

Title: General Relativity for Cosmology - Lecture 16B

Date: Nov 12, 2009 05:30 PM

URL: <http://pirsa.org/09110120>

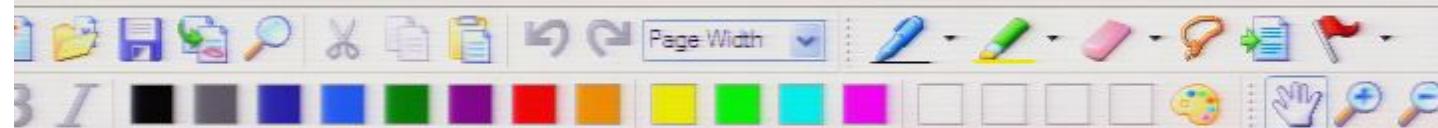
Abstract:



Singularities and Exact solutions

Recall the problem:

- Exact solutions to the Einstein equation can be obtained in practice only when restricting to highly symmetric cases.
- Examples, such as Schwarzschild black holes and FRW cosmological solutions clearly exhibit the presence of a curvature singularity.
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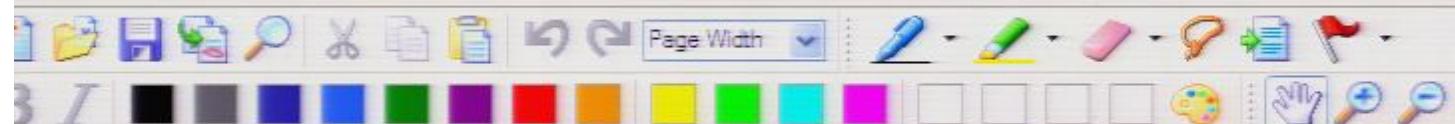
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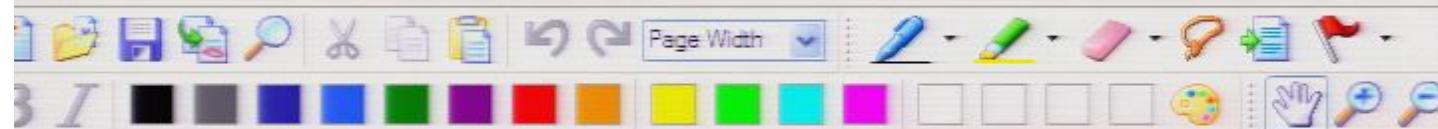
Singularity theorems \Rightarrow

Yes, the prediction of singularities is robust.

Note: Quantum Gravity, if it is to remove singularities, will have to overcome this robustness.

Strategy for singularity theorems:

- a) Focus attention on singularities that can be identified by the existence of incomplete inextendible timelike (or null) geodesics.



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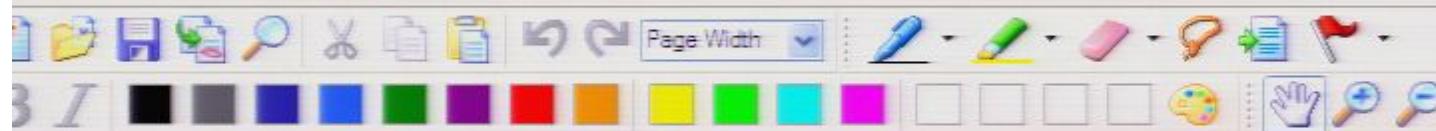
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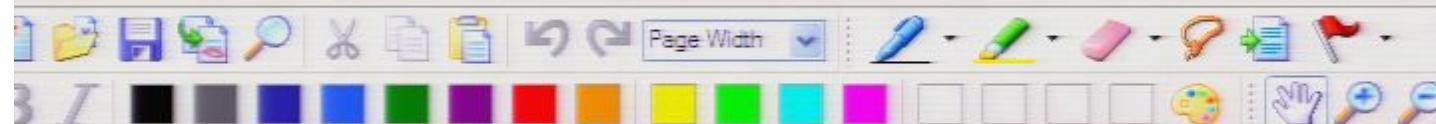
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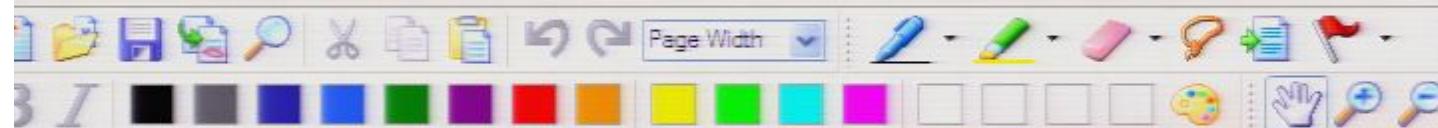
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(e.g. singularities identified through incomplete spacelike geodesics or singularities identified by some other criterion.)

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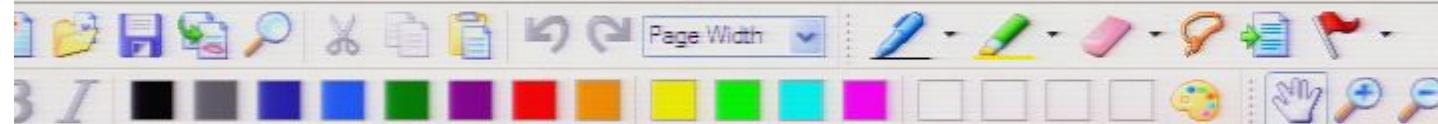
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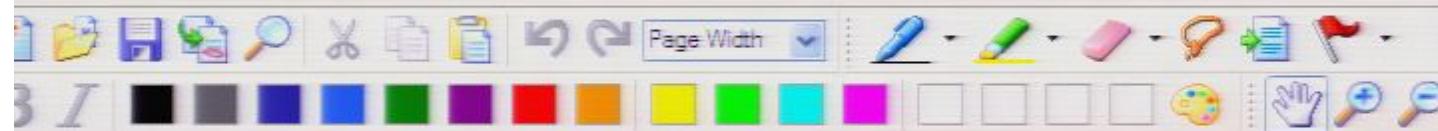
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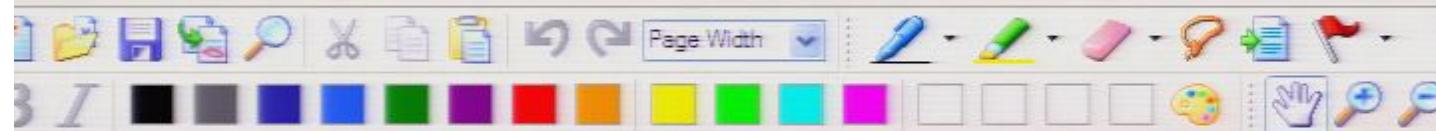
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Extremizing curve length \Rightarrow geodesic equation

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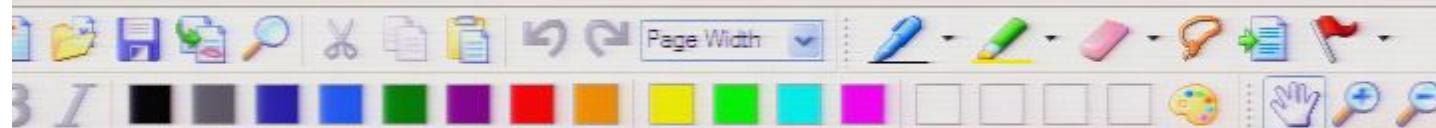
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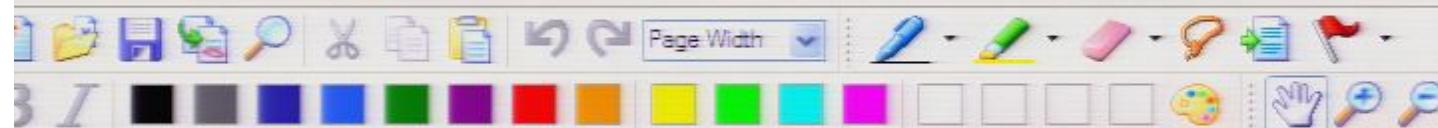


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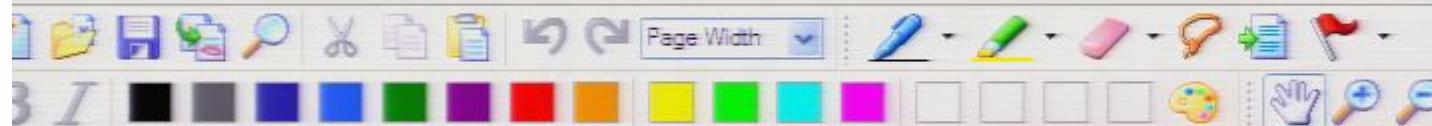
At least locally, geodesics are paths of extremal length:

- Space-like geodesics are curves of shortest proper distance.
- Time-like geodesics are curves of maximal proper time (i.e. of maximal signum).



Why maximal?

If there is a timelike curve between two events p, q , then there are timelike curves



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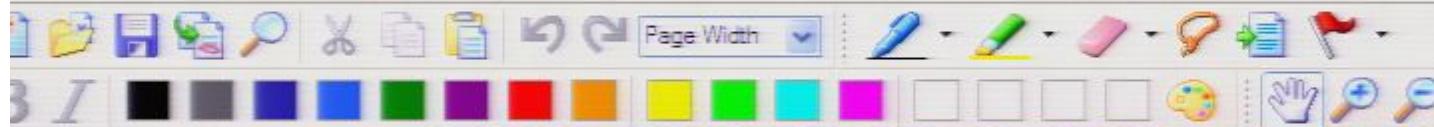
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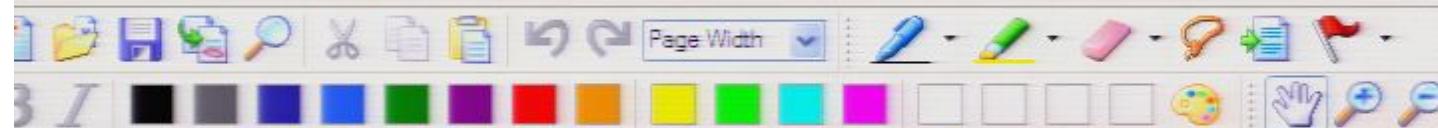
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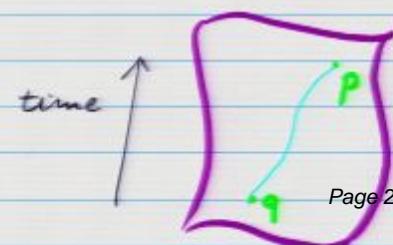
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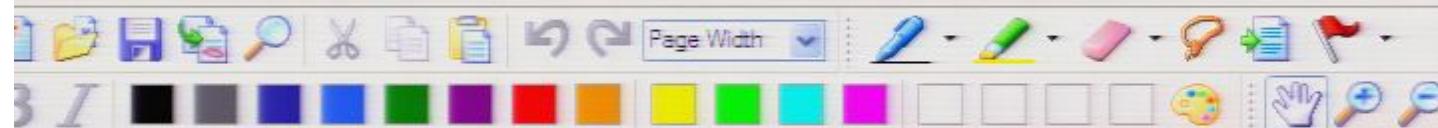
Why maximal?

If there is a timelike curve between two events p, q , then there are timelike curves with shorter signtime: just take a longer path and travel it faster.

d.) Prove that, even in generic spacetimes:

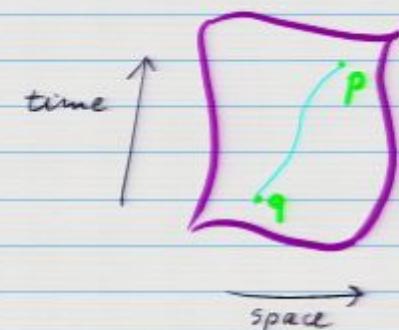
There always exist curves of maximal length between two events.





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Assumptions needed?

Yes, e.g. the assumption that spacetime is globally hyperbolic suffices.



Note: Space time may not be globally hyperbolic, let us for now disregard black holes. For the purpose, now, of studying the existence of a cosmological singularity, then space-time probably is well-described as globally hyperbolic.



e.) Prove that these extremal length curves cannot be geodesics with eigentime larger than a certain finite amount either into the past or future.

Assumptions needed?

Yes: Matter must be assumed to obey an energy condition.



f.) Conclude that there are incomplete geodesics, i.e., that we have a singularity in the past (or future) :



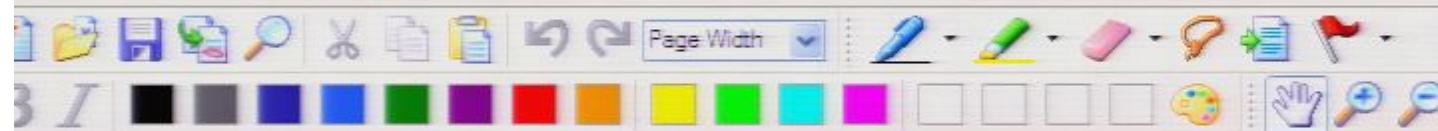
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A singularity theorem:

Assume that: $\square (M, g)$ is a globally hyperbolic spacetime

- \square The energy-momentum tensor of matter obeys the "Strong energy condition":

$$T_{\mu\nu}\xi^\mu\xi^\nu \geq -\frac{1}{2}T^0_0 \text{ for all timelike } \xi.$$

- \square There exists a C^2 spacelike Cauchy surface Σ , on which the trace of the extrinsic curvature, K , is bounded from above by a negative constant C :



