Title: Mathematical Physics (PHYS 624) - Lecture 6

Date: Nov 23, 2009 09:00 AM

URL: http://pirsa.org/09110100

Abstract:

Summing divergent series

Some examples

$$1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 + \dots$$

$$1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + \dots$$

$$1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + \dots$$

$$1 + 0 - 1 + 1 + 0 - 1 + 1 + 0 - 1 + \dots$$

Divergent series are not bad! They are useful, and they converge faster than convergent series!

$$\sum_{n=0}^{\infty} a_n$$

$$S_N = \sum_{n=0}^{N} a_n$$

$$S_N \to S \text{ as } N \to \infty.$$

Some examples

$$1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 + \ldots$$

$$1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + \ldots$$

$$1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + \ldots$$

$$1 + 0 - 1 + 1 + 0 - 1 + 1 + 0 - 1 + \ldots$$

Divergent series are not bad! They are useful, and they converge faster than convergent series!

A standard example: the Anharmonic Oscillator

$$-\tfrac{1}{2}\psi''(x) + \tfrac{1}{2}x^2\psi(x) + \epsilon x^4\psi(x) = E(\epsilon)\psi(x)$$

$$E(\epsilon) \sim \tfrac{1}{2} + \sum_{n=1}^{\infty} a_n \epsilon^n$$

$$a_1 = \tfrac{3}{4}$$

$$a_2 = -\tfrac{21}{8}$$

$a_3 = \frac{333}{\ldots}$

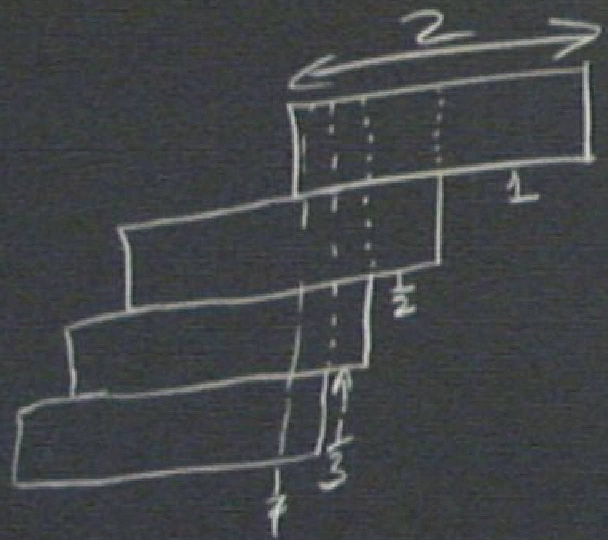
Some series are REALLY divergent --- and this is good!

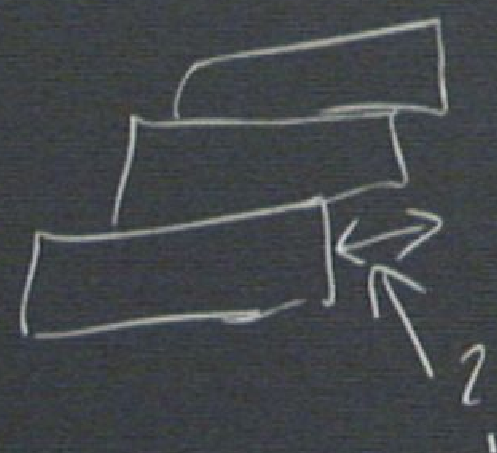
$$1 + \tfrac{1}{2} + 1/3 + \tfrac{1}{4} + 1/5 + \ldots = \zeta\,(1)$$

$$\sum_{n=0}^{\infty} a_n$$

$$S_N = \sum_{n=0}^{N} a_n$$

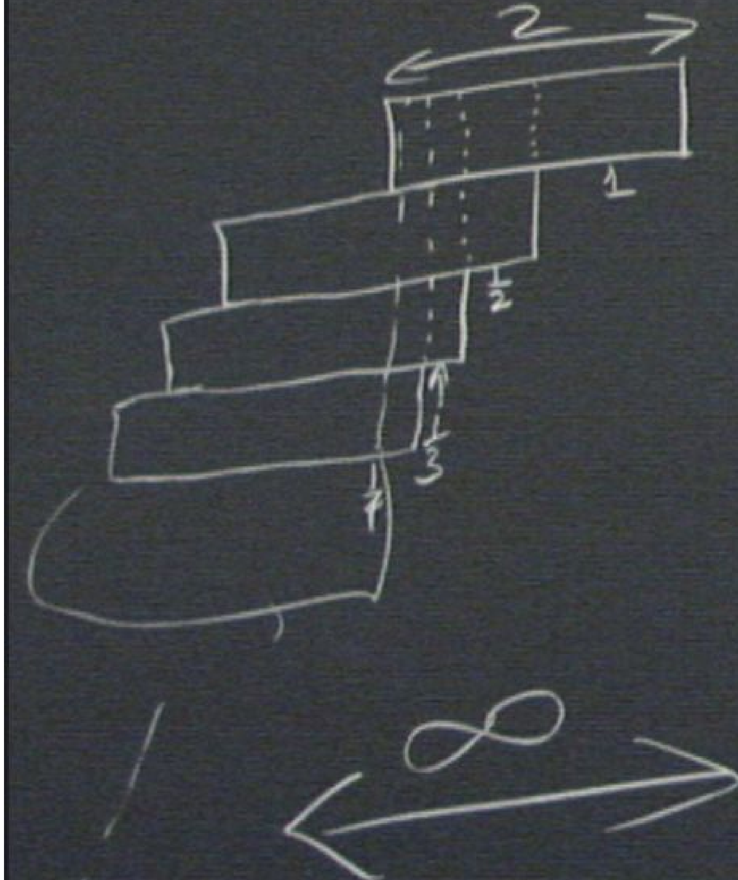
$S_N \to S$ as $N \to \infty$.

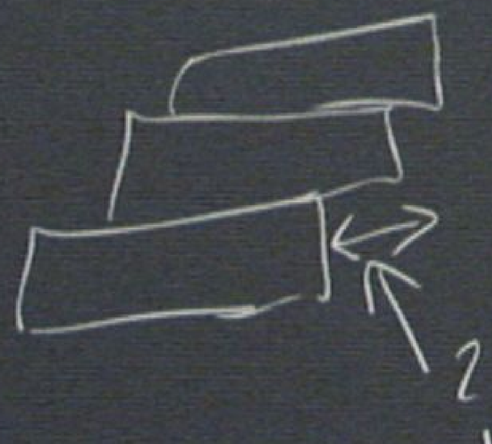




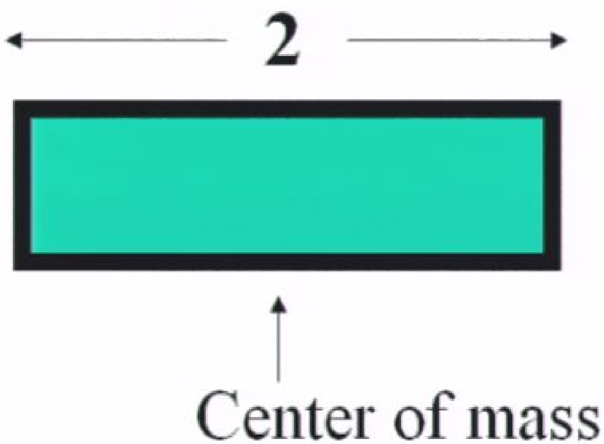
$$\sum_{n=0}^{\infty} a_n \qquad S_N = \sum_{n=0}^{N} a_n$$

