Title: Mathematical Physics (PHYS 624) - Lecture 1

Date: Nov 16, 2009 09:00 AM

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Abstract:

How to sum a series if it CONVERGES —and— How to sum a series if it DIVERGES

Carl Bender
Physics Department
Washington University in St. Louis


MY TALK

Series arise when you use perturbation theory

Perturbation theory for a HARD PROBLEM:

Step 1. Insert a small parameter ε:

HARD PROBLEM(ε)

Step 2. Expand answer as a perturbation series in powers of ε:

$$\text{ANSWER}(\varepsilon) = \sum_{n=0}^{\infty} a_n \varepsilon^n$$

Step 3. Set ε=1 and sum the series – **this is not so easy!**

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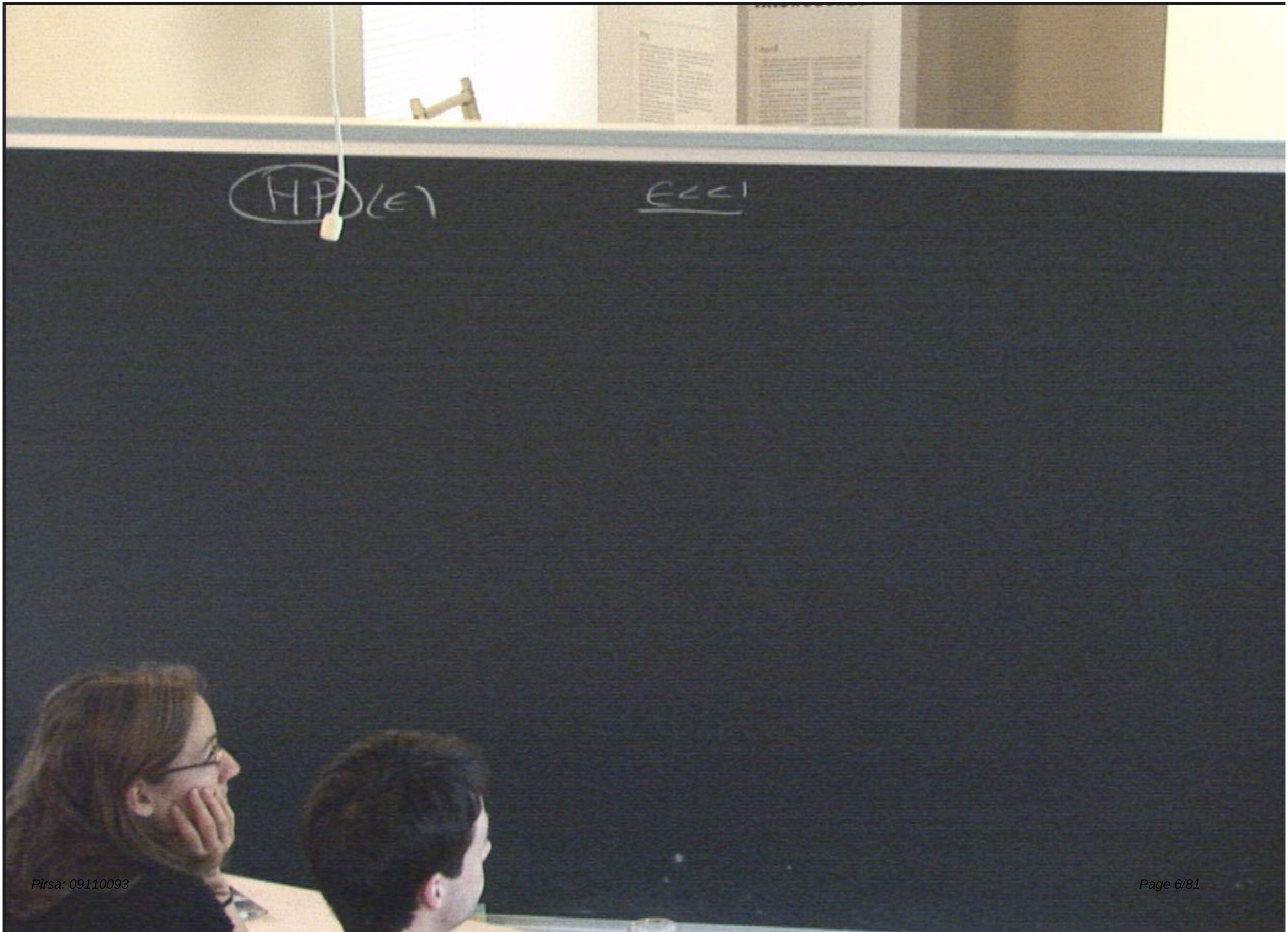
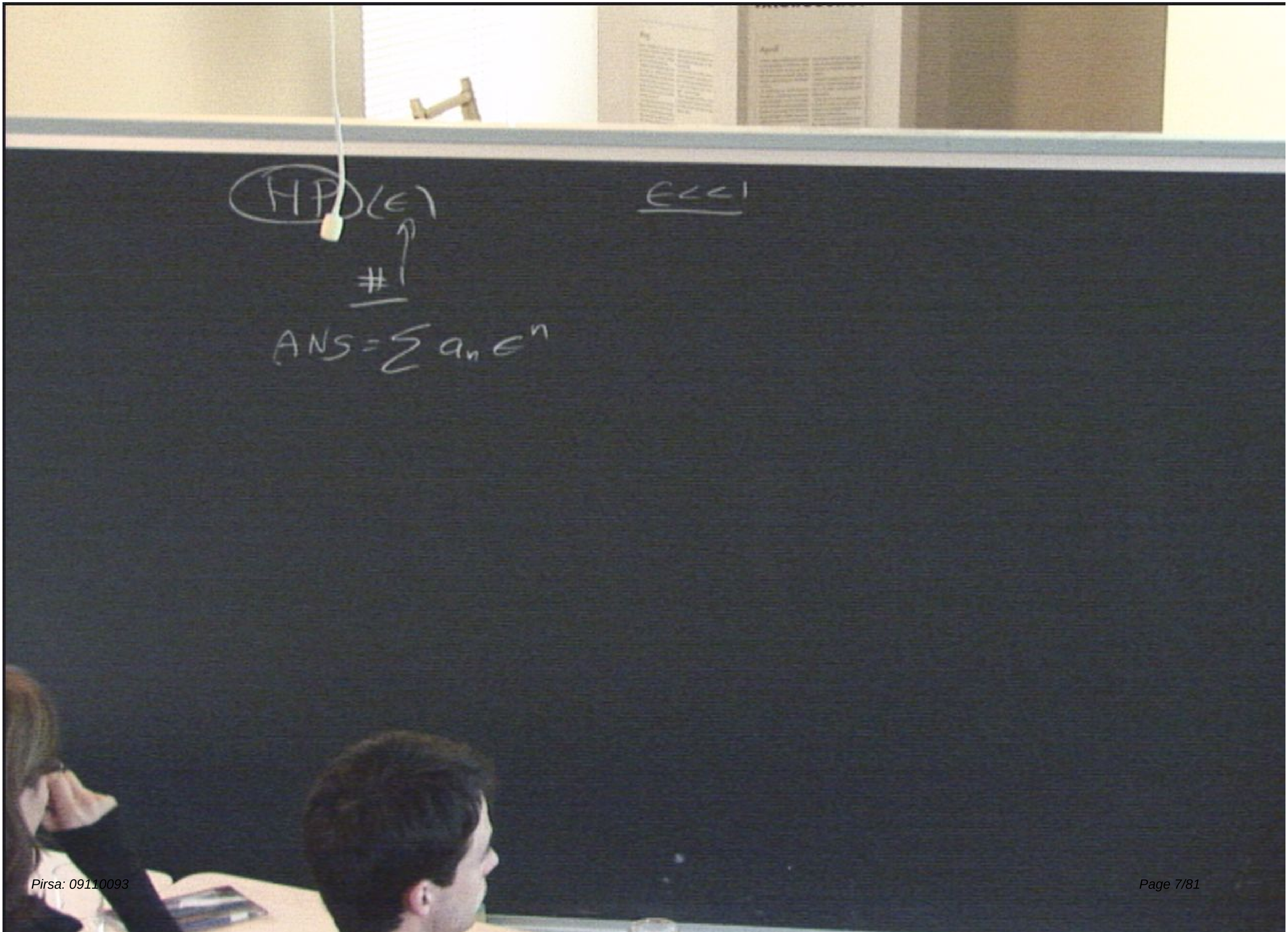
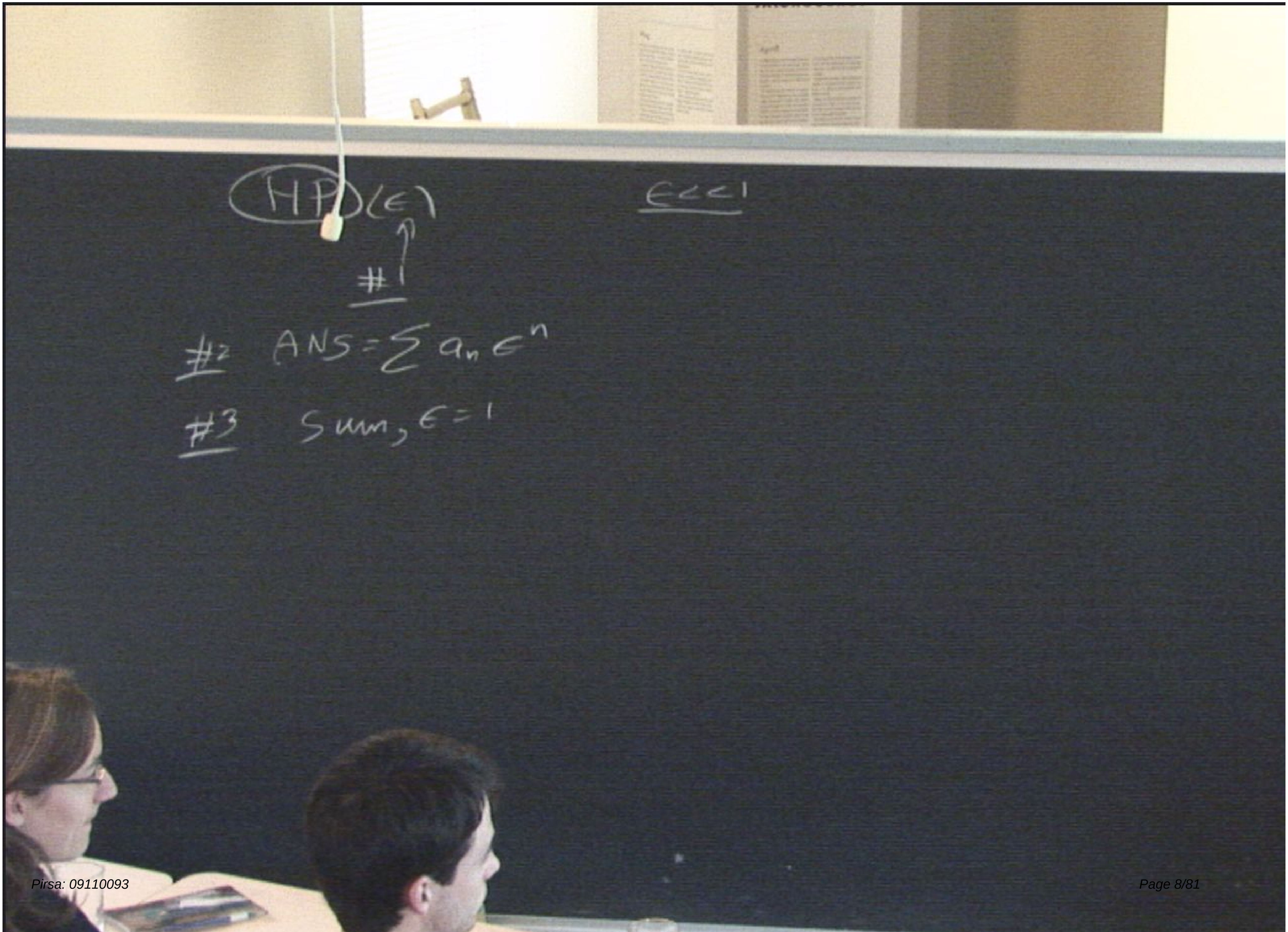
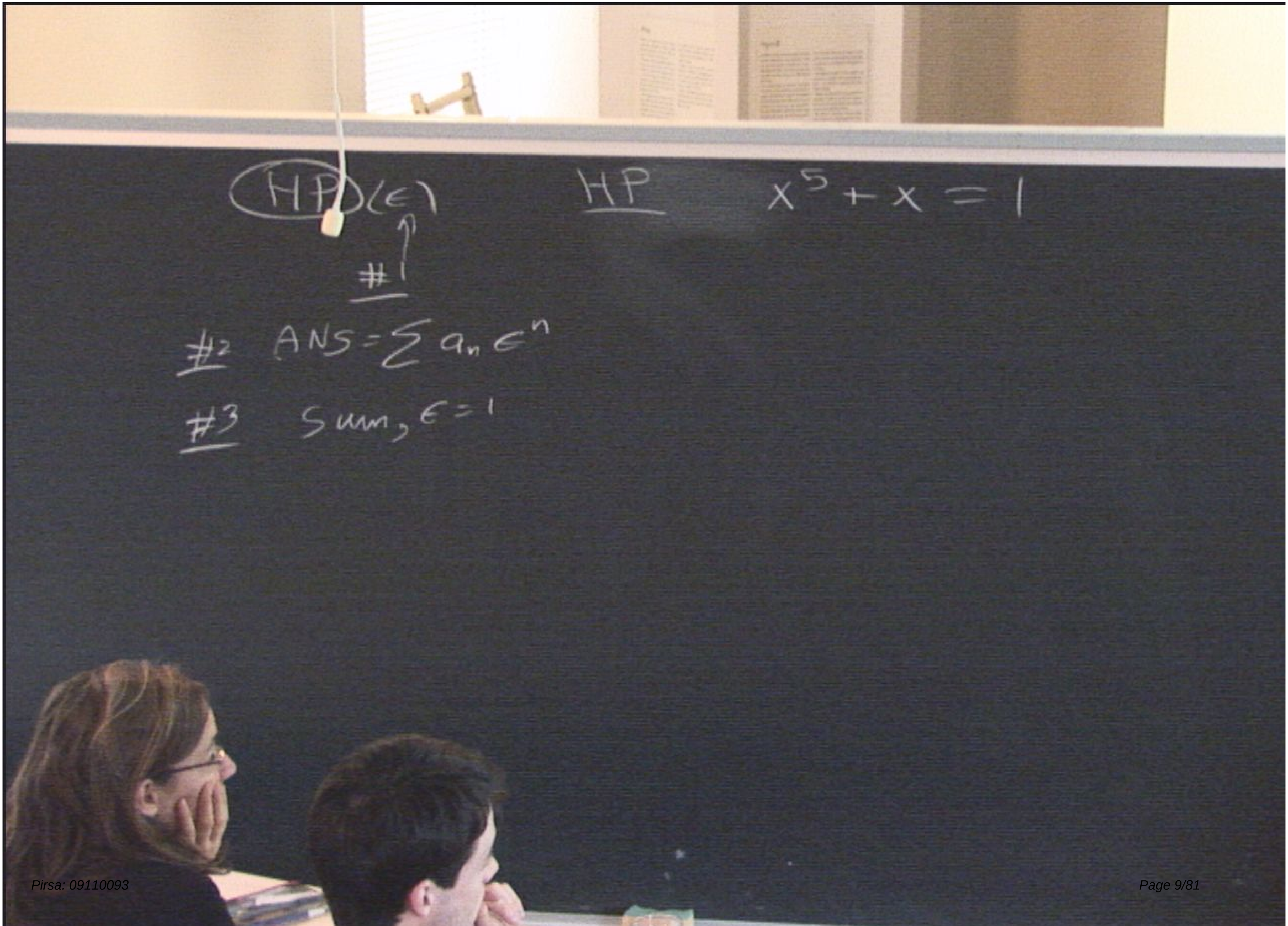



HP(ε)
$\epsilon << 1$
#1
$ANS = \sum a_n \epsilon^n$


HP (ε)
ε << 1
#1
#2 ANS = $\sum a_n \epsilon^n$
#3 Sum, ε = 1


HP(ε)
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x^5 + x = 1
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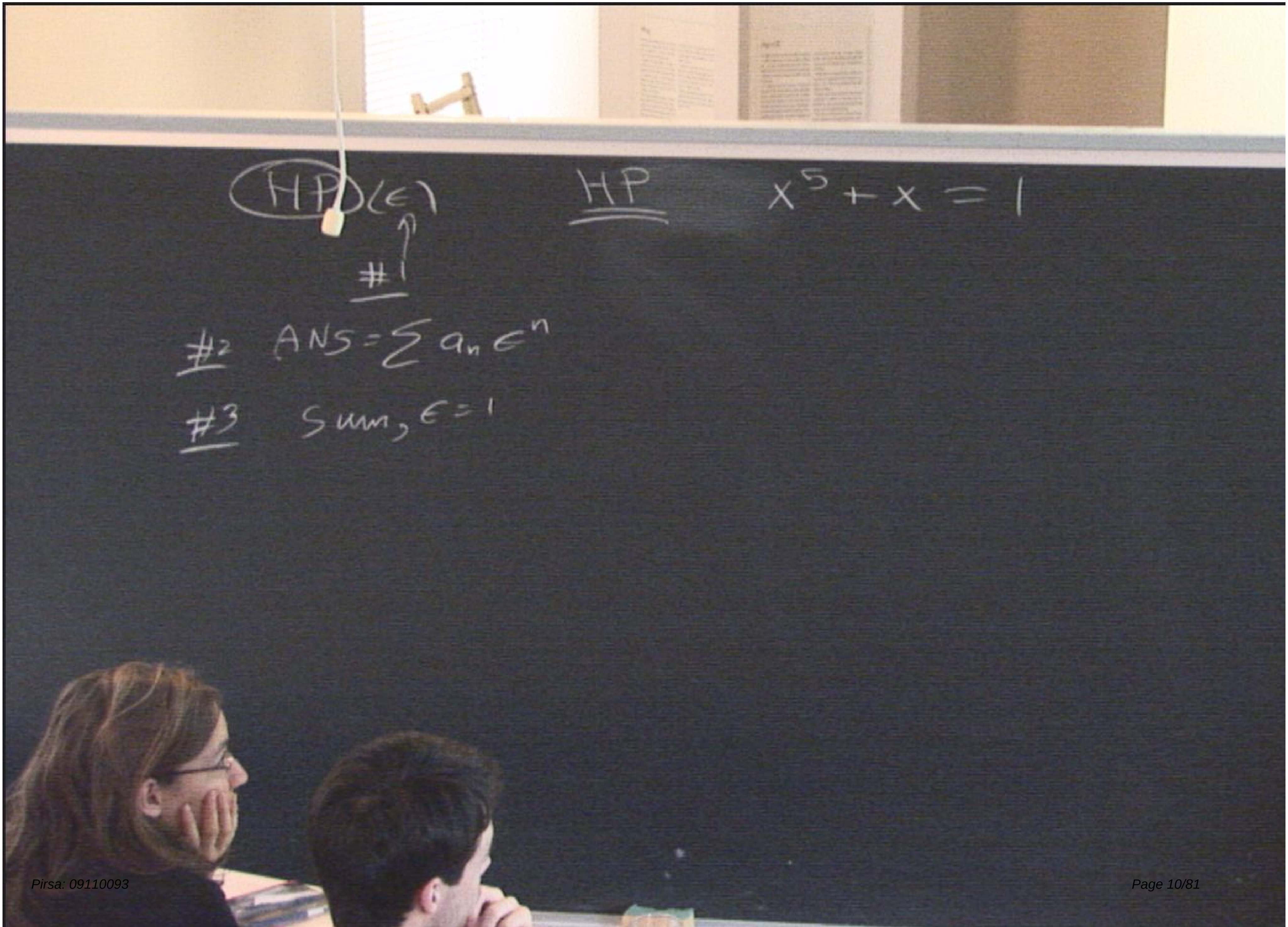
#3 Sum, ε = 1


HP(ε)
HP
$x^5 + x = 1$
#1
#2 ANS = $\sum a_n \epsilon^n$
#3 Sum, ε = 1

Simple example

HARD PROBLEM: Find the positive root of

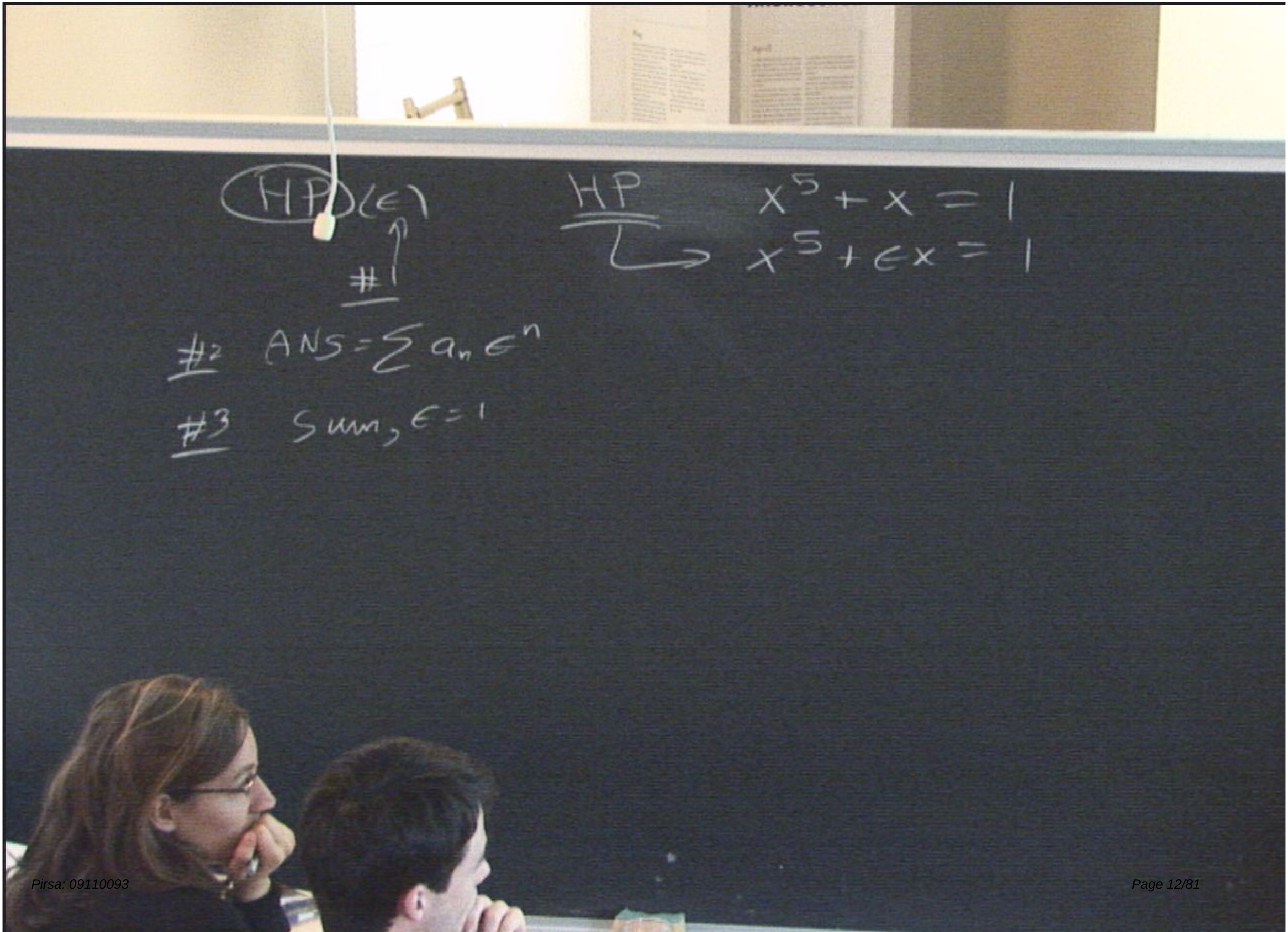
$$x^5 + x = 1$$

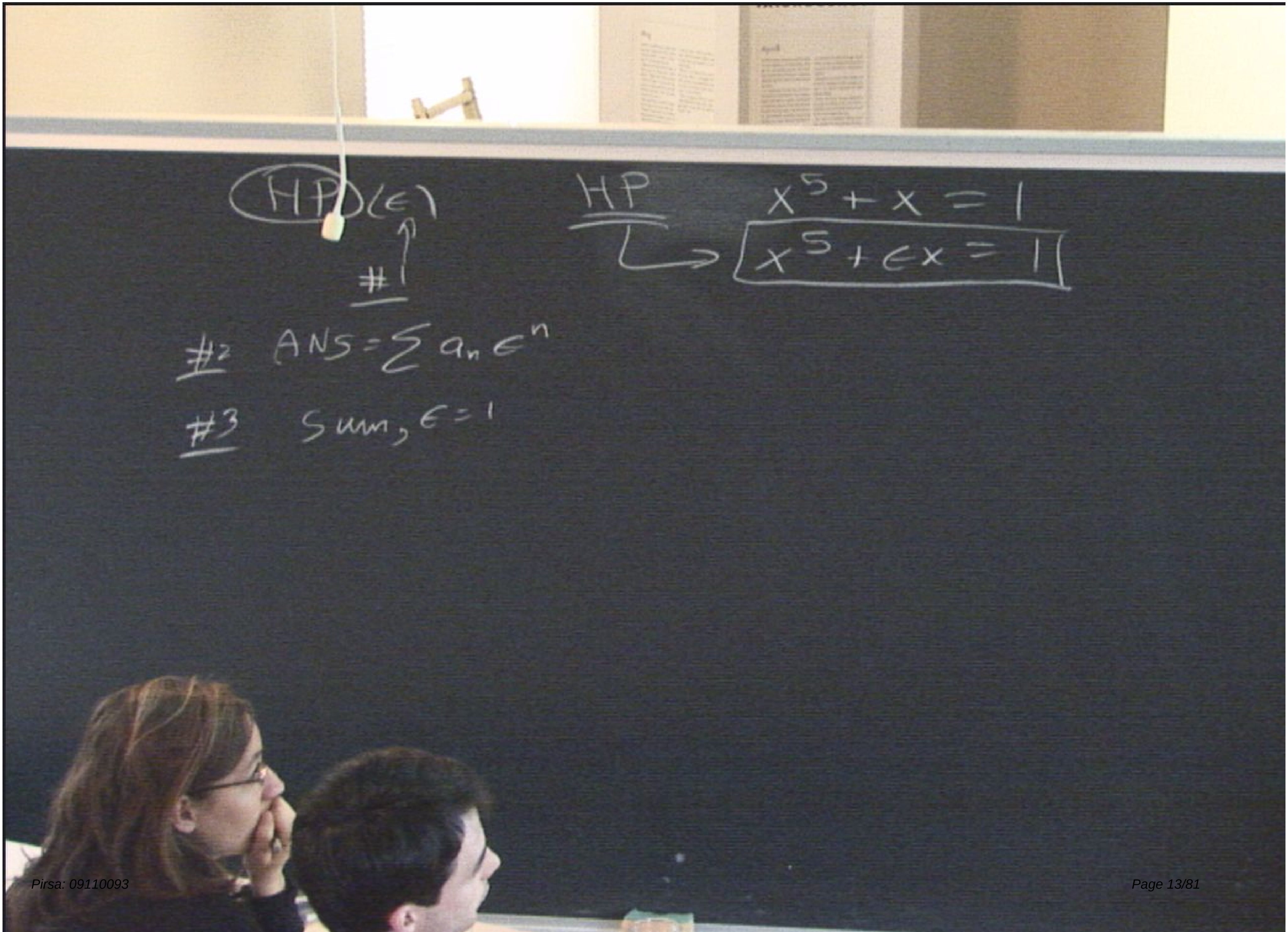
ANSWER: $x = 0.75487767\cdots$

Step 1. Insert ε: $x^5 + \epsilon x = 1$ (Strong coupling)

