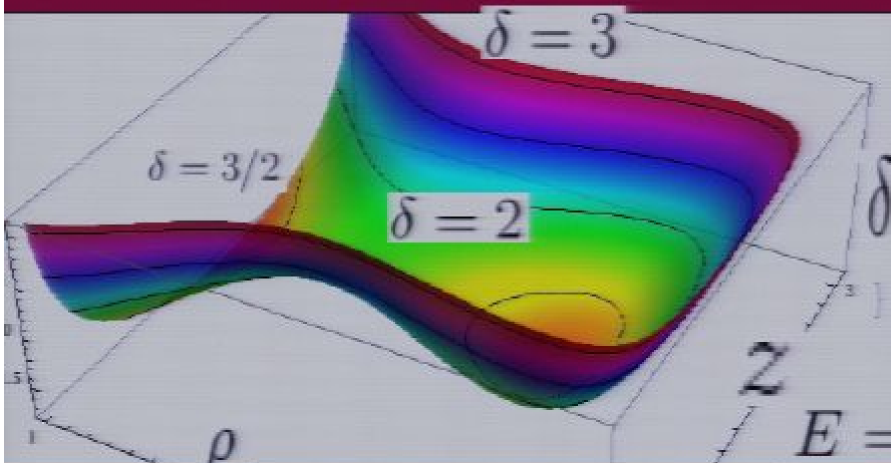

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$$+ \frac{4 (x^2)^{-2\delta^2}}{(x^2-1)^2} \left(\frac{x-1}{x+1}\right)^{-4\delta} \left(L_z^4 x^4 \left(\tfrac{x-1}{x+1}\right)^{4\delta} (x^2-1)^{2\delta^2} + L_z^2 \left(\tfrac{x-1}{x+1}\right)^{2\delta} \left(6\, C_2 (x^2)^{\delta^2+1} (x^2-1)^{\delta^2+1} \left(E^2 (x^2-1) - \left(\tfrac{x-1}{x+1}\right)^{\delta} \left(L_z^2 \left(\tfrac{x-1}{x+1}\right)^{\delta} + (x^2-1)\mu^2\right)\right)\right) + C_3 (x^2-1)^2 (x^2)^{2\delta^2} \left(e^2 (x^2-1) - \left(\tfrac{x-1}{x+1}\right)^{\delta} \left(Lz^2 \left(\tfrac{x-1}{x+1}\right)^{\delta} + (x^2-1)\mu^2\right)\right)^2 \right)$$

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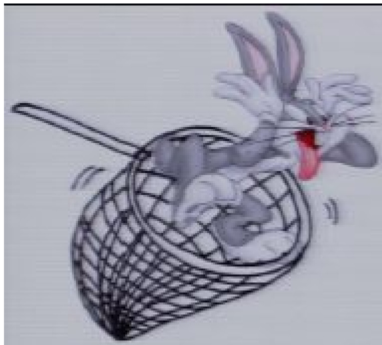

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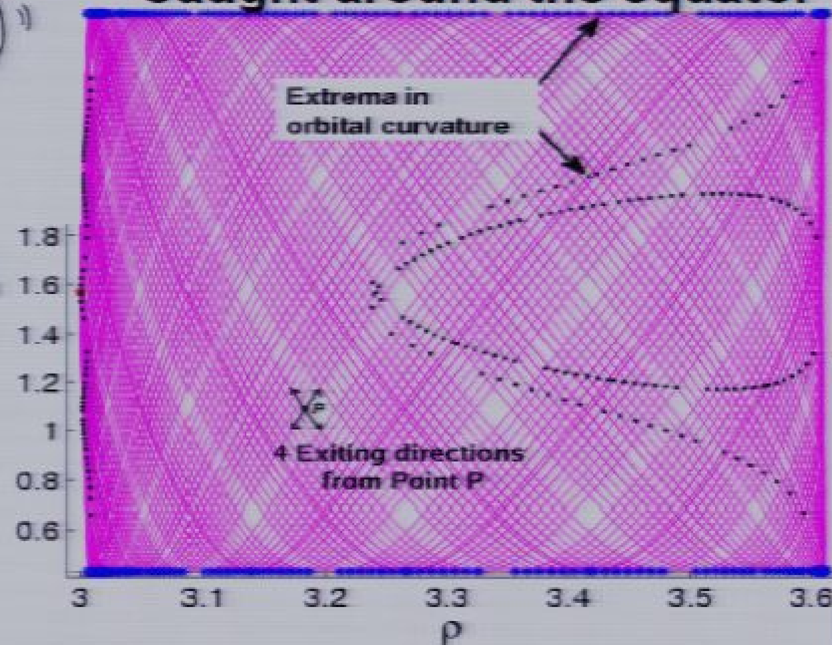

