Title: Aspects of symmetric product orbifolds

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Abstract: In the context of AdS/CFT correspondence the AdS_3/CFT_2 instance of the duality stands apart from other well studied cases, like AdS_5/CFT_4 or AdS_4/CFT_3. One of the reasons is that the CFT side of this duality is not a theory of matrices but rather a two dimensional orbifold based on the group of permutations. In this talk we will discuss some aspects of this theory. In particular a diagrammatic language, akin to Feynman diagrams used for gauge theories, will be developed. Moreover, we will compute a large set of protected quantities in a certain symmetric product orbifold CFT, and show that these are elegantly given in terms of Hurwitz numbers.

Aspects of Symmetric Product Orbifolds.

Shlomo S. Razamat

C.N. Yang Institute for Theoretical Physics
razamat@max2.physics.sunysb.edu

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Motivation.

- $AdS_5 \rightarrow CFT_4 \rightarrow SYM$
- $AdS_4 \rightarrow CFT_3 \rightarrow$ *Chern-Simons/Matter*
- $AdS_3 \rightarrow CFT_2 \rightarrow$ $\boxed{\textit{Symmetric product orbifold}}$ + *deformations*

Outline.

- Perturbative regime - Diagrams

  A. Pakman, L. Rastelli, SSR 0905.3448

- Protected quantities - Extremal Correlators

  A. Pakman, L. Rastelli, SSR 0905.3451

Symmetric product *CFT*

The action is given by

$$S = \frac{1}{2\pi} \int d\sigma d\tau \, G_{ij}(X) \left(\partial_\sigma X_I^i \partial_\sigma X_I^j - \partial_\tau X_I^i \partial_\tau X_I^j \right) + \ldots ,$$

where

$$I = 1 \ldots N,$$

and the fields are identified under the action of the symmetric group

$$h X_I^i \cong X_I^i, \qquad h \in S_N$$

Twist fields

We can have twisted boundary conditions

$$X_I^j(e^{2\pi i}z) = h\, X_I^j(z).$$

Twisted sectors are labeled by conjugacy classes of S_N, $[h]$

$$h' \in [h] \;\rightarrow\; \exists g \in S_N \quad ghg^{-1} = h'.$$

Twist fields, σ_g, create a vacuum of twisted sector g,

$$X_I^i(e^{2\pi i}\, z)\sigma_g(0) = g\, X_I^i(\, z)\sigma_g(0).$$

Gauge invariant twist fields are defined as,

$$\sigma_{[g]} = \sum_{h \in S(N)} \sigma_{h^{-1} g h}.$$

Correlators of twist fields

- An interesting quantity to consider is a correlator of twist fields

$$\langle \prod_{j=1}^{s} \sigma_{[g]_j} \rangle = \langle \prod_{j=1}^{s} \sum_{h_j \in S(N)} \sigma_{h_j g_j h_j^{-1}} \rangle.$$

- We can group different terms in the sum above into equivalence classes, identifying two terms differing only by relabeleing the colors

$$\langle \prod_{j=1}^{s} \sigma_{g_j} \rangle \sim \langle \prod_{j=1}^{s} \sigma_{g'_j} \rangle \quad \leftrightarrow \quad \exists h \in S_N, \qquad \forall i = 1 \dots s, \qquad \hat{g}_i = h \hat{g}'_i h^{-1}.$$

- Then we can write the above sum as

$$\boxed{\langle \prod_{j=1}^{s} \sigma_{[g]_j(z_j)} \rangle = \sum_{\alpha} \mathcal{N}_\alpha(N, \{g_j\}) \langle \prod_{j=1}^{s} \sigma_{g_j^{(\alpha)}}(z_j) \rangle.}$$

- Assuming $|z_1| < |z_2| < \cdots < |z_s|$ the group elements have to satisfy

$$g_1^{(\alpha)} g_2^{(\alpha)} \dots g_s^{(\alpha)} = 1.$$

Diagrams

- A cycle σ_n corresponds to a vertex with $2n$ (double line) edges emanating from it,

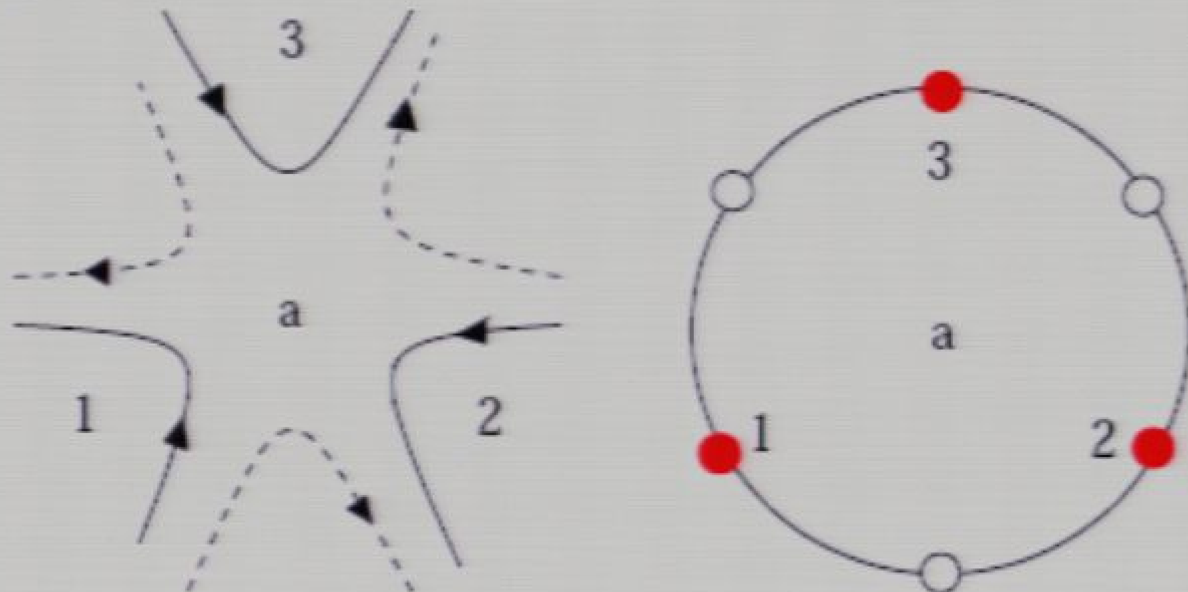


Figure: An example of a vertex corresponding to $\sigma_3 = (1\,2\,3)$.

- To obtain the diagrams we *Wick* contract in all possible ways ****modulo two simple rules.**