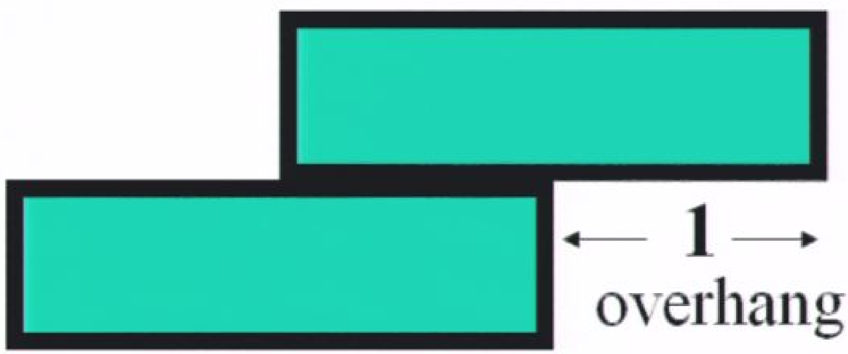
$$S_N \to S \text{ as } N \to \infty.$$





Building a bridge with bricks





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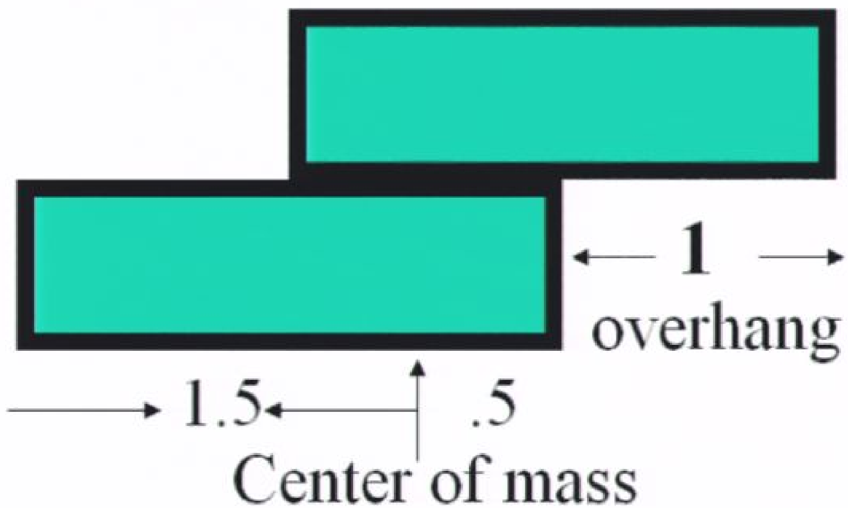
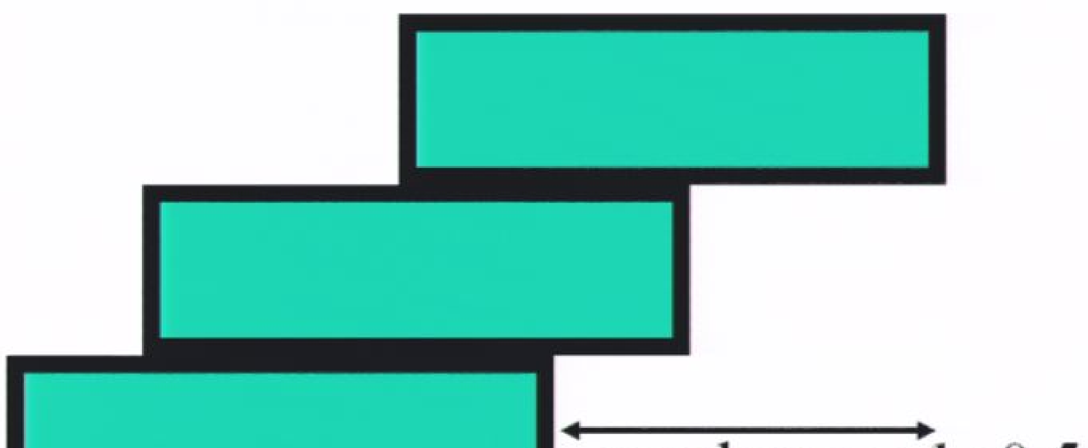
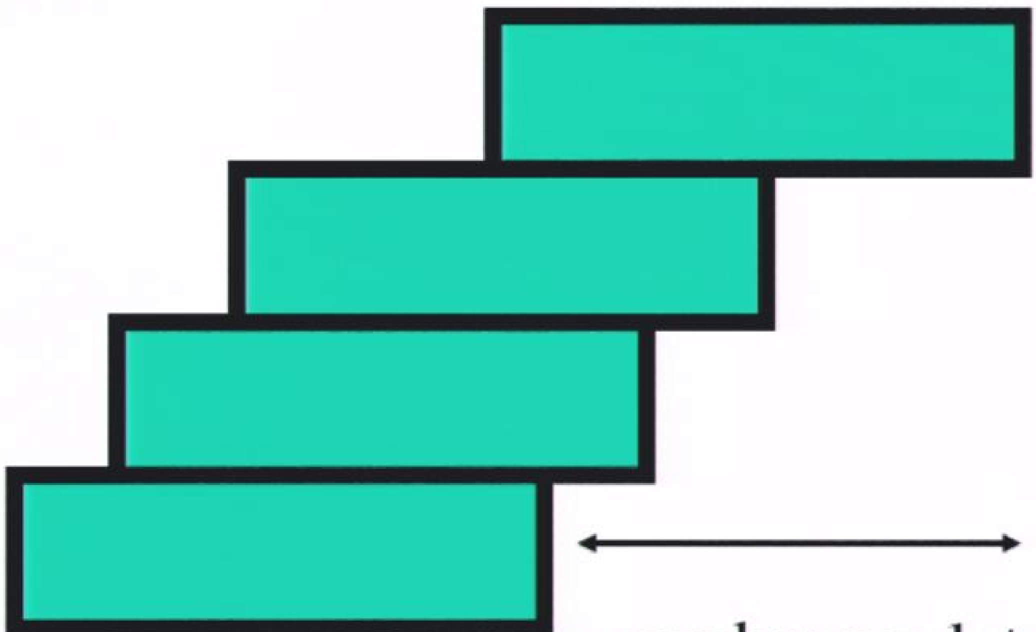




Building a bridge with bricks



$$\text{overhang} = 1 + \frac{1}{2} + \frac{1}{3}$$

Techniques for summing series

- Note: addition is not infinitely commutative
- Euler summation
- Borel summation
- Generic summation methods
- Note: addition is not infinitely associative
- Zeta summation
- Continued functions

$N \to \infty$.

C: $a+b=b+a$

A: $a+(b+c)=(a+b)+c$

D: $\underline{a(b+c)=ab+ac}$

Linear

Techniques for summing series

- Note: addition is not infinitely commutative
- Euler summation
- Borel summation
- Generic summation methods
- Note: addition is not infinitely associative
- Zeta summation
- Continued functions

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots = \ln 2$$

$S_1 = 1$

$S_2 = 1 - \frac{1}{2}$

$S_3 = 1 - \frac{1}{2} + \frac{1}{3}$

$\vdots$

$S_N \to \ln 2$ as $N \to \infty$.