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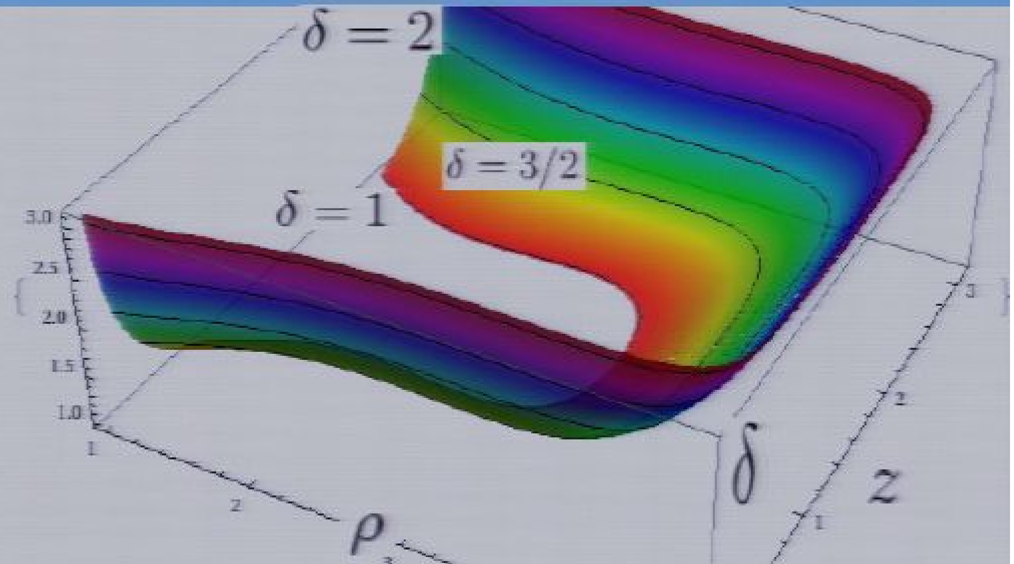

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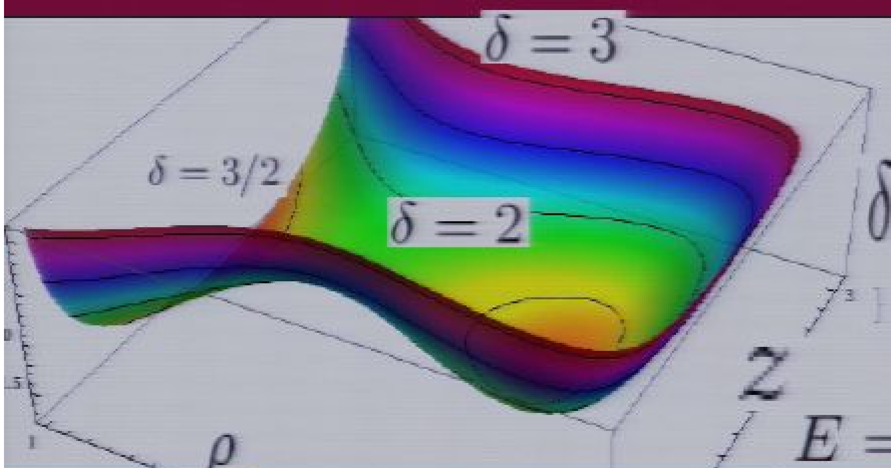