Example of a diagram

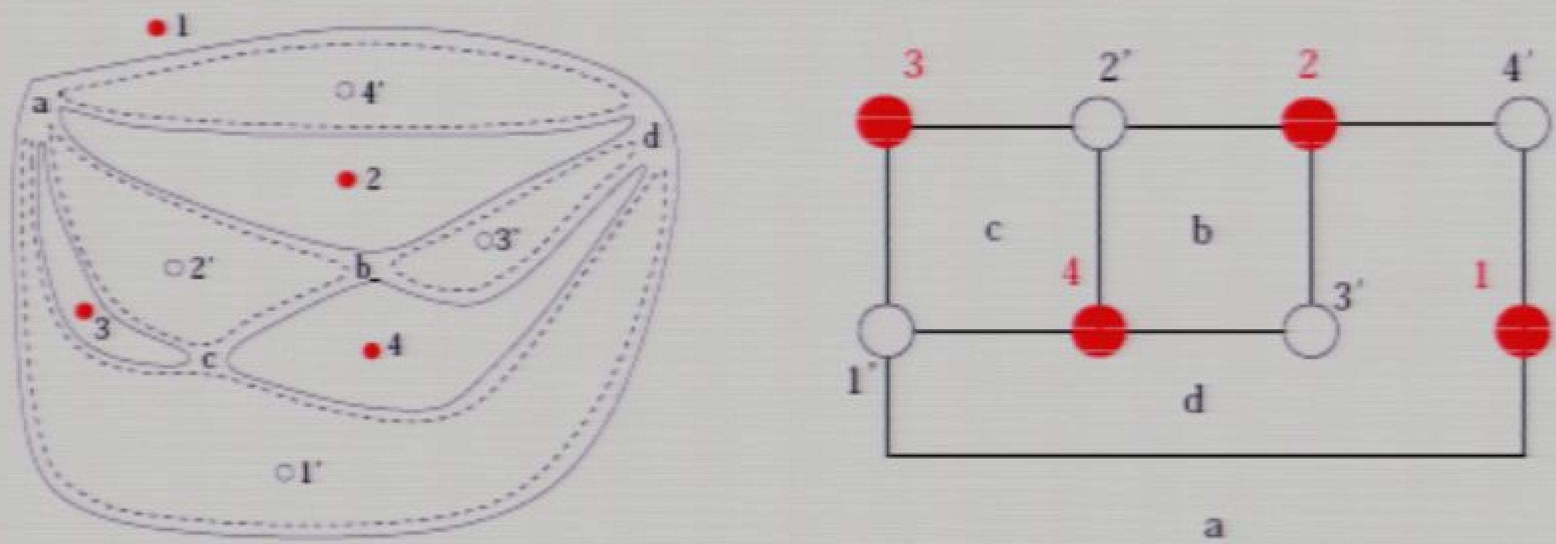
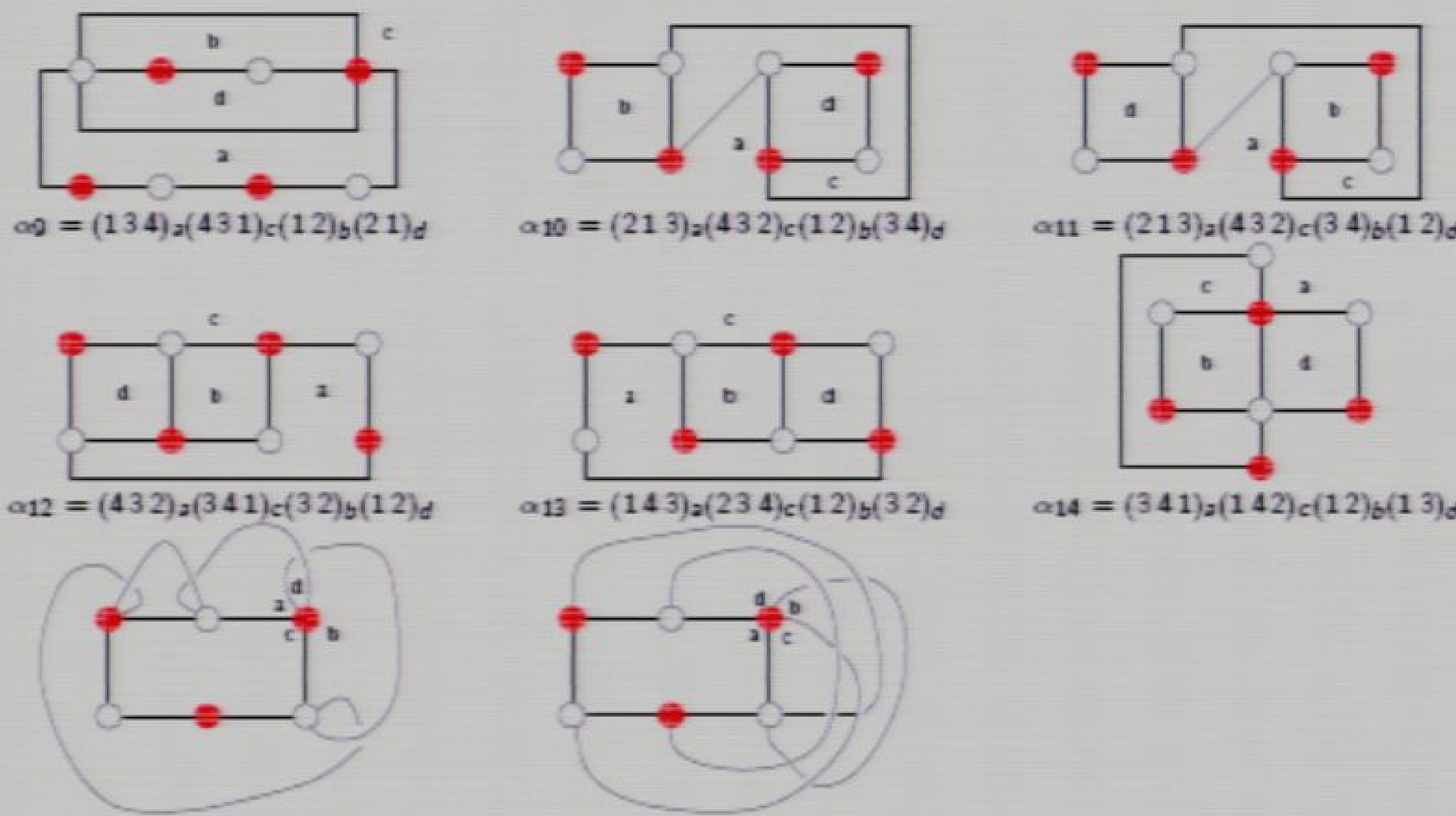


Figure: *On the left*, the diagram corresponding to $(132)_a(24)_b(34)_c(241)_d$. A red (solid) dot is drawn for clarity on the inside of each color (solid) loop and is labeled by a color index. Each *vertex* (letter) corresponds to a twist field: going around the vertex counterclockwise one reads off the color indices of the corresponding cyclic permutation. *On the right*, the (graph theoretic) dual diagram, obtained as usual by dualizing vertices into faces. Each *loop* in the dual graph corresponds to a twist field.

Example of a correlator with all its diagrams.



$\alpha_9 = (1\,3\,4)_a(4\,3\,1)_c(1\,2)_b(2\,1)_d$ $\alpha_{10} = (2\,1\,3)_a(4\,3\,2)_c(1\,2)_b(3\,4)_d$ $\alpha_{11} = (2\,1\,3)_a(4\,3\,2)_c(3\,4)_b(1\,2)_d$

$\alpha_{12} = (4\,3\,2)_a(3\,4\,1)_c(3\,2)_b(1\,2)_d$ $\alpha_{13} = (1\,4\,3)_a(2\,3\,4)_c(1\,2)_b(3\,2)_d$ $\alpha_{14} = (3\,4\,1)_a(1\,4\,2)_c(1\,2)_b(1\,3)_d$

$\alpha_{15} = (1\,2\,3)_a(3\,2\,1)_c(2\,3)_b(2\,3)_d$ $\alpha_{16} = (1\,2\,3)_a(1\,2\,3)_c(1\,3)_b(3\,2)_d$

Figure: Connected diagrams contributing to $\langle \sigma_{[3]}(a)\sigma_{[3]}(c)\sigma_{[2]}(b)\sigma_{[2]}(d)\rangle$.

