Title: Mathematical Physics (PHYS 624) - Lecture 9

Date: Nov 26, 2009 09:00 AM

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Abstract:

Table 8.13 Padé summation of the Taylor series for $z^{-1} \ln(1+z)$ about $z = 0$

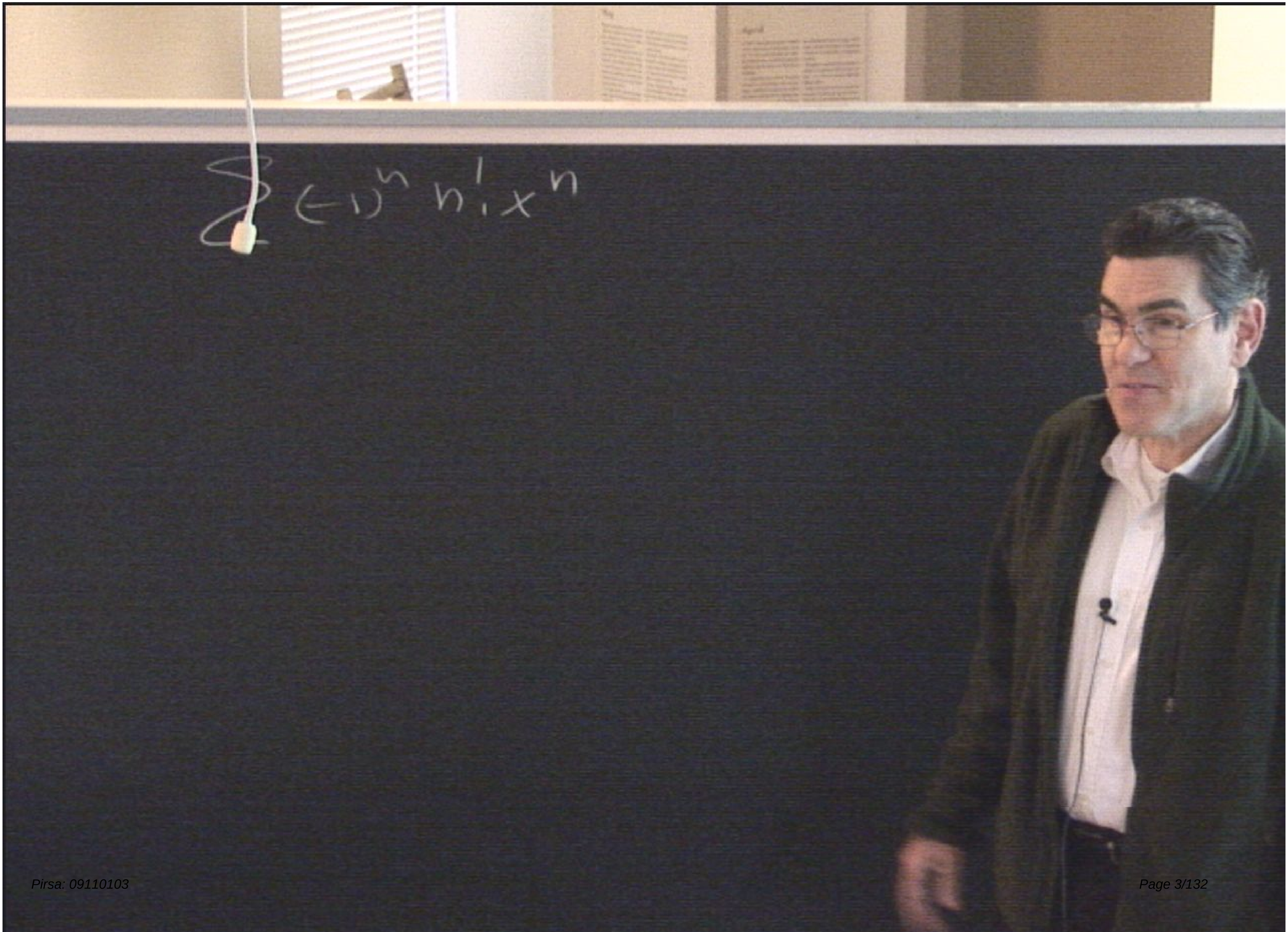
In Sec. 8.5, it will be shown that the sequences of Padé approximants $P_N^N(z)$ and $P_{N+1}^N(z)$ converge rapidly, even beyond the circle of convergence of the Taylor series $|z| < 1$. Observe that for real positive x the Padé approximants $P_N^N(x)$ monotonically decrease and $P_{N+1}^N(x)$ monotonically increase with N to the common limit $\ln(1+x)/x$. Thus, for any N, these Padé approximants supply upper and lower bounds on $\ln(1+x)/x$

$P_N^N(x)$

N	$x = 0.5$	$x = 1$	$x = 2$
1	0.812 500 000 0	0.700 000 000 0	0.571 428 571 4
2	0.810 945 273 6	0.693 333 333 3	0.550 724 637 7
3	0.810 930 365 3	0.693 152 454 8	0.549 402 823 0
4	0.810 930 217 7	0.693 147 332 4	0.549 312 879 5
5	0.810 930 216 2	0.693 147 185 0	0.549 306 618 4
6	0.810 930 216 2	0.693 147 180 7	0.549 306 177 9
7	0.810 930 216 2	0.693 147 180 6	0.549 306 146 7
8	0.810 930 216 2	0.693 147 180 6	0.549 306 144 5

$P_{N+1}^N(x)$

N	$x = 0.5$	$x = 1$	$x = 2$
1	0.810 810 810 8	0.692 307 692 3	0.545 454 545 5
2	0.810 928 961 7	0.693 121 693 1	0.549 019 607 8
3	0.810 930 203 2	0.693 146 417 4	0.549 285 176 8
4	0.810 930 216 1	0.693 147 157 9	0.549 304 620 9
5	0.810 930 216 2	0.693 147 179 9	0.549 306 034 1
6	0.810 930 216 2	0.693 147 180 5	0.549 306 136 4
7	0.810 930 216 2	0.693 147 180 6	0.549 306 143 8
8	0.810 930 216 2	0.693 147 180 6	0.549 306 144 3


$\sum (-1)^n n! x^n$

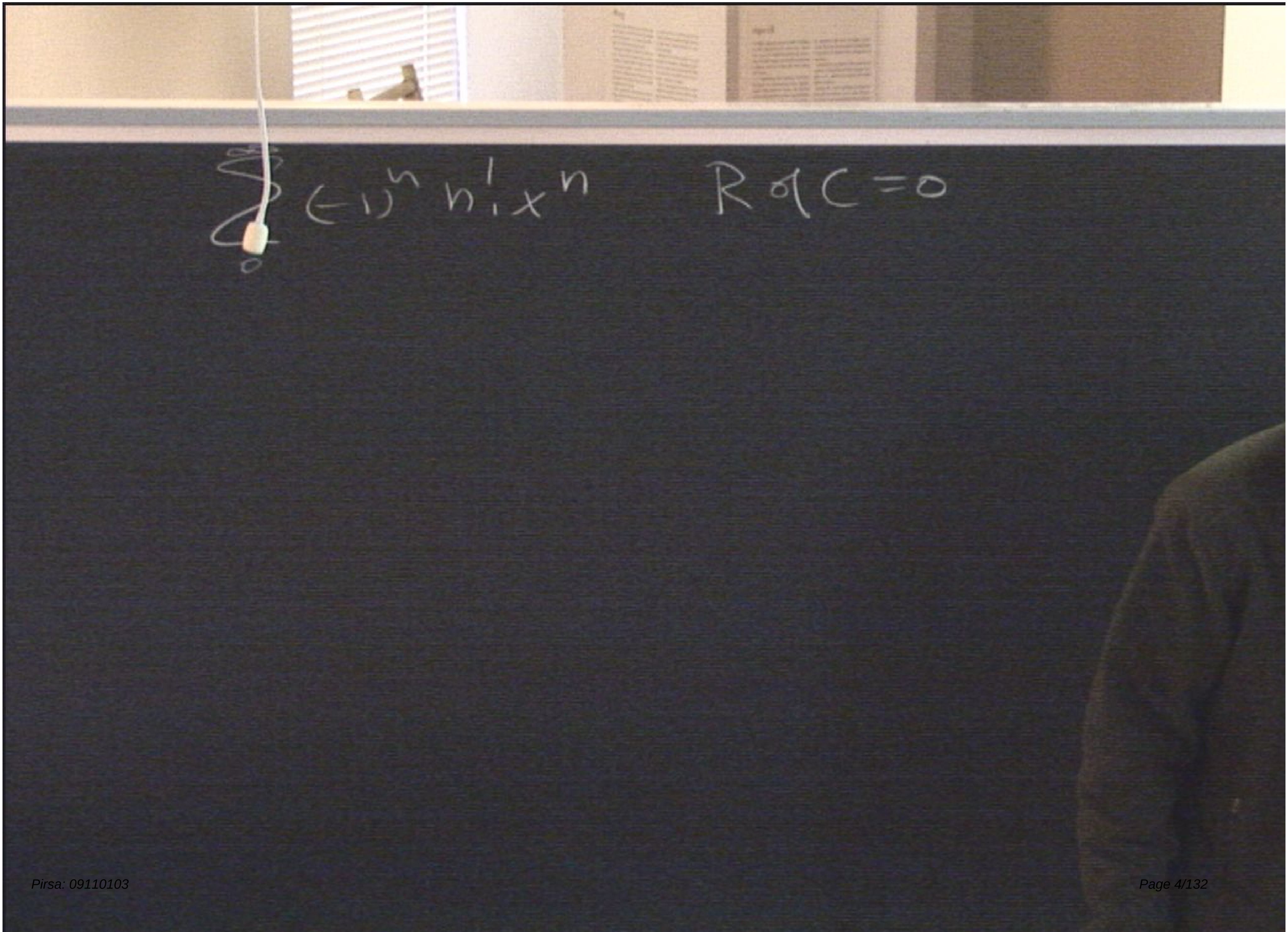

$\sum_{0}^{\infty} (-1)^n n! x^n$
RoC = 0

Table 8.15 Padé approximants $P_N^N(x)$ and $P_{N+1}^N(x)$ for the Stieltjes series $\sum_{n=0}^{\infty} (-x)^n n!$ evaluated at $x = 1$ and $x = 10$

In contrast to these good results, the optimal asymptotic approximation to the Stieltjes series at $x = 1$ and $x = 10$ is worthless (the optimal truncation contains no terms of the series). On the other hand, $P_{14}^{14}(1)$ and $P_{15}^{14}(1)$ agree with the "sum" of the series, $\int_0^\infty e^{-t}/(1 + t)\, dt$, to better than five decimal places, while $P_{50}^{50}(1)$ and $P_{51}^{50}(1)$ agree to ten decimal places

	$x = 1$		$x = 10$	
N	$P_N^N(1)$	$P_{N+1}^N(1)$	$P_N^N(10)$	$P_{N+1}^N(10)$
0	1.0	0.5	1.0	0.090 91
1	0.666 67	0.571 43	0.523 81	0.128 63
2	0.615 38	0.588 24	0.379 73	0.149 66
3	0.602 74	0.593 30	0.314 24	0.162 95
4	0.598 80	0.595 08	0.278 47	0.171 96
5	0.597 38	0.595 78	0.256 73	0.178 36
6	0.596 82	0.596 08	0.242 56	0.183 06
7	0.596 57	0.596 21	0.232 84	0.186 60
8	0.596 46	0.596 28	0.225 93	0.189 32
9	0.596 41	0.596 31	0.220 86	0.191 45
10	0.596 38	0.596 33	0.217 06	0.193 13
50	0.596 35	0.596 35	0.201 56	0.201 39
∞	0.596 35	0.596 35	0.201 46	0.201 46

Table 8.12 Padé summation of the Taylor series (5.4.4) for $1/\Gamma(x)$

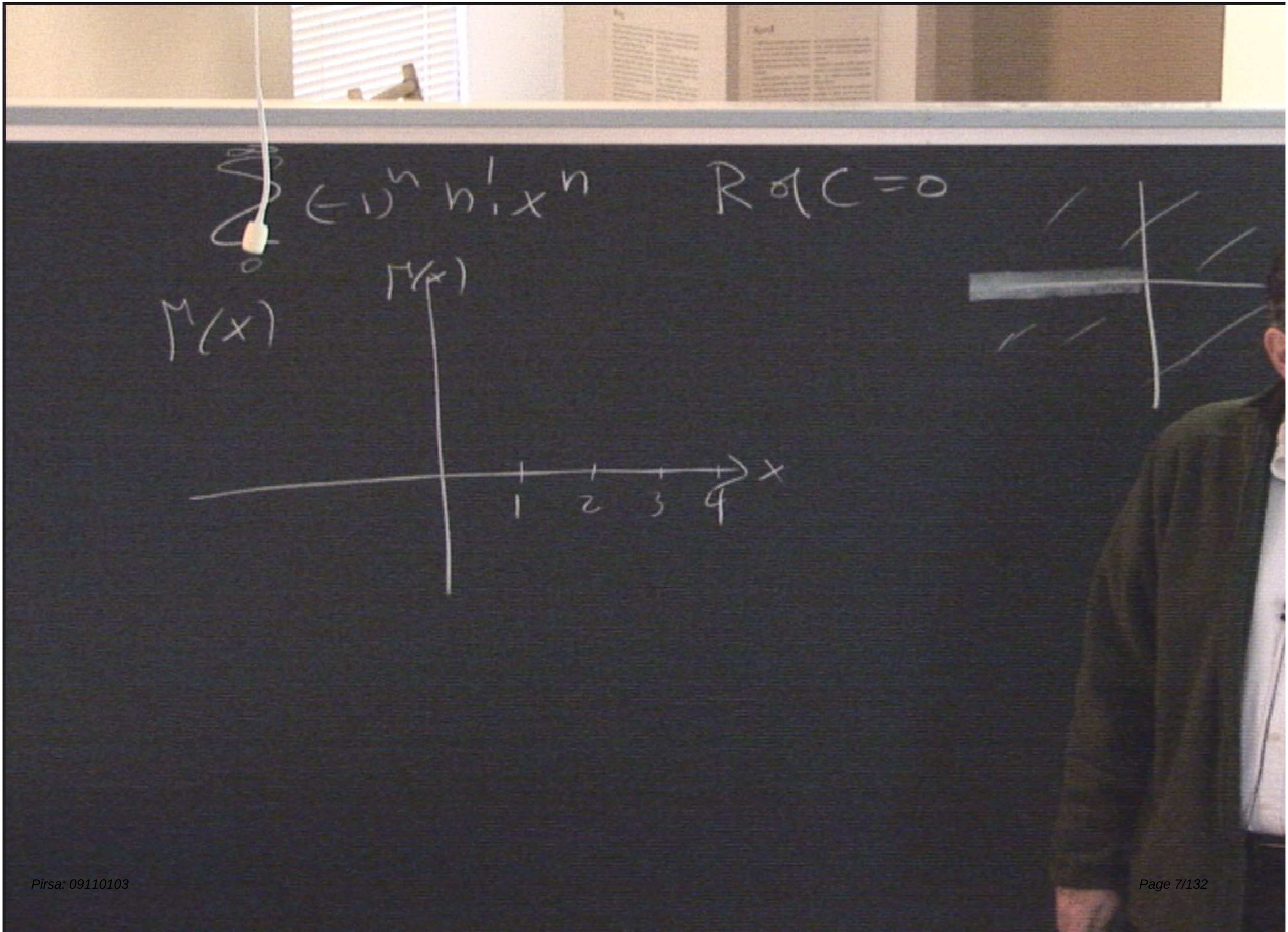
The entries in the table were computed as follows. The truncated Taylor series for $1/[x\Gamma(x)]$ was used to construct the Padé approximants $P_N^N(x)$ and $P_{N+1}^N(x)$; the resulting Padé approximants were then inverted and multiplied by x. It is clear from the above table that the entries in the three columns are approaching their correct limits: $\Gamma(1) = 1$, $\Gamma(2) = 1$, and $\Gamma(4) = 6$. Although Padé summation enhances the convergence of the Taylor series, the effect is not dramatic: the Padé sequence requires about half the number of terms that the Taylor series requires to achieve 1 percent accuracy (see Table 5.2). The slow convergence of the Taylor series is not due to the presence of singularities [$1/\Gamma(x)$ is entire], but rather to the enormous disparity of behavior of $1/\Gamma(x)$ for positive and negative x. Padé approximants improve the convergence of series by mimicking the singularities of the function to be represented; since $1/\Gamma(x)$ has no singularities, the benefits of Padé summation are marginal

$1/[xP_N^N(x)]$

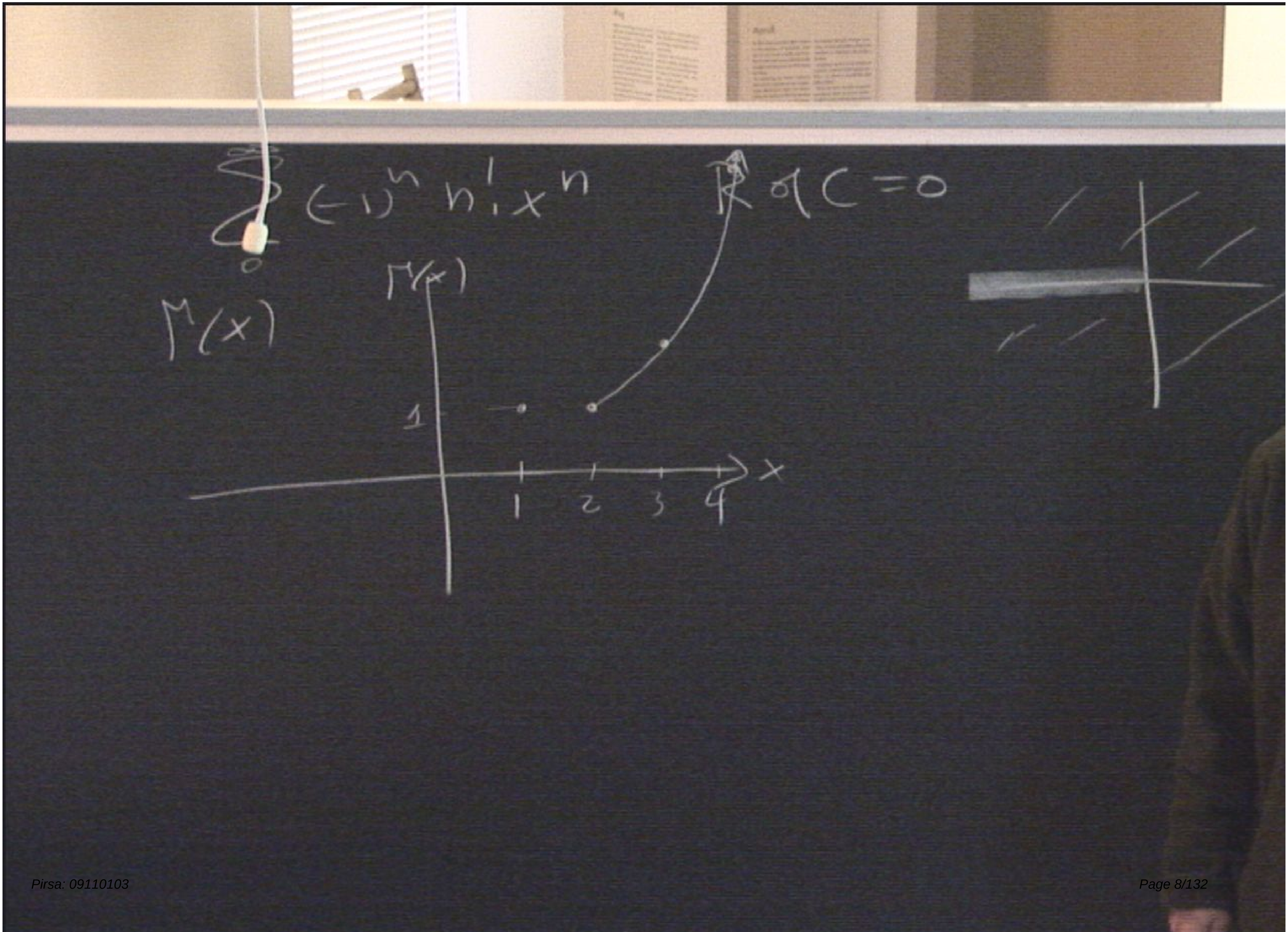
N	$x = 1$	$x = 2$	$x = 4$
1	0.787 279 606 6	0.369 614 402 3	0.176 506 593 3
2	0.993 980 121 1	0.983 777 381 1	−1.076 767 507 8
3	1.002 208 342 9	1.148 824 428 2	−0.501 000 407 3
4	0.999 993 179 2	0.996 095 931 1	1.662 143 630 6
5	0.999 992 416 9	0.995 977 024 8	1.648 553 231 8
6	1.000 000 023 8	1.000 087 294 7	9.332 982 540 2
7	0.999 999 998 8	0.999 982 355 2	4.271 068 387 3
8	1.000 000 000 0	0.999 999 507 8	5.891 861 437 4
9	1.000 000 000 0	1.000 000 001 6	6.016 004 972 1
10	1.000 000 000 0	1.000 000 000 1	6.001 556 368 7

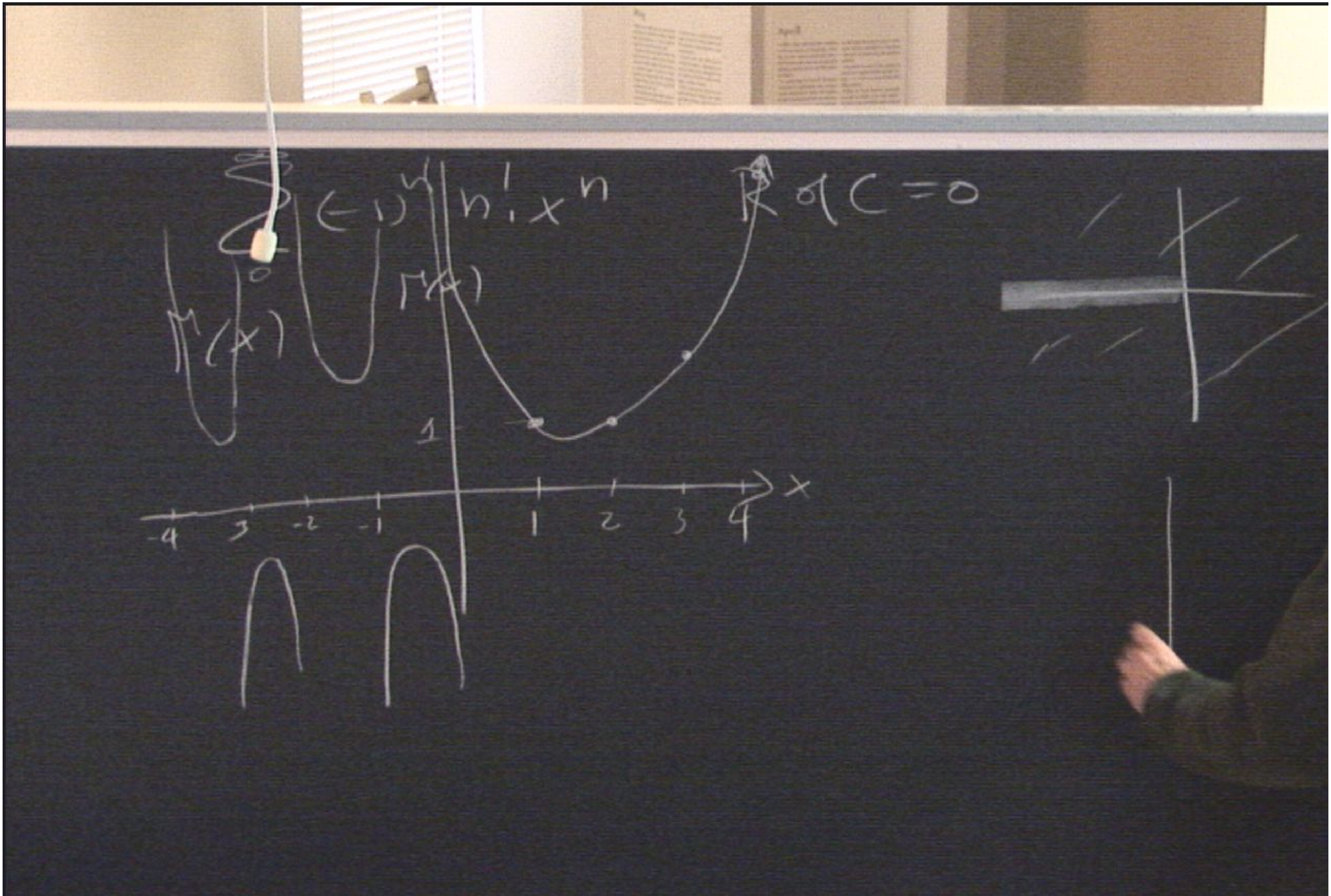
$1/[xP_{N+1}^N(x)]$

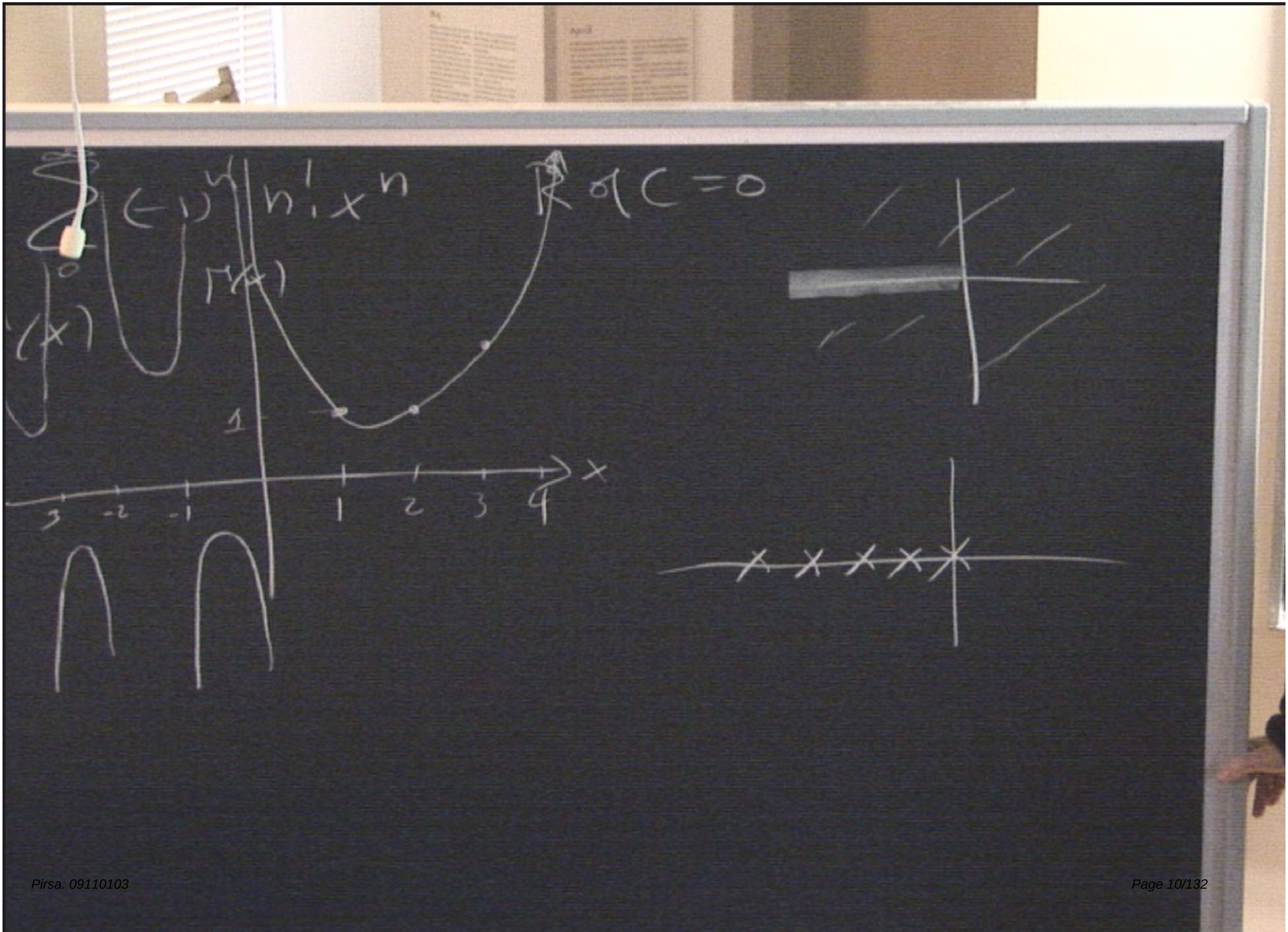
N	$x = 1$	$x = 2$	$x = 4$
1	0.938 583 793 1	0.620 521 015 8	0.519 926 684 9
2	1.012 480 430 2	4.152 436 398 7	0.085 466 832 0
3	1.000 317 915 9	1.035 343 862 7	−1.821 772 429 8
4	0.999 989 665 5	0.995 106 544 4	1.465 422 290 2
5	0.999 999 630 8	0.999 508 399 5	3.822 815 515 8



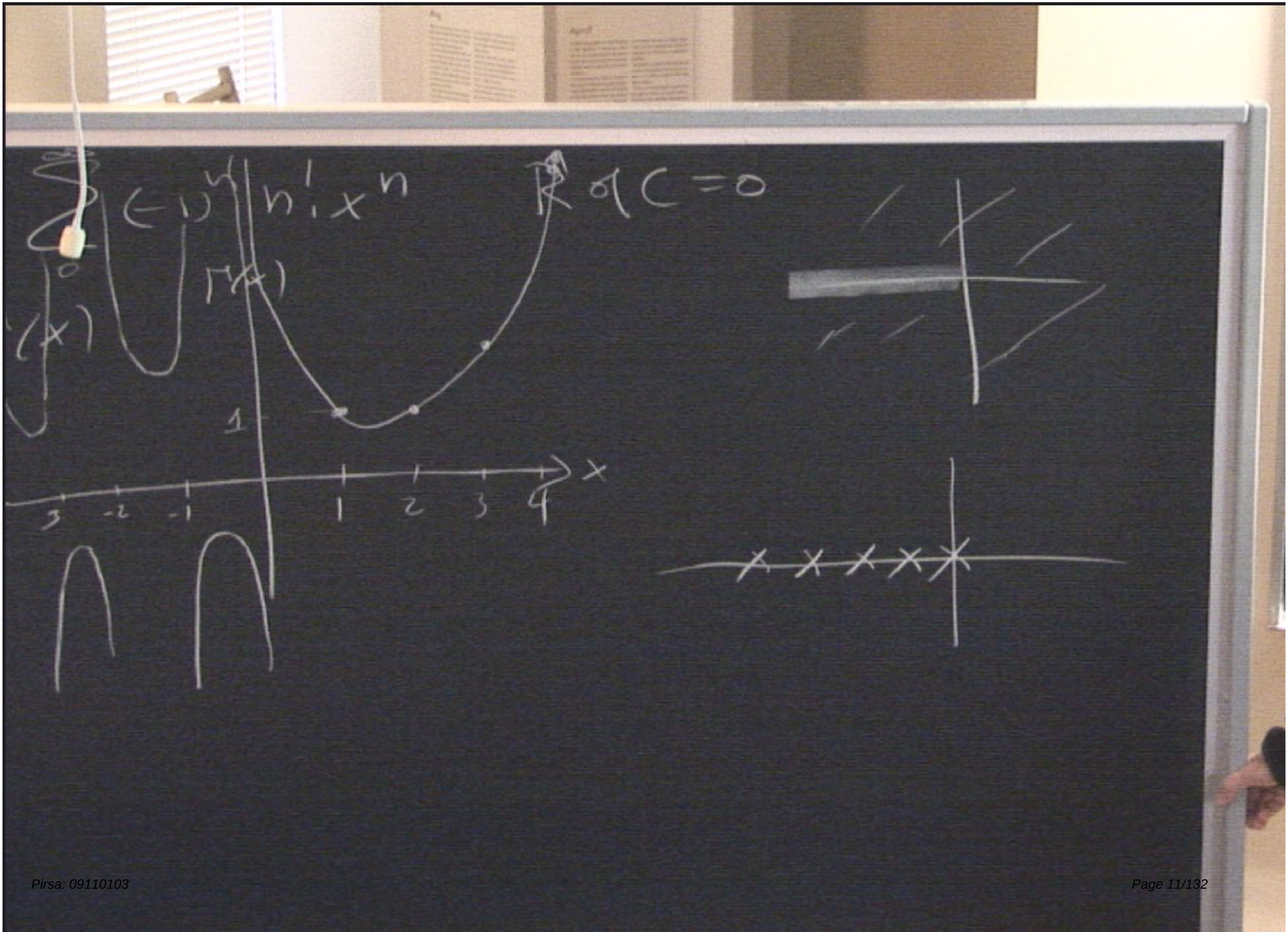
Σ (-1)^n n! x^n
0
R of C = 0
Γ(x)
Γ(x)
x
1
2
3
4