Step 2. $x(\epsilon) = \sum_{n=0}^{\infty} a_n \epsilon^n$

$$a_0 = 1$$


HP(ε)
HP
$x^5 + x = 1$
$x^5 + \epsilon x = 1$
#1
#2 ANS = $\sum a_n \epsilon^n$
#3 Sum, $\epsilon = 1$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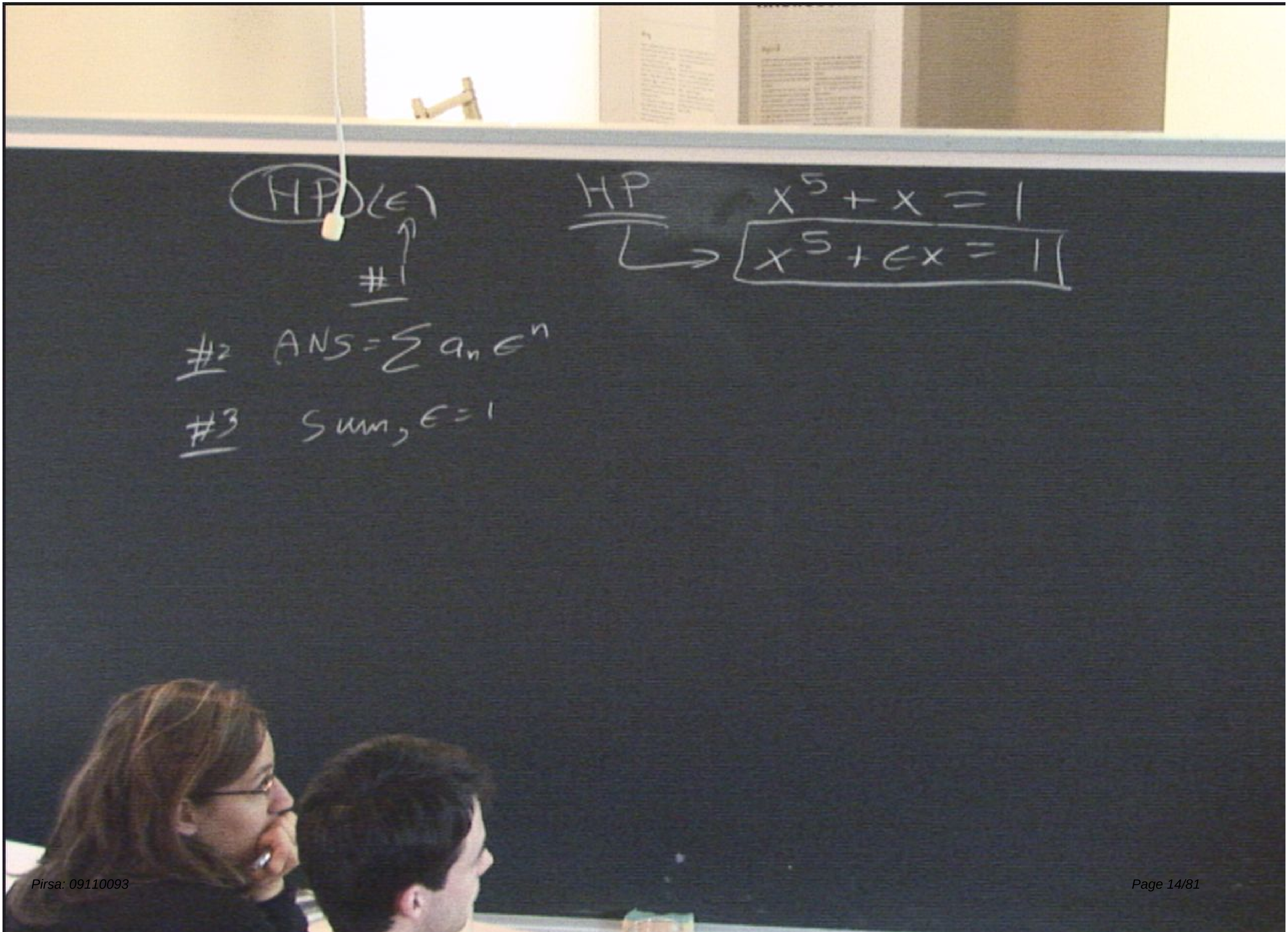
#3 Sum, $\epsilon = 1$


HP(ε)
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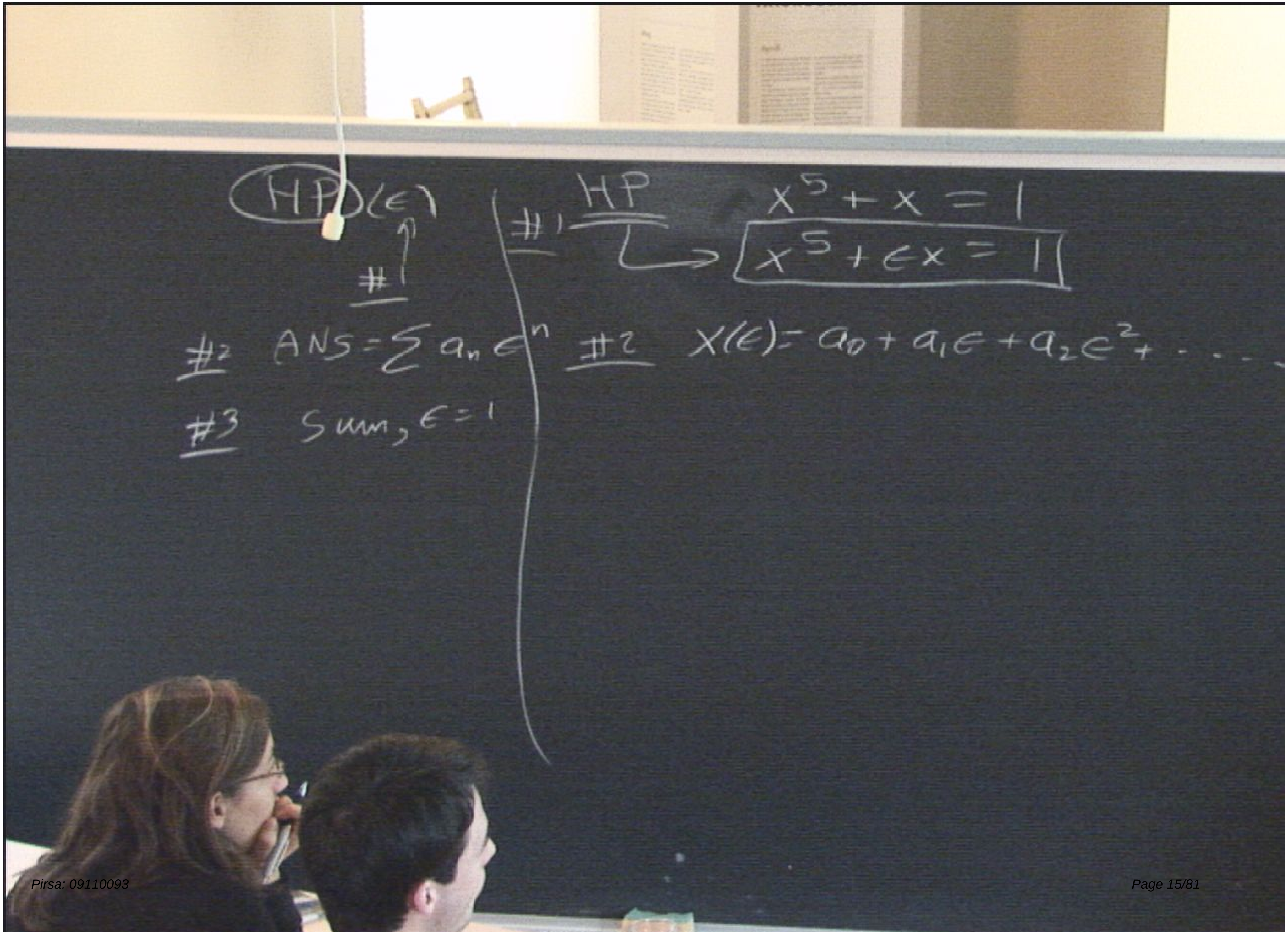
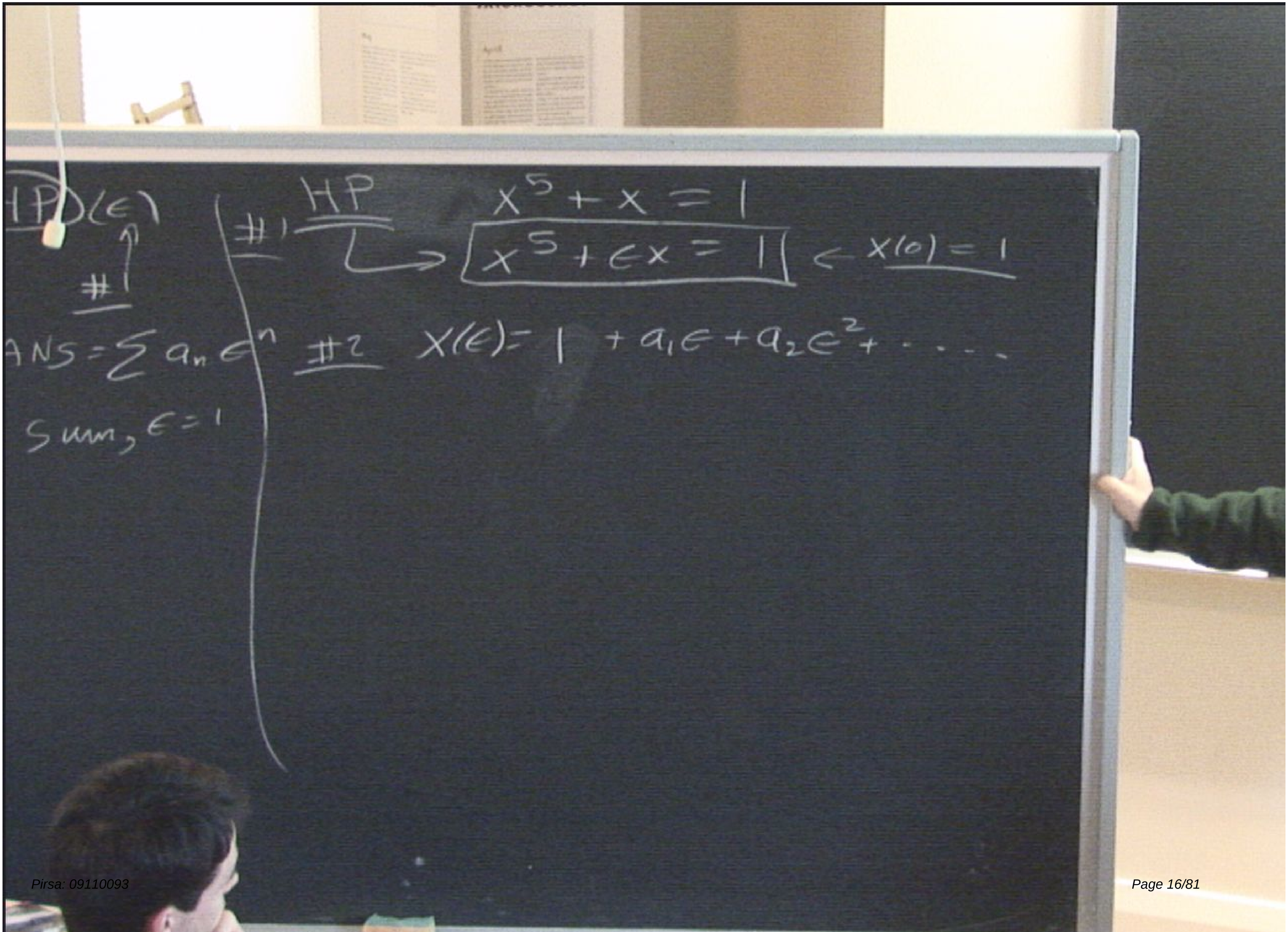
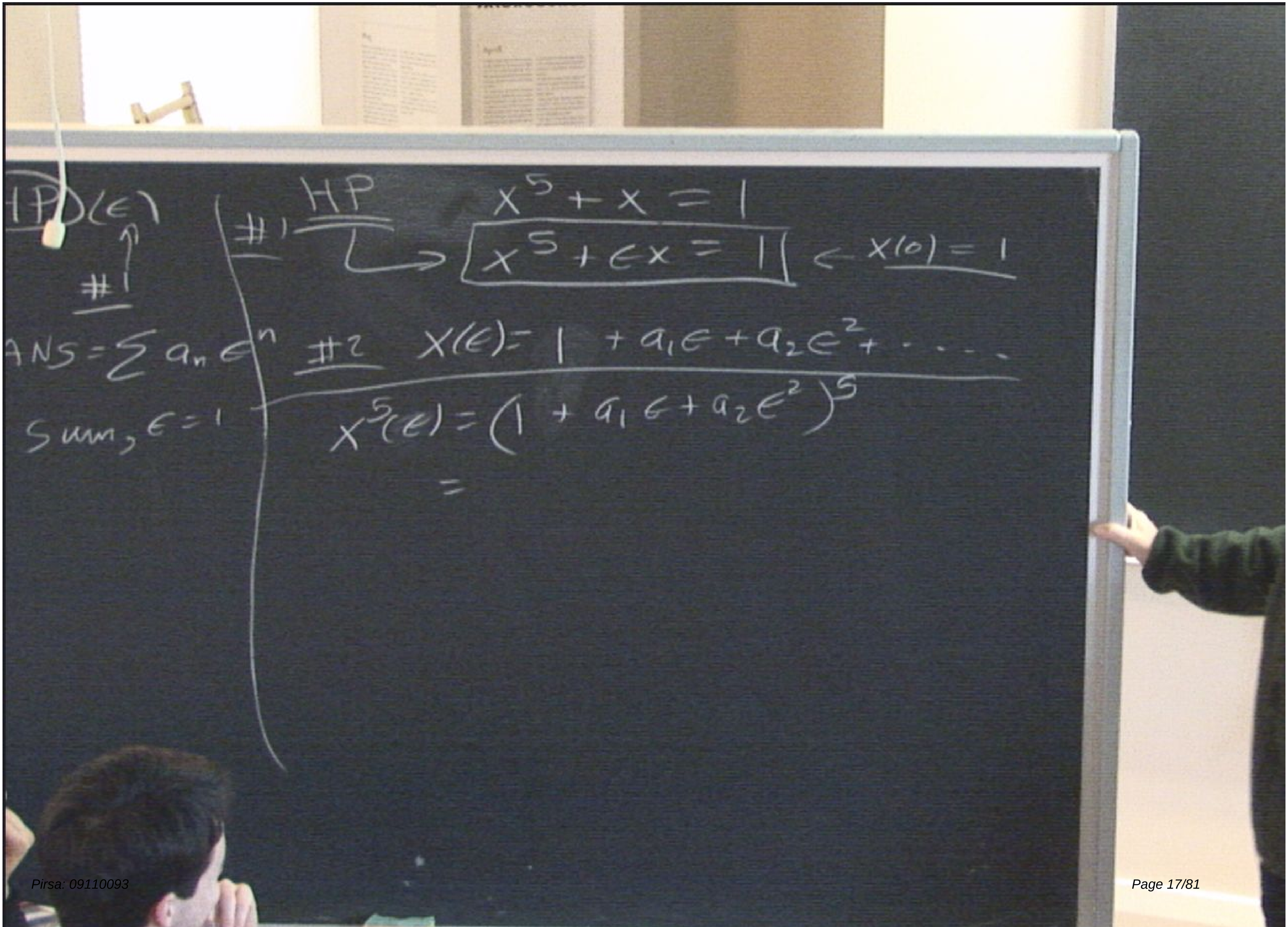
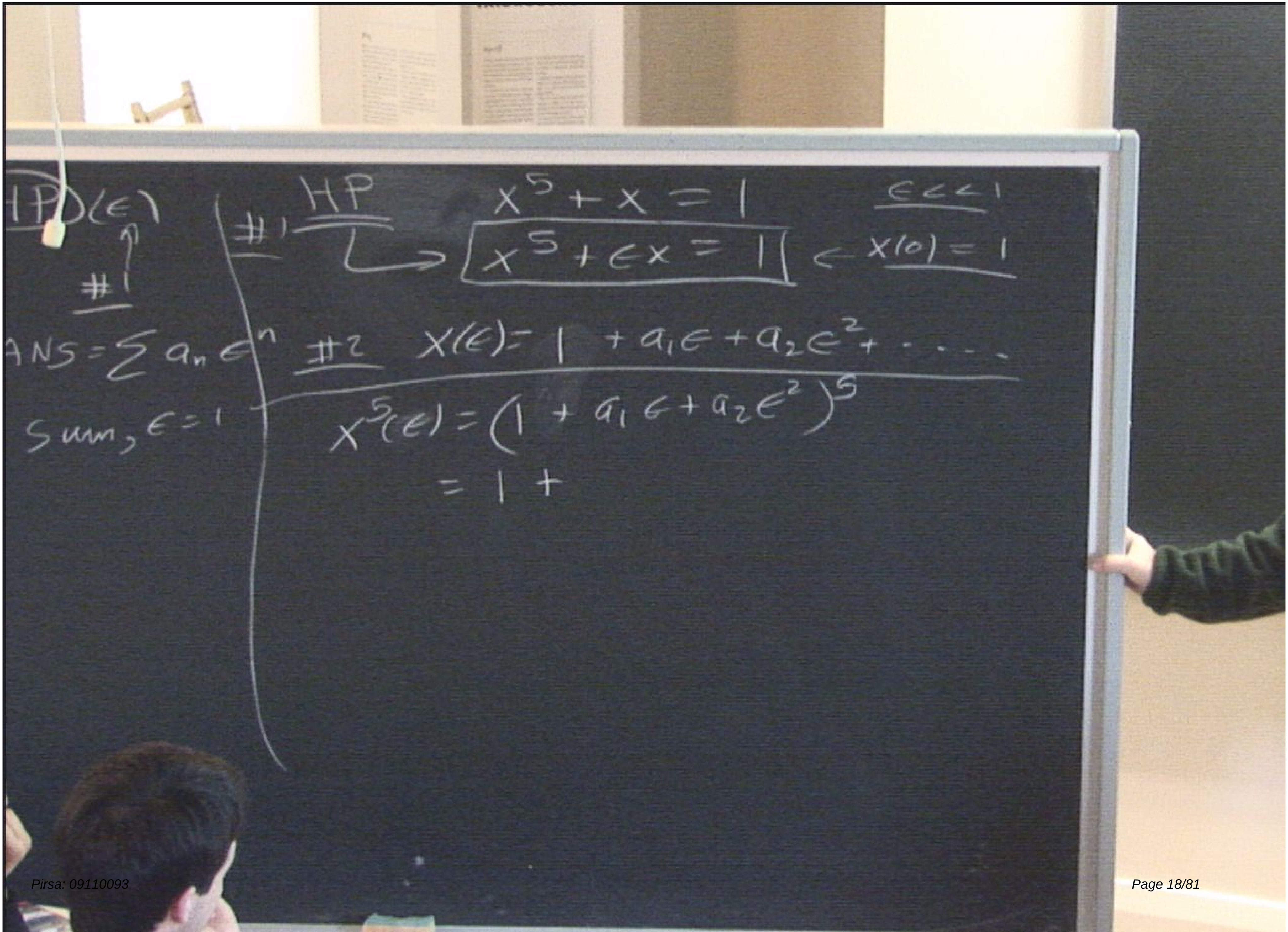
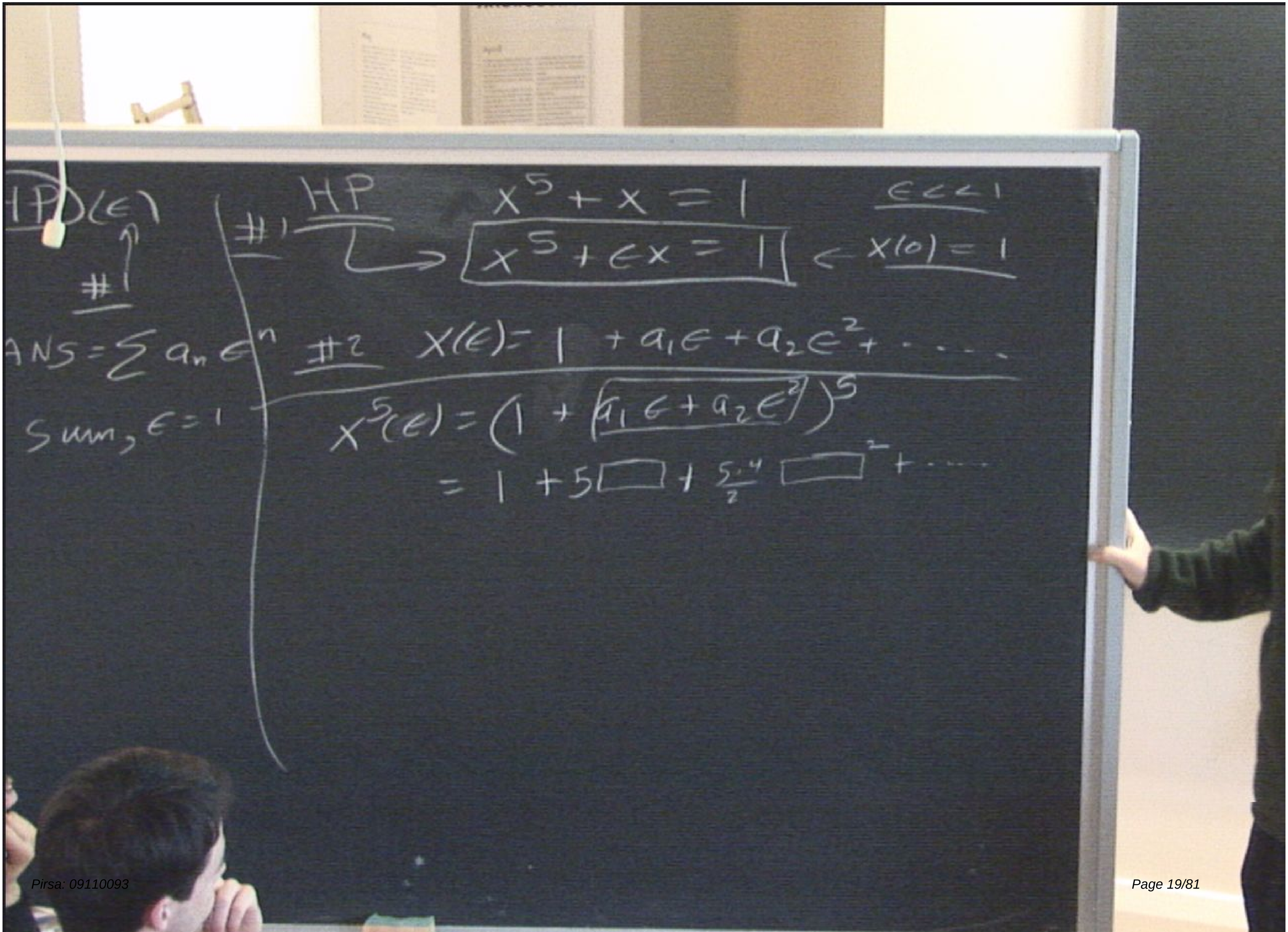
#2 ANS = $\sum a_n \epsilon^n$
#3 Sum, ε = 1

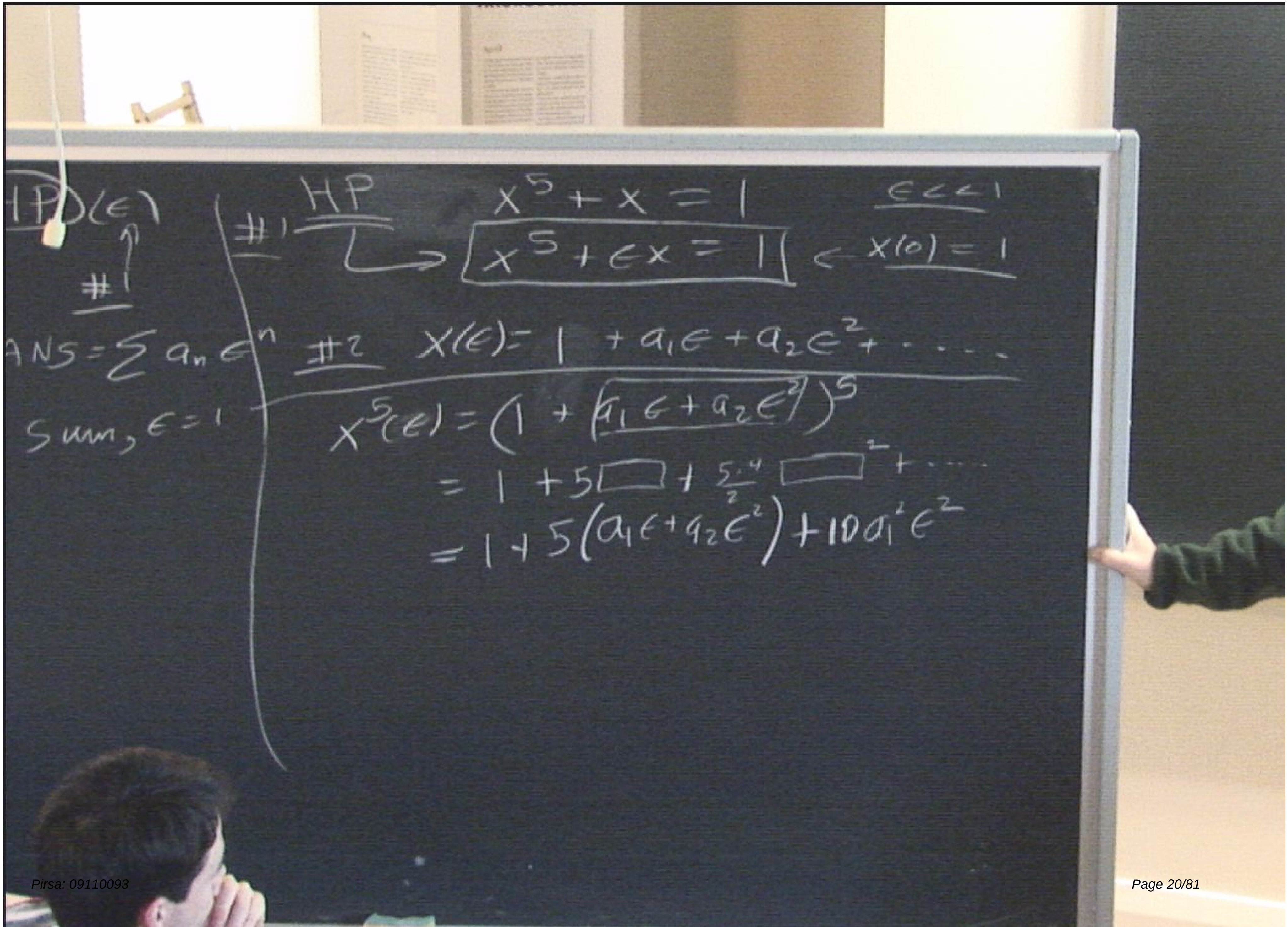

HP(ε)
#1
#2 ANS = Σ aₙ εⁿ
#3 Sum, ε = 1
#1 HP
x⁵ + x = 1
x⁵ + εx = 1
#2 x(ε) = a₀ + a₁ε + a₂ε² + . . .