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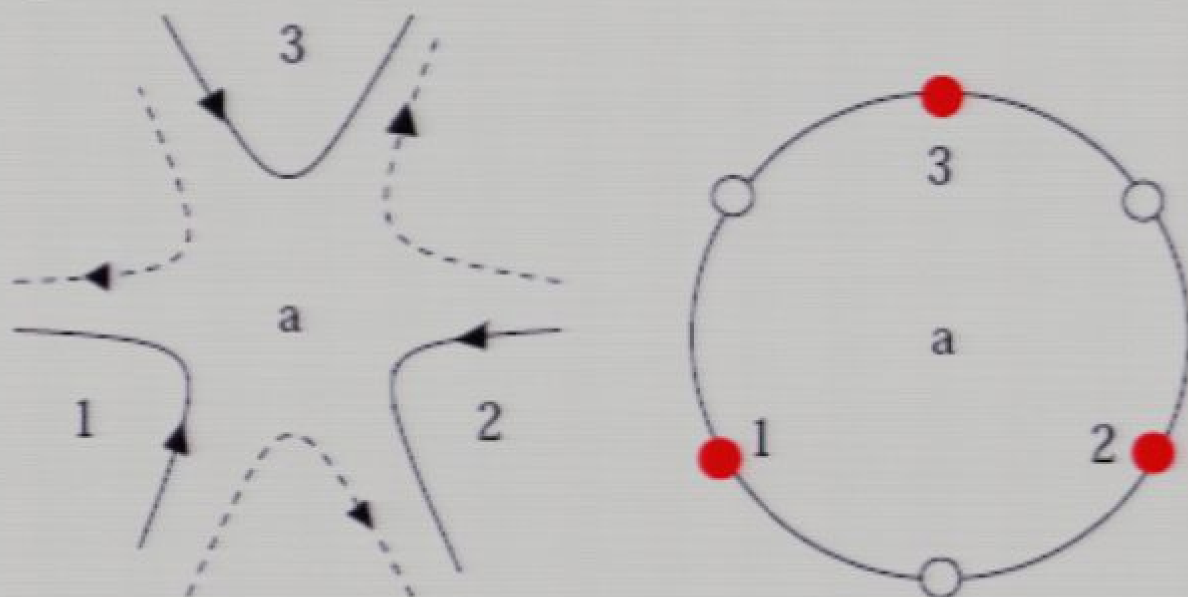

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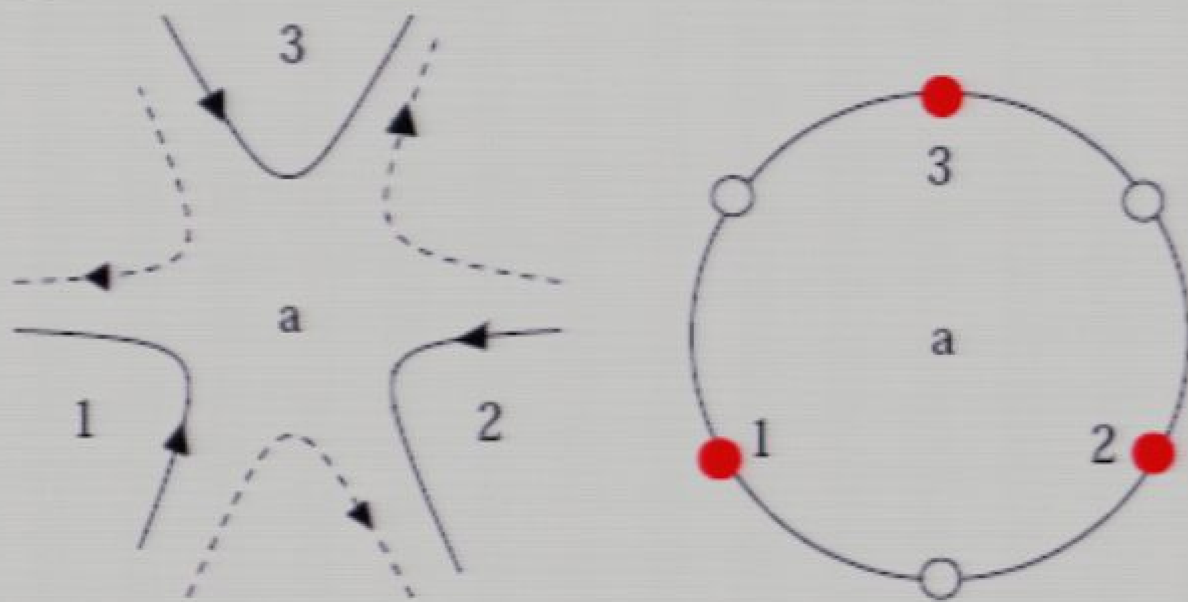

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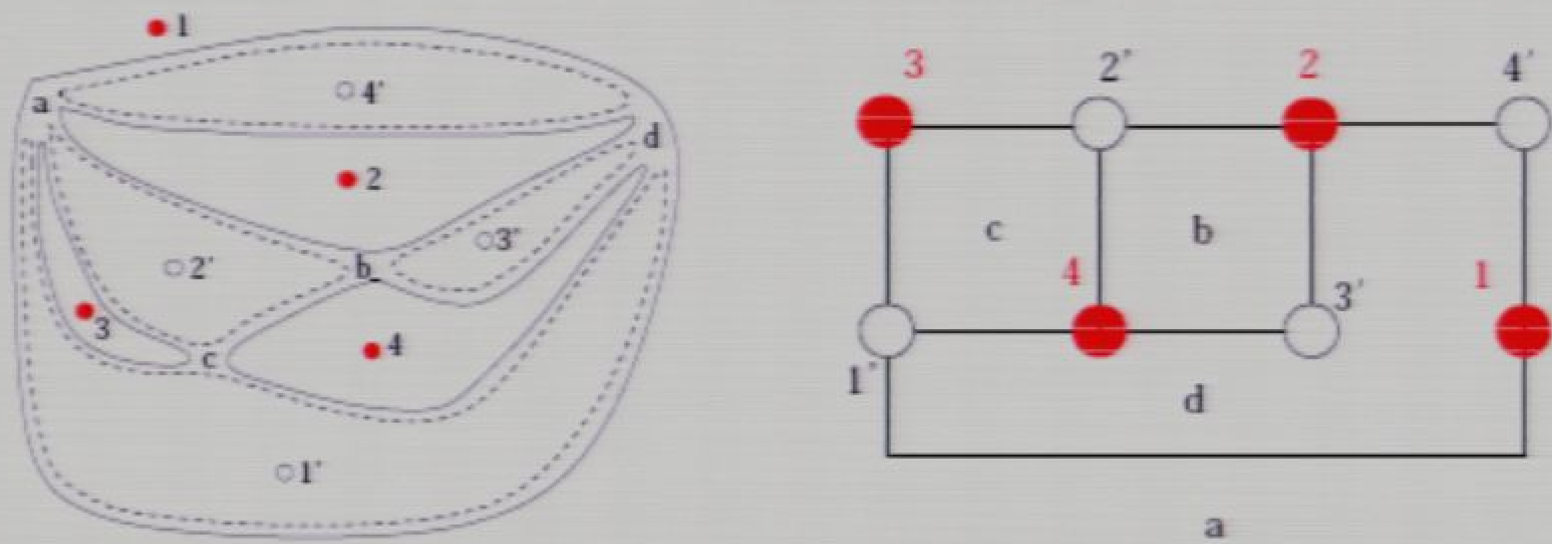