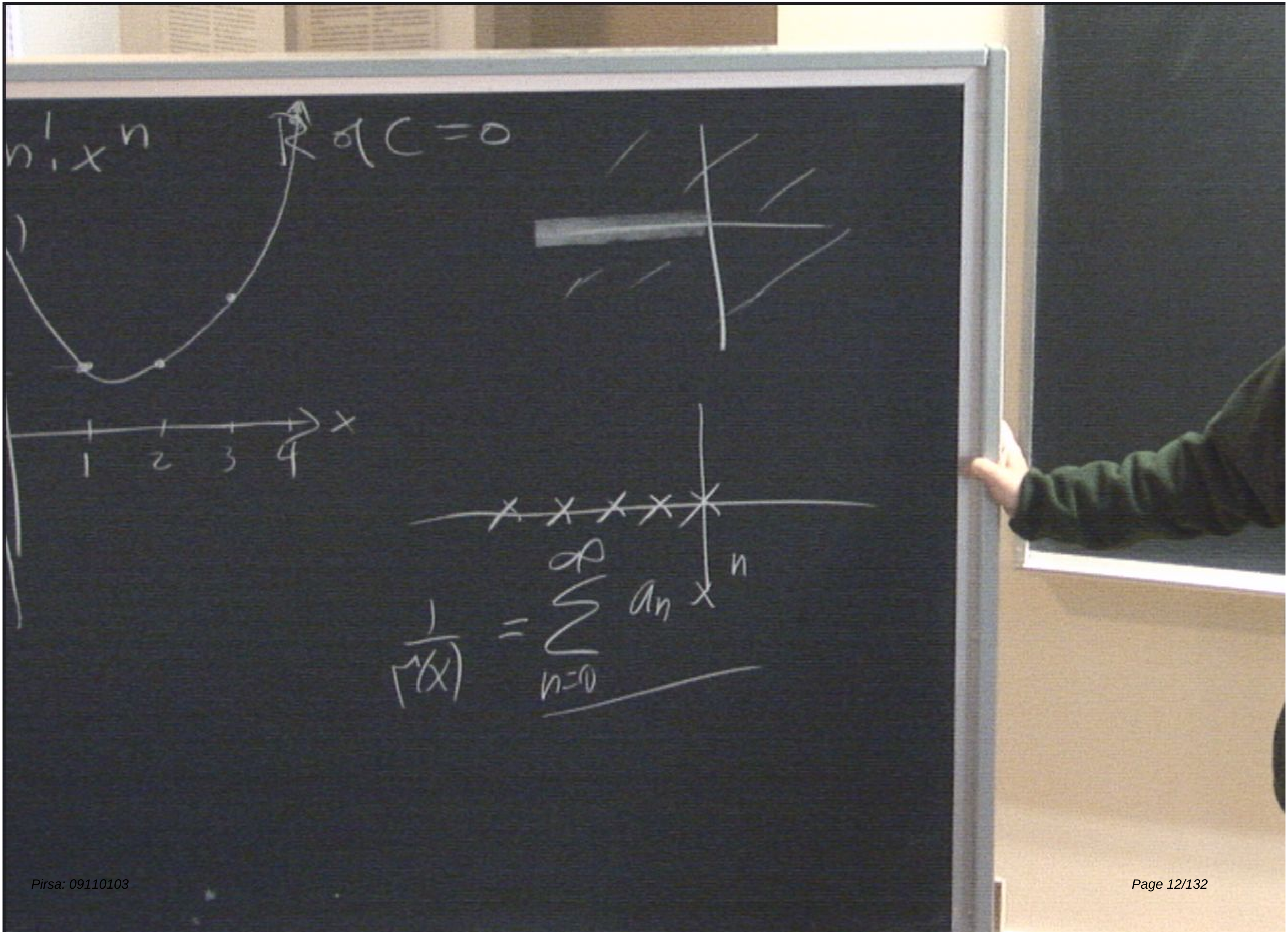

$\sum_0^\infty (-1)^n n! x^n$
$\Gamma(x)$
$\Gamma(x)$
1
1 2 3 4
x


(-1)^n n! x^n
Γ(x)
Γ(x)
1
-4 3 -2 -1 1 2 3 4
x

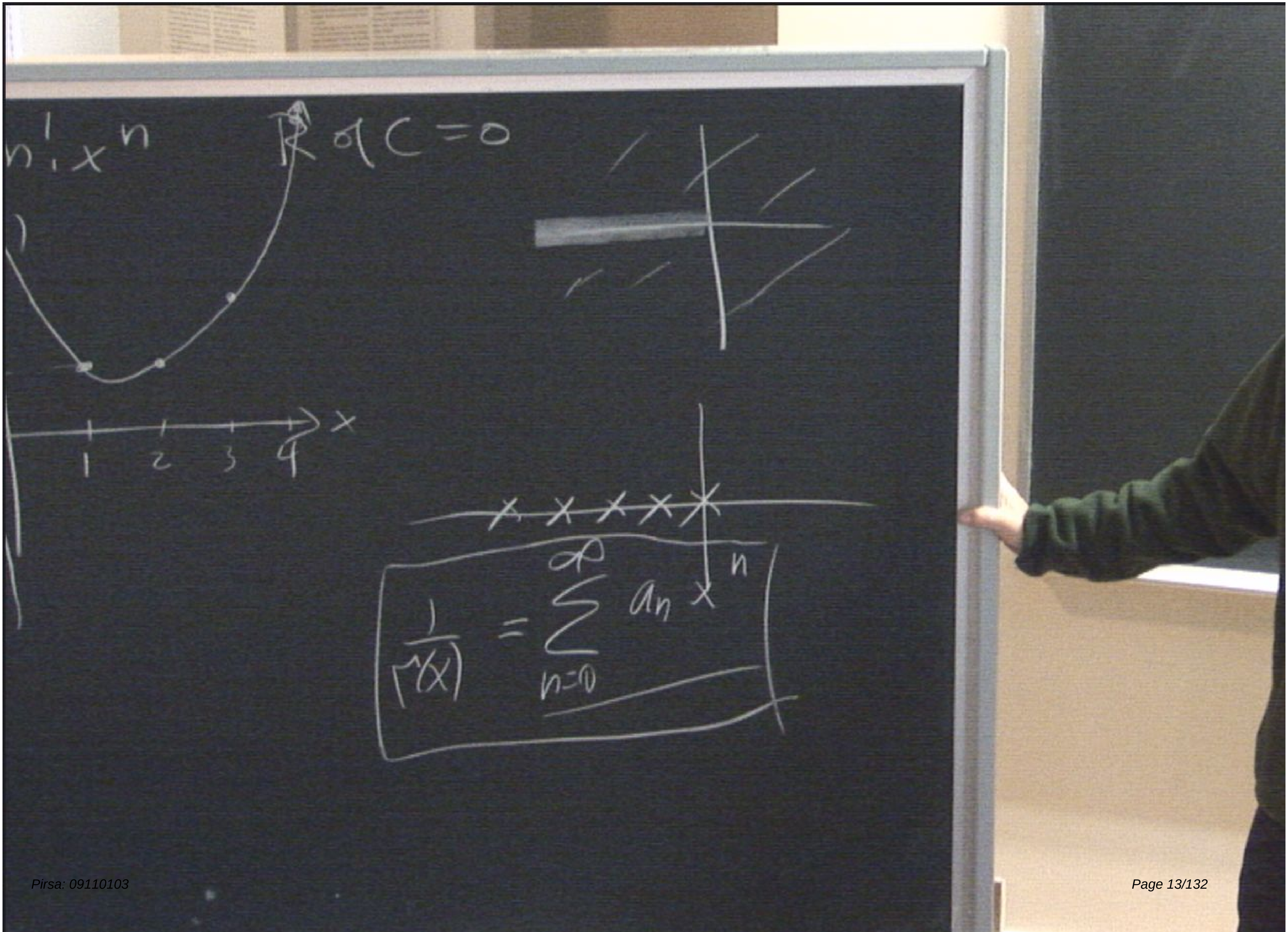


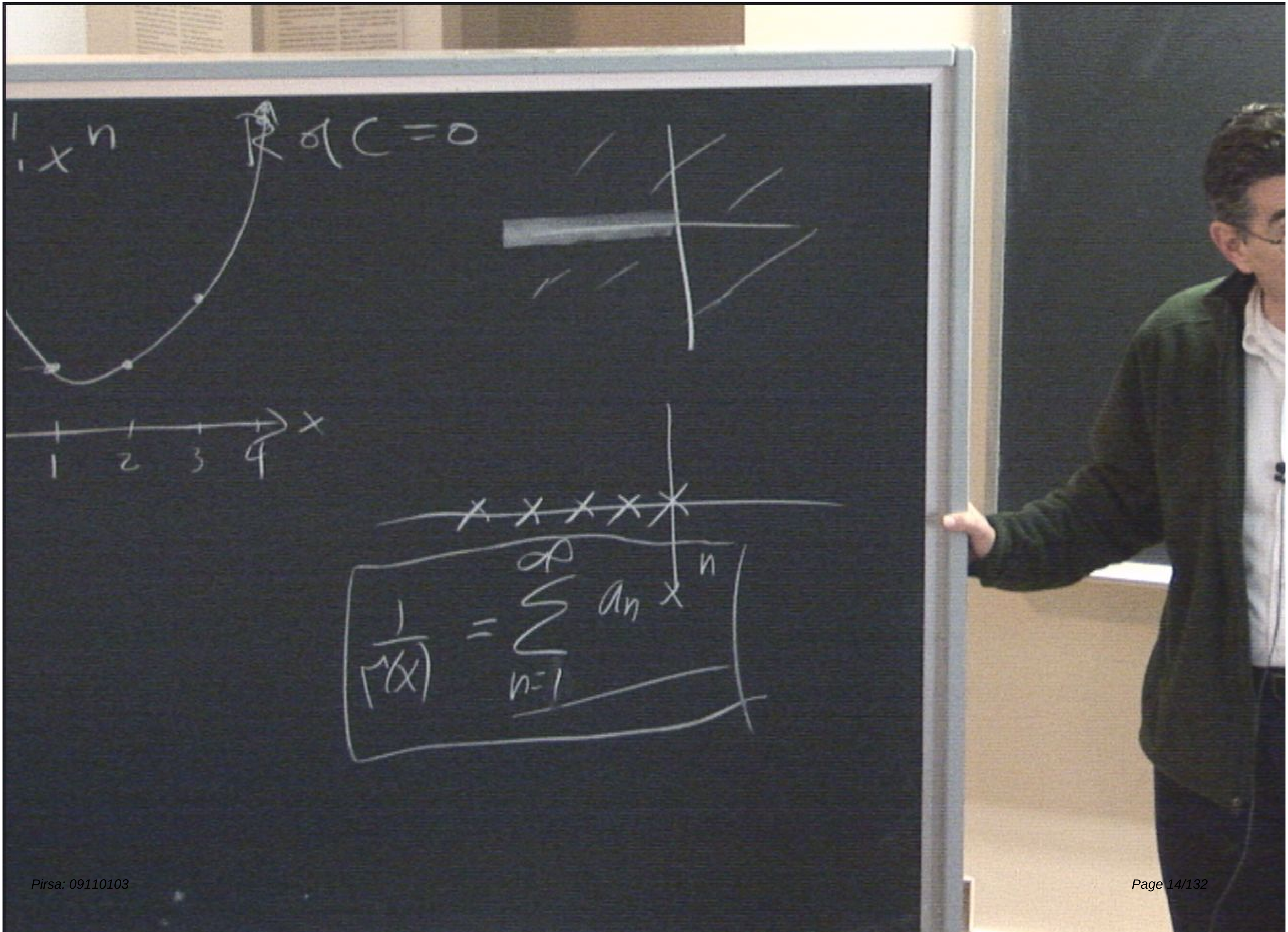
(-1)^n n! x^n
Γ(x)
1
x
-3 -2 -1 1 2 3 4


(-1)^n n! x^n
Γ(x)
x

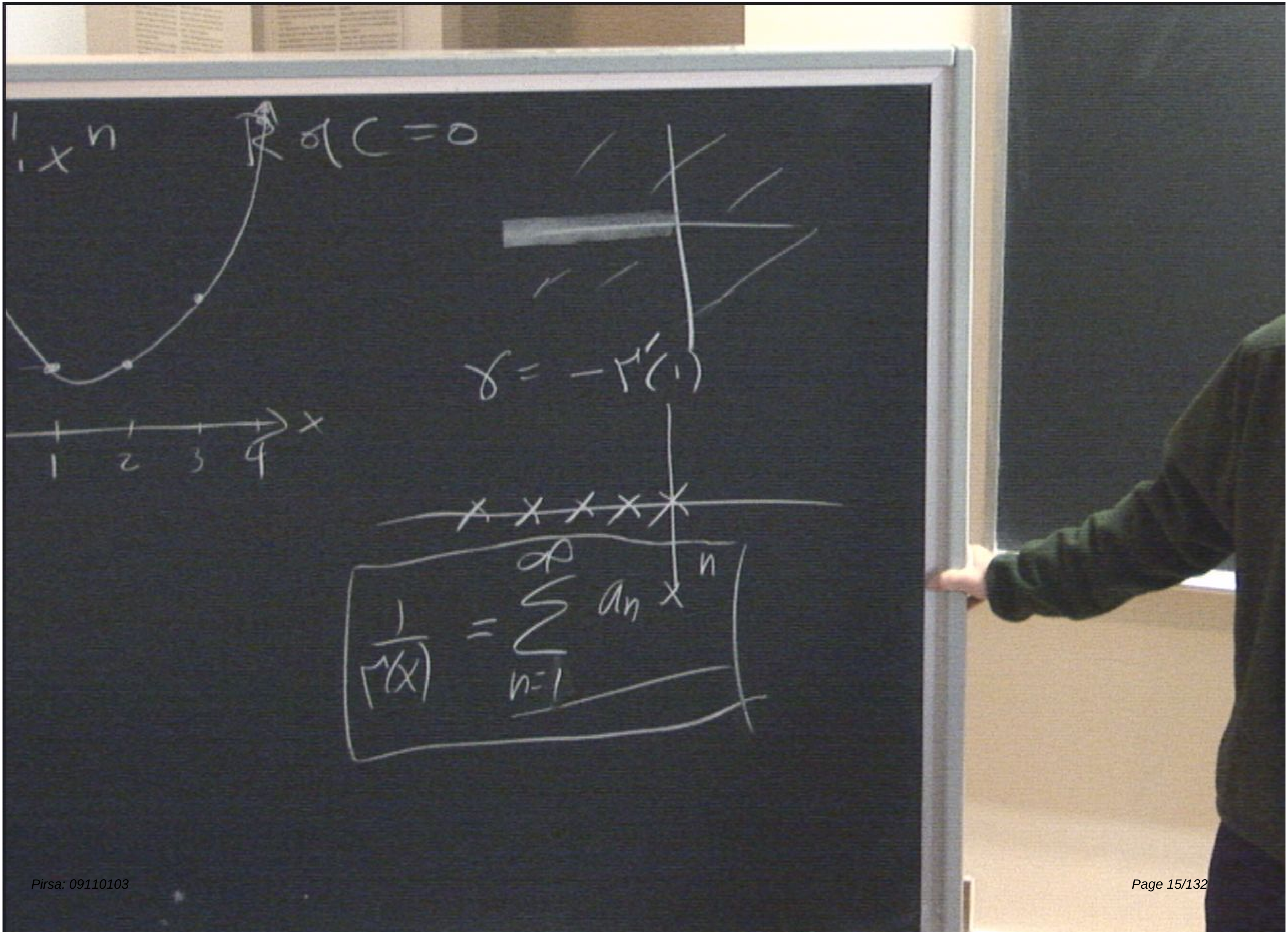


n! x^n
x
1 2 3 4
n
1/Γ(x) = Σ_{n=0}^∞ a_n x^n

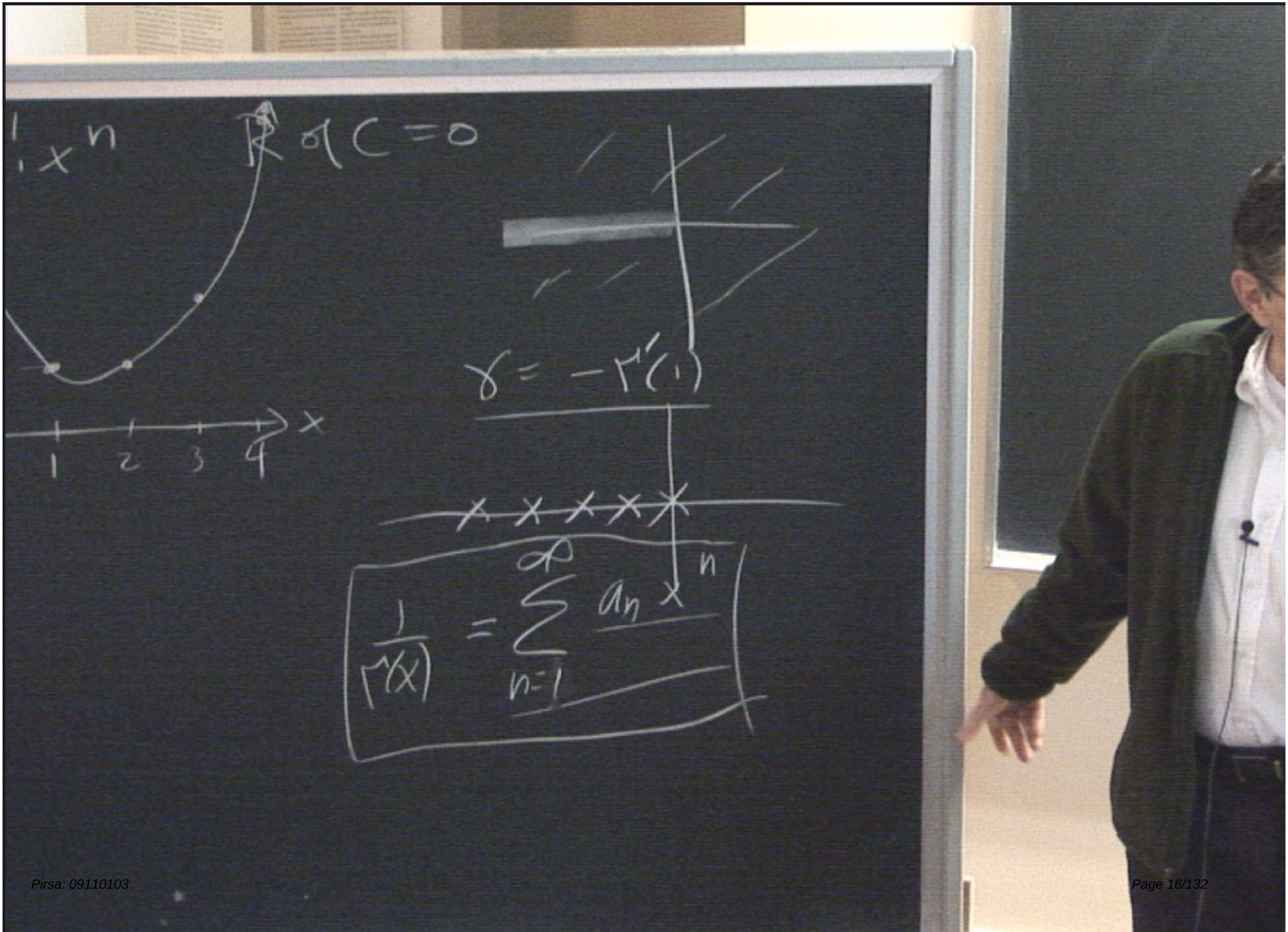

n! x^n
Re C = 0
x
1 2 3 4
1/Γ(x) = Σ_{n=0}^∞ a_n x^n

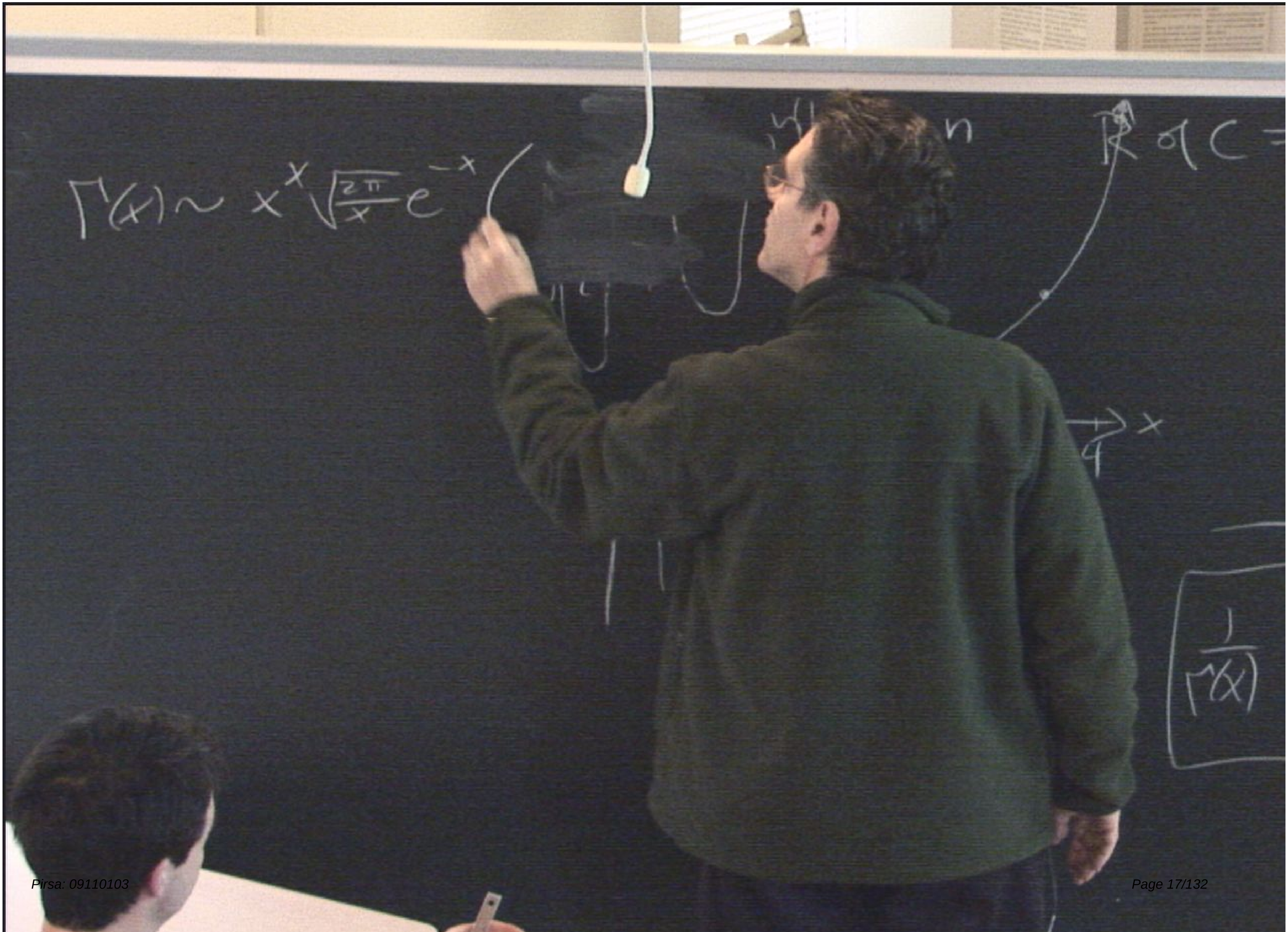


x^n
$= 0$
x
1 2 3 4
$\frac{1}{\Gamma(x)} = \sum_{n=1}^{\infty} a_n x^n$

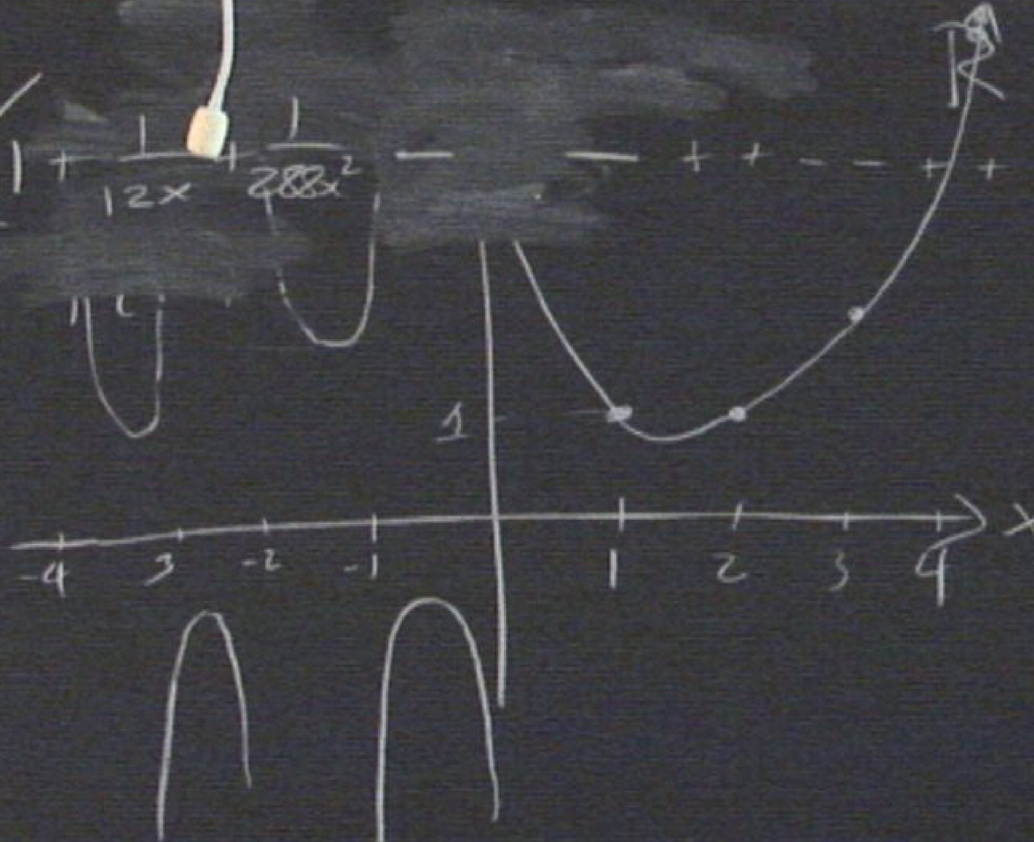


γ = −Γ'(1)
1/Γ(x) = Σ_{n=1}^{∞} a_n x^n


$\gamma = -\Gamma'(1)$
$\frac{1}{\Gamma(x)} = \sum_{n=1}^{\infty} a_n x^n$
1 2 3 4
x

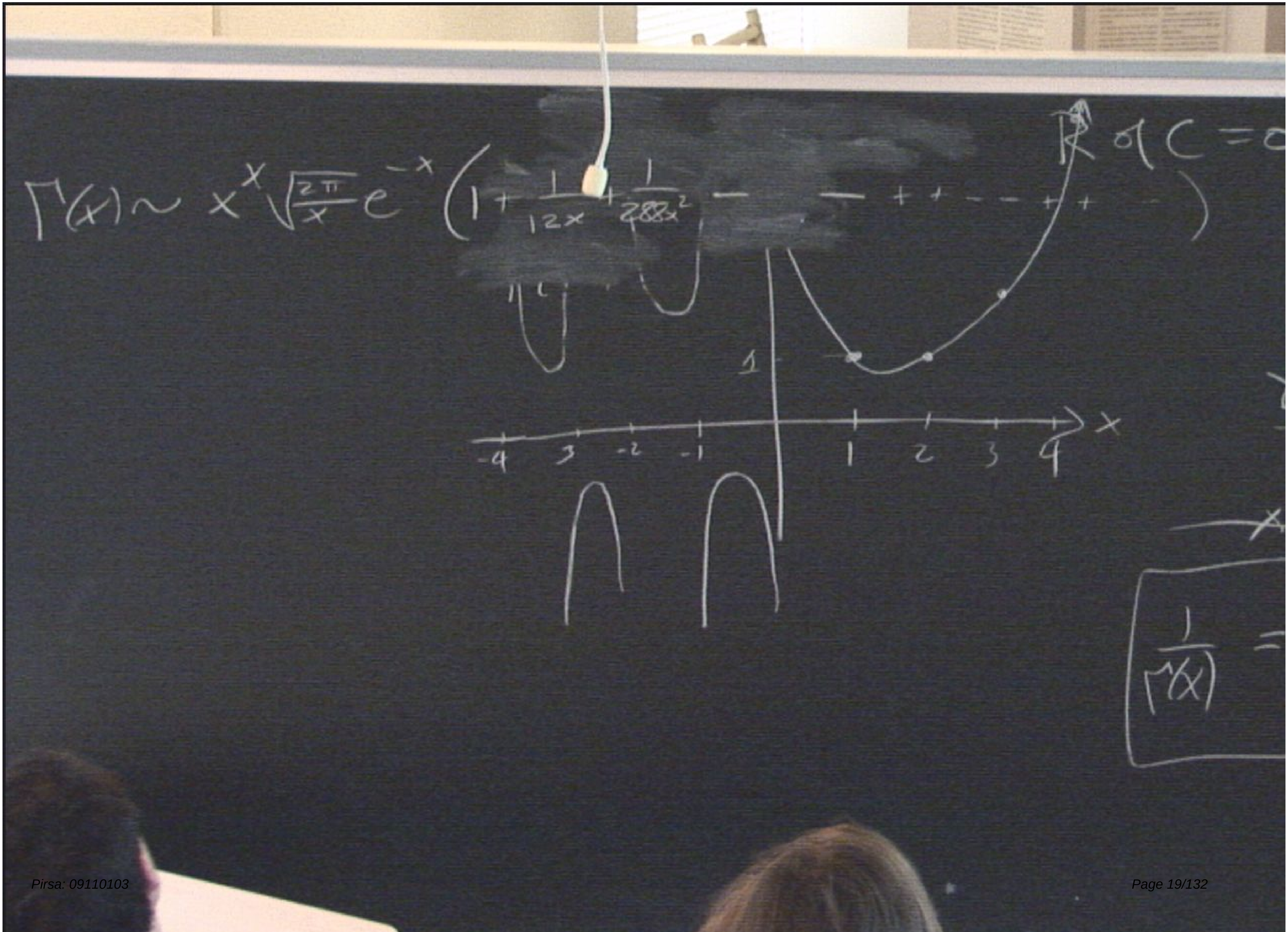

Γ(x) ~ x^x √(2π/x) e^(-x)
1/Γ(x)

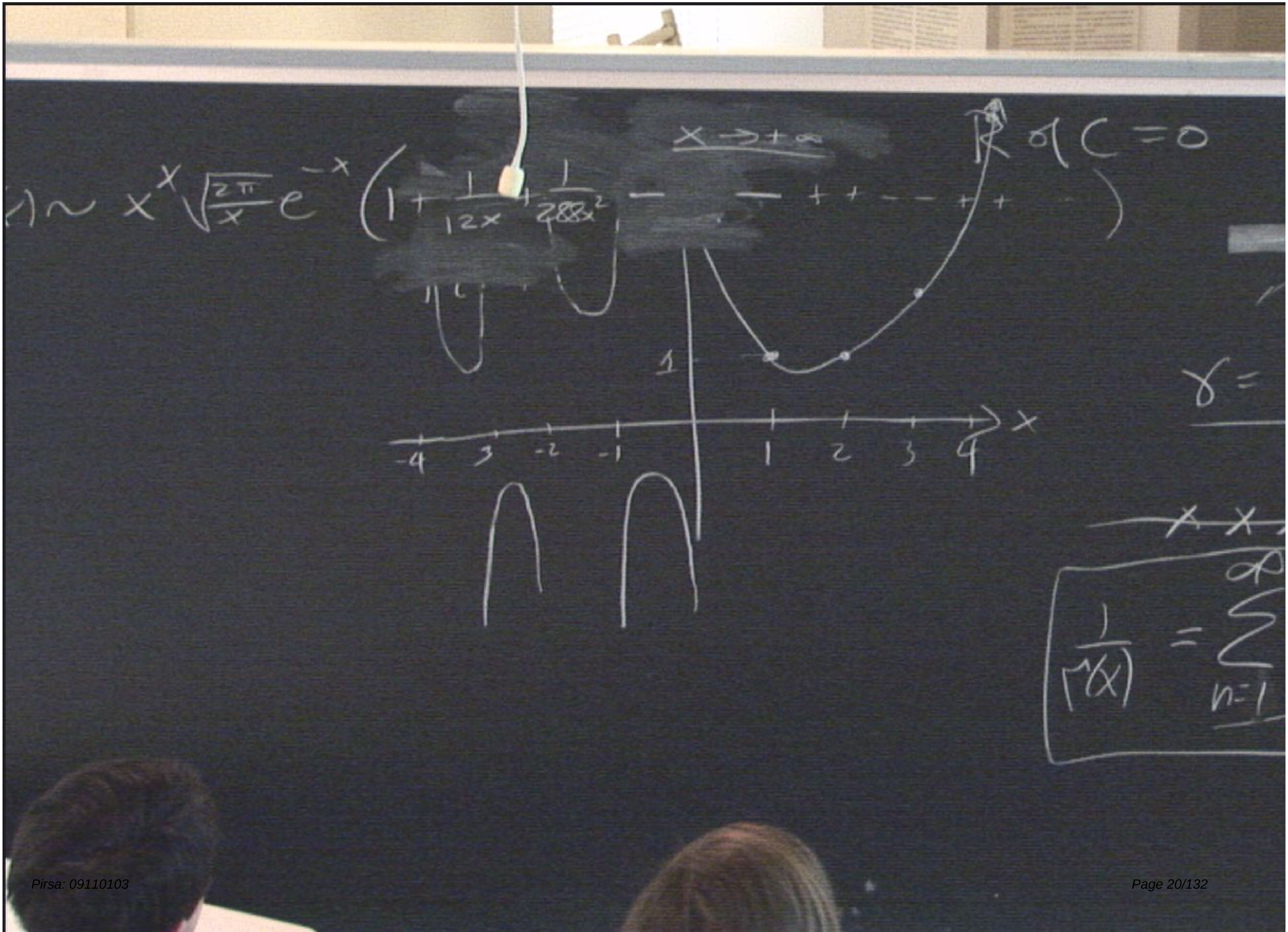
$$\Gamma(x) \sim x^x \sqrt{\frac{2\pi}{x}}\, e^{-x}\left(1 + \frac{1}{12x} + \frac{1}{288x^2} - - + + - - + + \cdots\right)$$