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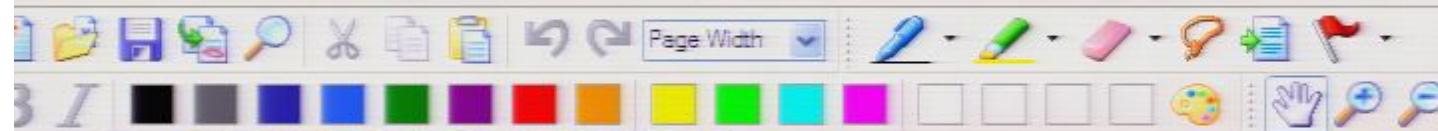
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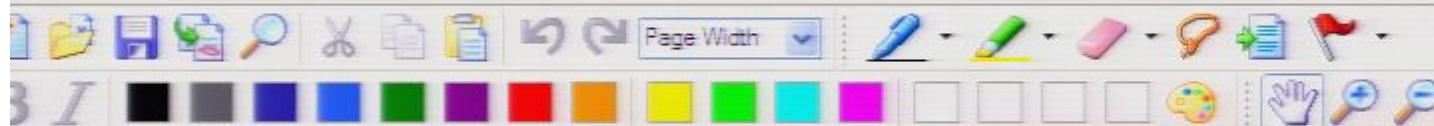
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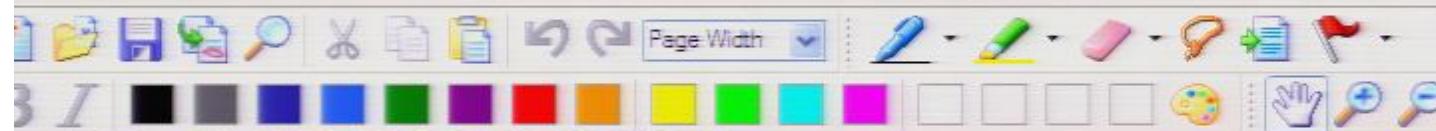
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because all past-directed paths end on it.
⇒ There is a cosmological singularity in
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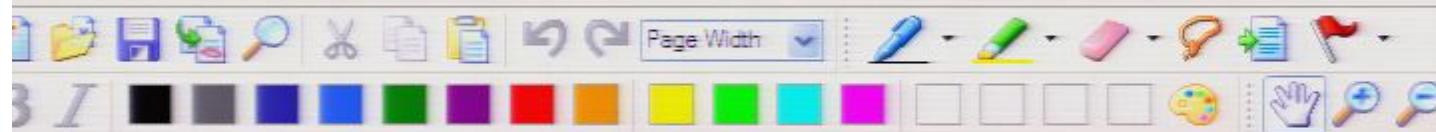
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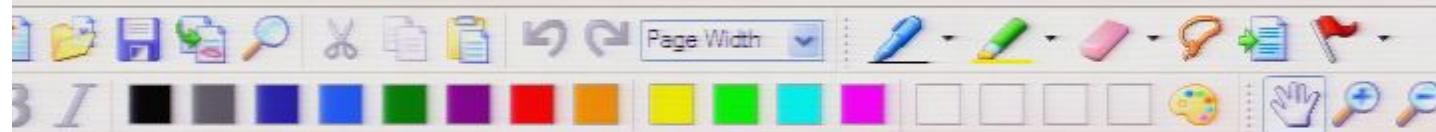
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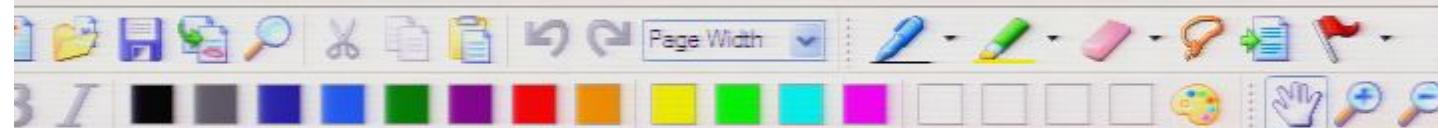
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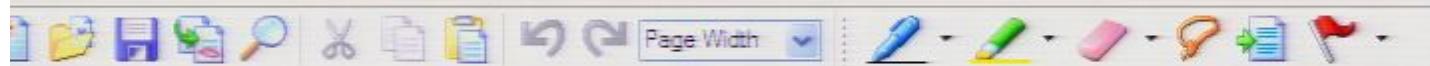
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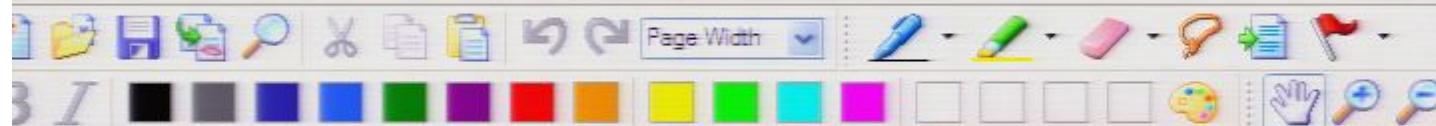
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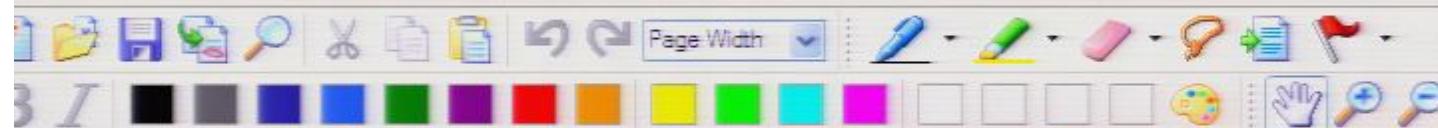
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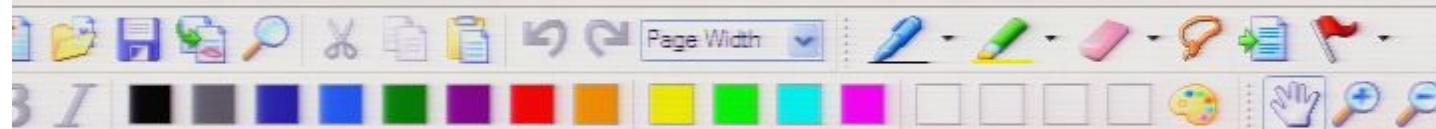
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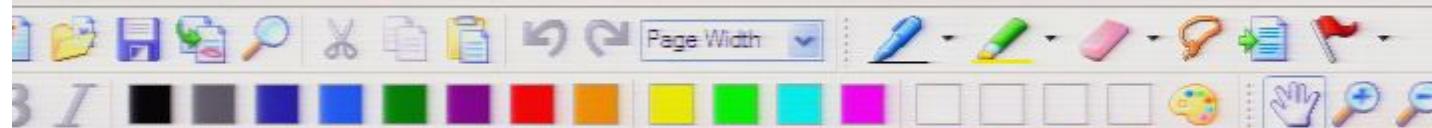
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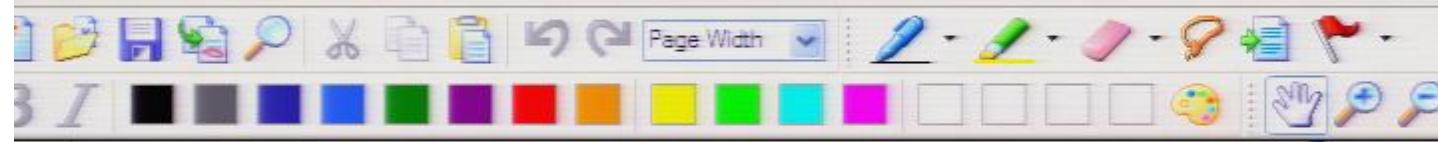
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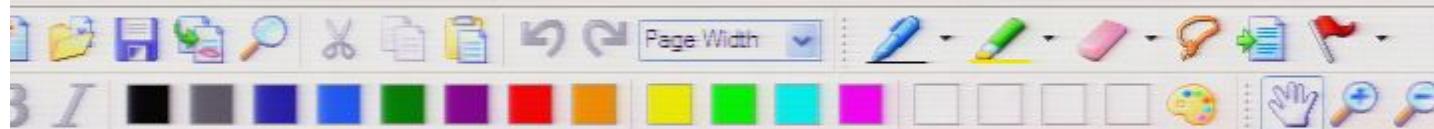
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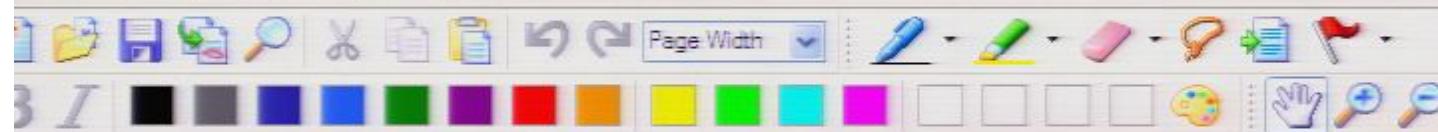
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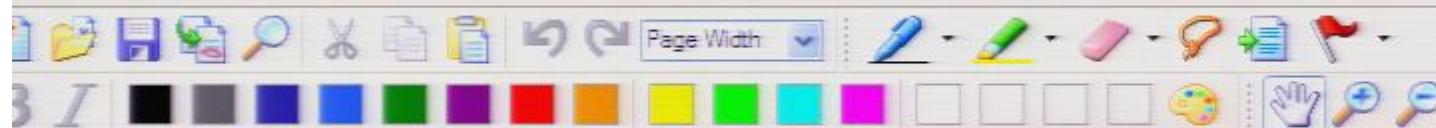
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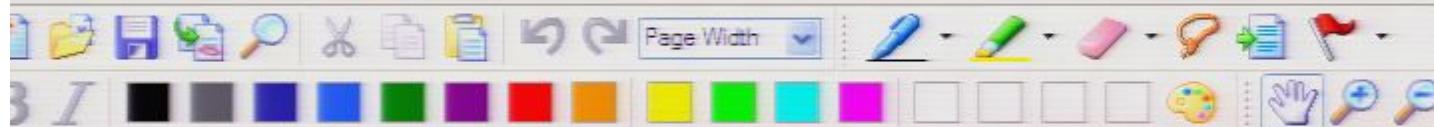
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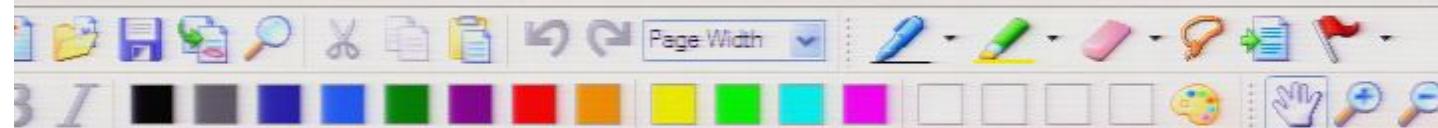
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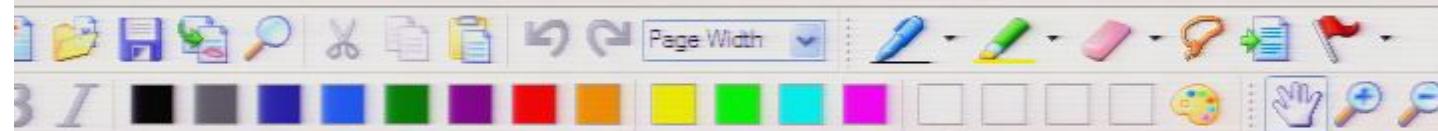
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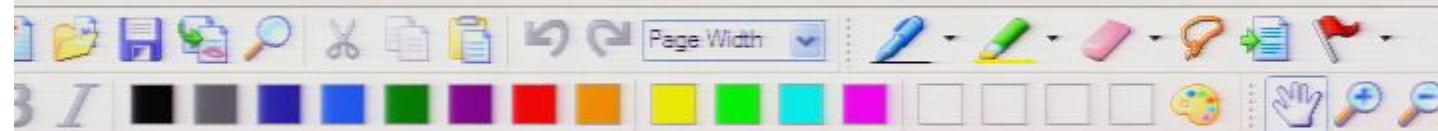
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$$\underbrace{T_{\mu\nu} v^\mu v^\nu \geq 0}_{\text{weak energy condition}} \text{ and } K_\mu K^\mu \leq 0$$

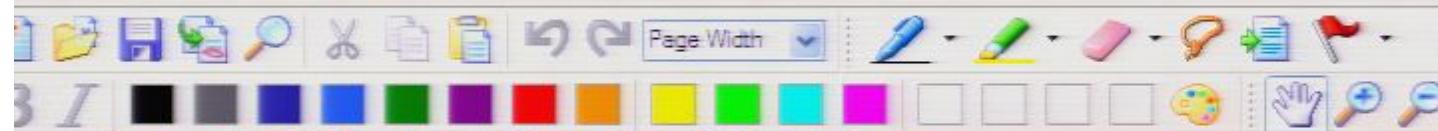
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vector K may not be conserved but

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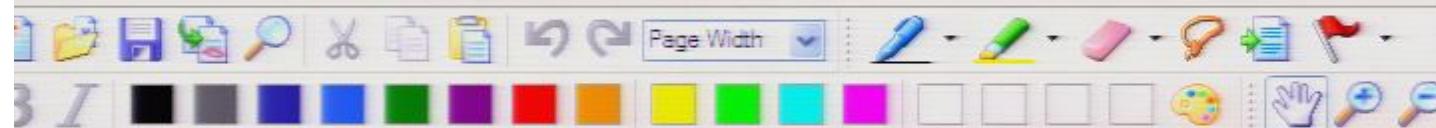
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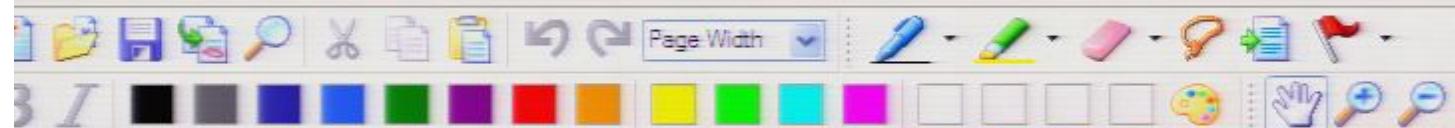
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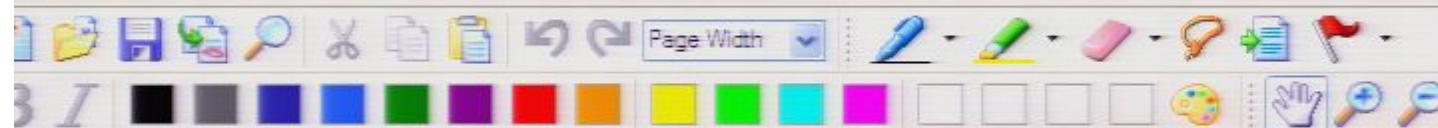
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Matter is said to obey the **strong energy condition** iff :



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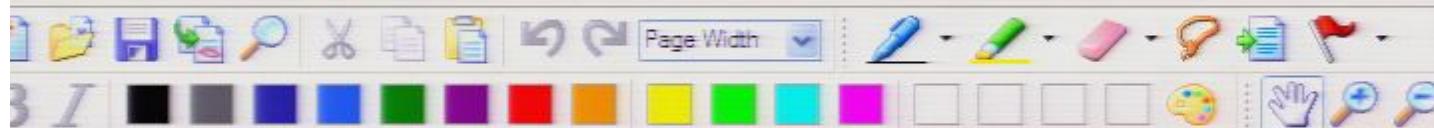
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energy density observed by comoving observer

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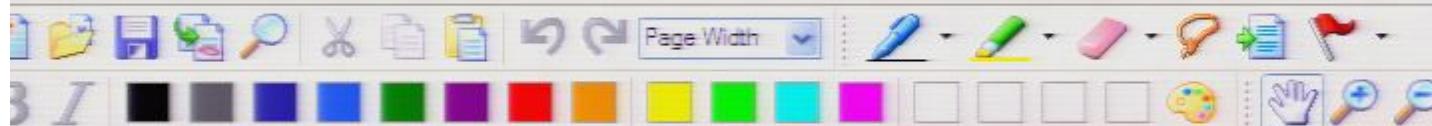
The energy conditions then read:

Weak: $\mathfrak{s} \geq 0$ and $\mathfrak{s} + p_i \geq 0$ for $i \in \{1, 2, 3\}$

Dominant: $\mathfrak{s} \geq |p_i|$ for $i \in \{1, 2, 3\}$

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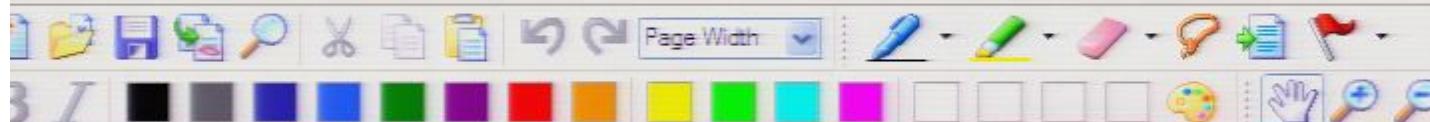
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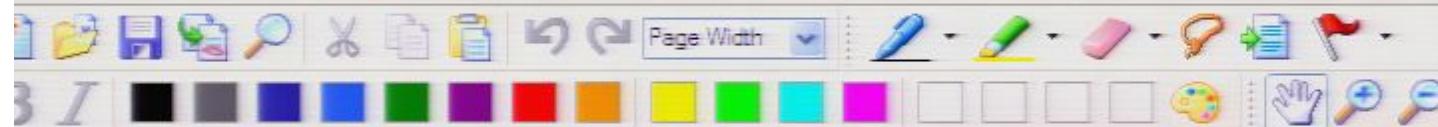
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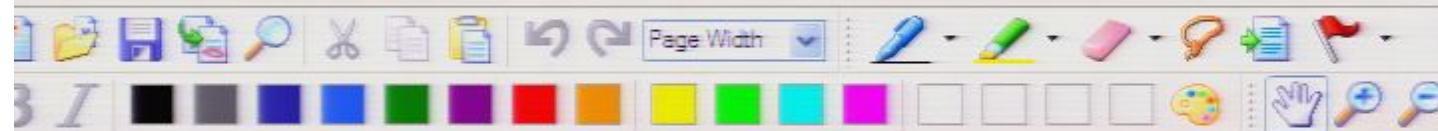
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Given, in particular, the strong energy condition, one can show that geodesics meet a divergence of a quantity called expansion, Θ , in finite proper time:

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- Consider a "congruence of timelike geodesics" through Σ , i.e., a smooth family of timelike geodesics, exactly one through each $p \in \Sigma$. If parametrized by proper

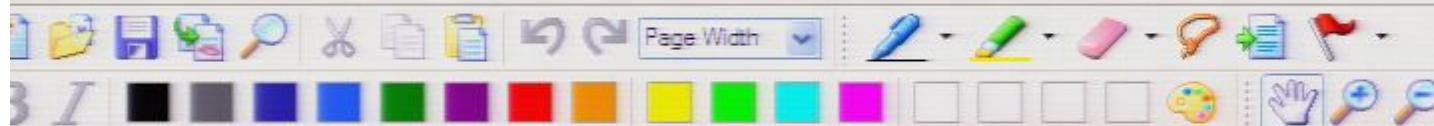


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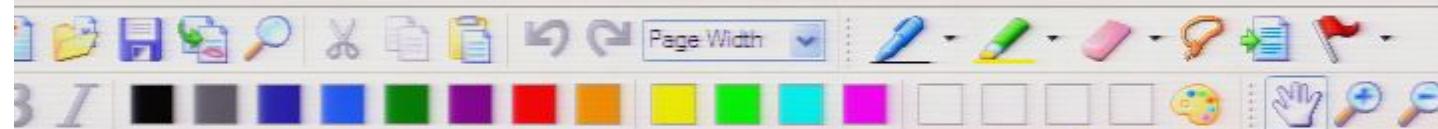
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expansion rate everywhere on Σ .

