Title: General Relativity for Cosmology - Lecture 14B

Date: Nov 05, 2009 05:30 PM

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Abstract:

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two problems in particular:

1) Dirac Equation

Tetrad formalism is useful because it allows $g_{\mu\nu} = \begin{pmatrix} -1 & & 0 \\ & 1 & \\ 0 & & \ddots \end{pmatrix}$ for all $x \in M$

2) Global conservation laws

Tetrad formalism is useful because uses (tensor-valued) differential forms which can be integrated and for which Stokes' theorem can relate local to global entities.

1) Dirac equation: (Brief treatment of basics only of Dirac spinors)

In special relativity: (with units such that $\hbar = 1$)

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## In special relativity:

(with units such that $\hbar = 1$)

$$\boxed{\left(i\gamma^\mu \frac{\partial}{\partial x^\mu} - m\right)\psi(x) = 0}$$

"Dirac equation"

where $\psi = \begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix}$ is a "Spinor"

describes spin ½ particles such as electrons and quarks

and the four $4\times 4$ matrices $\gamma^\mu$ obey:

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu} \qquad (*)$$

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- By choosing an orthonormal tetrad, $\{\theta^i\}$, we achieve

$$g^{\mu\nu} = \eta^{\mu\nu} \quad \forall p \in M$$

i.e. one set of matrices $\gamma^\mu$ obeying $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}$ suffices.

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- But what is the covariant derivative of a spinor?

$$\nabla_{e_\mu}\psi = ?$$

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**Idea:** The covariant derivative of a <u>spinor</u> should yield the rotation of the spinor by the same infinitesimal Lorentz transformation.

**Recall:** Infinitesimal parallel transport of a vector $e_\rho$ in direction $e_\mu$:

$$e_\rho \to e_\rho + \nabla_{e_\mu} e_\rho = e_\rho + \omega_\rho^{\ \sigma}(e_\mu)\, e_\sigma$$

<u>Recall:</u> the curvature 1-form takes values that are infinitesimal Lorentz transformations.

Recall intuition why parallel transport yields Lorentz transformation:
Parallel transport preserves the

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**To this end:** Recall from Special Relativity how an infinitesimal Lorentz transformation acts on a spinor:



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- Assume $\{s_i\}_{i=1}^{4}$ are ON basis in Spinor space, i.e.

$$\psi = \psi^i(x)\, s_i$$

these are Spinor indices: $i = 1,2,3,4$

- How do the $s_i$ transform under Lorentz transformations? I.e., what is $\nabla_{e_i} s_j = ?$ (In analogy to $\nabla_{e_a} e_\mu = \omega^\nu{}_\mu(e_a)\, e_\nu$)

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$$e_\mu \rightarrow e_\mu + \omega_\mu{}^\nu e_\nu$$

and the Dirac spinors transform as:

**Where does $[\gamma^\mu, \gamma^\nu]$ come from?**

Recall that e.g. translations in space are generated by momentum operators, $e^{-i\vec{a}\vec{p}} f(x) e^{i\vec{a}\vec{p}} = f(x+\vec{a})$, if they obey the commutation relations $[x_i, p_j] = i\delta_{ij}$.

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From special relativity it is known that under infinitesimal Lorentz transformations,

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Similarly, Lorentz transformations are generated by operators $M^{\mu\nu}$: antisym. antisym.

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# Apply to GR:

If a vector $e_\mu$ is infinitesimally parallel transported in the direction of $e_a$ then it obtains an infinitesimal "rotation", namely the infinitesimal Lorentz transformation

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which is the value of the connection 1-form, i.e.:

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$$(i\gamma^{\mu}\nabla_{e_{\mu}} - m)\Psi = 0$$

now takes the explicit form:

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The general relativistic Dirac equation

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**Remark:** In GR, the relationship between the Dirac operator $D = i\gamma^\mu \nabla_{e_\mu}$ and the Laplace or d'Alembert operator $\Box$ becomes nontrivial. This is a potential essay topic.

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## 2) Global conservation laws:

Recall:



$\Box$ In special relativity, energy and momenta

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Recall:

- In special relativity, energy and momentum conservation etc express the fact that the translation in time or space (or rotation etc) of a solution is a solution too.
  This doesn't hold in curved space-time, of course.

- Indeed, it is always true that

$$T^{\mu\nu}{}_{;\nu} = 0$$

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- In regions of significant gravitational effects,

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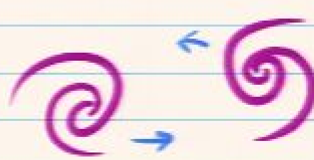

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- **Problems:**

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  Recall: Locally, gravity can always be eliminated, $\Gamma^{\mu}_{\nu\sigma}(p)=0$, i.e. there surely is

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# Problems:

a.) What is "gravitational potential energy in GR"?

Recall: Locally, gravity can always be eliminated, $\Gamma^{\mu}{}_{\nu\rho}(p)=0$, i.e. there surely is no local notion of gravitational potential energy.

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4 dim space-time region 3-dim space time submanifold

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differential forms

II Defining $\tau_\alpha := T_\alpha + t_\alpha$ we then have:

$$d(\text{something}) = 8\pi G \sqrt{g} * \tau_\alpha$$

III Then, from $d^2 = 0$ we obtain:

$$d\left(\sqrt{g} * \tau_\alpha\right) = 0$$

IV Using Stokes' theorem, $\int_D d\nu = \int_{\partial D} \nu$, we obtain:

$$0 = \int \overbrace{d\left(\sqrt{g} * \tau_\alpha\right)}^{4\text{-form}} = \int \overbrace{\sqrt{g} * \tau_\alpha}^{3\text{-form}}$$

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II Defining $\tau_\alpha := T_\alpha + t_\alpha$ we then have:

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IV Using Stokes' theorem, $\int_D d\nu = \int_{\partial D} \nu$, we obtain:

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$D$: 4 dim space-time region

$\partial D$: 3-dim space time sub manifold.