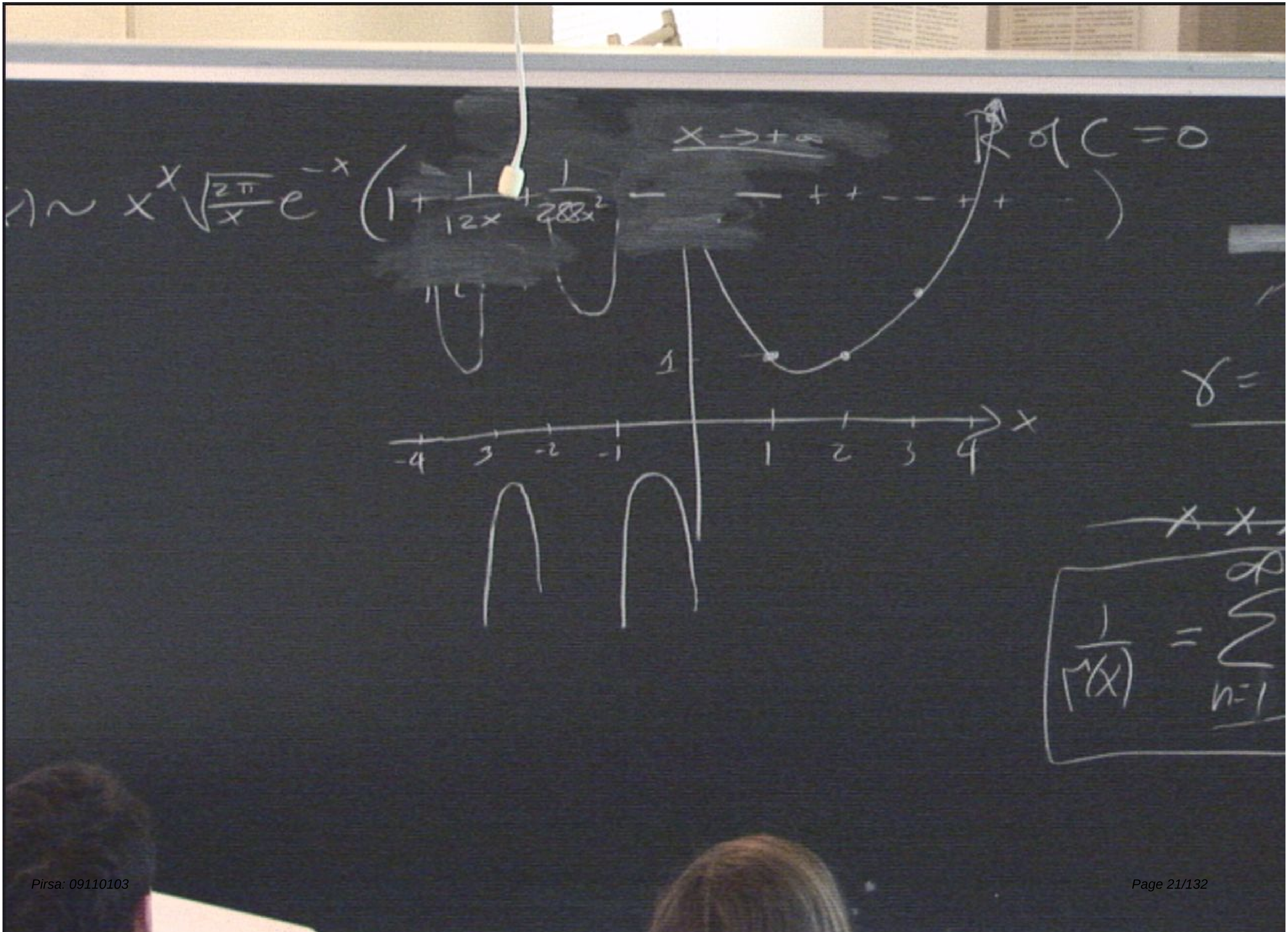


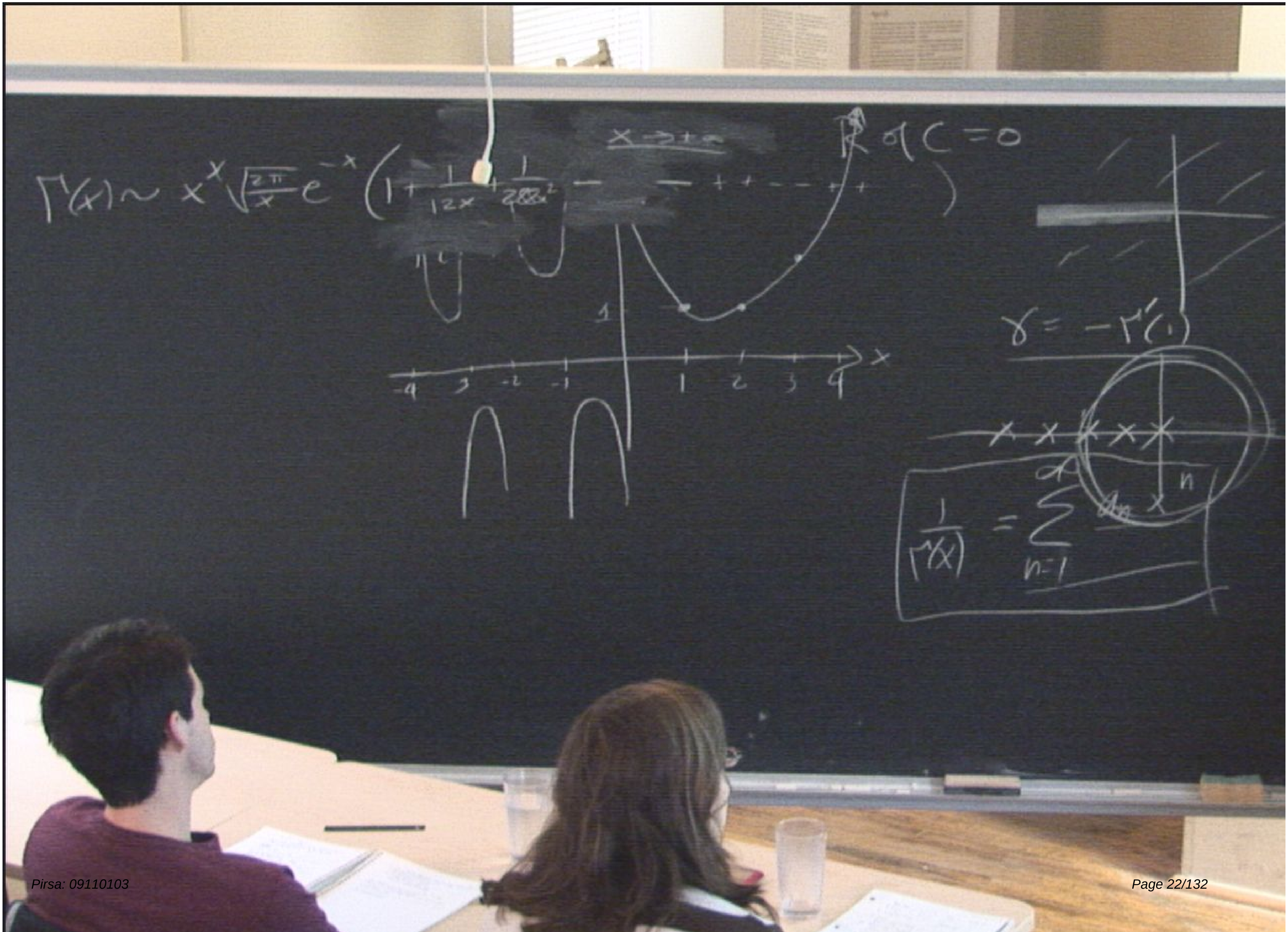
R σ C =

$$\boxed{\frac{1}{\Gamma(x)}}$$

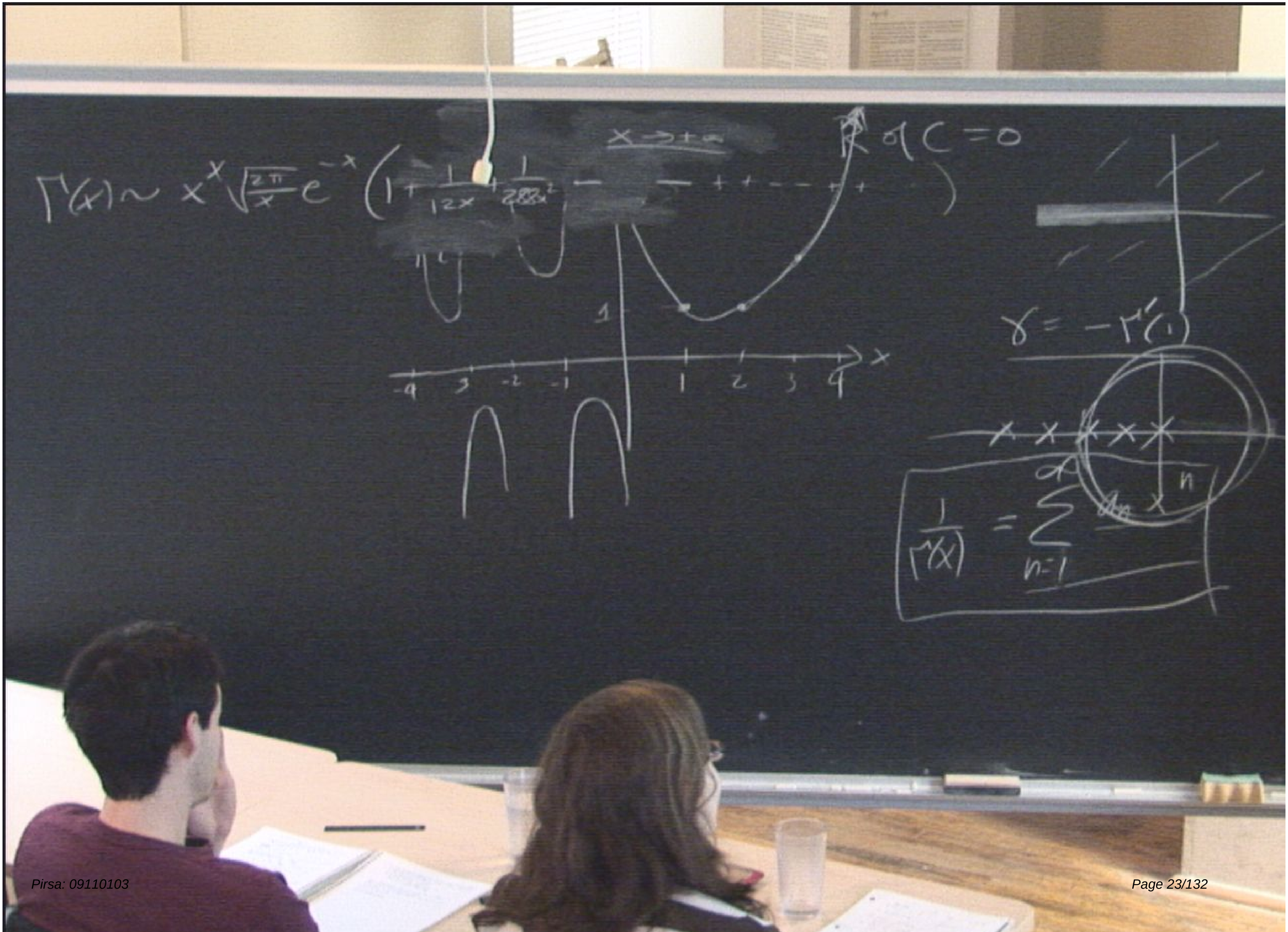

Γ(x) ~ x^x √(2π/x) e^(−x) (1 + 1/12x + 1/288x² − …)
1
x
-4 -3 -2 -1 1 2 3 4
1/Γ(x) =


x → +∞
~ x^x √(2π/x) e^{-x} (1 + 1/12x + 1/288x^2 ...)
C = 0
-4 3 -2 -1 1 2 3 4 x
1
1/Γ(x) = Σ n=1 ∞


x → +∞
~ x^x √(2π/x) e^(−x) (1 + 1/(12x) + 1/(288x²) −
1
−4 3 −2 −1 1 2 3 4 x
γ =
1/Γ(x) = Σ ∞ n=1



Γ(x) ~ x^x √(2π/x) e^(-x) (1 + 1/12x + 1/288x² ...)
γ = −Γ′(1)
1/Γ(x) = Σ n=1


$\Gamma(x) \sim x^x \sqrt{\frac{2\pi}{x}} e^{-x} \left(1 + \frac{1}{12x} + \frac{1}{288x^2} \ldots\right)$
$x \to +\infty$
$\gamma = -\Gamma'(1)$
$\frac{1}{\Gamma(x)} = \sum_{n=1} a_n x^n$

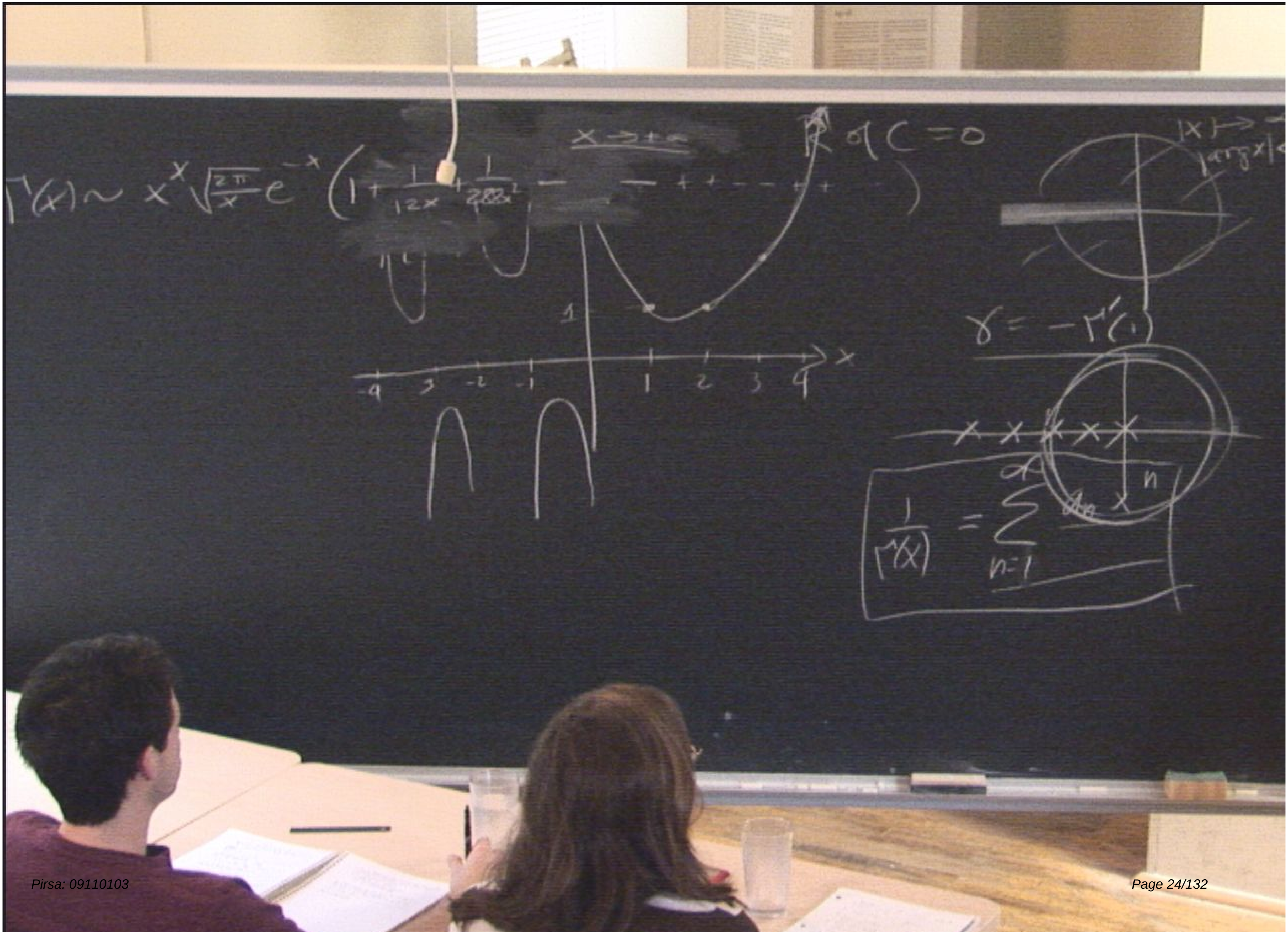

Γ(x) ~ x^x √(2π/x) e^{-x} (1 + 1/12x + 1/288x² ...)
x → +∞
γ = −Γ′(1)
1/Γ(x) = Σ_{n=1} a_n x^n

Table 8.12 Padé summation of the Taylor series (5.4.4) for $1/\Gamma(x)$

The entries in the table were computed as follows. The truncated Taylor series for $1/[x\Gamma(x)]$ was used to construct the Padé approximants $P_N^N(x)$ and $P_{N+1}^N(x)$; the resulting Padé approximants were then inverted and multiplied by x. It is clear from the above table that the entries in the three columns are approaching their correct limits: $\Gamma(1) = 1$, $\Gamma(2) = 1$, and $\Gamma(4) = 6$. Although Padé summation enhances the convergence of the Taylor series, the effect is not dramatic: the Padé sequence requires about half the number of terms that the Taylor series requires to achieve 1 percent accuracy (see Table 5.2). The slow convergence of the Taylor series is not due to the presence of singularities [$1/\Gamma(x)$ is entire], but rather to the enormous disparity of behavior of $1/\Gamma(x)$ for positive and negative x. Padé approximants improve the convergence of series by mimicking the singularities of the function to be represented; since $1/\Gamma(x)$ has no singularities, the benefits of Padé summation are marginal

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3	1.002 208 342 9	1.148 824 428 2	−0.501 000 407 3
4	0.999 993 179 2	0.996 095 931 1	1.662 143 630 6
5	0.999 992 416 9	0.995 977 024 8	1.648 553 231 8
6	1.000 000 023 8	1.000 087 294 7	9.332 982 540 2
7	0.999 999 998 8	0.999 982 355 2	4.271 068 387 3
8	1.000 000 000 0	0.999 999 507 8	5.891 861 437 4
9	1.000 000 000 0	1.000 000 001 6	6.016 004 972 1
10	1.000 000 000 0	1.000 000 000 1	6.001 556 368 7

$1/[xP_{N+1}^N(x)]$

N	$x = 1$	$x = 2$	$x = 4$
1	0.938 583 793 1	0.620 521 015 8	0.519 926 684 9
2	1.012 480 430 2	4.152 436 398 7	0.085 466 832 0
3	1.000 317 915 9	1.035 343 862 7	−1.821 772 429 8
4	0.999 989 665 5	0.995 106 544 4	1.465 422 290 2
5	0.999 999 639 8	0.999 508 399 5	3.822 815 515 8

3	1.002 208 342 9	[illegible]	[illegible]
4	0.999 993 179 2	0.996 095 931 1	1.662 143 630 6
5	0.999 992 416 9	0.995 977 024 8	1.648 553 231 8
6	1.000 000 023 8	1.000 087 294 7	9.332 982 540 2
7	0.999 999 998 8	0.999 982 355 2	4.271 068 387 3
8	1.000 000 000 0	0.999 999 507 8	5.891 861 437 4
9	1.000 000 000 0	1.000 000 001 6	6.016 004 972 1
10	1.000 000 000 0	1.000 000 000 1	6.001 556 368 7

$1/[xP^N_{N+1}(x)]$

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4	0.999 989 665 5	0.995 106 544 4	1.465 422 290 2
5	0.999 999 639 8	0.999 508 399 5	3.822 815 515 8
6	1.000 000 004 4	1.000 026 327 2	7.035 967 528 8
7	1.000 000 000 0	1.000 001 311 6	6.095 643 764 4
8	1.000 000 000 0	0.999 999 880 7	5.960 840 282 9
9	1.000 000 000 0	1.000 000 001 1	6.002 325 742 5
10	1.000 000 000 0	1.000 000 000 0	6.002 625 162 0

Table 8.16 Padé approximants $P_N^N(x)$ to the Stirling series for $\Gamma(x)/[(x/e)^x\sqrt{2\pi/x}] \sim 1 + 1/12x + 1/288x^2 - \cdots\ (x \to \infty)$

Notice that the sign pattern of the error in $P_N^N(x)$ is $+ - + - \cdots$. Therefore, more rapid convergence to $\Gamma(x)$ can be immediately achieved by averaging successive pairs of terms in the table. For comparison purposes, we have also listed in the table values of the optimal asymptotic approximation $\Gamma_{opt}(x)$ and the exact value of $\Gamma(x)$

	$(x/e)^x\sqrt{\frac{2\pi}{x}}\,P_N^N(1/x)$		
N	$x = 0.2$	$x = 0.5$	$x = 1.0$
0	3.326 00	1.520 35	0.922 137 01
1	5.076 52	1.796 77	1.002 322 84
2	4.171 29	1.761 14	0.999 582 32
3	4.850 31	1.777 68	1.000 113 00
4	4.329 20	1.768 90	0.999 954 49
5	4.774 21	1.774 64	1.000 020 05
6	4.397 08	1.770 75	0.999 989 13
7	4.735 03	1.773 64	1.000 005 97
8	4.435 34	1.771 46	0.999 996 23
9	4.710 81	1.773 20	1.000 002 35
10	4.460 10	1.771 80	0.999 998 37
11	4.694 19	1.772 97	1.000 001 10
12	4.477 53	1.771 99	0.999 999 18
13	4.682 03	1.772 83	1.000 000 58
14	4.490 52	1.772 11	0.999 999 55
15	4.672 69	1.772 74	1.000 000 34

Table 8.16 Padé approximants $P_N^N(x)$ to the Stirling series for $\Gamma(x)/[(x/e)^x\sqrt{2\pi/x}] \sim 1 + 1/12x + 1/288x^2 - \cdots\ (x \to \infty)$

Notice that the sign pattern of the error in $P_N^N(x)$ is $+ - + - \cdots$. Therefore, more rapid convergence to $\Gamma(x)$ can be immediately achieved by averaging successive pairs of terms in the table. For comparison purposes, we have also listed in the table values of the optimal asymptotic approximation $\Gamma_{opt}(x)$ and the exact value of $\Gamma(x)$

	$(x/e)^x\sqrt{\frac{2\pi}{x}}\,P_N^N(1/x)$		
N	$x = 0.2$	$x = 0.5$	$x = 1.0$
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1	5.076 52	1.796 77	1.002 322 84
2	4.171 29	1.761 14	0.999 582 32
3	4.850 31	1.777 68	1.000 113 00
4	4.329 20	1.768 90	0.999 954 49
5	4.774 21	1.774 64	1.000 020 05
6	4.397 08	1.770 75	0.999 989 13
7	4.735 03	1.773 64	1.000 005 97
8	4.435 34	1.771 46	0.999 996 23
9	4.710 81	1.773 20	1.000 002 35
10	4.460 10	1.771 80	0.999 998 37
11	4.694 19	1.772 97	1.000 001 10
12	4.477 53	1.771 99	0.999 999 18
13	4.682 03	1.772 83	1.000 000 58
14	4.490 52	1.772 11	0.999 999 55
15	4.672 69	1.772 74	1.000 000 34
$\Gamma_{opt}(x)$	4.711 83	1.762 24	0.999 499 47
$\Gamma(x)$	4.590 84	1.772 45	1.0

Table 8.17 Diagonal sequence of Padé approximants for the functions $D_\nu(x)$, $\nu = 3.5$ and $\nu = 11.5$, at $x = 1$ and $x = 2$

These Padé approximants were computed from the asymptotic expansion of $D_\nu(x)$ [see (3.5.13) and (3.5.14)] as follows. The expansion for $w(x)$ in (3.5.14) was converted to a rational function with argument x^2. The approximation to $D_\nu(x)$ was obtained by multiplying the Padé approximants by $x^\nu e^{-x^2/4}$. Observe that the Padé sequences P_N^N and P_{N+1}^N converge monotonically provided that N is large enough. The comparison between the Padé approximations in this table and the optimal asymptotic approximations in Table 3.2 is impressive. When $x = 2$, the optimal asymptotic approximations to $D_{3.5}(2)$ and $D_{11.5}(2)$ are in error by 0.3 and 0.005 percent, respectively, while the Padé approximants P_{11}^{11} are in error by about 1 part in 10^8

$\nu = 3.5$

	$x = 1$		$x = 2$	
N	P_N^N	P_{N+1}^N	P_N^N	P_{N+1}^N
4	−2.031 80	−2.039 34	−0.182 255 623	−0.182 267 983
5	−2.035 03	−2.038 24	−0.182 262 496	−0.182 265 165
6	−2.036 03	−2.037 63	−0.182 263 761	−0.182 264 459
7	−2.036 43	−2.037 28	−0.182 264 054	−0.182 264 262
8	−2.036 61	−2.037 09	−0.182 264 134	−0.182 264 201
9	−2.036 69	−2.036 97	−0.182 264 158	−0.182 264 182
10	−2.036 74	−2.036 91	−0.182 264 166	−0.182 264 175
11	−2.036 77	−2.036 87	−0.182 264 169	−0.182 264 172
Exact	−2.036 81	−2.036 81	−0.182 264 170	−0.182 264 170

$\nu = 11.5$

	$x = 1$		$x = 2$	
N	P_N^N	P_{N+1}^N	P_N^N	P_{N+1}^N
4	·1244.86	865.52	1438.851 67	1366.535 33
5	−2494.96	−2926.67	1332.749 03	1332.605 32

Table 8.17 Diagonal sequence of Padé approximants for the functions $D_\nu(x)$, $\nu = 3.5$ and $\nu = 11.5$, at $x = 1$ and $x = 2$

These Padé approximants were computed from the asymptotic expansion of $D_\nu(x)$ [see (3.5.13) and (3.5.14)] as follows. The expansion for $w(x)$ in (3.5.14) was converted to a rational function with argument x^2. The approximation to $D_\nu(x)$ was obtained by multiplying the Padé approximants by $x^\nu e^{-x^2/4}$. Observe that the Padé sequences P_N^N and P_{N+1}^N converge monotonically provided that N is large enough. The comparison between the Padé approximations in this table and the optimal asymptotic approximations in Table 3.2 is impressive. When $x = 2$, the optimal asymptotic approximations to $D_{3.5}(2)$ and $D_{11.5}(2)$ are in error by 0.3 and 0.005 percent, respectively, while the Padé approximants P_{11}^{11} are in error by about 1 part in 10^8

$\nu = 3.5$

	$x = 1$		$x = 2$	
N	P_N^N	P_{N+1}^N	P_N^N	P_{N+1}^N
4	−2.031 80	−2.039 34	−0.182 255 623	−0.182 267 983
5	−2.035 03	−2.038 24	−0.182 262 496	−0.182 265 165
6	−2.036 03	−2.037 63	−0.182 263 761	−0.182 264 459
7	−2.036 43	−2.037 28	−0.182 264 054	−0.182 264 262
8	−2.036 61	−2.037 09	−0.182 264 134	−0.182 264 201
9	−2.036 69	−2.036 97	−0.182 264 158	−0.182 264 182
10	−2.036 74	−2.036 91	−0.182 264 166	−0.182 264 175
11	−2.036 77	−2.036 87	−0.182 264 169	−0.182 264 172
Exact	−2.036 81	−2.036 81	−0.182 264 170	−0.182 264 170

$\nu = 11.5$

	$x = 1$		$x = 2$	
N	P_N^N	P_{N+1}^N	P_N^N	P_{N+1}^N
4	−1244.86	865.52	1438.851 67	1366.535 33
5	−2494.96	−2926.67	1332.749 03	1332.605 32

Stieltjes $\begin{cases} \text{Function} \\ \text{Series} \end{cases}$

Stieltjes $\begin{cases} \text{Function} \\ \text{Series} \end{cases}$

S Series: $\sum_{n=0}^{\infty} (-1)^n a_n x^n$

Stieltjes $\begin{cases} \text{Function} \\ \text{Series} \end{cases}$

<u>S Series</u>: $\sum_{n=0}^{\infty} (-1)^n a_n x^n$ <u>where</u> $a_n = \int_0^{\infty} dx\, W(x)\, x^n$

$\underline{W(x) \geq 0}$

Stieltjes $\begin{cases} \text{Function} \\ \text{Series} \end{cases}$

S Series: $\sum_{n=0}^{\infty} (-1)^n a_n x^n$ where $a_n = \int_0^{\infty} dx\, W(x)\, x^n$

$W(x) \geq 0$ $(0, \infty)$

Stieltjes $\begin{cases} \text{Function} \\ \text{Series} \end{cases}$

S Series: $\sum_{n=0}^{\infty} (-1)^n a_n x^n$ where $a_n = \int_0^{\infty} dx\, W(x)\, x^n$

$W(x) \geq 0 \quad (0 \leq x < \infty)$

Ex: $\sum_0^{\infty} (-1)^n n!\, x^n$

Stieltjes $\begin{cases} \text{Function} \\ \text{Series} \end{cases}$

<u>S Series</u>: $\sum_{n=0}^{\infty} (-1)^n a_n x^n$ <u>where</u> $a_n = \int_0^{\infty} dt\, w(t)\, t^n$

<u>$w(t) \geq 0$</u> $(0 \leq t < \infty)$

<u>Ex</u>: $\sum_0^{\infty} (-1)^n n!\, x^n$

$w(t) = e^{-t}$

Stieltjes $\begin{cases} \text{Function} \\ \text{Series} \end{cases}$

S Series: $\displaystyle\sum_{n=0}^{\infty} (-1)^n a_n x^n$ where $a_n = \int_0^{\infty} dt\, w(t)\, t^n$

$w(t) \geq 0 \quad (0 \leq t < \infty)$

Ex: $\displaystyle\sum_{0}^{\infty} (-1)^n n!\, x^n$

$w(t) = e^{-t}$

S Function: $\displaystyle\sum_{n=0}^{\infty} (-1)^n x^n \int_0^{\infty} dt\, w(t)\, t^n$

$\displaystyle = \int_0^{\infty} dt\, w(t) \sum_{n=0}^{\infty} (-1)^n x^n t^n$

Stieltjes $\begin{cases} \text{Function} \\ \text{Series} \end{cases}$

S Series: $\displaystyle\sum_{n=0}^{\infty} (-1)^n a_n x^n$ where $a_n = \int_0^{\infty} dt\, w(t)\, t^n$

$w(t) \geq 0 \quad (0 \leq t < \infty)$

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S Function: $\displaystyle\sum_{n=0}^{\infty} (-1)^n x^n \int_0^{\infty} dt\, w(t)\, t^n$

$$= \int_0^{\infty} dt\, w(t) \sum_{n=0}^{\infty} (-1)^n x^n t^n$$

$$F(x) = \int_0^{\infty} \frac{dt\, w(t)}{1 + xt}$$

Stieltjes $\begin{cases} \text{Function} \\ \text{Series} \end{cases}$

S Series: $\sum_{n=0}^{\infty} (-1)^n a_n x^n$ where $a_n = \int_0^{\infty} dt\, w(t)\, t^n$

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S Function: $\sum_{n=0}^{\infty} (-1)^n x^n \int_0^{\infty} dt\, w(t)\, t^n$

$= \int_0^{\infty} dt\, w(t) \sum_{n=0}^{\infty} (-1)^n x^n t^n$

$$F(x) = \int_0^{\infty} \frac{dt\, w(t)}{1 + xt}$$

Stieltjes $\begin{cases}\text{Function}\\ \text{Series}\end{cases}$

S Series: $\displaystyle\sum_{n=0}^{\infty}(-1)^n a_n x^n$ where $a_n=\int_0^{\infty}dt\, w(t)\, t^n$

$w(t)\geq 0 \quad (0\leq t<\infty)$

Ex: $\displaystyle\sum_0^{\infty}(-1)^n n!\, x^n$

$w(t)=e^{-t}$

S Function: $\displaystyle\sum_{n=0}^{\infty}(-1)^n x^n\int_0^{\infty}dt\, w(t)\, t^n$

$$=\int_0^{\infty}dt\, w(t)\sum_{n=0}^{\infty}(-1)^n x^n t^n$$

$$\boxed{F(x)\equiv\int_0^{\infty}\frac{dt\, w(t)}{1+xt}}$$

Stieltjes $\begin{cases} \text{Function} \\ \text{Series} \end{cases}$

S Series: $\sum_{n=0}^{\infty} (-1)^n a_n x^n$ where $a_n = \int_0^{\infty} dt\, w(t)\, t^n$

$w(t) \geq 0 \quad (0 \leq t < \infty)$

Ex: $\sum_0^{\infty} (-1)^n n!\, x^n$

$w(t) = e^{-t}$

S Function: $\sum_{n=0}^{\infty} (-1)^n x^n \int_0^{\infty} dt\, w(t)\, t^n$

$= \int_0^{\infty} dt\, w(t) \sum_{n=0}^{\infty} (-1)^n x^n t^n$

$$\boxed{F(x) \equiv \int_0^{\infty} \frac{dt\, w(t)}{1+xt}}$$

Stieltjes $\begin{cases} \text{Function} \\ \text{Series} \end{cases}$

<u>S Series</u>: $\sum_{n=0}^{\infty} (-1)^n a_n x^n$ <u>where</u> $a_n = \int_0^\infty dt\, w(t)\, t^n$

<u>$w(t) \geq 0$</u> $(0 \leq t < \infty)$

<u>Ex</u>: $\sum_0^\infty (-1)^n n!\, x^n$

$w(t) = e^{-t}$

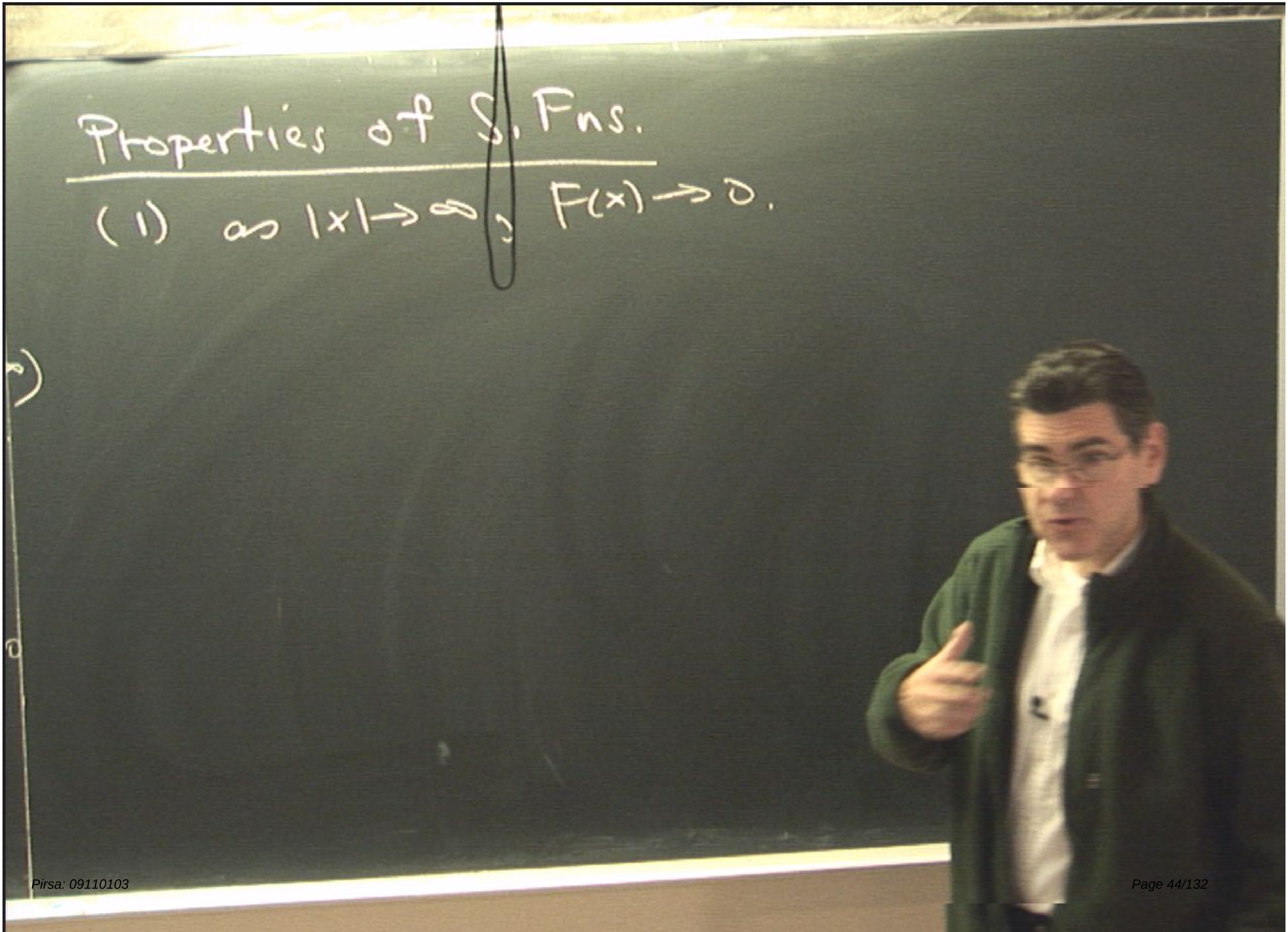
<u>S Function</u>: $\sum_{n=0}^{\infty} (-1)^n x^n \int_0^\infty dt\, w(t)\, t^n$

$= \int_0^\infty dt\, w(t) \sum_{n=0}^{\infty} (-1)^n x^n t^n$

$$\boxed{F(x) \equiv \int_0^\infty \frac{dt\, w(t)}{1+xt}}$$

Where $w(t) \geq 0$ for $t \geq 0$ and $\int_0^\infty w(t) t^n dt$ exists

Properties of S. Fns.