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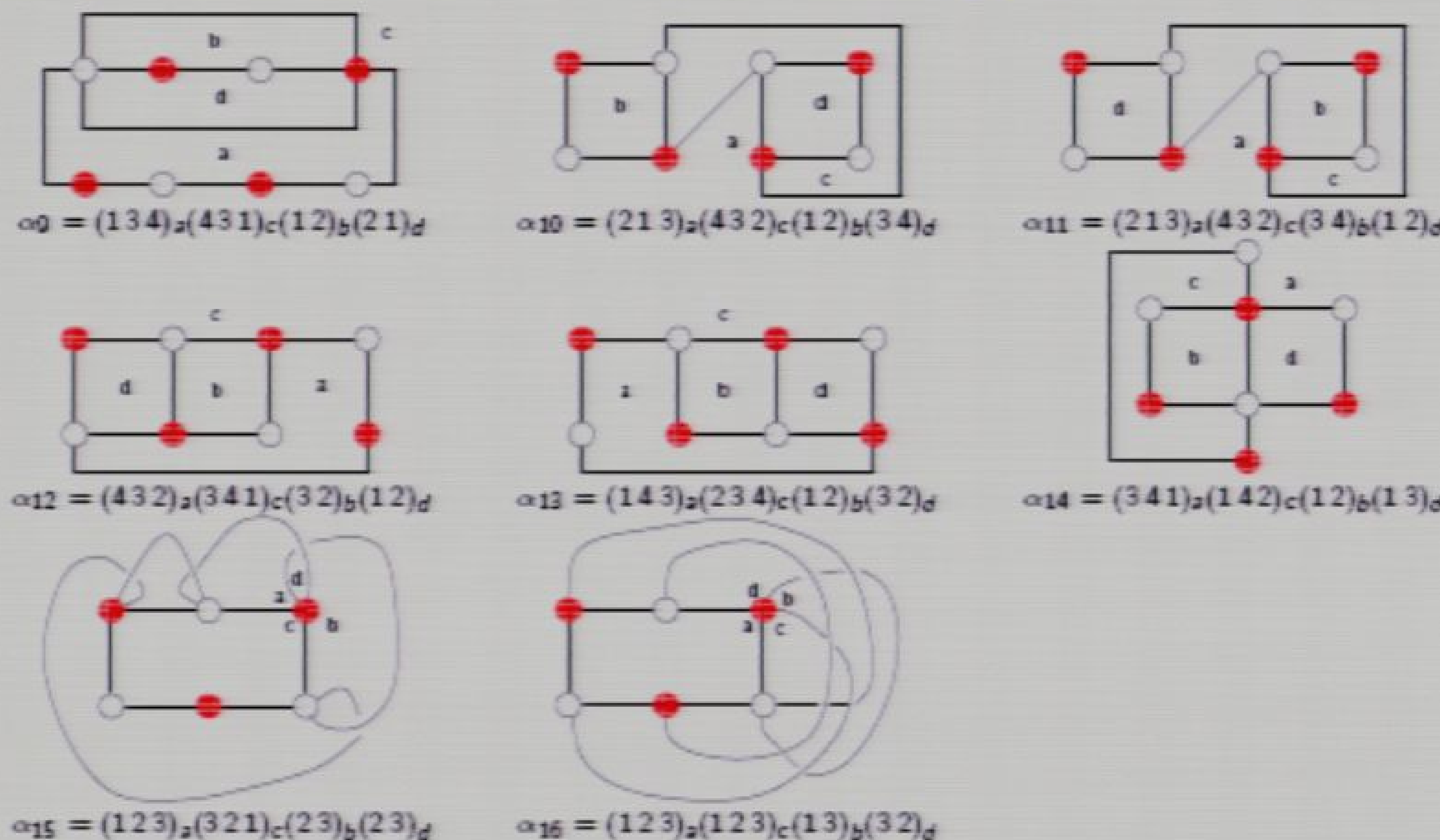

Figure: Connected diagrams contributing to $\langle \sigma_{[3]}(a)\sigma_{[3]}(c)\sigma_{[2]}(b)\sigma_{[2]}(d)\rangle$.

Relation to covering maps.

- We can compute the genus of each diagram ($2g - 2 = e - v - f$),

$$g = \frac{1}{2}\sum_{i=1}^{s}(n_i - 1) - c + 1.$$

  This is the *Riemann-Hurwitz* relation determining the genus of a c-sheeted cover of a sphere.

- The actual computation of the correlator of twist fields is usually done on a covering surface where the fields are single valued. Essentially, each diagram corresponds to a particular map to a covering surface.

- We thus can associate for each term in the expansion of the correlator a genus parameter,

$$\boxed{\langle\prod_{j=1}^{s}\sigma_{[h]_j}(z_j)\rangle = \sum_{g=0}^{g_{max}}\sum_{\alpha_g}\mathcal{N}_{g,\alpha_g}\langle\prod_{j=1}^{s}\sigma_{h_j^{(\alpha_g)}}(z_j)\rangle_g}\, , \quad \mathcal{N}_{g,\alpha_g} \sim N^{1-g-\frac{1}{2}s}.$$

More on covering maps

- Consider a sphere four point function

$$\langle \sigma_{[n_1]}(z_1)\sigma_{[n_2]}(z_2)\sigma_{[n_3]}(z_3)\sigma_{[n_4]}(z_4)\rangle \,,$$

defined on the base sphere S^2_{base}. By an $SL(2,C)$ transformation, we fix

$$z_1 = 0\,, \quad z_2 = 1\,, \quad z_3 = u\,, \quad z_4 = \infty\,.$$

- The goal is to find all the covering maps

$$t \in S^2_{cover} \rightarrow z(t) \in S^2_{base}$$

with four ramification points z_i of order n_i. The ramification points z_i have unique pre-images t_i on S^2_{cover}, which by another $SL(2,C)$ transformation we fix to

$$t_1 = 0\,, \quad t_2 = 1\,, \quad t_3 = x\,, \quad t_4 = \infty\,.$$

- We are looking for a map $z_x : S^2_{cover} \rightarrow S^2_{base}$ with the following branching behavior:

$$\begin{aligned}
\lim_{t\rightarrow 0} z_x(t) &\sim b_1 t^{n_1}\,,\\
\lim_{t\rightarrow 1} z_x(t) &\sim 1 + b_2(t-1)^{n_2}\,,\\
\lim_{t\rightarrow x} z_x(t) &\sim u + b_3(t-x)^{n_3}\,,\\
\lim_{t\rightarrow \infty} z_x(t) &\sim b_4 t^{n_4}\,.
\end{aligned}$$

More on covering maps 2

- From very generic considerations one can show that the covering map is given by a ratio of two polynomials $z_x(t) = f_1(t)/f_2(t)$ both of which are solutions of Heun's differential equation

$$f'' - \left[\frac{n_1-1}{t} + \frac{n_2-1}{(t-1)} + \frac{n_3-1}{(t-x)}\right] f' + \frac{(d_1\, d_2 t + q)}{t(t-1)(t-x)} f = 0\,.$$

- the parameters d_1 and d_2 are fixed by the sizes of the cycles and are respectively the degrees of polynomials f_1 and f_2,

$$d_1 = \frac{n_1+n_2+n_3+n_4}{2} - 1 = c\,, \qquad d_2 = d_1 - n_4$$

- There are two parameters we still have to fix, q and x. These are fixed by two polynomial conditions

$$P(q,x) = 0\,, \qquad z_x(x) = u\,.$$

- The first condition comes from demanding that the solutions to the Heun's equation are polynomial and the second by demanding that the pre-image of the ramification point at u is x.
- The different solutions to these polynomial equation give the different covering maps and are in one-to-one correspondence with the diagrams.