**III** [illegible]

$$d\left(\sqrt{g}\, * \tau_\alpha\right) = 0$$

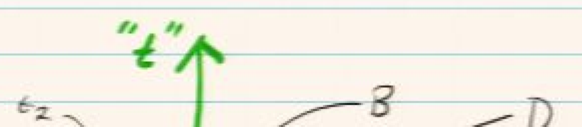
**IV** Using Stokes' theorem, $\int_D d\nu = \int_{\partial D} \nu$, we obtain:

$$0 = \int_D \overbrace{d\left(\sqrt{g}\, \underbrace{* \underbrace{\tau_\alpha}_{\text{1-form}}}_{\text{3-form}}\right)}^{\text{4-form}} = \int_{\partial D} \overbrace{\sqrt{g}\, * \tau_\alpha}^{\text{3-form}} \qquad \otimes$$

4 dim space-time region

3-dim space time sub manifold.

**V** Choose $D$ bounded by the large box $B$, i.e. on large scales we have:



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$$d\left(V_g * \tau_\alpha\right) = 0$$

**IV** Using Stokes' theorem, $\int_D d\nu = \int_{\partial D} \nu$, we obtain:

$$0 = \int_D \overbrace{d\left(V_g * \underbrace{\underbrace{\tau_\alpha}_{1\text{-form}}}_{3\text{-form}}\right)}^{4\text{-form}} = \int_{\partial D} \overbrace{V_g * \tau_\alpha}^{3\text{-form}} \qquad X$$

4 dim space-time region

3-dim space time sub manifold.

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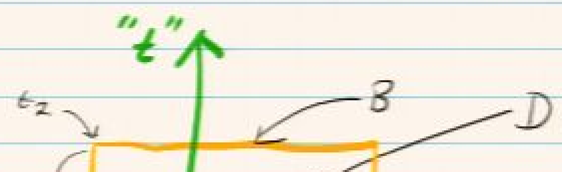

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**IV** Using Stokes' theorem, $\int_D dv = \int_{\partial D} v$, we obtain:

$$0 = \int_D \overbrace{d\big(\underbrace{V_g * \underbrace{\tau_\alpha}_{1\text{-form}}}_{3\text{-form}}\big)}^{4\text{-form}} = \int_{\partial D} \overbrace{V_g * \tau_\alpha}^{3\text{-form}} \qquad \otimes$$

$D$: 4dim space-time region

$\partial D$: 3-dim space time sub manifold.

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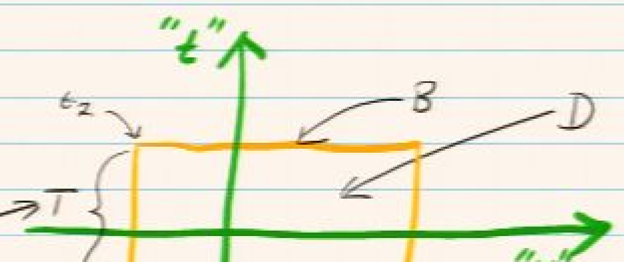



(T is assumed far out

**IV** Using Stokes' theorem, $\int_D d\nu = \int_{\partial D} \nu$, we obtain:

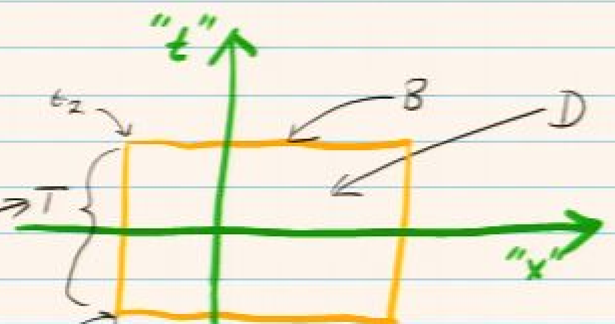
$$0 = \int_D \overbrace{d\big(\sqrt{g}\, \underbrace{* \underbrace{\tau_\alpha}_{1\text{-form}}}_{3\text{-form}}\big)}^{4\text{-form}} = \int_{\partial D} \overbrace{\sqrt{g} * \tau_\alpha}^{3\text{-form}} \qquad \otimes$$

$D$: 4-dim space-time region

$\partial D$: 3-dim space time submanifold.

**V** Choose $D$ bounded by the large box $B$, i.e. on large scales we have:

(T is assumed far out in space, where there is no matter, energy, momentum



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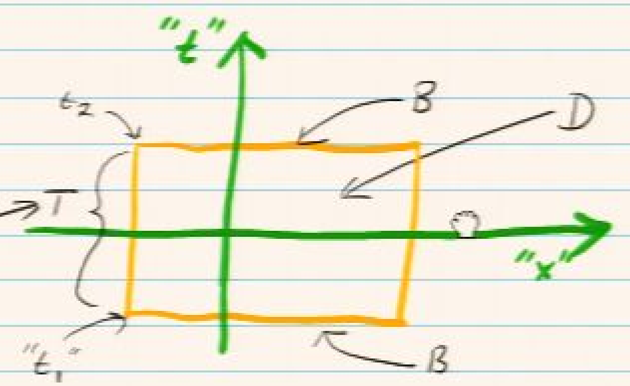
$$0 = \int_D d(V_g \ast \underbrace{\underbrace{\tau_\alpha}_{\text{1-form}})}_{\text{3-form}} = \int_{\partial D} V_g \ast \tau_\alpha \qquad \otimes$$

$D$: 4 dim space-time region

$\partial D$: 3-dim space time submanifold.

