Title: Recursion relations in spin foam models

Date: Dec 09, 2009 04:00 PM

URL: http://pirsa.org/09120011

Abstract: Spin foam models aim at defining non-perturbative and background independent amplitudes for quantum gravity. In this work, I argue that the dynamics and the geometric properties of spin foam models can be nicely studied using recursion relations. In 3d gravity and in the 4d Ooguri model, the topological invariance leads to recursion relations for the amplitudes. I also derive recursions from the action of holonomy operators on spin network functionals. Their geometric content is discussed in terms of elementary moves on simplices, and is related to the classical constraints of the underlying theories. Interestingly, our recurrence relations apply to any SU(2) invariant symbol. Another method is considered for non-topological objects, and applied to the 10j-symbol which defines the Barrett-Crane spin foam model.

Recursion relations in spin foam models

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Collaboration with E.R. Livine (ENS Lyon) and S. Speziale (CPT)

Introduction

3d gravity

4d BF - Ooguri's model

The Barrett-Crane model

conclusion

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Introduction

Spin foam amplitudes $\mathcal{A}$ provide a non-perturbative and background independent definition for the path integral :

$$\mathcal{A}[q_{ab}] \sim K[q_{ab}] \equiv \int_{q_{ab}} \mathcal{D}g_{\mu\nu}\, e^{iS_{\mathrm{GR}}[g_{\mu\nu}]}. \qquad (1)$$

$K[q_{ab}]$ projects onto the kernels of the diffeomorphism and hamiltonian constraints :

$$\hat{\mathcal{H}}_a(q_{ab}, \delta/\delta q_{ab})\, K[q_{ab}] = 0, \qquad \hat{\mathcal{H}}(q_{ab}, \delta/\delta q_{ab})\, K[q_{ab}] = 0. \quad (2)$$

Therefore, expect spin foam models to implement some version of these equations !

Spin foams

$$\mathcal{A}[q_{ab}] = \sum_{v^\star} \lambda^{v^\star} c_{v^\star}[q_{ab}], \tag{3}$$

- $c_{v^\star}$ are algebraic quantities coming from Lie groups.
- Shown in the past years : $c_{v^\star}$ built from path integrals for discrete geometries [Conrady-Freidel, VB] : a Regge-like action, with constraints ensuring geometricity on-shell.
- As a result, the (discrete) boundary metrics take discrete values.
- Relation to spin networks, define transition amplitudes.

Quantum constraints?

- Look for constraint operator $\hat{H}$ on spin networks, and implement it via spin foams.

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  Classically [Dittrich Bahr] : Tent moves define an evolution which reflects the dynamics of the covariant action.
  Symmetries are generically broken.
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  Symmetries are generically broken.
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- Otherwise, look at the equations satisfied by quantum amplitudes !
  The boundary metrics take discrete values $\longrightarrow$ Look at difference equations coming from representation theory.

Special relations in spin foams

Some special relations satisfied by the quantum spin foam amplitudes ?

- In topological field theories, all triangulations contribute the same. The invariance under Pachner moves leads to recursion relations !
- Well-known in the 3d Ponzano-Regge model. New relations for $\mathrm{SU}(2)$ 15j-symbols from 4d BF spin foams.

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Relations to the classical constraints?

- Geometric picture of the recursion : Elementary translation of a vertex of the triangulation.
- Using the action of holonomy operators on spin-networks, we got recursions for any $\{3nj\}$-symbols. Interpretation as quantum tent moves in 3d gravity!

For non-topological theories ?

- The symmetries are generically broken at the discrete level [Dittrich and Bahr].
- Proposal : sum over bulk triangulations.

Toy example : loop quantum cosmology

$\hat{H}_{LQC}$ is a difference operator.
Write the evolution operator as a vertex expansion :

$$\langle\psi_f|e^{it\hat{H}_{LQC}}|\psi_i\rangle = \sum_{M=0}^{\infty} A_M \quad (4)$$

Each A_M contains M volume transitions.

For non-topological models

- What can we expect from looking at only one triangulation ?
  $\longrightarrow$ Probe the geometric properties of the matrix elements.
- For isoceles 6j-symbols, found a recursion stating invariance under a combination of Pachner moves.
- For the 4d Barrett-Crane model, found a recursion probing the closure of 4-simplices.
- renormalisation !

The Biedenharn-Elliott identity

Tetrahedron triangulating S^2. Only one physical state satisfying $\mathrm{SU}(2)$ flatness : $\prod_{\text{dual triangles}} \delta(\text{holonomies})$, or :

$$|\psi\rangle = \sum_{\{j_i\}} \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} |j_i\rangle \quad (5)$$

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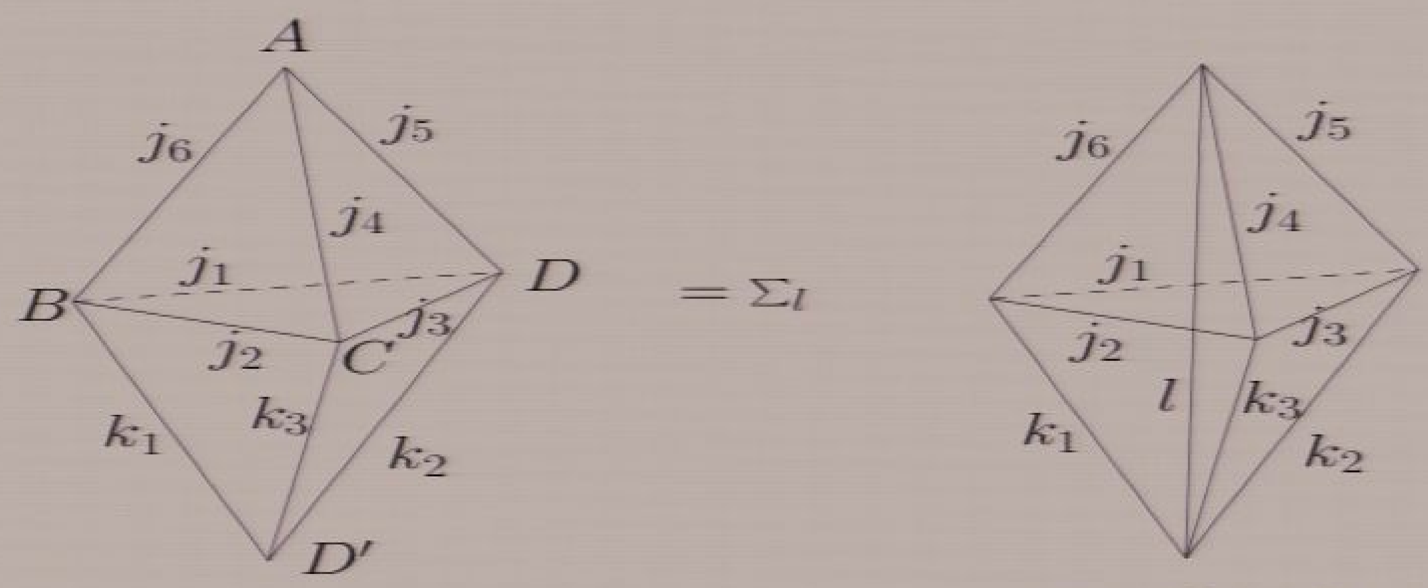
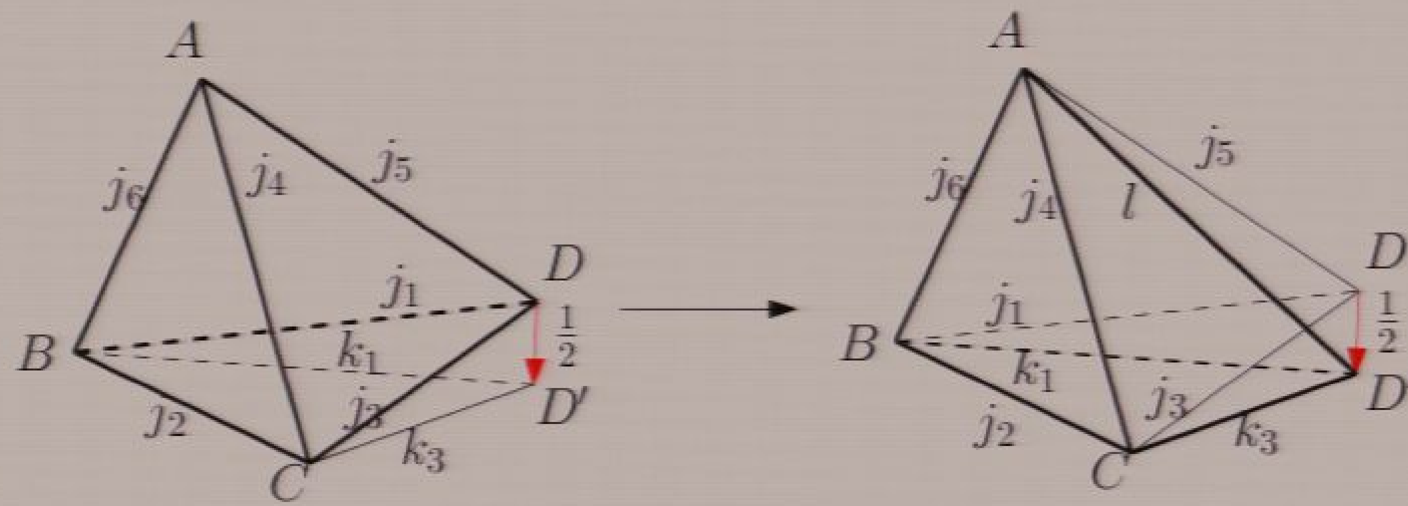
How can we characterize the 6j ?

Key ingredient in the topological Ponzano-Regge model : the Biedenharn-Elliott identity.



Interpretation of the BE identity

Take D close to D', i.e. the spin $k_2 = \frac{1}{2}$. Then k_1 and k_3 differ from j_1 and j_3 by shifts of $\frac{1}{2}$, and $l = j_5 \pm \frac{1}{2}$.

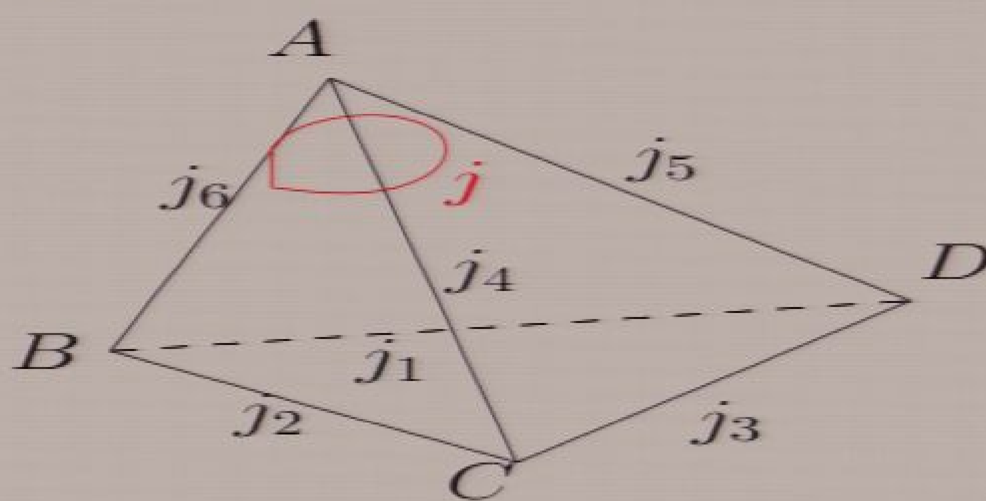


- It results in a small displacement of D to D', and a recursion relation of order 2.
- Sum over internal spin l : probe the dynamics of 3d gravity.