Map → Diagram

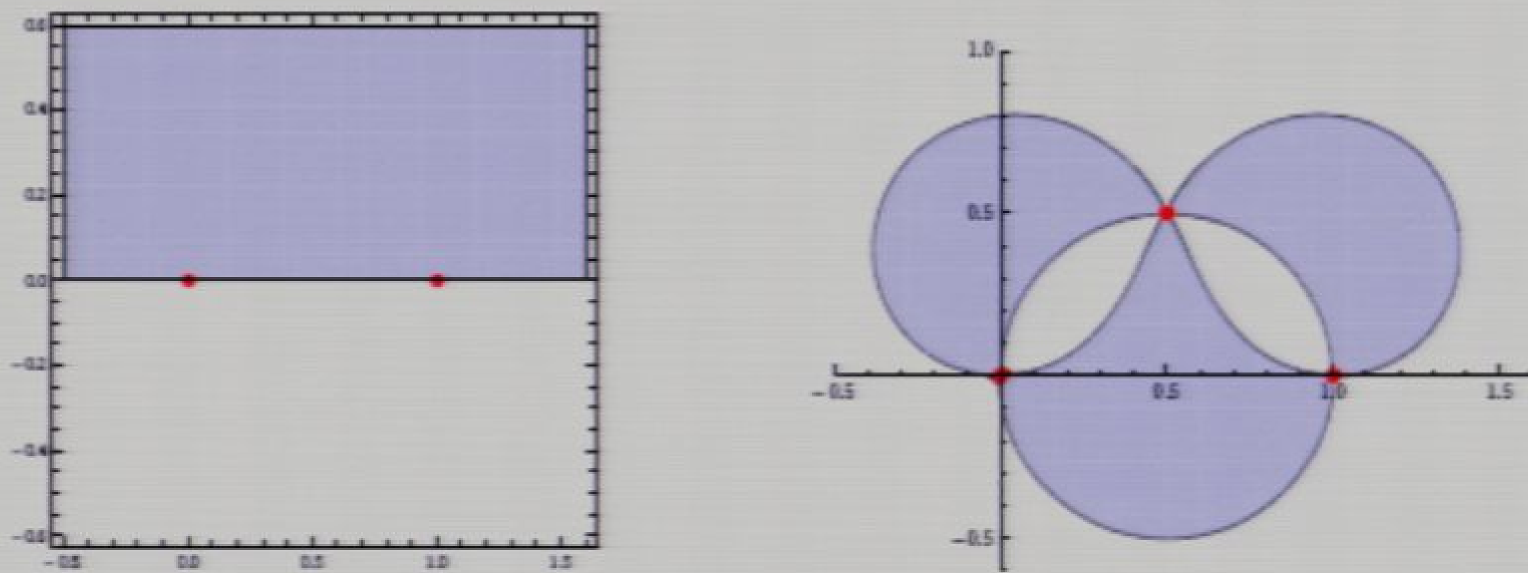

Figure: An example of the inverse-image method, applied to the correlator $\langle \sigma_{[2]}\sigma_{[2]}\sigma_{[3]} \rangle$. On the left we show the base sphere with the twist-two fields inserted at $z = 0, 1$ and the twist-three field at $z = \infty$. We connect the insertions by a line going through the real axis. On the right we show the pre-image of this line on the covering sphere. The insertions are now at $t = 0, 1$ and $t = \frac{1}{2} + \frac{1}{2}i$ respectively.The explicit branched covering map is $z(t) = t^2 \frac{-3i+(1+3i)t}{(-i+(1+i)t)^3}$.

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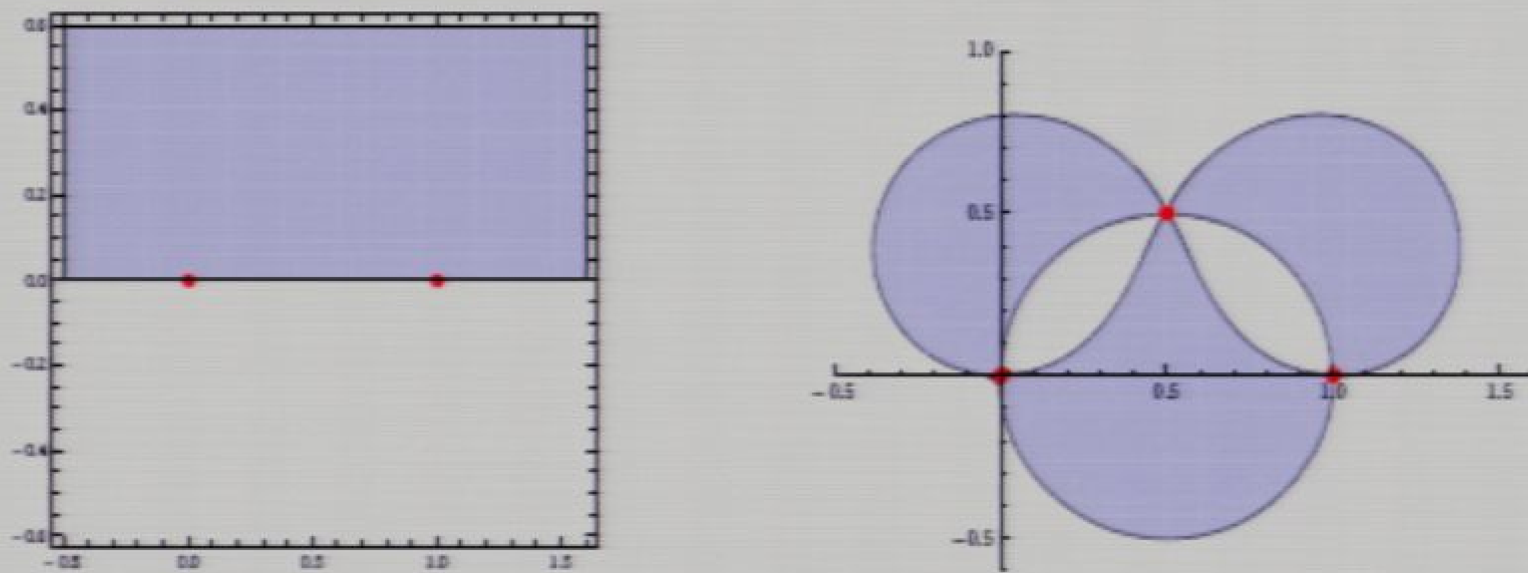

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Symmetric product of Supersymmetric T^4

- The symmetric product orbifold $\mathrm{Sym}^N\, T^4$ is obtained by considering N copies of T^4 and identifying the coordinates under the action of the permutation group S_N. Each copy of T^4 has bosonic coordinates X^i_I with real fermionic partners χ^i_I, where $i = 1, 2, 3, 4$ and $I = 1, \ldots, N$.
- The fermions in each copy can be combined into a complex pair and bosonized

$$\psi^1_I = e^{i\phi^1_I}\,, \qquad \psi^2_I = e^{i\phi^2_I}\,.$$

- The (anti)chiral operators are built by dressing the twist-fields with invariant contributions from the fermionic sector to satisfy the (anti)chiral relationship $\Delta = \pm Q$, where Q is the charge under the $U(1)$ of the $\mathcal{N} = 2$ subalgebra. There are three types of chiral operators: $O^{(0,0)}_n$, $O^{(a,\bar{a})}_n(a, \bar{a} = 1, 2)$ and $O^{(2,2)}_n$, corresponding to 0, 1 and 2-forms in T^4, respectively, with n being the length of the permutation cycle.
- Explicitly the operators are given by the following

$$o^{(0,0)}_{(12\ldots n)} = e^{i\frac{n-1}{2n}\sum_{l=1}^{n}(\phi^1_l+\phi^2_l+\bar{\phi}^1_l+\bar{\phi}^2_l)}\sigma_{(12\ldots n)} \quad\rightarrow\quad \frac{1}{\sqrt{n\,N!(N-n)!}}\sum_{h\in S(N)} o^{(0,0)}_{h^{-1}(12\ldots n)h}$$

$$o^{(a-1,\bar{a}-1)}_{(12\ldots n)} = e^{i\frac{n+1}{2n}\sum_{l=1}^{n}(\phi^1_l+\bar{\phi}^1_l)+i\frac{n-1}{2n}\sum_{l=1}^{n}(\phi^2_l+\bar{\phi}^2_l)}\sigma_{(12\ldots n)} \quad\rightarrow\quad \frac{1}{\sqrt{n\,N!(N-n)!}}\sum_{h\in S(N)} o^{(a-1,\bar{a}-1)}_{h^{-1}(12\ldots n)h}$$