$\underline{\mathrm{V}}$ Choose $D$ bounded by the large box $B$, i.e. on large scales we have:

(T is assumed far out in space, where there is no matter, energy, momentum and no curvature)




so that equation $\otimes$, namely

$$\int V_g \ast \tau_\alpha = 0$$

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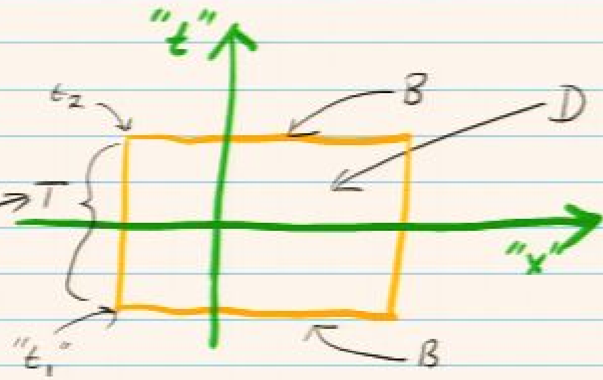
V Choose $D$ bounded by the large box $B$, i.e. on large scales we have:




(T is assumed far out in space, where there is no matter, energy, momentum and no curvature.)

so that equation ⊗, namely

$$\int_{\partial D} v_J * \tau_\alpha = 0$$

becomes:

$$0 = \int_{B(t_1)} v_J * \tau_\alpha + \int_{B(t_2)} v_J * \tau_\alpha + \int_{T} v_J * \tau_\alpha$$

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$$0 = \int_D d\big(\overbrace{\mathcal{V}_\xi * \underbrace{\underbrace{\tau_\alpha}_{1\text{-form}}}_{3\text{-form}}}^{3}\big) = \int_{\partial D} \overbrace{\mathcal{V}_\xi * \tau_\alpha}^{3\text{-form}} \qquad (\times)$$

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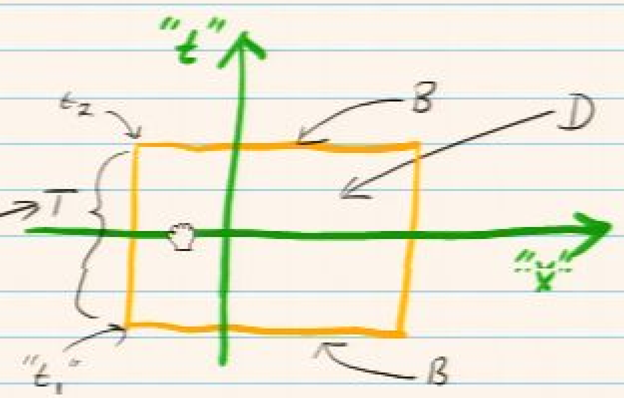
$\partial D$: 3-dim space time sub manifold.

**V** Choose $D$ bounded by the large box $B$, i.e. on large scales we have:




(T is assumed far out in space, where there is no matter, energy, momentum and no curvature.)

so that equation $(\times)$, namely

$$0 = \int_D \overbrace{d\big(\sqrt{g}\; \underbrace{* \underbrace{\tau_\alpha}_{1\text{-form}}}_{3\text{-form}}\big)}^{4\text{-form}} = \int_{\partial D} \overbrace{\sqrt{g} * \tau_\alpha}^{3\text{-form}} \qquad \otimes$$

$D$: 4dim space-time region

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**V** Choose $D$ bounded by the large box $B$, i.e. on large scales we have:


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so that equation ⊗, namely

( T is assumed far out in space, where there is no matter, energy, momentum and no curvature )


so that equation ⊗, namely

$$\int_{\partial D} V_g * \tau_\alpha = 0$$

becomes:

$$0 = \int_{B(t_1)} V_g * \tau_\alpha + \int_{B(t_2)} V_g * \tau_\alpha + \int_{T} V_g * \tau_\alpha$$

VI We notice that

$$\int V_g * \tau_\alpha = 0$$

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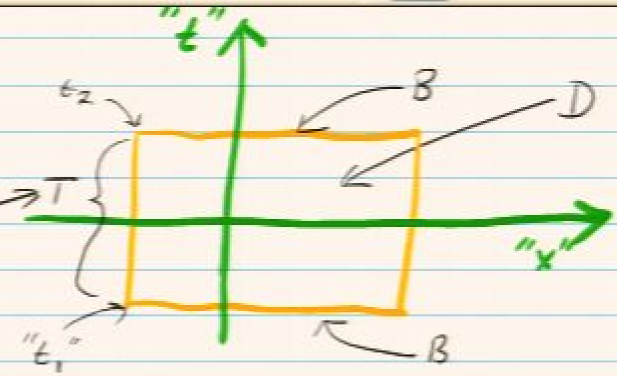
(T is assumed far out in space, where there is no matter, energy, momentum and no curvature)




so that equation ⊗, namely

$$\int_{\partial D} V_\beta * \tau_\alpha = 0$$

becomes:

$$0 = \int_{B(t_1)} V_\beta * \tau_\alpha + \int_{B(t_2)} V_\beta * \tau_\alpha + \int_{T} V_\beta * \tau_\alpha$$

VI We notice that

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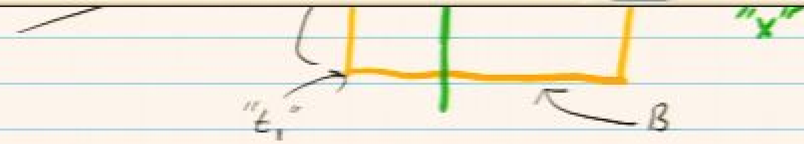
in space, where there is no matter, energy, momentum and no curvature




so that equation ⊗, namely

$$\int_{\partial D} v_g * \tau_\alpha = 0$$

becomes:

$$0 = \int_{B(t_1)} v_g * \tau_\alpha + \int_{B(t_2)} v_g * \tau_\alpha + \int_{T} v_g * \tau_\alpha$$