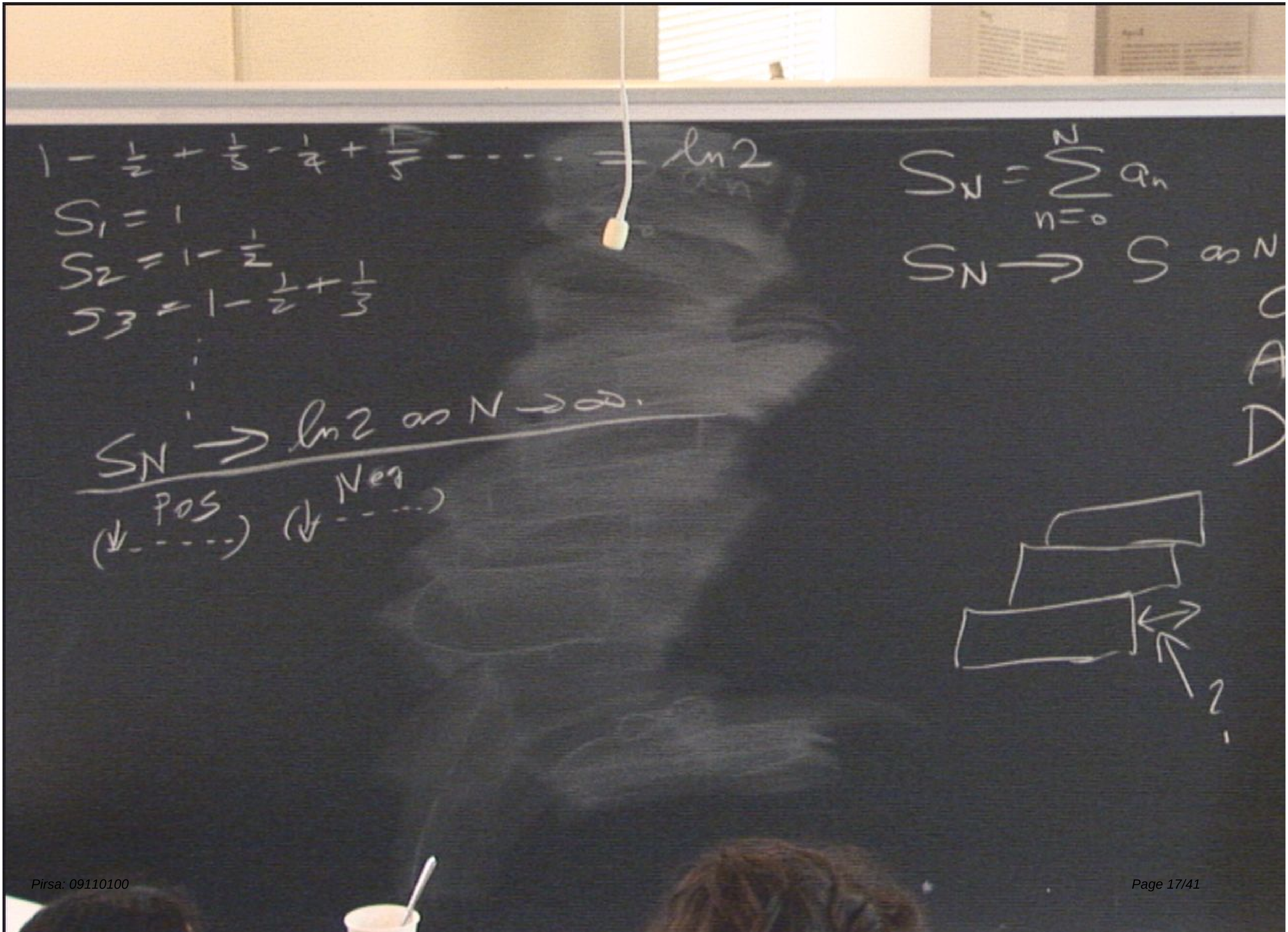
$$S_N = \sum_{n=0}^{N} a_n$$

$S_N \to S$ as N

C

A

D


1 − 1/2 + 1/3 − 1/4 + 1/5 - - - - - = ln 2
S1 = 1
S2 = 1 − 1/2
S3 = 1 − 1/2 + 1/3
SN → ln 2 as N → ∞.
Pos
Neg
SN = Σ an
n=0
N
SN → S as N
A
D
2

$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - - - - - = \ln 2$

$S_N = \sum_{n=0}^{N} a_n$

$S_1 = 1$

$S_2 = 1 - \frac{1}{2}$

$S_3 = 1 - \frac{1}{2} + \frac{1}{3}$

$\vdots$

$S_N \to \ln 2$ as $N \to \infty$.

$S_N \to S$ as N

C

A

D

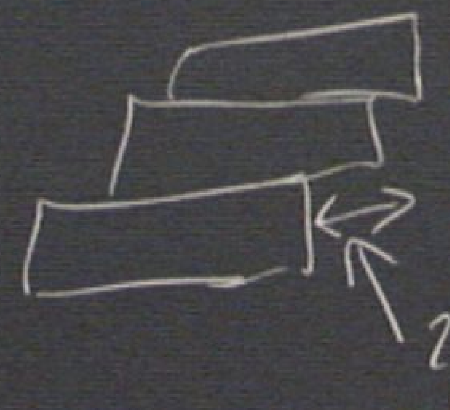
Pos Neg

$(\downarrow - - - - -)\ (\downarrow - - - - -)$

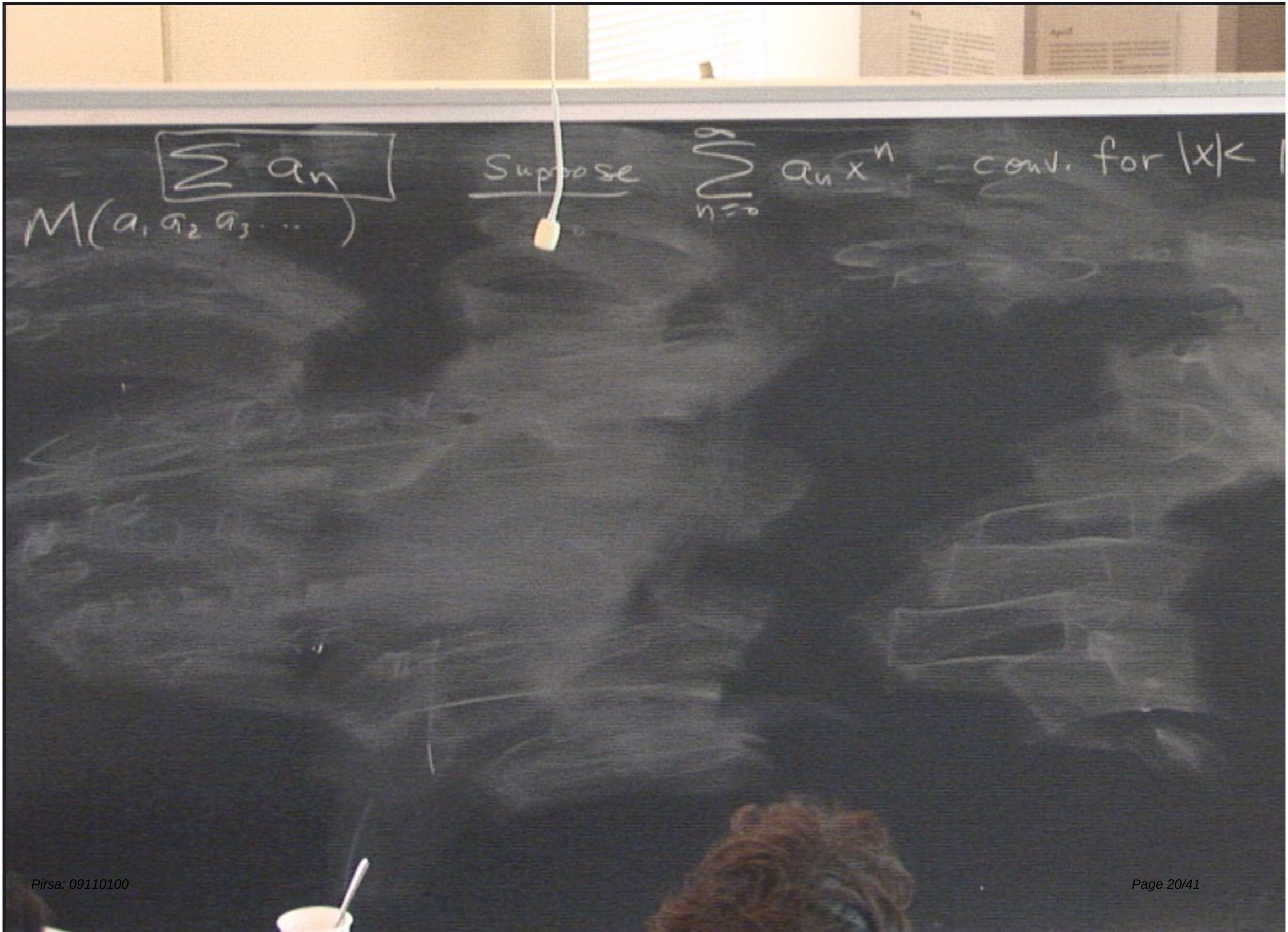
$(p_1 p_2 p_3 p_4)(n_1 n_2 n_3)(p_5 p_6 p_7)(n_4 n_5)$