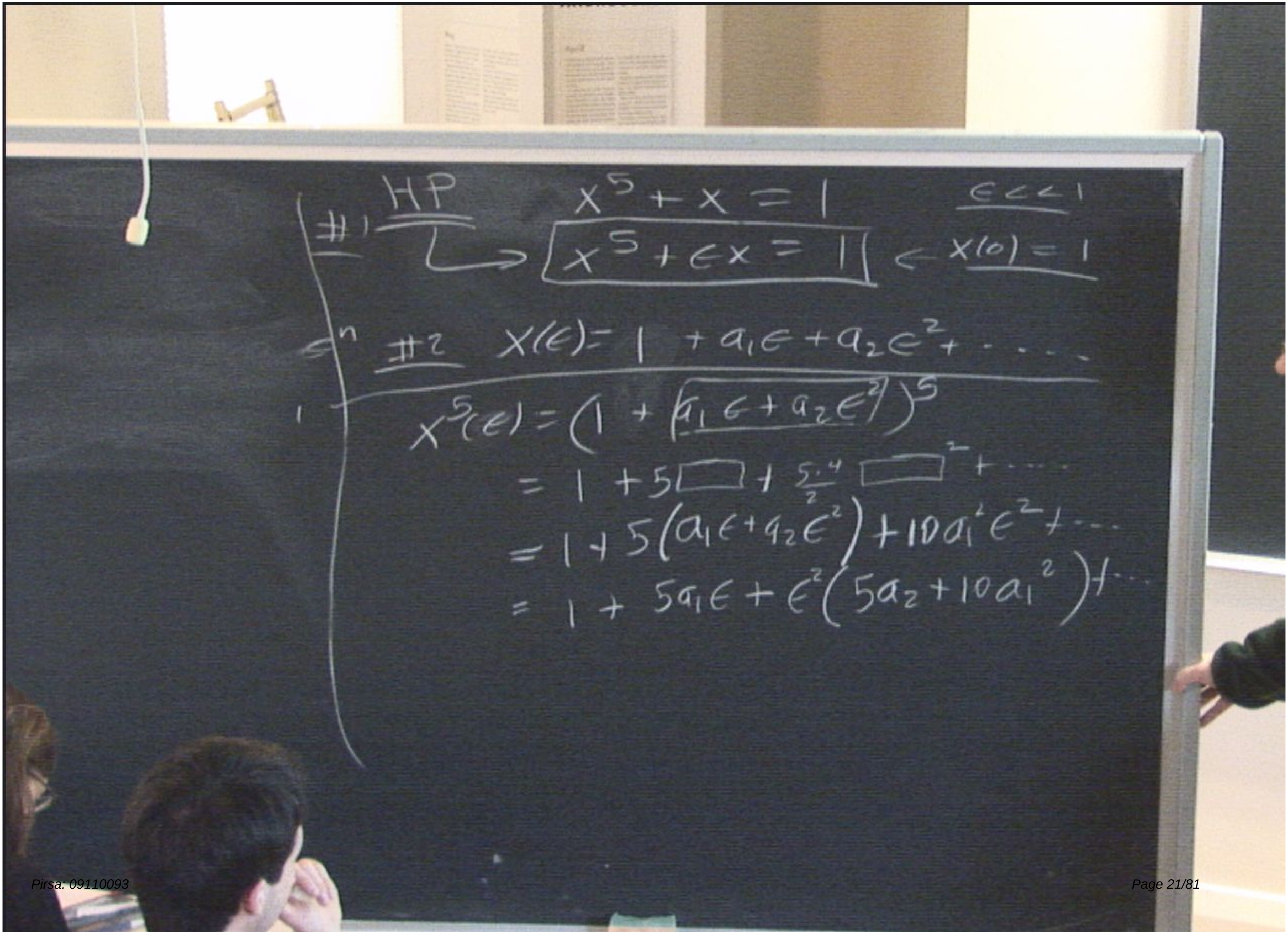

HP(ε)
#1
ANS = $\sum a_n \epsilon^n$
Sum, $\epsilon = 1$
#1 HP
$x^5 + x = 1$
$x^5 + \epsilon x = 1$
$x(0) = 1$
#2 $x(\epsilon) = 1 + a_1\epsilon + a_2\epsilon^2 + \dots$

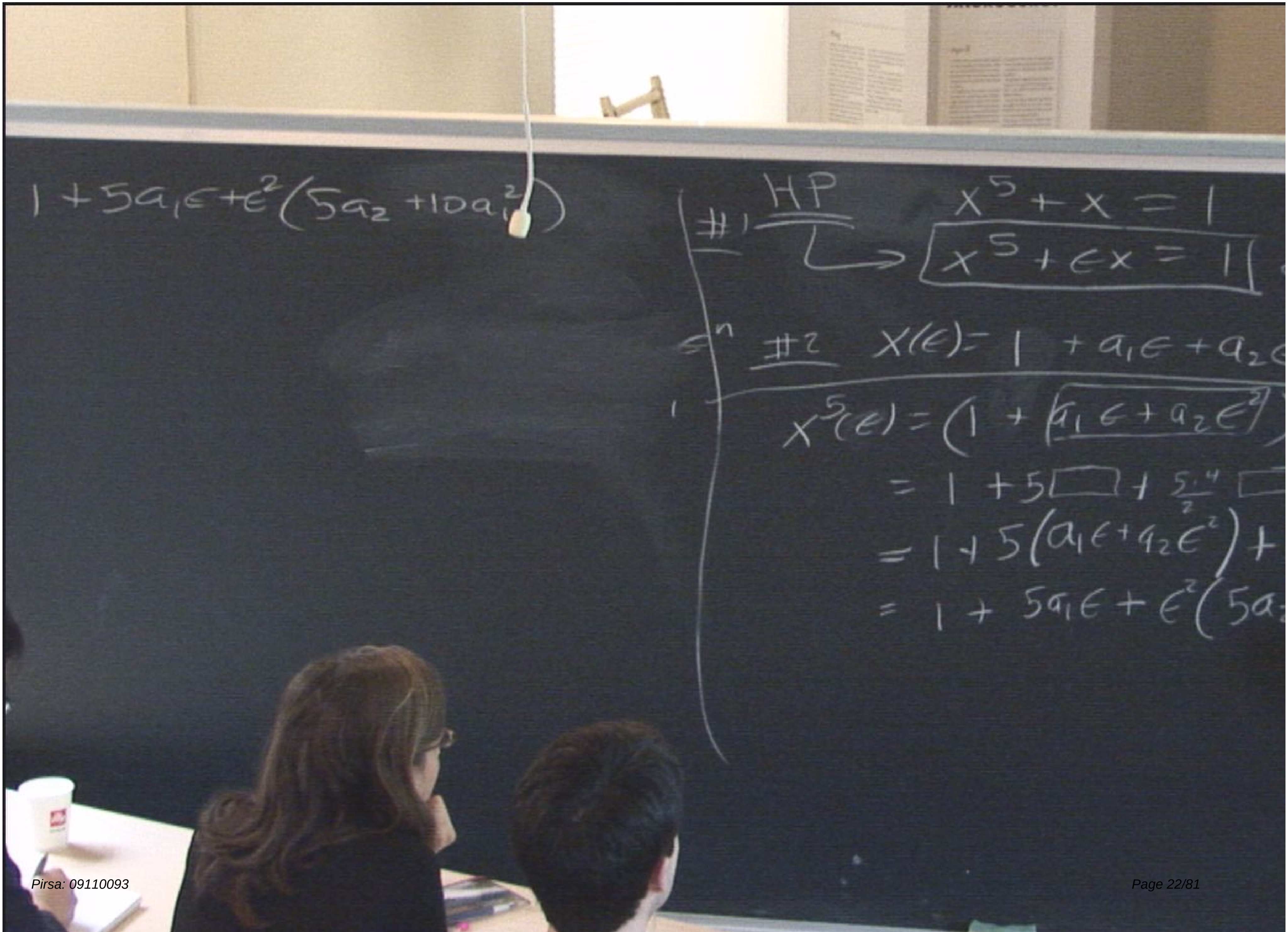

HP(ε)
#1
ANS = Σ aₙ εⁿ
Sum, ε = 1
#1 HP
x⁵ + x = 1
x⁵ + εx = 1
← x(0) = 1
#2 x(ε) = 1 + a₁ε + a₂ε² + ...
x⁵(ε) = (1 + a₁ε + a₂ε²)⁵
=

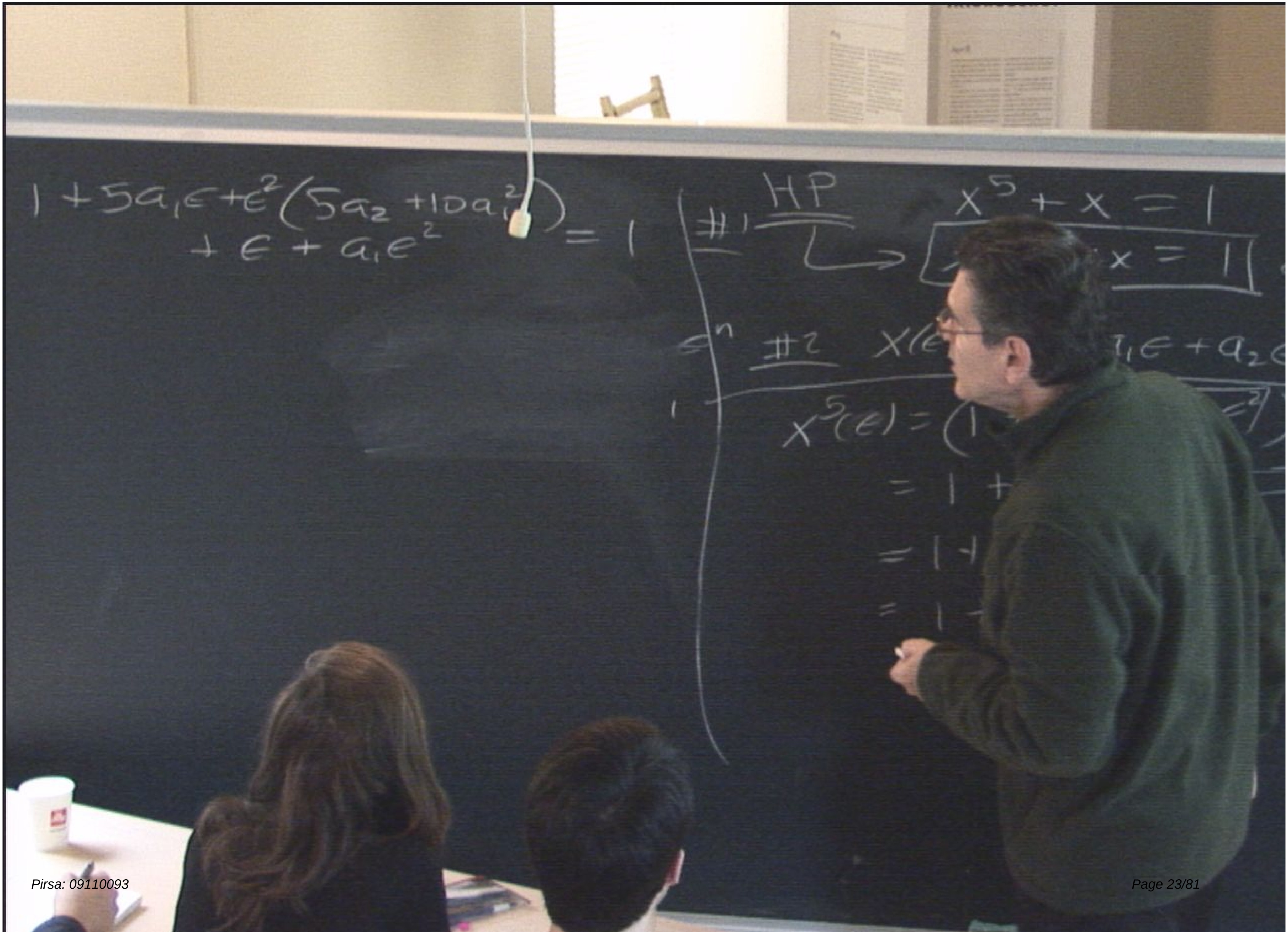

HP(ε)
#1
ANS = Σ aₙ εⁿ
Sum, ε = 1
#1 HP
x^5 + x = 1
ε << 1
x^5 + εx = 1
x(0) = 1
#2 x(ε) = 1 + a₁ε + a₂ε² + ...
x^5(ε) = (1 + a₁ε + a₂ε²)^5
= 1 +

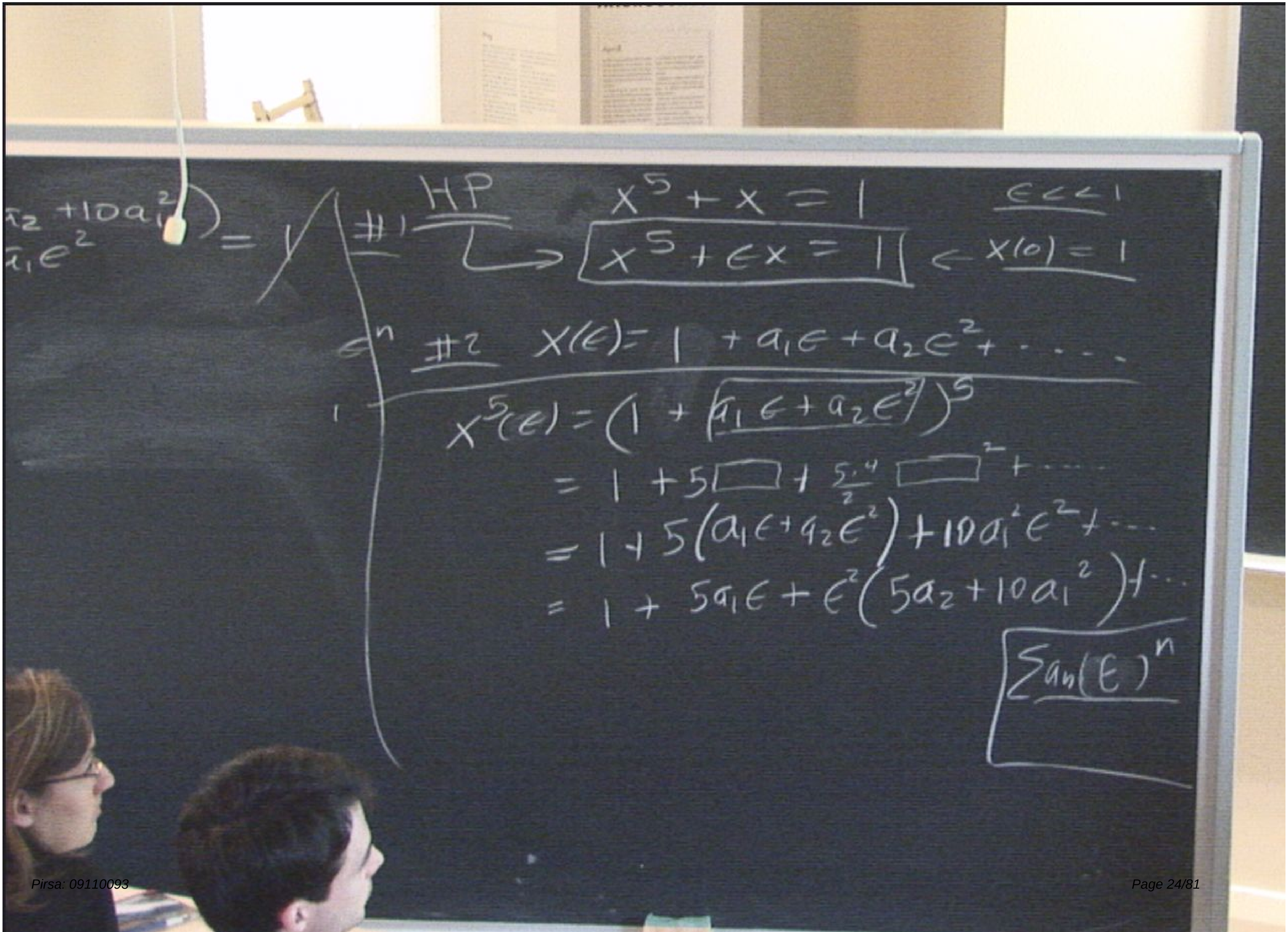

HP (ε)
#1
ANS = Σ aₙ εⁿ
Sum, ε = 1
#1 HP
x⁵ + x = 1
ε << 1
x⁵ + εx = 1
x(0) = 1
#2 x(ε) = 1 + a₁ε + a₂ε² + ...
x⁵(ε) = (1 + (a₁ε + a₂ε²))⁵
= 1 + 5☐ + 5·4/2 ☐² + ...


HP(ε)
#1
ANS = Σ a_n ε^n
Sum, ε = 1
#1 HP x^5 + x = 1
ε << 1
x^5 + εx = 1
x(0) = 1
#2 x(ε) = 1 + a_1 ε + a_2 ε^2 + ...
x^5(ε) = (1 + (a_1 ε + a_2 ε^2))^5
= 1 + 5 + 5·4/2 ^2 + ...
= 1 + 5(a_1 ε + a_2 ε^2) + 10 a_1^2 ε^2

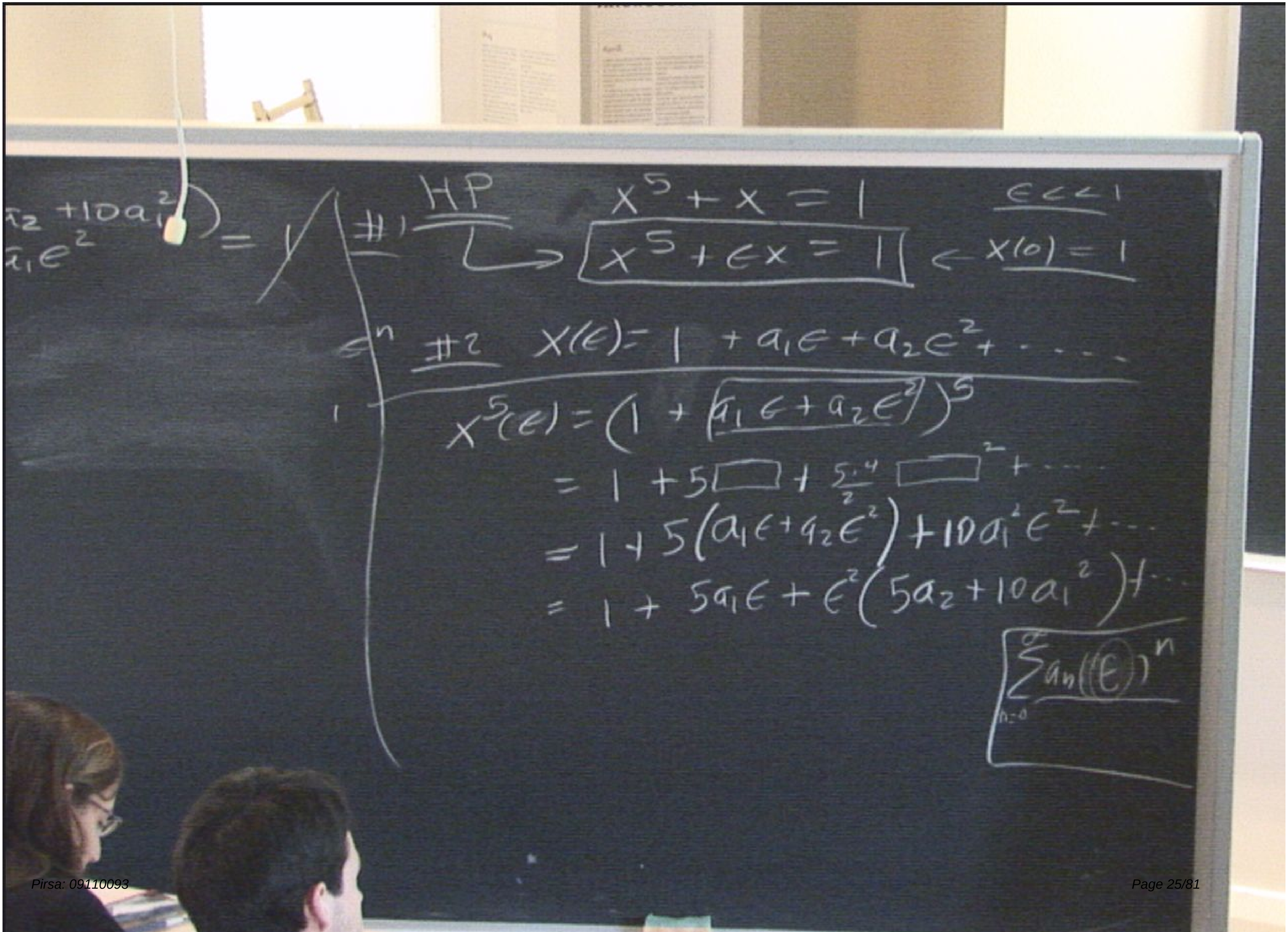

#1 HP
x5 + x = 1
ε << 1
x5 + εx = 1 ← x(0) = 1
#2 x(ε) = 1 + a1ε + a2ε2 + ...
x5(ε) = (1 + (a1ε + a2ε2))5
= 1 + 5[ ] + 5·4/2 [ ]2 + ...
= 1 + 5(a1ε + a2ε2) + 10a12ε2 + ...
= 1 + 5a1ε + ε2(5a2 + 10a12) + ...