VI We notice that

$$\int_{T} v_g * \tau_\alpha = 0$$



if, as we here assume, space is flat

$$0 = \int_{B(t_1)} v_\beta * \tau_\alpha + \int_{B(t_2)} v_\beta * \tau_\alpha + \int_T v_\beta * \tau_\alpha$$

VI We notice that

$$\int_T v_\beta * \tau_\alpha = 0$$

if, as we here assume, space is flat and empty far out in space. I.e., where events of $T$ are, there we have $T_\alpha = 0$ and $t_\alpha = 0$

VII Therefore:

$$\int_{B(t_1)} v_\beta * \tau_\alpha + \int_{B(t_2)} v_\beta * \tau_\alpha = 0$$

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$$0 = \int_{B(t_1)} v_j * \tau_\alpha + \int_{B(t_2)} v_j * \tau_\alpha + \int_T v_j * \tau_\alpha$$

VI We notice that

$$\int_T v_j * \tau_\alpha = 0$$

if, as we here assume, space is flat and empty far out in space. I.e., where events of $T$ are, there we have $T_\alpha = 0$ and $t_\alpha = 0$

VII Therefore:

$$\int_{B(t_1)} v_j * \tau_\alpha + \int_{B(t_2)} v_j * \tau_\alpha = 0$$

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$$0=\int_{B(t_1)} \nabla_j * \tau_\alpha + \int_{B(t_2)} \nabla_j * \tau_\alpha + \int_{T} \nabla_j * \tau_\alpha$$

VI We notice that

$$\int_{T} \nabla_j * \tau_\alpha = 0$$

if, as we here assume, space is flat and empty far out in space. I.e., where events of $T$ are, there we have $T_\alpha = 0$ and $t_\alpha = 0$

VII Therefore:

$$\int_{B(t_1)} \nabla_j * \tau_\alpha + \int_{B(t_2)} \nabla_j * \tau_\alpha = 0$$

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becomes:

$$0 = \int_{B(t_1)} v_j^{\,} * \tau_\alpha + \int_{B(t_2)} v_j * \tau_\alpha + \int_T v_j * \tau_\alpha$$

VI We notice that

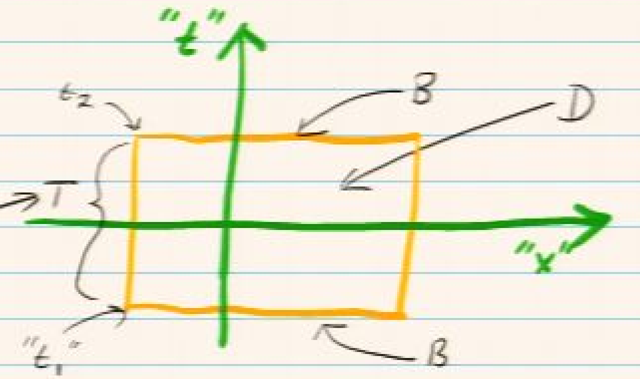
$$\int_T v_j * \tau_\alpha = 0$$

if, as we here assume, space is flat and empty far out in space. I.e., where events of $T$ are, there we have $T_\alpha = 0$ and $t_\alpha = 0$

VII Therefore:

$$\int v_j * \tau_\alpha + \int v_j * \tau_\alpha = 0$$

on large scales we have:




(T is assumed far out in space, where there is no matter, energy, momentum and no curvature.)

so that equation ⊗, namely

$$\int_{\partial D} v_g * \tau_\alpha = 0$$

becomes:

$$0 = \int_{B(t_1)} v_g * \tau_\alpha + \int_{B(t_2)} v_g * \tau_\alpha + \int_{T} v_g * \tau_\alpha$$

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becomes:

$$0 = \int_{B(t_1)} \sqrt{g} * \tau_\alpha + \int_{B(t_2)} \sqrt{g} * \tau_\alpha + \int_{T} \sqrt{g} * \tau_\alpha$$

VI We notice that

$$\int_{T} \sqrt{g} * \tau_\alpha = 0$$

if, as we here assume, space is flat and empty far out in space. I.e., where events of $T$ are, there we have $T_\alpha = 0$ and $t_\alpha = 0$

VII Therefore:

$$\int_{B(t_1)} \sqrt{g} * \tau_\alpha + \int_{B(t_2)} \sqrt{g} * \tau_\alpha = 0$$

## VII Therefore:

$$\int_{B(t_1)} \nabla\xi * \tau_\alpha + \int_{B(t_2)} \nabla\xi * \tau_\alpha = 0$$

But we notice that the 2nd integral has the timelike normal vector to $B(t_2)$ pointing to the past.

$\Rightarrow$ If we define the integrations both with respect to future-pointing normals, we obtain:

$$\int^{f}_{B(t_1)} \nabla\xi * \tau_\alpha = \int^{f}_{B(t_2)} \nabla\xi * \tau_\alpha$$

## VIII Define the total "ADM 4-momentum":

Arnowitt, Deser & Misner

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## VII Therefore:

$$\int_{B(t_1)} \nabla\xi * \tau_\alpha + \int_{B(t_2)} \nabla\xi * \tau_\alpha = 0$$

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## VIII Define the total "ADM 4-momentum":

Arnowitt, Deser & Misner

$$\int_{B(t_1)}^{f} \sqrt{g} * \tau_\alpha = \int_{B(t_2)}^{f} \sqrt{g} * \tau_\alpha$$