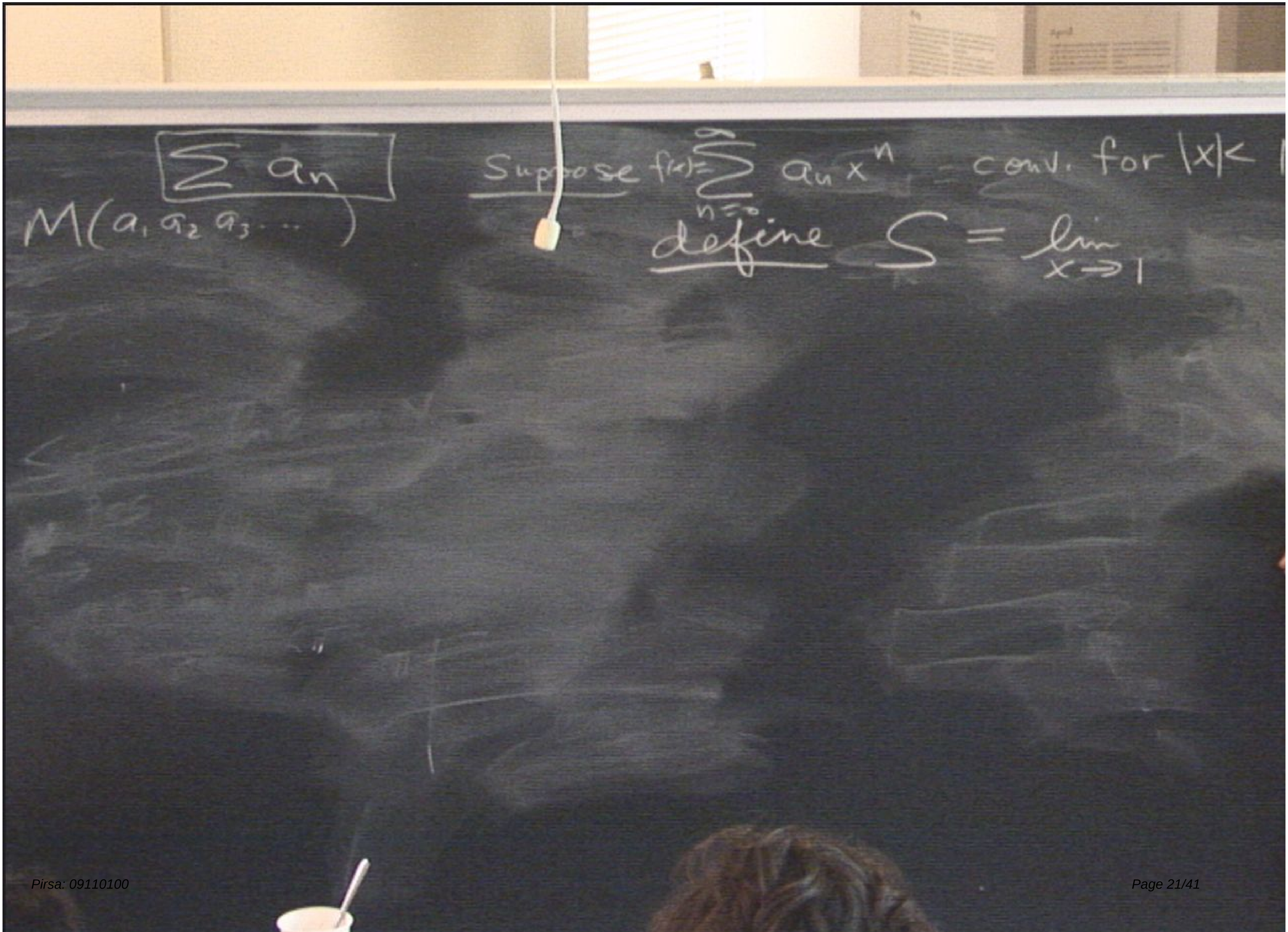
$>\pi \quad <\pi \quad >\pi \quad <\pi$

π



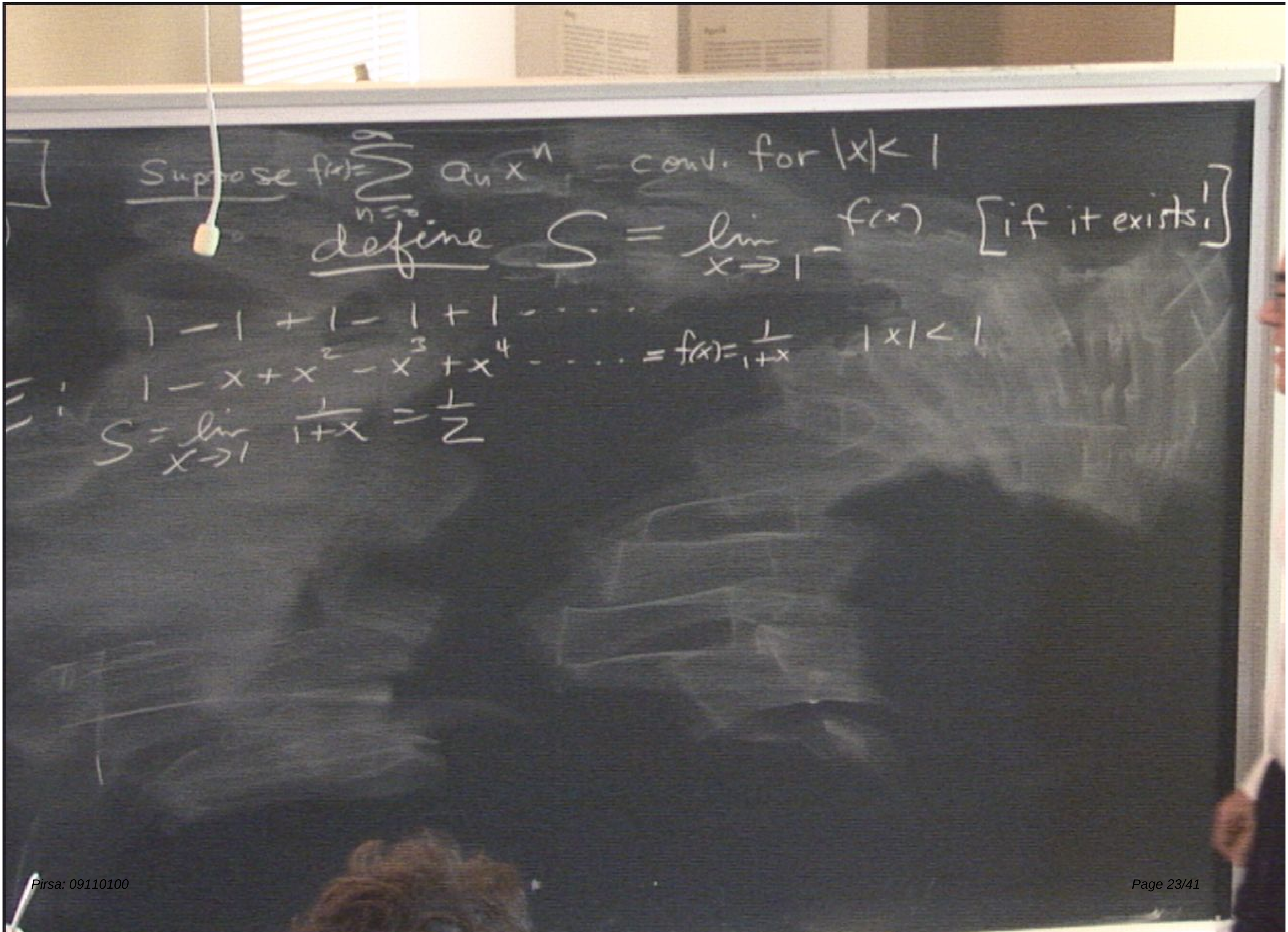
Euler summation

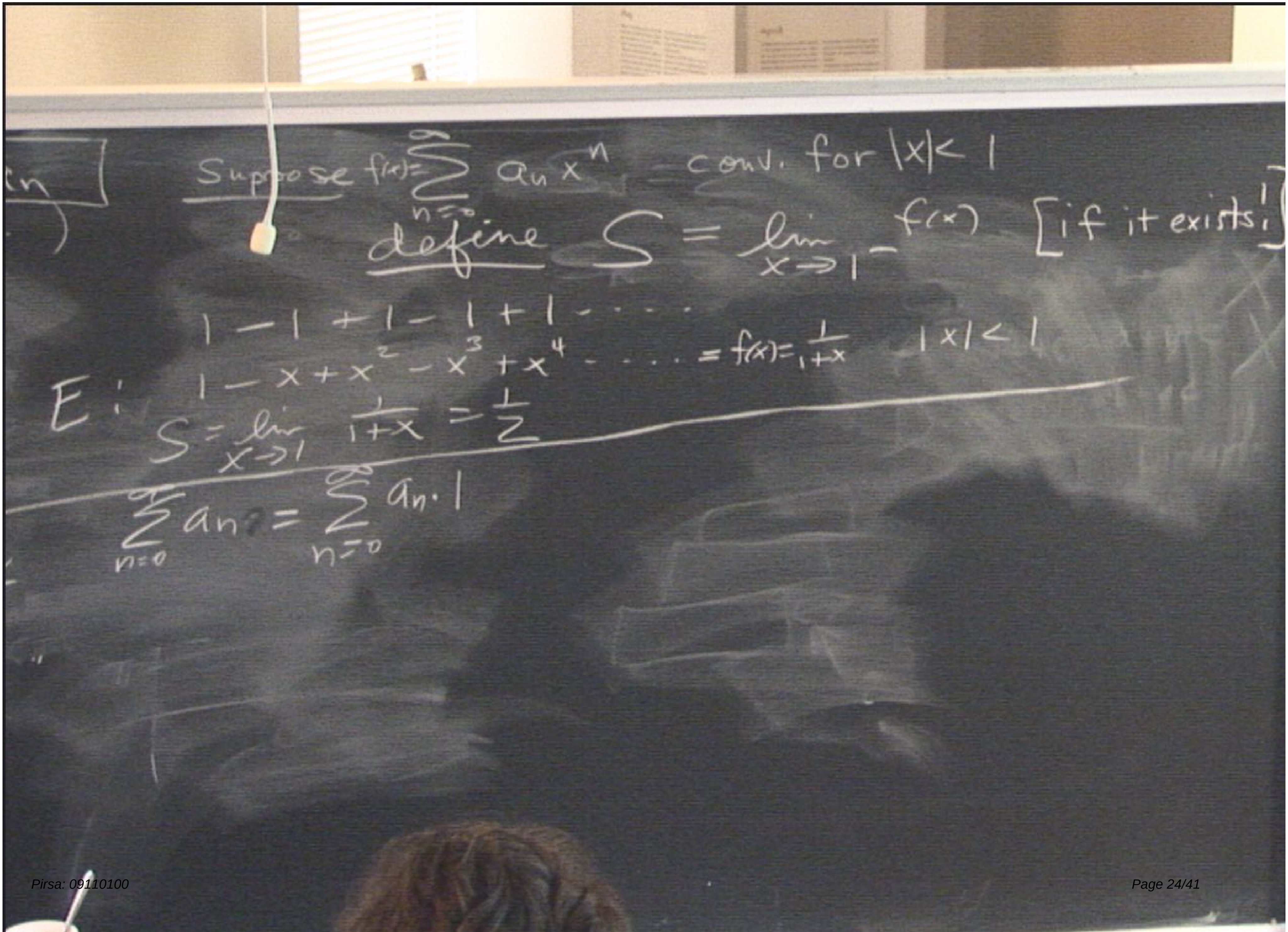

$\boxed{\sum a_n}$
$M(a_1, a_2, a_3 \ldots)$
Suppose $\sum_{n=0}^{\infty} a_n x^n$ conv. for $|x| <$ 1


$\boxed{\sum a_n}$
$M(a_1, a_2\, a_3 \ldots)$
Suppose $f(x) = \sum_{n=0}^{\infty} a_n x^n$ conv. for $|x| < 1$
define $S = \lim_{x \to 1}$

Suppose $f(x) = \sum_{n=0}^{\infty} a_n x^n$ conv. for $|x| < 1$

define $S = \lim_{x \to 1^-} f(x)$ [if it exists!]


Suppose $f(x) = \sum_{n=0}^{\infty} a_n x^n$ conv. for $|x| < 1$
define $S = \lim_{x \to 1^-} f(x)$ [if it exists!]
$1 - 1 + 1 - 1 + 1 - \cdots$
$1 - x + x^2 - x^3 + x^4 - \cdots = f(x) = \frac{1}{1+x}$ $|x| < 1$
$S = \lim_{x \to 1} \frac{1}{1+x} = \frac{1}{2}$