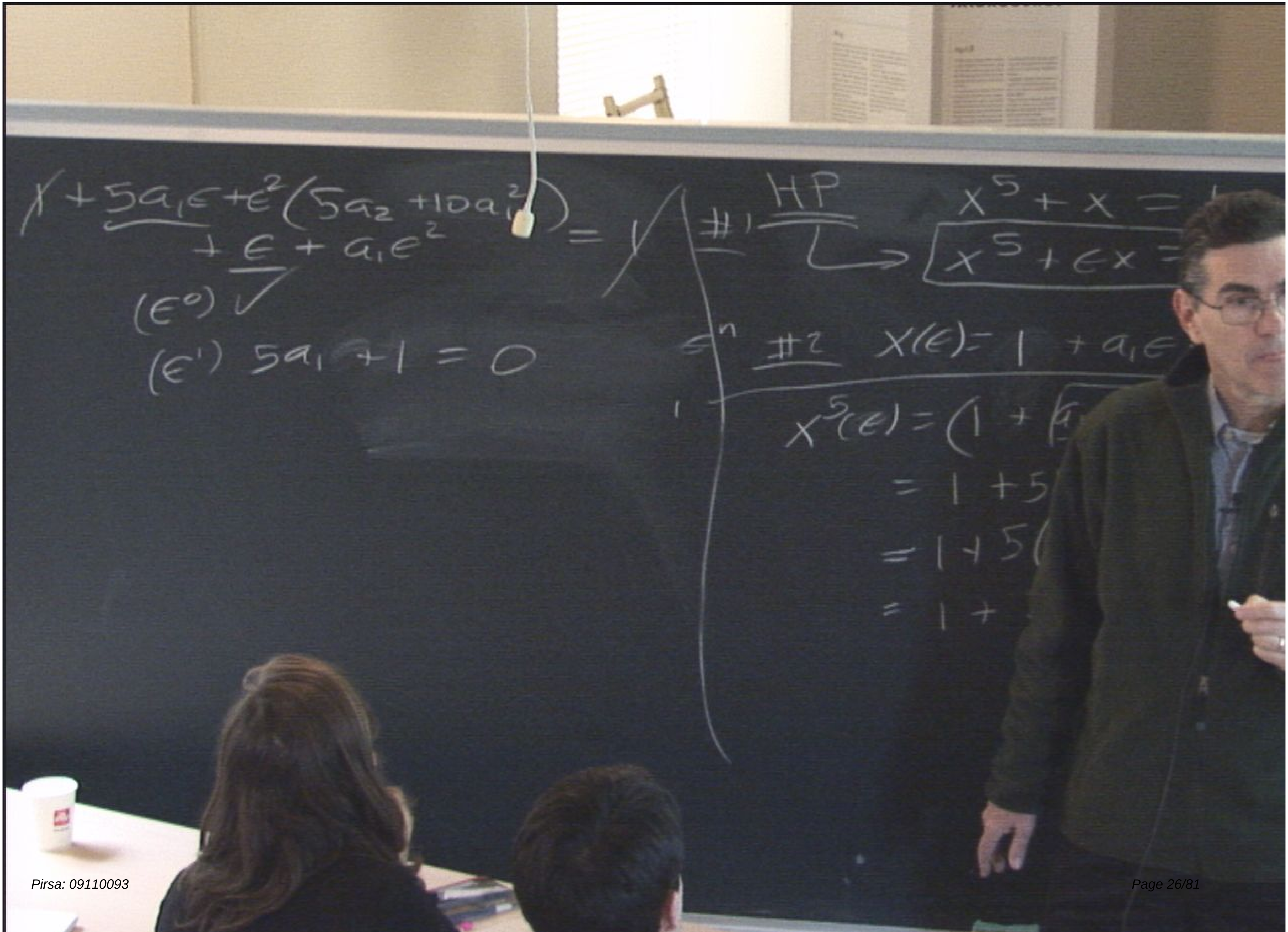

1 + 5a₁ε + ε²(5a₂ + 10a₁²)
#1 HP x⁵ + x = 1
x⁵ + εx = 1
εⁿ
#2 x(ε) = 1 + a₁ε + a₂
x⁵(ε) = (1 + (a₁ε + a₂ε²)
= 1 + 5 + 5·4/2
= 1 + 5(a₁ε + a₂ε²) +
= 1 + 5a₁ε + ε²(5a

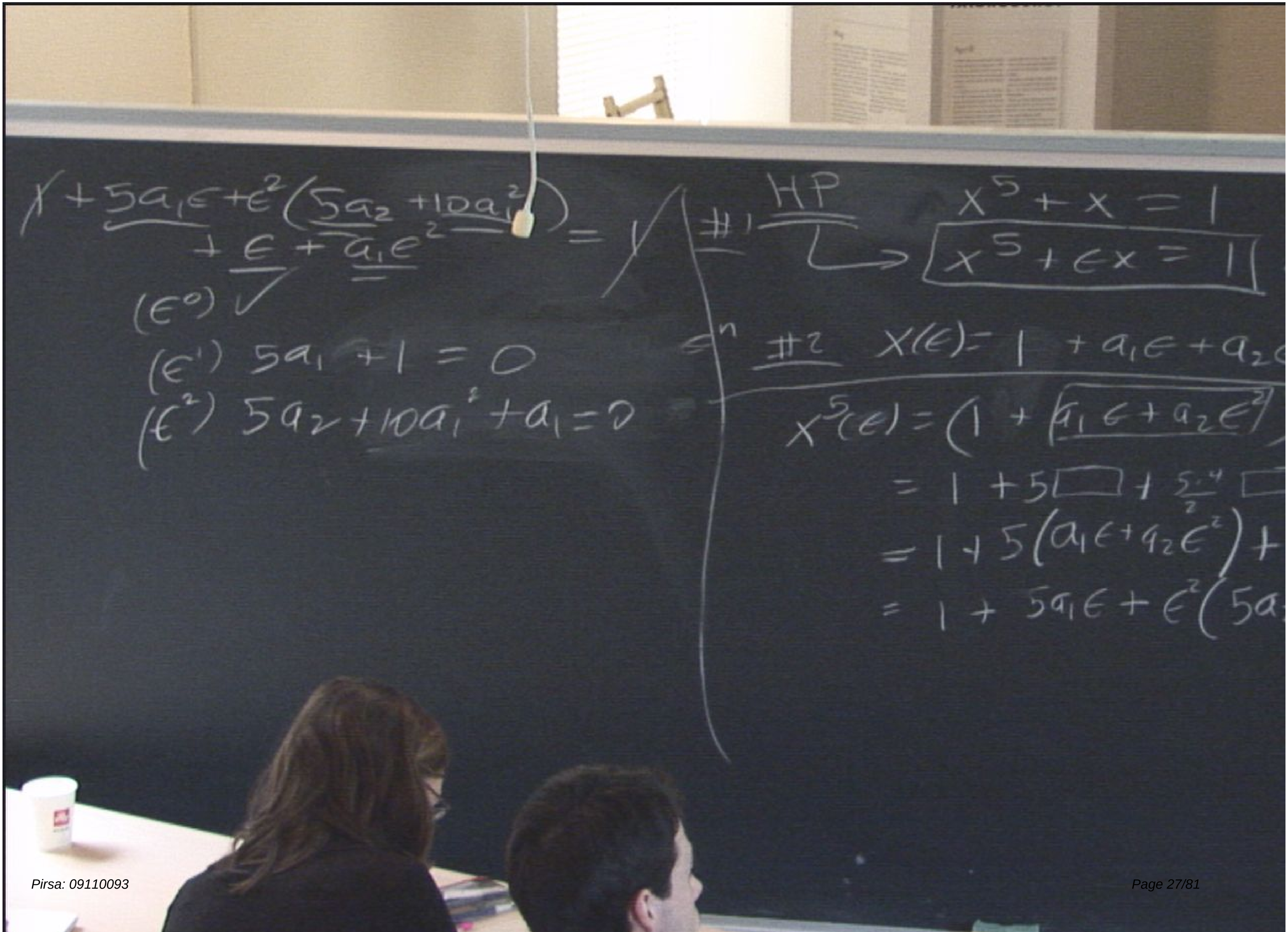

1 + 5a1ε + ε2(5a2 + 10a12) + ε + a1ε2 = 1
#1 HP
x5 + x = 1
#2
x5(ε) =

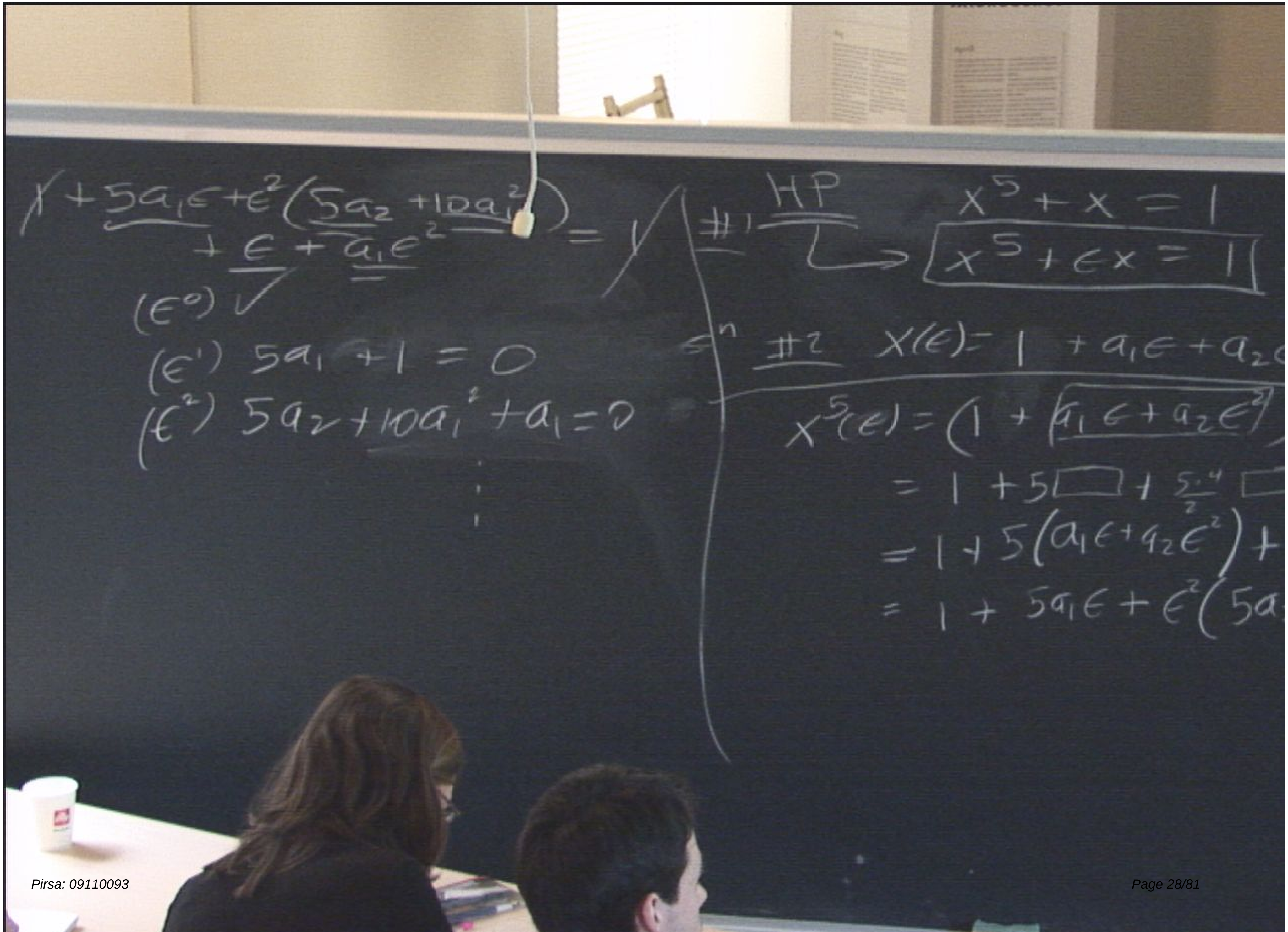


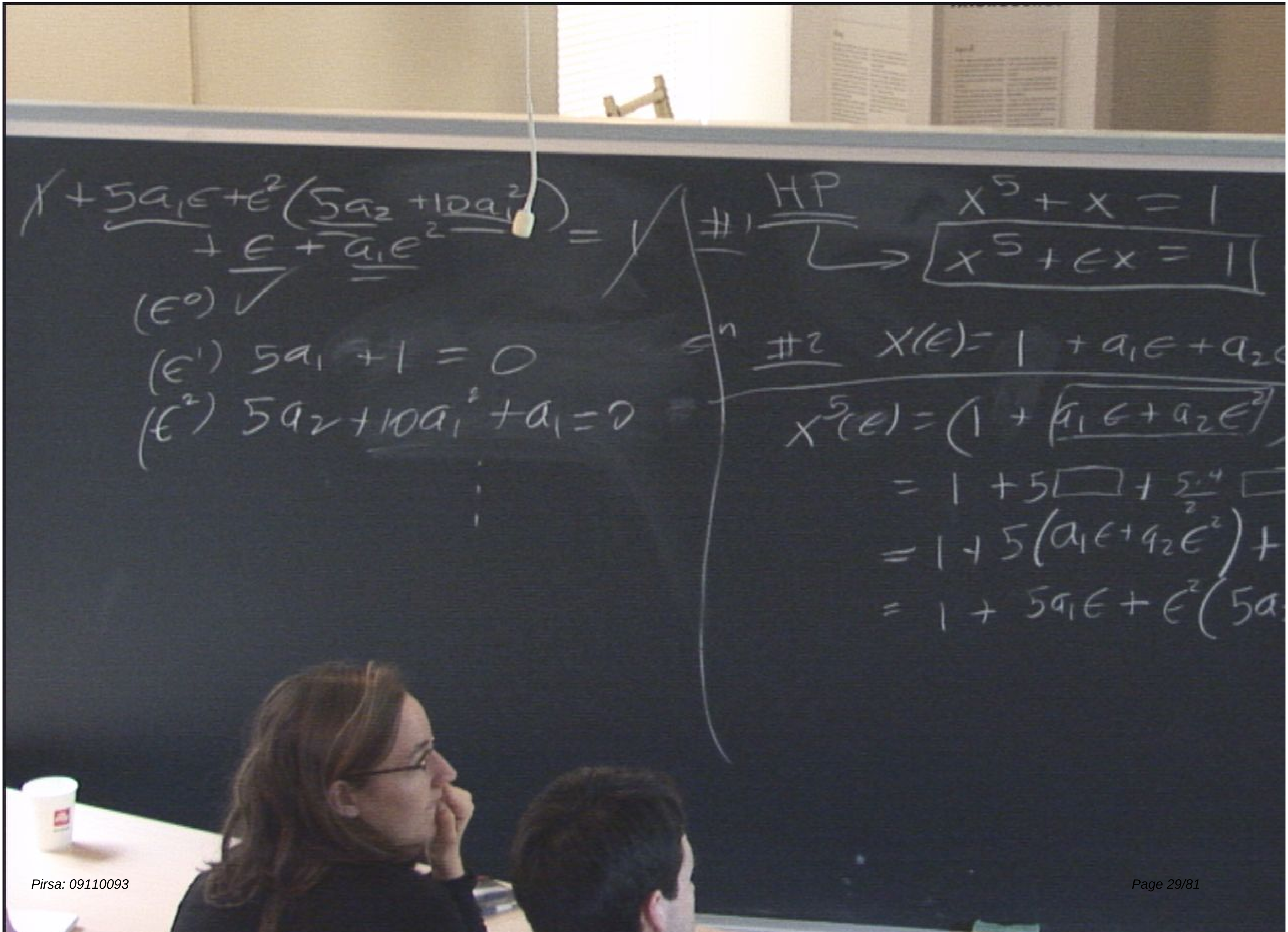
HP
#1
x^5 + x = 1
ε << 1
x^5 + εx = 1
x(0) = 1
#2 x(ε) = 1 + a_1ε + a_2ε^2 + ...
ε^n
x^5(ε) = (1 + (a_1ε + a_2ε^2))^5
= 1 + 5[] + 5·4/2 []^2 + ...
= 1 + 5(a_1ε + a_2ε^2) + 10a_1^2ε^2 + ...
= 1 + 5a_1ε + ε^2(5a_2 + 10a_1^2) + ...
Σ a_n (ε)^n

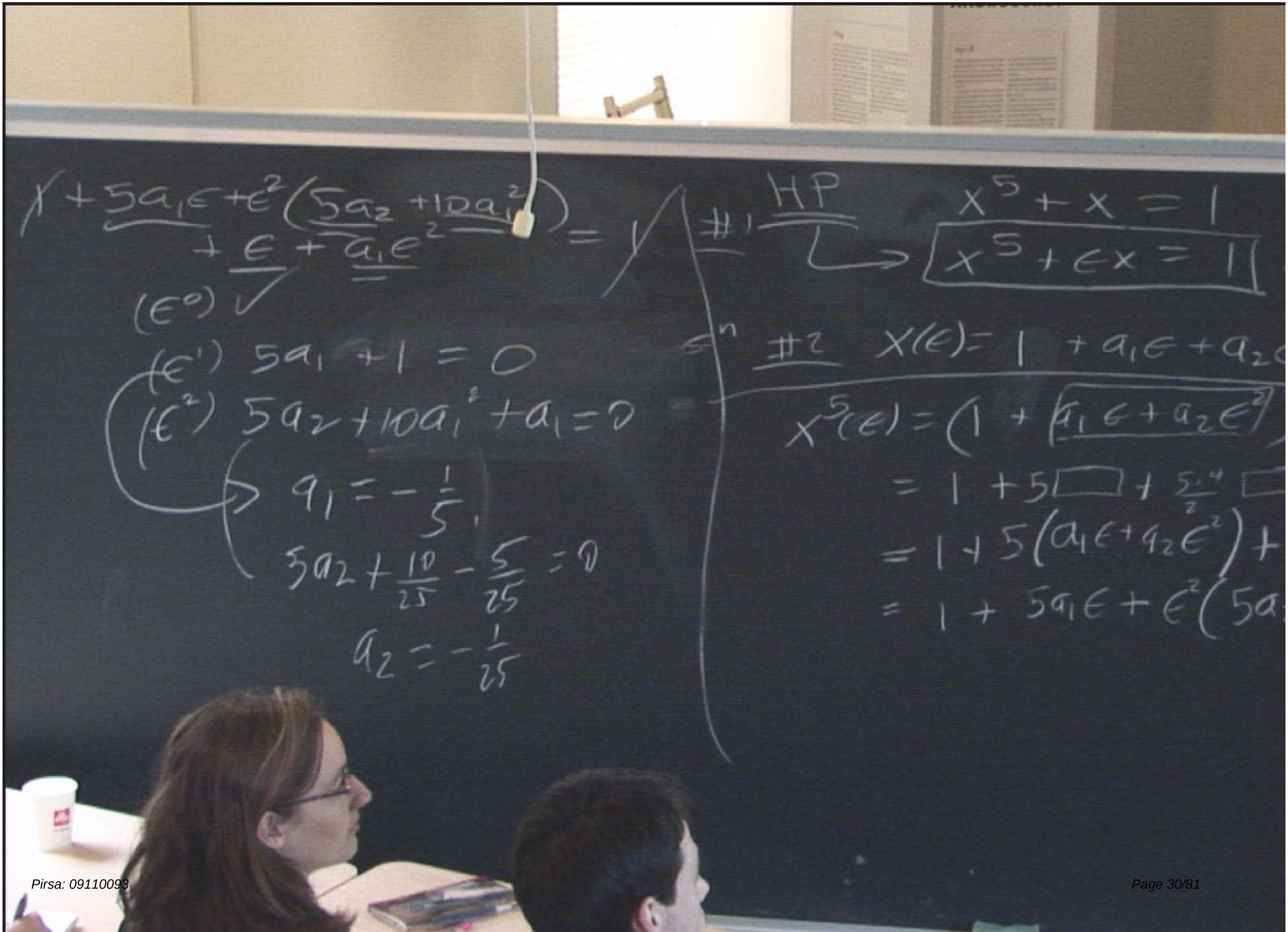

#1 HP
x^5 + x = 1
ε << 1
x^5 + εx = 1
x(0) = 1
#2 x(ε) = 1 + a1ε + a2ε^2 + ...
x^5(ε) = (1 + (a1ε + a2ε^2))^5
= 1 + 5[ ] + 5·4/2 [ ]^2 + ...
= 1 + 5(a1ε + a2ε^2) + 10a1^2ε^2 + ...
= 1 + 5a1ε + ε^2(5a2 + 10a1^2) + ...
Σ an (ε)^n
n=0

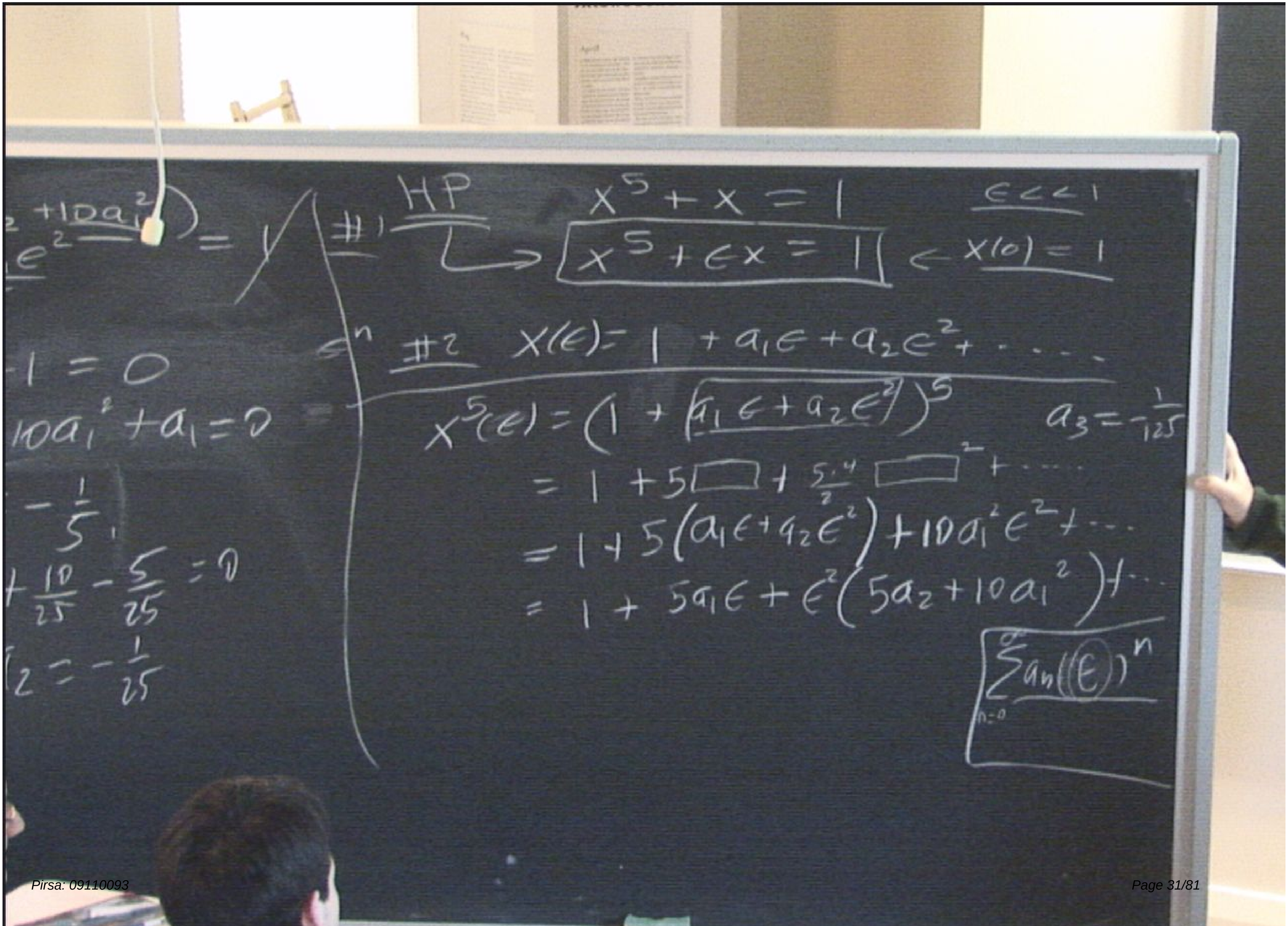

$1 + 5a_1\epsilon + \epsilon^2(5a_2 + 10a_1^2) = 1$
$+ \epsilon + a_1\epsilon^2$
(ϵ^0) ✓
$(\epsilon^1)\ 5a_1 + 1 = 0$
#1 HP
$x^5 + x = 1$
$x^5 + \epsilon x =$
ϵ^n
#2 $x(\epsilon) = 1 + a_1\epsilon$
$x^5(\epsilon) = (1 +$
$= 1 + 5$
$= 1 + 5($
$= 1 +$

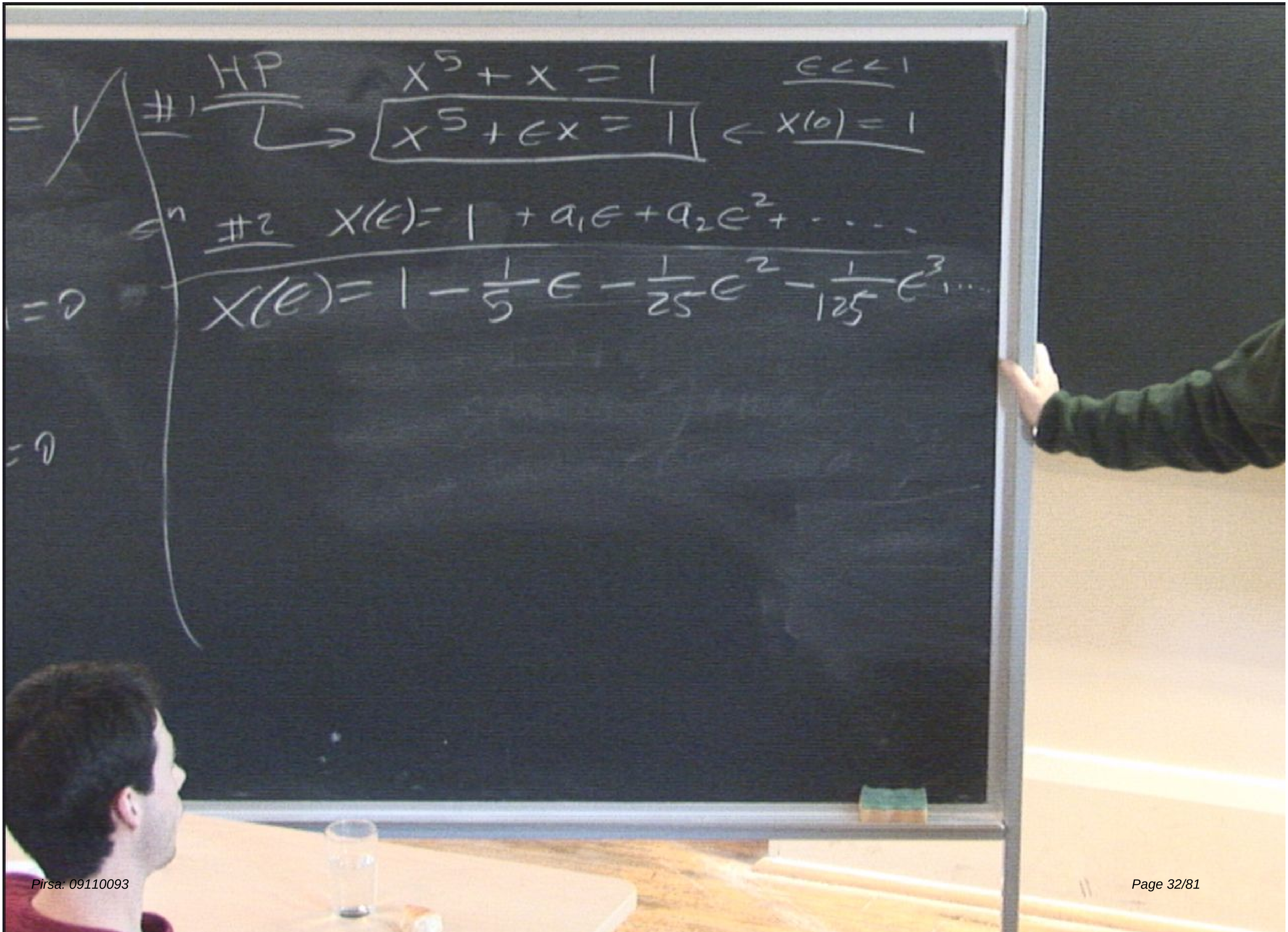

$x + 5a_1\epsilon + \epsilon^2(5a_2 + 10a_1^2)$
$+ \epsilon + a_1\epsilon^2 = 1$
(ϵ^0) ✓
$(\epsilon^1)\ 5a_1 + 1 = 0$
$(\epsilon^2)\ 5a_2 + 10a_1^2 + a_1 = 0$
HP
#1
$x^5 + x = 1$
$x^5 + \epsilon x = 1$
#2 $x(\epsilon) = 1 + a_1\epsilon + a_2$
$x^5(\epsilon) = (1 + a_1\epsilon + a_2\epsilon^2$
$= 1 + 5\square + \frac{5\cdot 4}{2}\square$
$= 1 + 5(a_1\epsilon + a_2\epsilon^2) +$
$= 1 + 5a_1\epsilon + \epsilon^2(5a$