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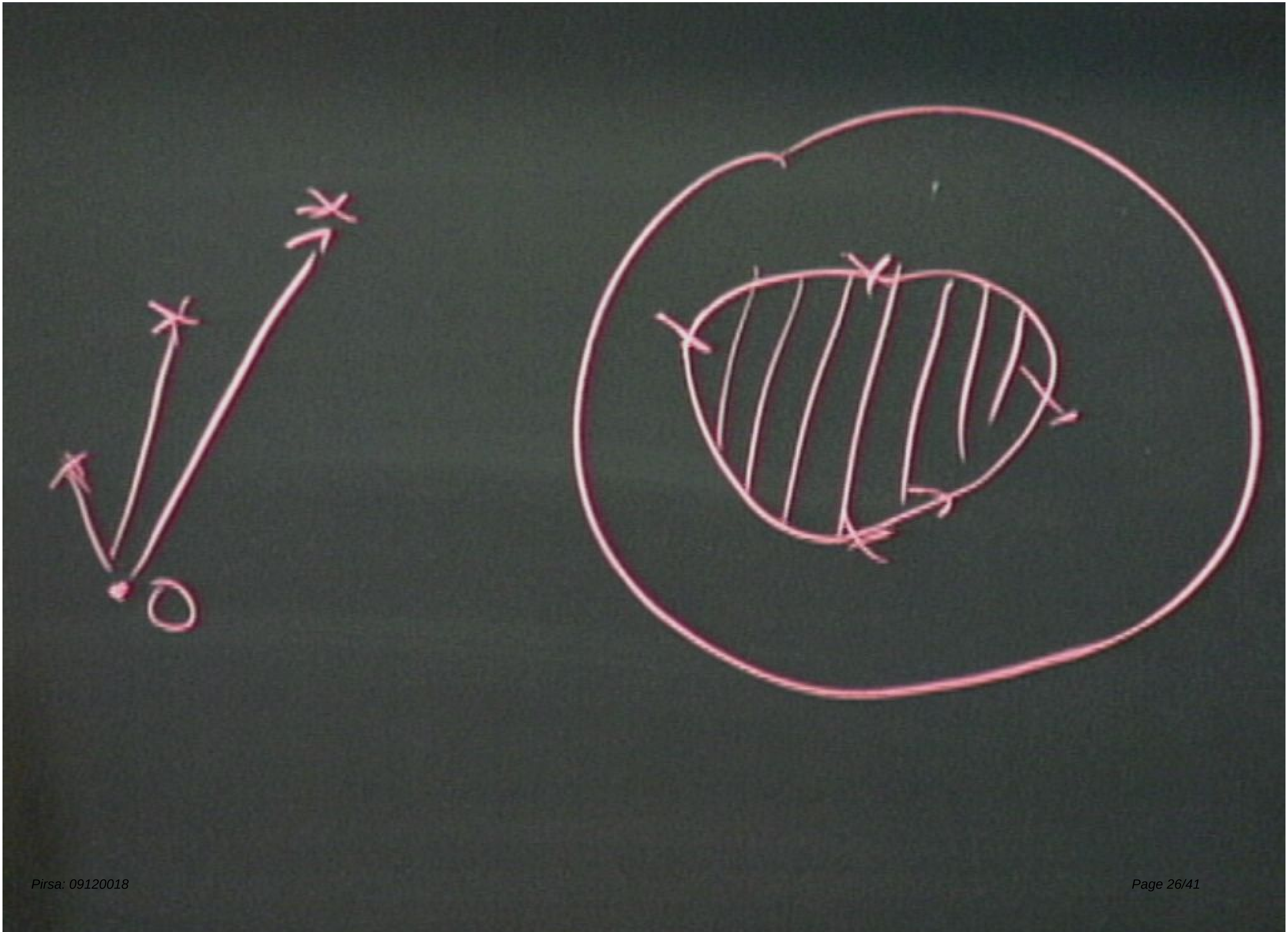
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3-point functions

- The charges and conformal dimensions of the chiral operators are given by

$$\Delta_n^0 = Q_n^0 = \frac{n-1}{2}, \quad \Delta_n^{\bar 1} = Q_n^{\bar 1} = \frac{n}{2}, \quad \Delta_n^2 = Q_n^2 = \frac{n+1}{2}.$$

- The three point functions are given by

$$\begin{aligned}
\langle O_{n_3}^{(0,0)\dagger} O_{n_2}^{(0,0)} O_{n_1}^{(0,0)} \rangle &= \left(\frac{1}{N}\right)^{\frac{1}{2}} \frac{(n_3)^{3/2}}{(n_2 n_1)^{1/2}}, \\
\langle O_{n_3}^{(2,2)\dagger} O_{n_2}^{(2,2)} O_{n_1}^{(0,0)} \rangle &= \left(\frac{1}{N}\right)^{\frac{1}{2}} \frac{(n_2)^{3/2}}{(n_3 n_1)^{1/2}}, \\
\langle O_{n_3}^{(b,\bar b)\dagger} O_{n_2}^{(a,\bar a)} O_{n_1}^{(0,0)} \rangle &= \delta^{ab}\delta^{\bar a \bar b} \left(\frac{1}{N}\right)^{\frac{1}{2}} \frac{(n_3 n_2)^{1/2}}{(n_1)^{1/2}}, \\
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\langle O_{n_3}^{(2,2)\dagger} O_{n_2}^{(0,0)} O_{n_1}^{(0,0)} \rangle &= \left(\frac{1}{N}\right)^{\frac{1}{2}} \frac{1}{(n_3 n_2 n_1)^{1/2}}.
\end{aligned}$$

- Let us define a more abstract set of operators

$$o_n^{\alpha} = e^{i\alpha \sum_{l=1}^{n}(\phi_l^1 + \phi_l^2 + \bar\phi_l^1 + \bar\phi_l^2)} \sigma_n, \qquad O_n^{\alpha} = \frac{1}{\sqrt{nN!(N-n)!}} \sum_{h \in S(N)} o^{\alpha}_{h^{-1}(12\ldots n)h}.$$

The chiral operators $O_n^{(0,0)}$, $O_n^{(2,2)}$ of the previous slide are obtained for special values of "momentum" α.

General method of computation

- A generic four point function of these operators is given by

$$\langle O_{n_4}^{\alpha_4}(z_4,\bar z_4) O_{n_3}^{\alpha_3}(z_3,\bar z_3) O_{n_2}^{\alpha_2}(z_2,\bar z_2) O_{n_1}^{\alpha_1}(z_1,\bar z_1)\rangle_{g=0} = G(u,\bar u)$$
$$\times z_{24}^{-2\Delta_2}\, z_{14}^{\Delta_2+\Delta_3-\Delta_1-\Delta_4}\, z_{34}^{\Delta_1+\Delta_2-\Delta_3-\Delta_4}\, z_{13}^{-\Delta_1-\Delta_2-\Delta_3+\Delta_4} \times c.c.\,,$$

where

$$u = \frac{z_{12} z_{34}}{z_{13} z_{24}}\,.$$

- We can write this as a sum over diagrams

$$\langle O_{n_4}^{\alpha_4}(\infty) O_{n_3}^{\alpha_3}(1) O_{n_2}^{\alpha_2}(u) O_{n_1}^{\alpha_1}(0)\rangle_{g=0} = G(u,\bar u) \sim \sum_{j=1}^{M} \langle o_{n_4}^{\alpha_4}(\infty) o_{n_3}^{\alpha_3}(1) o_{n_2}^{\alpha_2}(u) o_{n_1}^{\alpha_1}(0)\rangle_j = \sum_{j=1}^{M} G_j(u,\bar u)\,,$$

where in order to evaluate each term we use a different map $z_j(t)$ to go to the covering surface.

- To evaluate $G_j(u,\bar u)$ we use the stress-energy tensor method. One defines

$$g_j(z,u) = \frac{\langle T(z) o_{n_4}^{\alpha_4}(\infty) o_{n_3}^{\alpha_3}(1) o_{n_2}^{\alpha_2}(u,\bar u) o_{n_1}^{\alpha_1}(0)\rangle_j}{\langle o_{n_4}^{\alpha_4}(\infty) o_{n_3}^{\alpha_3}(1) o_{n_2}^{\alpha_2}(u) o_{n_1}^{\alpha_1}(0)\rangle_j}\,,$$

and takes the OPE of $T(z)$ with one of the operators to obtain a differential equation

$$T(z)\, o_{n_2}^{\alpha_2}(u) = \frac{\Delta_2}{(z-u)^2} o_{n_2}^{\alpha_2}(u) + \frac{1}{z-u}\partial o_{n_2}^{\alpha_2}(u) + \ldots \quad \rightarrow \quad \boxed{\partial_u \ln G_j(u) = \{g_j(z,u)\}_{\frac{1}{z-u}}}\,.$$

Example of a correlator

- A correlator of twist $n+2$, twist n and two twist 2 fields with generic dressing is given by

$$\langle O_{n+2}^{(\alpha_1)}(\infty) O_2^{(\alpha_2)}(1) O_2^{(\alpha_3)}(u,\bar{u}) O_n^{(\alpha_4)}(0)\rangle = C_4 \sum_{j=1}^{n+2} |x_j(u)-1|^{2\alpha} \, |x_j(u)|^{2\beta} \left|x_j(u) - \frac{n}{2+n}\right|^{2\gamma} .$$