Properties of S. Fns.
(1) as |x|→∞, F(x)→0.

Properties of S. Fns.

(1) as $|x| \to \infty$, in the cut plane, $F(x) \to 0$.

Stieltjes $\begin{cases}\text{Function}\\ \text{Series}\end{cases}$

Prop

(1)

S Series: $\sum_{n=0}^{\infty}(-1)^n a_n x^n$ where $a_n=\int_0^{\infty} dt\, w(t)\, t^n$

$w(t)\geq 0 \; (0\leq t<\infty)$

Ex: $\sum_0^{\infty}(-1)^n n!\, x^n$

$w(t)=e^{-t}$

Function: $\sum_{n=0}^{\infty}(-1)^n x^n \int_0^{\infty} dt\, w(t)\, t^n$

$=\int_0^{\infty} dt\, w(t) \sum_{n=0}^{\infty}(-1)^n x^n t^n$

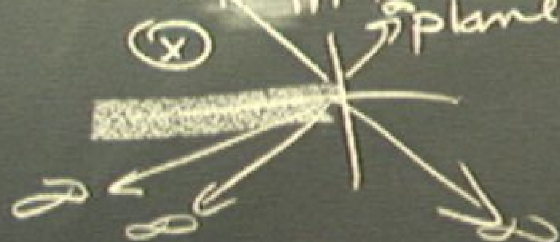
$$\boxed{F(x)\equiv \int_0^{\infty}\frac{dt\, w(t)}{1+xt}}$$

where $w(t)\geq 0$ for $t\geq 0$ and $\int_0^{\infty} w(t)t^n dt$ exists

defined for all x in ~~...~~ [illegible]

Properties of S. Fns.

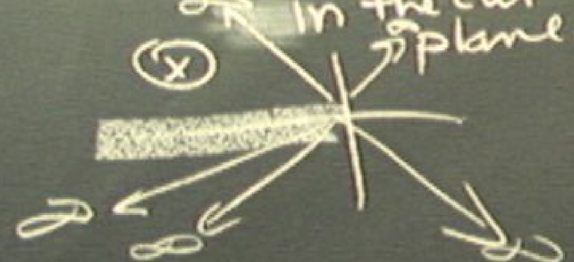
(1) as $|x| \to \infty$, $F(x) \to 0$.
in the cut plane.

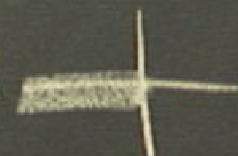


Properties of S. Fns.

(1) as $|x| \to \infty$, $F(x) \to 0$.

∞ in the cut plane.

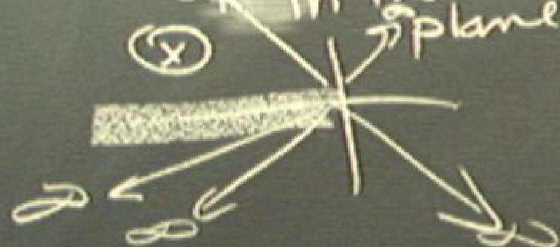


(2) $F(x)$ is ANALYTIC in 

Properties of S. Fns.

(1) as $|x| \to \infty$, $F(x) \to 0$.

∞ in the cut plane.

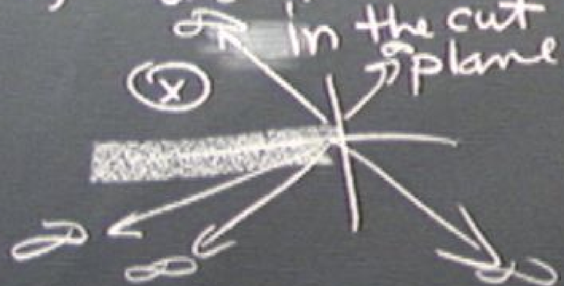


(2) $F(x)$ is <u>ANALYTIC</u> in 

Properties of S. Fns.

(1) as $|x| \to \infty$, $F(x) \to 0$.

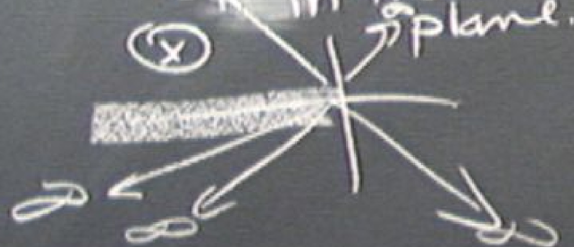
∞ in the cut plane.



(2) $F(x)$ is ANALYTIC in  $F'(x) = -\int_0^\infty \frac{t\,dt\,W(t)}{(1+xt)^2}$

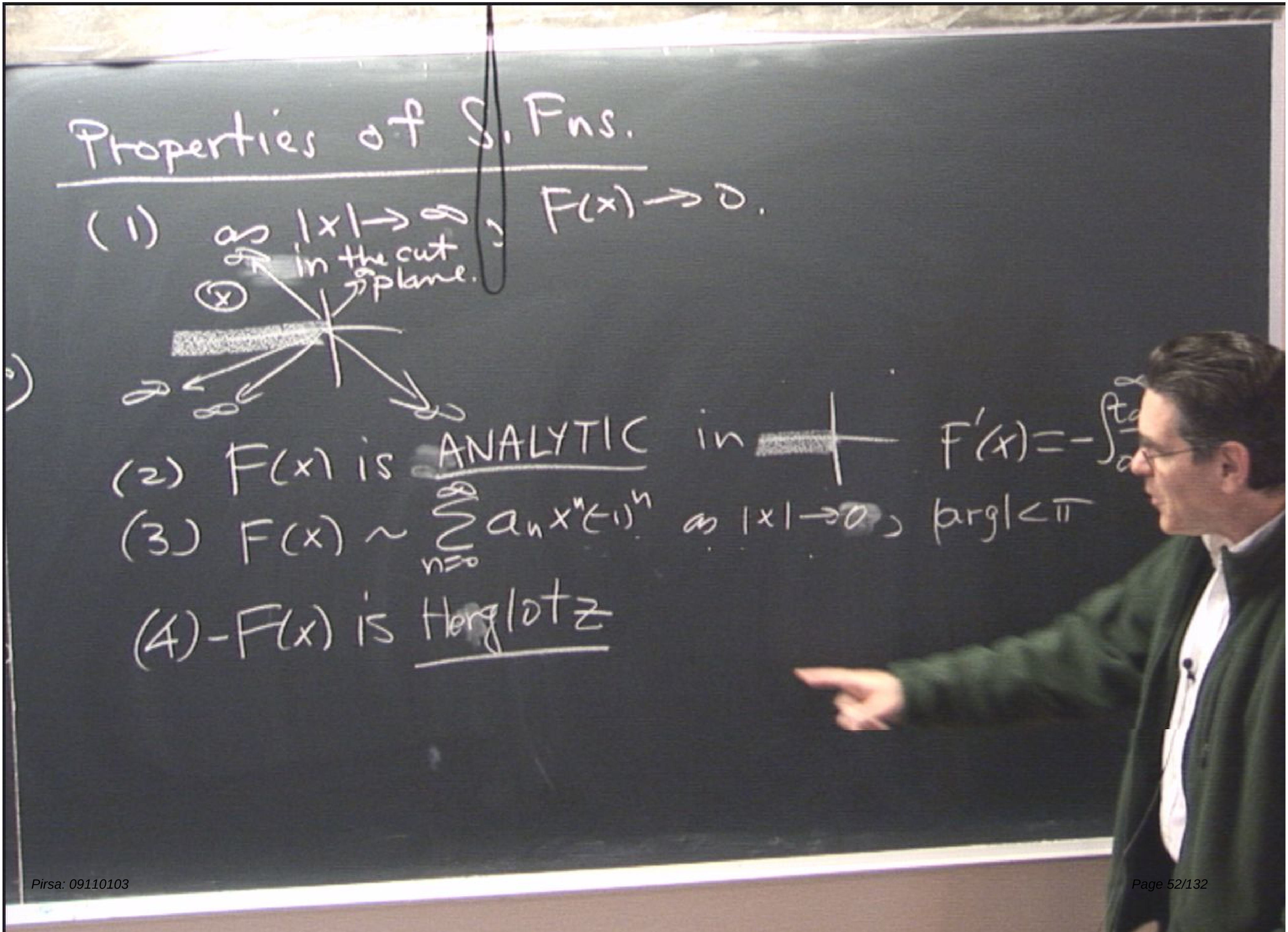
Properties of S. Fns.

(1) as $|x| \to \infty$ in the cut plane, $F(x) \to 0$.



(2) $F(x)$ is ANALYTIC in  $F'(x) = -\int_0^\infty \frac{t\,dt\,W(t)}{(1+xt)^2}$

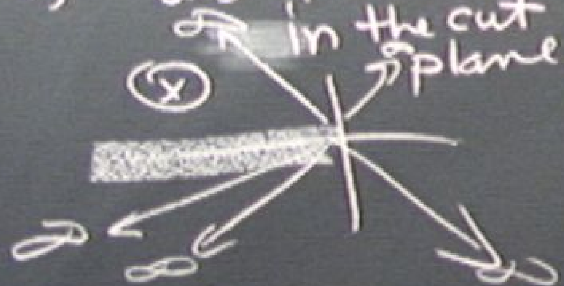
(3) $F(x) \sim \sum_{n=0}^{\infty} a_n x^n (-1)^n$ as $|x| \to 0$, $|\arg| < \pi$



Properties of S. Fns.
(1) as $|x| \to \infty$, $F(x) \to 0$.
in the cut plane.
(2) F(x) is ANALYTIC in
$F'(x) = -\int$
(3) $F(x) \sim \sum_{n=0}^{\infty} a_n x^n (-1)^n$ as $|x| \to 0$, $|\arg| < \pi$
(4) $-F(x)$ is Herglotz

Properties of S. Fns.

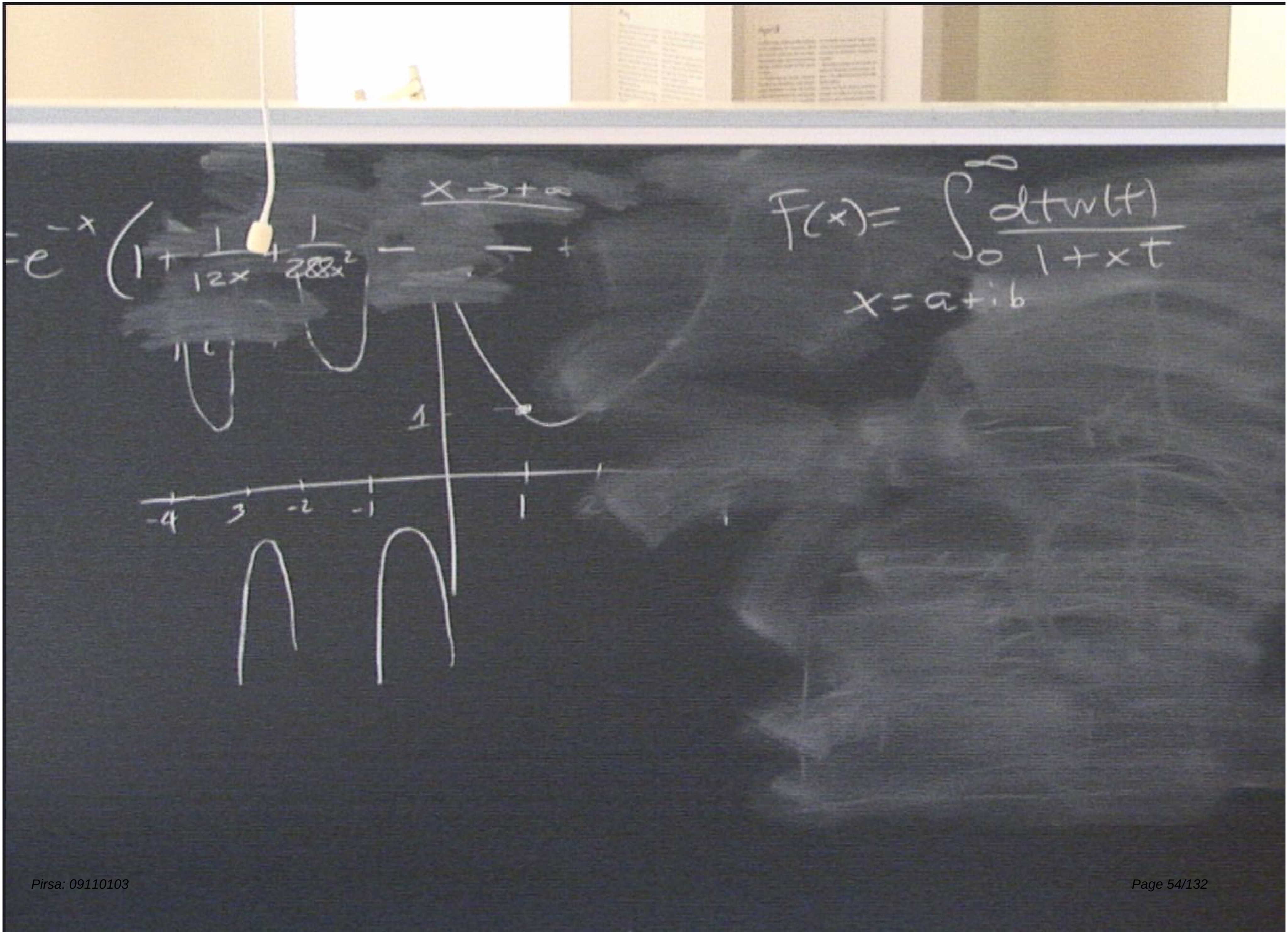
(1) as $|x| \to \infty$, $F(x) \to 0$.
in the cut plane.



(2) $F(x)$ is ANALYTIC in  $F'(x) = -\int_0^\infty \frac{t\,dt}{(1+\ldots}$

(3) $F(x) \sim \sum_{n=0}^{\infty} a_n x^n (-1)^n$ as $|x| \to 0$, $|\arg| < \pi$

(4) $-F(x)$ is Herglotz $\quad \mathrm{Im}(-F(x))$

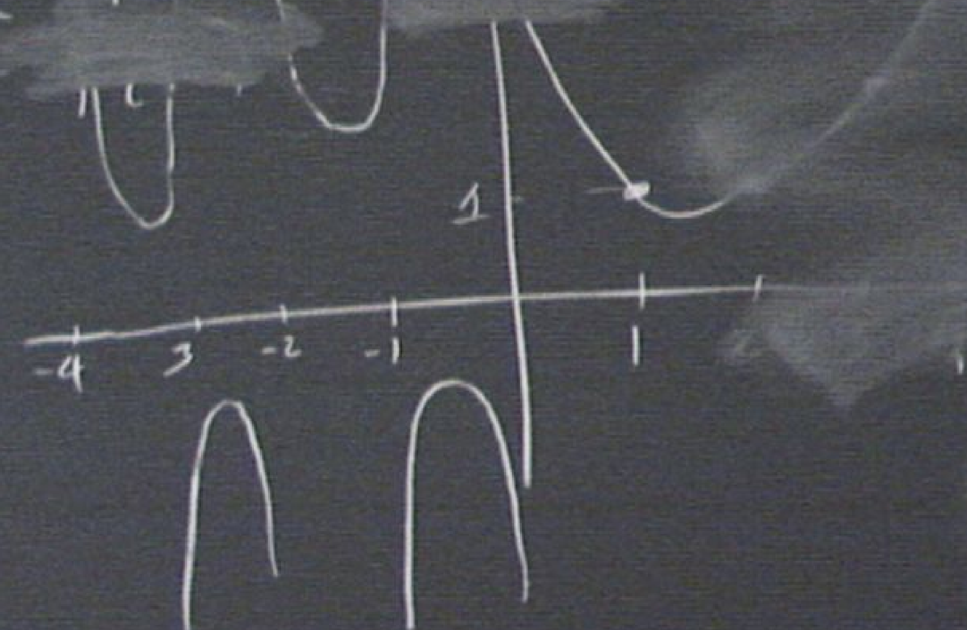

x → +∞
e^{-x}(1 + 1/12x + 1/288x^2 − ... + ...
F(x) = ∫_0^∞ dt w(t) / (1 + xt)
x = a + ib
1
-4 3 -2 -1 1

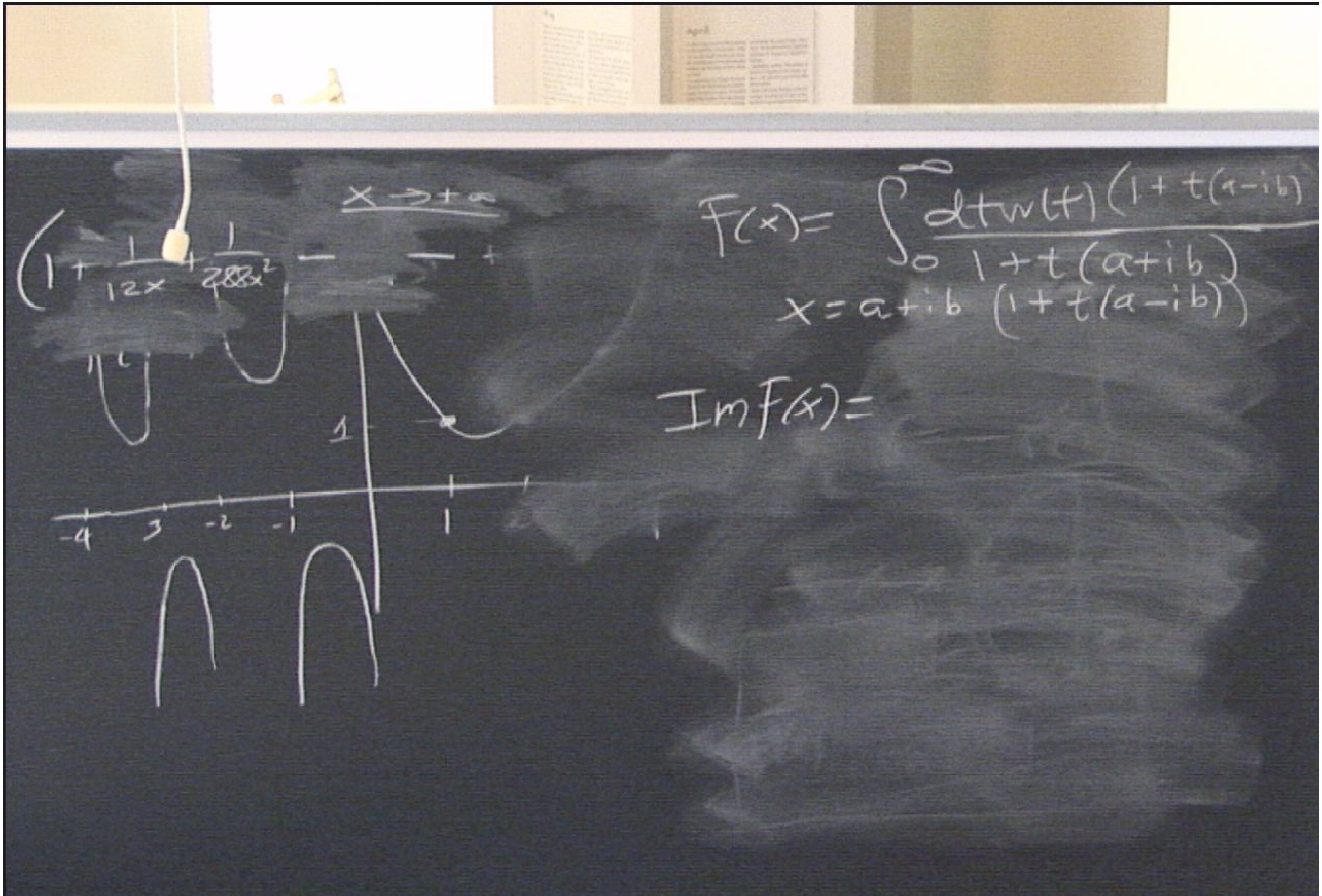
$x \to +\infty$

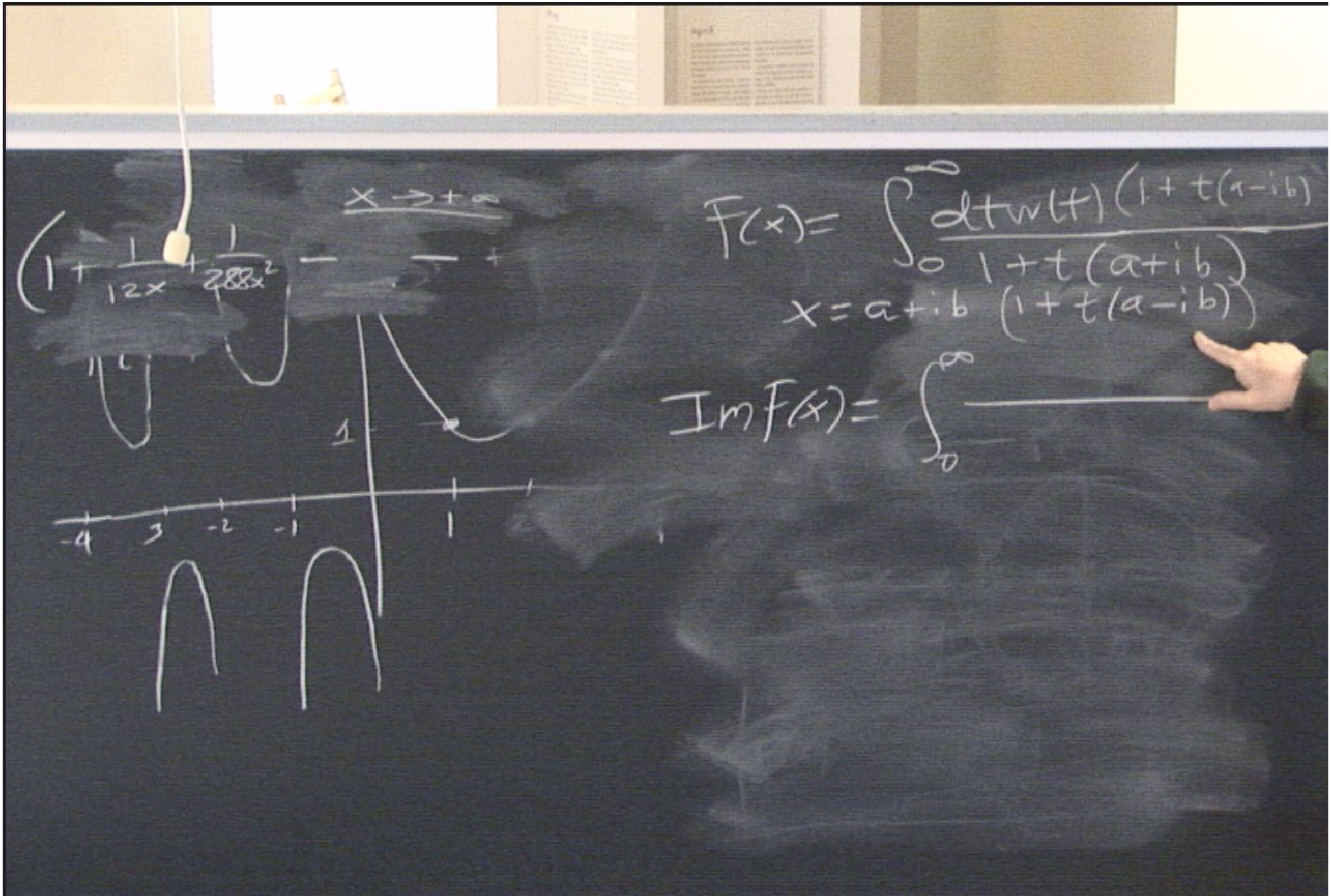
$\left(1 + \frac{1}{12x} + \frac{1}{288x^2} - \ldots - \ldots + \right.$

$$F(x) = \int_0^\infty \frac{dt\, w(t)}{1 + t(a+ib)}$$

$x = a + ib$



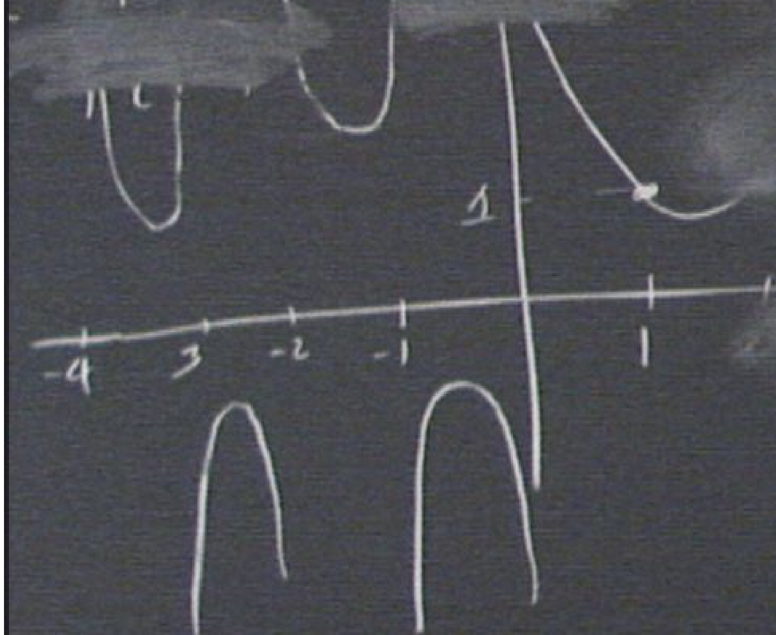

x → +∞
(1 + 1/12x + 1/288x² − ... − +
F(x) = ∫₀^∞ dt w(t) (1 + t(a−ib)) / (1 + t(a+ib))
x = a+ib (1 + t(a−ib))
Im F(x) =
1
−4 3 −2 −1 1



x → +∞
(1 + 1/12x + 1/288x² — — +
F(x) = ∫₀^∞ dt w(t) (1 + t(a−ib)) / (1 + t(a+ib))
x = a + ib (1 + t(a−ib))
Im F(x) = ∫₀^∞
1
-4 3 -2 -1 1

$\underline{x \to +\infty}$

$$1 + \frac{1}{12x} + \frac{1}{288x^2} - \qquad - \quad +$$



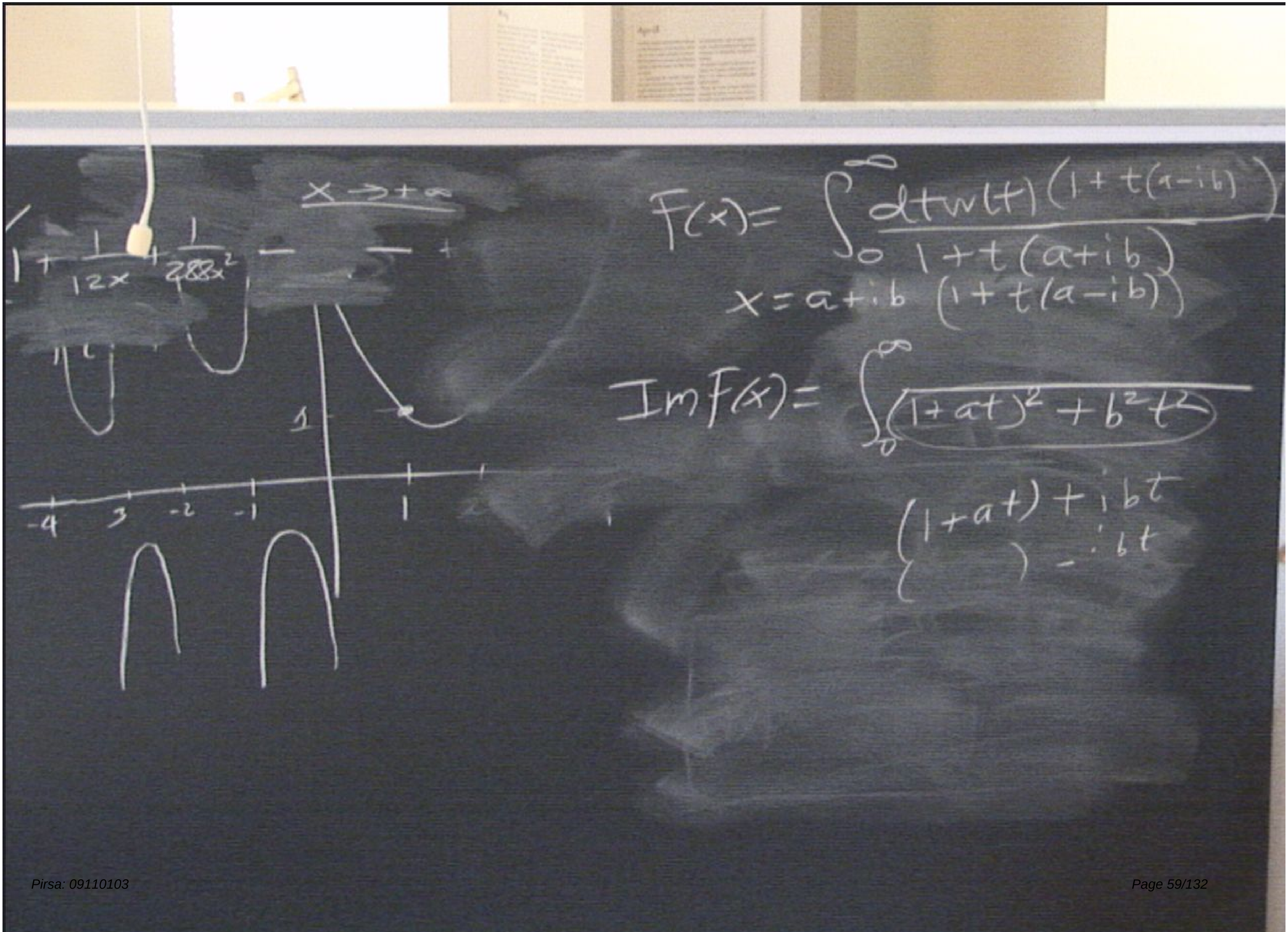
$$F(x) = \int_0^\infty \frac{dt\, w(t)\,\big(1 + t(a-ib)\big)}{1 + t(a+ib)}$$

$$x = a + ib \quad \big(1 + t(a-ib)\big)$$

$$\mathrm{Im}\, F(x) = \int_0^\infty \frac{\qquad}{(1+at)^2 + b^2t^2}$$

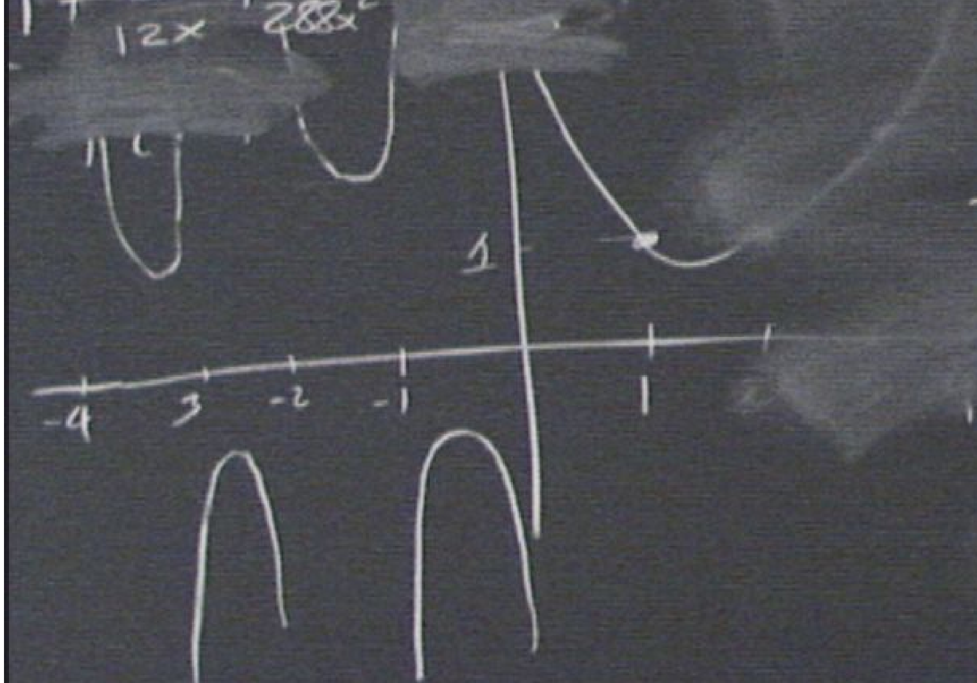
$$(1+at) + ibt$$

$$(\qquad) - ibt$$


x → +∞
1 + 1/12x + 1/288x²
F(x) = ∫₀^∞ dt w(t) (1 + t(a − ib)) / (1 + t(a + ib))
x = a + ib (1 + t(a − ib))
Im F(x) = ∫₀^∞ / ((1 + at)² + b²t²)
(1 + at) + ibt
( ) − ibt
1
-4 3 -2 -1 1