Suppose $f(x)=\sum_{n=0}^{\infty} a_n x^n$ conv. for $|x|<1$
define $S = \lim_{x\to 1^-} f(x)$ [if it exists!]
$1 - 1 + 1 - 1 + 1 - \dots$
$1 - x + x^2 - x^3 + x^4 - \dots = f(x) = \frac{1}{1+x}$ $|x|<1$
E: $S = \lim_{x\to 1} \frac{1}{1+x} = \frac{1}{2}$
$\sum_{n=0}^{\infty} a_n ? = \sum_{n=0}^{\infty} a_n \cdot 1$

Suppose $f(x)=\sum_{n=0}^{\infty} a_n x^n$ conv. for $|x|<1$

define $S = \lim_{x\to 1^-} f(x)$ [if it exists!]

E: $1-1+1-1+1-\cdots$

$1-x+x^2-x^3+x^4-\cdots = f(x) = \frac{1}{1+x}$ $|x|<1$

$S = \lim_{x\to 1} \frac{1}{1+x} = \frac{1}{2}$

$1 = \frac{\int_0^\infty e^{-t} t^n \, dt}{n!}$

$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} a_n \cdot 1$

Suppose $f(x)=\sum_{n=0}^{\infty} a_n x^n$ conv. for $|x|<1$

define $S = \lim_{x\to 1^-} f(x)$ [if it exists!]