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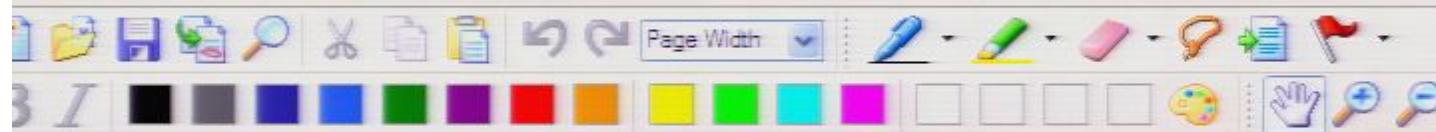
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Then:

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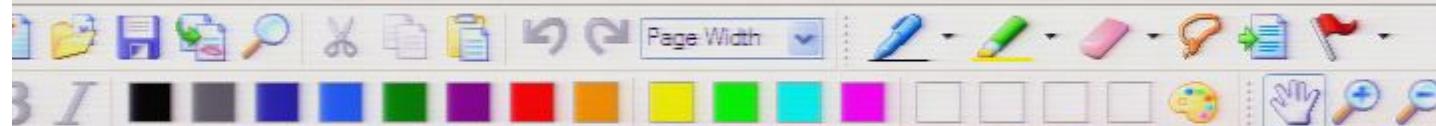
because all past-directed paths end on it.

\Rightarrow There is a cosmological singularity in
the finite past!

Assumptions needed?

Yes: Matter must be assumed to obey an energy condition.

f.) Conclude that there are incomplete geodesics, i.e.,
that we have a singularity in the past (or future) :



then space-time probably is well-described as
globally hyperbolic.

e) Prove that these extremal length curves cannot be geodesics with eigentime larger than a certain finite amount either into the past or future.

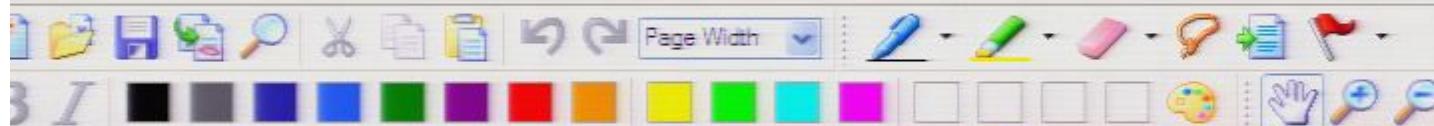
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curvature, K , is bounded from above by
a negative constant C :

$$K(p) \leq C < 0 \text{ for all } p \in \Sigma$$

Then:

No past-directed timelike curve from
 Σ can have eigentime, i.e. proper length, larger than $\frac{3}{C}$.

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of spacetime, more precisely its negative.

Thus: Assuming $K(p) \leq G < 0$ meant that spacetime has a finite minimum expansion rate everywhere on Σ .

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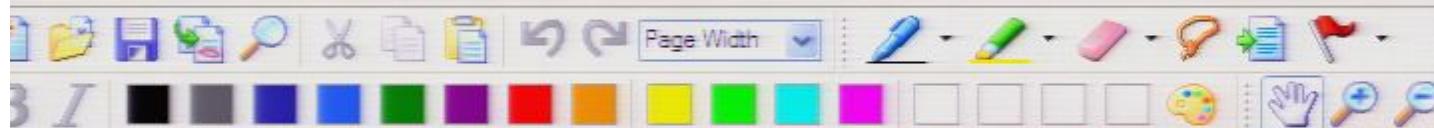
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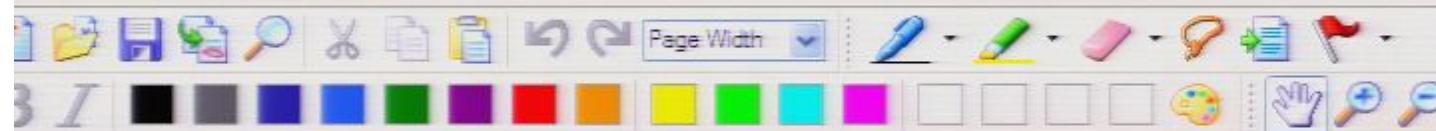
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Given, in particular, the strong energy condition, one can show that geodesics meet a divergence of a quantity called expansion, Θ , in finite proper time:

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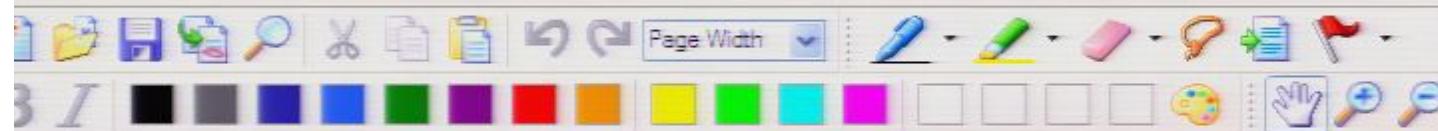
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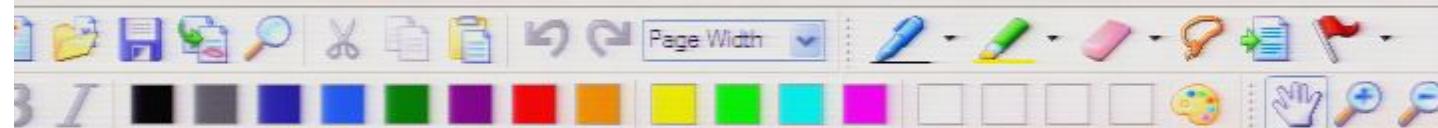
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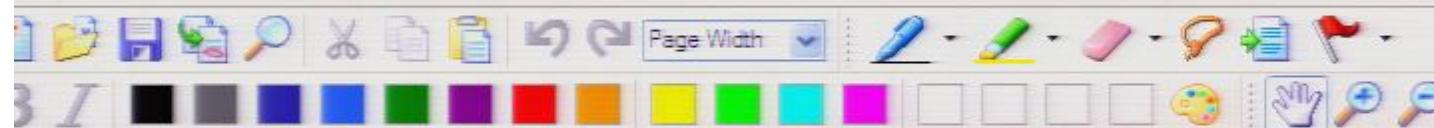
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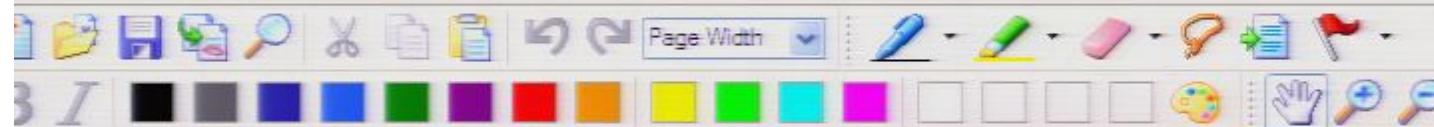
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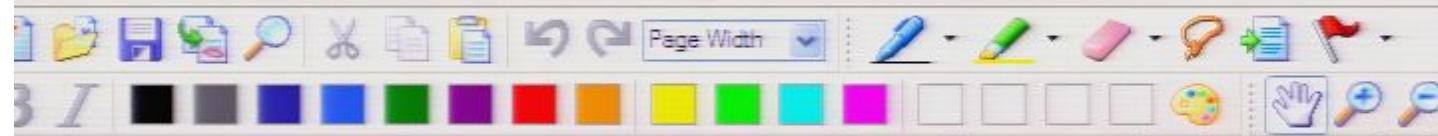
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$$\gamma(\tau, s)$$

↗ parameter of family of neighboring geodesics



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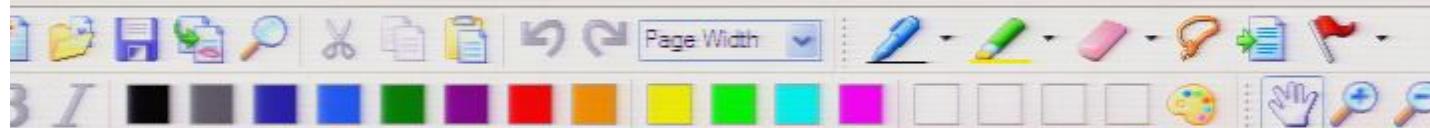
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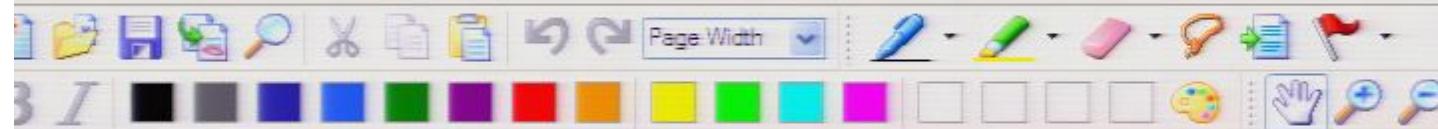
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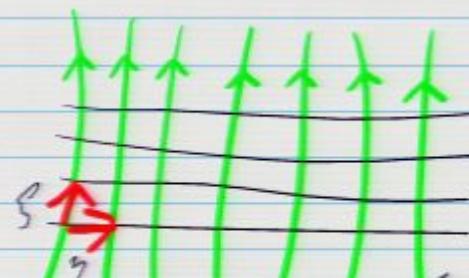
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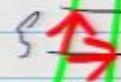
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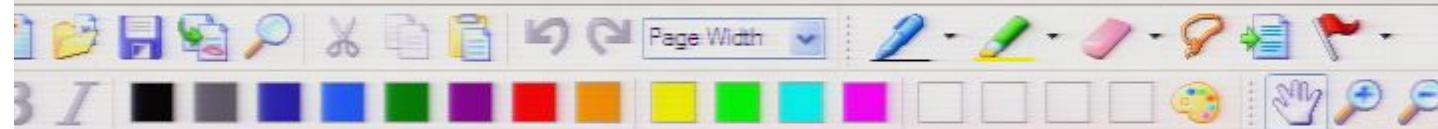
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a line of constant τ value



a geodesic, i.e., a line of constant ζ value



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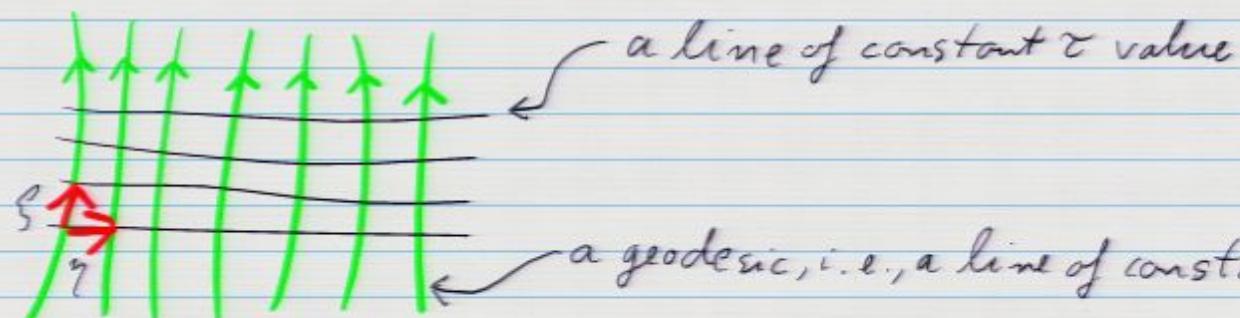
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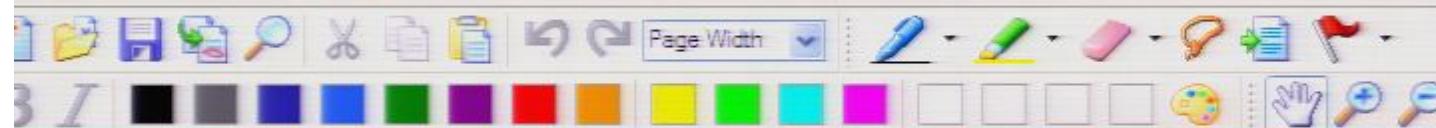
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□ How does η change along a geodesic?