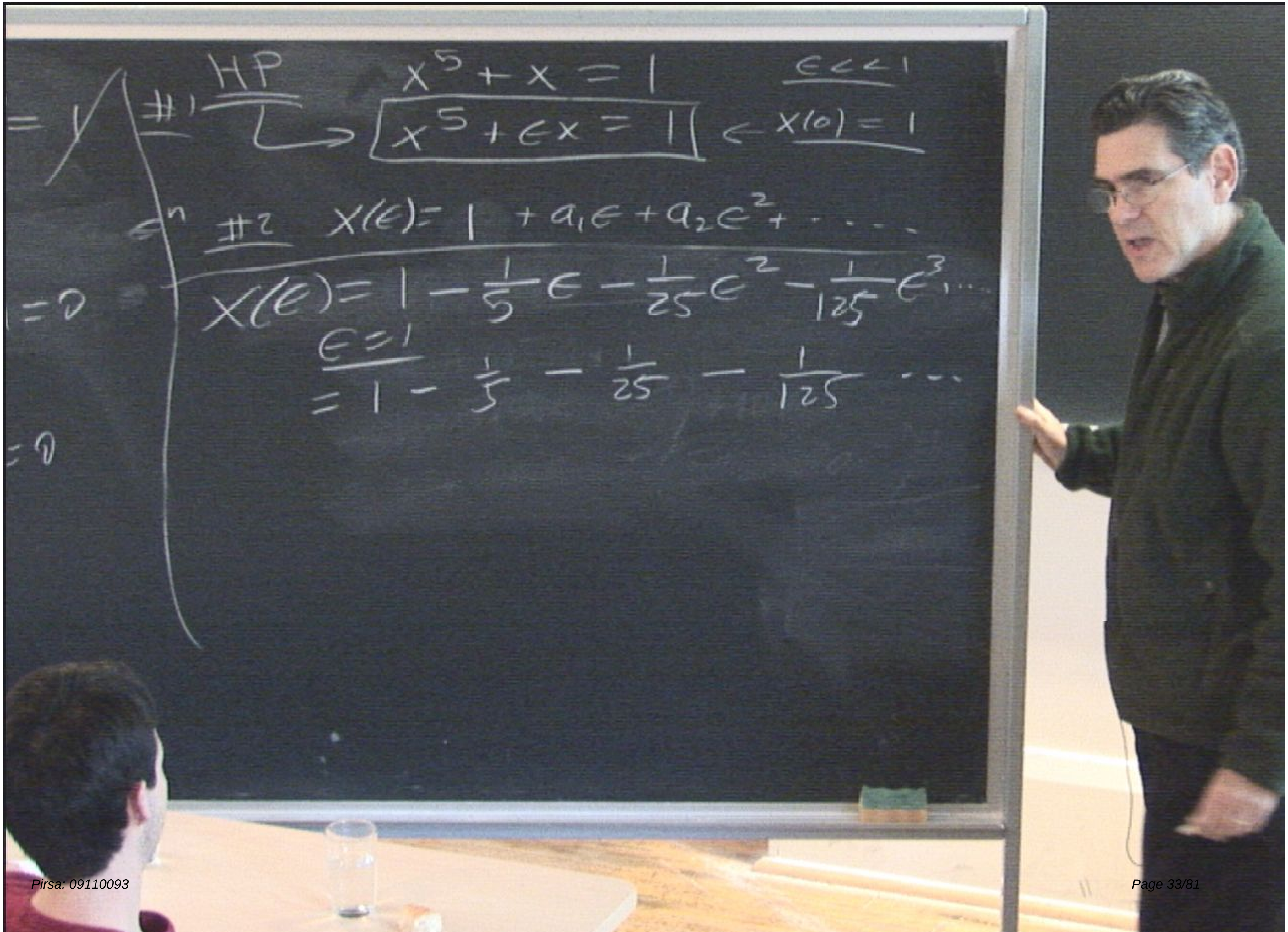

1 + 5a₁ε + ε²(5a₂ + 10a₁²) + ε + a₁ε² = 1
(ε⁰) ✓
(ε¹) 5a₁ + 1 = 0
(ε²) 5a₂ + 10a₁² + a₁ = 0
HP
#1 x⁵ + x = 1
x⁵ + εx = 1
#2 x(ε) = 1 + a₁ε + a₂ε
x⁵(ε) = (1 + a₁ε + a₂ε²
= 1 + 5 + 5·4/2
= 1 + 5(a₁ε + a₂ε²) +
= 1 + 5a₁ε + ε²(5a

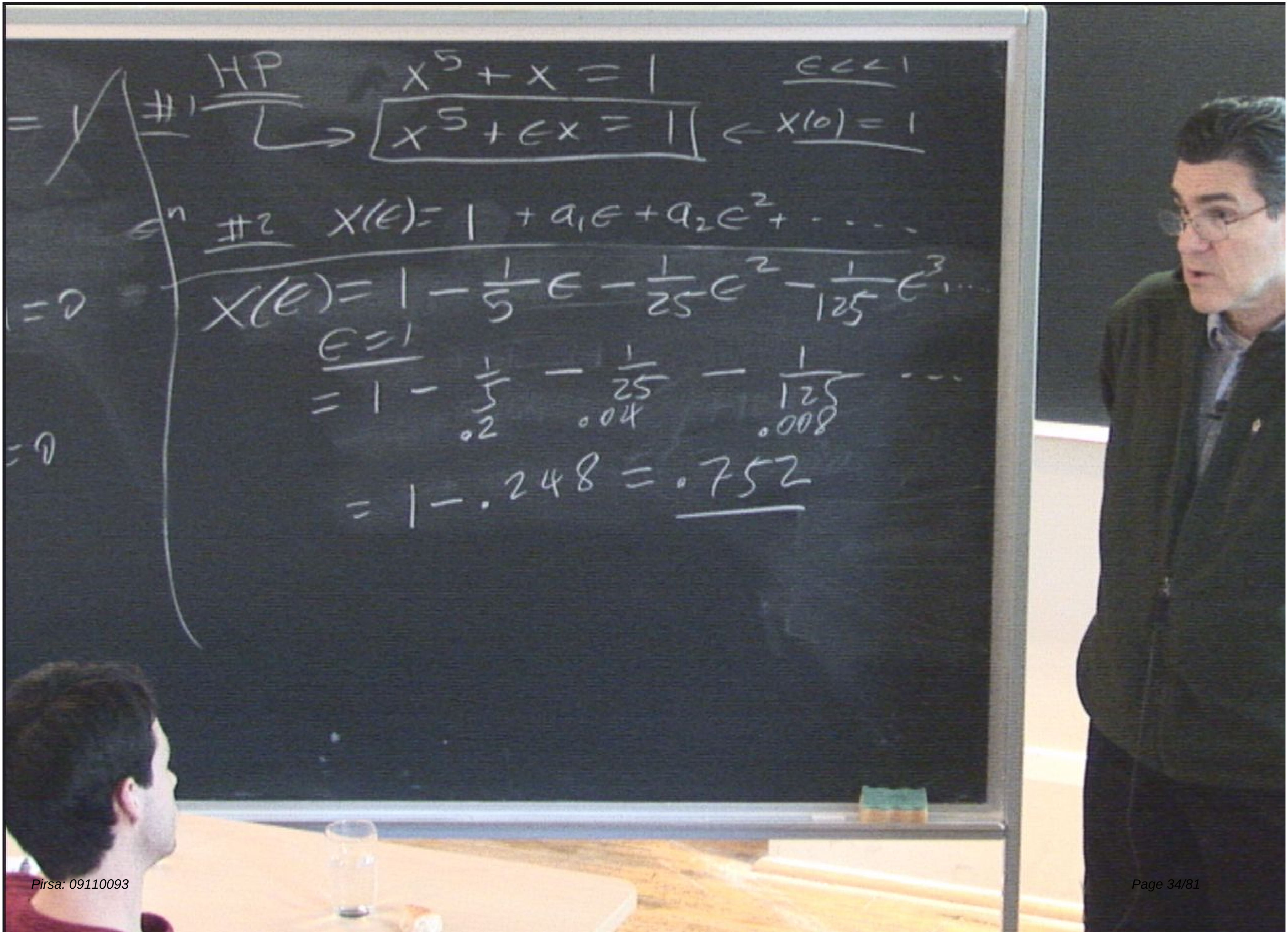

HP
#1
x^5 + x = 1
x^5 + εx = 1
#2
x(ε) = 1 + a_1 ε + a_2
x^5(ε) = (1 + a_1 ε + a_2 ε^2)
= 1 + 5 + (5·4)/2
= 1 + 5(a_1 ε + a_2 ε^2) +
= 1 + 5a_1 ε + ε^2 (5a
1 + 5a_1 ε + ε^2 (5a_2 + 10a_1^2)
+ ε + a_1 ε^2
= 1
(ε^0) ✓
(ε^1) 5a_1 + 1 = 0
(ε^2) 5a_2 + 10a_1^2 + a_1 = 0


HP
#1
x^5 + x = 1
x^5 + εx = 1
#2
x(ε) = 1 + a_1 ε + a_2
x^5(ε) = (1 + a_1 ε + a_2 ε^2
= 1 + 5a_1 ε + ε^2(5a
(ε^0) ✓
(ε^1) 5a_1 + 1 = 0
(ε^2) 5a_2 + 10a_1^2 + a_1 = 0
a_1 = −1/5
5a_2 + 10/25 − 5/25 = 0
a_2 = −1/25


HP
#1
x^5 + x = 1
ε << 1
x^5 + εx = 1
x(0) = 1
#2
x(ε) = 1 + a_1ε + a_2ε^2 + ...
x^5(ε) = (1 + (a_1ε + a_2ε^2))^5
a_3 = -1/125
= 1 + 5[ ] + 5·4/2 [ ]^2 + ...
= 1 + 5(a_1ε + a_2ε^2) + 10a_1^2ε^2 + ...
= 1 + 5a_1ε + ε^2(5a_2 + 10a_1^2) + ...
Σ_{n=0}^{∞} a_n (ε)^n
+10a_1^2
= 1
-1 = 0
10a_1^2 + a_1 = 0
-1/5
+ 10/25 - 5/25 = 0
a_2 = -1/25


HP
x^5 + x = 1
ε << 1
#1
x^5 + εx = 1
x(0) = 1
#2 x(ε) = 1 + a_1 ε + a_2 ε^2 + ...
x(ε) = 1 − 1/5 ε − 1/25 ε^2 − 1/125 ε^3 ....


HP
x^5 + x = 1
ε << 1
#1
x^5 + εx = 1
x(0) = 1
#2 x(ε) = 1 + a₁ε + a₂ε² + ...
x(ε) = 1 − 1/5 ε − 1/25 ε² − 1/125 ε³, . . .
ε = 1
= 1 − 1/5 − 1/25 − 1/125 . . .


HP
x^5 + x = 1
ε << 1
#1
x^5 + εx = 1
x(0) = 1
#2 x(ε) = 1 + a_1 ε + a_2 ε^2 + ...
x(ε) = 1 − 1/5 ε − 1/25 ε^2 − 1/125 ε^3 ...
ε = 1
= 1 − 1/5 − 1/25 − 1/125 − . . .
.2 .04 .008
= 1 − .248 = .752

Match powers of ε

$$
\begin{aligned}
5a_1 + 1 &= 0, \\
5a_2 + 10a_1^2 + a_1 &= 0, \\
5a_3 + 20a_1a_2 + a_2 + 10a_1^3 &= 0, \\
5a_4 + 20a_1a_3 + a_3 + 10a_2^2 + 30a_1^2a_2 + 5a_1^4 &= 0.
\end{aligned}
$$

$$a_1 = -\frac{1}{5}, \quad a_2 = -\frac{1}{25}, \quad a_3 = -\frac{1}{125},$$

$$a_4 = 0, \quad a_5 = \frac{21}{15625}, \quad a_6 = \frac{78}{78125}$$

The perturbation series...

$$x(\epsilon) \;=\; 1-\frac{1}{5}\epsilon-\frac{1}{25}\epsilon^2-\frac{1}{125}\epsilon^3+\frac{21}{15625}\epsilon^5+\frac{78}{78125}\epsilon^6+\cdots$$

Step 3. Sum the series at $x = 1$

Radius of convergence of this series: 1.64938...

Sixth-order result $x(1) = 0.75434$

Exact answer $x = 0.75488$

Another way to insert ε

Step 1. Insert ε: $\epsilon x^5 + x = 1$ (weak coupling)

Step 2. Perturbation series

$$x(\epsilon) = 1 - \epsilon + 5\epsilon^2 - 35\epsilon^3 + 285\epsilon^4 - 2530\epsilon^5 + 23751\epsilon^6 - \cdots$$

Step 3. Sum the series at $x = 1$

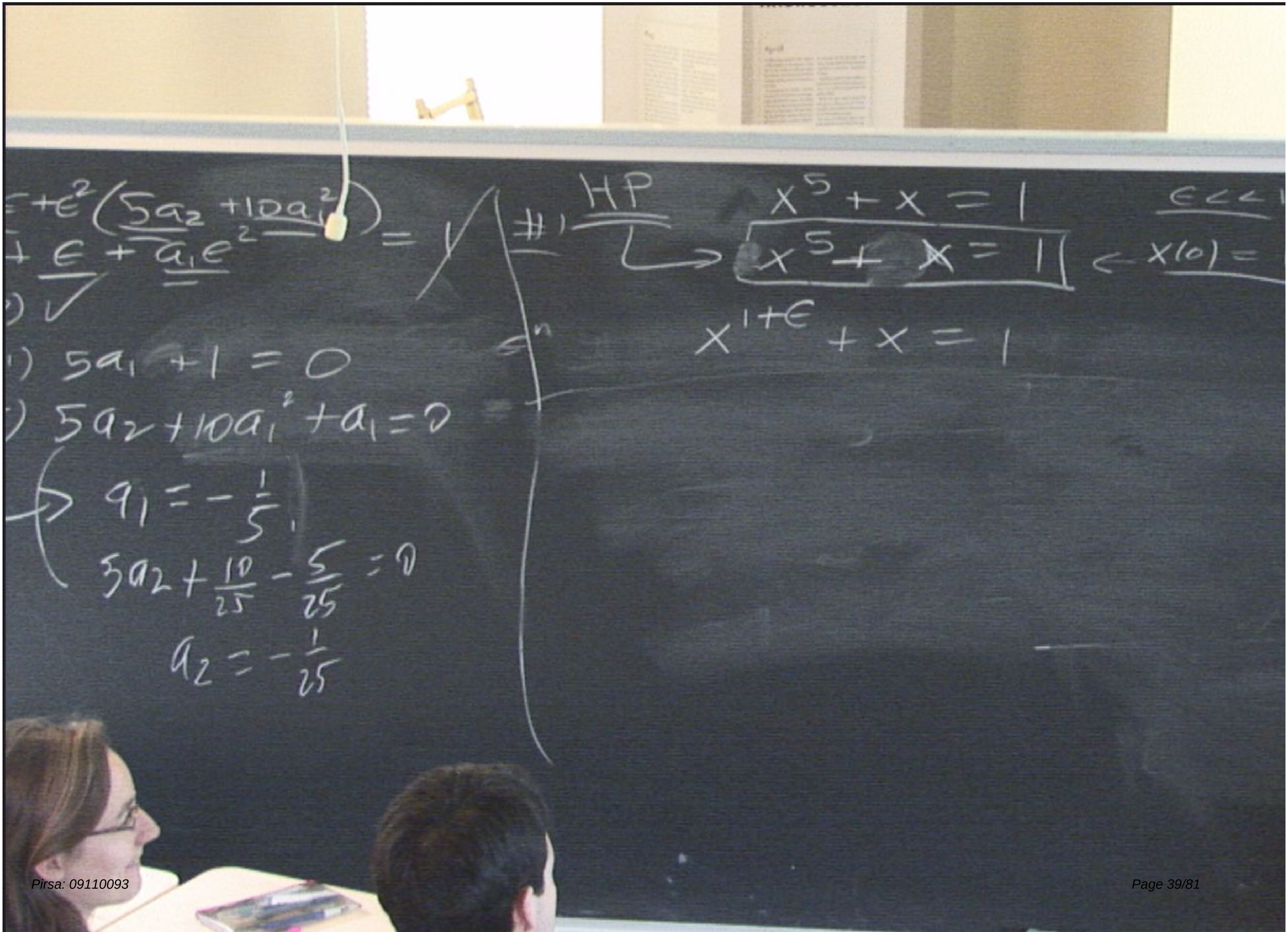
Radius of convergence of this series 0.08192