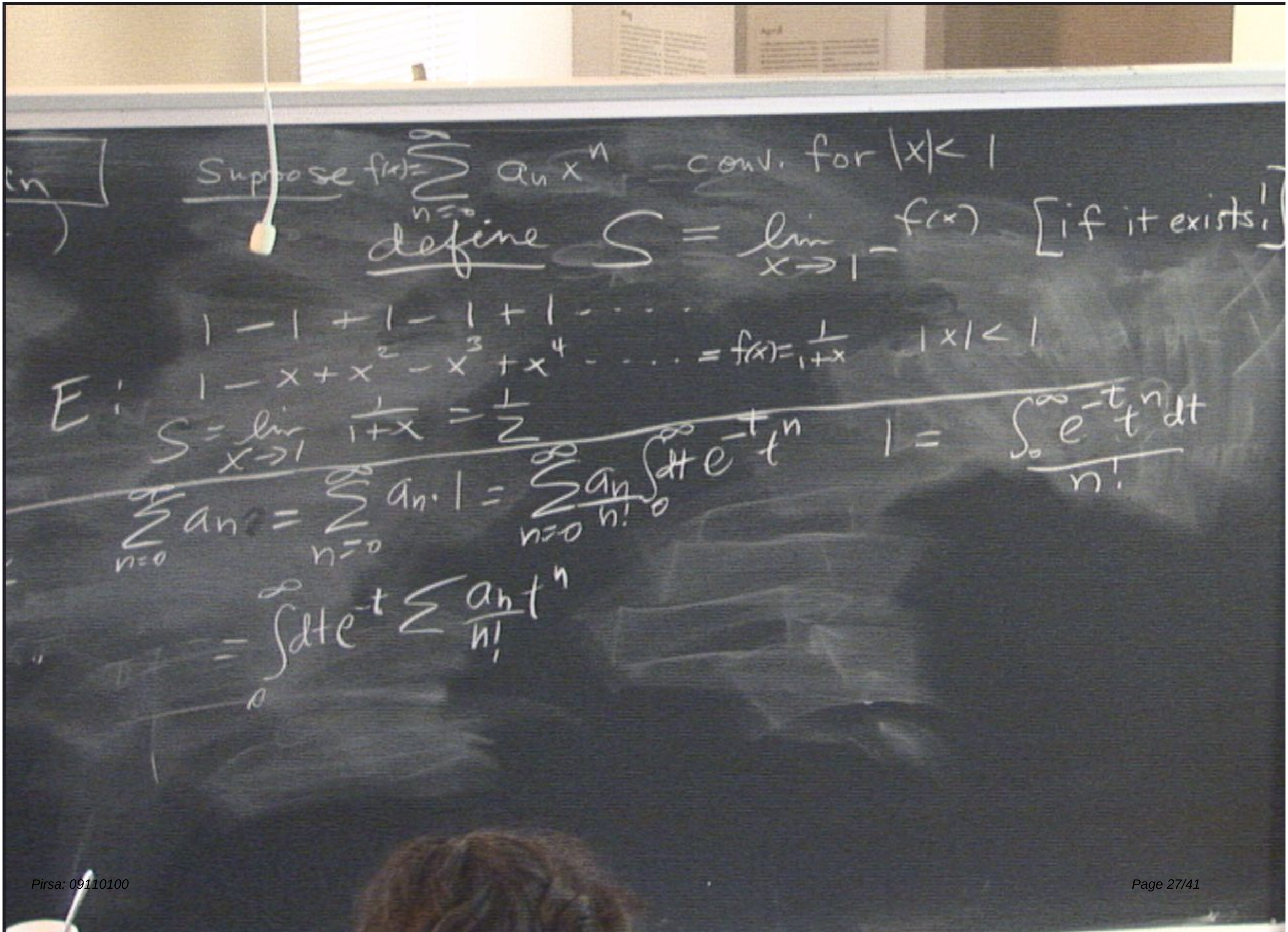
E: $1-1+1-1+1-\cdots$

$1-x+x^2-x^3+x^4-\cdots = f(x)=\frac{1}{1+x}$ $\quad |x|<1$

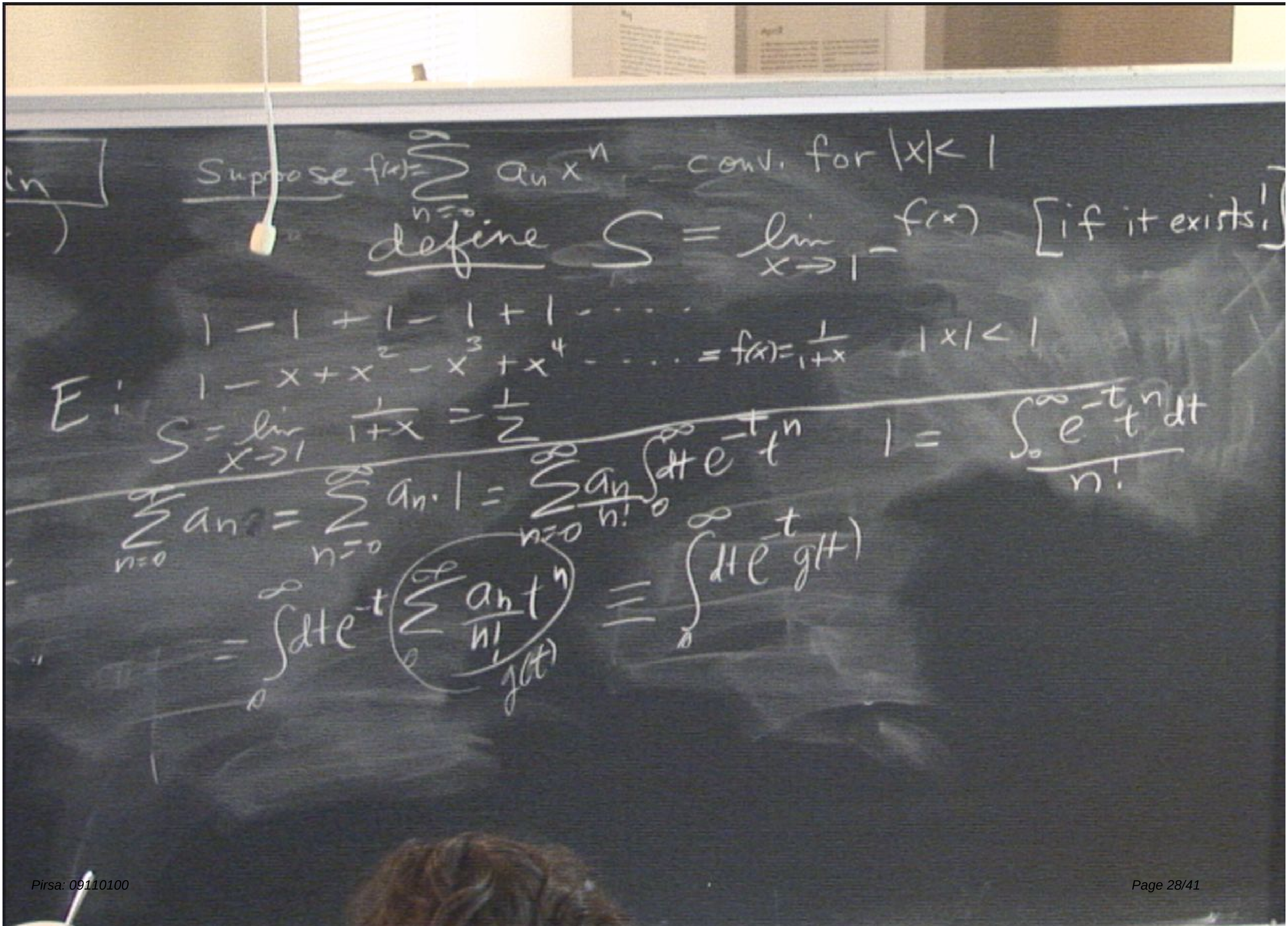
$S=\lim_{x\to 1}\frac{1}{1+x}=\frac{1}{2}$

$1 = \frac{\int_0^{\infty} e^{-t}t^n\,dt}{n!}$

$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} a_n\cdot 1 = \sum_{n=0}^{\infty}\frac{a_n}{n!}\int_0^{\infty} dt\, e^{-t}t^n$



Suppose $f(x)=\sum_{n=0}^{\infty} a_n x^n$ conv. for $|x|<1$
define $S=\lim_{x\to 1^-} f(x)$ [if it exists!]
$1-1+1-1+1 - \cdots$
$1-x+x^2-x^3+x^4 - \cdots = f(x) = \frac{1}{1+x}$ $|x|<1$
E: $S=\lim_{x\to 1}\frac{1}{1+x}=\frac{1}{2}$
$1=\frac{\int_0^\infty e^{-t}t^n\,dt}{n!}$
$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} a_n\cdot 1 = \sum_{n=0}^{\infty}\frac{a_n}{n!}\int_0^\infty dt\, e^{-t} t^n$
$=\int_0^\infty dt\, e^{-t}\sum \frac{a_n}{n!}t^n$