## VIII Define the total "ADM 4-momentum":

Arnowitt, Deser & Misner

$$P_\mu := \int_{B} \sqrt{g} * \tau_\mu$$

B ← big box

It is conserved in time:

$$\boxed{P_\mu(t_1) = P_\mu(t_2)}$$

Because under $\theta(x)^\mu \to \Lambda^\mu{}_\nu(x)\,\theta^\nu(x)$ we have generally [illegible] $\Lambda(x)\,d\Lambda^{-1}(x) + (d\Lambda)\Lambda$ but far out in space we now have:

**Note:** It is a Minkowski tensor with respect to local Lorentz transformations that approach a

$$\int_{B(t_1)}^{f} \sqrt{g} \, * \tau_\alpha = \int_{B(t_2)}^{f} \sqrt{g} \, * \tau_\alpha$$

## VIII Define the total "ADM 4-momentum":

Arnowitt, Deser & Misner

$$P_\mu := \int_{B} \sqrt{g} \, * \tau_\mu$$

big box

It is conserved in time:

$$\boxed{P_\mu(t_1) = P_\mu(t_2)}$$

Because under $\theta(x)^\nu \to \Lambda^\nu{}_\rho(x)\,\theta^\rho(x)$ we have generally $\omega(x) \to \Lambda(x)\,\omega\,\Lambda^{-1}(x) - (d\Lambda)\Lambda$ but far out in space we now have: $\Lambda(x) \to$ ...

**Note:** It is a Minkowski tensor with respect to local Lorentz transformations that approach a

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$$\int_{B(t_1)} \sqrt{g} * \tau_\alpha + \int_{B(t_2)} \sqrt{g} * \tau_\alpha = 0$$

But we notice that the 2nd integral has the timelike normal vector to $B(t_2)$ pointing to the past.

$\Rightarrow$ If we define the integrations both with respect to future-pointing normals, we obtain:

$$\int^{f}_{B(t_1)} \sqrt{g} * \tau_\alpha = \int^{f}_{B(t_2)} \sqrt{g} * \tau_\alpha$$

## VIII Define the total "ADM 4-momentum":

Arnowitt, Deser & Misner

$$P_\mu := \int \sqrt{g} * \tau_\mu$$

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$$\int_{B(t_1)} \sqrt{g} * \tau_\alpha + \int_{B(t_2)} \sqrt{g} * \tau_\alpha = 0$$

But we notice that the 2nd integral has the timelike normal vector to $B(t_2)$ pointing to the past.

$\Rightarrow$ If we define the integrations both with respect to future-pointing normals, we obtain:

$$\int_{B(t_1)}^{f} \sqrt{g} * \tau_\alpha = \int_{B(t_2)}^{f} \sqrt{g} * \tau_\alpha$$

$\underline{\text{VIII}}$ Define the total "ADM 4-momentum":

Arnowitt, Deser & Misner

$$P_\mu := \int_B \sqrt{g} * \tau_\mu$$

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to future-pointing normals, we obtain:

$$\int_{B(t_1)}^{f} \sqrt{g} * \tau_\alpha = \int_{B(t_2)}^{f} \sqrt{g} * \tau_\alpha$$

**VIII** Define the total "ADM 4-momentum":

Arnowitt, Deser & Misner

$$P_\mu := \int_B \sqrt{g} * \tau_\mu$$

B ← big box

It is conserved in time:

$$\boxed{P_\mu(t_1) = P_\mu(t_2)}$$

Because under $\theta(x)^\mu \to A^\mu{}_\nu(x)\,\theta^\nu(x)$ we have generally $\omega(x) \to A(x)\,\omega(x)\,A^{-1}(x) - (dA)A$

Note: It is a Minkowski tensor with respect to

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# VIII Define the total "ADM 4-momentum":

Arnowitt, Deser & Misner

$$P_\mu := \int_B \sqrt{g} * \tau_\mu$$

big box

It is conserved in time:

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(Because under
$\theta(x)^\mu \to \Lambda^\mu{}_\nu(x)\,\theta^\nu(x)$
we have generally
$\omega(x) \to \Lambda(x)\omega(x)\Lambda^{-1}(x) - (d\Lambda)\Lambda$
but far out in space we now have:
$\Lambda(x) \to$ const. i.e.
$\omega(x) \to \Lambda(x)\omega(x)\Lambda^{-1}(x) - 0$)

**Note:** It is a Minkowski tensor with respect to local Lorentz transformations that approach a constant Lorentz transformation far out in space.

## Determination of $\tau_\alpha$

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to future-pointing normals, we obtain:

$$\int_{B(t_1)}^{f} \sqrt{g} * \tau_\alpha = \int_{B(t_2)}^{f} \sqrt{g} * \tau_\alpha$$

VIII Define the total "ADM 4-momentum":

Arnowitt, Deser & Misner

$$P_\mu := \int_B \sqrt{g} * \tau_\mu$$

big box

It is conserved in time:

$$\boxed{P_\mu(t_1) = P_\mu(t_2)}$$

Because under $\theta(x) \to A^\nu_\mu(x)\theta^\mu(x)$

Note: It is a Minkowski tensor with ...

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$$P_\mu := \int_B \sqrt{g}\, * \tau_\mu$$

$B$ ← big box

It is conserved in time:

$$\boxed{P_\mu(t_1) = P_\mu(t_2)}$$

(Because under
$\theta(x)^\mu \to \Lambda^\mu{}_\nu(x)\,\theta^\nu(x)$
we have generally
$\omega(x) \to \Lambda(x)\,\omega(x)\,\Lambda^{-1}(x) - (d\Lambda)\Lambda$
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<u>Note:</u> It is a Minkowski tensor with respect to local Lorentz transformations that approach a constant Lorentz transformation far out in space.