Recursion from holonomy operator

Consider a tetrahedron and its dual tetrahedral spin network functional, $\varphi_{\{j_l\}}(g_1, \ldots, g_6) = \prod_{l=1}^{6} D^{(j_l)}_{a_l b_l}(g_l) \prod_{n=1}^{4} i_n$.

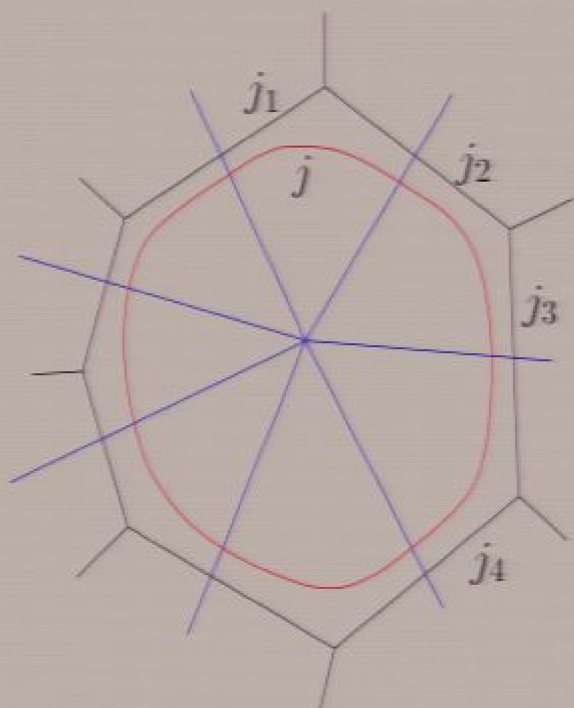
- Act with a character along the dual triangle, $\chi_j(g_4 g_5 g_6)$. Use standard recoupling, and evaluate on flat connections.



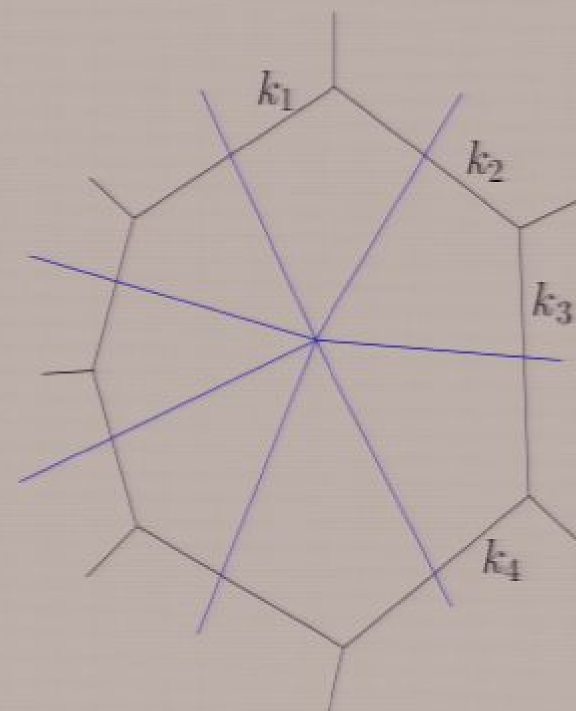
- A regularized 1-4 Pachner move! Get the naive one by acting with the flatness constraint, $\delta(g_4 g_5 g_6)$. The Freidel-Louapre gauge fixing is $j = 0$.

n-valent tent moves

Consider a n-valent vertex of the (2d) boundary triangulation. Act with holonomy operator in spin j around the vertex.



$$= \sum_{k_1,\dots,k_n} C^{j}_{j_i,k_i}$$



- Extract a 6j-symbol at each dual node.

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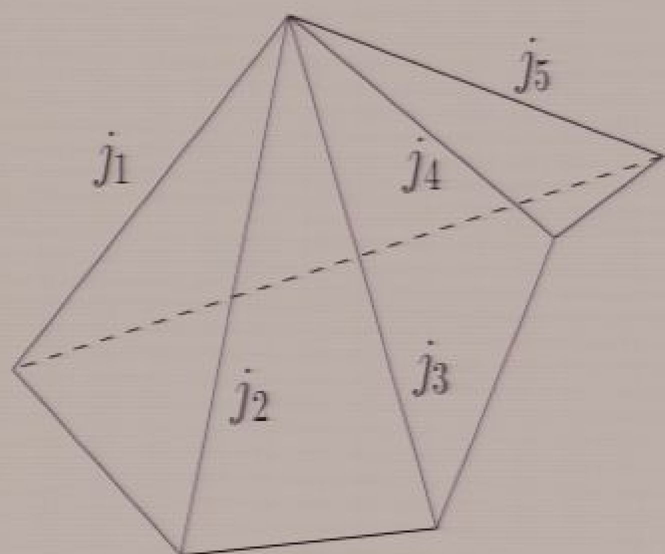

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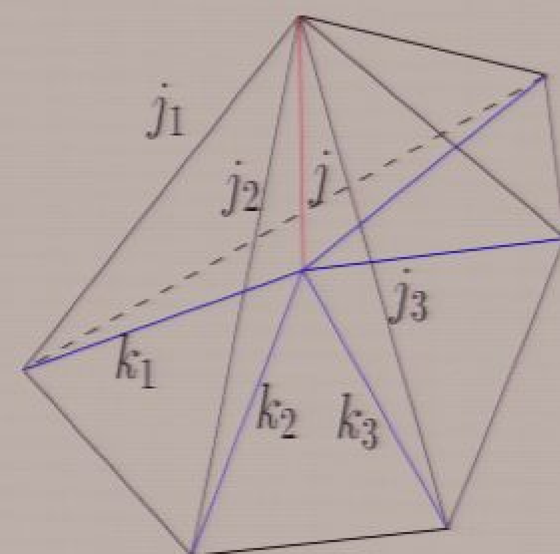
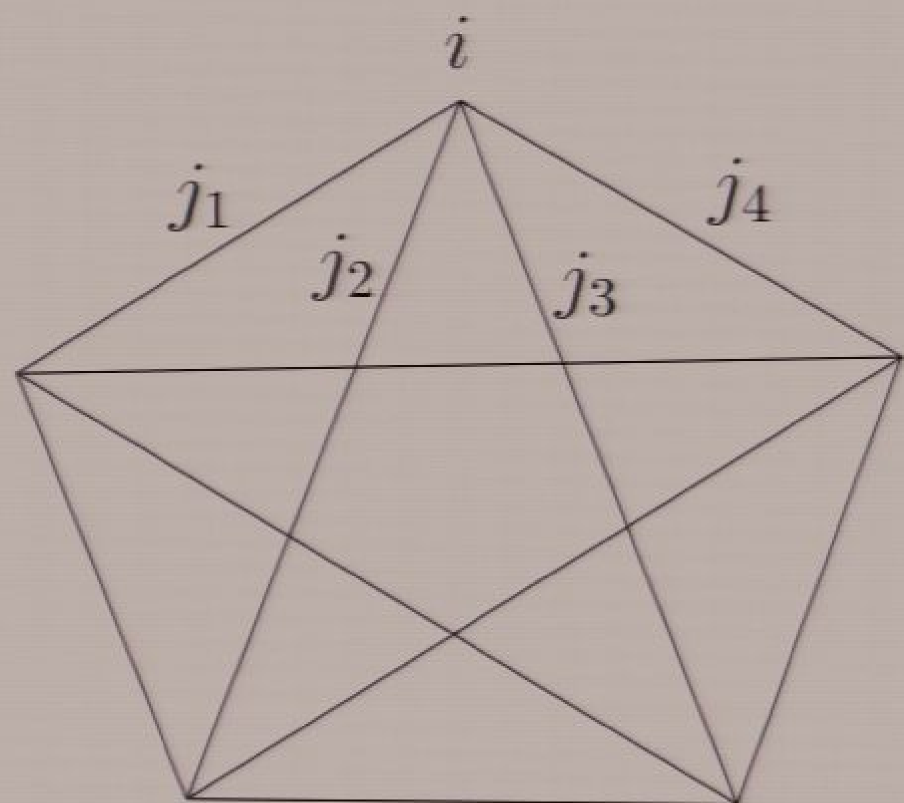


- Ponzano-Regge amplitude of the right hand side, with a fixed tent pole length $j + \frac{1}{2}$.
- Recursion of order 2 for $j = \frac{1}{2}$. The coefficients are 6j-symbols with a coefficient being $\frac{1}{2}$. Geometric content?

Topological 4d BF spin foams

- Assign $\mathrm{SU}(2)$ spins to triangles, intertwiners to tetrahedra.
- Amplitude of a 4-simplex : evaluation of a 15j-symbol (10 spins, 5 intertwiners).



- Classical constraint : $\mathrm{SU}(2)$ flatness.
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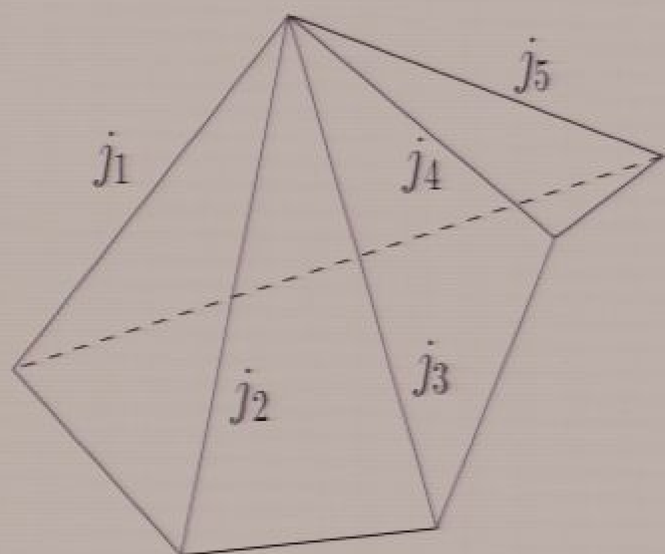

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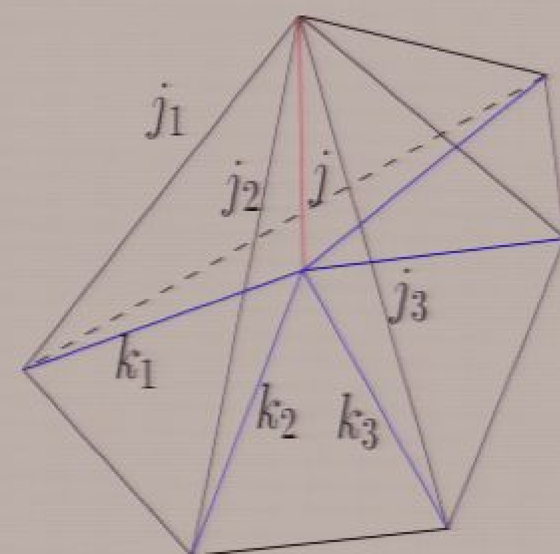