τ
 s

We have $\frac{d}{d\tau} \frac{d}{ds} = \frac{d}{ds} \frac{d}{d\tau}$ in a coordinate system



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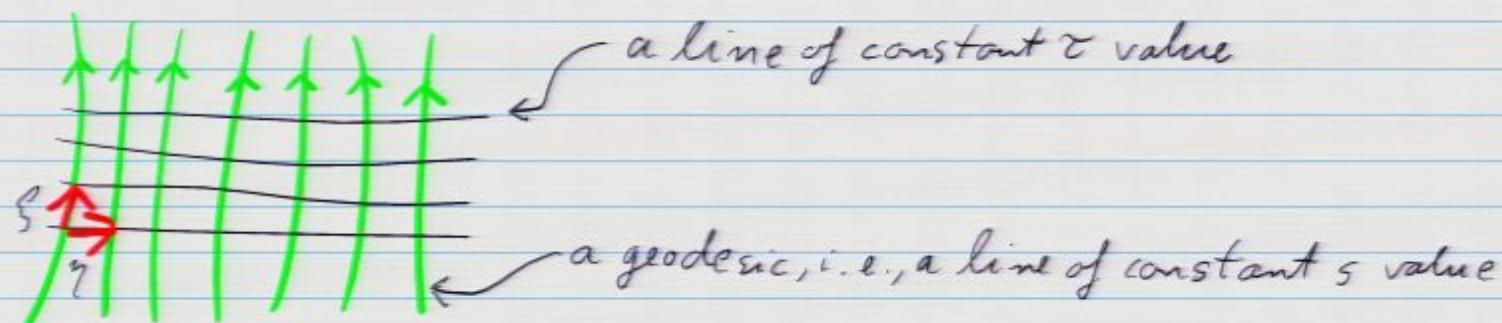
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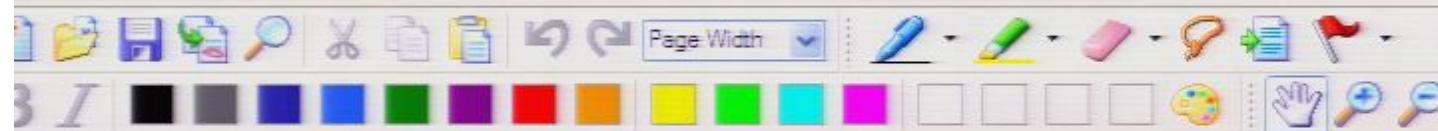
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→ to see this, have them act on a scalar function

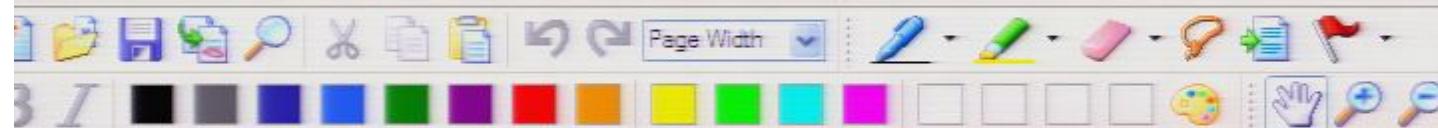
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⇒ Along the geodesic, ξ , the deviation vector η^μ changes its direction and length by $B^\nu_\mu \eta^\mu$.

□ The tensor B^ν_μ can be decomposed covariantly



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$$\nabla_{\xi} \eta = \nabla_{\eta} \xi$$

$$\Rightarrow \xi^{\mu} \nabla_{e_{\mu}} \eta^{\nu} e_{\nu} = \eta^{\mu} \nabla_{e_{\mu}} \xi^{\nu} e_{\nu}$$

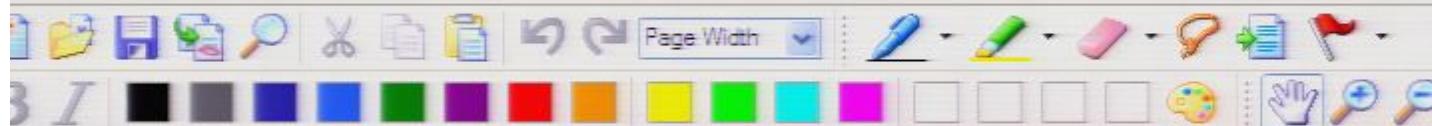
$$\Rightarrow \xi^{\mu} \eta^{\nu} e_{\mu} e_{\nu} = \eta^{\mu} \xi^{\nu} e_{\mu} e_{\nu}$$

$$\Rightarrow \xi^{\mu} \eta^{\nu} = \eta^{\mu} \xi^{\nu} = \eta^{\mu} B^{\nu}_{\mu} \text{ for}$$

$$B^{\nu}_{\mu} := \xi^{\nu} \eta^{\mu}$$

⇒ Along the geodesic, ξ , the deviation vector η^{μ} changes its direction and length by $B^{\nu}_{\mu} \eta^{\mu}$.

□ The tensor B^{ν}_{μ} can be decomposed covariantly and uniquely into:



□ How does η change along a geodesic?

We have $\frac{d}{ds} \frac{d}{d\tau} = \frac{d}{d\tau} \frac{d}{ds}$, i.e., in arb. coordinate system:

→ to see this, have them act on a scalar function

$$\nabla_{\xi} \eta = \nabla_{\eta} \xi$$

$$\Rightarrow \xi^{\mu} \nabla_{e_r} \eta^{\nu} e_s = \eta^{\mu} \nabla_{e_s} \xi^{\nu} e_r$$

$$\Rightarrow \xi^{\mu} \eta^{\nu} e_r e_s = \eta^{\mu} \xi^{\nu} e_r e_s$$

$$\Rightarrow \xi^{\mu} \eta^{\nu} = \eta^{\mu} \xi^{\nu} = \eta^{\mu} B^{\nu}_{\mu} \text{ for}$$

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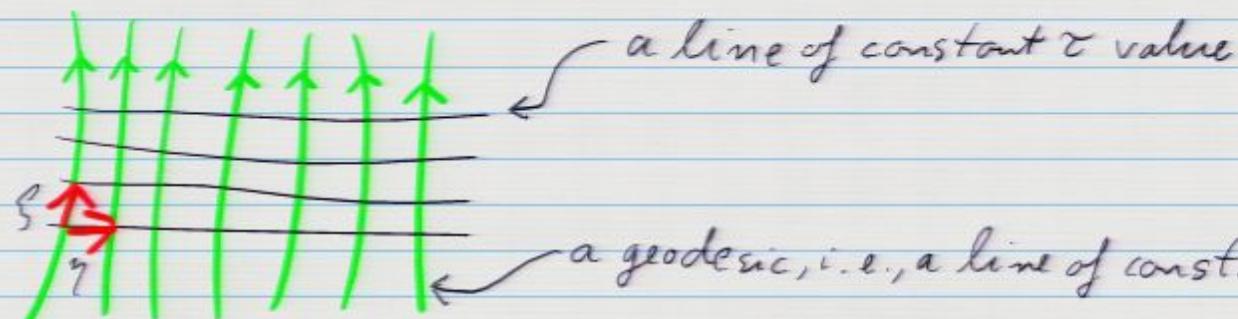
$\gamma(\tau, s)$

paramete of family of neighboring geodesics.

a "connecting vector field"

Then, we define the deviation vector:

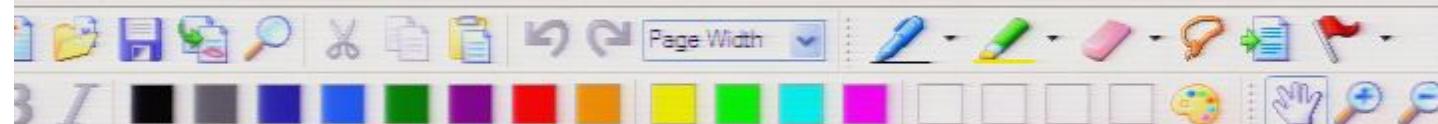
$$\eta := \frac{d}{ds}$$



□ How does η change along a geodesic?

We have $\frac{d}{ds} \frac{d}{d\tau} = \frac{d}{d\tau} \frac{d}{ds}$, i.e., in arb. coordinate system:

↗ to see this, have them act on a scalar function



□ How does η^μ change along a geodesic?

We have $\frac{d}{ds} \frac{d}{d\tau} = \frac{d}{d\tau} \frac{d}{ds}$, i.e., in arb. coordinate system:

to see this, have them act on a scalar function

$$\nabla_\xi \eta^\mu = \eta^\nu \nabla_\xi \xi^\mu$$

$$\Rightarrow \xi^\nu \nabla_{e_\mu} \eta^\mu e_\nu = \eta^\nu \nabla_{e_\mu} \xi^\mu e_\nu$$

$$\Rightarrow \xi^\nu \tilde{\eta}^\mu e_\mu e_\nu = \eta^\nu \xi^\mu e_\mu e_\nu$$

$$\Rightarrow \xi^\nu \tilde{\eta}^\mu = \eta^\nu \xi^\mu = \eta^\nu B^\mu_\nu \text{ for } B^\mu_\nu := \xi^\mu e_\nu$$

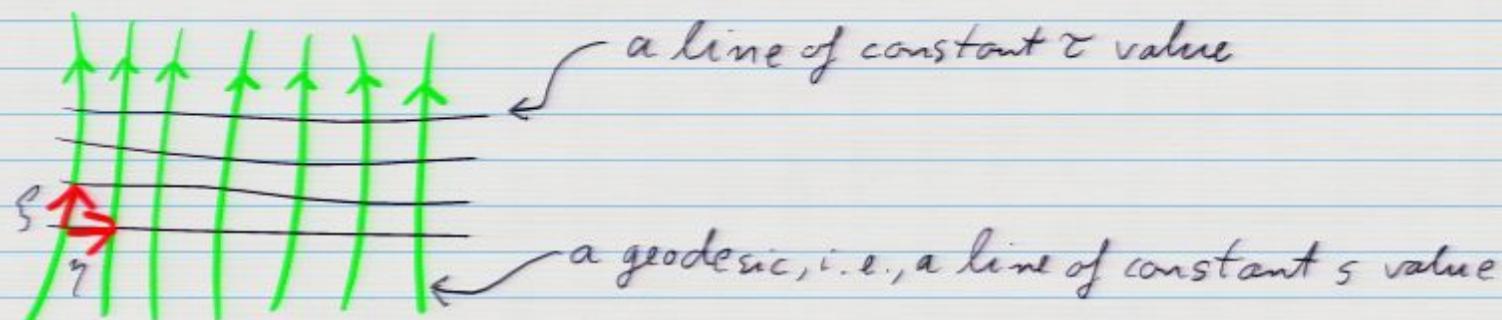
$$B^\mu_\nu := \xi^\mu e_\nu$$

\Rightarrow Along the geodesic, ξ , the deviation

vector η^μ changes its direction and length by $B^\mu_\nu \eta^\nu$.



$$\gamma := \frac{d}{ds}$$



□ How does γ change along a geodesic?

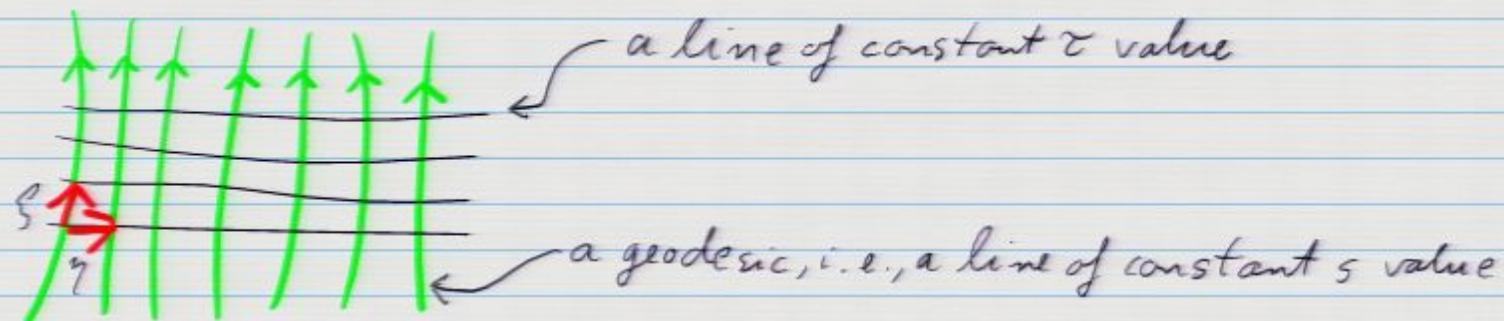
We have $\frac{d}{ds} \frac{d}{d\tau} = \frac{d}{d\tau} \frac{d}{ds}$, i.e., in arb. coordinate system:

$$\nabla_{\dot{\gamma}} \gamma = \nabla_{\gamma} \dot{\gamma}$$

→ to see this, have them act on a scalar function

$$\Rightarrow \xi^a \nabla_{e_a} \gamma^b e_b = \gamma^a \nabla_{e_a} \xi^b e_b$$

$$\Rightarrow \xi^a \gamma^b e_a = \gamma^a \xi^b e_a$$



□ How does η change along a geodesic?

We have $\frac{d}{ds} \frac{d}{d\tau} = \frac{d}{d\tau} \frac{d}{ds}$, i.e., in arb. coordinate system:

$$\nabla_{\dot{\gamma}} \eta = \nabla_{\dot{\gamma}} \eta$$

$$\Rightarrow \xi^\mu \nabla_{e_\mu} \eta^\nu e_\nu = \eta^\mu \nabla_{e_\mu} \xi^\nu e_\nu$$

$$\Rightarrow \xi^\mu \eta^\nu e_\mu e_\nu = \eta^\mu \xi^\nu e_\mu e_\nu$$

$$\Rightarrow \xi^\mu \eta^\nu = \eta^\mu \xi^\nu$$

$$R^\nu{}_\mu := \eta^\nu \xi_\mu$$

→ to see this, have them act on a scalar function



□ How does η change along a geodesic?

We have $\frac{d}{ds} \frac{d}{d\tau} = \frac{d}{d\tau} \frac{d}{ds}$, i.e., in arb. coordinate system:

to see this, have them act on a scalar function

$$\nabla_{\xi} \eta = \nabla_{\eta} \xi$$

$$\Rightarrow \xi^{\mu} \nabla_{e_{\mu}} \eta^{\nu} e_{\nu} = \eta^{\mu} \nabla_{e_{\mu}} \xi^{\nu} e_{\nu}$$

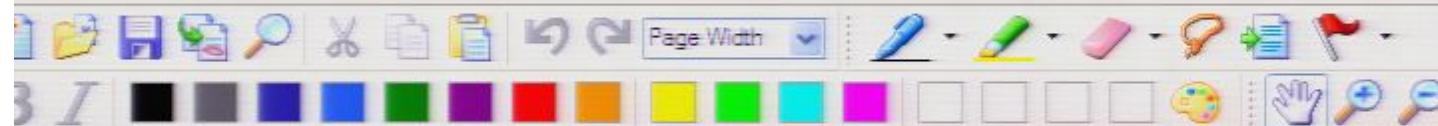
$$\Rightarrow \xi^{\mu} \eta^{\nu} e_{\mu} e_{\nu} = \eta^{\mu} \xi^{\nu} e_{\mu} e_{\nu}$$

$$\Rightarrow \xi^{\mu} \eta^{\nu} = \eta^{\mu} \xi^{\nu} = \eta^{\mu} B^{\nu}_{\mu} \text{ for}$$

$$B^{\nu}_{\mu} := \xi^{\nu}_{,\mu}$$

\Rightarrow Along the geodesic, ξ , the deviation

vector η^{μ} changes its direction and length by $B^{\nu}_{\mu} \eta^{\mu}$



□ How does η change along a geodesic?

We have $\frac{d}{ds} \frac{d}{d\tau} = \frac{d}{d\tau} \frac{d}{ds}$, i.e., in arb. coordinate system:

→ to see this, have them act on a scalar function

$$\nabla_{\dot{\gamma}} \eta = \nabla_{\dot{\gamma}} \xi$$

$$\Rightarrow \xi^\mu \nabla_{e_\mu} \eta^\nu e_\nu = \eta^\mu \nabla_{e_\mu} \xi^\nu e_\nu$$

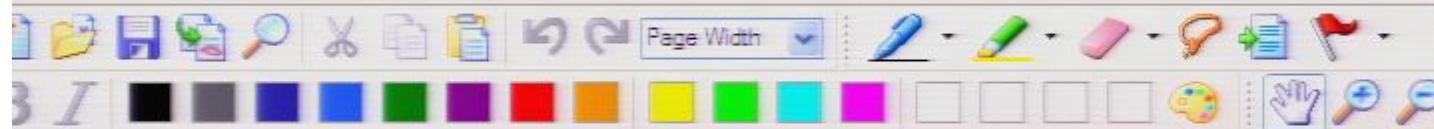
$$\Rightarrow \xi^\mu \eta^\nu e_\mu e_\nu = \eta^\mu \xi^\nu e_\mu e_\nu$$

$$\Rightarrow \xi^\mu \eta^\nu = \eta^\mu \xi^\nu = \eta^\mu B^\nu_\mu \text{ for}$$

$$B^\nu_\mu := \xi^\nu_{;\mu}$$

⇒ Along the geodesic, ξ , the deviation vector η^μ changes its direction and length by $B^\nu_\mu \eta^\mu$.