## Determination of $\tau_\alpha$

- Recall starting assumption, namely that we can reformulate the Einstein equation

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It is conserved in time:

$$\boxed{P_\mu(t_1) = P_\mu(t_2)}$$

(Because under
$\theta(x)^\mu \to A^\mu{}_\nu(x)\,\theta^\nu(x)$
we have generally
$\omega(x) \to A(x)\,\omega(x)\,A^{-1}(x) - (dA)A$
but far out in space we now have:
$A(x) \to$ const. i.e.
$\omega(x) \to A(x)\,\omega(x)\,A^{-1}(x) - 0$)

**Note:** It is a Minkowski tensor with respect to local Lorentz transformations that approach a constant Lorentz transformation far out in space.

## Determination of $\tau_\alpha$

- Recall starting assumption, namely that we can reformulate the Einstein equation

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It is conserved in time:

$$\boxed{P_\gamma(t_1) = P_\gamma(t_2)}$$

(Because under
$\theta(x)^\beta \to \Lambda^\beta{}_\delta(x)\,\theta^\delta(x)$
we have generally
$\omega(x) \to \Lambda(x)\,\omega(x)\,\Lambda^{-1}(x) - (d\Lambda)\Lambda$
but far out in space we now have:
$\Lambda(x) \to$ const. i.e.
$\omega(x) \to \Lambda(x)\,\omega(x)\,\Lambda^{-1}(x) - 0$)

Note: It is a Minkowski tensor with respect to local Lorentz transformations that approach a constant Lorentz transformation far out in space.

## Determination of $\tau_\alpha$

- Recall starting assumption, namely that we can reformulate the Einstein equation

$$-\frac{1}{2} H_{\alpha\beta\gamma} \wedge \Omega^{\beta\gamma} = 8\pi G \, {*T_\alpha}$$

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constant Lorentz transformation far out in space.

# Determination of $\tau_\alpha$

- Recall starting assumption, namely that we can reformulate the Einstein equation

$$-\frac{1}{2} H_{\alpha\beta\gamma} \wedge \Omega^{\beta\gamma} = 8\pi G \ast T_\alpha$$

so that it reads:

$$d(\text{something}) = 8\pi G \sqrt{g} \overbrace{\left(\ast T_\alpha + \ast t_\alpha\right)}^{\ast \tau_\alpha \,=}$$

**Note:** (recall: $T_{\mu\nu}$ too is only unique if we require $T_{\mu\nu} = T_{\nu\mu}$)

$$\boxed{P_\gamma(t_1) = P_\gamma(t_2)}$$

Because under
$\theta^\alpha(x) \to A^\alpha{}_\beta(x)\,\theta^\beta(x)$
we have generally
$\omega(x) \to A(x)\,\omega(x)\,A^{-1}(x) - (dA)A^{-1}$
but far out in space we now have:
$A(x) \to \text{const.}$ i.e.
$\omega(x) \to A(x)\,\omega(x)\,A^{-1}(x) - 0$

<u>Note:</u> It is a Minkowski tensor with respect to local Lorentz transformations that approach a constant Lorentz transformation far out in space.

## Determination of $\tau_\alpha$

- Recall starting assumption, namely that we can reformulate the Einstein equation

$$-\frac{1}{2} H_{\alpha\beta\gamma} \wedge \Omega^{\beta\gamma} = 8\pi G \, {*T_\alpha}$$

so that it reads:

$$d(\text{something}) = 8\pi G \left( \underbrace{{*T_\alpha} + {*t_\alpha}}_{= {*\tau_\alpha}} \right)$$

## Determination of $\tau_\alpha$

- Recall starting assumption, namely that we can reformulate the Einstein equation

$$-\frac{1}{2} H_{\alpha\beta\gamma} \wedge \Omega^{\beta\gamma} = 8\pi G * T_\alpha$$

so that it reads:

$$d(\text{something}) = 8\pi G \sqrt{g} \, \big( \overbrace{* T_\alpha + * t_\alpha}^{* \tau_\alpha \,=} \big)$$

**Note:** (recall: $T_{\mu\nu}$ too is only unique if we require $T_{\mu\nu} = T_{\nu\mu}$)

This decomposition is not unique, but there is a unique decomposition so that $t_\alpha = t_{\alpha\beta}\,\theta^\beta$ is symmetric:

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# Determination of $\tau_\alpha$

- Recall starting assumption, namely that we can reformulate the Einstein equation

$$-\frac{1}{2} H_{\alpha\beta\gamma} \wedge \Omega^{\beta\gamma} = 8\pi G \, {*T_\alpha}$$

so that it reads:

$$d(\text{something}) = 8\pi G \sqrt{g} \, \overbrace{\left({*T_\alpha} + {*t_\alpha}\right)}^{{*\tau_\alpha} \,=}$$

**Note:** (recall: $T_{\mu\nu}$ too is only unique if we require $T_{\mu\nu} = T_{\nu\mu}$)

This decomposition is not unique, but there is a unique decomposition so that $t_\alpha = t_{\alpha\beta}\,\theta^\beta$ is symmetric:

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so that it reads:

$$d(\text{something}) = 8\pi G\sqrt{g}\,(\overbrace{*T_\alpha + *t_\alpha}^{"})$$

**Note:** (recall: $T_{\mu\nu}$ too is only unique if we require $T_{\mu\nu} = T_{\nu\mu}$)

This decomposition is not unique, but there is a unique decomposition so that $t_\alpha = t_{\alpha\beta}\,\theta^\beta$ is symmetric:

**Proposition:** (for proof see e.g. text by Straumann, page 112. It is not too difficult.)

The Einstein eqn. $-\frac{1}{2} H_{\alpha\beta\gamma} \wedge \Omega^{\beta\gamma} = 8\pi G *T_\alpha$ can be reformulated as

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$d(\text{something}) = 8\pi G \sqrt{g}\,(\ast T_\alpha + \ast t_\alpha)$