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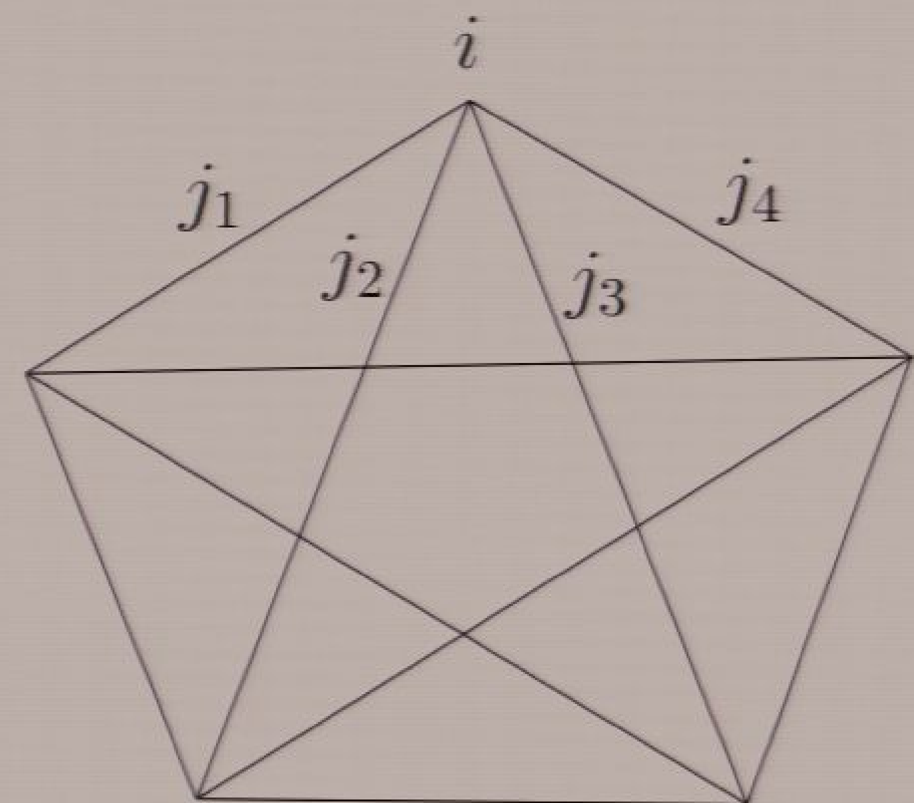

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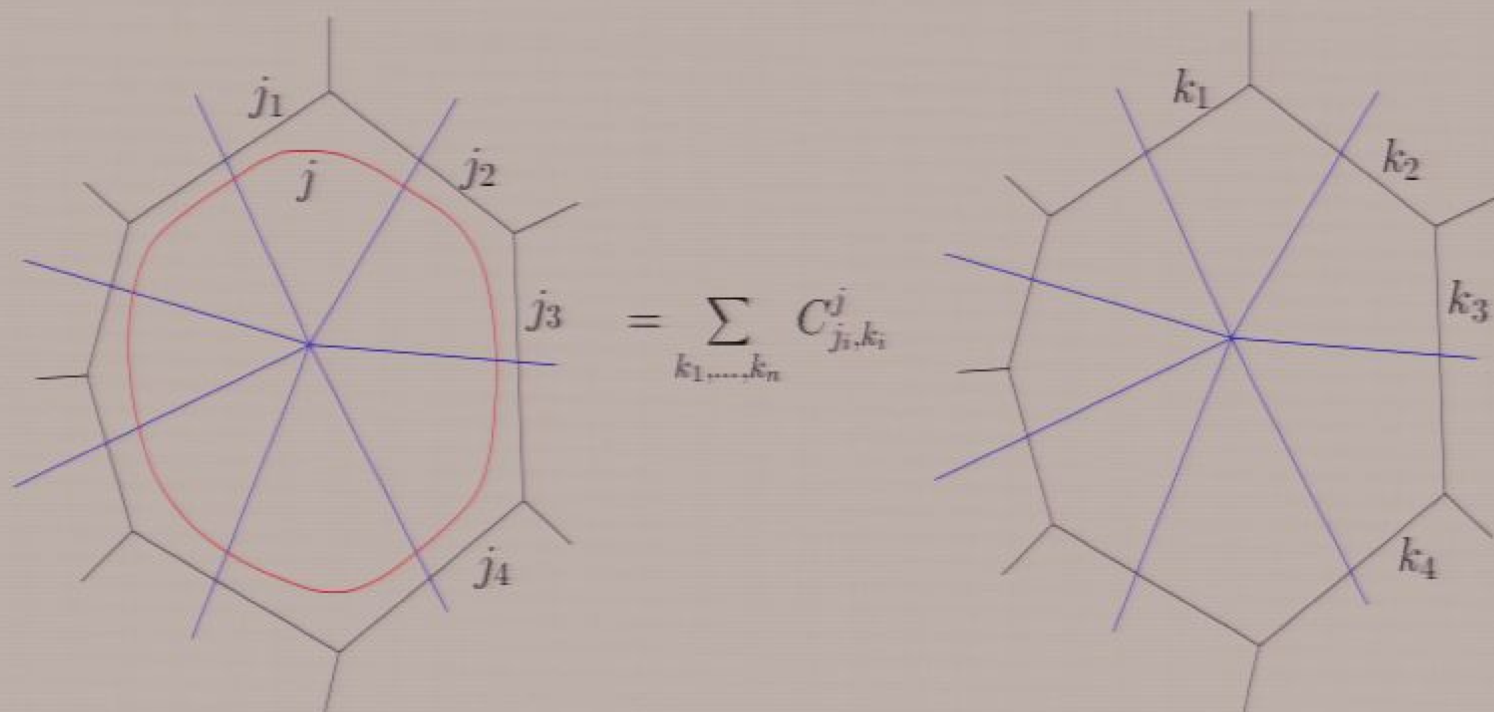

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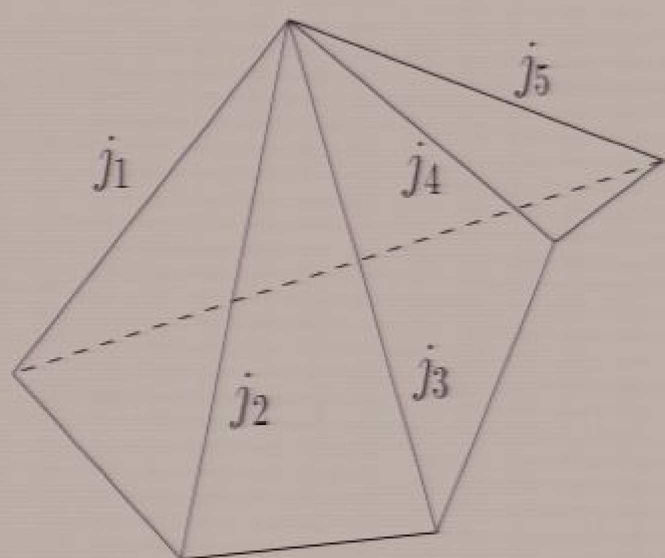

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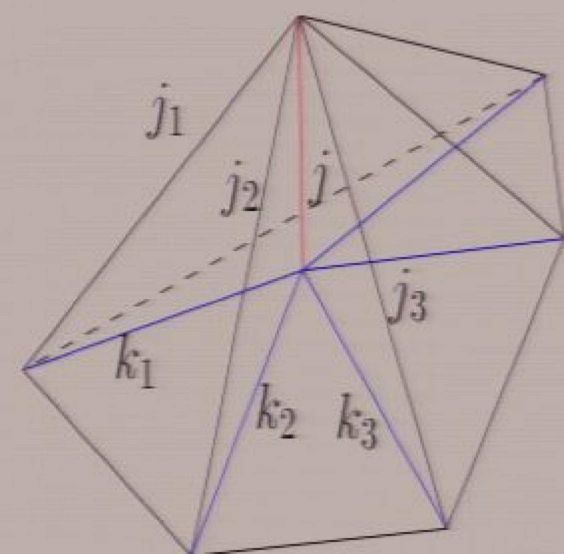

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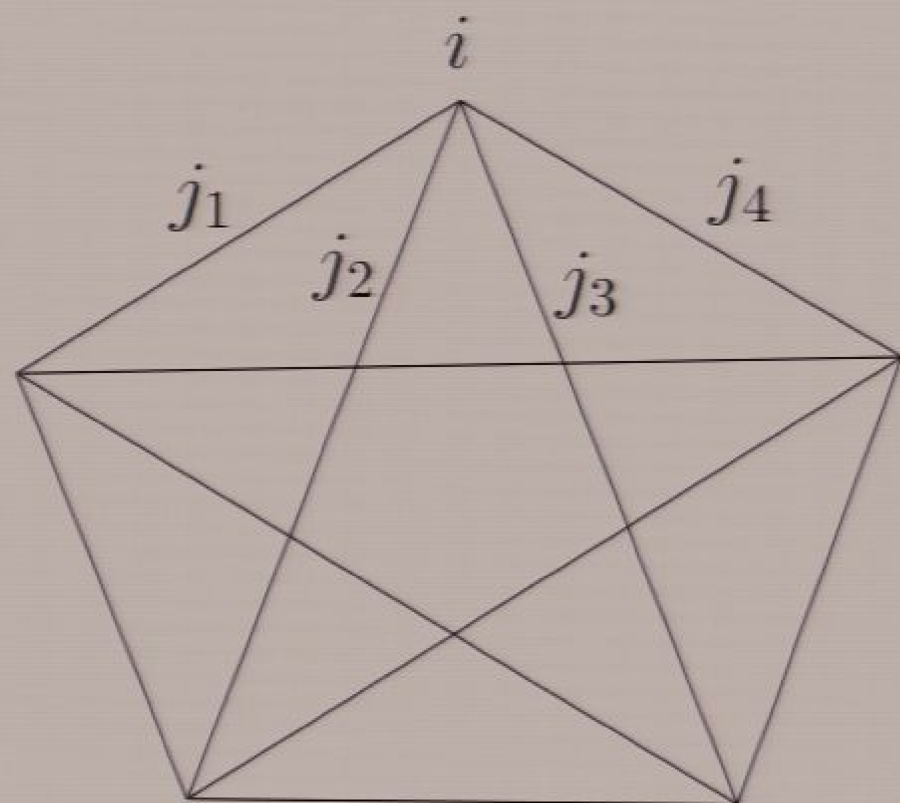

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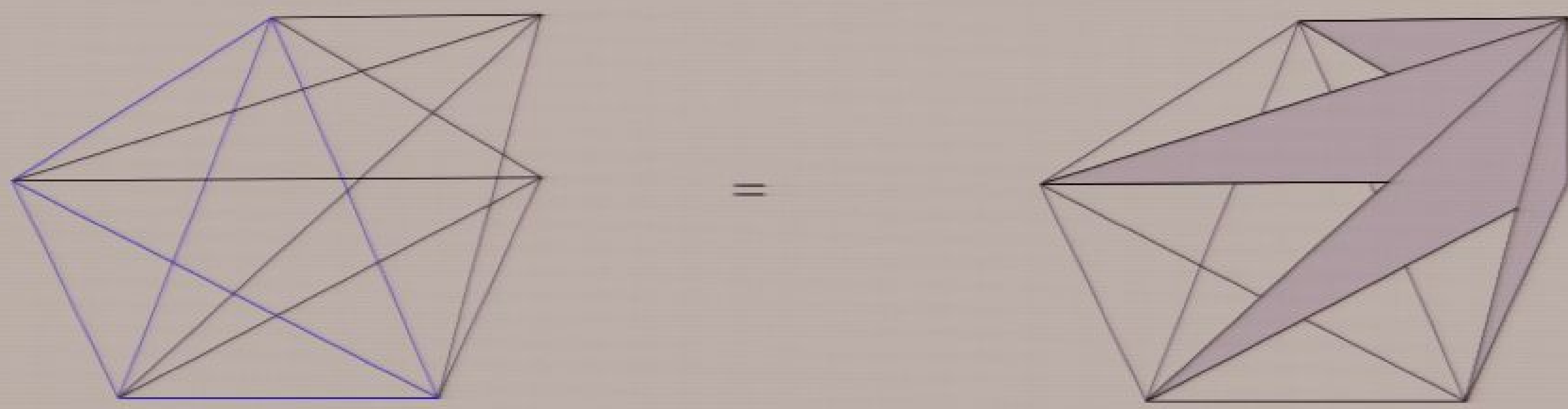
Recursion from topological invariance