$x \to +\infty$

$1 + \frac{1}{12x} + \frac{1}{288x^2} - \ldots$



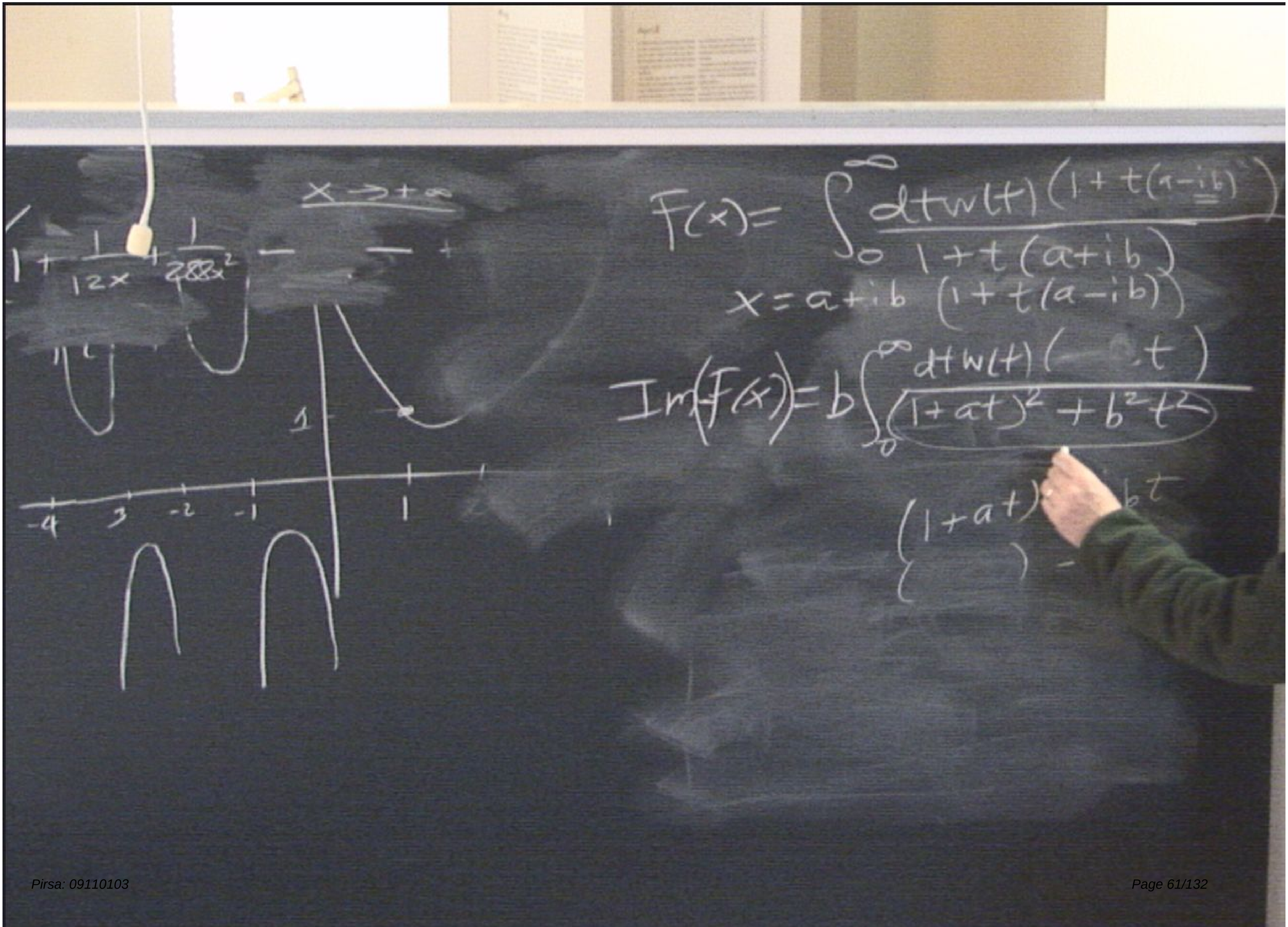
$$F(x) = \int_0^\infty \frac{dt\, w(t)\, (1+t(a-ib))}{1+t(a+ib)}$$

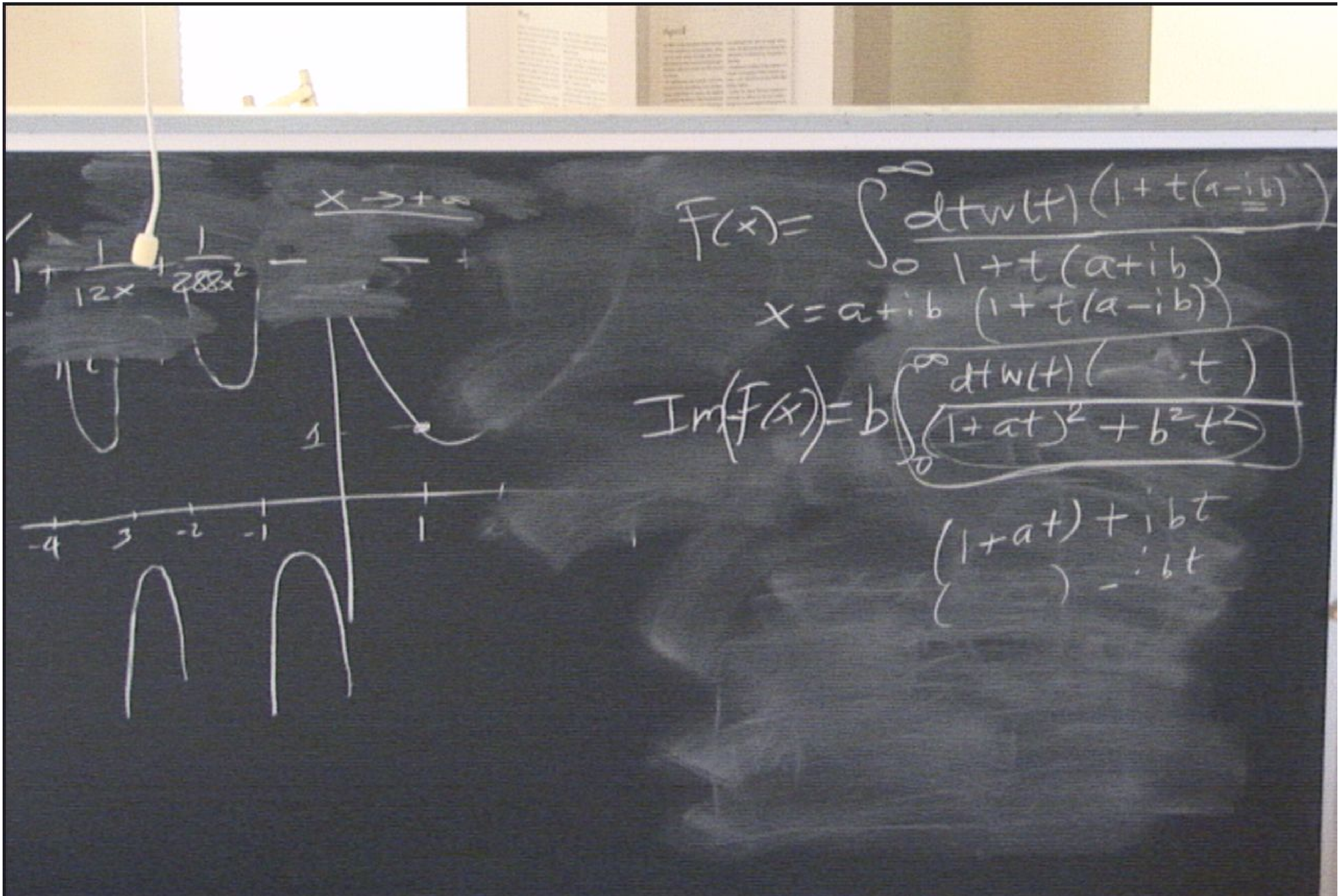
$x = a+ib \qquad (1+t(a-ib))$

$$\mathrm{Im}\, F(x) = \int_0^\infty \frac{dt\, w(t)\, (-bt)}{(1+at)^2 + b^2 t^2}$$

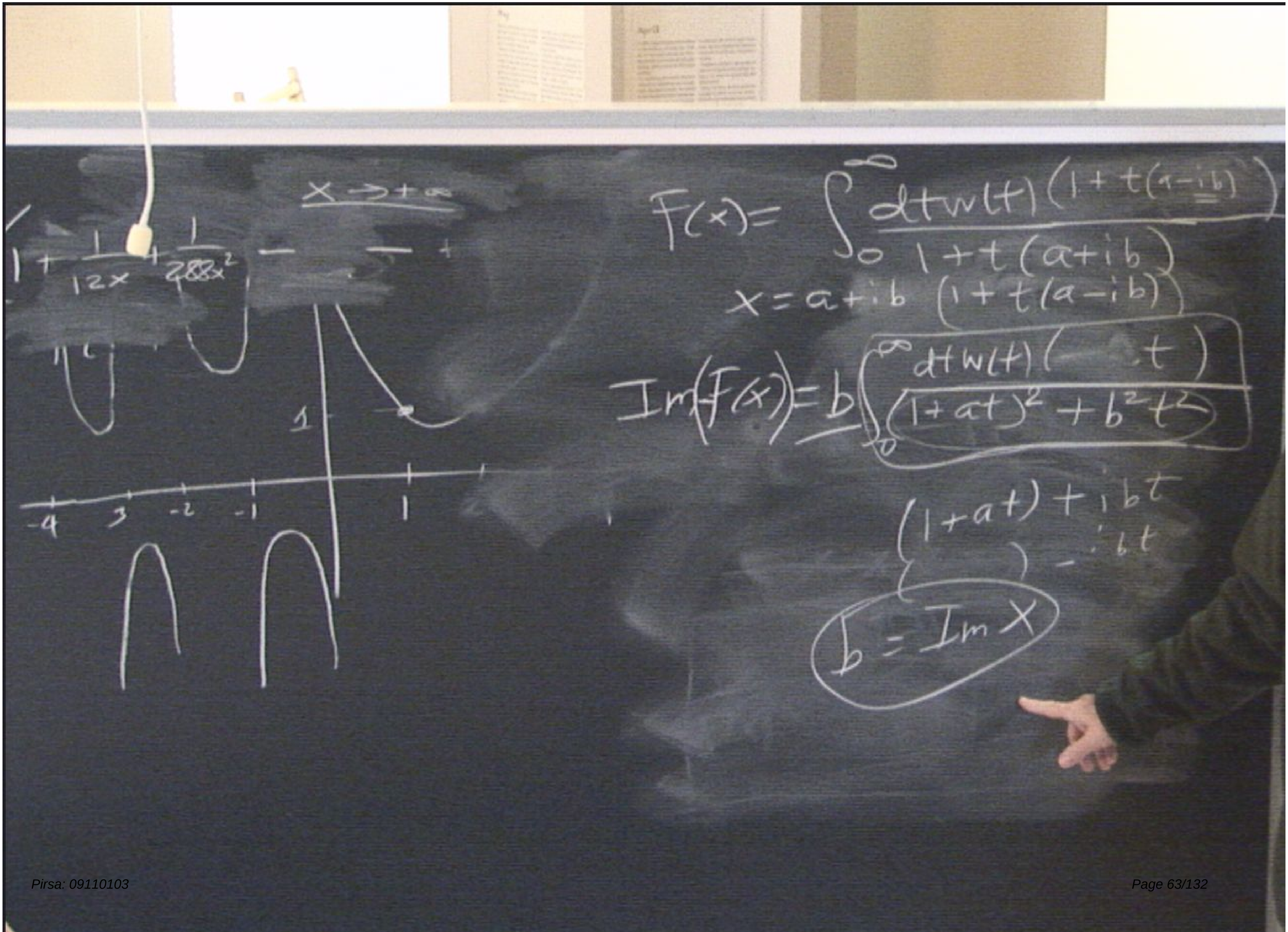
$(1+at) + ibt$

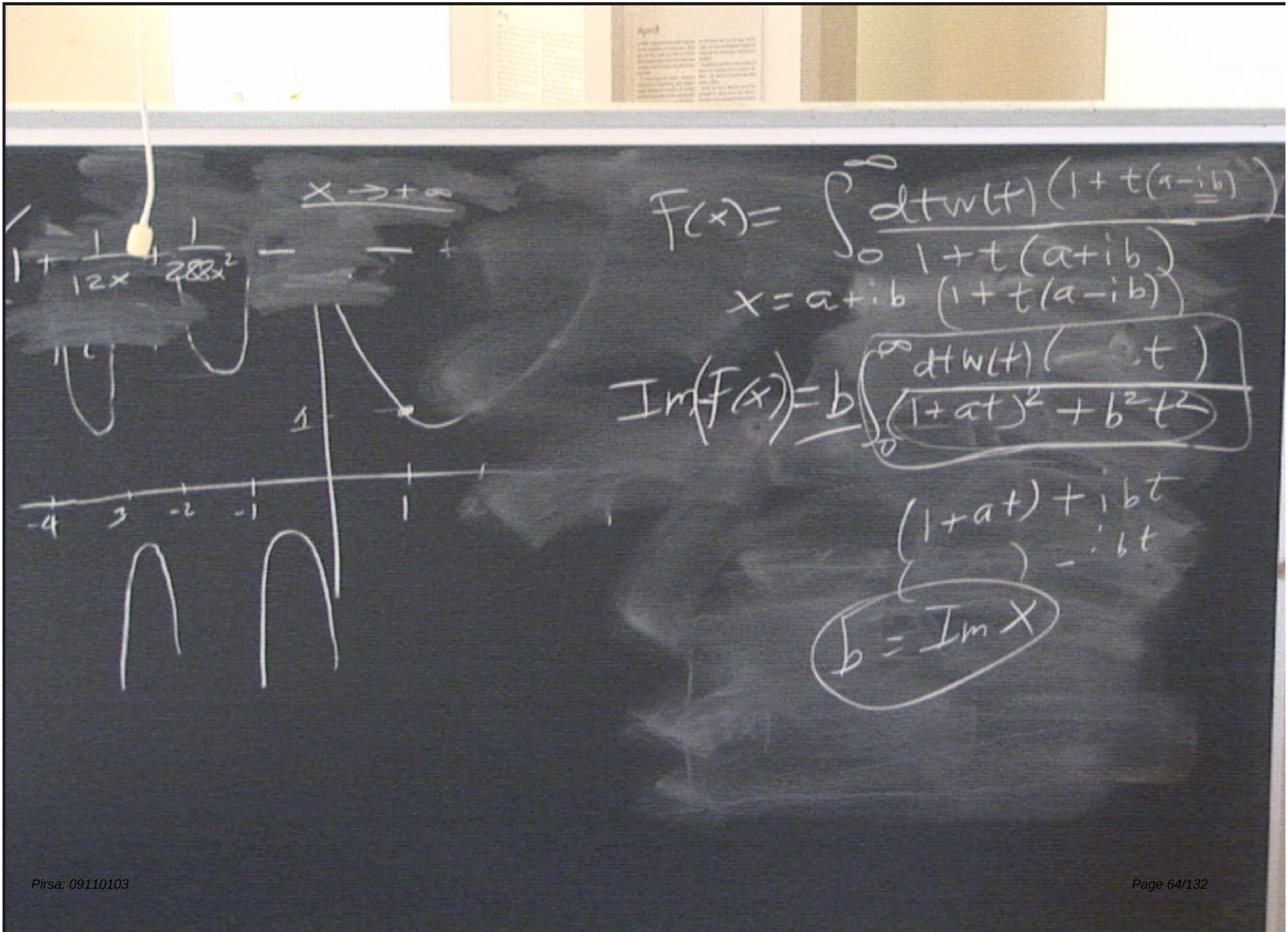
$(\quad) - ibt$


x → +∞
1 + 1/12x + 1/288x²
F(x) = ∫₀^∞ dt w(t) (1 + t(a − ib)) / (1 + t(a + ib))
x = a + ib
(1 + t(a − ib))
Im(F(x)) = b ∫₀^∞ dt w(t) (t) / ((1 + at)² + b²t²)
(1 + at)
-4 -3 -2 -1 1



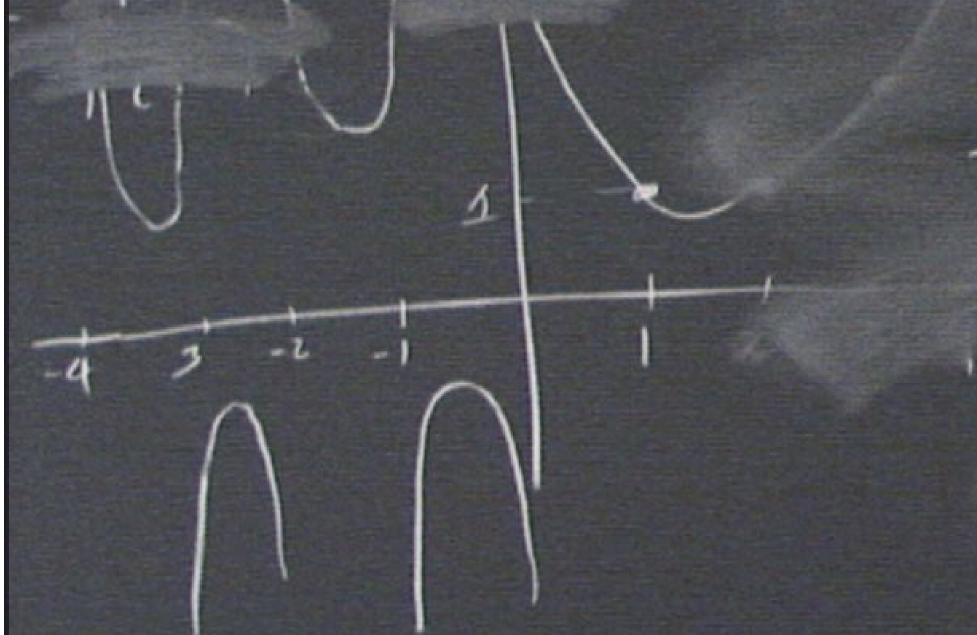
$x \to +\infty$
$1 + \frac{1}{12x} + \frac{1}{288x^2}$
$F(x) = \int_0^\infty \frac{dt\, w(t)\,(1+t(a-ib))}{1+t(a+ib)}$
$x = a+ib$
$(1+t(a-ib))$
$\mathrm{Im}(F(x)) = b \int_0^\infty \frac{dt\, w(t)\,(\quad t)}{(1+at)^2 + b^2t^2}$
$(1+at) + ibt$
$(\quad) - ibt$
-4 3 -2 -1 1
1


x → +∞
F(x) = ∫₀^∞ dt w(t) (1 + t(a − ib)) / (1 + t(a + ib))
x = a + ib
Im F(x) = b ∫₀^∞ dt w(t) t / ((1 + at)² + b²t²)
b = Im x


x → +∞
F(x) = ∫₀^∞ dt w(t) (1 + t(a − ib)) / (1 + t(a + ib))
x = a + ib
(1 + t(a − ib))
Im(F(x)) = b ∫₀^∞ dt w(t) (t) / ((1+at)² + b²t²)
(1+at) + ibt
b = Im x

$x \to +\infty$

$1 + \frac{1}{12x} + \frac{1}{288x^2} - \quad - \quad +$



$$F(x) = \int_0^\infty \frac{dt\, w(t)\,(1+t(a-ib))}{1+t(a+ib)}$$

$$x = a+ib \qquad (1+t(a-ib))$$

$$\mathrm{Im}(F(x)) = b \int_0^\infty \frac{dt\, w(t)\,(\quad t)}{(1+at)^2+b^2t^2}$$

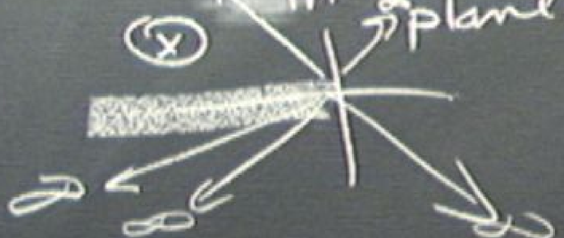
$$(1+at)+ibt$$

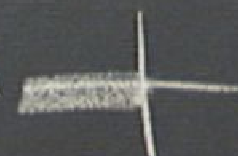
$$-ibt$$

$$b = \mathrm{Im}\, x$$

Properties of S. Fns.

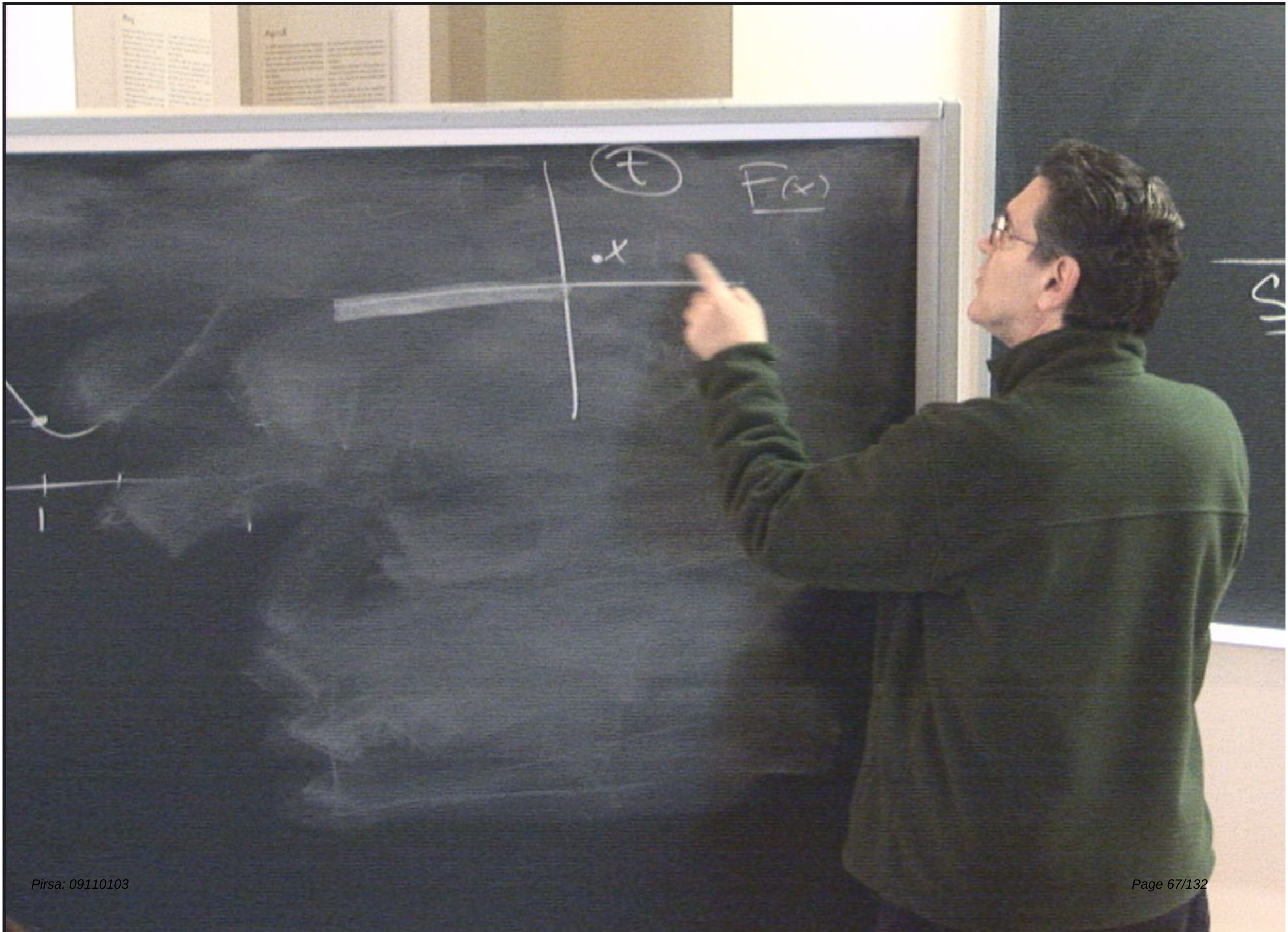
(1) as $|x| \to \infty$, in the cut plane. $F(x) \to 0$.

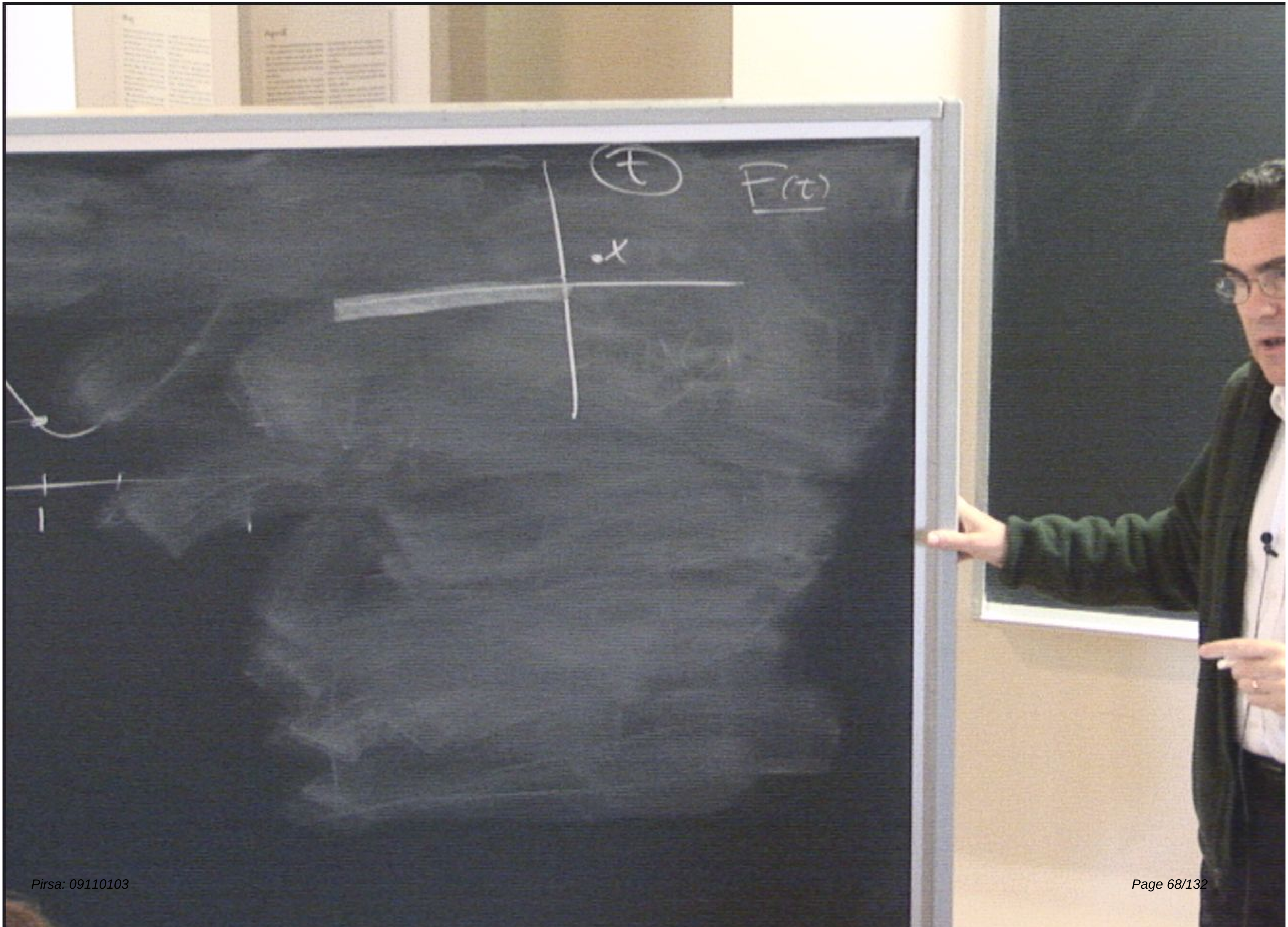


(2) $F(x)$ is ANALYTIC in  $F'(x) = -\int_0^\infty \frac{t\,dt\,w(t)}{(1+xt)^2}$

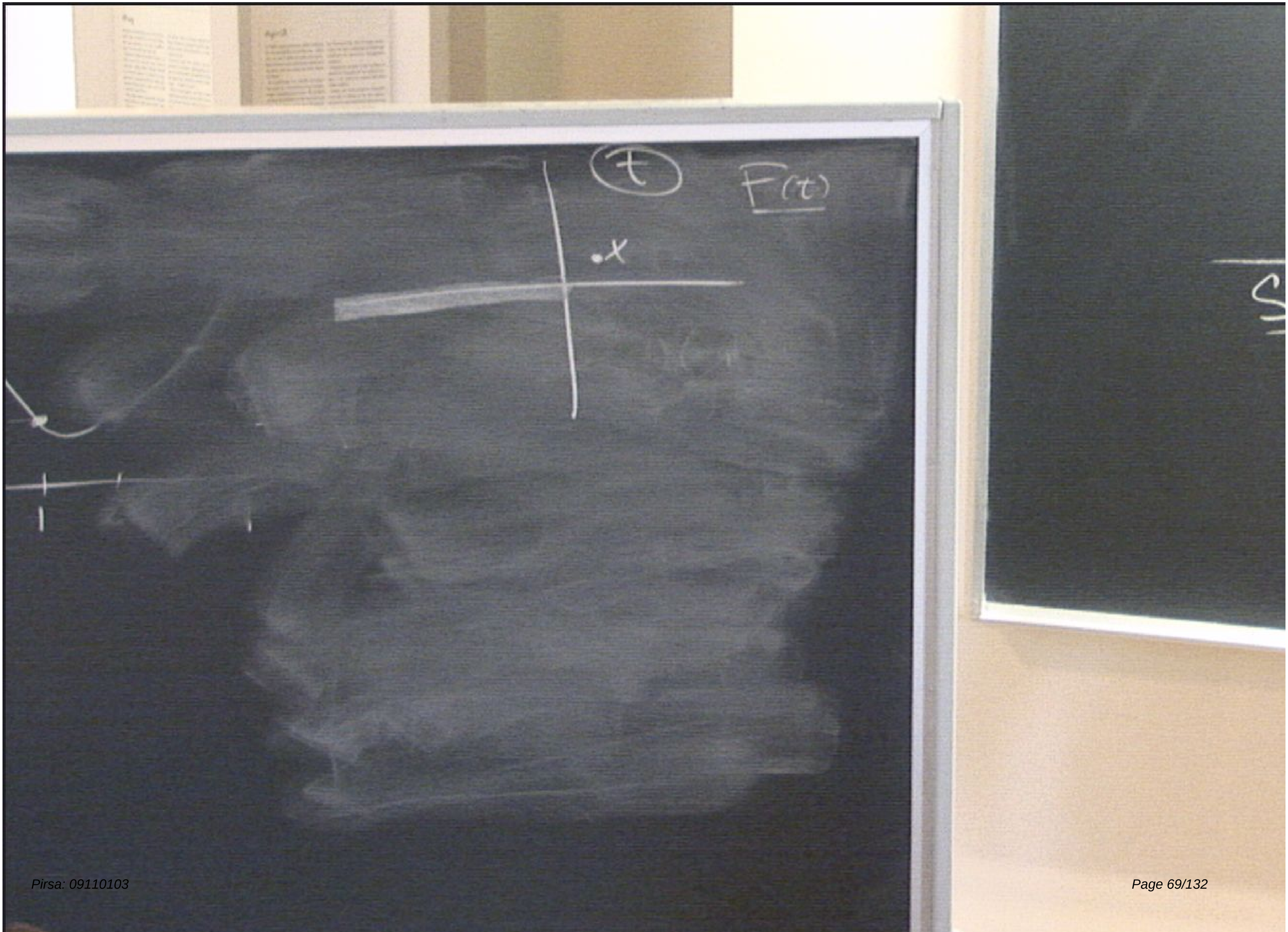
(3) $F(x) \sim \sum_{n=0}^{\infty} a_n x^n (-1)^n$ as $|x| \to 0$, $|\arg| < \pi$

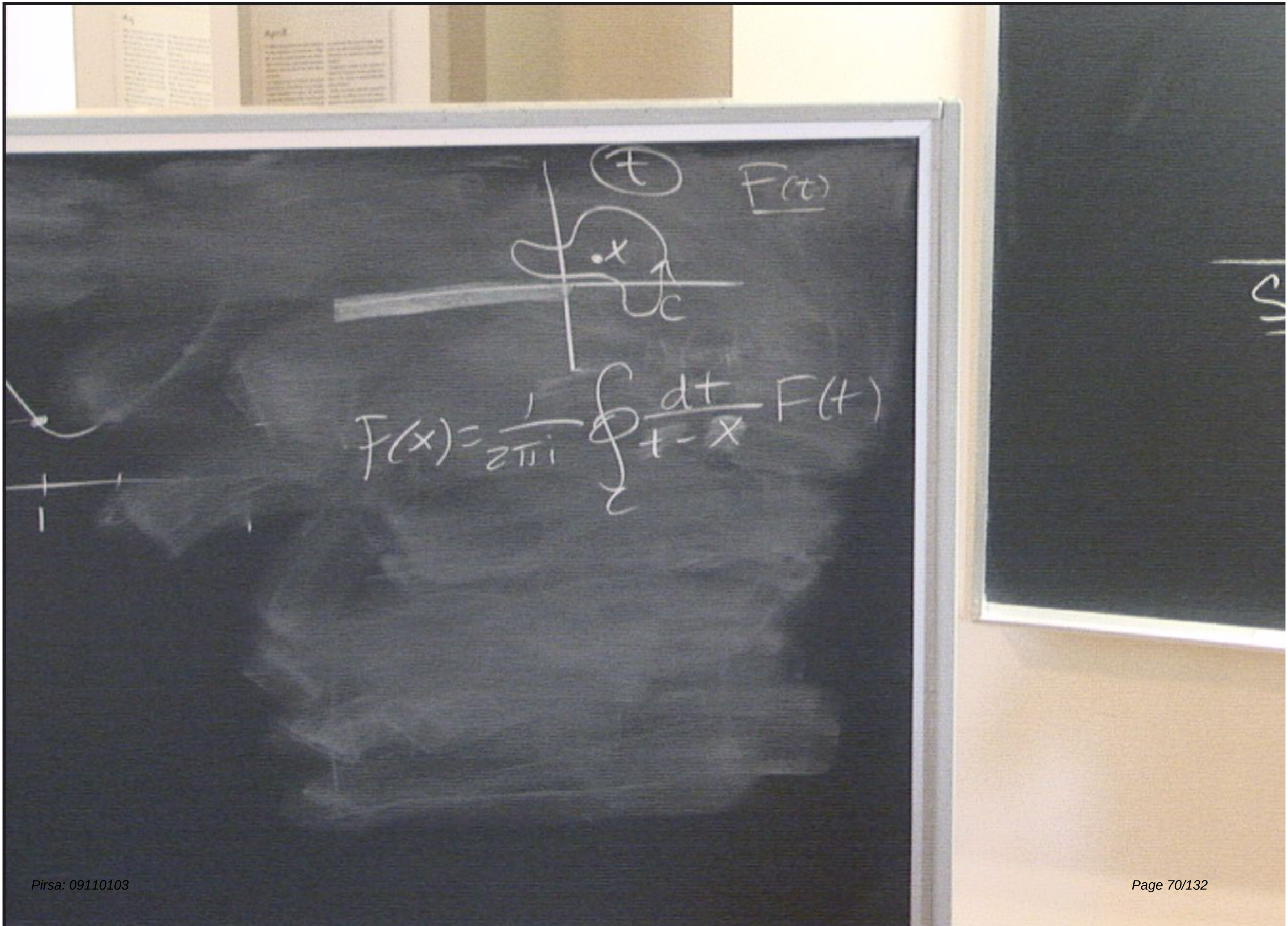
(4) $-F(x)$ is Herglotz $\quad \mathrm{Im}(-F(x))$ same sign as $\mathrm{Im}(x)$.