- The coefficient C_4 can be fixed in general by taking OPE limits, and the powers α, β, γ are given by the following

$$\begin{aligned}
\alpha &= -\frac{1}{2}\left[\frac{1}{2} + 4(\alpha_3 - \alpha_2)^2 - 8\alpha_3\alpha_2\right] , \\
\beta &= -\left[\frac{-2+n+n^2}{8n} + n\left(2\alpha_2^2 - 4\alpha_2\alpha_1 + \alpha_1^2\right) - 2\alpha_2^2\right] , \\
\gamma &= \frac{1}{4n(n+2)}\left[n^2 + 2n - 2 + 8n^2\left(\alpha_1^2 + \alpha_2^2 + \alpha_3^2\right) - 16\left(\alpha_1 n^2(\alpha_2 + \alpha_3) + 2n\alpha_2\alpha_3\right)\right] .
\end{aligned}$$

- The functions $x_j(u)$ are solutions to

$$z_x(x) = x^{1+n}\frac{2+n-nx}{(2+n)x-n} = u .$$

A Non-Extremal correlator

- Let us apply the results to the following correlator ($\alpha_1 = \frac{n+1}{2n}$, $\alpha_2 = -\alpha_3 = -\frac{1}{4}$)

$$\langle O_{n+2}^{(0,0)\dagger}(\infty) O_2^{(0,0)}(1) O_2^{(0,0)\dagger}(u,\bar{u}) O_n^{(2,2)}(0)\rangle = C_4 \sum_{i=1}^{n+2} |x_i(u)-1|^{-2}\, |x_i(u)|^{-2n-2} \left| x_i(u) - \frac{n}{2+n}\right|^2 .$$

- To determine C_4 consider the OPE limit $u \to 0$. In this limit

$$u \sim -\frac{n+2}{n} x^{n+1}, \qquad \text{or} \qquad x \sim \frac{n+2}{n} .$$

The first solution corresponds to the n-cycle and the 2-cycle joining to an $n+1$-cycle and the second to these joining to a double cycle operator.

- From here we get

$$C_4\,(n+1) = \langle O_{n+2}^{(0,0)\dagger} O_2^{(0,0)} O_{n+1}^{(0,0)}\rangle \langle O_n^{(2,2)} O_2^{(0,0)\dagger} O_{n+1}^{(0,0)\dagger}\rangle ,$$

and using the known three point functions we get

$$C_4 = \frac{(n+2)^{3/2}}{2(n+1)^2 n^{1/2}} \sqrt{\frac{(N-n)(N-n-1)}{N^2(N-1)^2}} .$$

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$$\langle O^{(0,0)\dagger}_{n+2}(\infty) O^{(0,0)}_{2}(1) O^{(0,0)\dagger}_{2}(u,\bar{u}) O^{(2,2)}_{n}(0)\rangle = C_4 \sum_{i=1}^{n+2} |x_i(u) - 1|^{-2} \, |x_i(u)|^{-2n-2} \left| x_i(u) - \frac{n}{2+n} \right|^2 .$$

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$$C_4 = \frac{(n+2)^{3/2}}{2(n+1)^2 n^{1/2}} \sqrt{\frac{(N-n)(N-n-1)}{N^2(N-1)^2}} .$$

Extremal correlators

- Let us compute an extremal correlator ($\alpha_1 = \frac{n-1}{2n}$, $\alpha_2 = \alpha_3 = \frac{1}{4}$)

$$\langle O_{n+2}^{(0,0)\dagger}(\infty) O_2^{(0,0)}(1) O_2^{(0,0)}(u,\bar{u}) O_n^{(0,0)}(0)\rangle = C_4 \sum_{j=1}^{n+2} |x_j(u) - 1|^{2\alpha} \, |x_j(u)|^{2\beta} \left| x_j(u) - \frac{n}{2+n} \right|^{2\gamma} .$$

- However, here one gets that $\alpha = \beta = \gamma = 0$!!. This is to be expected as the OPEs of chirals are not singular and thus the four point function should be a constant. Extremal correlator does not depend on the cross ratio u.
- The general statement for extremal four point functions is ($n_4 = n_3 + n_2 + n_1 - 2$)

$$\langle O_{n_4}^{(0,0)\dagger}(\infty) O_{n_3}^{(0,0)}(1) O_{n_2}^{(0,0)}(u,\bar{u}) O_{n_1}^{(0,0)}(0)\rangle = C_4 \,\#(\text{Diagrams}) .$$

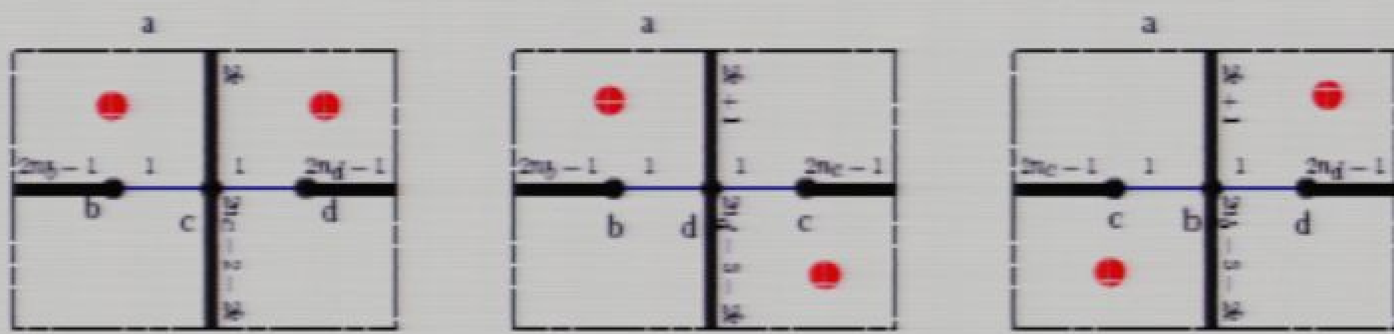
- To fix C_4 we can again try compute OPE limits. However, now the OPEs are not singular and in principle it is not clear how to disentangle the single cycle from multi cycle contributions.
- A natural "eduacted" guess is that

$$C_4 \, \tilde{n} = \langle \text{3-point} \rangle_1 \, \langle \text{3-point} \rangle_2 ,$$

where $\tilde{n}$ is the number of diagrams producing the relevant three point functions in the OPE limit. This guess can be rigorously motivated by deforming the operator "away" from the extremal/chiral point.

Extremal correlators 2.

- How many maps/diagrams contribute in an *OPE* limit?
- A diagram is expected to contribute in an *OPE* limit if two twist fields brought together share a color and do not commute. We can just count the diagrams with this property.
- The structure of the twist fields of an extremal correlator of the previous slide is $\langle \sigma_{n_b}\sigma_{n_c}\sigma_{n_d}\sigma_{n_b+n_c+n_d-2}\rangle$
-



$(I)\ :\ n_c$ $\qquad (II)\ :\ n_d-1$ $\qquad (III)\ :\ n_b-1$

- The diagrams which contribute to *OPE* of σ_{n_b} with σ_{n_c} are (I) and (III) above. Thus, the number of different *OPE* limits is $\bar{n}=n_b+n_c-1$.
- From here we get that

$$\langle O^{(0,0)\dagger}_{n_4}(\infty)O^{(0,0)}_{n_3}(1)O^{(0,0)}_{n_2}(u,\bar{u})O^{(0,0)}_{n_1}(0)\rangle = C_4\,\#(\text{Diagrams}) =$$

$$= \left[\frac{(N-n_1)!(N-n_2)!(N-n_3)!}{(N-n_4)!(N!)^2}\right]^{1/2}\frac{n_4^{5/2}}{(n_1n_2n_3)^{1/2}}.$$

p-point extremal correlators

- This picture generalizes to any p-point extremal correlator.
- By OPE arguments one gets recurssion relations of the following form

$$C_p \,\#(\text{Number of OPE diagrams}) = \langle\text{3-point}\rangle\, C_{p-1}\, H_{p-1},$$

where $H_p = n_p^{p-3}$ ($n_p = \sum_{i=1}^{p-1} n_i - p + 2$) is the number of diagrams for the relevant p-point extremal correlator.