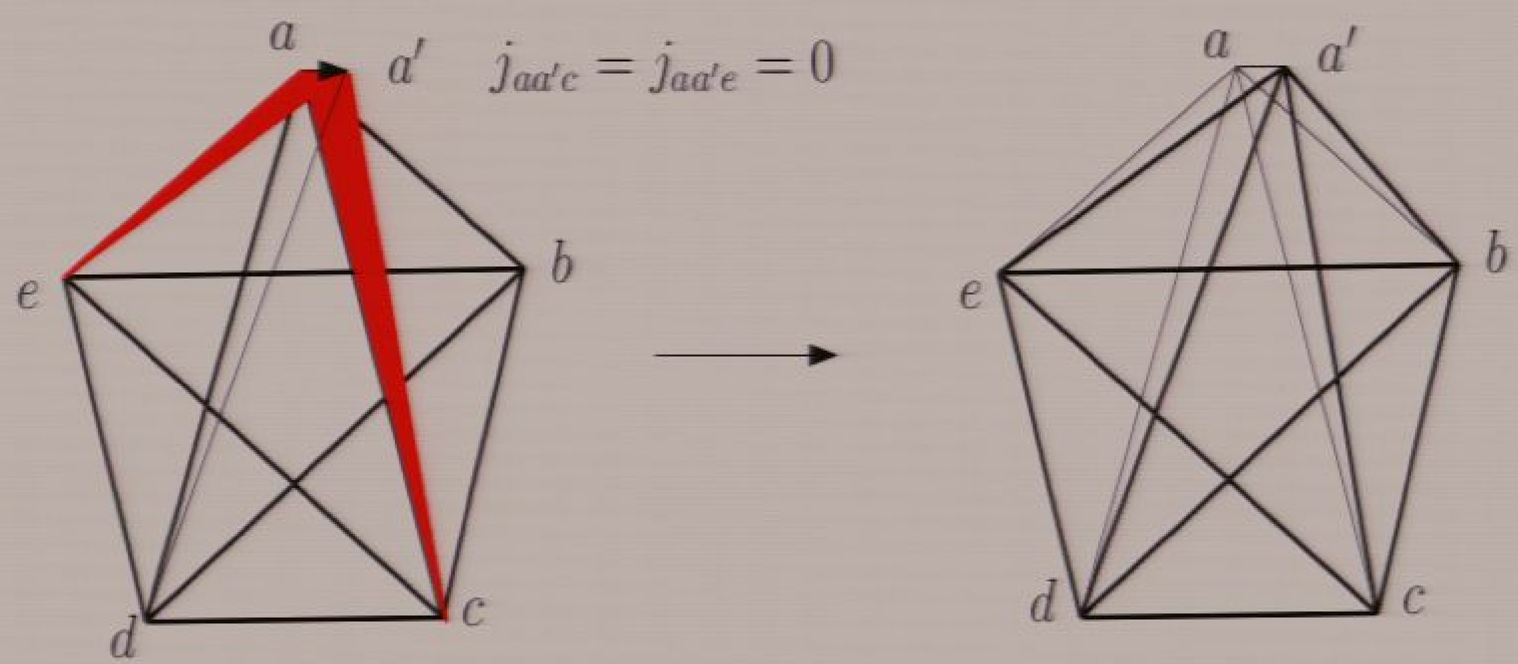
2-4 Pachner move



- Glue two 4-simplices on the left hand side. Four 4-simplices on the rhs, with four inner triangles.
- It is divergent! Too many $\delta(\textit{holonomies})$. Remove one redundant.

Recursion and 4d interpretation

Choose specific boundary data.



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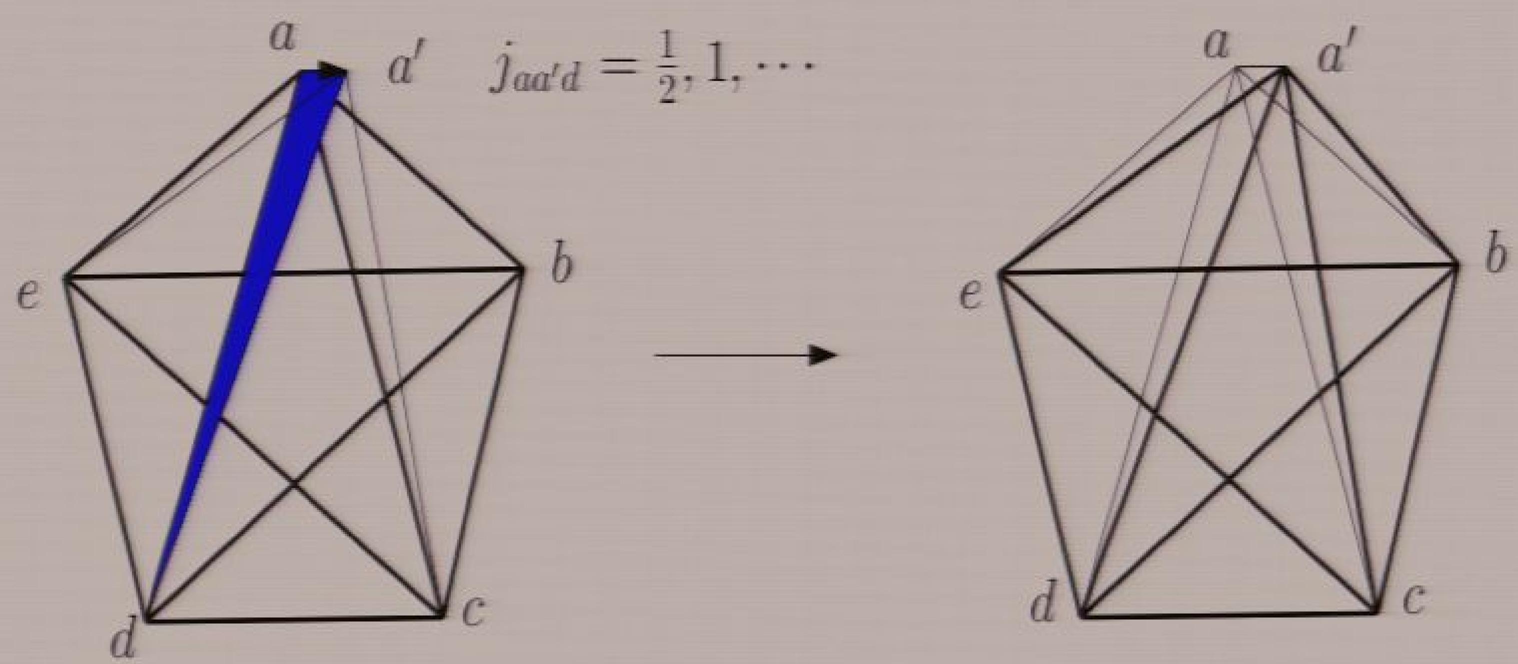

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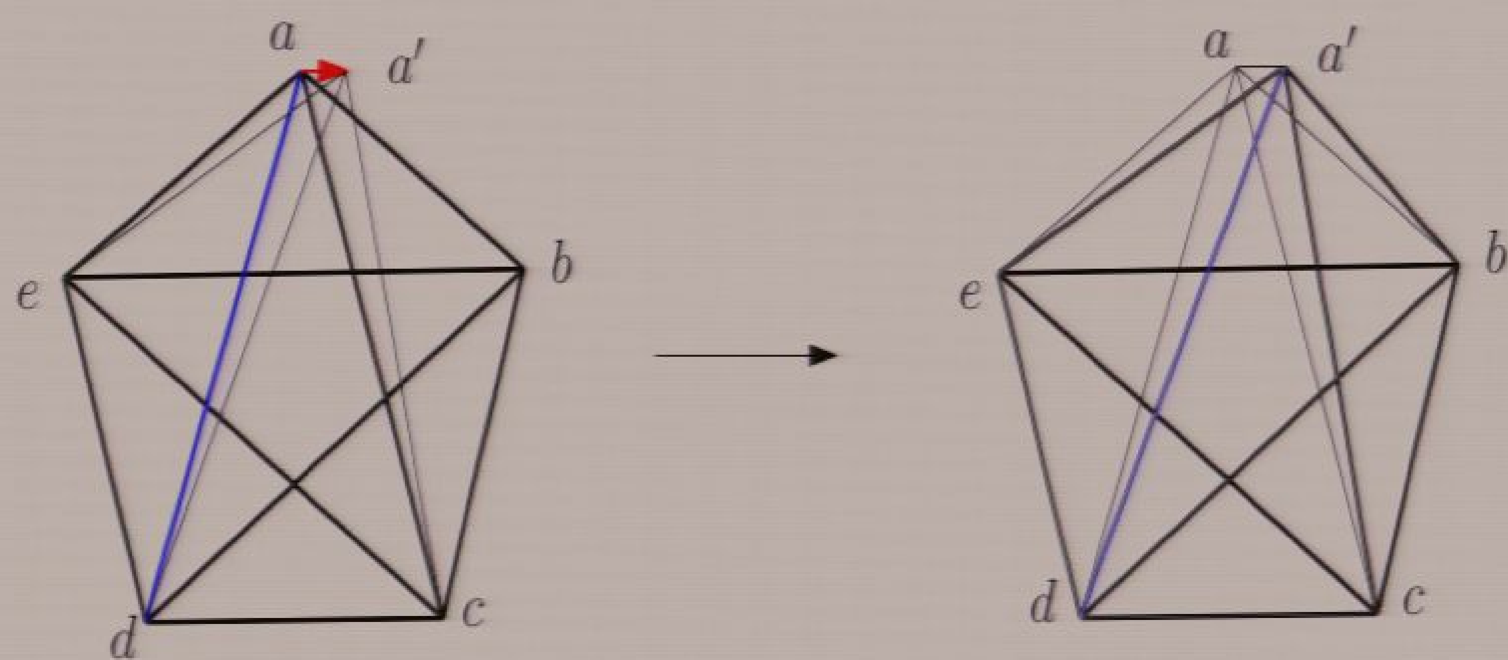

- Choose a' "close" to a.
- Shifts the spins of the three triangles meeting at the edge $(ad) \longrightarrow (a'd)$.
- Recursion relations of order 2 for 15j-symbols, acting on cycles. In fact, apply for any invariant graphs with such cycles.