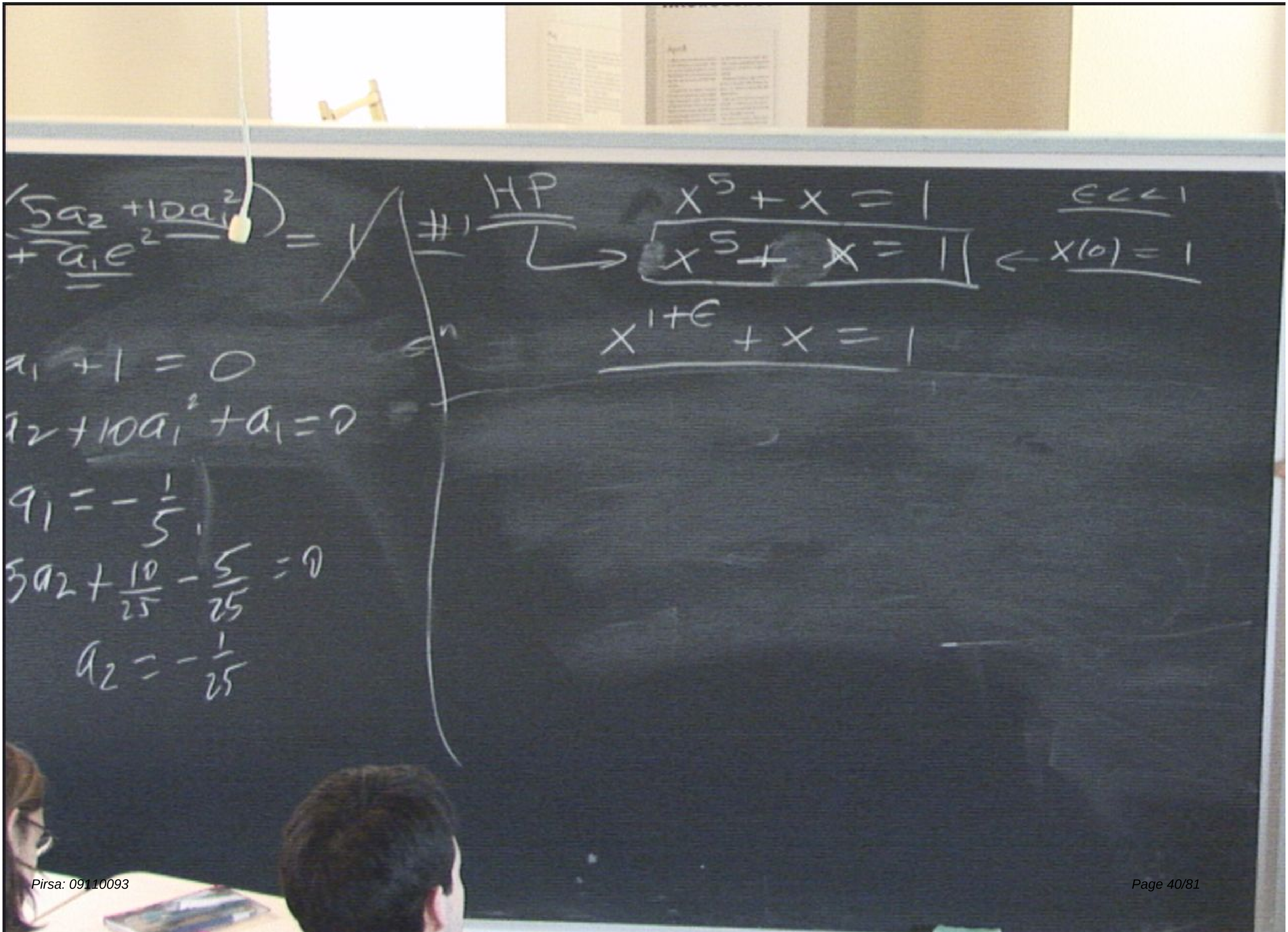
Result: $x(1) = 21476$

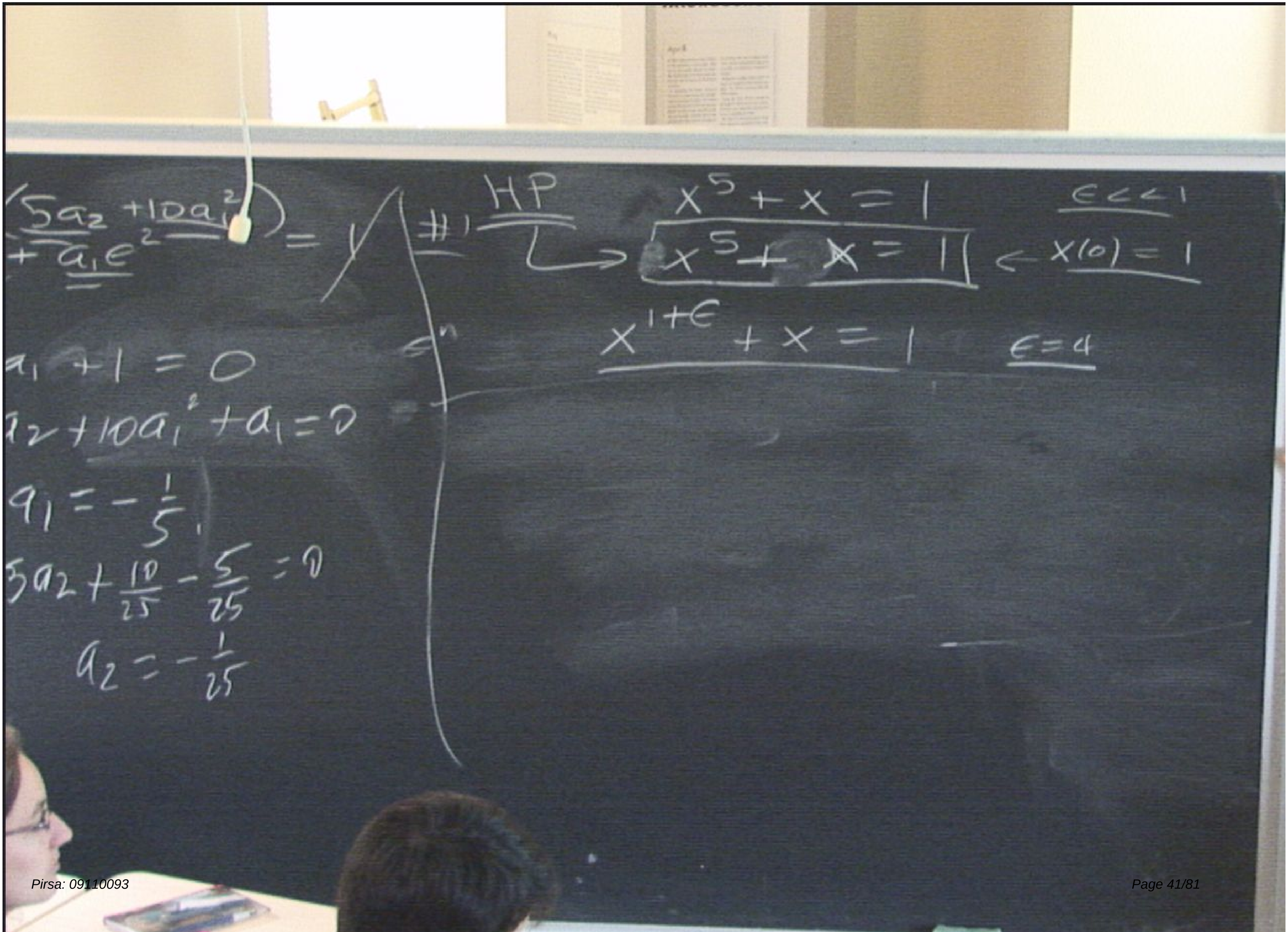
Yet another way to insert ε

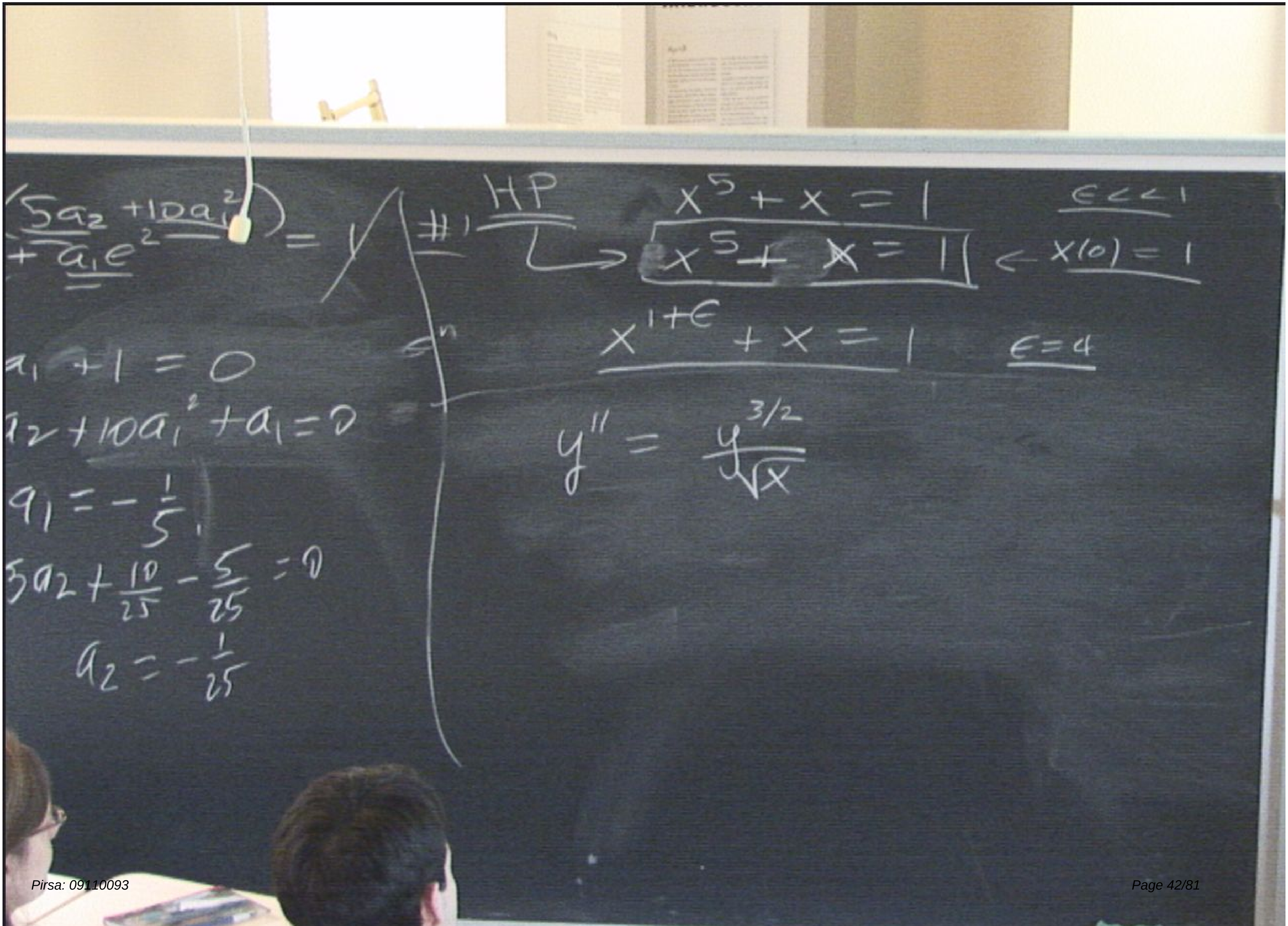
Put it in the exponent…

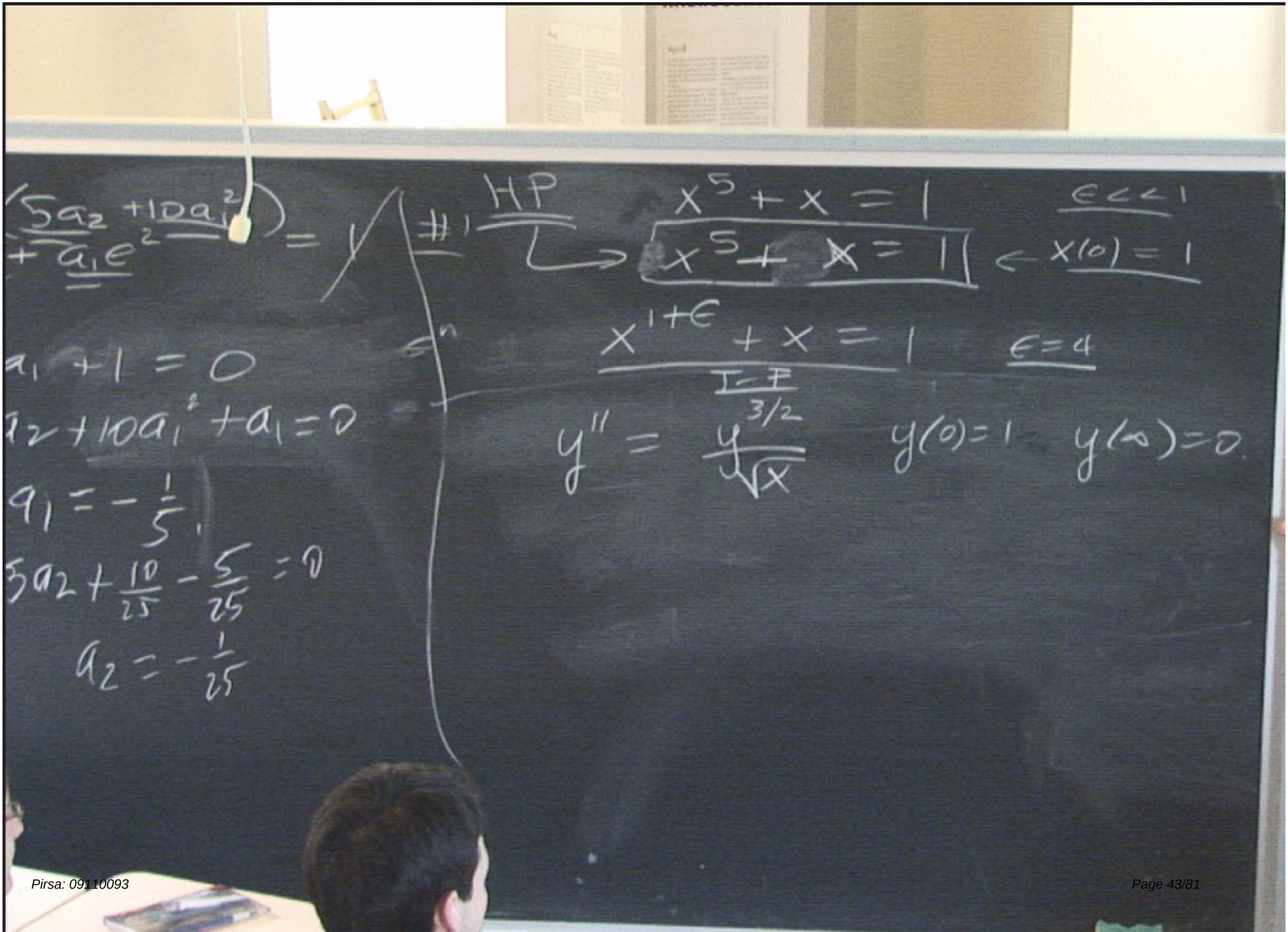
Example: Thomas-Fermi equation

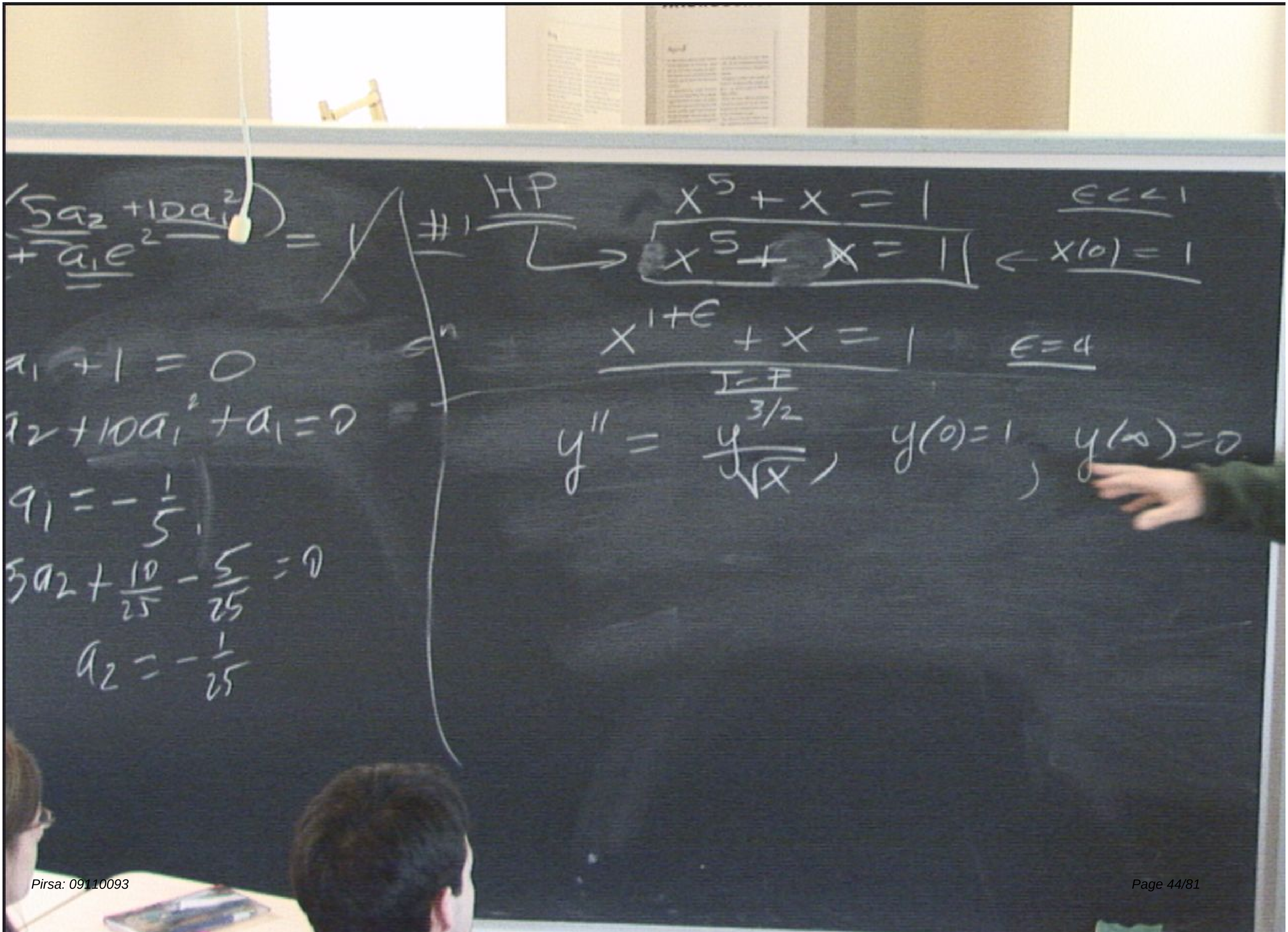

HP
x5 + x = 1
#1
x5 + x = 1
x1+ε + x = 1
5a1 + 1 = 0
5a2 + 10a1^2 + a1 = 0
a1 = - 1/5
5a2 + 10/25 - 5/25 = 0
a2 = - 1/25

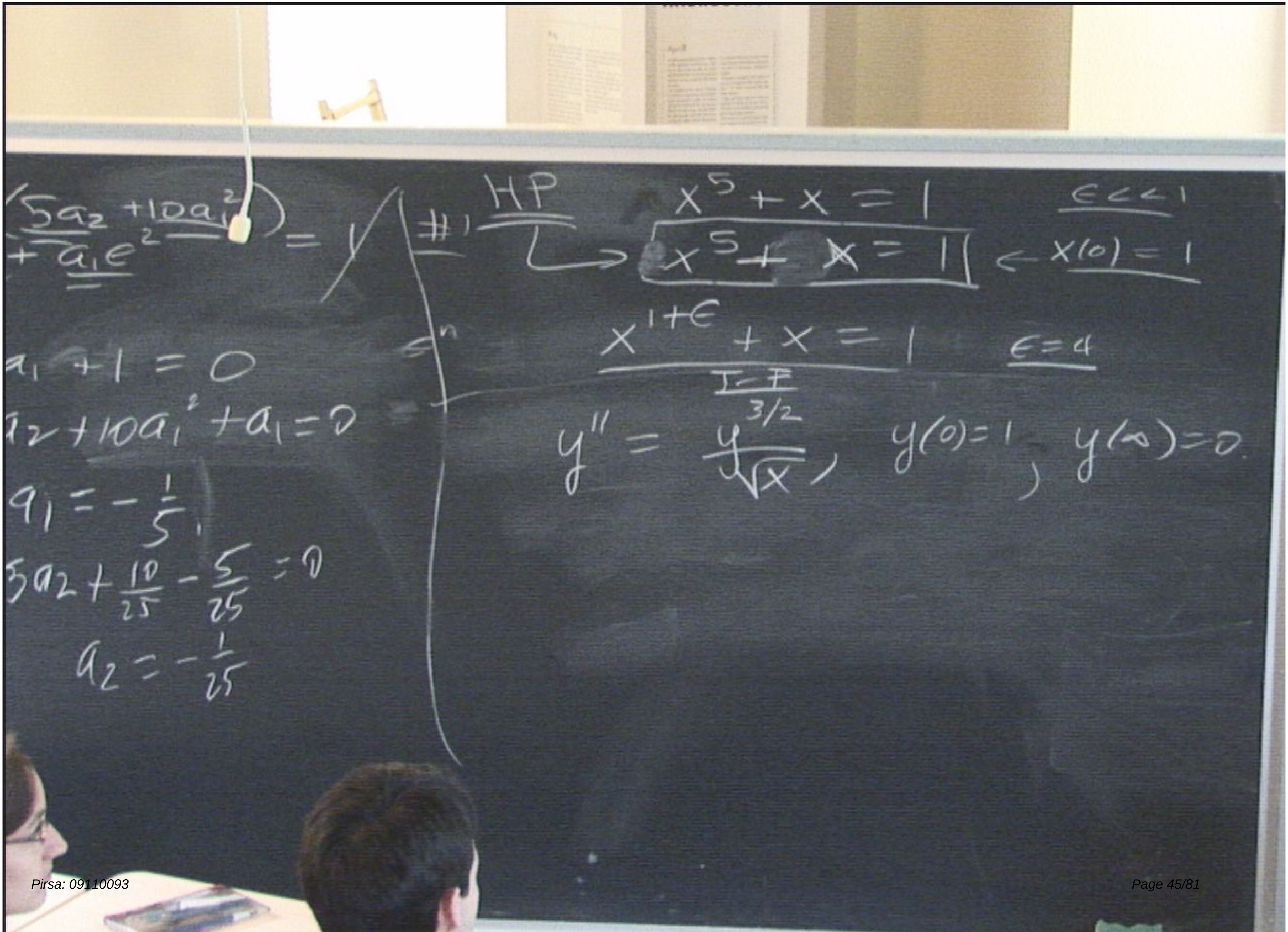

(5a2 + 10a1²) = 1
+ a1ε²
a1 + 1 = 0
5a2 + 10a1² + a1 = 0
a1 = -1/5
5a2 + 10/25 - 5/25 = 0
a2 = -1/25
#1 HP
x⁵ + x = 1
ε << 1
x⁵ + x = 1
← x(0) = 1
x^(1+ε) + x = 1

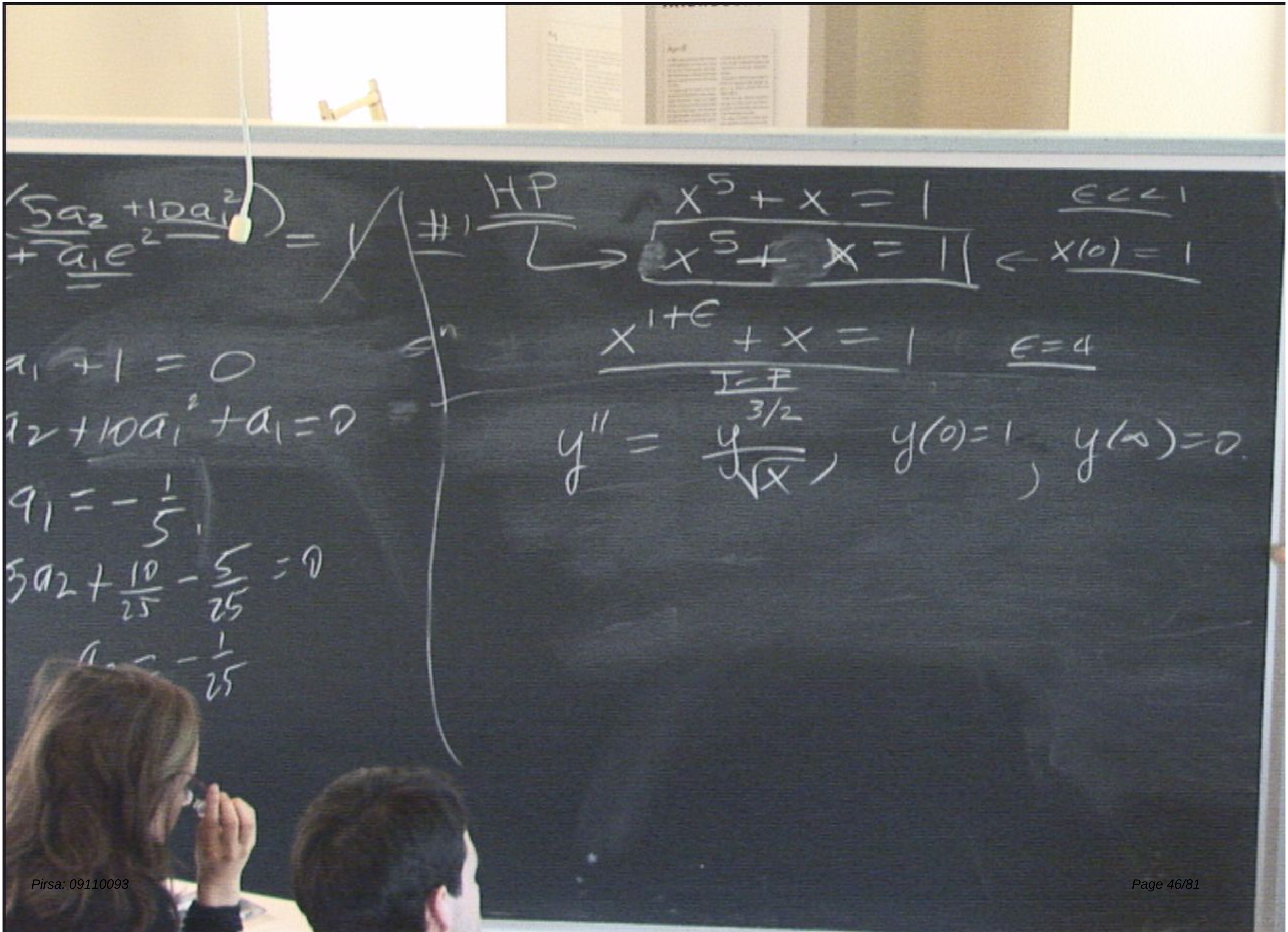

HP
#1
x^5 + x = 1
ε << 1
x^5 + x = 1
x(0) = 1
x^{1+ε} + x = 1
ε = 4
a_1 + 1 = 0
a_2 + 10a_1^2 + a_1 = 0
a_1 = -1/5
5a_2 + 10/25 - 5/25 = 0
a_2 = -1/25

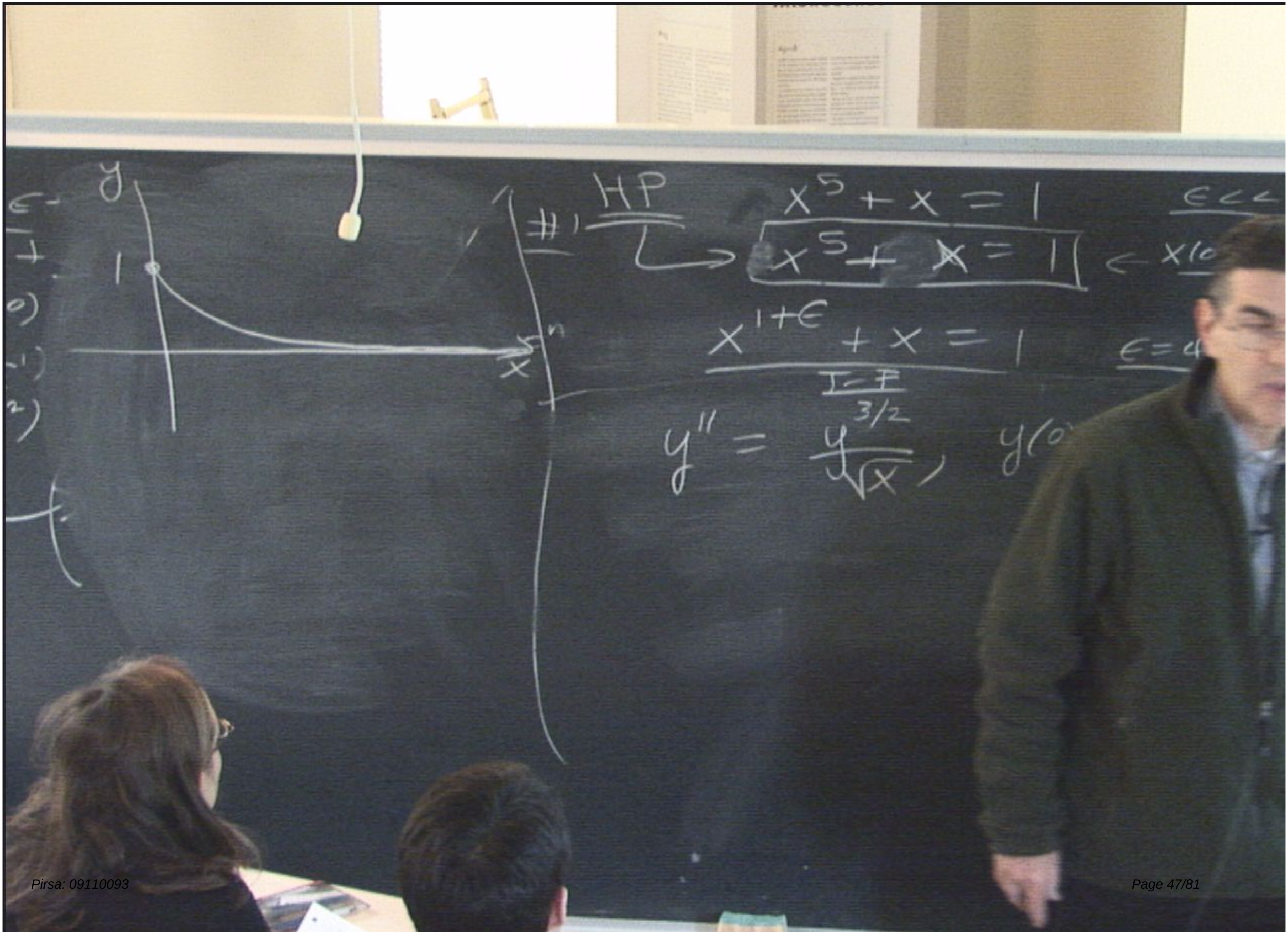

HP
#1
x^5 + x = 1
ε << 1
x^5 + x = 1
x(0) = 1
x^{1+ε} + x = 1
ε = 4
a_1 + 1 = 0
a_1 = -1/5
5a_2 + 10/25 - 5/25 = 0
a_2 = -1/25
y'' = y^{3/2}/√x

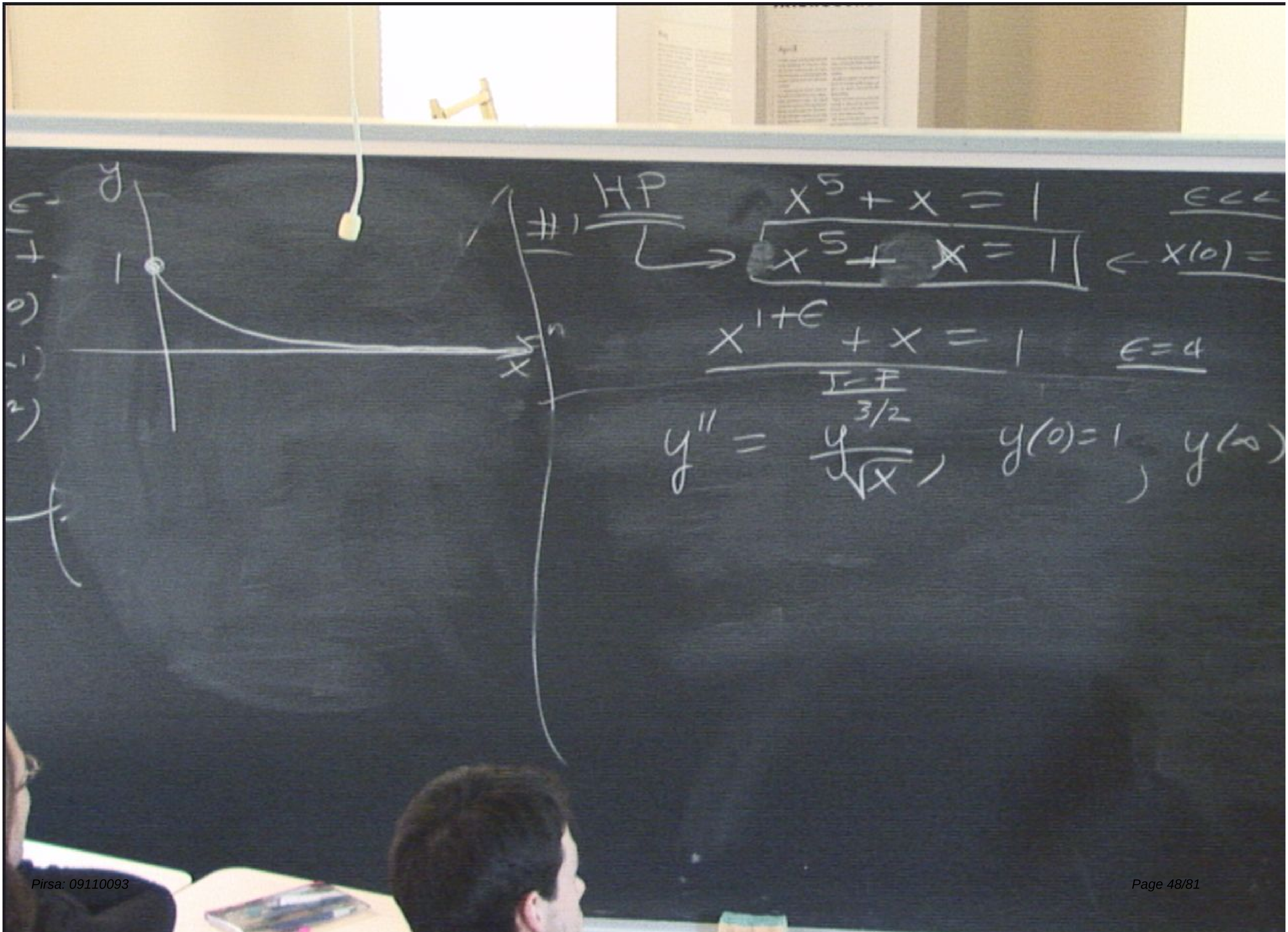

HP
#1
x^5 + x = 1
ε << 1
x^5 + x = 1
x(0) = 1
x^{1+ε} + x = 1
ε = 4
I-F
y'' = y^{3/2} / √x
y(0) = 1
y(∞) = 0
a_1 + 1 = 0
a_1 = -1/5
3a_2 + 10/25 - 5/25 = 0
a_2 = -1/25