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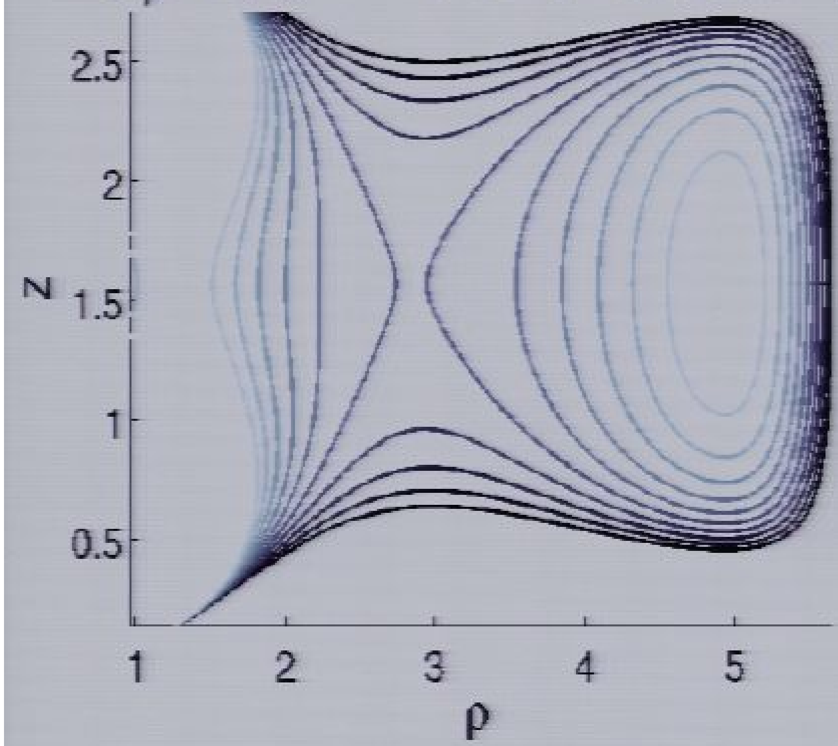
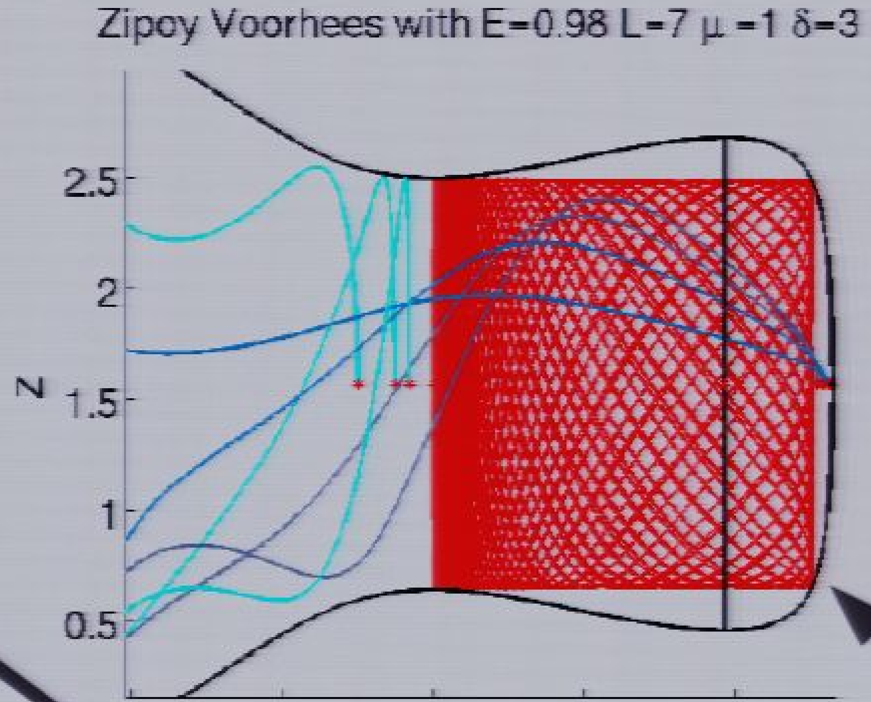
Hamiltonian Constraint sets allowed region for orbit