The Barrett-Crane model

- Assign ten spins to triangles, their quantum areas, and built a 10j-symbol for each 4-simplex :

$$\{10j\} = \int_{\mathrm{SU}(2)^5} \prod_{a=1}^{5} dg_a \prod_{a<b} \chi_{j_{ab}}(g_a^{-1} g_b) \tag{6}$$

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- Rewrite it as an integral over ten angles :

$$\{10j\} = \int \prod_{a<b} d\theta_{ab}\ \sin\big((2j_{ab}+1)\theta_{ab}\big)\ \delta\big(\det\cos\theta_{ab}\big), \tag{7}$$

  Closure of the 4-simplex : the θ_{ab} are dihedral angles of a 4-simplex. Geometric saddle point, areas $2j_{ab}+1$.

- Contributions from degenerate configurations which hide the geometric part in the asymptotics.

Recursion for the BC model

- Insert the closure condition into the integral :

$$\int \prod_{a<b} d\theta_{ab}\ \sin\big((2j_{ab}+1)\theta_{ab}\big)\ \delta\big(\det\cos\theta_{ab}\big)\ \det\big(\cos\theta_{ab}\big) = 0.$$

- Expand the determinant, and recast the integral as a sum of 10j-symbols with shifted spins ! Typically, shifts of links along a cycle of the graph.
- Get a recursion of order 4, with trivial coefficients, which probes the closure of the 4-simplex.

Interpretation

A solution in the asymptotics

Natural candidate : the Regge action as function of areas, $S_R = \sum_{a<b}(2j_{ab}+1)\bar{\theta}_{ab}(j)$. Dihedral angles $\bar{\theta}_{ab}(j)$. Show that :

$$R \triangleright \cos(S_R(j_{ab})+\alpha) = \cos(S_R(j_{ab})+\alpha)\ \det\left(\cos\bar{\theta}_{ab}(j)\right) = 0. \quad (8)$$

- In 3d, same scaling for geometric and degenerate configurations, recursion of order 2.

Different scalings in the asymptotics $\longrightarrow$ higher order recursion.

Another example

Isoceles 6j-symbols :

$$\begin{Bmatrix} j & J & K \\ k & J & K \end{Bmatrix} = (-1)^{2j} \int_{\mathrm{SU}(2)^2} dgdh\, \chi_j(g)\, \chi_k(h)\, \chi_J(gh)\, \chi_K(gh^{-1})$$

Get a recursion directly, interpreted as an invariance under a combination of Pachner moves.

Conclusion

- In 3d and 4d BF, recursions from Pachner moves. Sum over spins : dynamical criterion !
- Geometric interpretation : vertex translation. More precise relation between classical and quantum tent moves ?

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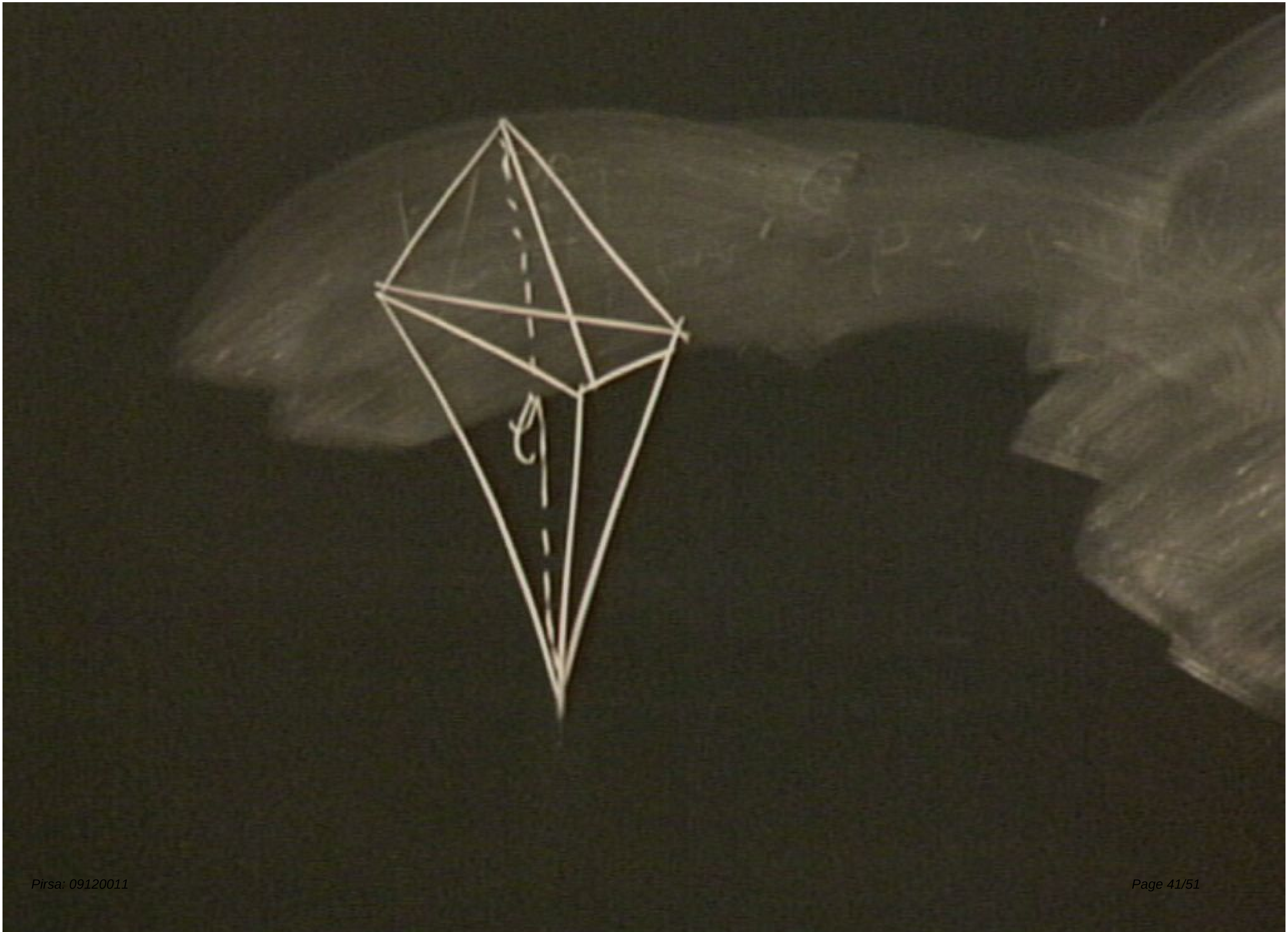
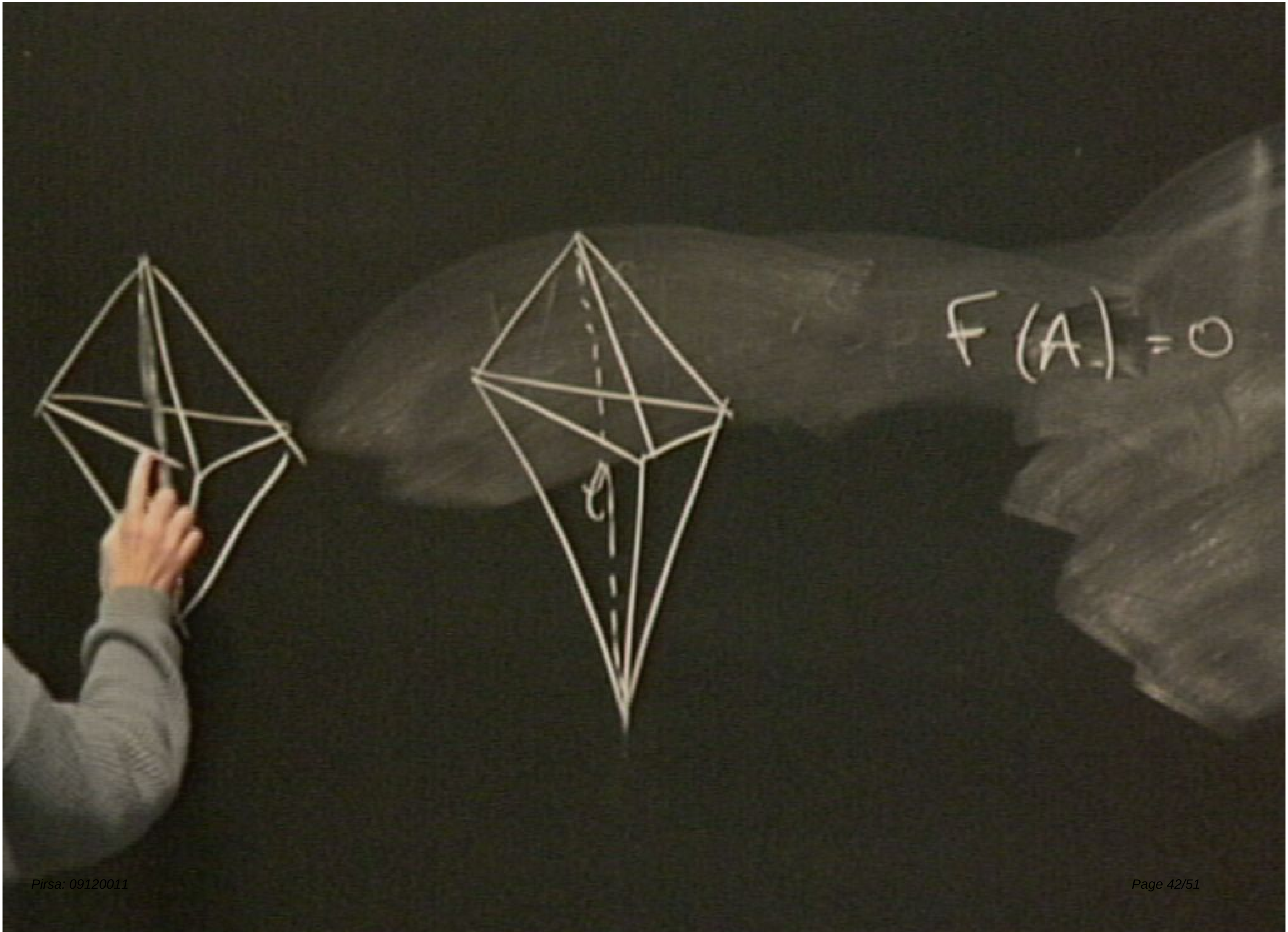
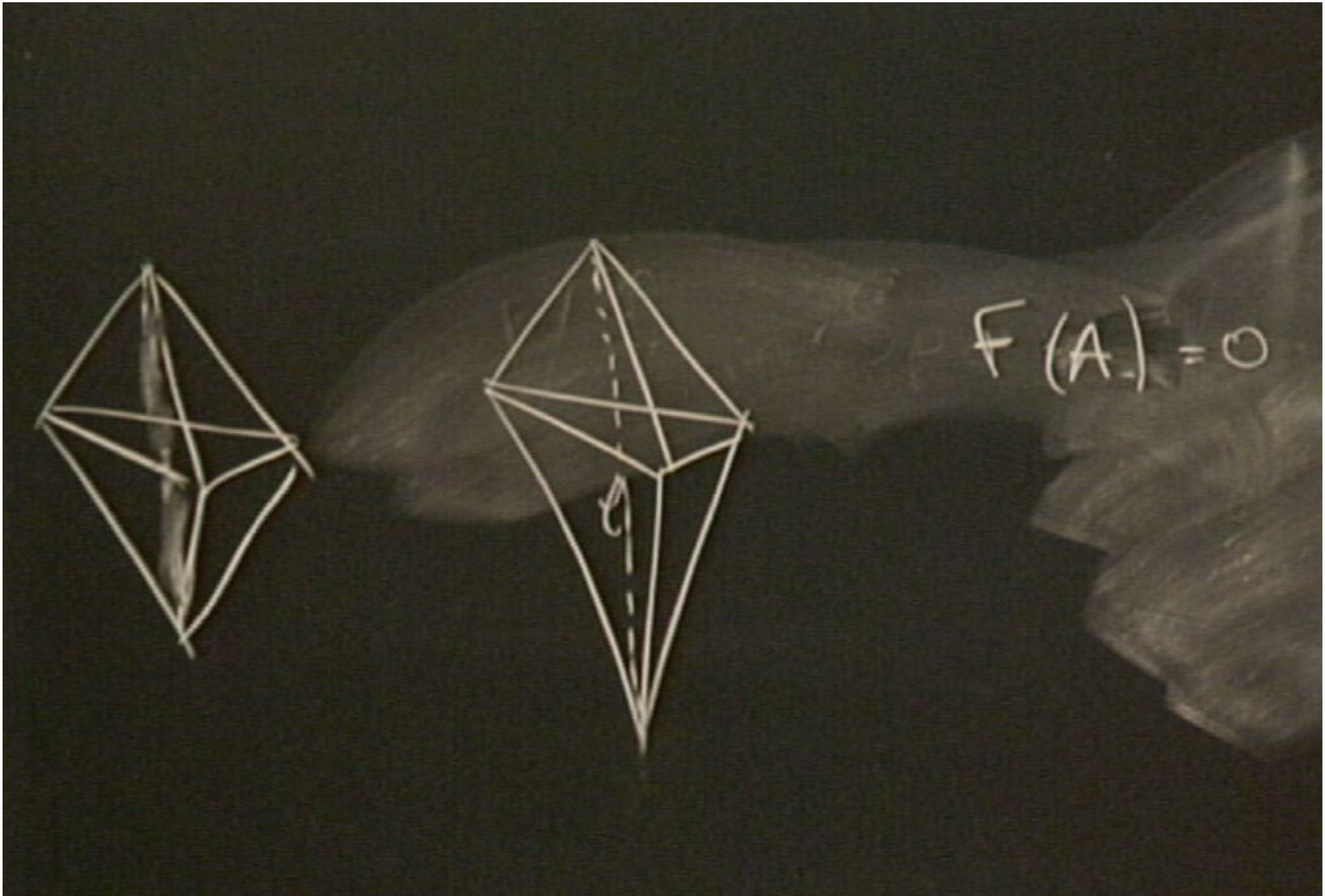
- In renormalization : classify counter-terms, extend recursion relations to group field theories !