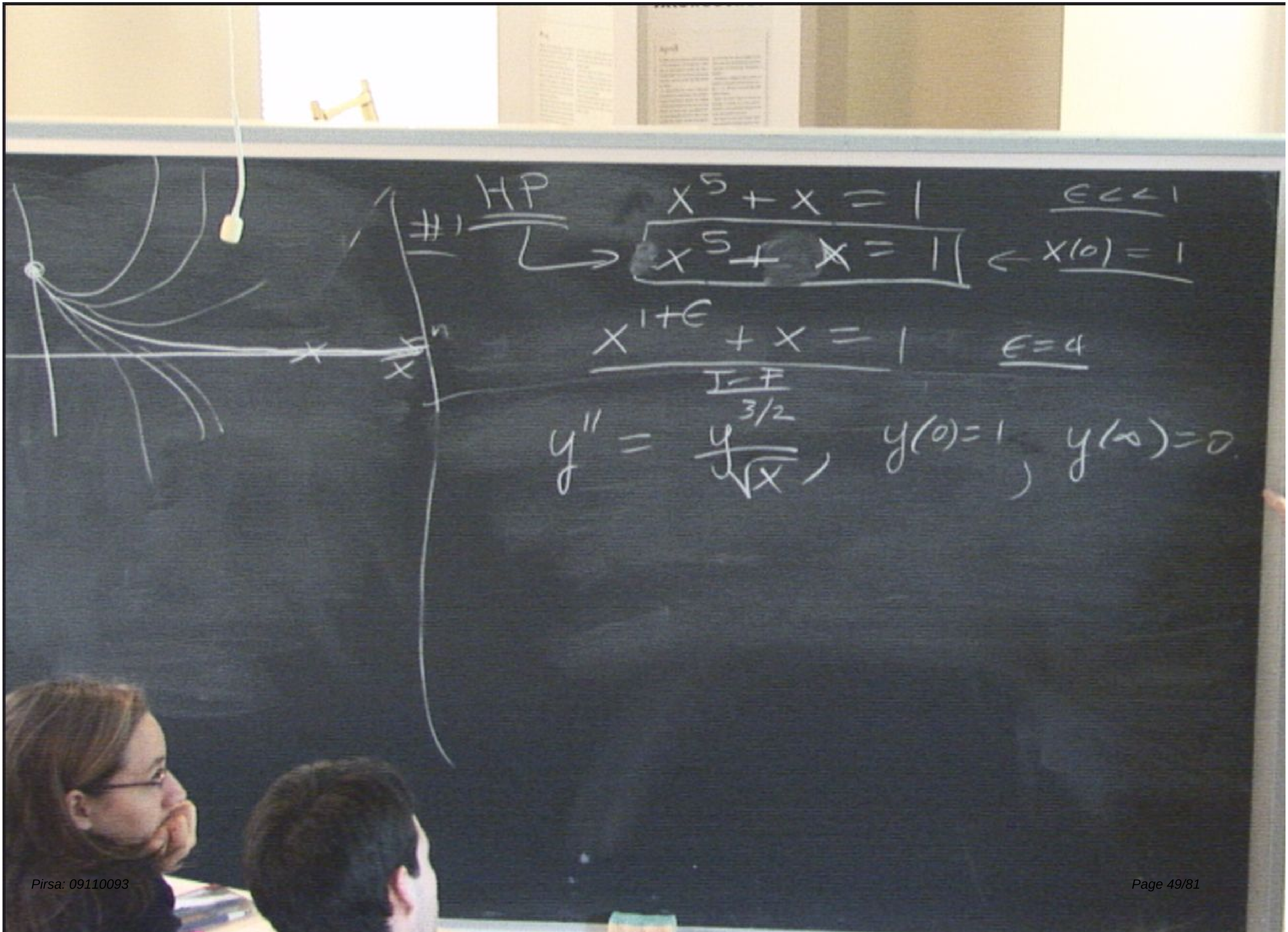

HP
#1
x^5 + x = 1
ε << 1
x^5 + x = 1
x(0) = 1
x^{1+ε} + x = 1
ε = 4
y'' = y^{3/2}/√x , y(0) = 1 , y(∞) = 0
a_1 + 1 = 0
a_1 = -1/5
5a_2 + 10/25 - 5/25 = 0
a_2 = -1/25

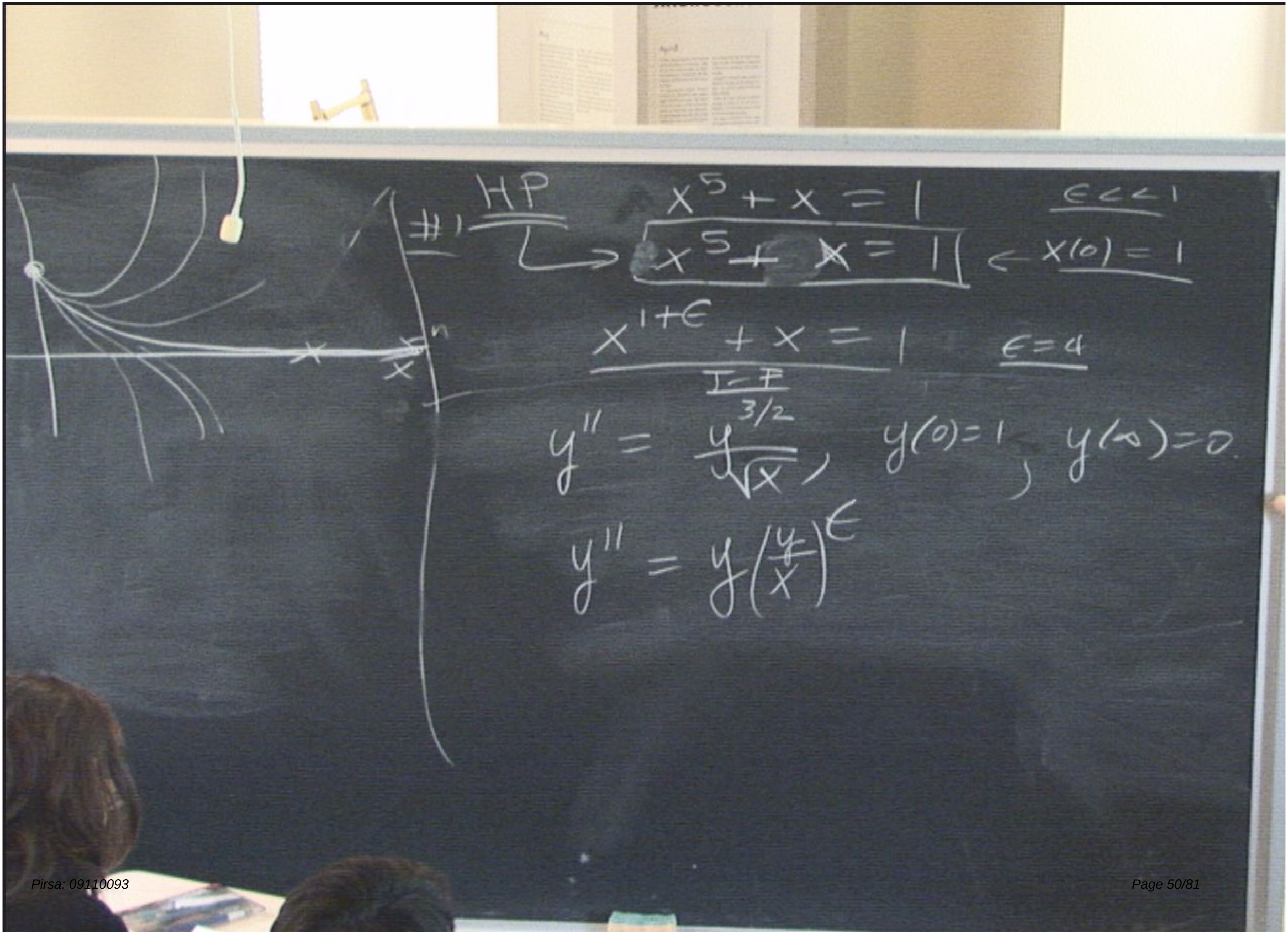

HP
#1
x^5 + x = 1
ε << 1
x^5 + εx = 1
x(0) = 1
x^{1+ε} + x = 1
ε = 4
I-F
y'' = y^{3/2}/√x, y(0) = 1, y(∞) = 0
a_1 + 1 = 0
a_2 + 10a_1^2 + a_1 = 0
a_1 = -1/5
5a_2 + 10/25 - 5/25 = 0
a_2 = -1/25

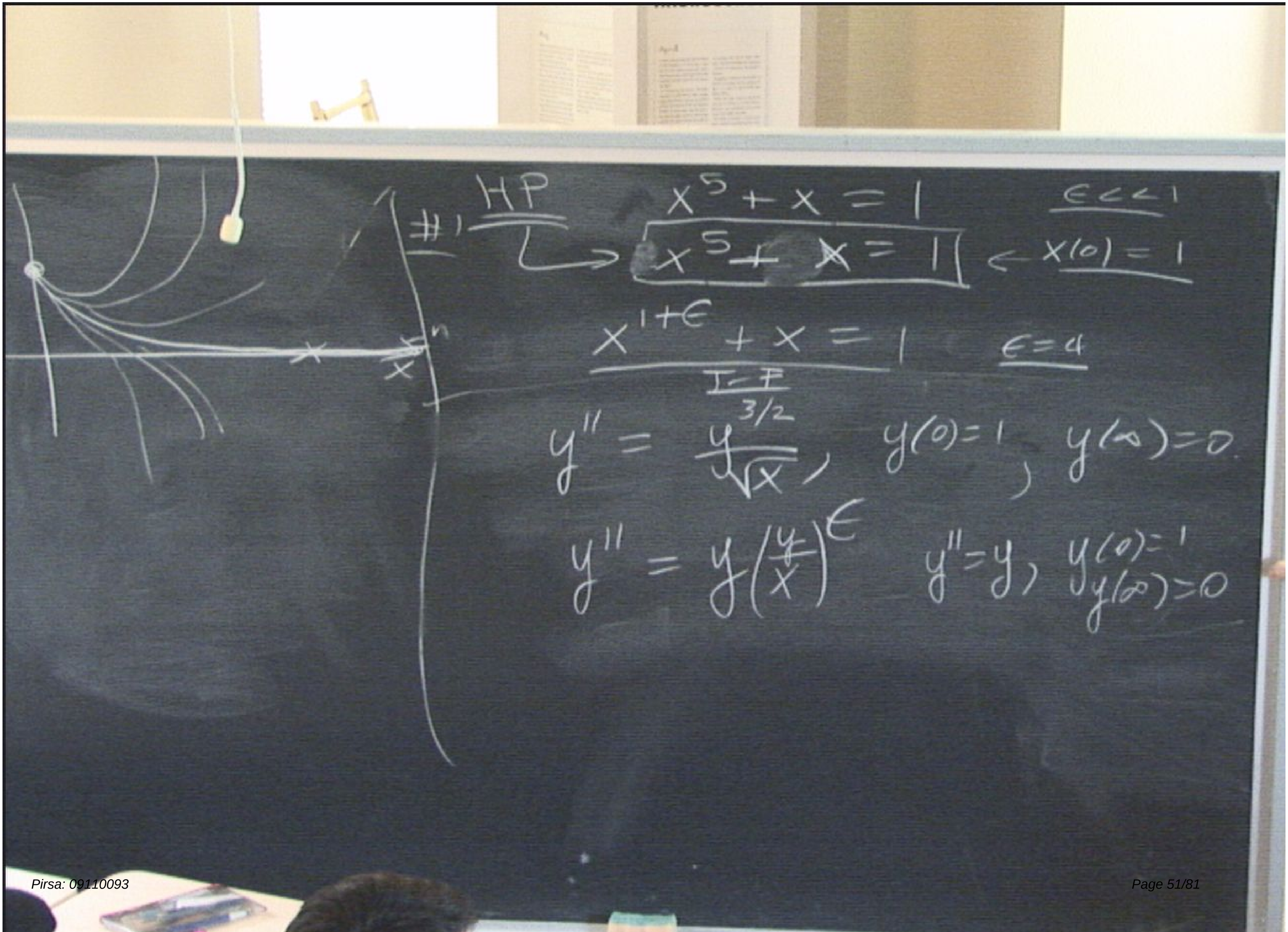

HP
#1
x^5 + x = 1
ε << 1
x^5 + x = 1
x(0) = 1
x^{1+ε} + x = 1
ε = 4
a_1 + 1 = 0
a_1 = -1/5
5a_2 + 10/25 - 5/25 = 0
y'' = y^{3/2}/√x, y(0) = 1, y(∞) = 0

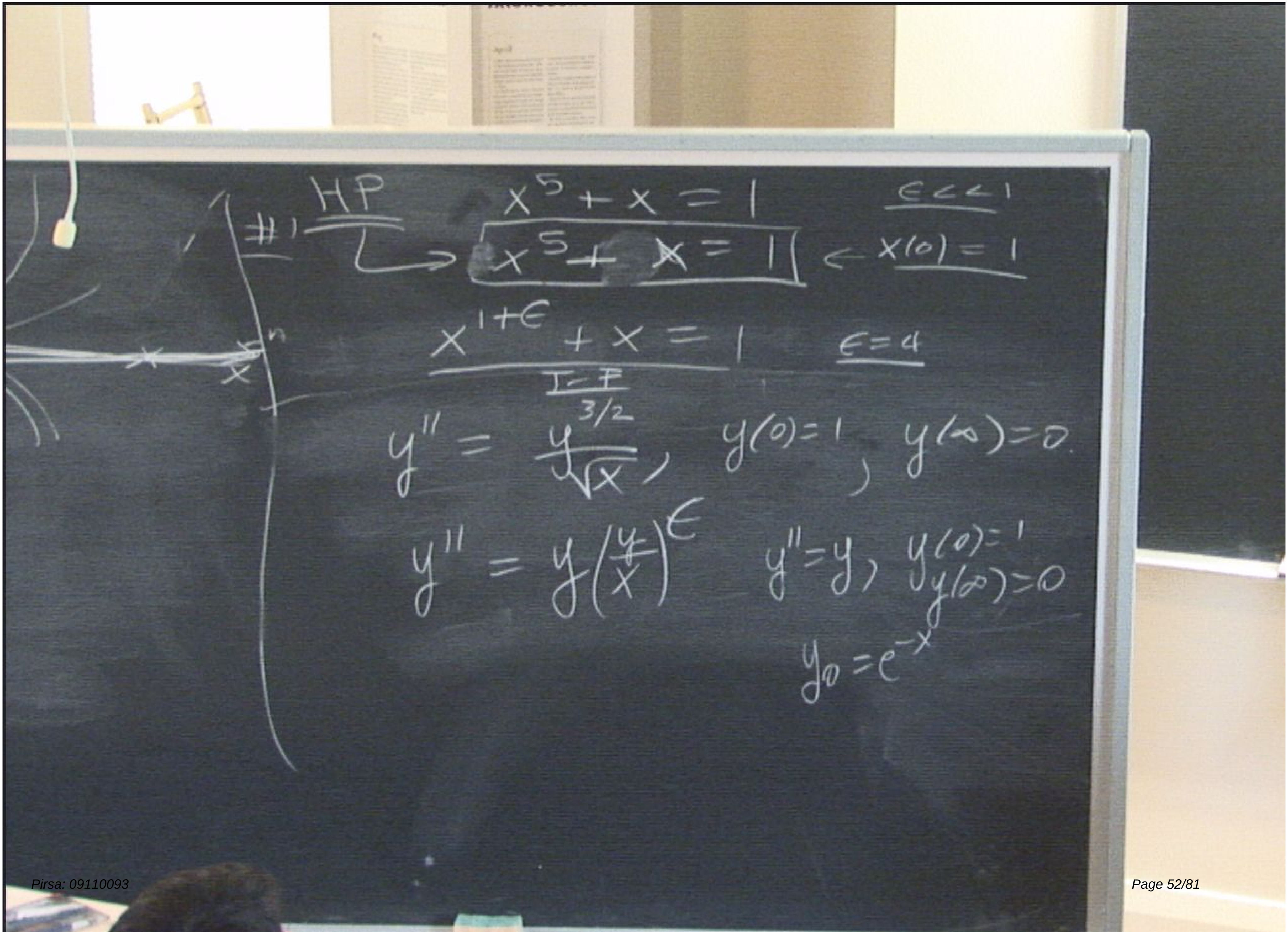

HP
x^5 + x = 1
x^5 + x = 1
x^{1+ε} + x = 1
y'' = y^{3/2}/√x

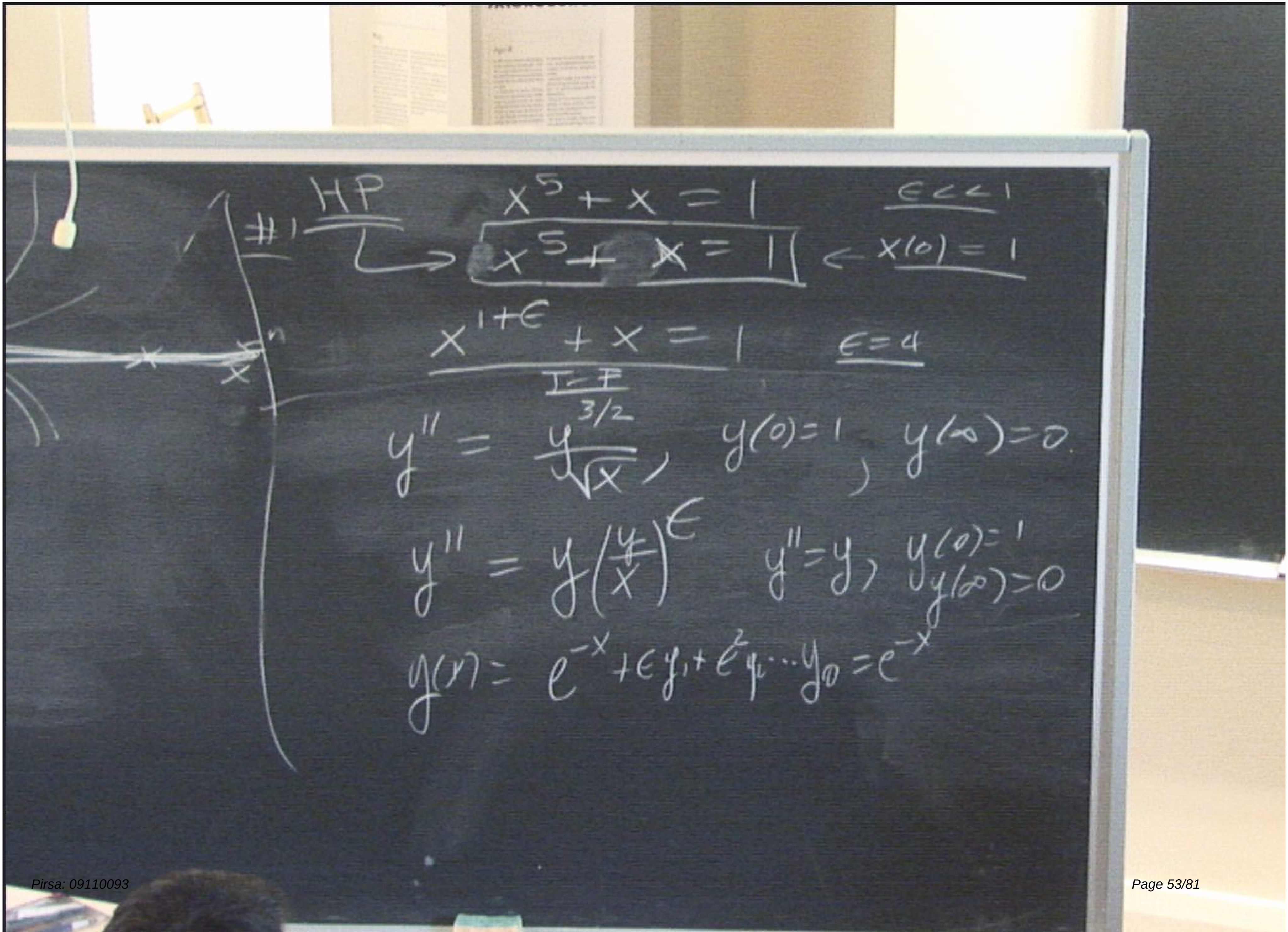

HP
#1
x^5 + x = 1
x^5 + x = 1
← x(0) =
x^{1+ε} + x = 1
ε = 4
I-F
y'' = y^{3/2}/√x, y(0) = 1, y(∞)

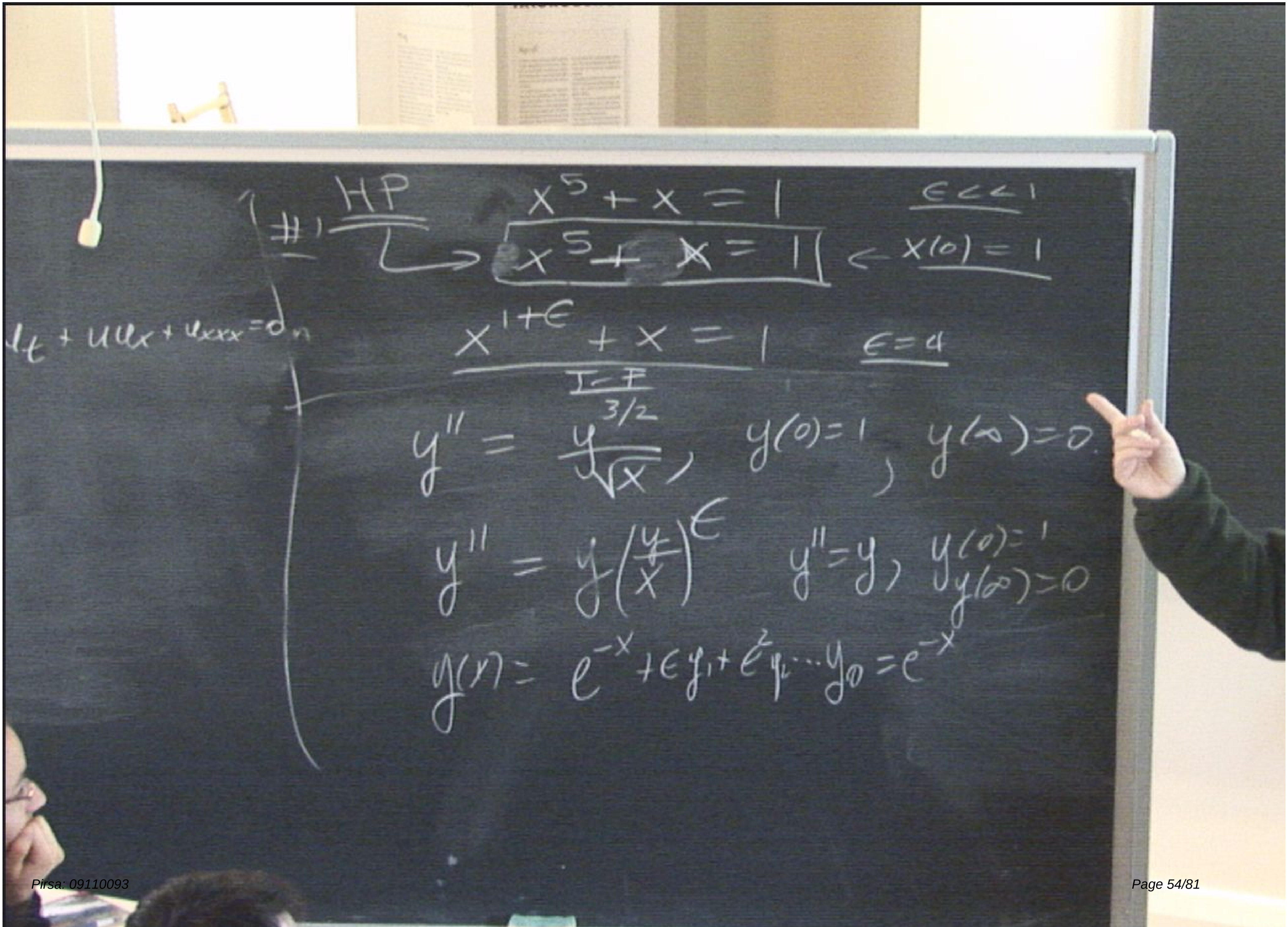

HP
#1
x^5 + x = 1
ε << 1
x^5 + x = 1
x(0) = 1
x^{1+ε} + x = 1
ε = 4
y'' = y^{3/2}/√x, y(0) = 1, y(∞) = 0


HP
#1
x5 + x = 1
ε << 1
x5 + x = 1
x(0) = 1
x1+ε + x = 1
ε = 4
3/2
y'' = y/√x , y(0) = 1 , y(∞) = 0
y'' = y(y/x)ε

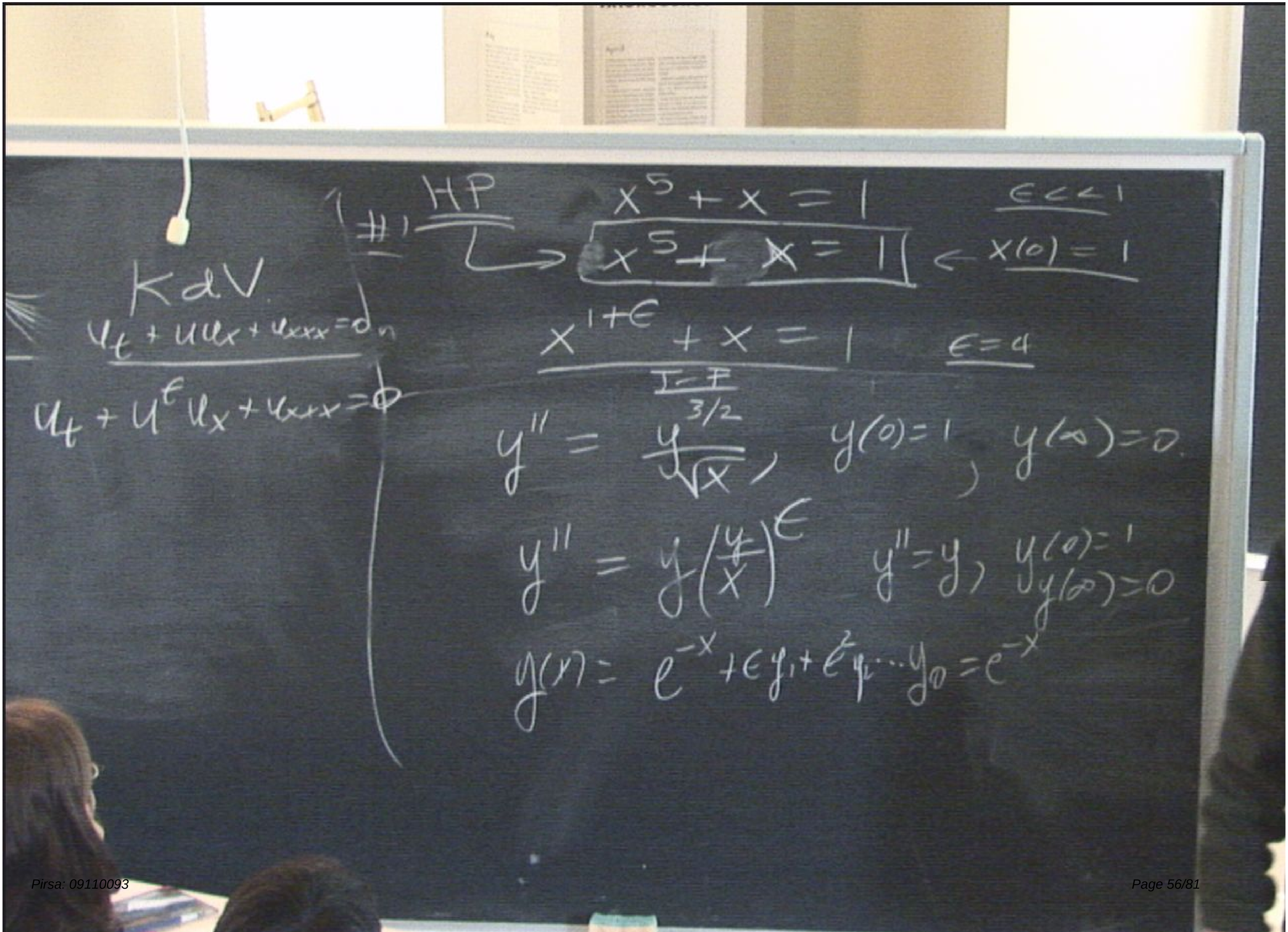

HP
#1
$x^5 + x = 1$
$\epsilon << 1$
$x^5 + x = 1$
$x(0) = 1$
$x^{1+\epsilon} + x = 1$
$\epsilon = 4$
3/2
$y'' = \frac{y}{\sqrt{x}}$, $y(0) = 1$, $y(\infty) = 0$
$y'' = y\left(\frac{y}{x}\right)^\epsilon$
$y'' = y$, $y(0) = 1$, $y(\infty) = 0$