□ The tensor B^ν_μ can be decomposed covariantly and uniquely into:



$$\nabla_{\mathbf{q}} \gamma = \nabla_{\mathbf{q}} \beta$$

$$\Rightarrow \zeta^{\nu} \nabla_{e_{\mu}} \eta^{\lambda} e_{\lambda} = \eta^{\lambda} \nabla_{e_{\mu}} \zeta^{\nu} e_{\nu}$$

$$\Rightarrow \tilde{\zeta}^{\mu} \tilde{\gamma}_{\nu\rho} e_{\sigma} = \tilde{\gamma}^{\mu} \tilde{\zeta}_{\nu\rho}^{\sigma} e_{\rho}$$

$$\Rightarrow \gamma^\mu \gamma^\nu = \gamma^\mu \gamma^\nu = \gamma^\mu B^\nu \text{ for } \sim$$

$$B^\sim_\mu := \{^\sim_j\mu$$

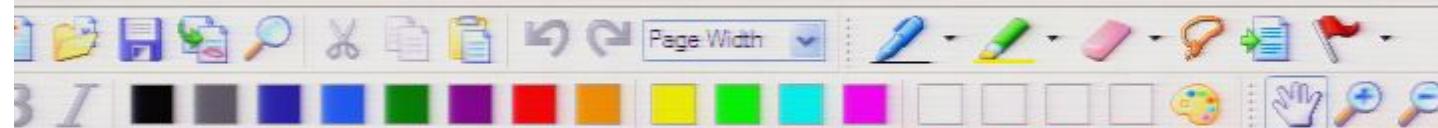
\Rightarrow Along the geodesic, ξ , the deviation vector η^r changes its direction and length by $B^r \cdot \eta^r$.

□ The tensor B^{γ}_{μ} can be decomposed covariantly and uniquely into:

$$B_{\mu\nu} = \omega_{\mu\nu} + g_{\mu\nu} + t_{\mu\nu}$$

↓
↑ ↑
symmetric and trace = 0 rest

(all 3 terms are tensors
because the split is covariant)



\Rightarrow Along the geodesic, ξ , the deviation vector γ^r changes its direction and length by $B^r_\mu \gamma^r$.

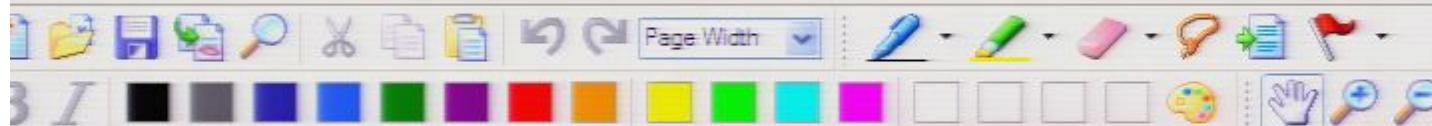
□ The tensor B^r_μ can be decomposed covariantly and uniquely into:

$$B_{\mu\nu} = \omega_{\mu\nu} + \overset{\downarrow}{\sigma_{\mu\nu}} + \overset{\uparrow}{t_{\mu\nu}}$$

Symmetric and trace = 0

↑ ↑ rest

(all 3 terms are tensors
because the split is covariant)



$$B^\nu_\mu := \xi^\nu \eta^\mu$$

\Rightarrow Along the geodesic, ξ , the deviation vector η^ν changes its direction and length by $B^\nu_\mu \eta^\mu$.

□ The tensor B^ν_μ can be decomposed covariantly and uniquely into:

$$B_{\mu\nu} = \underset{\substack{\text{antisymmetric} \\ \uparrow}}{\omega_{\mu\nu}} + \underset{\substack{\downarrow \\ \text{symmetric and trace=0}}}{G_{\mu\nu}} + \underset{\substack{\uparrow \\ \text{rest}}}{t_{\mu\nu}}$$

(all 3 terms are tensors
because the split is covariant)

We have: $\omega_{\mu\nu} = \frac{1}{2}(B_{\mu\nu} - B_{\nu\mu})$, clearly.



$$\Rightarrow \xi^\nu \gamma^\mu_{;\nu} = \gamma^\mu \xi^\nu_{;\nu} = \gamma^\mu B^\nu_\nu \text{ for } B^\nu_\mu := \xi^\nu_{;\mu}$$

\Rightarrow Along the geodesic, ξ , the deviation vector γ^μ changes its direction and length by $B^\nu_\mu \gamma^\mu$.

□ The tensor B^ν_μ can be decomposed covariantly and uniquely into:

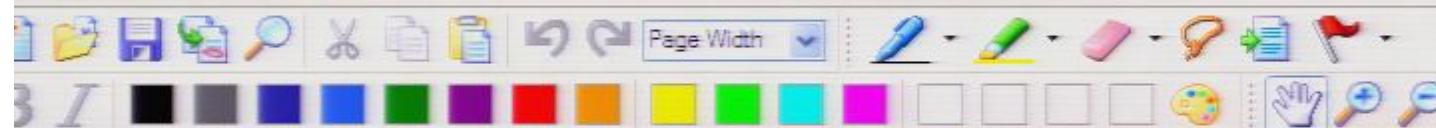
$$B_{\mu\nu} = \overset{\text{Symmetric and trace=0}}{\omega_{\mu\nu}} + \overset{\downarrow}{G_{\mu\nu}} + \overset{\uparrow}{t_{\mu\nu}}$$

(all 3 terms are tensors
because the split is covariant)

rest

We have: $\omega_{\mu\nu} = \frac{1}{2} (B_{\mu\nu} - B_{\nu\mu})$, clearly.

But $G_{\mu\nu}, t_{\mu\nu} = ?$



But $g_{\mu\nu}, t_{\mu\nu} = ?$

In preparation: define the projector $h_{\mu\nu}$ onto $(R\zeta)^\perp$ i.e.
onto the spatial components:
↑ timelike

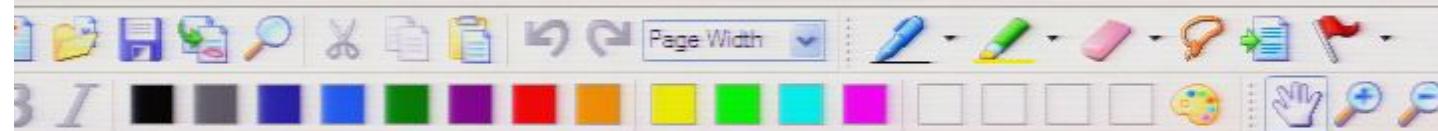
$$h_{\mu\nu} := g_{\mu\nu} + \zeta_\mu \zeta_\nu$$

Check: is $h_{\mu\nu} w^\nu$ really always \perp to ζ ?



Indeed: $\zeta^\mu h_{\mu\nu} w^\nu = (\zeta, w) + \tilde{(\zeta, \zeta)} (\zeta, w) = 0$

Define: The "expansion", θ , is defined as the magnitude of the spatial part of B :



and uniquely into:

$$B_{\mu\nu} = \underset{\substack{\text{antisymmetric} \\ \uparrow}}{\omega_{\mu\nu}} + \underset{\substack{\downarrow \\ \text{rest}}}{G_{\mu\nu}} + \underset{\substack{\uparrow \\ \text{Symmetric and trace=0}}}{t_{\mu\nu}}$$

(all 3 terms are tensors
because the split is covariant)

We have: $\omega_{\mu\nu} = \frac{1}{2}(B_{\mu\nu} - B_{\nu\mu})$, clearly.

But $G_{\mu\nu}, t_{\mu\nu} = ?$

In preparation: define the projector $h_{\mu\nu}$ onto $(R\vec{s})^\perp$ i.e.
onto the spatial components:

$$h_{\mu\nu} := g_{\mu\nu} + \vec{s}_\mu \vec{s}_\nu$$

\vec{s} is timelike



But $g_{\mu\nu}, t_{\mu\nu} = ?$

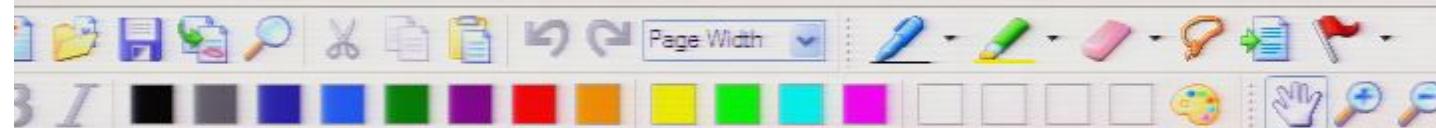
In preparation: define the projector $h_{\mu\nu}$ onto $(R\zeta)^\perp$ i.e.
onto the spatial components:

$$h_{\mu\nu} := g_{\mu\nu} + \zeta_\mu \zeta_\nu$$

Check: is $h_{\mu\nu} w^\nu$ really always \perp to ζ ?

Indeed: $\zeta^\mu h_{\mu\nu} w^\nu = (\zeta, w) + \tilde{(\zeta, \zeta)} (\zeta, w) = 0$

Define: The "expansion", θ , is defined as the magnitude of the spatial part of B :



$$h_{\mu\nu} := g_{\mu\nu} + \xi_\mu \xi_\nu$$

Check: is $h_{\mu\nu} w^\nu$ really always \perp to ξ ?

$$\text{Indeed: } \xi^\mu h_{\mu\nu} w^\nu = (\xi, w) + \overset{\approx -1}{(\xi, \xi)} (\xi, w) = 0$$

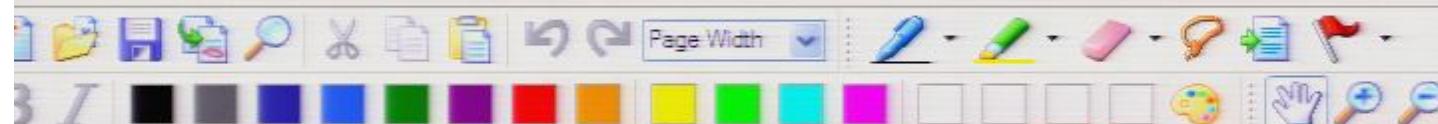
Define: The "expansion", θ , is defined as the magnitude of the spatial part of B :

$$\theta := B^{\mu\nu} h_{\mu\nu}$$

Claim: $\text{Tr}(B) = \theta$

$$\begin{aligned} \text{Indeed: } \theta &= B^{\mu\nu} h_{\mu\nu} = B^{\mu\nu} g_{\mu\nu} + \xi^\mu \xi_\nu B_\mu^\nu \\ &= \text{Tr}(B) + \xi^\mu \xi_\nu \nabla_\mu \xi^\nu \end{aligned}$$

$= 0$ because $\xi^\mu \xi_\nu = 0$
for geodesics.



Define: The "expansion", θ , is defined as the magnitude of the spatial part of B :

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Claim: $\text{Tr}(B) = \theta$

Indeed: $\theta = B^{\mu\nu} h_{\mu\nu} = B^{\mu\nu} g_{\mu\nu} + \xi^\mu \xi_\nu B_\mu^\nu$
 $= \text{Tr}(B) + \xi^\mu \xi_\nu \nabla_\mu \xi^\nu$ (=0 because $\nabla_\mu \xi^\nu = 0$ for geodesics.)

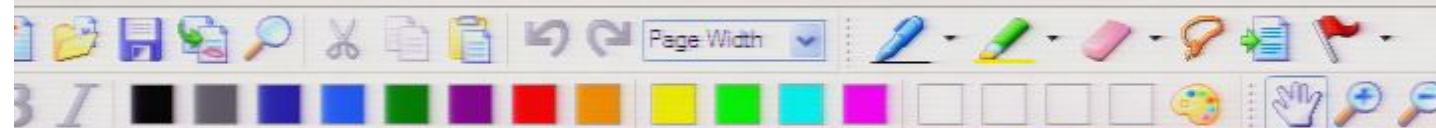
Therefore:

$$\sigma_{\mu\nu} = \frac{1}{2} (B_{\mu\nu} + B_{\nu\mu}) - \frac{1}{3} \theta h_{\mu\nu} \quad \left(\begin{array}{l} \text{because:} \\ \text{Tr}(h_{\mu\nu}) = g^{\mu\nu} h_{\mu\nu} \\ = g^{\mu\nu} (\delta_{\mu\nu} + \xi_\mu \xi_\nu) \\ = 4 - 1 \end{array} \right)$$

and:

$$t_{\mu\nu} = \frac{1}{3} \theta h_{\mu\nu}$$

← the "rest term".



vector η^μ changes its direction and length by $B^\nu_\mu \eta^\mu$.

□ The tensor B^ν_μ can be decomposed covariantly and uniquely into:

Symmetric and trace=0

Symmetric and trace=0

$$B_{\mu\nu} = \omega_{\mu\nu} + \overset{\downarrow}{G_{\mu\nu}} + \overset{\uparrow}{t_{\mu\nu}} \quad \begin{pmatrix} \text{all 3 terms are tensors} \\ \text{because the split is covariant} \end{pmatrix}$$

rest

↑
antisymmetric



We have: $\omega_{\mu\nu} = \frac{1}{2}(B_{\mu\nu} - B_{\nu\mu})$, clearly.

But $G_{\mu\nu}, t_{\mu\nu} = ?$

In preparation: define the projector $h_{\mu\nu}$ onto $(\mathbb{R}\xi)^\perp$



\Rightarrow Along the geodesic, ξ , the deviation vector γ^{ν} changes its direction and length by $B^{\nu}_{\mu} \gamma^{\mu}$.

□ The tensor B^{ν}_{μ} can be decomposed covariantly and uniquely into:

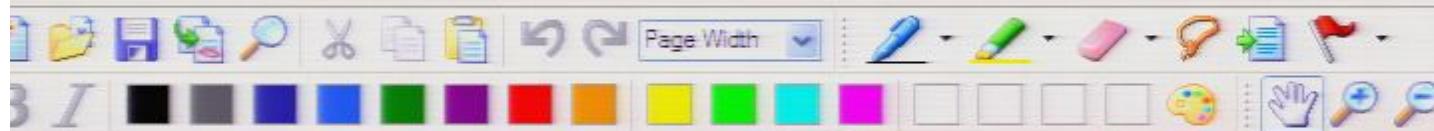
$$B_{\mu\nu} = \omega_{\mu\nu} + \overset{\downarrow}{g_{\mu\nu}} + \overset{\uparrow}{t_{\mu\nu}}$$

Symmetric and trace = 0

↑
antisymmetric
rest

(all 3 terms are tensors
because the split is covariant)

We have: $\omega_{\mu\nu} = \frac{1}{2} (B_{\mu\nu} - B_{\nu\mu})$, clearly.



$$\Rightarrow \xi^\nu \gamma_{\nu r} e_\nu = \gamma^\nu \xi^\rho e_\rho$$

$$\Rightarrow \xi^\nu \gamma_{\nu r} = \gamma^\nu \xi^\rho = \gamma^\nu B^\rho_\mu \text{ for } B^\rho_\mu := \xi^\rho$$

\Rightarrow Along the geodesic, ξ , the deviation vector η^ν changes its direction and length by $B^\nu_\mu \eta^\mu$.

□ The tensor B^ν_μ can be decomposed covariantly and uniquely into:

$$B_{\mu\nu} = \overset{\text{Symmetric and trace=0}}{\omega_{\mu\nu}} + \overset{\downarrow}{G_{\mu\nu}} + \overset{\uparrow}{t_{\mu\nu}}$$

↑
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rest

(all 3 terms are tensors
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We have: $\omega_{\mu\nu} = \frac{1}{2} (B_{\mu\nu} - B_{\nu\mu})$, clearly.