F (A) = 0
q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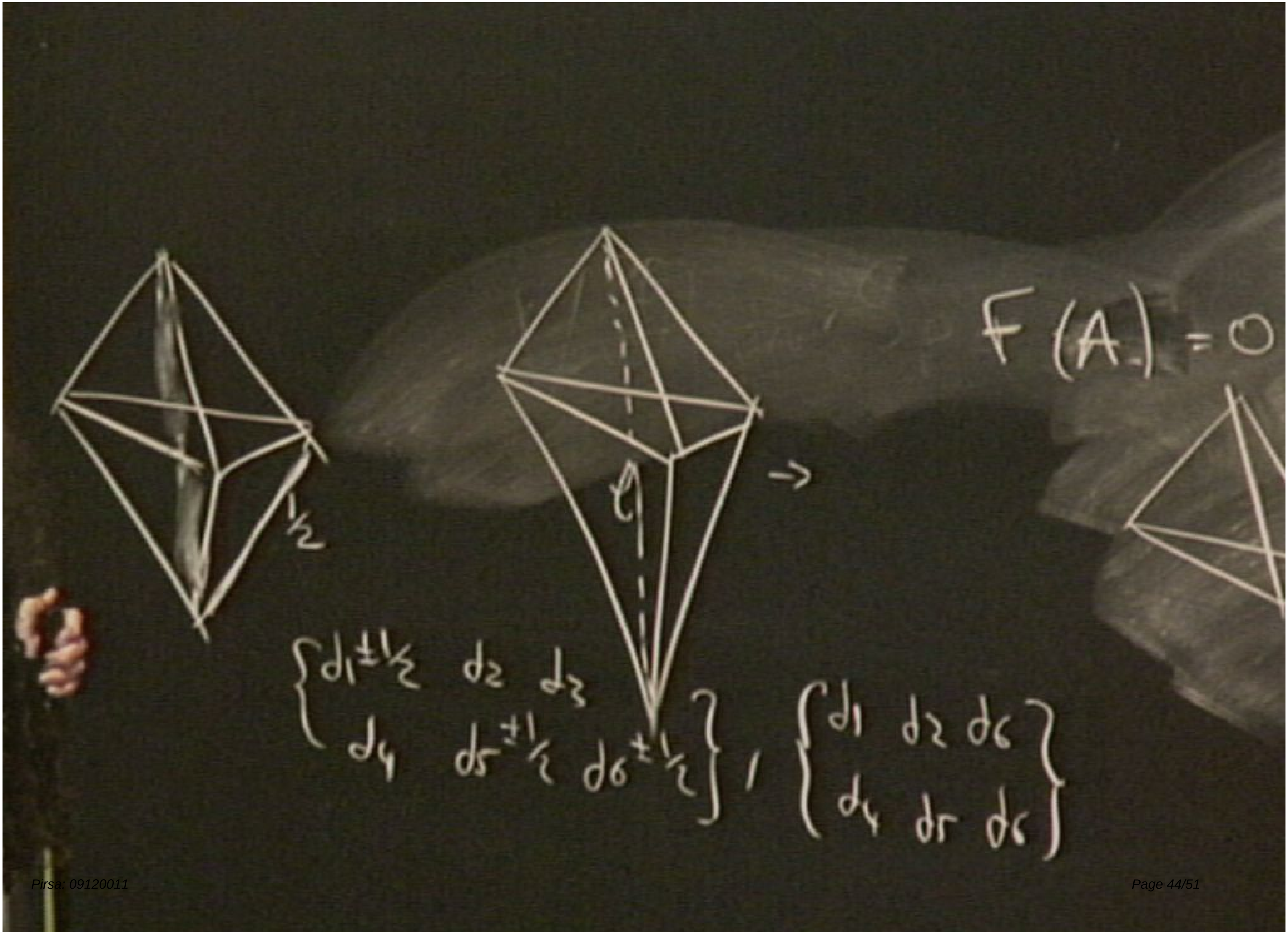
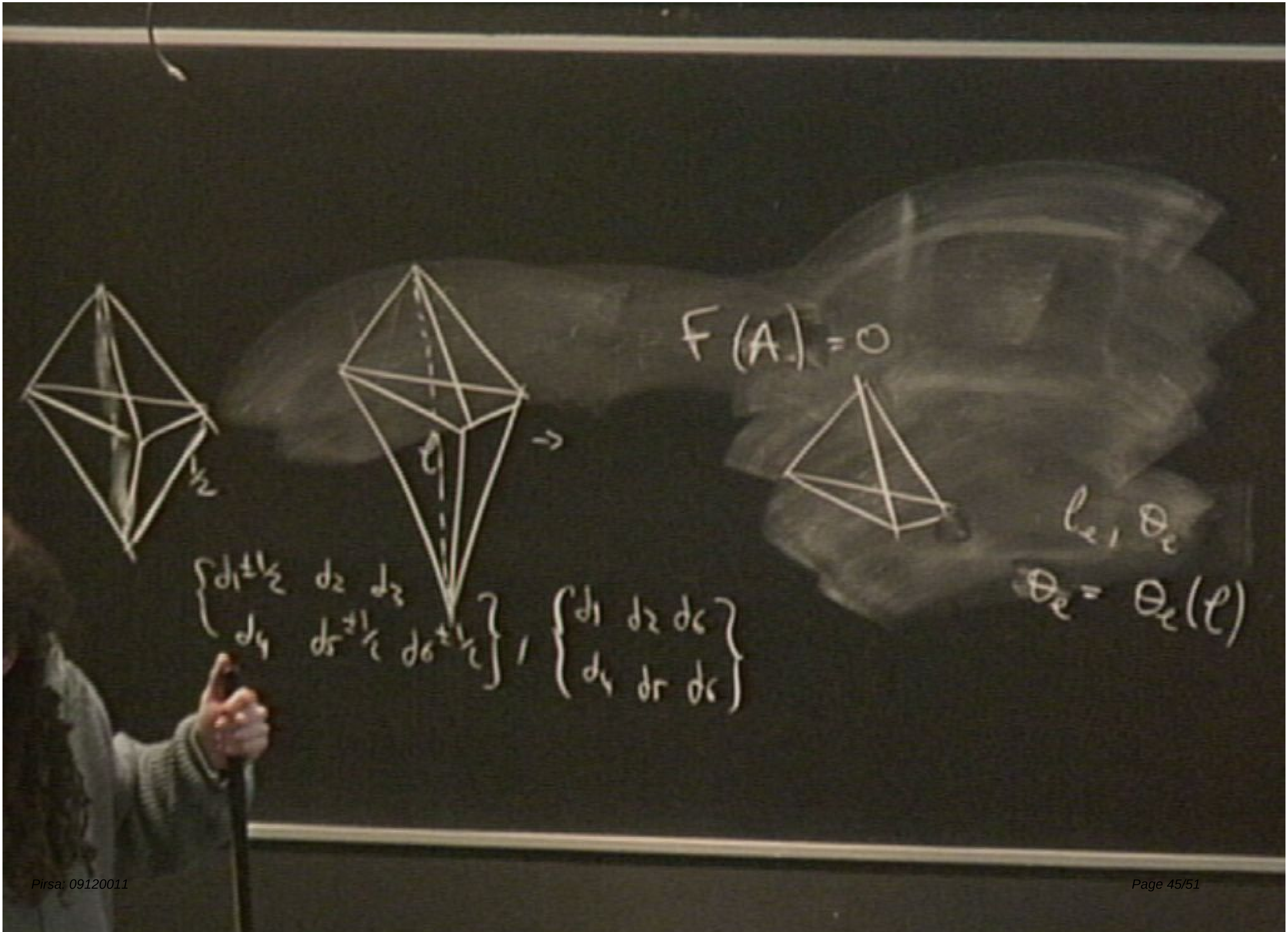