**Note:** (recall: $T_{\mu\nu}$ too is only unique if we require $T_{\mu\nu} = T_{\nu\mu}$)

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with



$:= \star(\theta^\alpha \wedge \theta^\beta \wedge \theta^\gamma \wedge \theta^\delta)$

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with

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so that it reads:

$$\overbrace{\qquad\qquad}^{*\tau_\alpha \,=}$$

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with

$$\underbrace{*t_\alpha}_{3\text{-form}} = -\frac{1}{16\pi G} \underbrace{H^{\alpha\beta\gamma\delta}}_{0\text{-form},} \underbrace{\left(\omega_{\sigma\beta} \wedge \omega^{\sigma}{}_{\gamma} \wedge \theta_\delta - \omega_{\beta\gamma} \wedge \omega_{\sigma\delta} \wedge \theta^\sigma\right)}_{3\text{-form}}$$

$H^{\alpha\beta\gamma\delta} := *(\theta^\alpha \wedge \theta^\beta \wedge \theta^\gamma \wedge \theta^\delta)$

$t_\alpha$: 1-form

(6)



which is the unique $t_\alpha$ for which $t_{\alpha\beta}\,\theta^\beta$ is symmetric

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1-form

$*(\theta^\alpha \wedge \theta^\beta \wedge \theta^\gamma \wedge \theta^\delta)$

(6)

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$$P_\mu := \int \sqrt{g}\, *\tau_\mu$$

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B ← big box



(The "positive ) with $\tau_\mu = T_\mu + t_\mu$

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$T_\mu$ from matter; $t_\mu$ from gravity using Eqn. (6) above.

**Theorem:** If the dominant energy condition holds, then $P_0 \geq 0$

with

1-form

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big box

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Angular momentum?

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3-form     0-form,     3-form     (G)

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## Angular momentum?



□ Choose coordinates that become cartesian Minkowski far out

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3-form (G)

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## Angular momentum?

- Choose coordinates that become cartesian Minkowski far out.

- Define: $*M^{\alpha\beta} := x^\alpha * \tau^\beta - x^\beta * \tau^\alpha$ → Note: For this it is necessa...

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(The "positive energy theorem") with $\tau_\mu = T_\mu + t_\mu$

← from gravity using Eqn. (G) above.

← from matter

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- Define: $*M^{\alpha\beta} := x^\alpha * \tau^\beta - x^\beta * \tau^\alpha$
- Proposition: $d(\sqrt{g} * M^{\alpha\beta}) = 0$

→ Note: For this it is necessary to have chosen the definition of $t$ which has $t_{\alpha\beta}$, and therefore also $\tau_{\alpha\beta}$, symmetric.

⇒ ADM 4-angular momentum $J^{\alpha\beta} := \int \sqrt{g} * M^{\alpha\beta}$ is conserved!

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$\Rightarrow$ ADM 4-angular momentum $J^{\alpha\beta} := \int_B \sqrt{g} \, {*M^{\alpha\beta}}$ is conserved!

b.) Taking account of grav. waves:

Do we have to account for possible energy-momentum loss due to radiation from the region of strong gravity, in particular, grav. wave radiation?

This depends on how we define our "box".

If the box is large enough for our assumptions to hold, then grav. radiation cannot escape the region between $t_1, t_2$

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If the box is large enough for our assumptions to hold, then grav. radiation cannot escape the region between $t_1, t_2$

But also: We can choose space-like hypersurfaces which at large distances "bend up" to become asymptotically light-like.