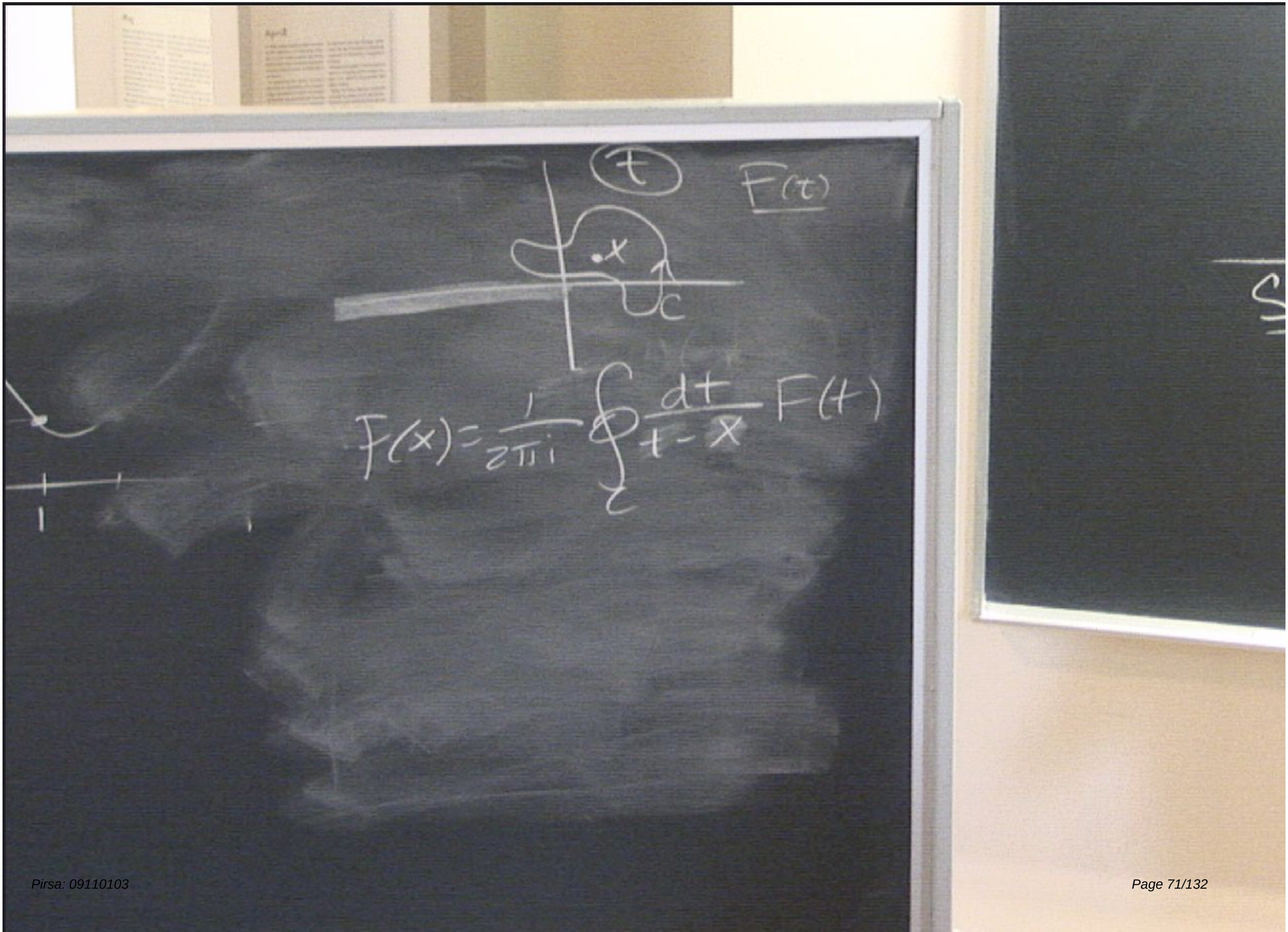

F(x)
x

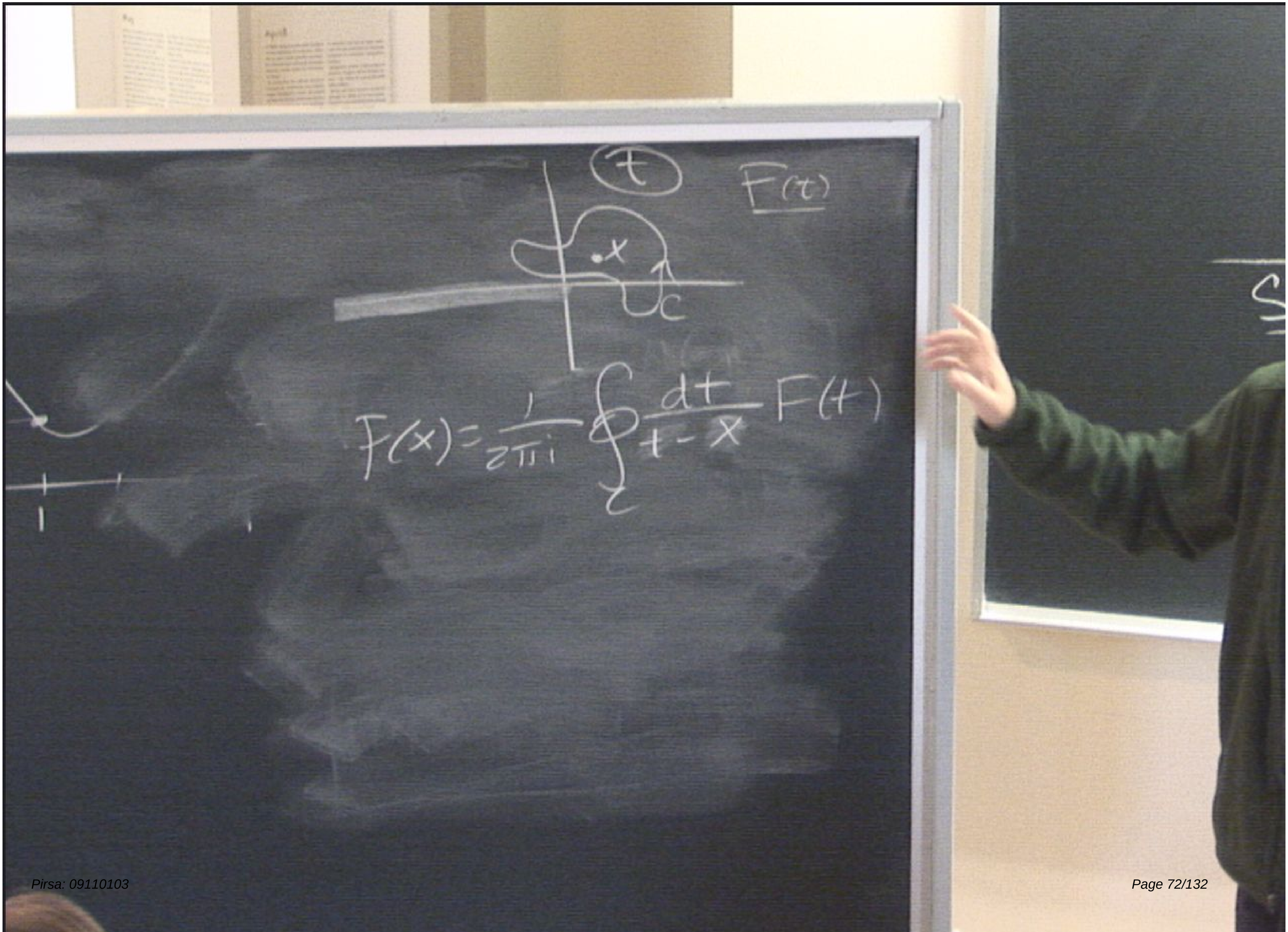


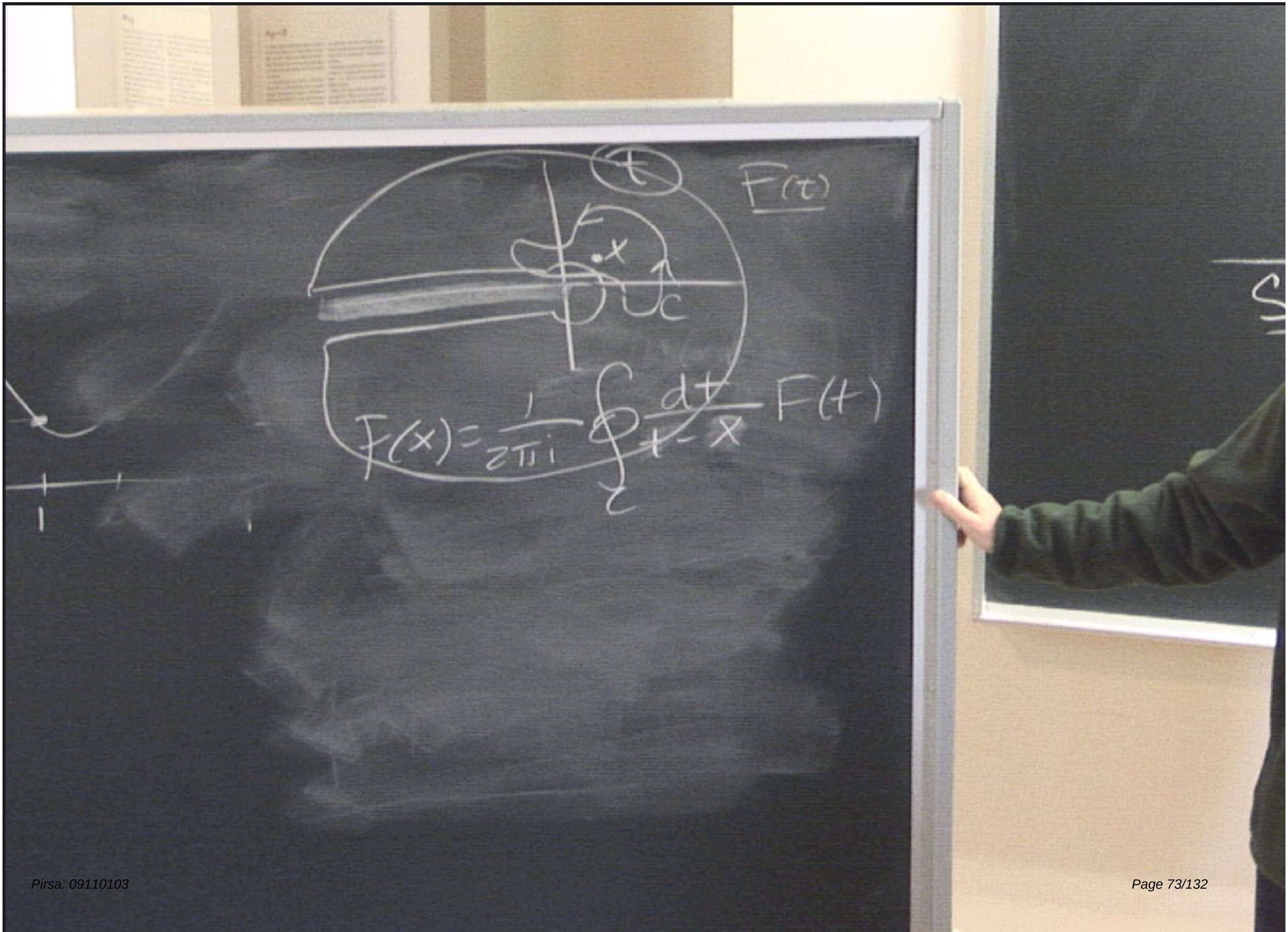
t
E(τ)
x

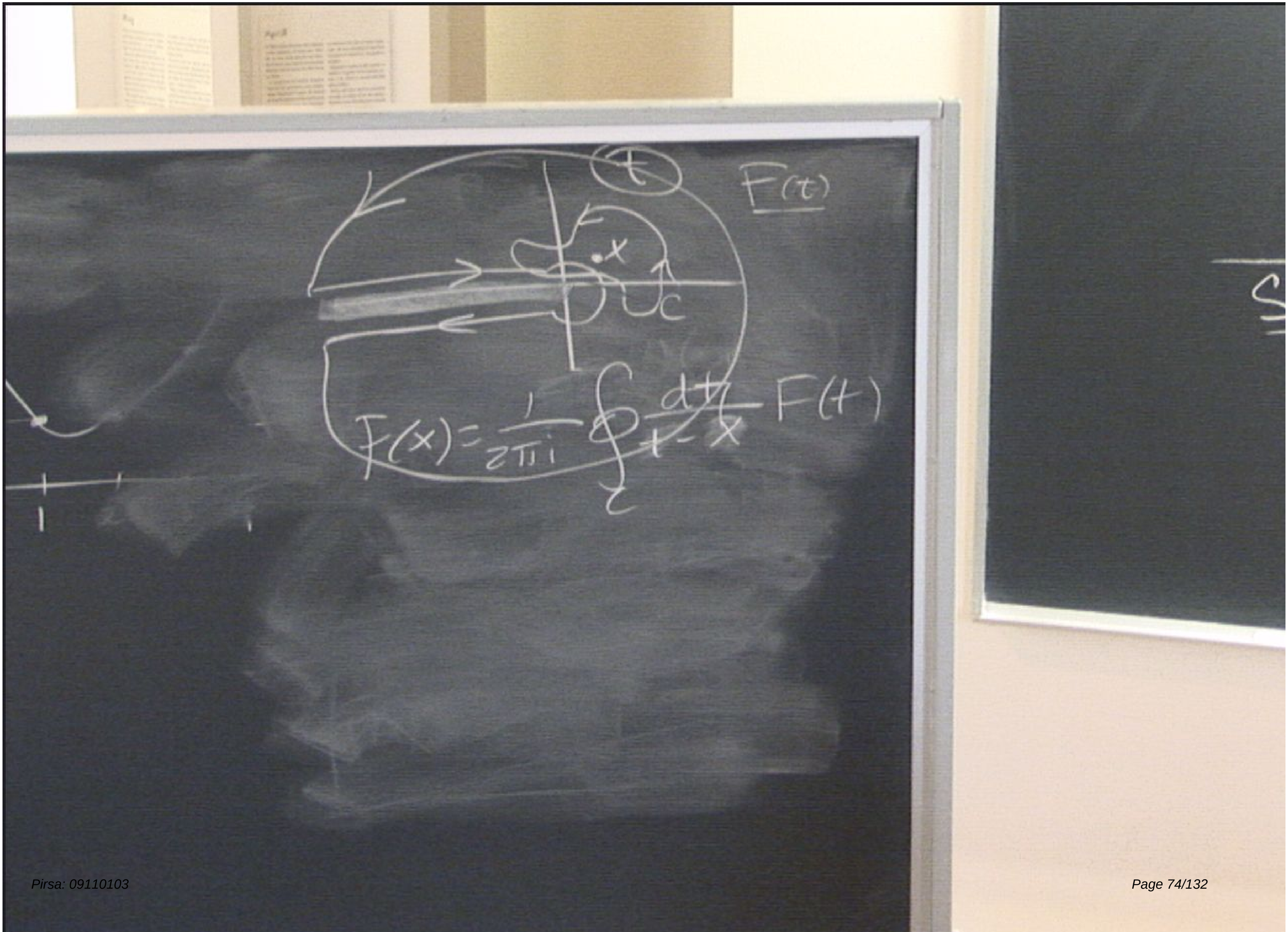

t
F(t)
x


t
F(t)
x
c
$F(x) = \frac{1}{2\pi i} \oint_c \frac{dt}{t-x} F(t)$


t
F(t)
x
C
F(x) = 1/2πi ∮_C dt/(t - x) F(t)


t
F(t)
x
C
F(x) = 1/2πi ∮_C dt/(t − x) F(t)


t
F(t)
x
C
$F(x) = \frac{1}{2\pi i} \oint_C \frac{dt}{t-x} F(t)$


t
F(t)
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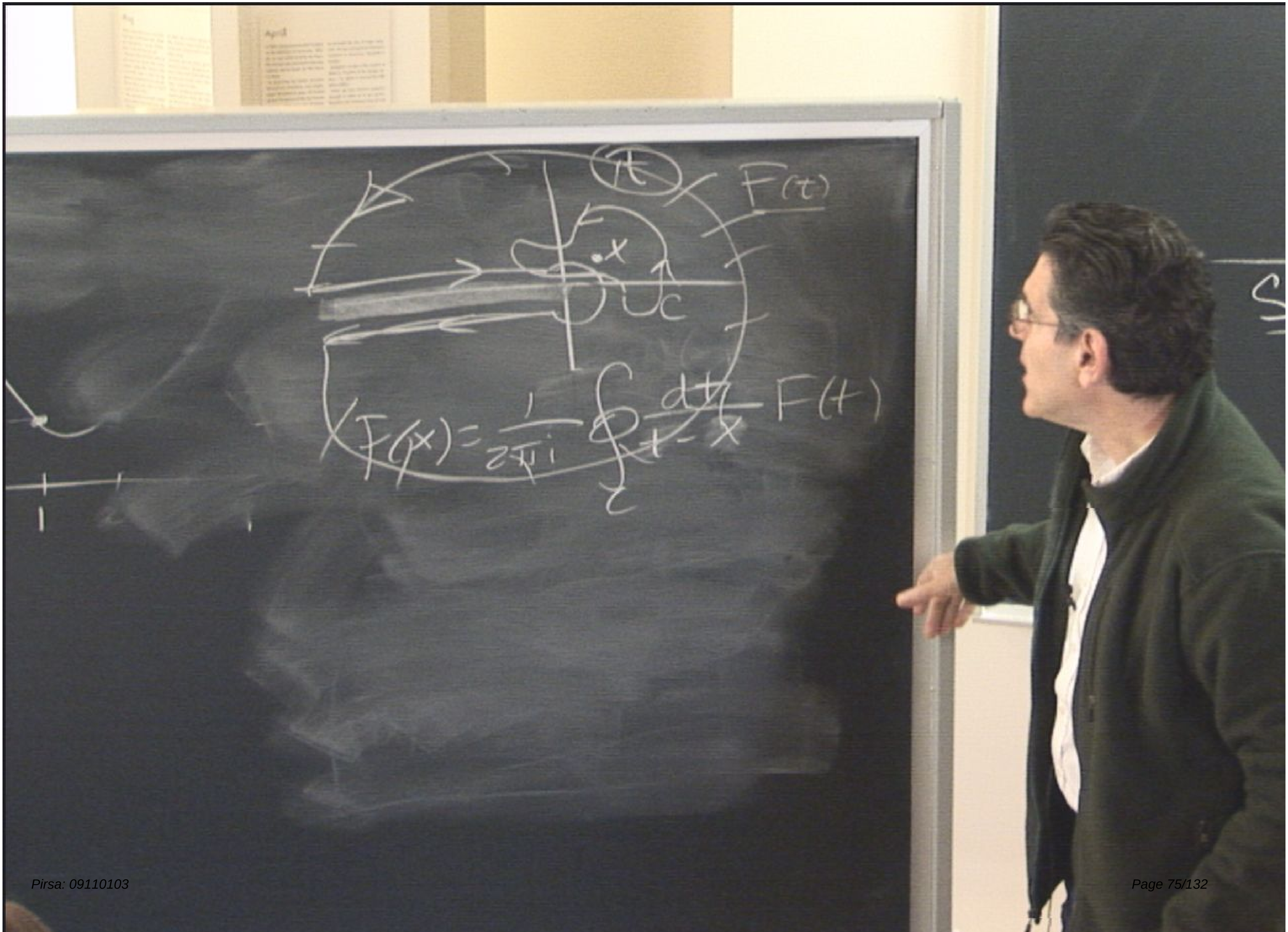
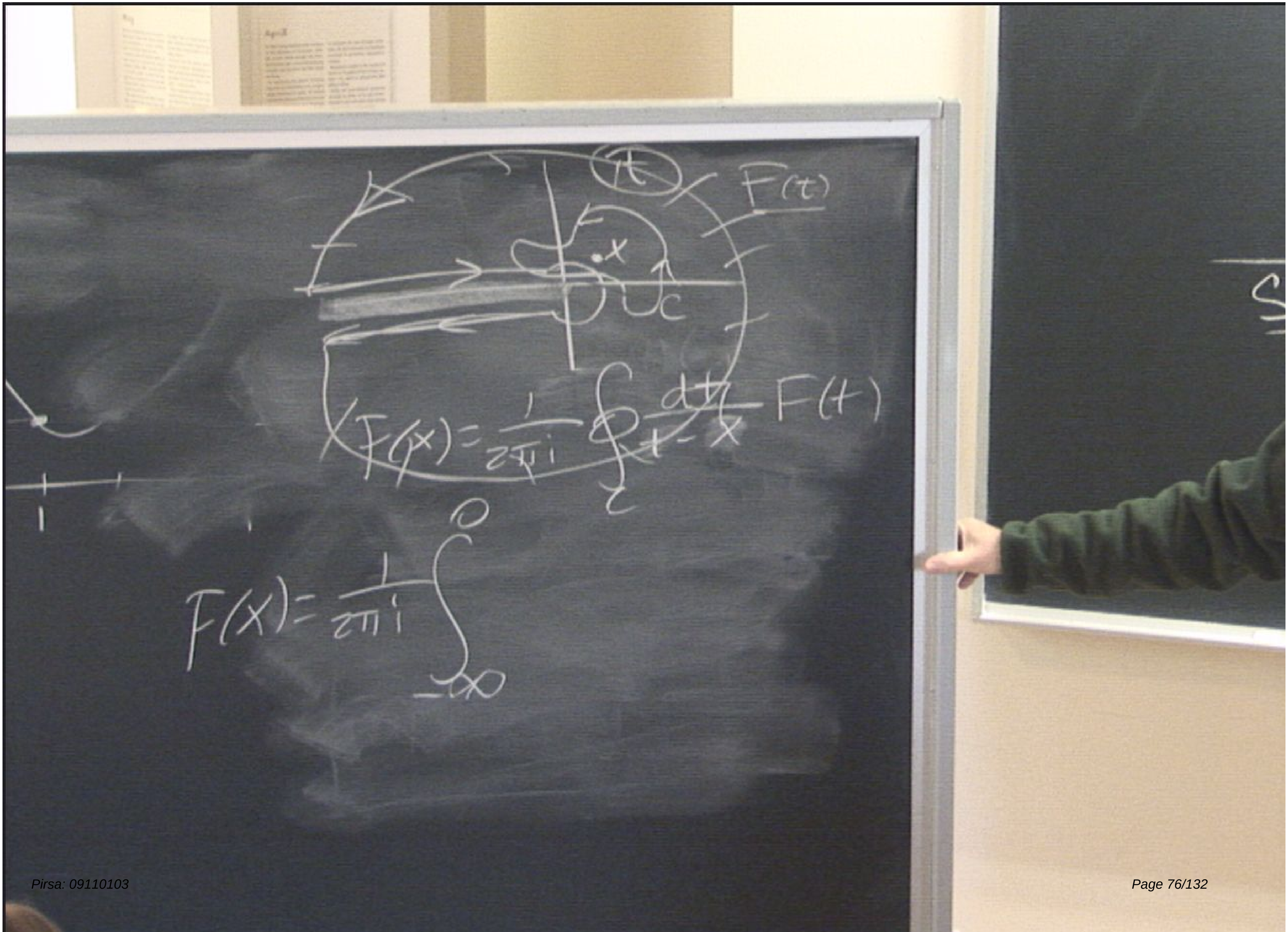
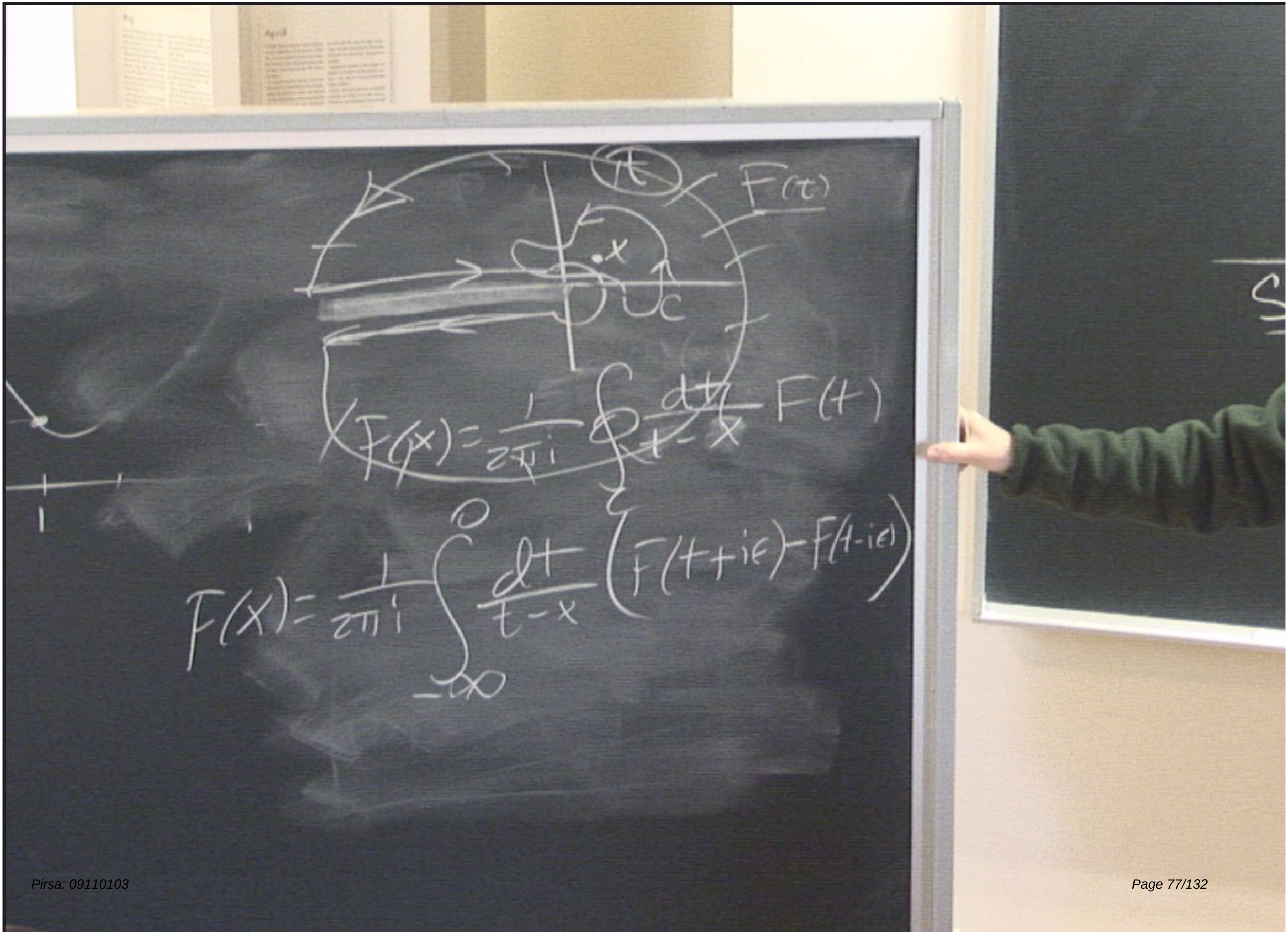
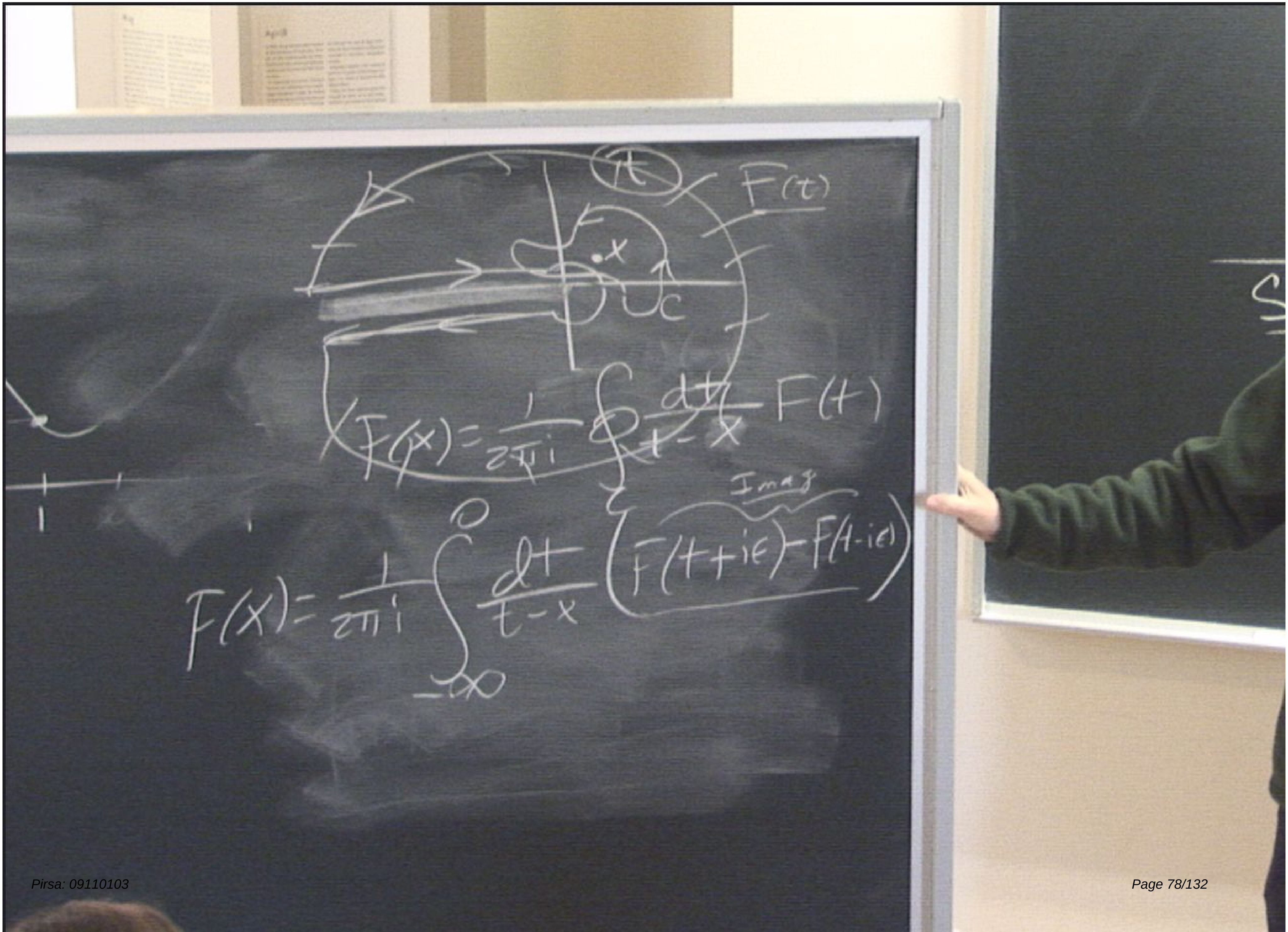
F(x) = 1/2πi ∮ dt/(t - x) F(t)

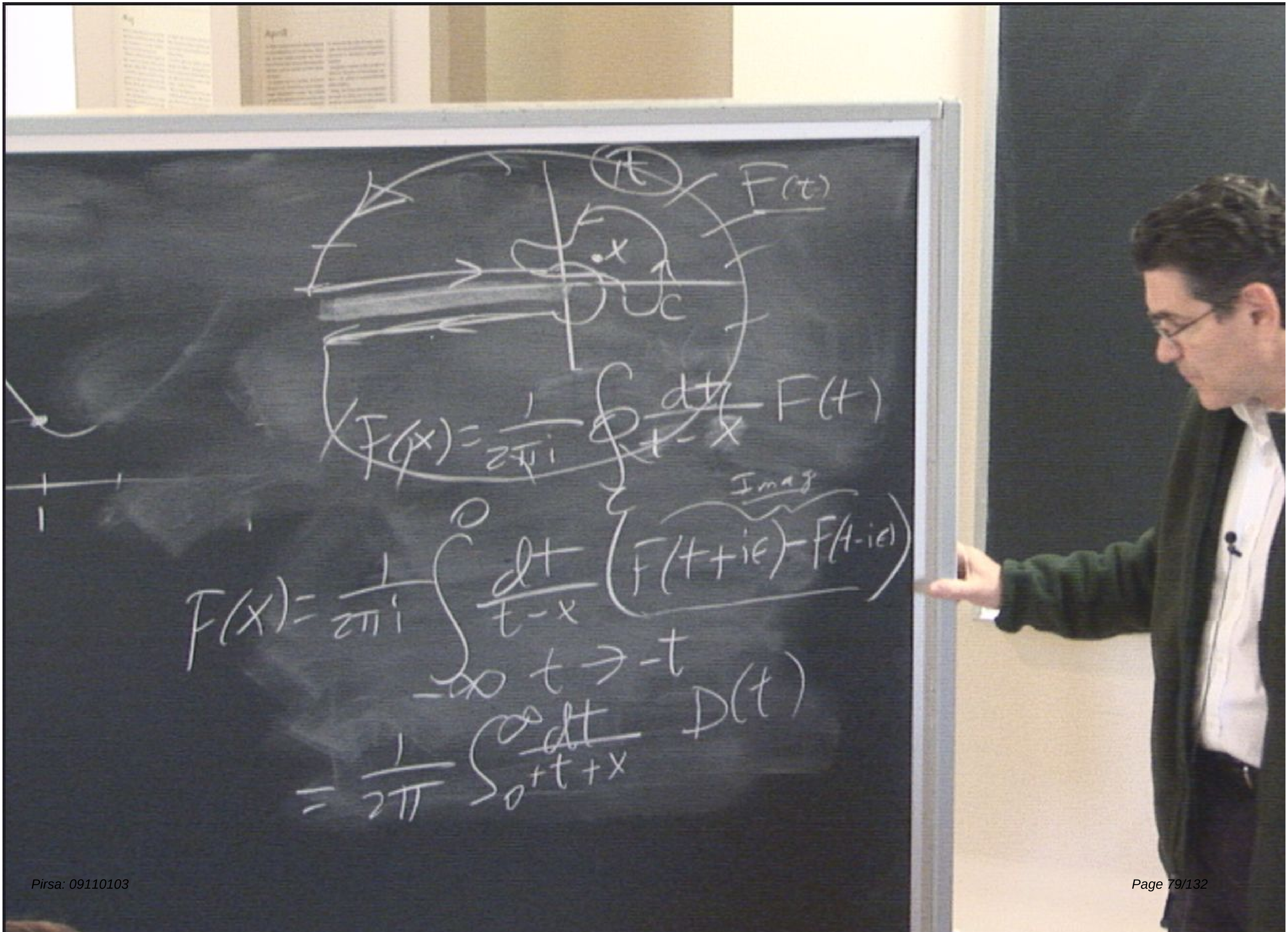


F(t)
c
F(x) = 1/(2πi) ∮ dt/(t − x) F(t)