F(A) = 0
1/2
{d1 ±1/2 d2 d3
d4 d5 ±1/2 d6 ±1/2} , {d1 d2 d6
d4 d5 d6}


F(A) = 0
1/2
l
l_e, θ_e
θ_e = θ_e(l)
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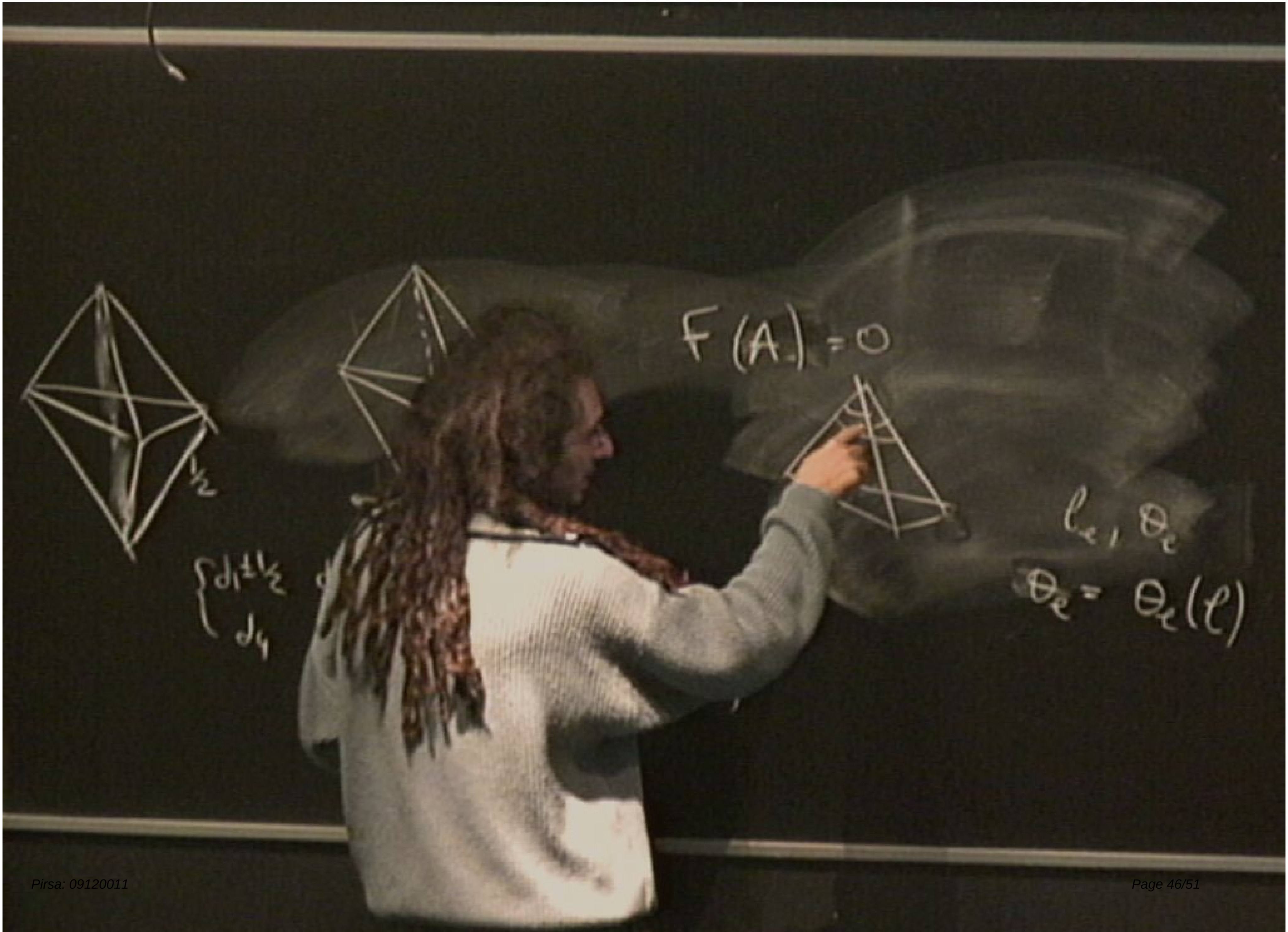
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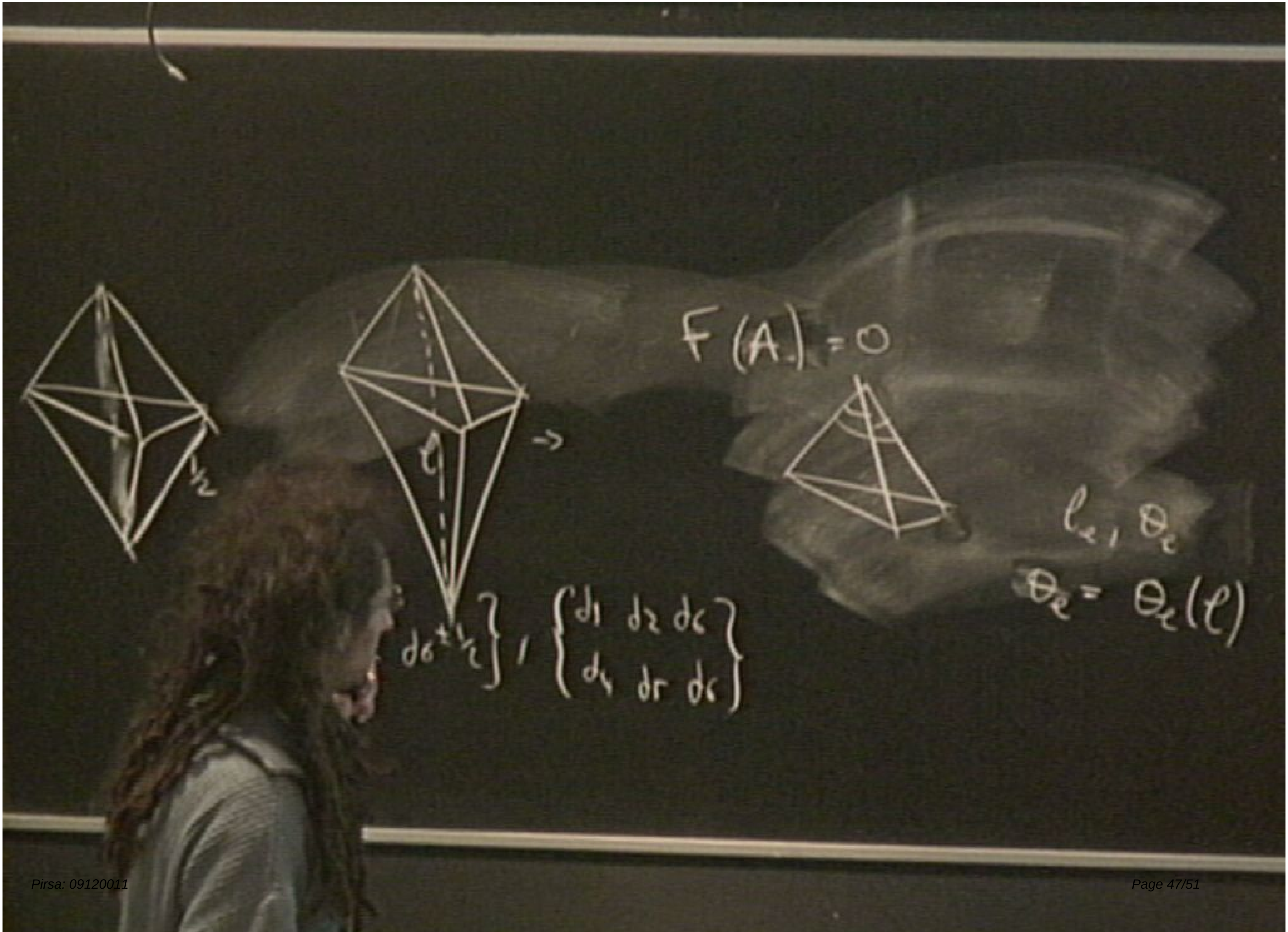
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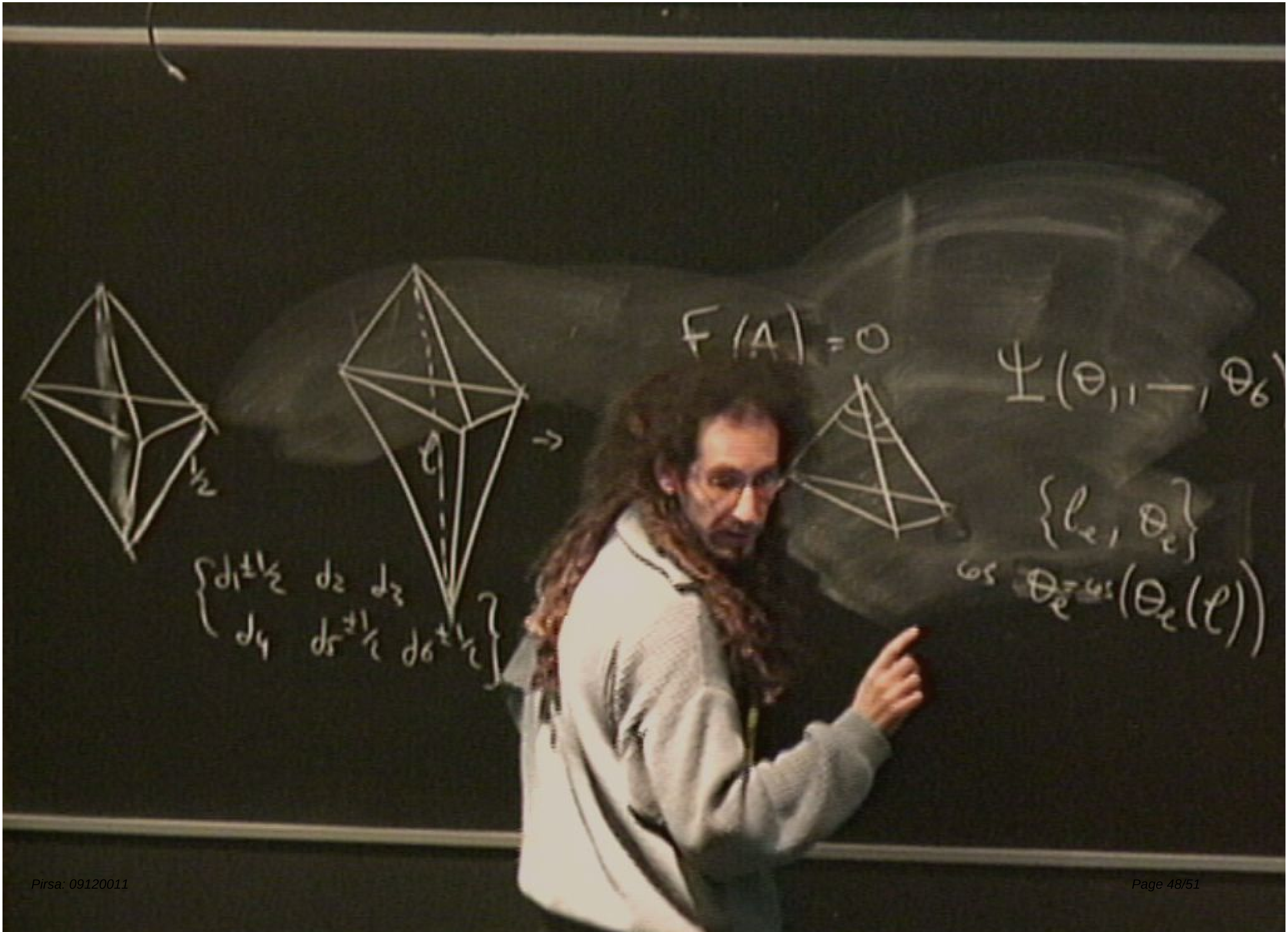
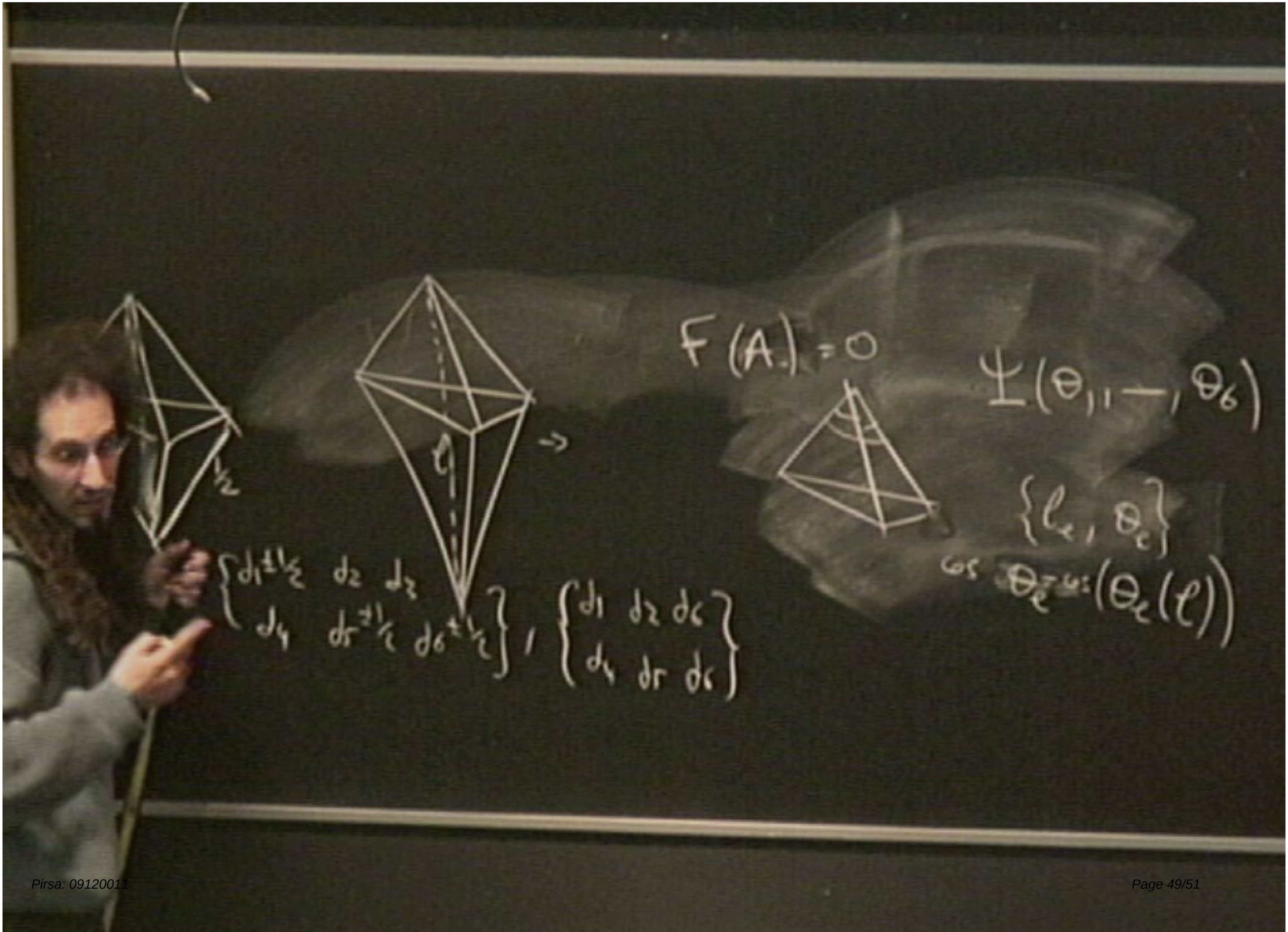