$$Q = P_\rho^4 \frac{4}{x^4} C3 \left(x^2\right)^{2\delta^2} \left(x^2-1\right)^{2-2\delta^2} - P_\rho^2 \frac{8}{x^2} \left(\frac{x-1}{x+1}\right)^{-2\delta} \left(x^2-1\right)^{-\delta^2} \left(\begin{array}{c} 3C_2 Lz^2 x^2 \left(\frac{x-1}{x+1}\right)^{2\delta} (x^2-1)^{\delta^2} \\ + C_3 (x^2-1)(x^2)^{\delta^2} \left(E^2 (x^2-1) - \left(\frac{x-1}{x+1}\right)^\delta \left(Lz^2 \left(\frac{x-1}{x+1}\right)^\delta + (x^2-1)\mu^2\right)\right) \end{array} \right)$$

$$+ \frac{4 (x^2)^{-2\delta^2}}{(x^2-1)^2} \left(\frac{x-1}{x+1}\right)^{-4\delta} \left(\begin{array}{c} L_z^4 x^4 \left(\frac{x-1}{x+1}\right)^{4\delta} (x^2-1)^{2\delta^2} \\ + L_z^2 \left(\frac{x-1}{x+1}\right)^{2\delta} \left(6 C_2 (x^2)^{\delta^2+1} (x^2-1)^{\delta^2+1} \left(E^2 (x^2-1) - \left(\frac{x-1}{x+1}\right)^\delta \left(L_z^2 \left(\frac{x-1}{x+1}\right)^\delta + (x^2-1)\mu^2\right)\right)\right) \\ + C_3 (x^2-1)^2 (x^2)^{2\delta^2} \left(e^2 (x^2-1) - \left(\frac{x-1}{x+1}\right)^\delta \left(Lz^2 \left(\frac{x-1}{x+1}\right)^\delta + (x^2-1)\mu^2\right)\right)^2 \end{array} \right)$$

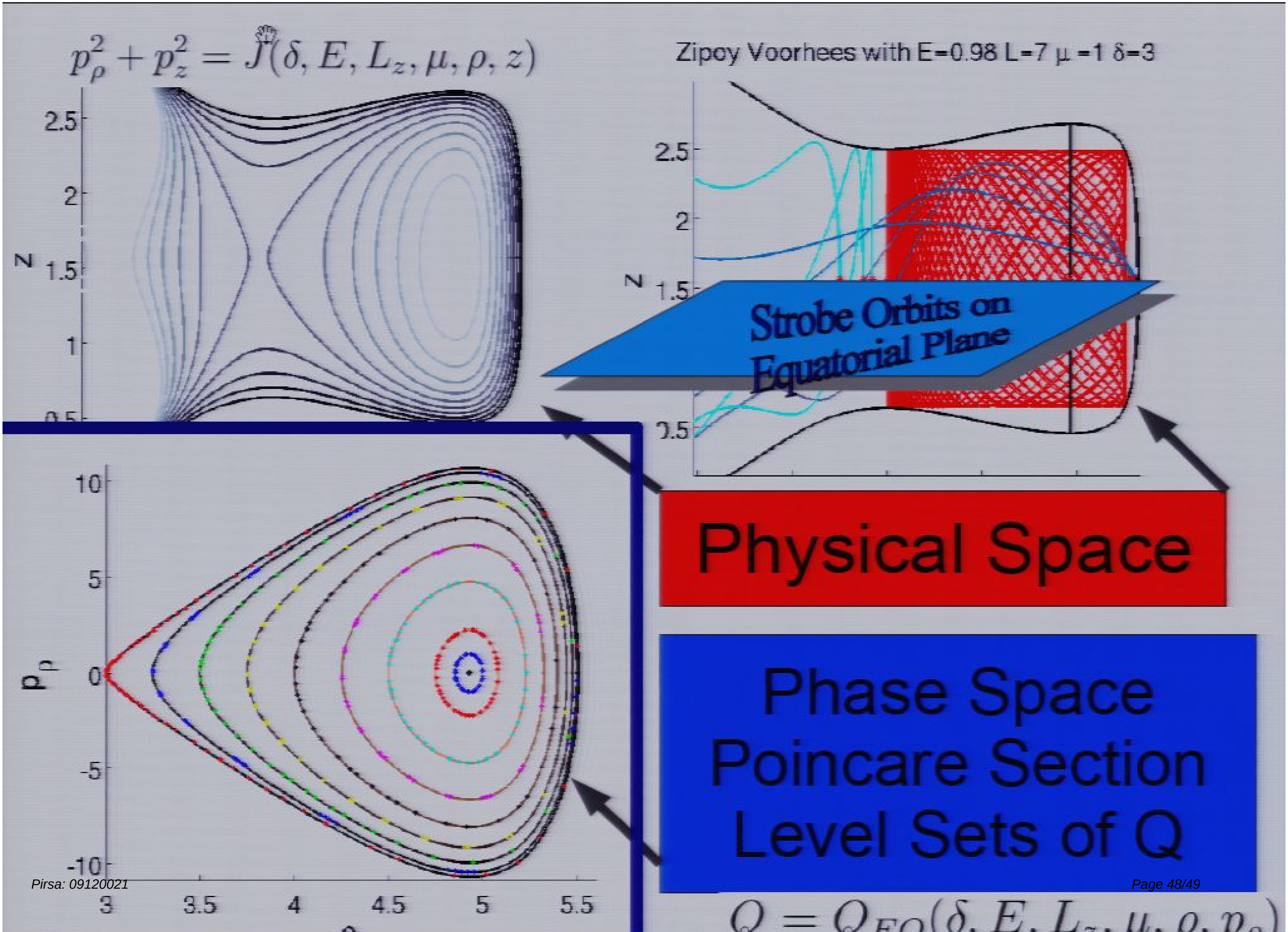
$$p_\rho^2 + p_z^2 = J(\delta, E, L_z, \mu, \rho, z)$$