$\gamma_{\mu\nu} = g_{\mu\nu} + \zeta_{\mu\nu}$

Check: is $h_{\mu\nu} w^\nu$ really always \perp to ζ ?

$$\text{Indeed: } \zeta^\mu h_{\mu\nu} w^\nu = (\zeta, w) + \underbrace{(\zeta, \zeta)}_{=0} (\zeta, w) = 0$$

Define: The "expansion", Θ , is defined as the magnitude of the spatial part of B :

$$\Theta := B^{\mu\nu} h_{\mu\nu}$$

Claim: $\text{Tr}(B) = \Theta$

$$\begin{aligned} \text{Indeed: } \Theta &= B^{\mu\nu} h_{\mu\nu} = B^{\mu\nu} g_{\mu\nu} + \zeta^\mu \zeta_\nu B_\nu^\nu \\ &= \text{Tr}(B) + \zeta^\mu \zeta_\nu \nabla_\mu \zeta^\nu \end{aligned}$$

(=0 because $\nabla_\mu \zeta^\nu = 0$
for $\mu \neq \nu$)



Indeed: $\xi^\mu h_{\mu\nu} w^\nu = (\xi, w) + \overset{\text{not}}{(B, \xi)} (\xi, w) = 0$

Define: The "expansion", Θ , is defined as the magnitude of the spatial part of B :

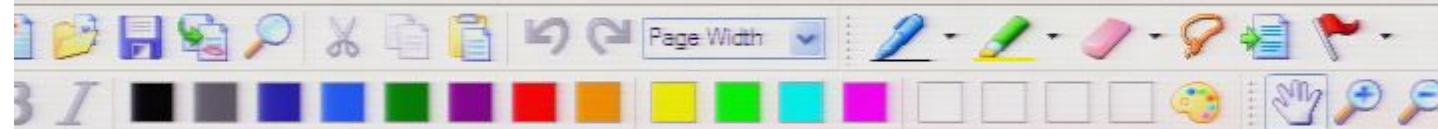
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Claim: $\text{Tr}(B) = \Theta$

Indeed: $\Theta = B^{\mu\nu} h_{\mu\nu} = B^{\mu\nu} g_{\mu\nu} + \xi^\mu \xi_\nu B_\mu^\nu$
 $= \text{Tr}(B) + \xi^\mu \xi_\nu \nabla_\mu \xi^\nu$ ($= 0$ because $\nabla_\mu \xi^\nu = 0$ for geodesics.)

Therefore: $\sigma_{\mu\nu} = \frac{1}{2} (B_{\mu\nu} + B_{\nu\mu}) - \frac{1}{3} \Theta h_{\mu\nu}$

(because:
 $\text{Tr}(h_{\mu\nu}) = g^{\mu\nu} h_{\mu\nu}$
 $= g^{\mu\nu} \overset{\text{not}}{(h_{\mu\nu})}$
 $= \frac{1}{4} \Theta$)



Define: The "expansion", θ , is defined as the magnitude of the spatial part of B :

$$\theta := B^{\mu\nu} h_{\mu\nu}$$

Claim: $\text{Tr}(B) = \theta$

Indeed: $\theta = B^{\mu\nu} h_{\mu\nu} = B^{\mu\nu} g_{\mu\nu} + \xi^\mu \xi_\nu B_\mu^\nu$

$$= \text{Tr}(B) + \xi^\mu \xi_\nu \underbrace{\nabla_\mu \xi^\nu}_{(=0 \text{ because } \nabla_\mu \xi^\nu = 0 \text{ for geodesics. } \text{Hand icon})}$$

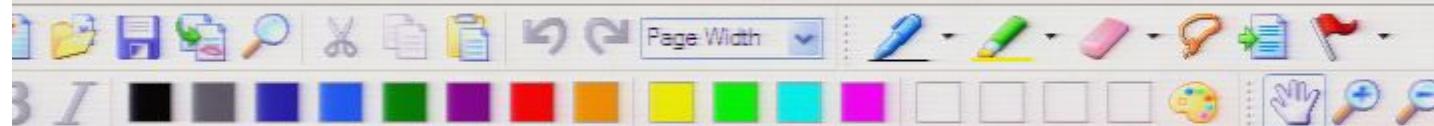
Therefore:

$$d_{\mu\nu} = \frac{1}{2} (B_{\mu\nu} + B_{\nu\mu}) - \frac{1}{3} \theta h_{\mu\nu}$$

(because:
 $\text{Tr}(h_{\mu\nu}) = g^{\mu\nu} h_{\mu\nu}$
 $= g^{\mu\nu} (g_{\mu\nu} + \xi_\mu \xi_\nu)$
 $= 4 - 1$)

and:

$$t_{\mu\nu} = \frac{1}{3} \theta h_{\mu\nu} \quad \leftarrow \text{the "rest term".}$$



Define: The "expansion", θ , is defined as the magnitude of the spatial part of B :

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Indeed: $\theta = B^{\mu\nu} h_{\mu\nu} = B^{\mu\nu} g_{\mu\nu} + \xi^\mu \xi_\nu B_\mu^\nu$

$$= \text{Tr}(B) + \xi^\mu \xi_\nu \nabla_\mu \xi^\nu$$

(=0 because $\nabla_\mu \xi^\nu = 0$ for geodesics.)

Therefore:

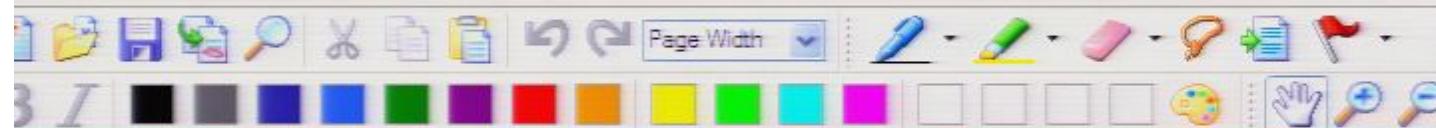
$$\sigma_{\mu\nu} = \frac{1}{2} (B_{\mu\nu} + B_{\nu\mu}) - \frac{1}{3} \theta h_{\mu\nu}$$

(because:
 $\text{Tr}(h_{\mu\nu}) = g^{\mu\nu} h_{\mu\nu}$
 $= g^{\mu\nu} (g_{\mu\nu} + \xi_\mu \xi_\nu)$
 $= \frac{1}{3} \theta$)

and:

$$t_{\mu\nu} = \frac{1}{3} \theta h_{\mu\nu}$$

← the "rest term".



Indeed: $\theta = B^{\mu\nu} h_{\mu\nu} = B^{\mu\nu} g_{\mu\nu} + \xi^\mu \xi_\nu B_{\mu}{}^\nu$

$$= \text{Tr}(B) + \xi^\mu \xi_\nu \nabla_\mu \xi^\nu$$

(= 0 because $\nabla_\mu \xi^\nu = 0$ for geodesics.)

Therefore: $\sigma_{\mu\nu} = \frac{1}{2} (B_{\mu\nu} + B_{\nu\mu}) - \frac{1}{3} \theta h_{\mu\nu}$

(because:
 $\text{Tr}(h_{\mu\nu}) = g^{\mu\nu} h_{\mu\nu}$
 $= g^{\mu\nu} (\xi_{\mu\nu} + \xi_\mu \xi_\nu)$
 $= \frac{1}{3} \theta$)

and:

$$t_{\mu\nu} = \frac{1}{3} \theta h_{\mu\nu} \quad \leftarrow \text{the "rest term".}$$



□ Interpretation:

a.) $w_{\mu\nu}$ is antisymmetric: $w_{\mu\nu} = -w_{\nu\mu}$

\Rightarrow it generates Lorentz transformation for γ



Therefore:

$$\sigma_{\mu\nu} = \frac{1}{2} (B_{\mu\nu} + B_{\nu\mu}) - \frac{1}{3} \Theta h_{\mu\nu}$$

$$\begin{aligned} & \text{(because:)} \\ & T^{\rho}_{\rho}(h_{\mu\nu}) = g^{\mu\rho} g^{\nu\sigma} h_{\mu\nu} \\ & = g^{\mu\nu} (g_{\mu\nu} + g_{\rho\sigma} g^{\rho\sigma}) \\ & = 4 - 1 \end{aligned}$$

↑ the part of $B_{\mu\nu}$ which is symmetric and traceless.

and:

$$t_{\mu\nu} = \frac{1}{3} \Theta h_{\mu\nu} \quad \leftarrow \text{the "rest term".}$$

□ Interpretation:



a) $w_{\mu\nu}$ is anti-symmetric: $w_{\mu\nu} = -w_{\nu\mu}$

\Rightarrow it generates Lorentz transformation for η .

but all η are \perp to the time direction

$\Rightarrow w_{\mu\nu}$ generates spatial rotations of neighboring



□ Interpretation:

a.) $\omega_{\mu\nu}$ is antisymmetric: $\omega_{\mu\nu} = -\omega_{\nu\mu}$
 \Rightarrow it generates Lorentz transformation for γ .

but all γ are \perp to the time direction

$\Rightarrow \omega_{\mu\nu}$ generates spatial rotations of neighboring geodesics around another. So, $\omega_{\mu\nu}$ is called

ω : "Twists tensor"

One can prove: (nontrivial)

If one chooses the congruence at



□ Interpretation:

a.) $w_{\mu\nu}$ is antisymmetric: $w_{\mu\nu} = -w_{\nu\mu}$

\Rightarrow it generates Lorentz transformation for η .

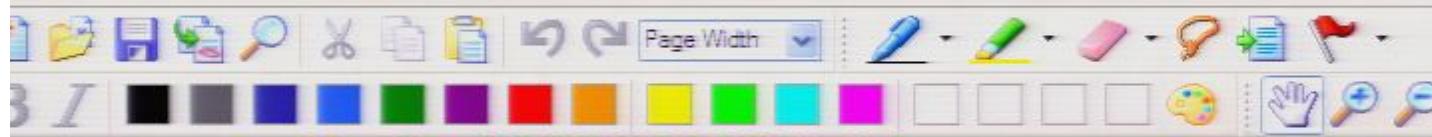
but all η are \perp to the time direction

$\Rightarrow w_{\mu\nu}$ generates spatial rotations of neighboring geodesics around another. So, $w_{\mu\nu}$ is called

w = "Twists tensor"

One can prove: (nontrivial)

If one chooses the congruence of geodesics \perp to Σ then $w_{\mu\nu} = 0$.



Interpretation:

a) $\omega_{\mu\nu}$ is antisymmetric: $\omega_{\mu\nu} = -\omega_{\nu\mu}$
 \Rightarrow it generates Lorentz transformation for η .

but all η are \perp to the time direction

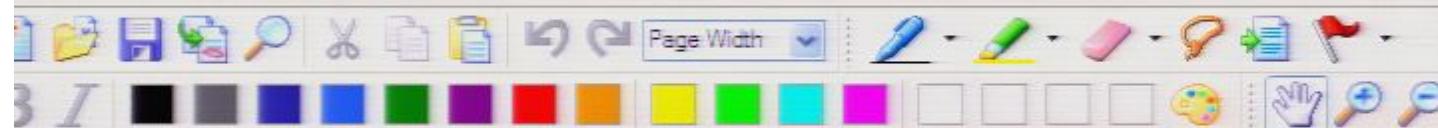
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One can prove: (nontrivial)

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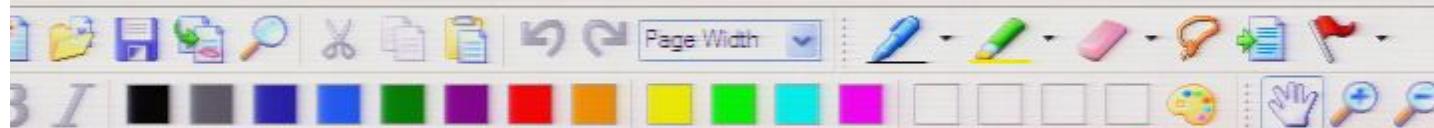
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One can prove: (nontrivial)

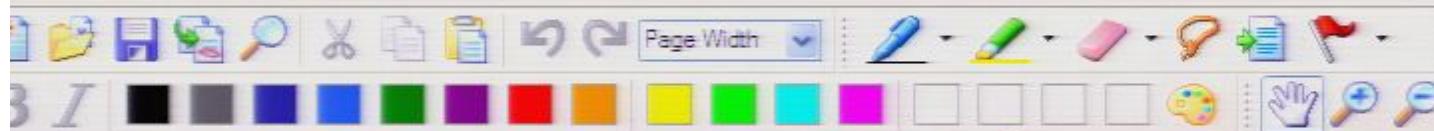
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Consider "diagonalized", by suitable choice of cd basis.

$\Rightarrow \sigma_{\mu\nu}$ changes the relative lengths of the basis vectors, by multiplying them with its eigenvalues.

i.e. points on a sphere will under geodesic flow \rightarrow become points on an ellipsoid.



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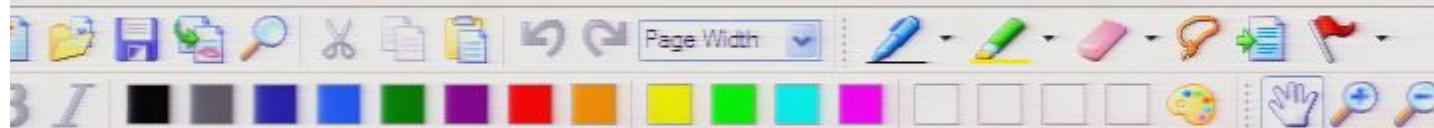
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Note: Since $\text{Tr}(\sigma) = 0$ we have $\det(e^{\sigma}) = 1$

infinitesimal transport along geodesics

finite transport

\Rightarrow The volume spanned by basis vectors stays the same under the action of σ .



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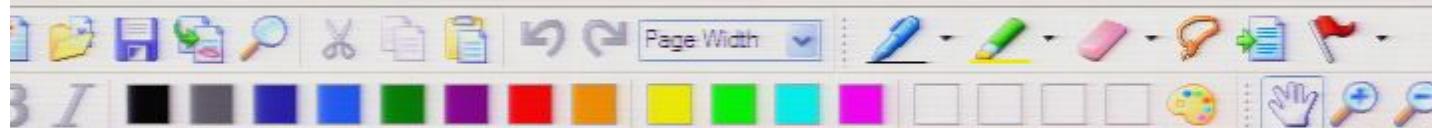
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Definition:

$\sim \therefore$ "Shear tensor"



b.) $G_{\mu\nu}$ is symmetric, $\sigma_{\mu\nu} = G_{\mu\nu}$. (i.e. hermitian)

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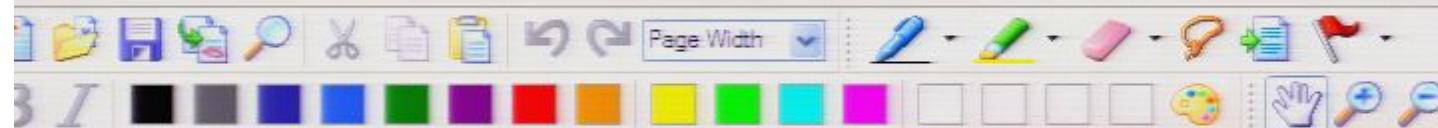
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Definition:

$G_{\mu\nu} =$ "Shear tensor"





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\rightsquigarrow Definition: $G_{\mu\nu} =:$ "Shear tensor" $\square \rightarrow \square$



c.) While the twist and shear tensors are both traceless and therefore volume-preserving, we see that the trace part, Θ , i.e., more precisely

$$t_{\mu\nu} = \frac{1}{3} \Theta h_{\mu\nu} =: \text{"Expansion tensor"}$$

↑
recall: is projector on spatial part.



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Evolution of Θ along a geodesic?