F(A) = 0
Ψ(θ1, —, θ6)
{le, θe}
cos θe = cos(θe(l))
{d1 ±1/2 d2 d3
d4 d5 ±1/2 d6 ±1/2}


F(A) = 0
Ψ(θ1, —, θ6)
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{d1±½ d2 d3 / d4 d5±½ d6±½}, {d1 d2 d6 / d4 d5 d6}
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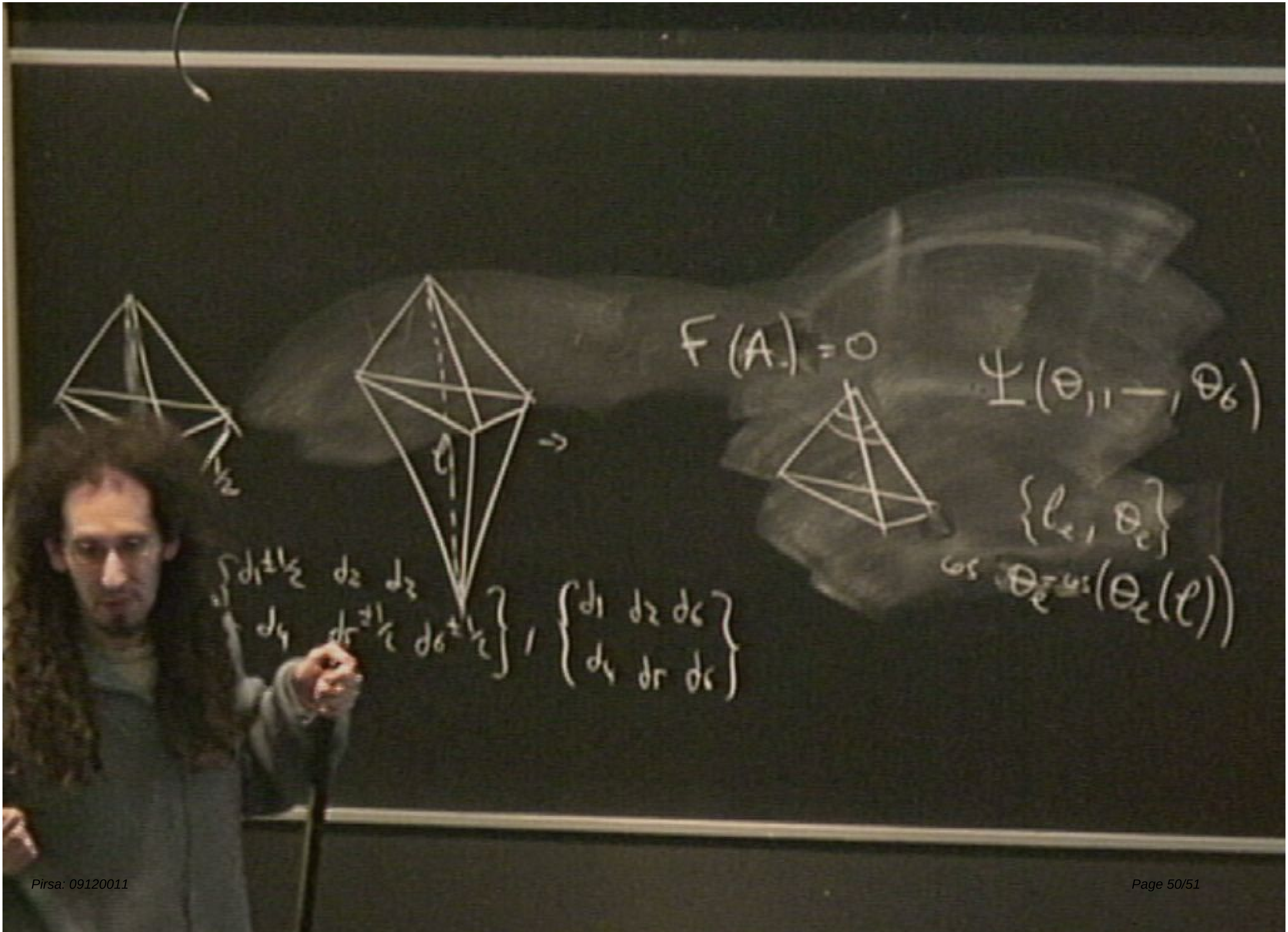
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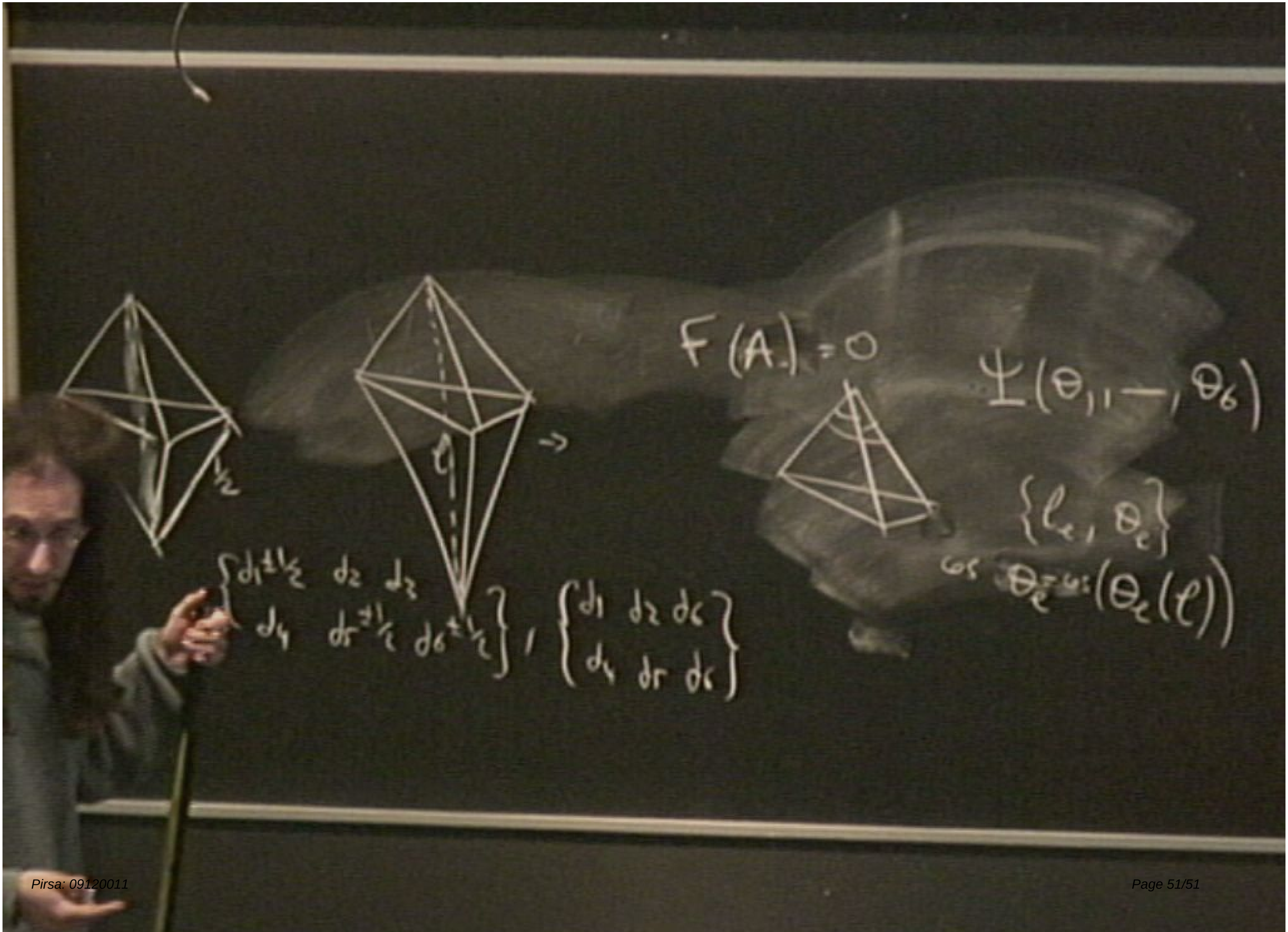
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