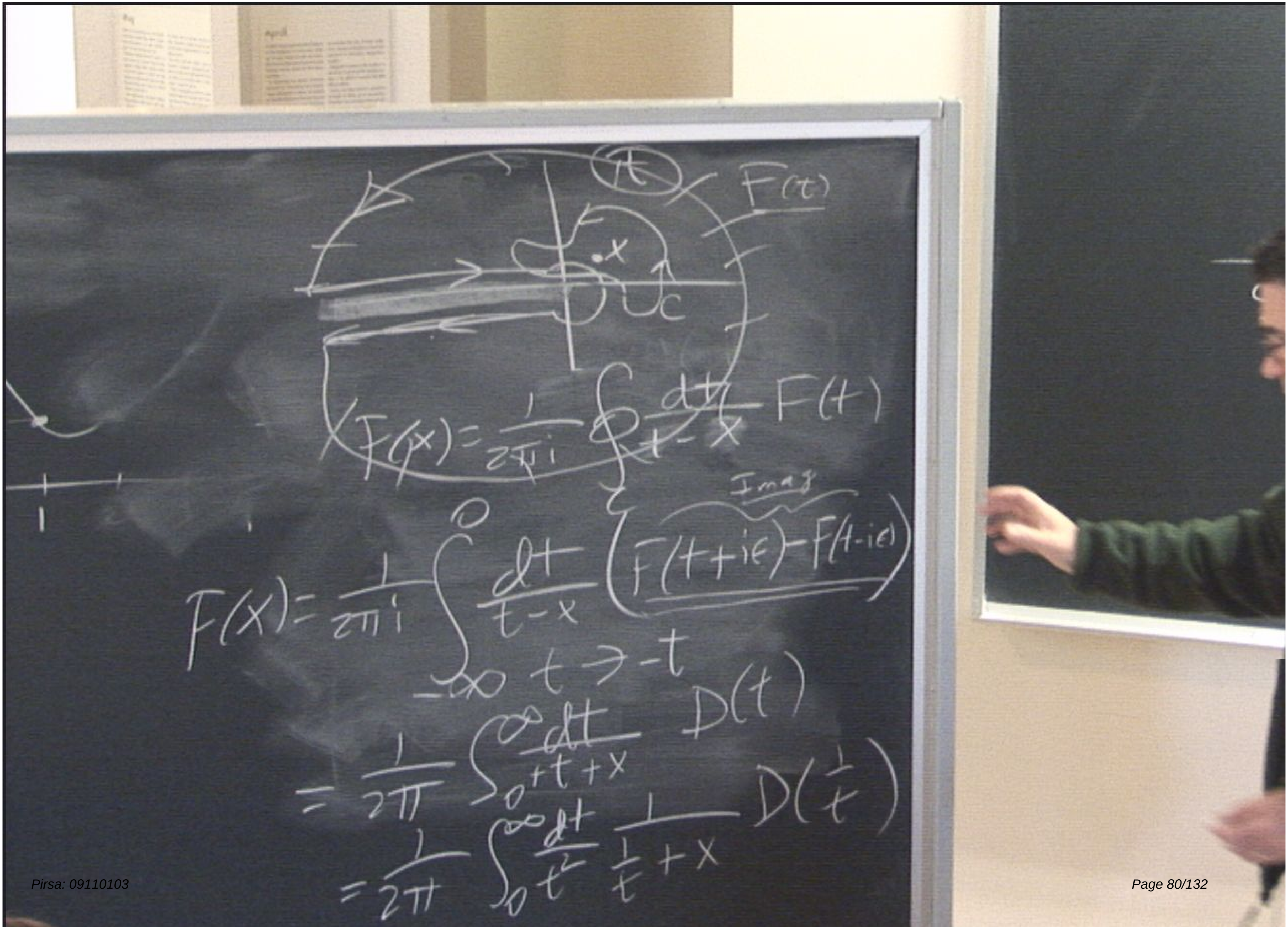

t
F(t)
x
C
$F(x) = \frac{1}{2\pi i} \oint_C \frac{dt}{t-x} F(t)$
$F(x) = \frac{1}{2\pi i} \int_{-\infty}^{0}$


t
F(t)
x
C
F(x) = 1/2πi ∮ dt/(t−x) F(t)
F(x) = 1/2πi ∫_{−∞}^{0} dt/(t−x) (F(t+iε) − F(t−iε))


t
F(t)
x
C
F(x) = 1/2πi ∮ dt/(t−x) F(t)
Imag
F(x) = 1/2πi ∫_{−∞}^{∞} dt/(t−x) (F(t+iε) − F(t−iε))



F(t)
C
F(x) = 1/(2πi) ∮ dt/(t − x) F(t)
Imag
F(x) = 1/(2πi) ∫_{−∞}^{0} dt/(t − x) (F(t+iε) − F(t−iε))
t → −t
D(t)
= 1/(2π) ∫_0^∞ dt/(t + x)



F(t)
x
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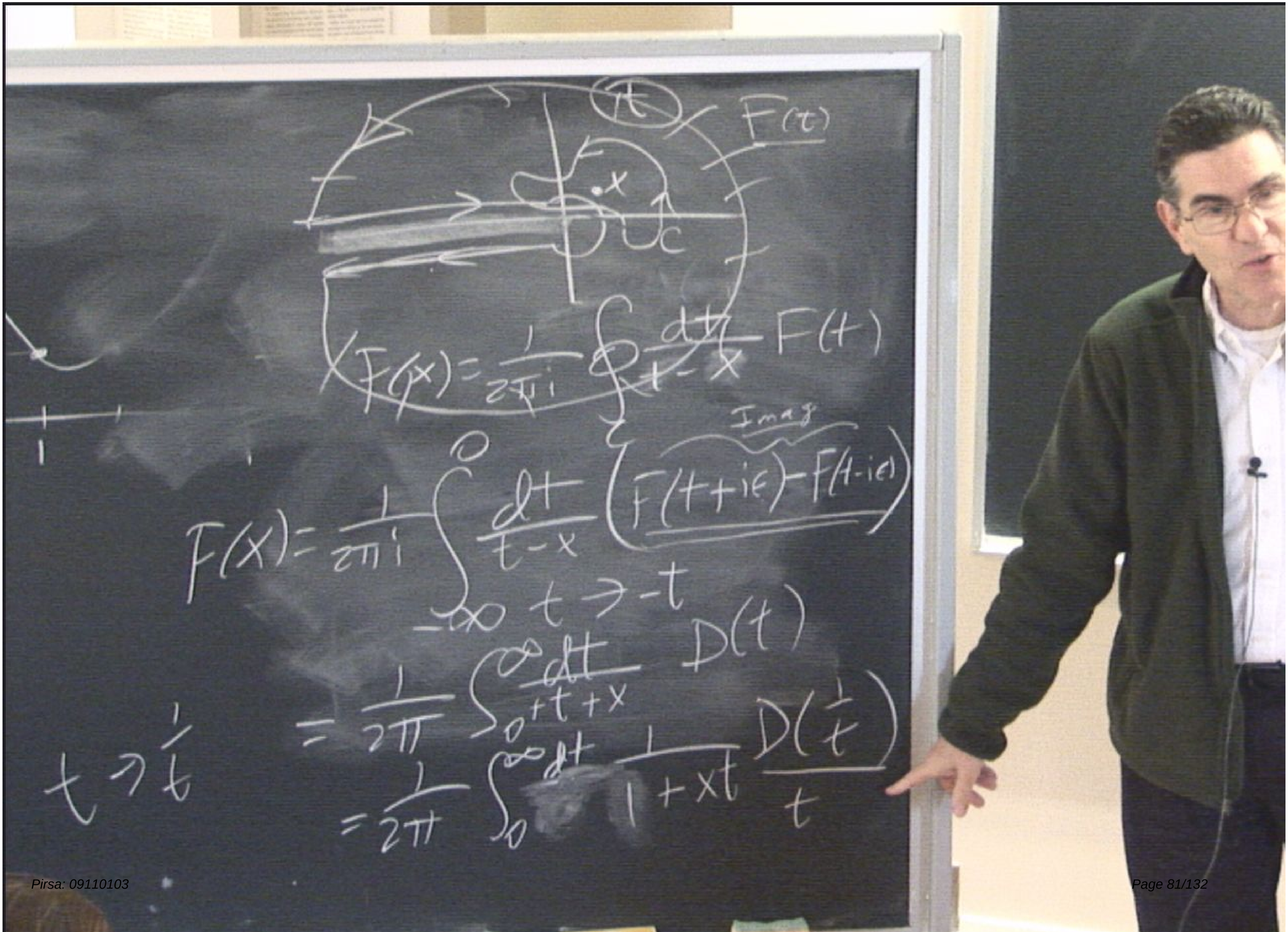
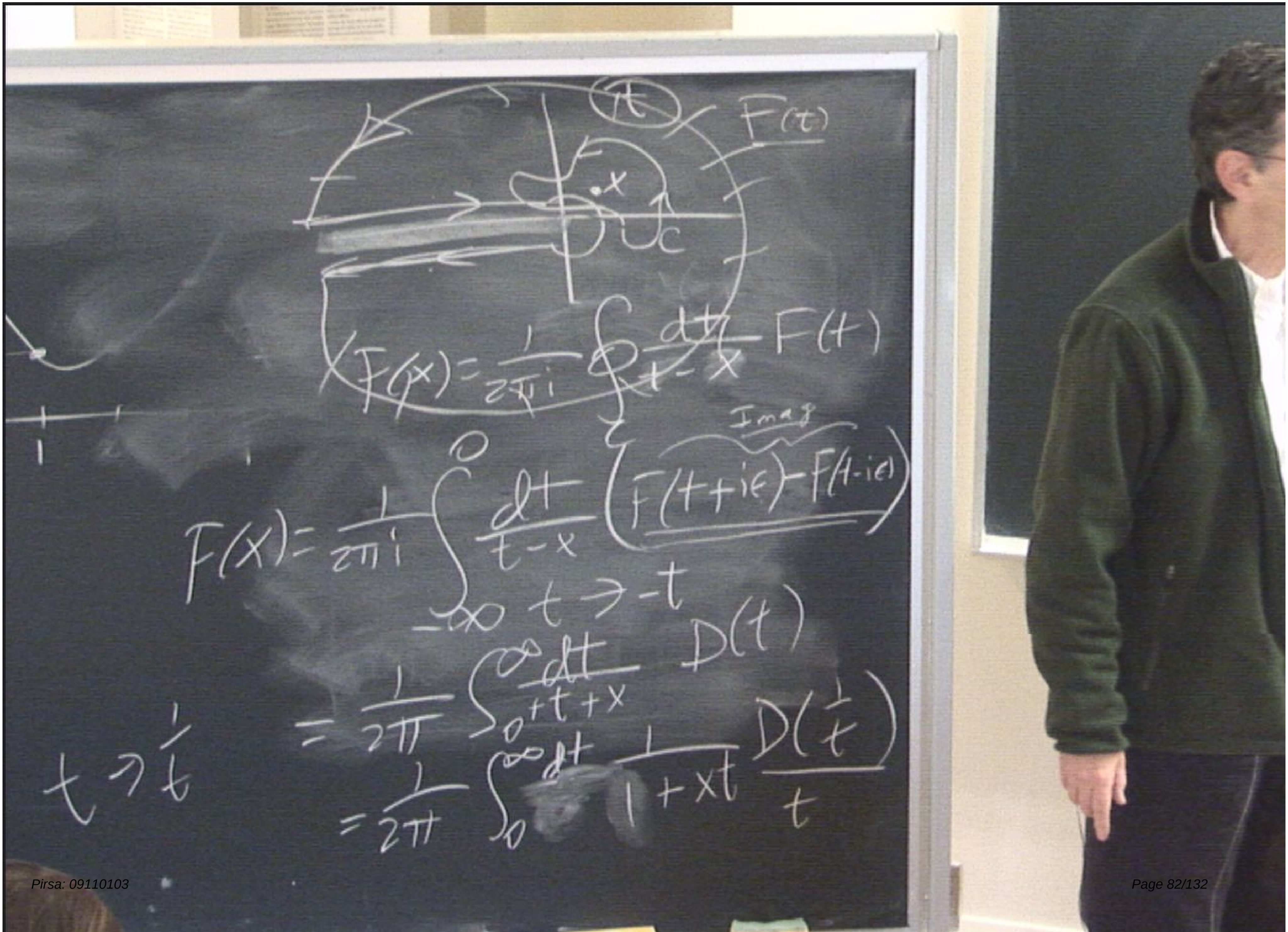
F(x) = 1/(2πi) ∮ dt/(t−x) F(t)
Imag
F(x) = 1/(2πi) ∫_{−∞}^{0} dt/(t−x) (F(t+iε) − F(t−iε))
t → −t
D(t)
= 1/(2π) ∫_0^∞ dt/(t+x) D(t)
= 1/(2π) ∫_0^∞ dt/t² · 1/(1/t + x) D(1/t)



F(t)
x
C
F(x) = 1/2πi ∮ dt/(t − x) F(t)
Imag
F(x) = 1/2πi ∫_{−∞}^{0} dt/(t − x) (F(t+iε) − F(t−iε))
t → −t
D(t)
= 1/2π ∫_0^∞ dt/(t + x) D(t)
t → 1/t
= 1/2π ∫_0^∞ dt 1/(1 + xt) D(1/t)/t


F(t)
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F(x) = 1/2πi ∮ dt/(t−x) F(t)
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D(t)
= 1/2π ∫_0^∞ dt/(t+x) D(t)
t → 1/t
= 1/2π ∫_0^∞ dt 1/(1+xt) D(1/t)/t

Stieltjes $\begin{cases}\text{Function} \\ \text{Series}\end{cases}$

S Series: $\sum_{n=0}^{\infty} (-1)^n a_n x^n$ where $a_n = \int_0^{\infty} dt\, w(t)\, t^n$

$w(t) \geq 0 \quad (0 \leq t < \infty)$

$\frac{D(t)}{t}$

Ex: $\sum_0^{\infty} (-1)^n n!\, x^n$

$w(t) = e^{-t}$

S Function: $\sum_{n=0}^{\infty} (-1)^n x^n \int_0^{\infty} dt\, w(t)\, t^n$

$= \int_0^{\infty} dt\, w(t) \sum_{n=0}^{\infty} (-1)^n x^n t^n$

$F(x) \equiv \int_0^{\infty} \frac{dt\, w(t)}{1+xt}$ where $w(t) \geq 0$ for $t \geq 0$ and $\int_0^{\infty} w(t) t^n dt$ exists

defined for all x in [illegible]

$$H = \underbrace{\frac{p^2}{2} + V(x)}_{H_0} + \epsilon W(x)$$

$$H = \underbrace{\frac{p^2}{2} + V(x)}_{H_0} + \epsilon W(x)$$

$$H\psi = E\psi$$

$$\left(-\frac{1}{2}\frac{d^2}{dx^2} + V(x) + \epsilon W(x)\right)\psi(x) = E\,\psi(x)$$

$$H = \frac{p^2}{2} + V(x) + \epsilon W(x)$$

$$\underbrace{\frac{p^2}{2} + V(x)}_{H_0}$$

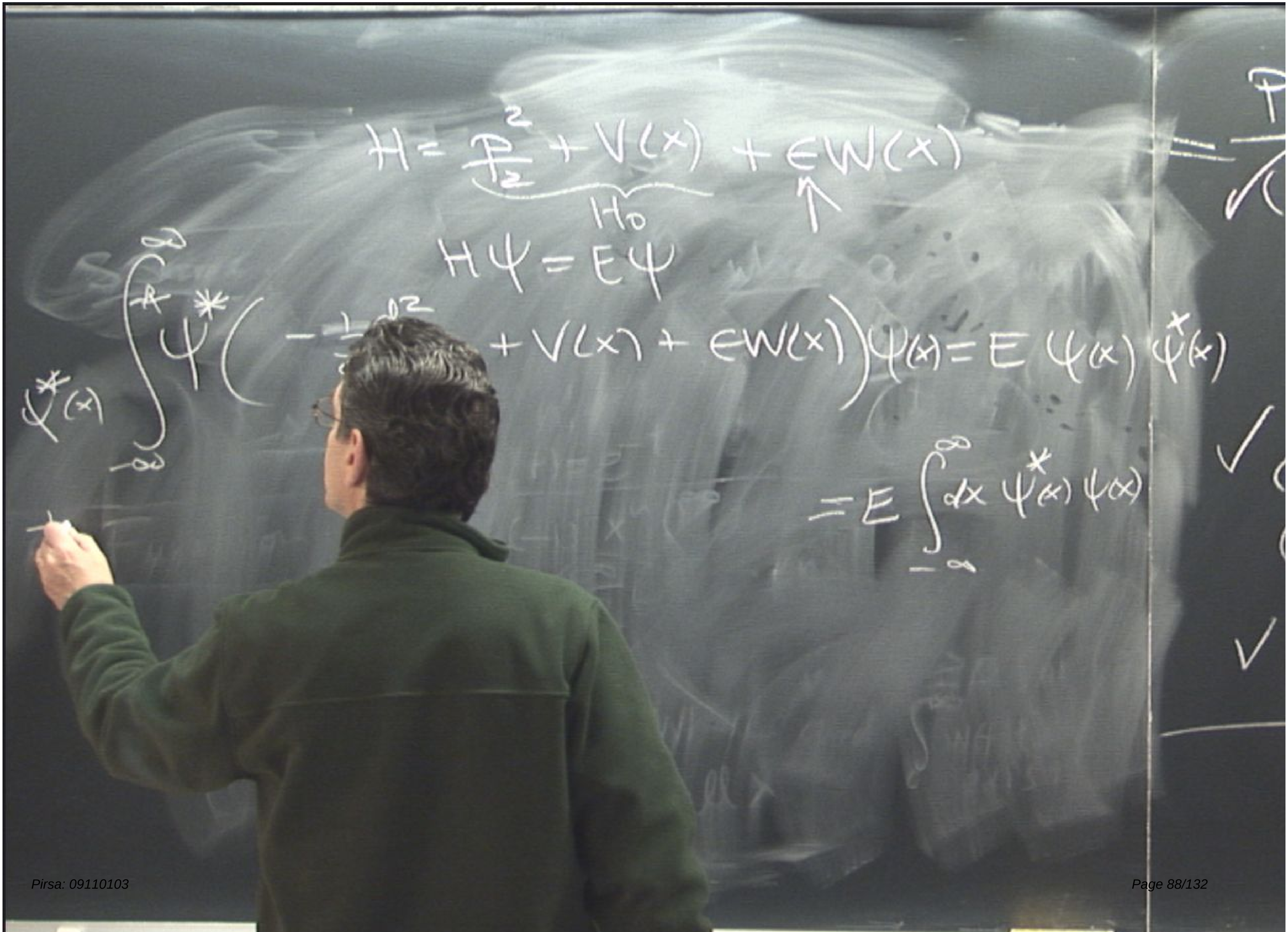
$$H\psi = E\psi$$

$$\left(-\frac{1}{2}\frac{d^2}{dx^2} + V(x) + \epsilon W(x)\right)\psi(x) = E\,\psi(x)$$

$$H = \underbrace{\frac{P^2}{2} + V(x)}_{H_0} + \epsilon W(x)$$

$$H\psi = E\psi$$

$$\psi^*(x) \quad \psi^*\left(-\frac{1}{2}\frac{d^2}{dx^2} + V(x) + \epsilon W(x)\right)\psi(x) = E\,\psi(x)\,\psi^*(x)$$



$H = \underbrace{\frac{P^2}{2} + V(x)}_{H_0} + \epsilon W(x)$
$H\psi = E\psi$
$= E \int_{-\infty}^{\infty} dx\, \psi^*(x)\, \psi(x)$

$$H = \underbrace{\frac{p^2}{2} + V(x)}_{H_0} + \epsilon W(x)$$

$$H\psi = E\psi$$

$$\psi^*(x)\int_{-\infty}^{\infty} \psi^* \left(-\frac{1}{2}\frac{d^2}{dx^2} + V(x) + \epsilon W(x) \right)\psi(x) = E\,\psi(x)\,\psi^*(x)$$

$$+\frac{1}{2}\int_{-\infty}^{\infty} dx\, \psi^{*\prime}(x)\,\psi'(x) \qquad = E\int_{-\infty}^{\infty} dx\, \psi^*(x)\,\psi(x)$$

$$H = \underbrace{\frac{P^2}{2} + V(x)}_{H_0} + \epsilon W(x)$$

$$H\psi = E\psi$$

$$\psi^*(x) \int_{-\infty}^{\infty} \psi^* \left(-\frac{1}{2}\frac{d^2}{dx^2} + V(x) + \epsilon W(x) \right) \psi(x) = E\, \psi(x)\, \psi^*(x)$$

$$+ \frac{1}{2}\int_{-\infty}^{\infty} dx\, \psi^{*\prime}(x)\, \psi'(x) + \int_{-\infty}^{\infty} V(x)\, \psi^* \psi + \epsilon \int_{-\infty}^{\infty} \psi^* \psi\, W(x) = E \int_{-\infty}^{\infty} dx\, \psi^*(x)\, \psi(x)$$

$$H = \underbrace{\frac{P^2}{2} + V(x)}_{H_0} + \epsilon W(x)$$

$$H\psi = E\psi$$

$$\psi^*(x) \int_{-\infty}^{\infty} \psi^* \left(-\frac{1}{2}\frac{d^2}{dx^2} + V(x) + \epsilon W(x) \right) \psi(x) = E\, \psi(x)\, \psi^*(x)$$

$$+\frac{1}{2}\int_{-\infty}^{\infty} dx\, \psi^{*\prime}(x)\, \psi'(x) + \int_{-\infty}^{\infty} V(x)\psi^*\psi + \epsilon \int_{-\infty}^{\infty} \psi^*\psi\, W(x) = E \boxed{\int_{-\infty}^{\infty} dx\, \psi^*(x)\, \psi(x)}$$

real

+

$$H = \frac{p^2}{2} + V(x) + \epsilon W(x)$$

$$\underbrace{\frac{p^2}{2} + V(x)}_{H_0}$$

$$H\psi = E\psi$$

$$\psi^*(x) \int_{-\infty}^{\infty} \psi^* \left(-\frac{1}{2}\frac{d^2}{dx^2} + V(x) + \epsilon W(x) \right) \psi(x) = E\,\psi(x)\,\psi^*(x)$$

$$+\frac{1}{2}\int_{-\infty}^{\infty} dx\, \psi^{*\prime}(x)\,\psi'(x) + \boxed{\int_{-\infty}^{\infty} V(x)\,\psi^*\psi} + \epsilon \boxed{\int_{-\infty}^{\infty} \psi^*\psi\, W(x)} = E \boxed{\int_{-\infty}^{\infty} dx\, \psi^*(x)\,\psi(x)}$$

real pos. $\quad x^2$ real pos. $\quad x^4$ real pos. $\quad$ real +

$$H = \underbrace{\frac{p^2}{2} + V(x)}_{H_0} + \epsilon W(x)$$

$$H\psi = E\psi$$

$$\psi^*(x) \quad \int_{-\infty}^{\infty} \psi^* \left(-\frac{1}{2}\frac{d^2}{dx^2} + V(x) + \epsilon W(x) \right) \psi(x) = E\,\psi(x)\,\psi^*(x)$$

$$+\frac{1}{2}\int_{-\infty}^{\infty} dx\, \psi^{*\prime}(x)\,\psi'(x) + \boxed{\int_{-\infty}^{\infty} V(x)\,\psi^*\psi} + \epsilon \boxed{\int_{-\infty}^{\infty} \psi^*\psi\, W(x)} = E \boxed{\int_{-\infty}^{\infty} dx\, \psi^*(x)\,\psi(x)}$$

real pos. $\quad x^2$ real pos. $\quad x^4$ real pos. $\quad$ real +

$$(\mathrm{Im}\,\epsilon)\left(\int_{-\infty}^{\infty} \psi^*\psi\, W(x)\right) = (\mathrm{Im}\,E)\left(\int \psi^*\psi\right)$$

pos

$$H = \underbrace{\frac{p^2}{2} + V(x)}_{H_0} + \epsilon W(x)$$

$$H\psi = E\psi$$

$$\psi^*(x)\int_{-\infty}^{\infty}\psi^*\left(-\frac{1}{2}\frac{d^2}{dx^2} + V(x) + \epsilon W(x)\right)\psi(x) = E\,\psi(x)\,\psi^*(x)$$

$$+\frac{1}{2}\int_{-\infty}^{\infty}dx\,\psi^{*\prime}(x)\,\psi'(x) + \boxed{\int_{-\infty}^{\infty}V(x)\,\psi^*\psi} + \epsilon\boxed{\int_{-\infty}^{\infty}\psi^*\psi\,W(x)} = E\boxed{\int_{-\infty}^{\infty}dx\,\psi^*(x)\,\psi(x)}$$

real pos. x^2 real pos. x^4 real pos. real +

$$(\mathrm{Im}\,\epsilon)\left(\int_{-\infty}^{\infty}\psi^*\psi\,W(x)\right) = (\mathrm{Im}\,E)\left(\int\psi^*\psi\right)$$

pos

E(ϵ) is Herglotz!

$$H = \underbrace{\frac{p^2}{2} + V(x)}_{H_0} + \epsilon W(x)$$

$$H\psi = E\psi$$

$$\psi^*(x)\int_{-\infty}^{\infty} \psi^*\left(-\frac{1}{2}\frac{d^2}{dx^2} + V(x) + \epsilon W(x)\right)\psi(x) = E\,\psi(x)\,\psi^*(x)$$

$$+\frac{1}{2}\int_{-\infty}^{\infty} dx\, \psi^{*\prime}(x)\,\psi'(x) + \boxed{\int_{-\infty} V(x)\,\psi^*\psi} + \epsilon\boxed{\int_{-\infty}^{\infty}\psi^*\psi\, W(x)} = E\boxed{\int_{-\infty}^{\infty} dx\, \psi^*(x)\,\psi(x)}$$

real pos.

x^2 real pos.

real x^4 pos.

real +

$$(\mathrm{Im}\,\epsilon)\left(\int_{-\infty}^{\infty}\psi^*\psi\, W(x)\right) = (\mathrm{Im}\,E)\left(\int \psi^*\psi\right)$$

pos

$\underline{E(\epsilon) \text{ is Herglotz!}}$

Properties of S. Fns.

$$H = \frac{p^2}{2} + \frac{x^2}{2} + \epsilon x^4$$

✓ (1) as $|x| \to \infty$ in the cut plane, $F(x) \to 0$.

$\psi(x)$

✓ (2) $F(x)$ is <u>ANALYTIC</u> in [cut plane] $\quad F'(x) = -\int_0^{\infty} \frac{t\,d\ldots}{(1+\ldots}$

(3) $F(x) \sim \sum_{n=0}^{\infty} a_n x^n (-1)^n$ as $|x| \to 0$, $|\arg| < \pi$

✓ (4) $-F(x)$ is <u>Herglotz</u> $\quad \mathrm{Im}(-F(x))$ same sign as I...

Properties of S. Fns.

$$H = \frac{p^2}{2} + \frac{x^2}{2} + \epsilon x^4$$

$$E(\epsilon) = \sum a_n \epsilon^n$$

✓ (1) as $|x| \to \infty$, $F(x) \to 0$.
in the cut plane.

$\psi^*(x)$

✓ (2) $F(x)$ is ANALYTIC in [cut plane] $\quad F'(x) = -\int_0^\infty \frac{t\,dt\, w(t)}{(1+xt)^2}$

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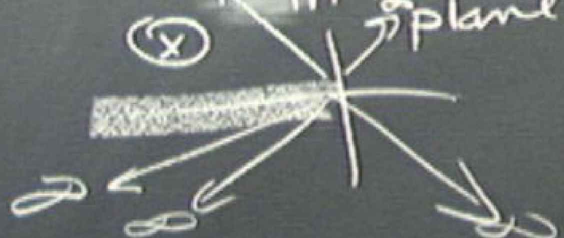
✓ (4) $-F(x)$ is Herglotz $\quad \mathrm{Im}(-F(x))$ same sign as $\mathrm{Im}(x)$.

$$H = \frac{p^2}{2} + \frac{x^2}{2} + \epsilon x^4$$

$$\underline{E(\epsilon)} = \sum a_n \epsilon^n$$

Properties of S. Fns.

(1) as $|x| \to \infty$, $F(x) \to 0$.
in the cut plane.



$\overset{*}{\psi}(x)$

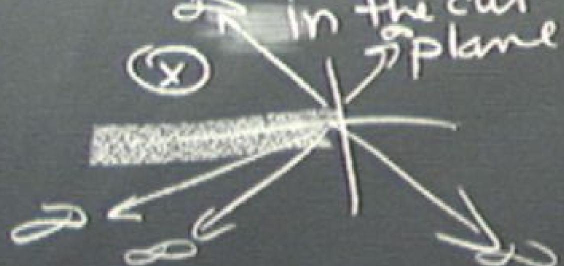
(2) $F(x)$ is ANALYTIC in  $F'(x) = -\int_0^\infty \frac{t\,dt\, W(t)}{(1+xt)^2}$

(3) $F(x) \sim \sum_{n=0}^{\infty} a_n x^n (-1)^n$ as $|x| \to 0$, $|\arg| < \pi$

(4) $-F(x)$ is Herglotz $\quad \mathrm{Im}(-F(x))$ same sign as $\mathrm{Im}(x)$.

Properties of S. Fns.

✓(1) as $|x| \to \infty$, $F(x) \to 0$.
in the cut plane.



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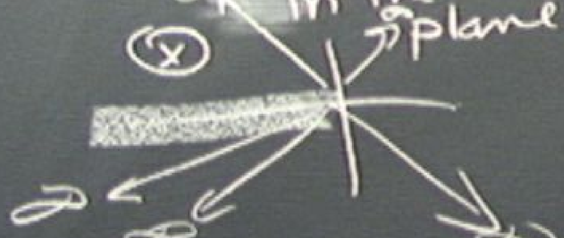
$H = \frac{p^2}{2} + \frac{x^2}{2} + \epsilon x^4$

$E(\epsilon) = \sum a_n \epsilon^n$

Herg.
Anal.
Asymp

Properties of S. Fns.

$\checkmark$ (1) as $|x| \to \infty$, $F(x) \to 0$.
∞ in the cut plane.



$\psi^*(x)$

$\checkmark$ (2) $F(x)$ is ANALYTIC in [cut plane] $\quad F'(x) = -\int_0^\infty \frac{t\,dt\, w(t)}{(1+xt)^2}$

(3) $F(x) \sim \sum_{n=0}^{\infty} a_n x^n (-1)^n$ as $|x| \to 0$, $|\arg| < \pi$

$\checkmark$ (4) $-F(x)$ is Herglotz $\quad \mathrm{Im}(-F(x))$ same sign as $\mathrm{Im}(x)$.

$H = \frac{p^2}{2} + \frac{x^2}{2} + \epsilon x^4$

$E(\epsilon) = \sum a_n \epsilon^n$

Herg.
Anal.
Asymp

$\frac{E(\epsilon)}{\epsilon} \to 0$ as $|\epsilon| \to \infty$

Properties of S. Fns.

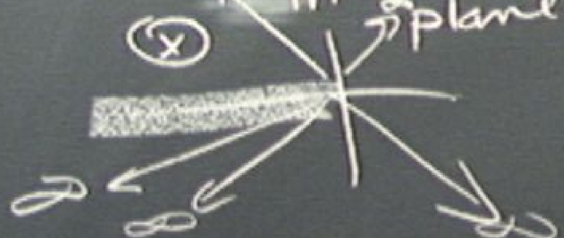
$H = \frac{p^2}{2} + \frac{x^2}{2} + \epsilon x^4$

$\boxed{E(\epsilon)} = \sum a_n \epsilon^n$

Herg.
Anal.
Asymp

$\frac{E(\epsilon)}{\epsilon} \to 0$ as $|\epsilon| \to \infty$

✓ (1) as $|x| \to \infty$, $F(x) \to 0$.
in the cut plane.