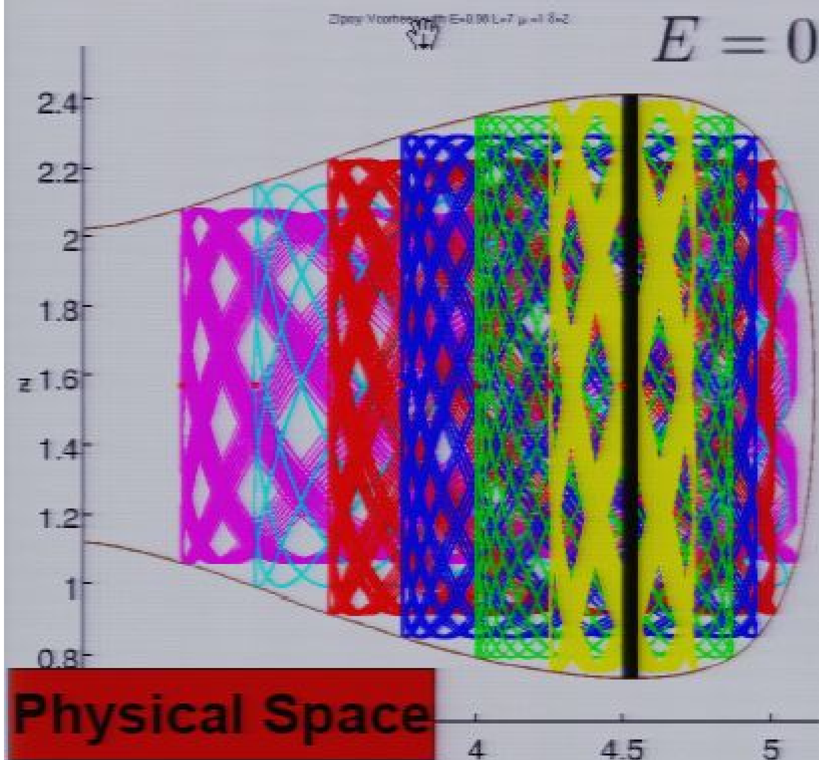


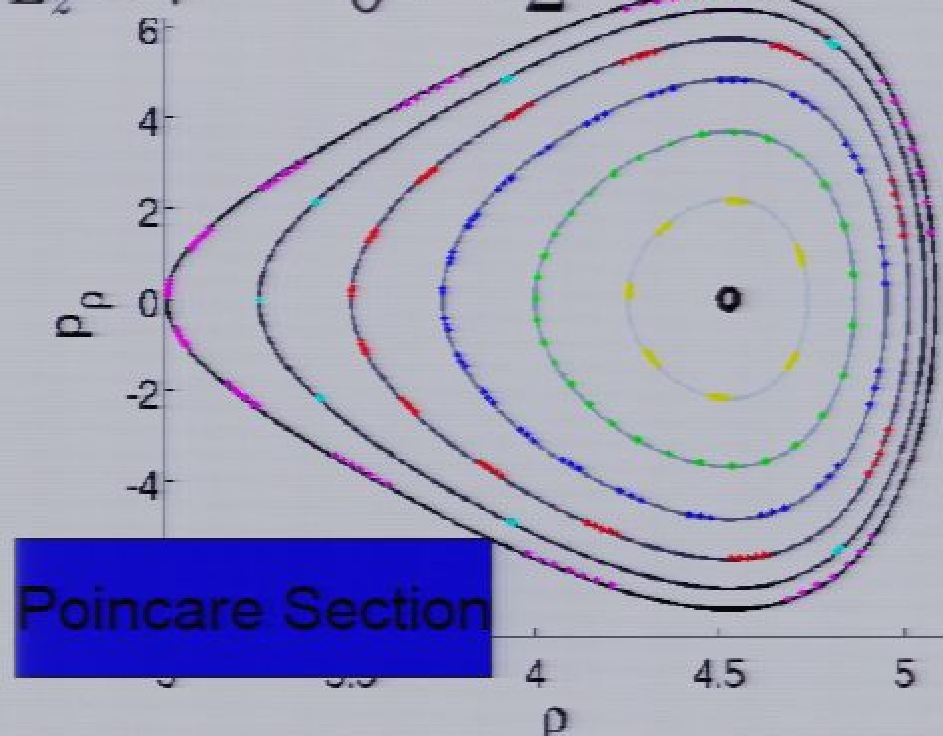
Physical Space

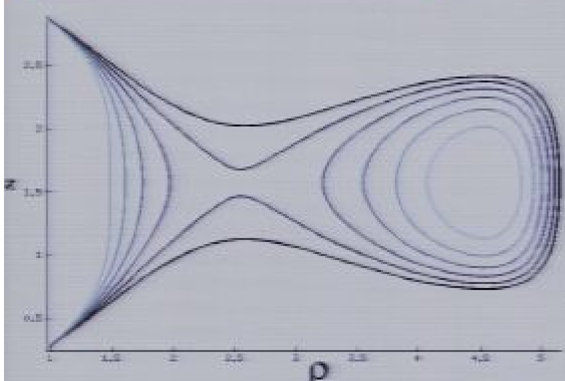
$$E = 0.98,\ L_z = 7 \quad \delta = 3$$


$p_\rho^2 + p_z^2 = J(\delta, E, L_z, \mu, \rho, z)$
Zipoy Voorhees with E=0.98 L=7 μ =1 δ=3
Strobe Orbits on Equatorial Plane
Physical Space
Phase Space
Poincare Section
Level Sets of Q
$Q = Q_{EQ}(\delta, E, L_z, \mu, \rho, p_\rho)$
z
p_ρ

$$E = 0.98,\ L_z = 7 \qquad \delta = 2$$







Knowledge of Q on the equatorial plane,
allows the radial period of geodesic
motion to be determined analytically for any orbit
For a given Q associated with the orbit use :

$$Q = Q_{EQ}(\delta, E, L_z, \mu, \rho, p_\rho)$$

To solve for the momentum p_ρ in terms of ρ

$$\frac{d\rho}{d\tau} = e^{2\psi - 2\gamma} p_\rho$$

Substitute into one of Hamilton's equations