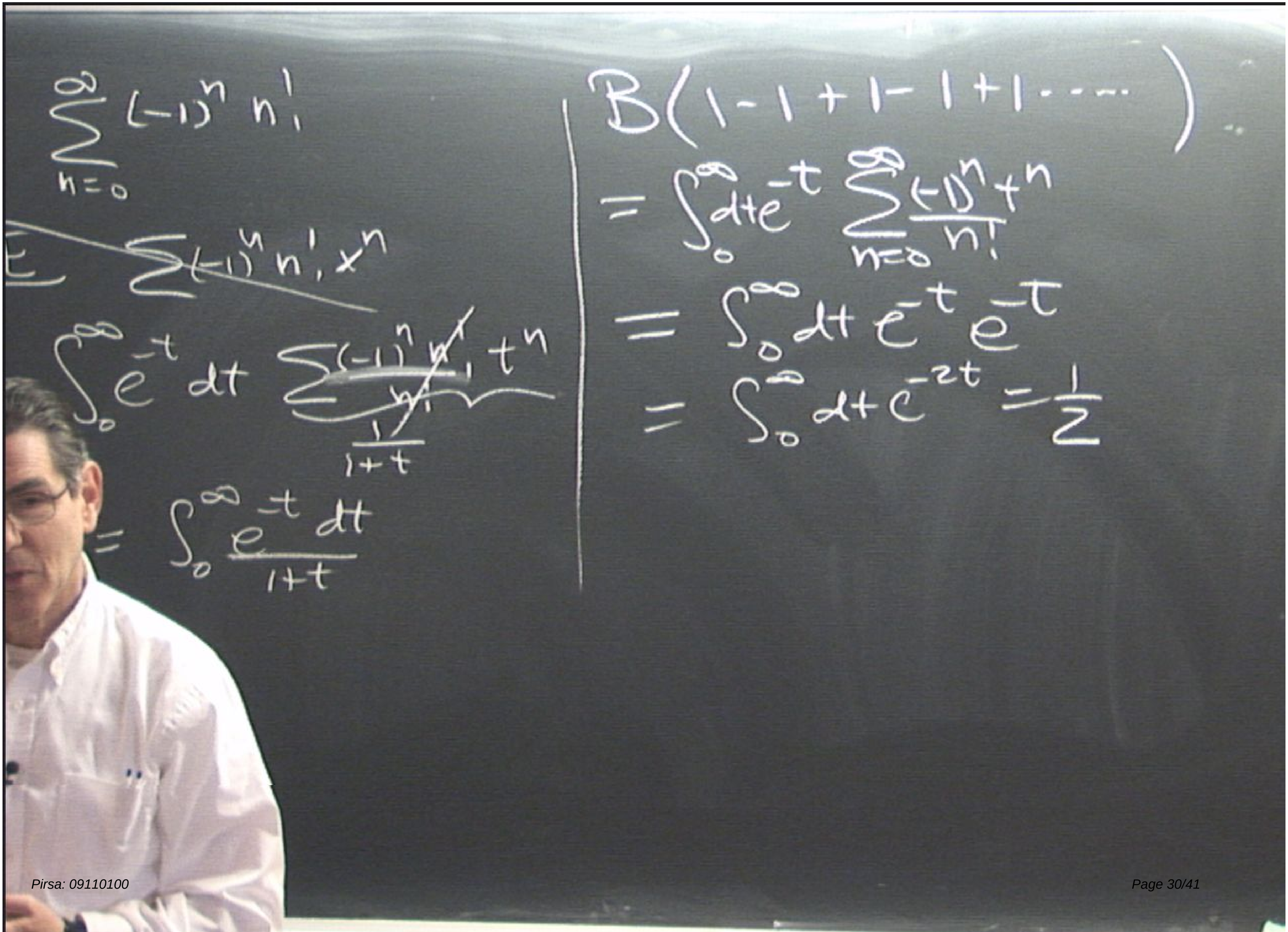

Suppose $f(x) = \sum_{n=0}^{\infty} a_n x^n$ conv. for $|x| < 1$
define $S = \lim_{x \to 1^-} f(x)$ [if it exists!]
$1 - 1 + 1 - 1 + 1 - \dots$
$1 - x + x^2 - x^3 + x^4 - \dots = f(x) = \frac{1}{1+x}$ $|x| < 1$
E:
$S = \lim_{x \to 1} \frac{1}{1+x} = \frac{1}{2}$
$1 = \frac{\int_0^\infty e^{-t} t^n dt}{n!}$
$\sum_{n=0} a_n = \sum_{n=0}^{\infty} a_n \cdot 1 = \sum_{n=0} \frac{a_n}{n!} \int_0^\infty dt\, e^{-t} t^n$
$= \int_0^\infty dt\, e^{-t} \underbrace{\sum_0^\infty \frac{a_n t^n}{n!}}_{g(t)} \equiv \int_0^\infty dt\, e^{-t} g(t)$

$$\sum_{n=0}^{\infty} (-1)^n n!$$

$$\underline{E} \quad \cancel{\sum (-1)^n n! \, x^n}$$

$$\int_0^{\infty} e^{-t} dt \underbrace{\sum \frac{(-1)^n \cancel{n!}\, t^n}{\cancel{n!}}}_{\frac{1}{1+t}}$$

$$B = \int_0^{\infty} \frac{e^{-t}\, dt}{1+t}$$


$\sum_{n=0}^{\infty} (-1)^n n!$
$\sum (-1)^n n!\, x^n$
$\int_0^\infty e^{-t}\, dt \underbrace{\sum \frac{(-1)^n n!}{n!} t^n}_{\frac{1}{1+t}}$
$= \int_0^\infty \frac{e^{-t}\, dt}{1+t}$
$B(1-1+1-1+1\cdots\cdots)$
$= \int_0^\infty dt\, e^{-t} \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{n!}$
$= \int_0^\infty dt\, e^{-t} e^{-t}$
$= \int_0^\infty dt\, e^{-2t} = \frac{1}{2}$

$$\sum_{n=0}^{\infty} (-1)^n n!$$

$$\underline{E} \quad \cancel{\sum (-1)^n n!\, x^n}$$

$$g(t)$$

$$\int_0^\infty e^{-t}\, dt \sum \frac{(-1)^n \cancel{n!}}{\cancel{n!}} t^n$$

$$\frac{1}{1+t}$$

$$B = \int_0^\infty \frac{e^{-t}\, dt}{1+t}$$

$$B(1-1+1-1+1\cdots)$$

$$= \int_0^\infty dt\, e^{-t} \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{n!}$$

$$= \int_0^\infty dt\, e^{-t} e^{-t}$$

$$= \int_0^\infty dt\, e^{-2t} = \frac{1}{2}$$

$$M(a_0, a_1, a_2, \ldots) = S$$

#1

$$M(a_0, a_1, a_2, \dots) = S$$

#1 $$M\left(\sum_{0}^{\infty} a_n\right) = a_0 + M\left(\sum_{1}^{\infty} a_n\right)$$

$$M(a_0, a_1, a_2, \dots) = S$$

#1 $$M\left(\sum_0^\infty a_n\right) = a_0 + M\left(\sum_1^\infty a_n\right)$$

#2 $$M\left(\sum_0^\infty (\alpha a_n + \beta b_n)\right) = \alpha M\left(\sum a_n\right) + \beta M\left(\sum b_n\right)$$

$$M(a_0, a_1, a_2, \ldots) = S$$

$$\underline{\#1} \quad M\left(\sum_0^\infty a_n\right) = a_0 + M\left(\sum_1^\infty a_n\right)$$