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$$H = \underbrace{\frac{p^2}{2} + V(x)}_{H_0} + \epsilon W(x)$$

$$H\Psi = E\Psi$$

Stieltjes :

$$H = \underbrace{\frac{p^2}{2} + V(x)}_{H_0} + \epsilon W(x)$$

$$H\psi = E\psi$$

Stieltjes:

The Padés for a St. series

$P^N_N(x) \longrightarrow L$ as $N \to \infty$

$P^N_{N+1}(x) \longrightarrow M$

$$H = \underbrace{\frac{p^2}{2} + V(x)}_{H_0} + \epsilon W(x)$$

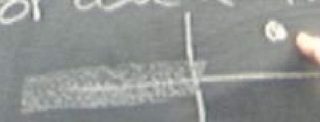
$$H\psi = E\psi$$

<u>Stieltjes</u>:

The Padés for a St. series

$P^N_N(x) \longrightarrow L_1$ as $N \to \infty$

$P^N_{N+1}(x) \longrightarrow L_2$ " for all x in



$$H = \underbrace{\frac{p^2}{2} + V(x)}_{H_0} + \epsilon W(x)$$

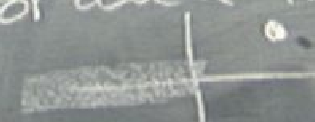
$$H\psi = E\psi$$

<u>Stieltjes:</u>

The Padés for a St. series

$P^N_N(x) \longrightarrow L_1$ as $N \to \infty$

$P^N_{N+1}(x) \longrightarrow L_2$ " for all x in



$$H = \underbrace{\frac{p^2}{2} + V(x)}_{H_0} + \epsilon W(x)$$

$$H\psi = E\psi$$

Stieltjes:

The Padés for a St. series

$P^N_N(x) \longrightarrow L_1$ as $N \to \infty$

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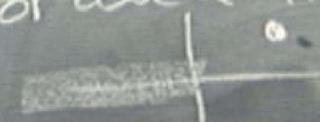


For real pos x $P^N_N(x) \downarrow$

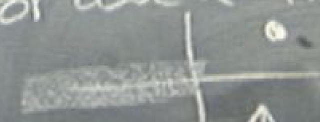
$$H = \underbrace{\frac{p^2}{2} + V(x)}_{H_0} + \epsilon W(x)$$

$$H\psi = E\psi$$

Stieltjes:

The Padés for a St. series

$P^N_N(x) \longrightarrow L_1$ as $N \to \infty$ for all x in



$P^N_{N+1}(x) \longrightarrow L_2$ "

For real pos x $P^N_N(x) \downarrow$, $P^N_{N+1}(x) \uparrow$

$<$ $<$ ...... $P^2_2 < P^1_1 < P^0_0$

$$H = \underbrace{\frac{p^2}{2} + V(x)}_{H_0} + \epsilon W(x)$$

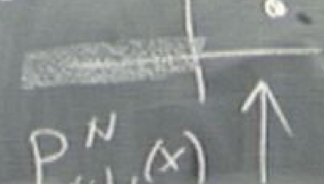
$$H\psi = E\psi$$

Stieltjes:

The Padés for a St. series

$P^N_N(x) \longrightarrow L_1$ as $N \to \infty$ for all x in

$P^N_{N+1}(x) \longrightarrow L_2$ "



For real pos x $P^N_N(x) \downarrow$, $P^N_{N+1}(x) \uparrow$

$$< \quad < L_1 < \dots < P^2_2 < P^1_1 < P^0_0$$

$$H = \underbrace{\frac{p^2}{2} + V(x)}_{H_0} + \epsilon W(x)$$

$$H\psi = E\psi$$

Stieltjes:

The Padés for a St. series

$P^N_N(x) \longrightarrow L_1$ as $N \to \infty$

$P^N_{N+1}(x) \longrightarrow L_2$ " for all x in

For real pos x $P^N_N(x)\downarrow$, $P^N_{N+1}(x)\uparrow$

$$P_1^0 < P_2^1 < P_3^2 \cdots \to L_2 \le F(x) < L_1 < \cdots < P_2^2 < P_1^1 < P_0^0$$

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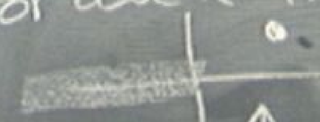

$P^N_{N+1}(x) \longrightarrow L_2$ "

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$$H = \underbrace{\frac{p^2}{2} + V(x)}_{H_0} + \epsilon W(x)$$

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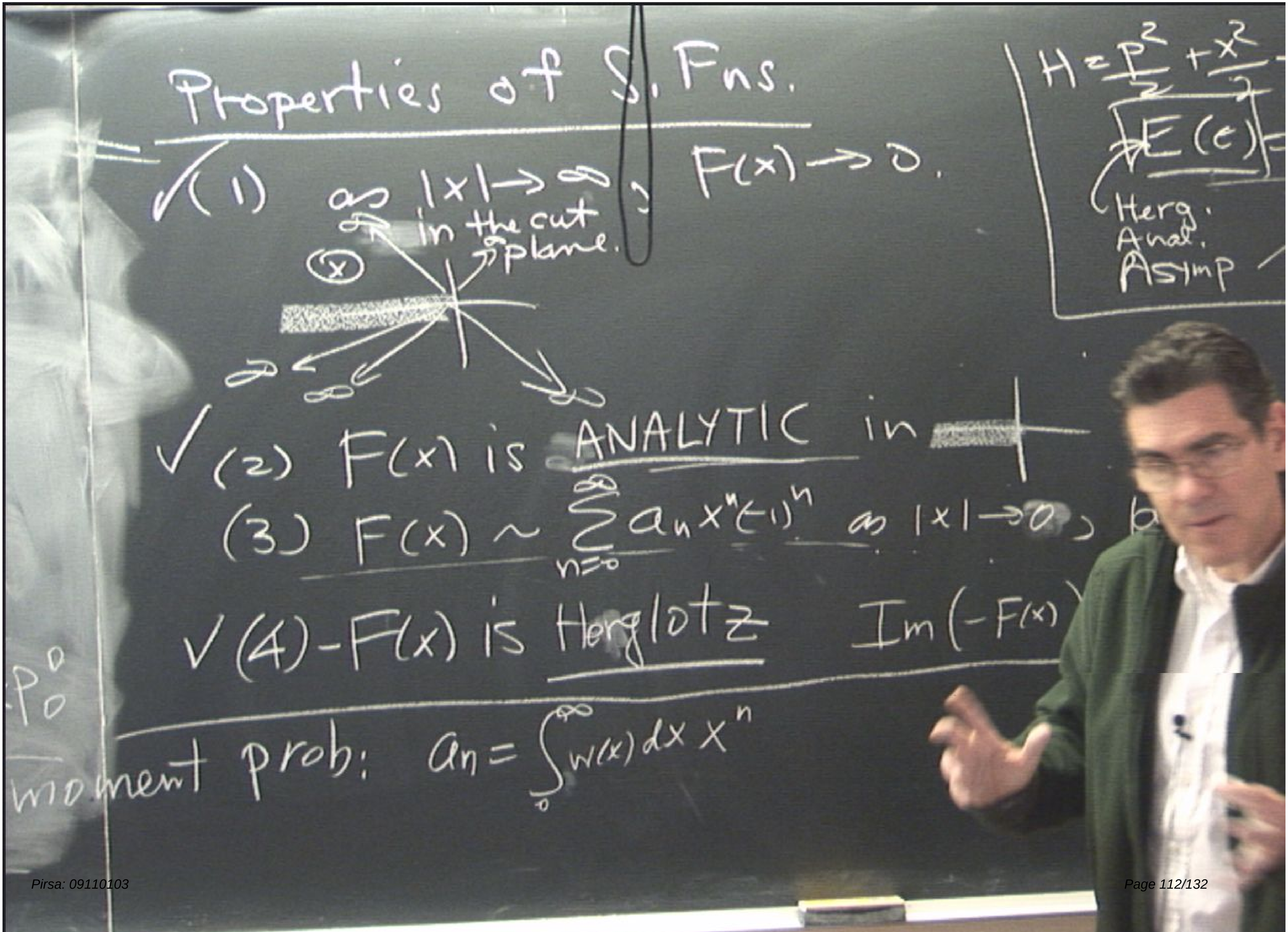
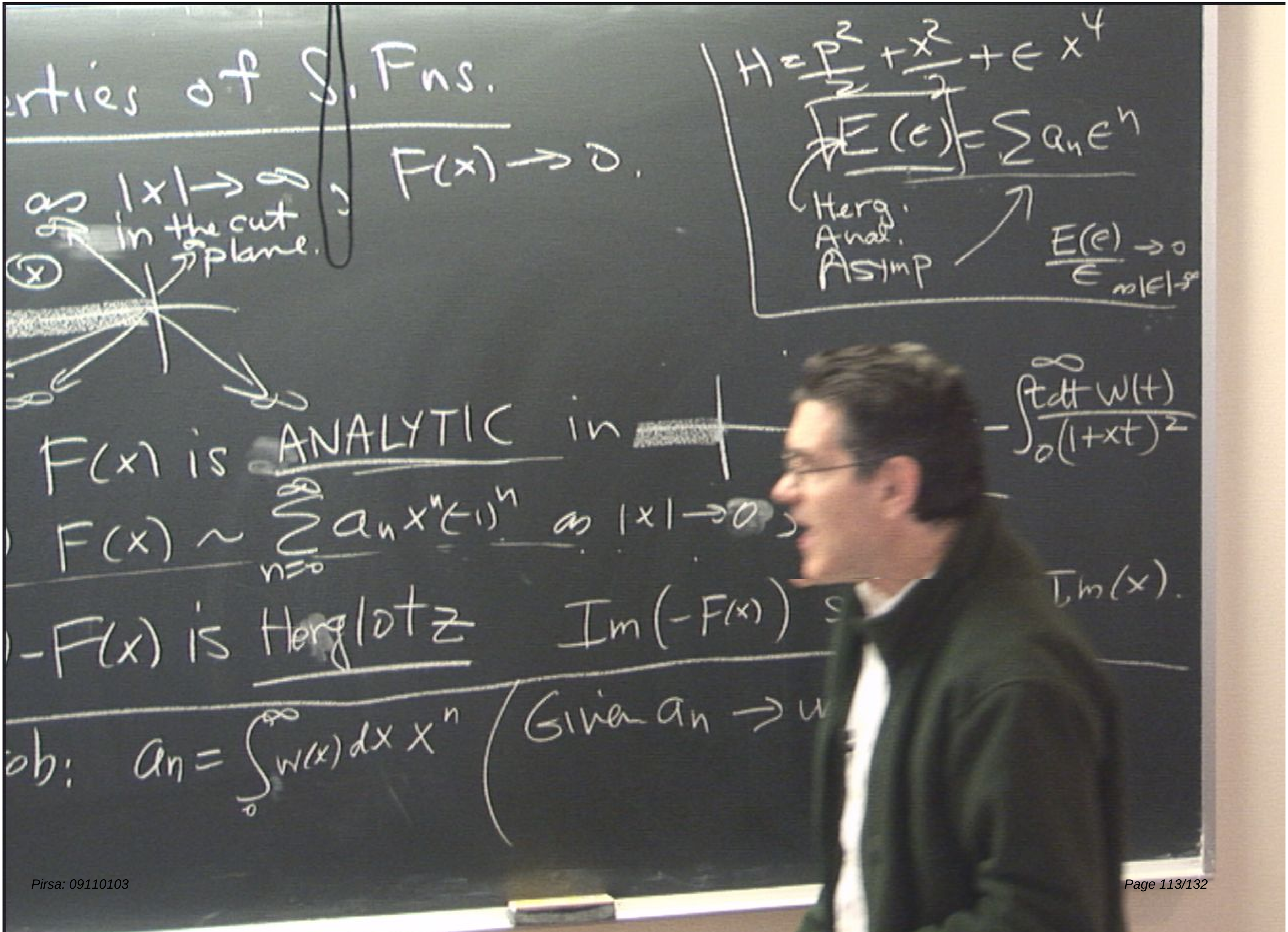
$$P^0_1 < P^1_2 < P^2_3 \cdots \to L_2 \leq F(x) < L_1 < \cdots < P^2_2 < P^1_1 < P^0_0$$

(5) if there is a unique soln to momen


Properties of S. Fns.
(1) as |x|→∞ , F(x)→0.
in the cut plane.
(2) F(x) is ANALYTIC in
(3) F(x) ~ Σ_{n=0}^∞ a_n x^n (-1)^n as |x|→0 ,
(4) -F(x) is Herglotz Im(-F(x)
moment prob: a_n = ∫_0^∞ w(x) dx x^n
H = p²/2 + x²/2
E(ε)
Herg.
Anal.
Asymp


erties of S. Fns.
F(x) → 0.
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F(x) is ANALYTIC in
F(x) ~ Σ a_n x^n (-1)^n as |x| → 0
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Im(x).
ob: a_n = ∫ w(x) dx x^n
Given a_n → w
H = p²/2 + x²/2 + ε x⁴
E(ε) = Σ a_n ε^n
Herg. Anal. Asymp
E(ε)/ε → 0
-∫ t dt w(t)/(1+xt)²

...erties of S. Fns.

as $|x| \to \infty$, $F(x) \to 0$.
∞ in the cut plane.

$$H = \frac{p^2}{2} + \frac{x^2}{2} + \epsilon x^4$$

$$\boxed{E(\epsilon)} = \sum a_n \epsilon^n$$

Herg.
Anal.
Asymp

$$\frac{E(\epsilon)}{\epsilon} \to 0 \quad \text{as } |\epsilon| \to \infty$$

...) $F(x)$ is ANALYTIC in

$$F'(x) = -\int_0^\infty \frac{t\,dt\, W(t)}{(1+xt)^2}$$

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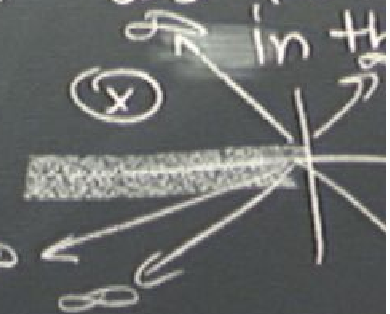
if there is a unique soln to moment prob: $a_n =$

then $L_1 = L_2 = F(x)$

Properties o

(1) as $|x|$ in th



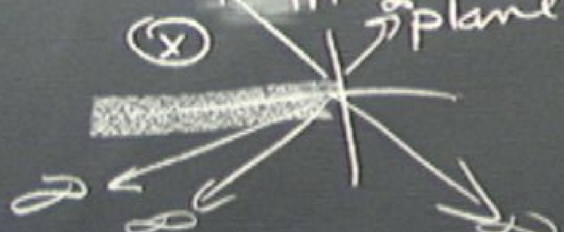
(2) $F(x)$ i

(3) $F(x)$

(4) $-F(x)$ is

Properties of S. Fns.

✓ (1) as $|x| \to \infty$, $F(x) \to 0$.
as in the cut plane.



✓ (2) $F(x)$ is ANALYTIC in [cut plane]

$$F'(x) = -\int_0^\infty \frac{t\,dt\, W(t)}{(1+xt)^2}$$

(3) $F(x) \sim \sum_{n=0}^{\infty} a_n x^n (-1)^n$ as $|x| \to 0$, $|\arg| < \pi$

✓ (4) $-F(x)$ is Herglotz $\quad \mathrm{Im}(-F(x))$ same sign as $\mathrm{Im}(x)$.

$$H = \frac{p^2}{2} + \frac{x^2}{2} + \epsilon x^4$$

$$\boxed{E(\epsilon)} = \sum a_n \epsilon^n$$

Herg.
Anal.
Asymp

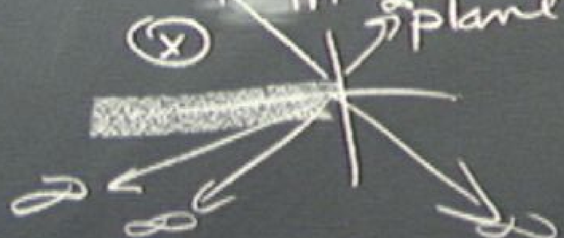
$$\frac{E(\epsilon)}{\epsilon} \to 0 \quad \text{as } |\epsilon| \to \infty$$

Moment prob: $a_n = \int_0^\infty W(x)\,dx\, x^n$ (Given $a_n \to$ unique $W(x)$)

Carleman: $a_n \lesssim (2n)!$

Properties of S. Fns.

✓ (1) as $|x| \to \infty$ in the cut plane, $F(x) \to 0$.



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Herg.
Anal.
Asymp

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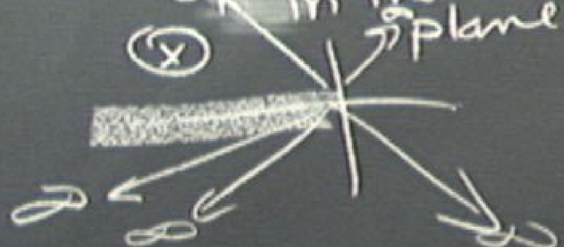
Properties of S. Fns.

✓(1) as $|x| \to \infty$, $F(x) \to 0$. in the cut plane.



✓(2) $F(x)$ is ANALYTIC in $\quad F'(x) = -\int_0^\infty \frac{t\,dt\, W(t)}{(1+xt)^2}$

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Herg.
Anal.
Asymp

$\frac{E(\epsilon)}{\epsilon} \to 0$ as $|\epsilon| \to \infty$

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Herg.
Anal.
Asymp

$\frac{E(\epsilon)}{\epsilon} \to 0$ as $|\epsilon| \to \infty$

...ment prob: $a_n = \int_0^\infty w(x)\,dx\, x^n$ / Given $a_n \to$ unique W...

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Properties of S. Fns.

$$H = \frac{p^2}{2} + \frac{x^2}{2} + \epsilon x^4$$

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Herg.
Anal.
Asymp

$$\frac{E(\epsilon)}{\epsilon} \to 0 \quad \text{as } |\epsilon| \to \infty$$

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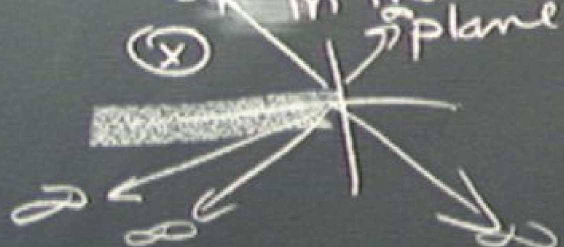

✓ (2) $F(x)$ is ANALYTIC in [cut plane] $\qquad F'(x) = -\int_0^\infty \frac{t\,dt\, W(t)}{(1+xt)^2}$

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$\checkmark$ (4) $-F(x)$ is Herglotz $\mathrm{Im}(-F(x))$ same sign as

moment prob: $a_n = \int_0^\infty w(x)\,dx\, x^n$ Given $a_n \to$ un

Carleman: $a_n \lesssim (2n)!$

$H = \frac{p^2}{2} + \frac{x^2}{2} + \epsilon$

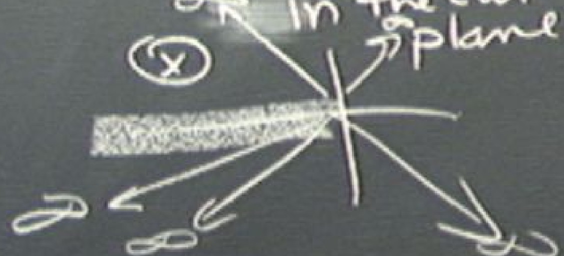
$\boxed{E(\epsilon)} = \sum$

Herg.
Anal.
Asymp

$F'(x) = -\int$

Properties of S. Fns.

✓(1) as $|x| \to \infty$, $F(x) \to 0$. in the cut plane.



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$\boxed{E(\epsilon)} = \sum a_n \epsilon^n$

Herg.
Anal.
Asymp

$\frac{E(\epsilon)}{\epsilon} \to 0$ as $|\epsilon| \to \infty$

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$$H = \frac{p^2}{2} + \frac{x^2}{2} + \epsilon x^4$$

$$\boxed{E(\epsilon)} = \sum a_n \epsilon^n$$

Herg.
Anal.
Asymp

$$\frac{E(\epsilon)}{\epsilon} \to 0 \quad {}_{\text{as } |\epsilon| \to \infty}$$

∞

$$H = \frac{p^2}{2} + \frac{x^2}{2} + \epsilon x^{\cancel{4}\,6}$$

$(2n)!$

$$\boxed{E(\epsilon)} = \sum a_n \epsilon^n$$

Herg.

Anal.

Asymp

$$\frac{E(\epsilon)}{\epsilon} \to 0 \text{ as } |\epsilon| \to \infty$$

$$H = \frac{p^2}{2} + \frac{x^2}{2} + \epsilon x^{\cancel{4}\,\cancel{6}\,8}$$

$\cancel{(2n)!}$ $(3n)!$

$$\boxed{E(\epsilon)} = \sum a_n \epsilon^n$$

Herg:

Anal:

Asymp

$$\frac{E(\epsilon)}{\epsilon} \to 0 \quad \text{as } |\epsilon| \to \infty$$

0.

~~6~~ 8

Fns.

$F(x) \to 0.$

P_N^N . . . . L_1

P_{N+1}^N . . . . L_2

...LYTIC in

$$H = \frac{p^2}{2} + \frac{x^2}{2} + \epsilon x^{\cancel{4}}$$

$$\boxed{E(\epsilon)} = \sum a_n \epsilon^n$$

Herg.
Anal.
Asymp

$$\frac{E(\epsilon)}{\epsilon}$$

$$F'(x) = -\int_0^\infty \frac{t\,dt\,W}{(1+xt}$$

Fns.

$F(x) \to 0.$

$P_N^N \cdots\cdots L_1$

$P_{N+1}^N \cdots\cdots L_2$

...LYTIC in

$$H = \frac{p^2}{2} + \frac{x^2}{2} + \epsilon x^4$$

$$\boxed{E(\epsilon)} = \sum a_n \epsilon^n$$

Herg.
Anal.
Asymp

$$\frac{E(\epsilon)}{\epsilon}$$

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Fns.

$F(x) \to 0.$

P^N_N . . . . L_1

P^N_{N+1} . . . . L_2

...LYTIC in

$6\ 8$

$$H = \frac{p^2}{2} + \frac{x^2}{2} + \epsilon x^4$$

$(3n)$ ~~(2n)!~~

$$\boxed{E(\epsilon)} = \sum a_n \epsilon^n$$

Herg.
Anal.
Asymp

$$\frac{E(\epsilon)}{\epsilon}$$

$$F'(x) = -\int_0^\infty \frac{t\,dt\, W(\ldots)}{(1+xt\ldots}$$

68

$H = \frac{p^2}{2} + \frac{x^2}{2} + \epsilon x^4$

$\boxed{E(\epsilon)} = \sum a_n \epsilon^n$, $a_n \sim \frac{(3n)!}{}$

Herg. Anal. Asymp

$\frac{E(\epsilon)}{\epsilon} \to 0$ as $|\epsilon| \to \infty$

Properties of S. Fns.

✓ (1) as $|x| \to \infty$ in the cut plane, $F(x) \to 0$.

P^N_N L_1 L_2 P^N_{N+1}

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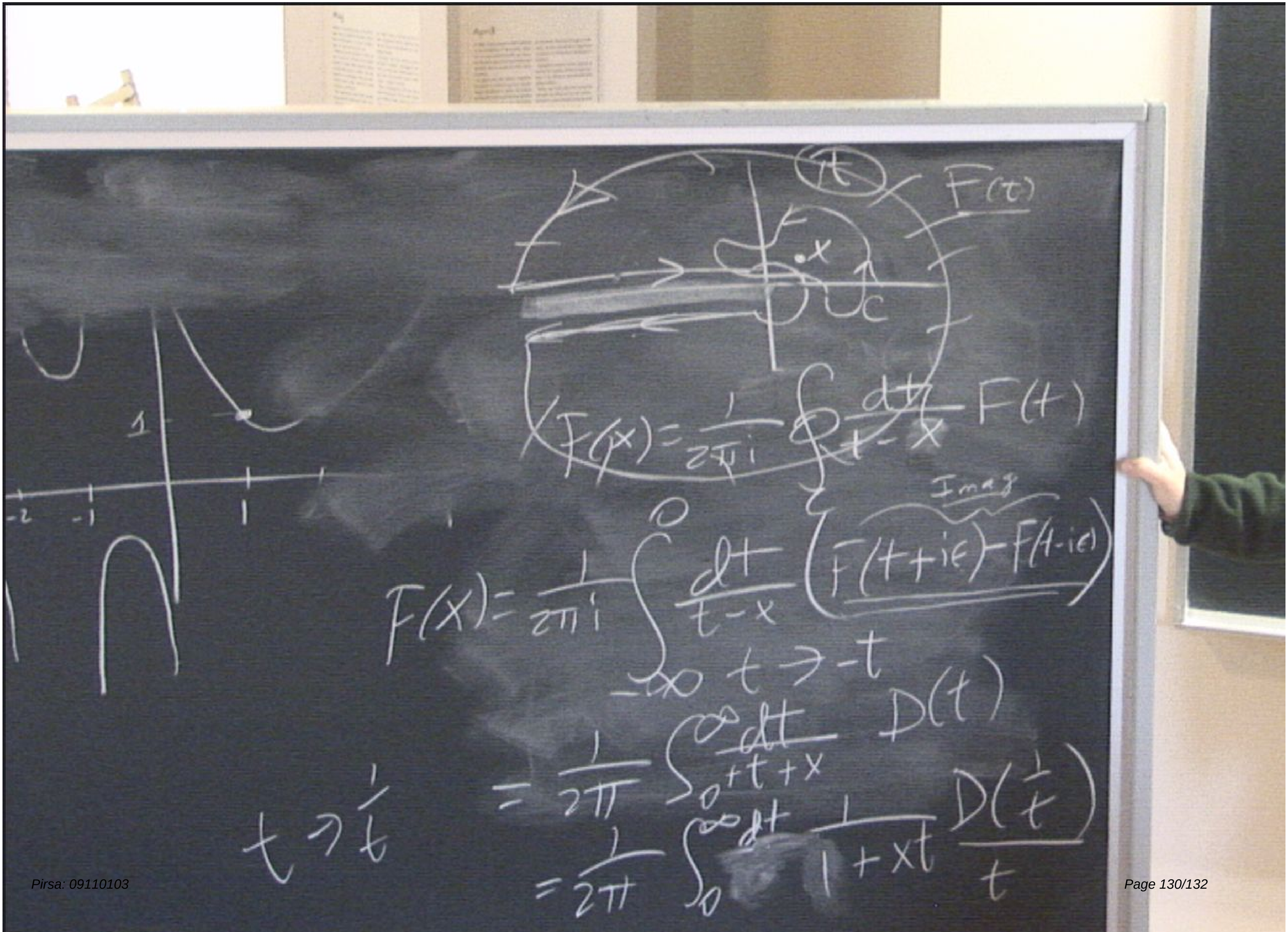
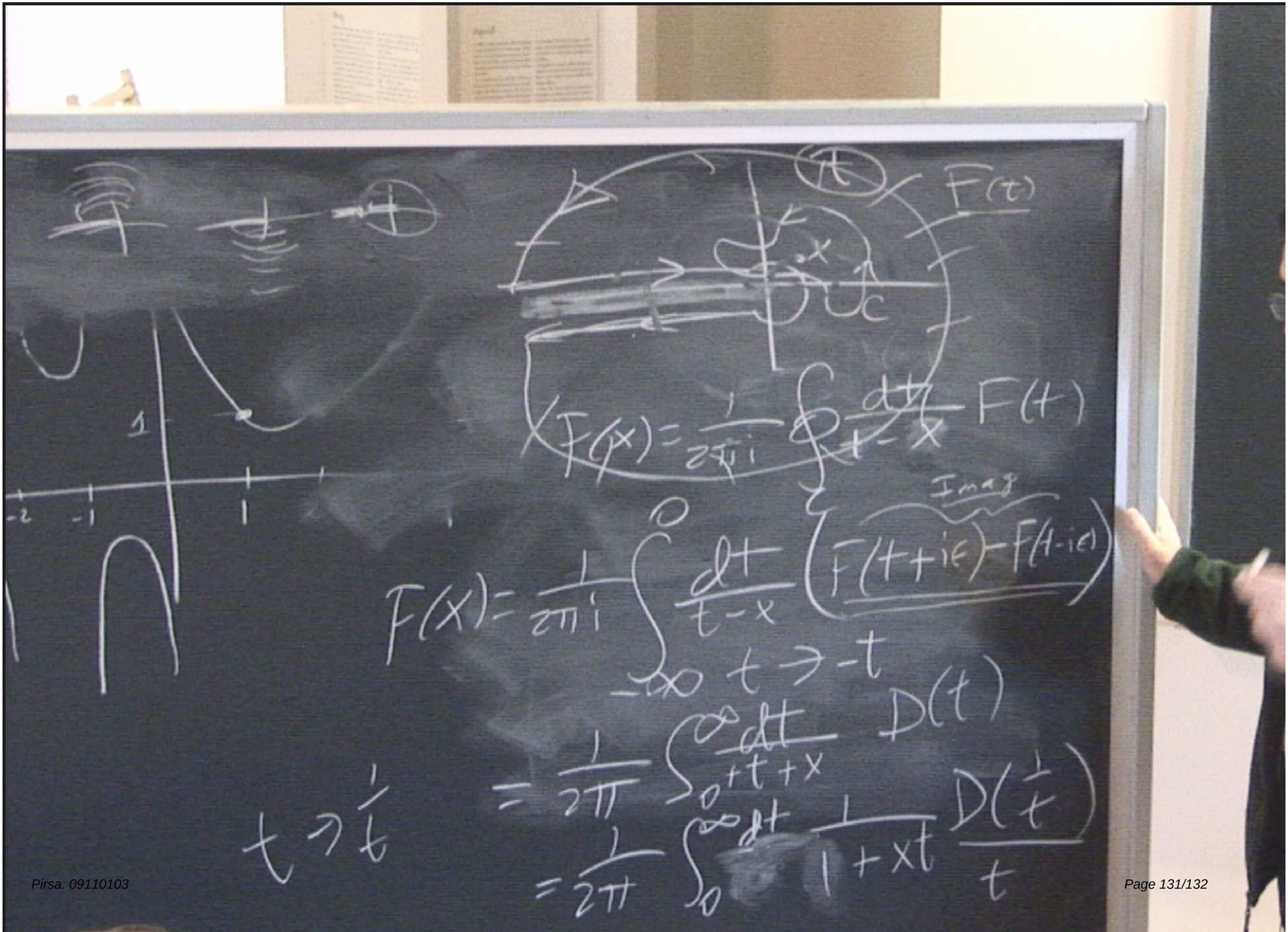
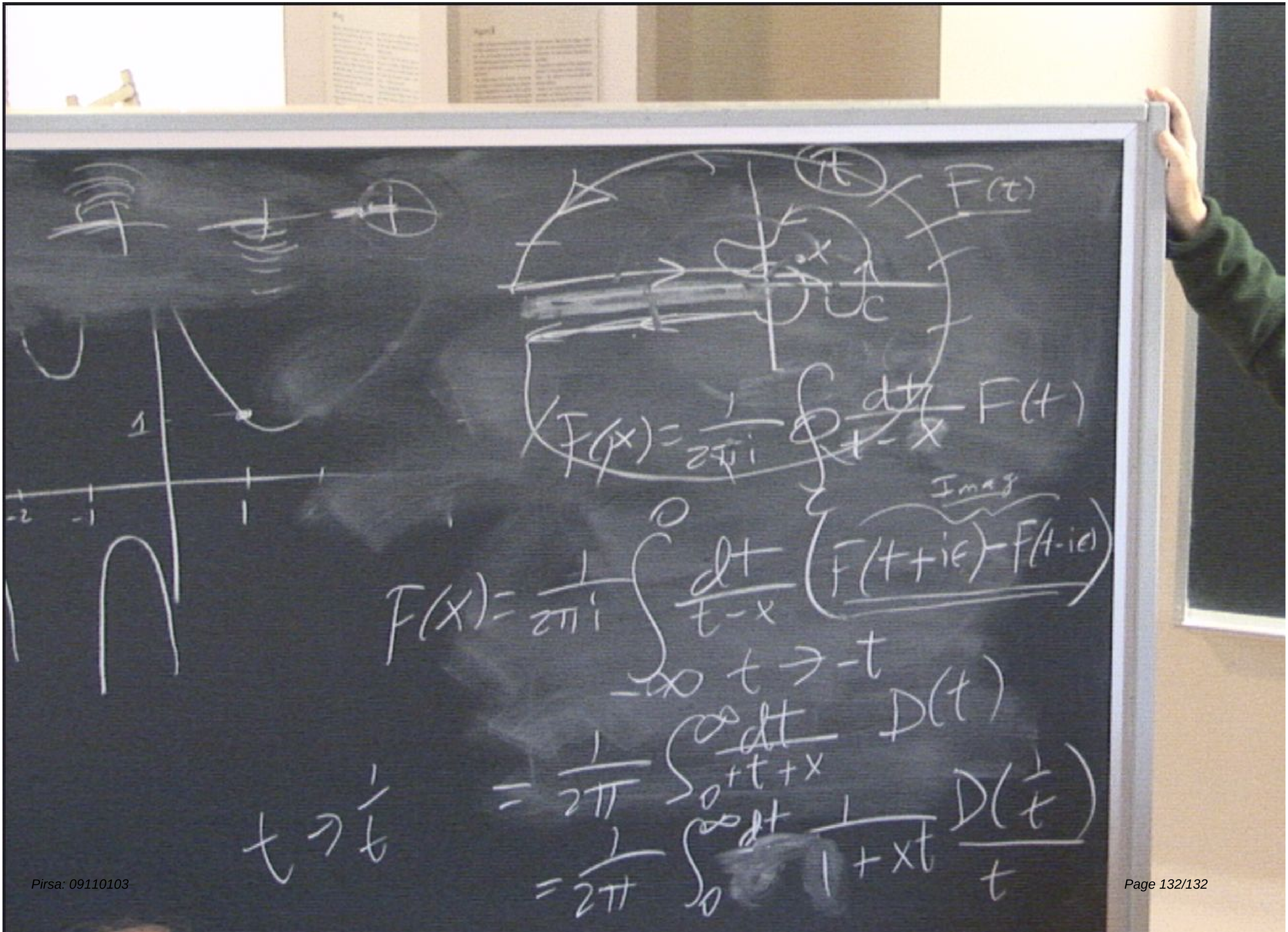


$F(\tau)$
$F(x) = \frac{1}{2\pi i} \oint \frac{dt}{t-x} F(t)$
Imag
$F(x) = \frac{1}{2\pi i} \int_{-\infty}^{0} \frac{dt}{t-x} \left(F(t+i\epsilon) - F(t-i\epsilon) \right)$
$t \to -t$
$= \frac{1}{2\pi} \int_0^\infty \frac{dt}{t+x} D(t)$
$t \to \frac{1}{t}$
$= \frac{1}{2\pi} \int_0^\infty dt \frac{1}{1+xt} \frac{D(\frac{1}{t})}{t}$


F(t)
F(x) = 1/2πi ∮ dt/(t−x) F(t)
Imag
F(x) = 1/2πi ∫ dt/(t−x) (F(t+iε) − F(t−iε))
t → −t
D(t)
= 1/2π ∫ dt/(t+x) D(t)
t → 1/t
= 1/2π ∫ dt 1/(1+xt) D(1/t)/t


F(t)
c
F(x) = 1/2πi ∮ dt/(t − x) F(t)
Imag
F(x) = 1/2πi ∫_{-∞}^{0} dt/(t − x) (F(t+iε) − F(t−iε))
t → −t
D(t)
= 1/2π ∫_0^∞ dt/(t + x) D(t)
t → 1/t
= 1/2π ∫_0^∞ dt 1/(1 + xt) D(1/t)/t
1
-2
-1
1