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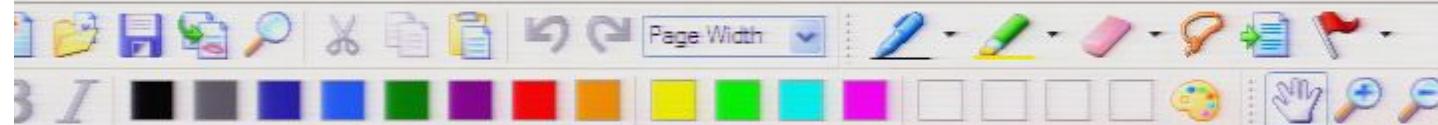
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Evolution of θ along a geodesic?

The Raychaudhuri equation:



Consider:

$$\xi^c \nabla_c B_{ab} \stackrel{\text{by def.}}{=} \xi^c \nabla_c \nabla_b \xi_a = \xi^c \nabla_b \nabla_c \xi_a + R_{cba}^{\quad d} \xi^c \xi_d$$

by definition of the curvature tensor

$$\xi^c B_{abc} = \nabla_a (\xi^c \nabla_b \xi_c) - (\nabla_a \xi^c)(\nabla_b \xi_c) + R_{cba}^{\quad d} \xi^c \xi_d$$

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Evolution of θ along a geodesic?

The Raychaudhuri equation:



Consider:

$$\begin{aligned} \xi^c \nabla_c B_{ab} &= \xi^c \nabla_c \nabla_b \xi_a && \text{by definition of the curvature tensor} \\ &\stackrel{\text{def. of } B}{=} \xi^c \nabla_b \nabla_c \xi_a + R_{cba}^{\quad d} \xi^c \xi_d \\ &\stackrel{\text{because geodesic}}{=} \nabla_b (\xi^c \nabla_c \xi_a) - (\nabla_b \xi^c) (\nabla_c \xi_a) + R_{cba}^{\quad d} \xi^c \xi_d \end{aligned}$$



The Raychaudhuri equation:

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 \xi^c \nabla_c B_{ab} &\stackrel{\text{by def.}}{=} \xi^c \nabla_c \nabla_b \xi_a = \xi^c \nabla_b \nabla_c \xi_a + R_{cba}^{\quad d} \xi^c \xi_d \\
 \xi^c B_{abc} &= \\
 &\stackrel{\text{Lorentz rule}}{=} \nabla_b (\underbrace{\xi^c \nabla_c \xi_a}_{\text{=0 because geodesic}}) - (\nabla_b \xi^c) (\nabla_c \xi_a) + R_{cba}^{\quad d} \xi^c \xi_d \\
 &\stackrel{\text{by def.}}{=} -B^c{}_b B_{ac} + R_{cba}^{\quad d} \xi^c \xi_d
 \end{aligned}$$

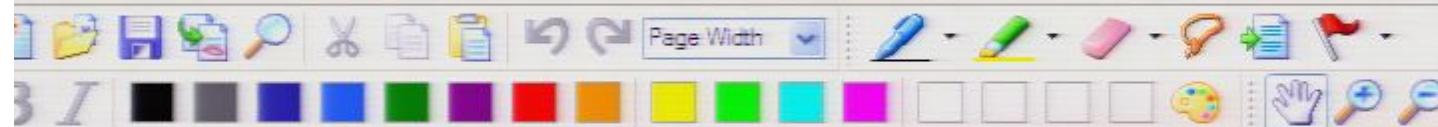
↙ use next: $B_{\mu\nu} = \frac{1}{3} \theta h_{\mu\nu} + g_{\mu\nu} + w_{\mu\nu}$

The Raychaudhuri equation is the trace of this equation:

recall: $\frac{d\theta}{d\tau} = \nabla \theta = -\frac{1}{3} \theta^2 - G_{ab} \xi^{ab} + w_{ab} w^{ab} - R_{cd} \xi^c \xi^d$

recall: Ricci tensor is
 \downarrow
 $R_{cd} = R_{cad}{}^a$

$$\frac{d\theta}{d\tau} = \nabla \theta = -\frac{1}{3} \theta^2 - \underbrace{G_{ab} \xi^{ab}}_{\substack{\text{always positive}}} + \underbrace{w_{ab} w^{ab}}_{\substack{\text{vanishes if}}} - \underbrace{R_{cd} \xi^c \xi^d}_{\substack{\text{PSS. or ne-} \rightarrow 22/25}}$$



The Raychaudhuri equation:

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 \xi^c \nabla_c B_{ab} & \stackrel{\text{by def.}}{=} \xi^c \nabla_c \nabla_b \xi_a & & \text{by definition of the curvature tensor} \\
 & \stackrel{\text{def. of } B}{=} \xi^c \nabla_b \nabla_c \xi_a + R_{cba}{}^d \xi^c \xi_d \\
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 & \stackrel{\text{Lancut rule}}{=} \nabla_b (\xi^c \nabla_c \xi_a) - (\nabla_b \xi^c) (\nabla_c \xi_a) + R_{cba}{}^d \xi^c \xi_d \\
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 & \quad \uparrow \text{use next: } B_{\mu\nu} = \frac{1}{3} \theta h_{\mu\nu} + g_{\mu\nu} + \omega_{\mu\nu}
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 $\int R_{cd} = R_{da}{}^a$
 always positive vanishes if
 choose congruence $\perp \Sigma$ pos. or neg.

Pirma: 09110120
 $\Rightarrow \frac{d}{dt} = \theta$ and $\text{Tr}(B) = \theta$

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$$\{ \nabla_c B_{ab} = \{ \nabla_c \nabla_b \xi_a = \{ \nabla_b \nabla_c \xi_a + R_{cba} \xi_d \xi_d$$

$$\{^c B_{abc} =$$

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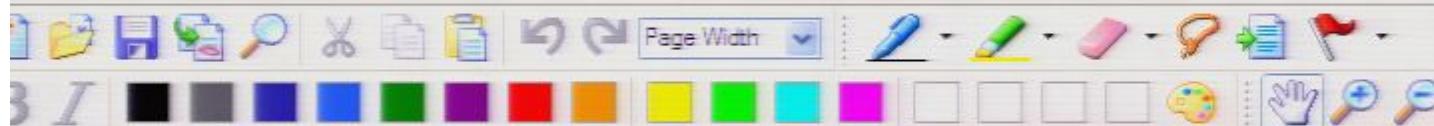
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 $\downarrow R_{cd} = R_{cda}{}^a$

pos. or neg?

Dynamics?

Assume that



Consider:

by definition of the curvature tensor

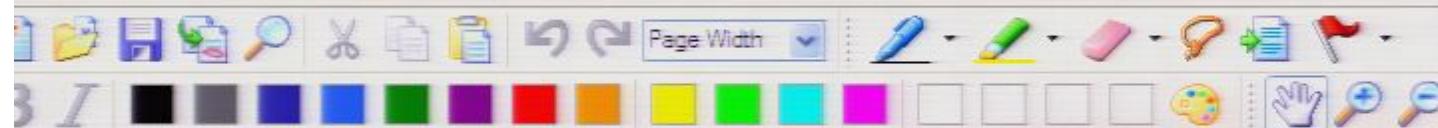
$$\begin{aligned}
 \xi^c \nabla_c B_{ab} &= \xi^c \nabla_c \nabla_b \xi_a = \xi^c \nabla_b \nabla_c \xi_a + R_{cba}^{\quad d} \xi^c \xi_d \\
 \xi^c B_{abc} &= \\
 \text{Lambert rule} &= \nabla_b (\underbrace{\xi^c \nabla_c \xi_a}_{\text{because geodesic}}) - (\nabla_b \xi^c) (\nabla_c \xi_a) + R_{cba}^{\quad d} \xi^c \xi_d \\
 &\stackrel{\text{by def. of } B}{=} -B^c{}_b B_{ac} + R_{cba}^{\quad d} \xi^c \xi_d \\
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$\frac{d}{d\tau} = \theta$ and $\text{Tr}(B) = \theta$

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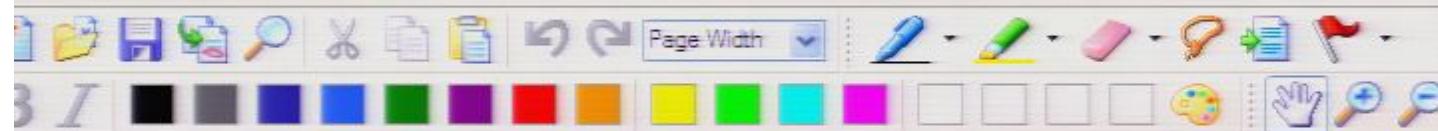
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 &\stackrel{\text{by def of } B}{=} -B^c{}_b B_{ac} + R_{cba}^d \xi^c \xi_d \\
 &\quad \left[\text{use next: } B_{\mu\nu} = \frac{1}{3} \theta h_{\mu\nu} + g_{\mu\nu} + \omega_{\mu\nu} \right]
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Dynamics?

Assume that



Dynamics?

Assume that

$$R_{\mu\nu}\xi^\mu\xi^\nu \geq 0 \text{ for all timelike } \xi$$

i.e., using the Einstein equation

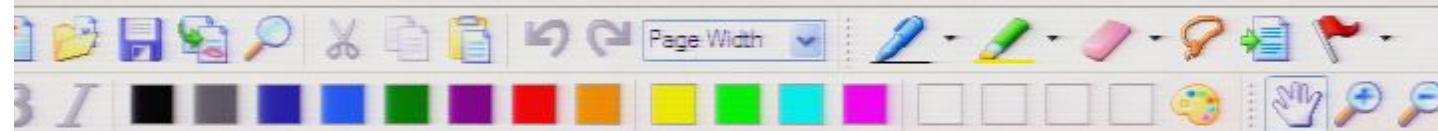
$$R_{\mu\nu} = 8\pi G (T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^a_a)$$



we are assuming that

$$T_{\mu\nu}\xi^\mu\xi^\nu - \frac{1}{2}\xi^\mu\xi_\mu T \stackrel{=}{>} 0$$

i.e. that $T_{\mu\nu}\xi^\mu\xi^\nu \geq -\frac{1}{2}T$ whenever $\xi^\mu\xi_\mu < 0$



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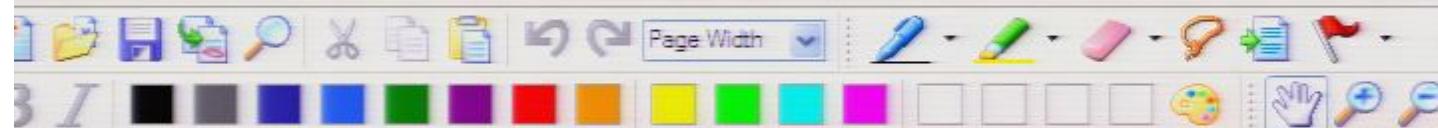
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The Raychaudhuri equation is the trace of this equation:

recall: $\frac{d\theta}{dt} = \nabla_\xi \theta = -\frac{1}{3}\theta^2 - \underbrace{G_{ab}\zeta^{ab}}_{\text{always positive}} + \underbrace{\omega_{ab}\omega^{ab}}_{\text{vanishes if}} - \underbrace{R_{cd}\xi^c\xi^d}_{\text{pos. or neg?}}$

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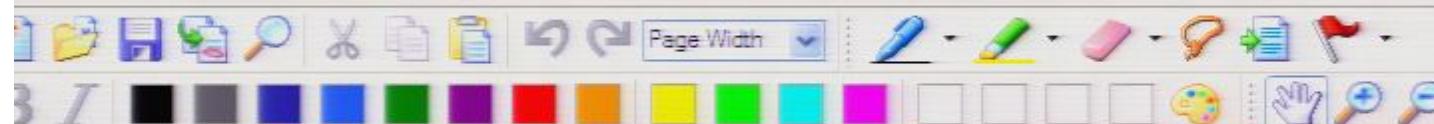
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Dynamics?

□ Assume that

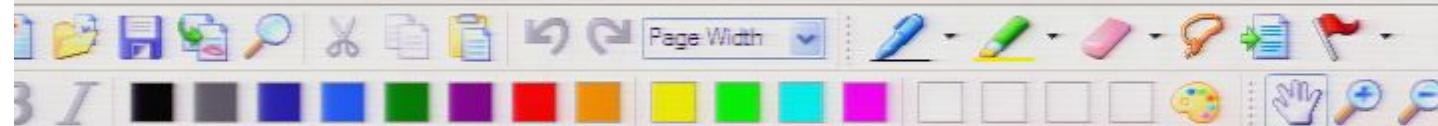
$$R_{\mu\nu}\zeta^\mu\zeta^\nu \geq 0 \quad \text{for all timelike } \zeta$$

i.e., using the Einstein equation

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$$T_{\mu\nu}\zeta^\mu\zeta^\nu - \frac{1}{2}\zeta^\mu\zeta_\mu T \stackrel{= -1}{\approx} 0$$



The Raychaudhuri equation is the trace of this equation:

recall: $\frac{d\theta}{d\tau} = \nabla_\xi \theta = -\frac{1}{3}\theta^2 - G_{ab}\zeta^{ab} + w_{ab}w^{ab} - R_{cd}\xi^c\xi^d$

$$\frac{d}{d\tau} = \xi^\mu \partial_\mu \text{ and } \text{Tr}(\mathcal{B}) = \theta$$

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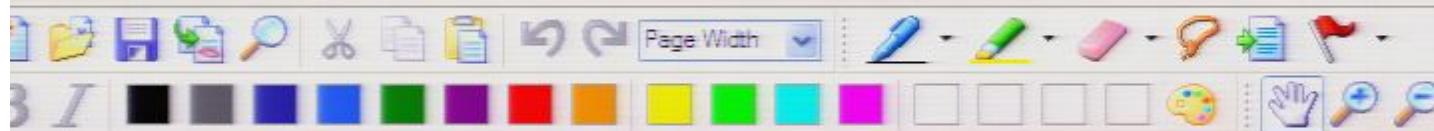
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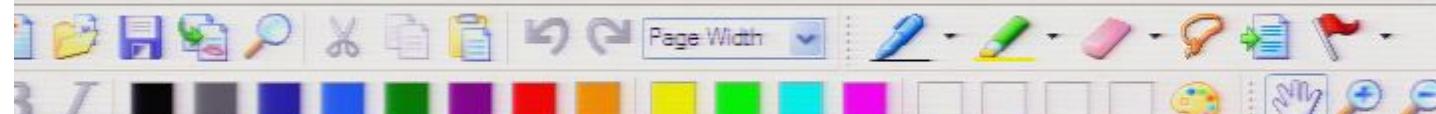
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i.e. the Strong Energy Condition.

□ Then:

$$\frac{d\theta}{dt} + \frac{1}{2}\theta^2 \leq 0$$



i.e. the Strong Energy Condition.