$$\underline{\#2} \quad M\left(\sum_0^\infty (\alpha a_n + \beta b_n)\right) = \alpha M\left(\sum a_n\right) + \beta M\left(\sum b_n\right)$$

$$= M(1 - 1 + 1 - 1 + 1 - \cdots)$$

$$\overset{\#1}{=} 1 + M(-1 + 1 - 1 + 1 - \cdots)$$

$$= 1 - M(1 - 1 + 1 - 1 \cdots)$$

$$= 1 - S$$

$$2S = 1$$

$$S = \frac{1}{2}$$

$$1+0-1+1+0-1+1+0-1-----$$

$\underline{E}$: $1-x^2+x^3-x^5+x^6+\cdots$

$$=\underbrace{\left[1+x^3+x^6+\cdots\right]}_{\frac{1}{1-x^3}}-x^2\left[1+x^3+x^6\cdots\cdots\right] \quad |x|<1$$

$$=\frac{1}{1-x^3}-x^2\frac{1}{1-x^3}$$

$$=\frac{1-x^2}{1-x^3}=\frac{1+x}{1+x+x^2}\xrightarrow[x\to 1]{}\underline{\frac{2}{3}}$$

$$M(a_0, a_1, a_2, \cdots) = S$$

#1 $$M\left(\sum_0^\infty a_n\right) = a_0 + M\left(\sum_1^\infty a_n\right)$$

A#2 $$M\left(\sum_0^\infty (\alpha a_n + \beta b_n)\right) = \alpha M\left(\sum a_n\right) + \beta M\left(\sum b_n\right)$$

$$S = M(1 + 0 - 1 + 1 + 0 - 1 + 1 + 0 - 1 \cdots)$$

$$S = 1 + M(0 - 1 + 1 + 0 - 1 + 1 + 0 - 1 \cdots)$$

$$S = 1 + M(-1 + 1 + 0 - 1 + 1 + 0 - 1 \cdots)$$

$$3S = 2 + M($$

$$M(a_0, a_1, a_2, \ldots) = S$$

#1 $$M\left(\sum_0^\infty a_n\right) = a_0 + M\left(\sum_1^\infty a_n\right)$$

A#2 $$M\left(\sum_0^\infty (\alpha a_n + \beta b_n)\right) = \alpha M\left(\sum a_n\right) + \beta M\left(\sum b_n\right)$$

$$S = M(1 + 0 - 1 + 1 + 0 - 1 + 1 + 0 - 1 \ldots)$$

$$S = 1 + M(0 - 1 + 1 + 0 - 1 + 1 + 0 - 1 \ldots)$$

$$S = 1 + M(-1 + 1 + 0 - 1 + 1 + 0 - 1 \ldots)$$

$$3S = 2 + M(0 + 0 + 0 + 0 + \ldots)$$

$$S = M(1 + 2 + 4 + 8 + \cdots)$$

#1

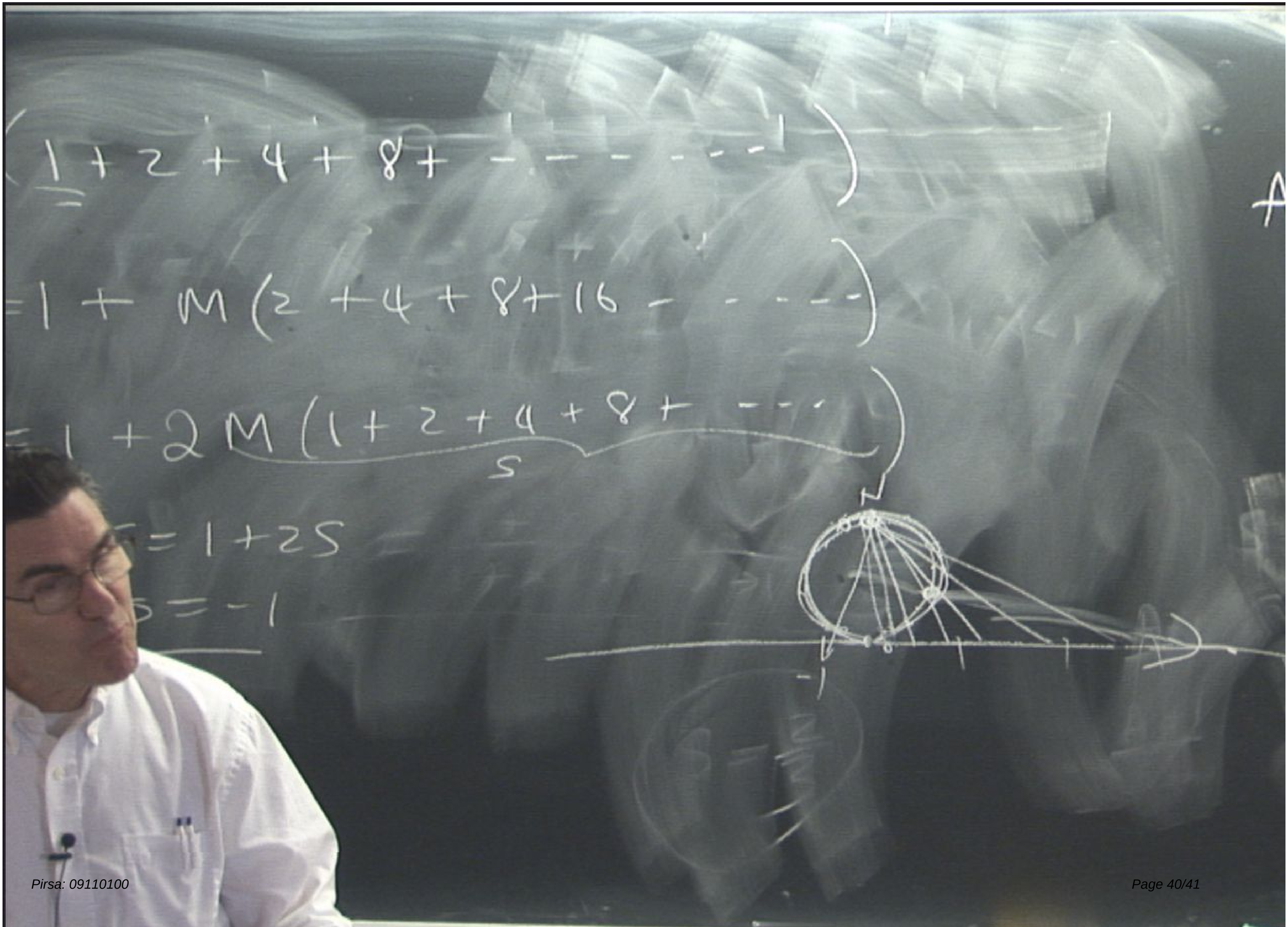
$$S = 1 + M(2 + 4 + 8 + 16 \cdots)$$

#2

$$S = 1 + 2M\underbrace{(1 + 2 + 4 + 8 + \cdots)}_{S}$$

$$S = 1 + 2S$$

$$S = -1$$


1 + 2 + 4 + 8 + - - - - - -
=1 + M (2 + 4 + 8 + 16 - - - - -)
1 + 2 M (1 + 2 + 4 + 8 + - - -)
S
= 1 + 2S
= -1
N
0
1
-1

$$M(a_0, a_1, a_2, \ldots) = S$$

$\underline{\#1}$ $$M\left(\sum_0^\infty a_n\right) = a_0 + M\left(\sum_1^\infty a_n\right)$$

$\underline{A\#2}$ $$M\left(\sum_0^\infty (\alpha a_n + \beta b_n)\right) = \alpha M\left(\sum a_n\right) + \beta M\left(\sum b_n\right)$$

$$= M(1 + 1 + 1 + \cdots)$$

$$= 1 + M\underbrace{(1 + 1 + 1 + \cdots)}_{S}$$

$S = 1 + S$

$\underline{S = \infty}$