- $\#$(Number of OPE diagrams) can be shown to be equal to $(n_1 + n_2 - 1)\, n_p^{p-4}$.
- These recursions can be solved to give

$$\langle O_{n_p}^{(0,0)\dagger} O_{n_{p-1}}^{(0,0)} \cdots O_{n_2}^{(0,0)} O_{n_1}^{(0,0)} \rangle = \left(\frac{1}{N}\right)^{\frac{p-2}{2}} \frac{(n_p)^{p-3/2}}{(n_1 n_2 \cdots n_{p-1})^{1/2}},$$

$$\langle O_{n_p}^{(2,2)\dagger} O_{n_{p-1}}^{(2,2)} O_{n_{p-2}}^{(0,0)} \cdots O_{n_1}^{(0,0)} \rangle = \left(\frac{1}{N}\right)^{\frac{p-2}{2}} \frac{(n_p)^{p-7/2}(n_{p-1})^{3/2}}{(n_{p-2} \cdots n_1)^{1/2}},$$

$$\langle O_{n_p}^{(b,\bar b)\dagger} O_{n_{p-1}}^{(a,\bar a)} O_{n_{p-2}}^{(0,0)} \cdots O_{n_1}^{(0,0)} \rangle = \delta^{ab}\delta^{\bar a\bar b} \left(\frac{1}{N}\right)^{\frac{p-2}{2}} \frac{(n_p)^{p-5/2}(n_{p-1})^{1/2}}{(n_{p-2} \cdots n_1)^{1/2}},$$

$$\langle O_{n_p}^{(2,2)\dagger} O_{n_{p-1}}^{(a,\bar a)} O_{n_{p-2}}^{(b,\bar b)} O_{n_{p-3}}^{(0,0)} \cdots O_{n_1}^{(0,0)} \rangle = \epsilon^{ab}\epsilon^{\bar a\bar b} \left(\frac{1}{N}\right)^{\frac{p-2}{2}} \frac{(n_p)^{p-7/2}(n_{p-1} n_{p-2})^{1/2}}{(n_{p-3} \cdots n_1)^{1/2}}.$$

Extremal correlators compute Hurwitz numbers

- Rescaling the operators,

$$\begin{aligned}
O_n^{(0,0)} &\rightarrow \hat{O}_n^{(0,0)} = n^{1/2}\, O_n^{(0,0)}, \\
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\end{aligned}$$

- the p-point extremal correlators are proportional to Hurwitz numbers

$$\boxed{\left(\frac{1}{N}\right)^{\frac{p-2}{2}} H_p(\{n_i\})\ .}$$

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$$\langle \mathcal{O}^{(0,0)\dagger}_{n_p} \mathcal{O}^{(0,0)}_{n_{p-1}} \cdots \mathcal{O}^{(0,0)}_{n_2} \mathcal{O}^{(0,0)}_{n_1} \rangle = \left(\frac{1}{N}\right)^{\frac{p-2}{2}} \frac{(n_p)^{p-3/2}}{(n_1 n_2 \cdots n_{p-1})^{1/2}},$$

$$\langle \mathcal{O}^{(2,2)\dagger}_{n_p} \mathcal{O}^{(2,2)}_{n_{p-1}} \mathcal{O}^{(0,0)}_{n_{p-2}} \cdots \mathcal{O}^{(0,0)}_{n_1} \rangle = \left(\frac{1}{N}\right)^{\frac{p-2}{2}} \frac{(n_p)^{p-7/2}(n_{p-1})^{3/2}}{(n_{p-2} \cdots n_1)^{1/2}},$$

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\boxed{\left(\frac{1}{N}\right)^{\frac{p-2}{2}} H_p(\{n_i\})\ .}
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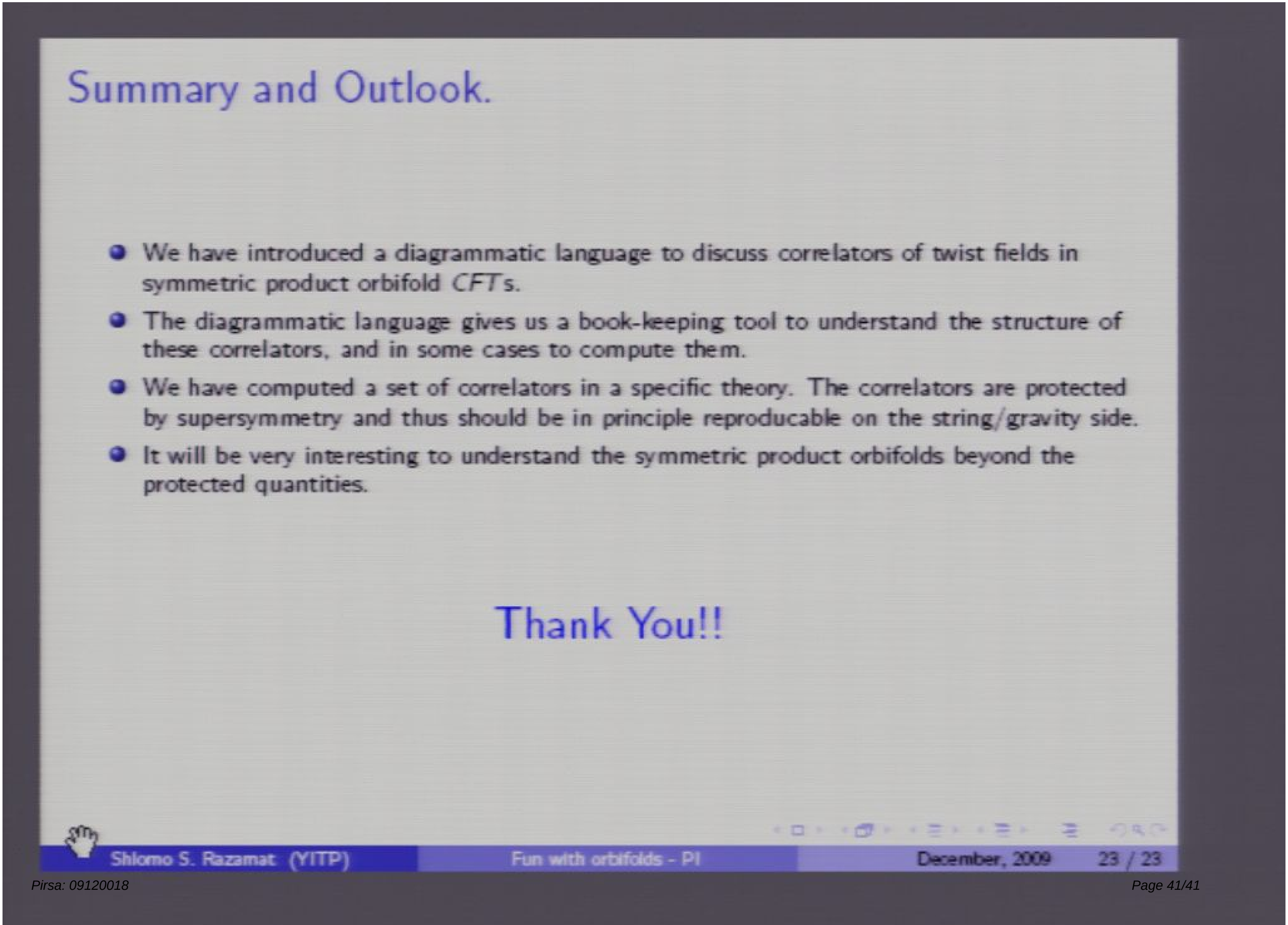
Summary and Outlook.

- We have introduced a diagrammatic language to discuss correlators of twist fields in symmetric product orbifold *CFT*s.
- The diagrammatic language gives us a book-keeping tool to understand the structure of these correlators, and in some cases to compute them.
- We have computed a set of correlators in a specific theory. The correlators are protected by supersymmetry and thus should be in principle reproducable on the string/gravity side.
- It will be very interesting to understand the symmetric product orbifolds beyond the protected quantities.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Thank You!!
Shlomo S. Razamat (YITP)
Fun with orbifolds - PI
December, 2009
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