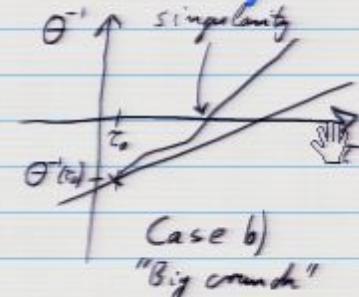
□ Then:

$$\frac{d\theta}{d\tau} + \frac{1}{3}\theta^2 \leq 0$$

$\Rightarrow \frac{d}{d\tau} \theta^{-1} \geq \frac{1}{3}$ Rewrite as $d\theta^{-1} \geq \frac{1}{3} d\tau$ and integrate:

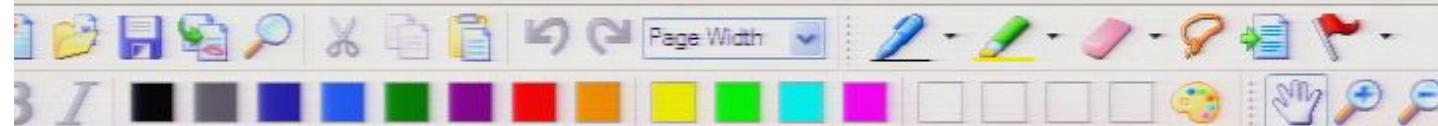
$$\Rightarrow \theta^{-1}(\tau) - \theta^{-1}(\tau_0) \geq \frac{1}{3}(\tau - \tau_0)$$

$$\theta^{-1}(\tau) \geq \frac{1}{3}(\tau - \tau_0) + \theta^{-1}(\tau_0) \quad (*)$$



□ Consider the cases when the geodesics are initially

- a.) diverging, i.e., $\theta(\tau_0) > 0$ (expanding universe)
- b.) converging, i.e., $\theta(\tau_0) < 0$ (contracting universe)



The Raychaudhuri equation is the trace of this equation:

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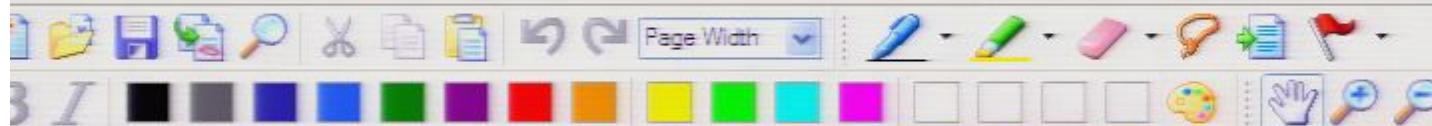
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i.e. that $T_{\mu\nu}\xi^\mu\xi^\nu \geq -\frac{1}{2}T$ whenever $\xi^\mu\xi_\mu < 0$

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□ Then:

$$\frac{d\theta}{d\tau} + \frac{1}{3}\theta^2 \leq 0$$

$$\Rightarrow \frac{d}{d\tau}\theta^{-1} \geq \frac{1}{3}$$

Rewrite as $d\theta^{-1} \geq \frac{1}{3}d\tau$ and integrate:

on Σ at a future time t_0



we are assuming that

$$T_{\mu\nu} \xi^\mu \xi^\nu - \frac{1}{2} \xi^\mu \xi_\mu T > 0$$

i.e. that $T_{\mu\nu} \xi^\mu \xi^\nu \geq -\frac{1}{2} T$ whenever $\xi^\mu \xi_\mu < 0$

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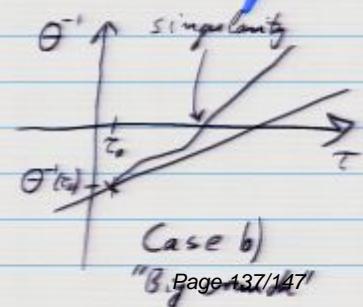
□ Then:

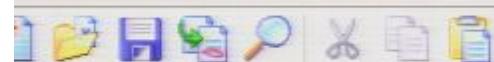
$$\frac{d\theta}{d\tau} + \frac{1}{3} \theta^2 \leq 0$$

$\Rightarrow \frac{d}{d\tau} \theta^{-1} \geq \frac{1}{3}$ Rewrite as $d\theta^{-1} \geq \frac{1}{3} d\tau$ and integrate:

$$\Rightarrow \theta^{-1}(\tau) - \theta^{-1}(\tau_0) \stackrel{\text{on } \Sigma \text{ at eventime } \tau_0}{\geq} \frac{1}{3}(\tau - \tau_0)$$

$$\theta^{-1}(\tau) \geq \frac{1}{3}(\tau - \tau_0) + \theta^{-1}(\tau_0) \quad (*)$$





Page Width



$$\begin{cases} d = 0 & \text{if } \Sigma \text{ is a manifold} \\ \frac{d}{dt} = \beta \text{ and } \text{Tr}(B) = 0 & \end{cases}$$

always positive
vanishes if
closed congruence

vanishes if
closed congruence $\perp \Sigma$

pos. or neg?
pos. or neg?

Dynamics?

□ Assume that

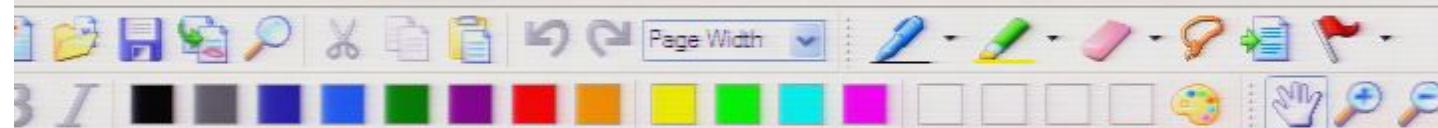
$$R_{\mu\nu}\xi^\mu\xi^\nu \geq 0 \quad \text{for all timelike } \xi$$

i.e., using the Einstein equation

$$R_{\mu\nu} = 8\pi G (T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^a_a)$$

we are assuming that

$$T_{\mu\nu}\xi^\mu\xi^\nu - \frac{1}{2}\xi^\mu\xi_\mu T \stackrel{= -1}{>} 0$$



of 5

$$\text{Lanczos: } B_{\mu\nu} = \frac{1}{3} \theta h_{\mu\nu} + g_{\mu\nu} + \omega_{\mu\nu}$$

The Raychaudhuri equation is the trace of this equation:

recall: $\frac{d\theta}{dt} = \nabla^{\alpha}\theta = -\frac{1}{3}\theta^2 - G_{ab}G^{ab} + \omega_{ab}\omega^{ab} - R_{cd}\delta^c\delta^d$

$\frac{d}{dt} = \delta$ and $\text{Tr}(B) = \theta$

recall: Ricci tensor is
 $R_{cd} = R_{da}{}^a$

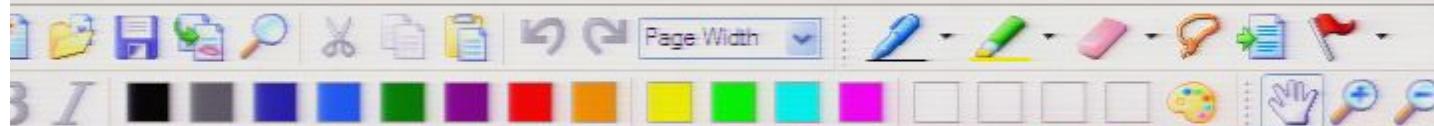
$G_{ab}G^{ab}$ always positive $\omega_{ab}\omega^{ab}$ vanishes if
 close congruence $\perp \Sigma$ $R_{cd}\delta^c\delta^d$ pos. or neg?

Dynamics?

Assume that

$$R_{\mu\nu}\delta^c\delta^d \geq 0 \text{ for all timelike } \delta$$

i.e., using the Einstein equation



i.e. that $T_{\mu\nu}\xi^\mu\xi^\nu \geq -\frac{1}{2}T$ whenever $\xi^\mu\xi_\mu < 0$

i.e. the Strong Energy Condition.

□ Then:

$$\frac{d\theta}{d\tau} + \frac{1}{3}\theta^2 \leq 0$$

$\Rightarrow \frac{d}{d\tau} \theta^{-1} \geq \frac{1}{3}$ Rewrite as $d\theta^{-1} \geq \frac{1}{3}d\tau$ and integrate:

$$\Rightarrow \theta^{-1}(\tau) - \theta^{-1}(\tau_0) \geq \frac{1}{3}(\tau - \tau_0)$$

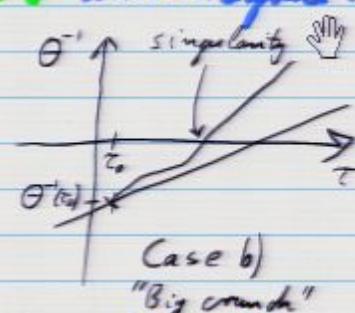
on Σ at signature t_0

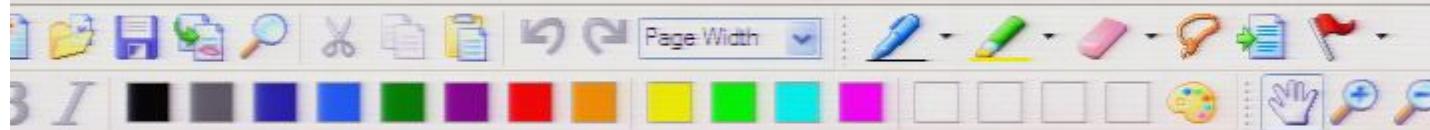
$$\theta^{-1}(\tau) \geq \frac{1}{3}(\tau - \tau_0) + \theta^{-1}(\tau_0) \quad (*)$$

□ Consider the cases when the geodesics are initially

a.) diverging, i.e., $\theta(\tau_0) > 0$ (expanding universe)

b.) converging, i.e., $\theta(\tau_0) < 0$ (contracting universe)





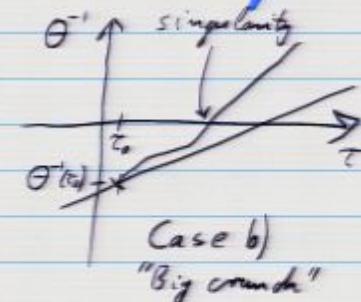
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Assuming that

$\frac{d}{d\tau} \theta^{-1}$ does not change

before zero crossing
of θ^{-1} could occur.

□ Now, follow the evolution in the two cases

a.) backwards in time (which changes the sign of $\frac{d\theta}{d\tau}$)

b.) forward in time:

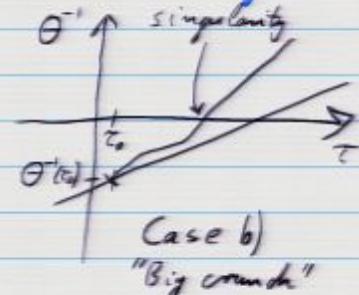


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- a.) diverging, i.e., $\theta(\tau_*) > 0$ (expanding universe)
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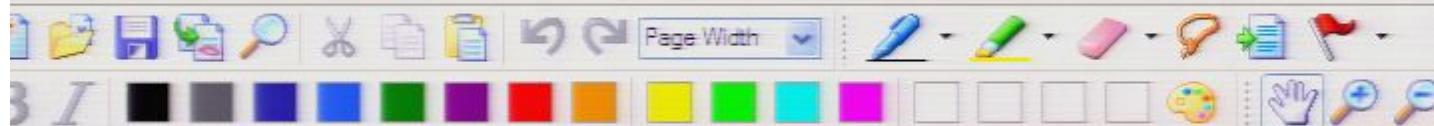
Assuming that

$\frac{d}{d\tau} \theta^{-1}$ does not change before a zero crossing of θ^{-1} could occur.

□ Now, follow the evolution in the two cases

- a.) backwards in time (which changes the sign of $\frac{d\theta}{d\tau}$)

- b.) forward in time:



- Consider the cases when the geodesics are initially
 - a.) diverging, i.e., $\dot{\Theta}(\tau_0) > 0$ (expanding universe)
 - b.) converging, i.e., $\dot{\Theta}(\tau_0) < 0$ (contracting universe)

Assuming that
 $\frac{d}{dt}\Theta^*$ does not change
 before a zero crossing
 of Θ^* could occur.

- Now, follow the evolution in the two cases

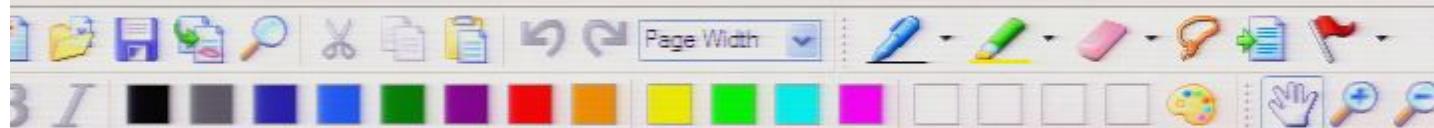
- a.) backwards in time (which changes the sign of $\frac{d\Theta}{d\tau}$)
- b.) forward in time:

Conclusion: Eq. (*) implies that $\Theta(\tau)$ must go through 0, i.e.:

- a.) for sufficiently early τ , have $\Theta \rightarrow +\infty$, i.e.: Big Bang
- b.) for sufficiently late τ , have $\Theta \rightarrow -\infty$, i.e.: Big Crunch

Note:

This type of reasoning leads also to further cosmological singularity theorems



- a.) diverging, i.e., $\dot{\Theta}(\tau_0) > 0$ (expanding universe)
- b.) converging, i.e., $\dot{\Theta}(\tau_0) < 0$ (contracting universe)

Assuming that

$\frac{d}{dt}\dot{\Theta}$ does not diverge
before a zero crossing
of $\dot{\Theta}$ could occur.

Now, follow the evolution in the two cases

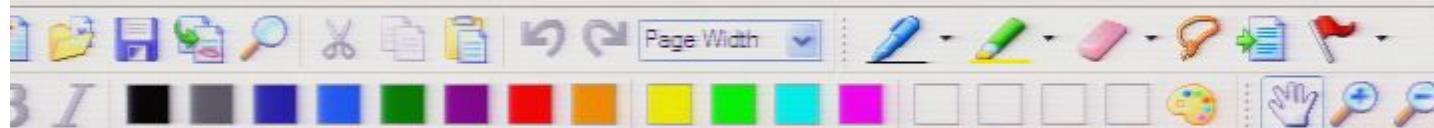
- a.) backwards in time (which changes the sign of $\frac{d\Theta}{d\tau}$)
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This type of reasoning leads also to further cosmological singularity theorems.



b) converging, i.e., $\Theta(\tau_0) < 0$ (contracting universe)

Assuming that

$\frac{d\Theta}{d\tau}$ does not change
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Now, follow the evolution in the two cases

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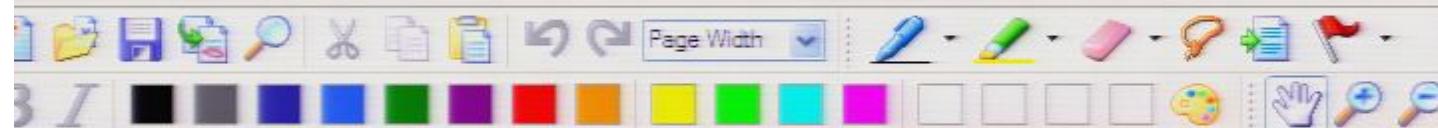
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Note: This type of reasoning leads also to further cosmological singularity theorems.

E.g., another cosmological singularity theorem does not



Conclusion: Eq. (*) implies that $\dot{\Theta}(\tau)$ must go through 0, i.e.:

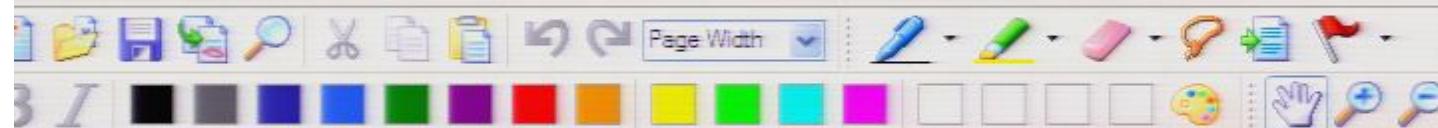
- a.) for sufficiently early τ , have $\Theta \rightarrow +\infty$, i.e.: Big Bang
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Note: This type of reasoning leads also to further cosmological singularity theorems.

E.g., another cosmological singularity theorem does not assume global hyperbolicity, and its conclusion is weaker:

There is at least one incomplete timelike geodesic.

Remark: Singularity theorems suitable for black hole case also assume a trapped surface, i.e., a surface



Conclusion: Eq. (*) implies that $\Theta(\tau)$ must go through 0, i.e.:

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Remark:

Singularity theorems suitable for black hole case also assume a trapped surface, i.e., a surface