This leads to the Sachs-Bondi 4-momentum $P^{(SB)}$

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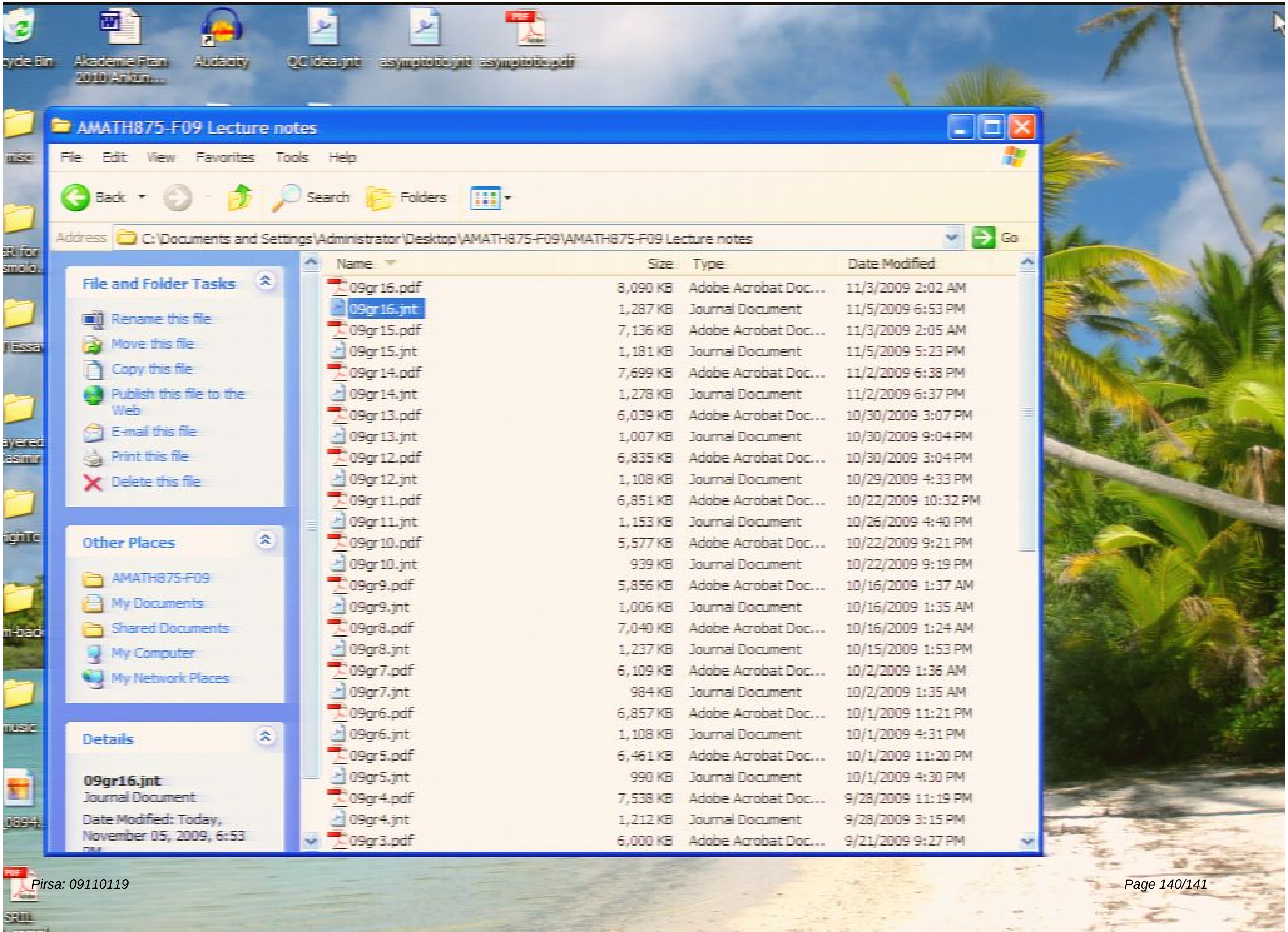
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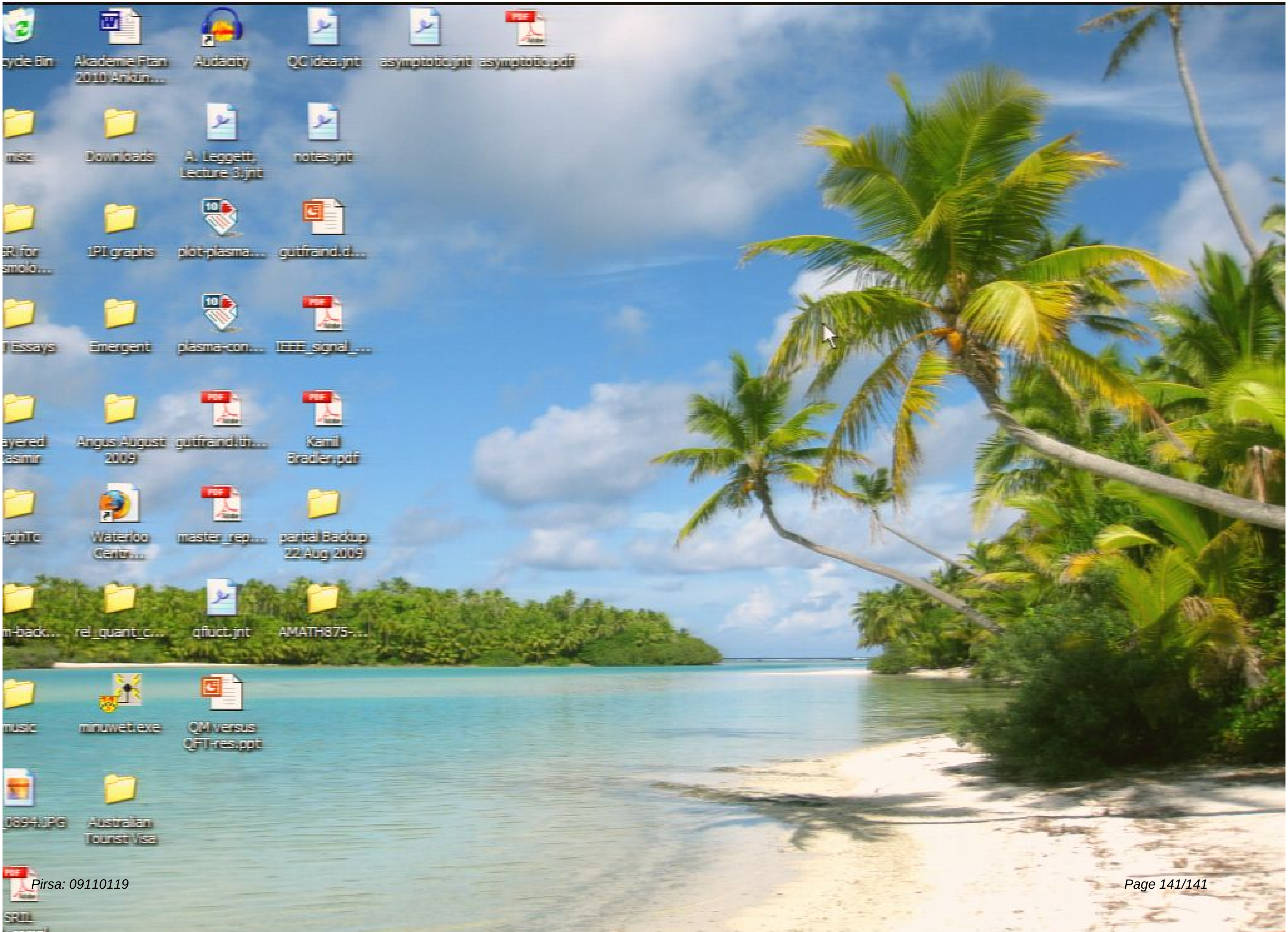
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master_rep...
partial Backup 22 Aug 2009
rel_quant_c...
qfluct.jnt
AMATH875-...
music
minuwet.exe
QM versus QFT-res.ppt
Australian Tourist Visa