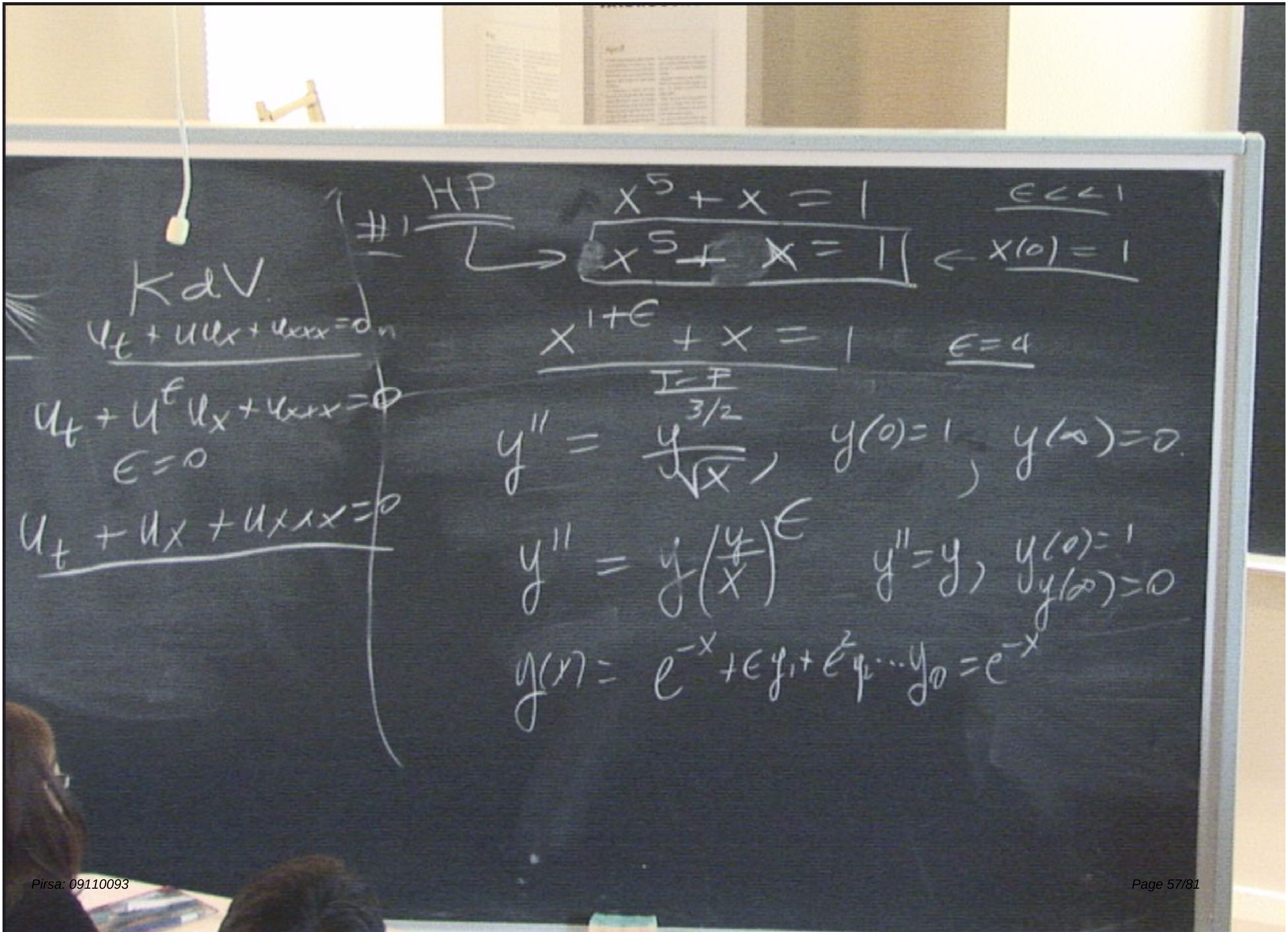

HP
#1
x^5 + x = 1
ε << 1
x^5 + x = 1
x(0) = 1
x^(1+ε) + x = 1
ε = 4
I-F
3/2
y'' = y^(3/2)/√x , y(0) = 1 , y(∞) = 0
y'' = y(y/x)^ε
y'' = y, y(0) = 1
y(∞) = 0
y_0 = e^(-x)

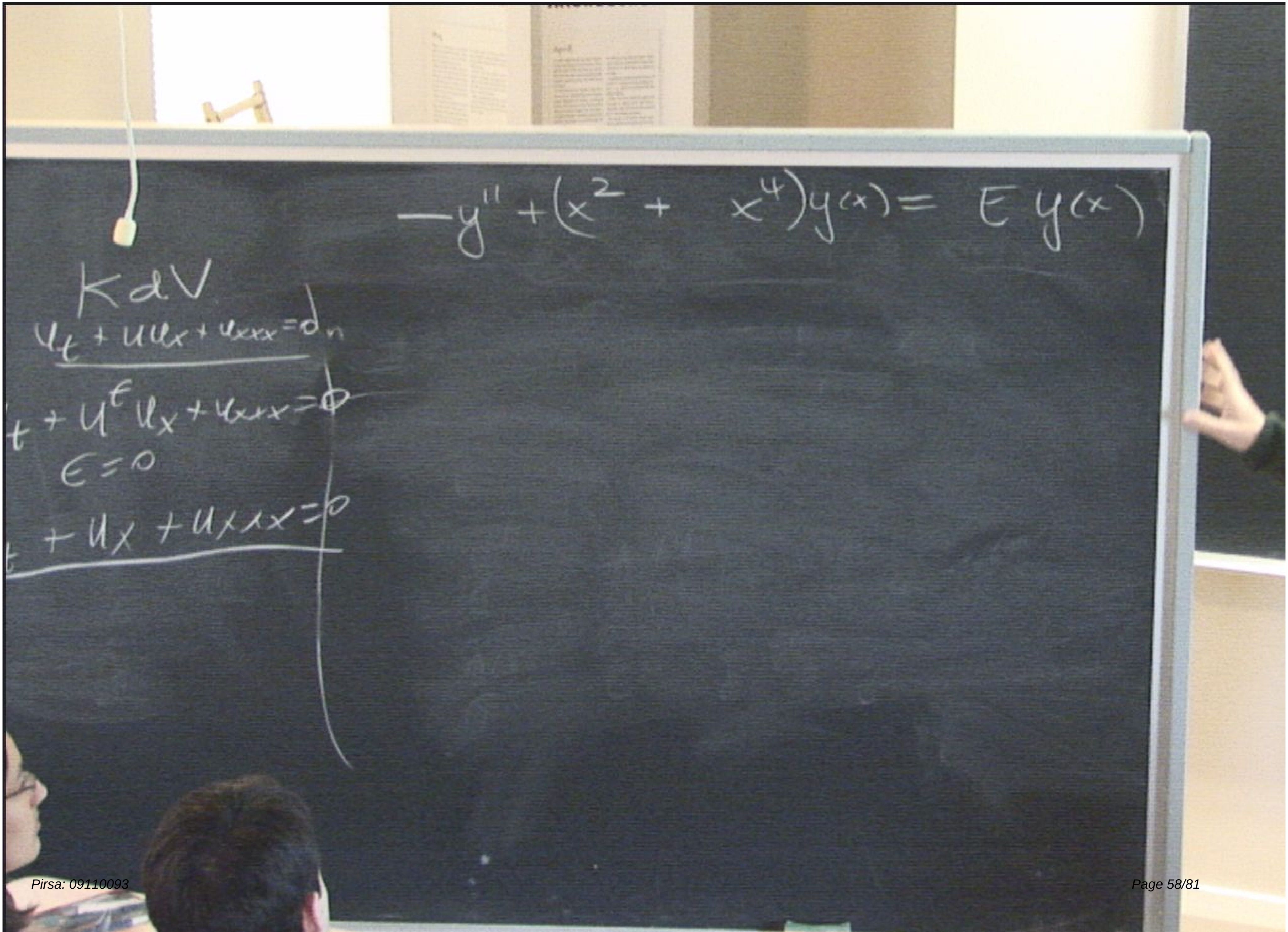

HP
#1
x^5 + x = 1
ε << 1
x^5 + x = 1
x(0) = 1
x^{1+ε} + x = 1
ε = 4
y'' = y^{3/2}/√x , y(0) = 1 , y(∞) = 0
y'' = y (y/x)^ε
y'' = y , y(0) = 1
y(∞) = 0
y(x) = e^{-x} + ε y_1 + ε^2 y_2 ... y_0 = e^{-x}

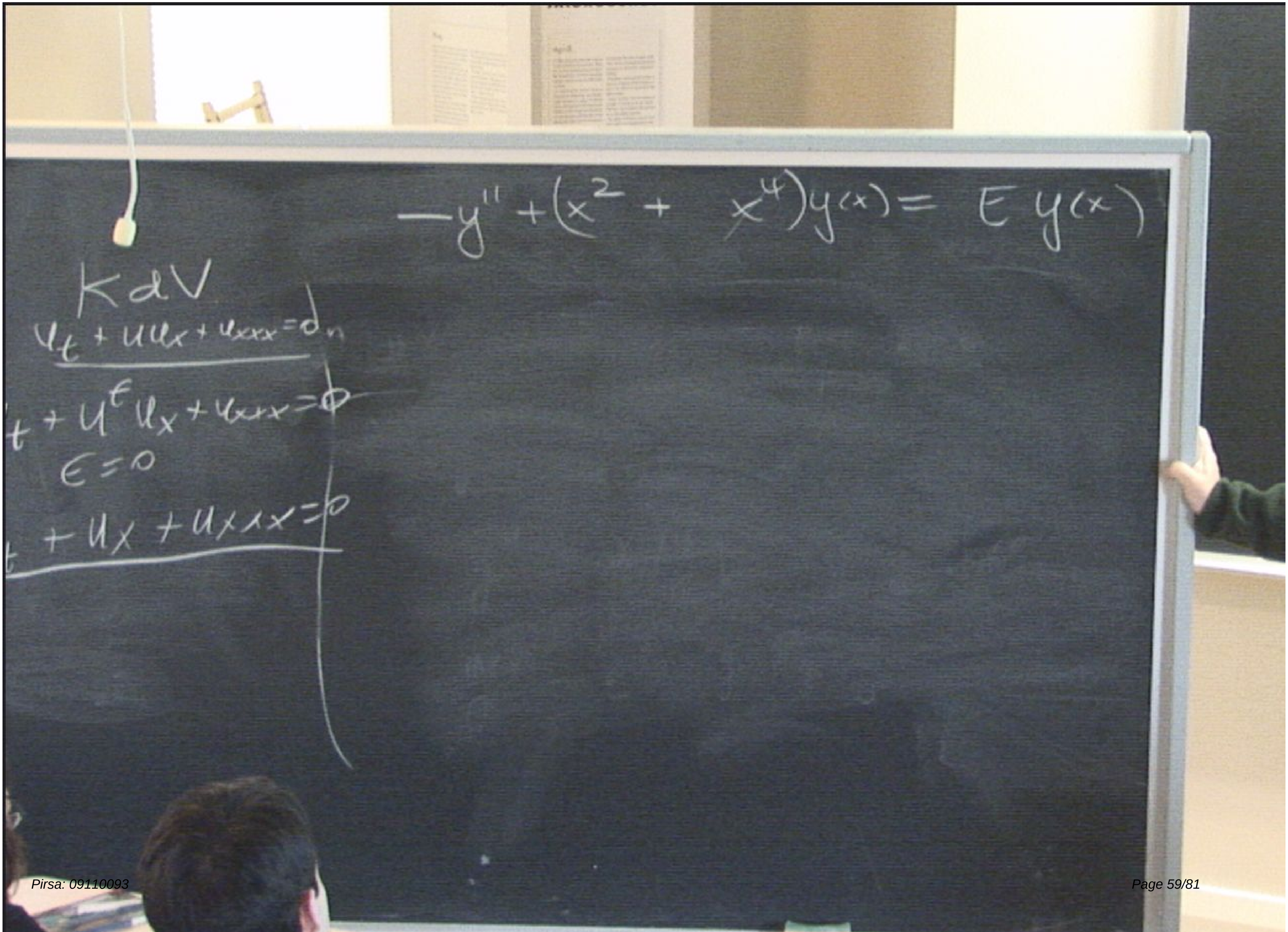

#1 HP
x^5 + x = 1
ε << 1
x^5 + x = 1
x(0) = 1
u_t + uu_x + u_xxx = 0
x^{1+ε} + x = 1
ε = 4
3/2
y'' = y/√x , y(0) = 1 , y(∞) = 0
y'' = y(y/x)^ε
y'' = y , y(0) = 1
y(∞) = 0
y(x) = e^{-x} + εy_1 + ε^2 y_2 ··· y_0 = e^{-x}

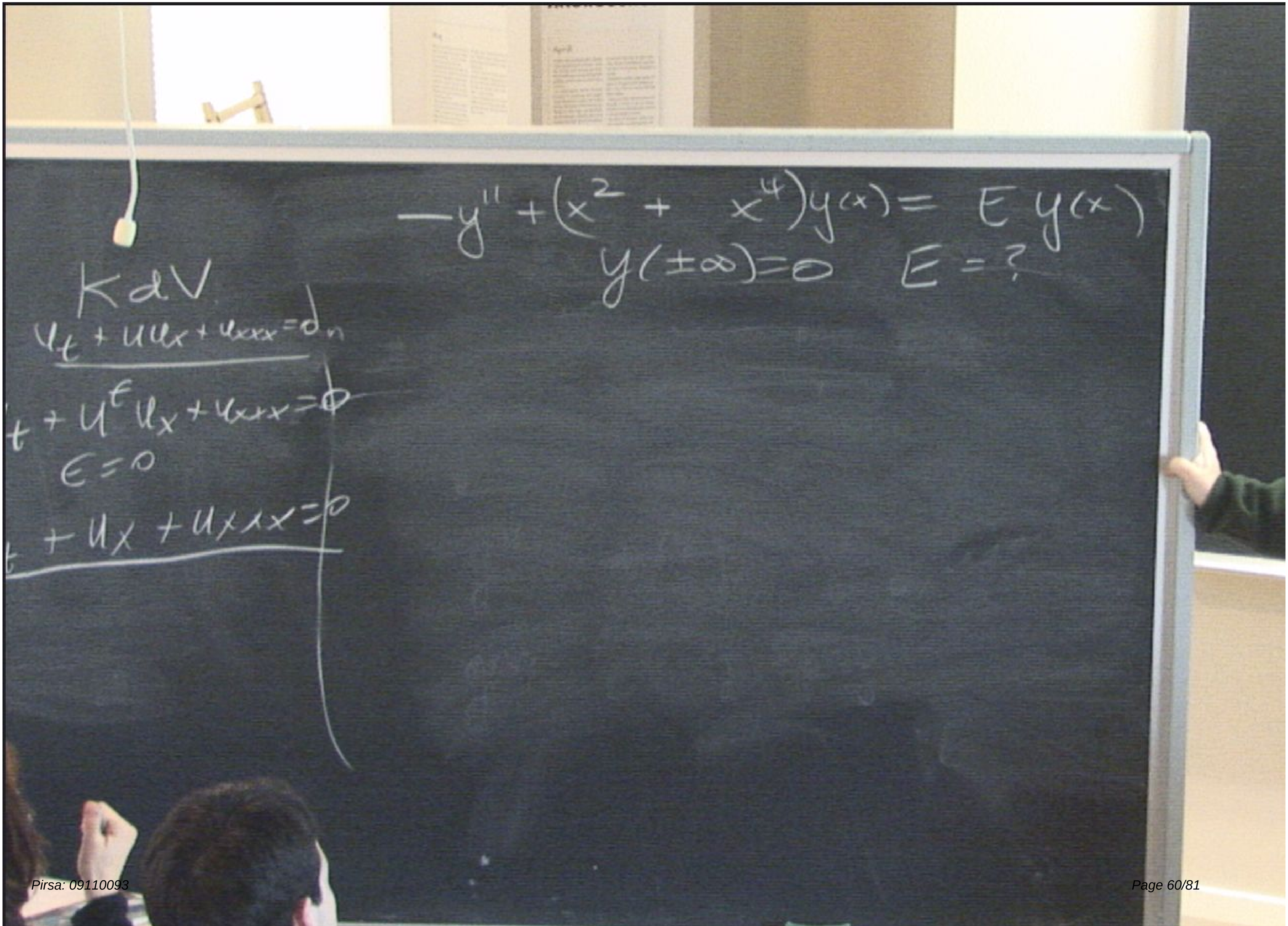

KdV
HP
ε << 1
x(0) = 1
ε = 4
y(∞) = 0.
y(0) = 1
y(∞) = 0

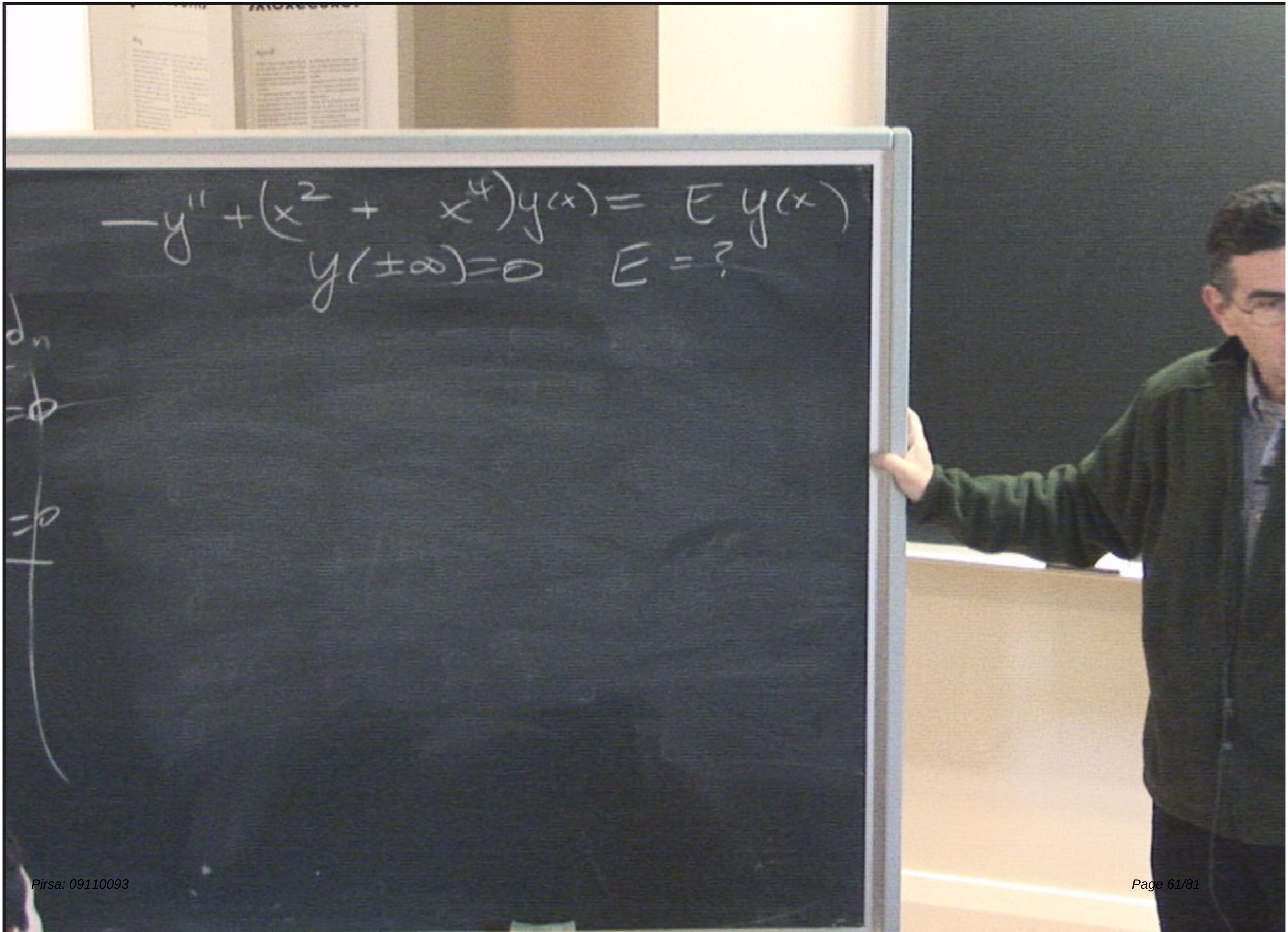

HP
#1
x^5 + x = 1
ε << 1
x^5 + x = 1
x(0) = 1
KdV
u_t + u u_x + u_xxx = 0
u_t + u^ε u_x + u_xxx = 0
x^(1+ε) + x = 1
ε = 4
I-F
3/2
y'' = y^(3/2)/√x , y(0)=1 , y(∞)=0
y'' = y (y/x)^ε
y''=y, y(0)=1
y(∞)=0
y(x) = e^(-x) + ε y_1 + ε^2 y_2 ... y_0 = e^(-x)

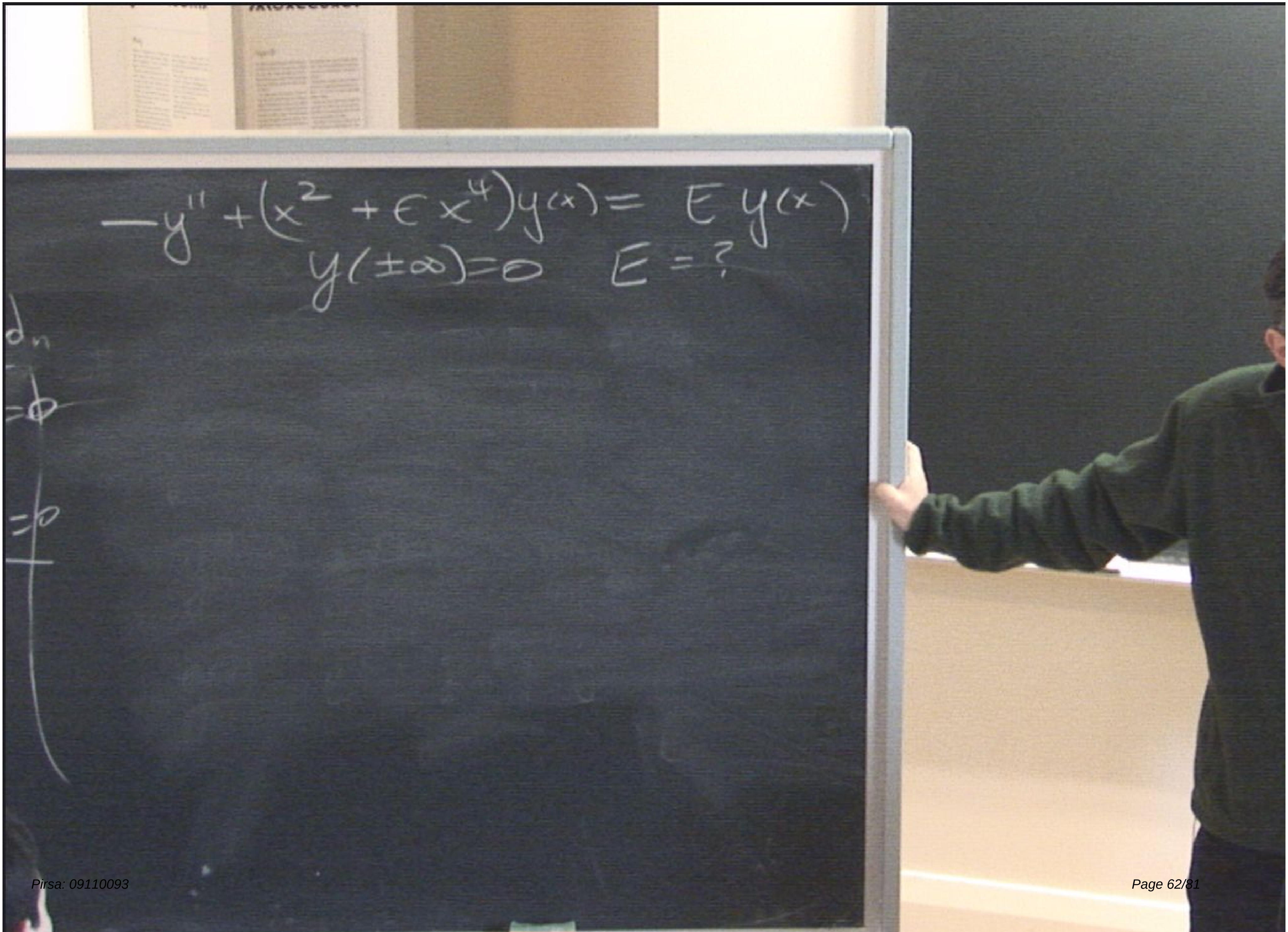

#1 HP
x^5 + x = 1
ε << 1
x^5 + x = 1
x(0) = 1
KdV
u_t + u u_x + u_xxx = 0
u_t + u^ε u_x + u_xxx = 0
ε = 0
u_t + u_x + u_xxx = 0
x^(1+ε) + x = 1
ε = 4
T-F
y'' = y^(3/2) / √x , y(0) = 1 , y(∞) = 0
y'' = y (y/x)^ε
y'' = y , y(0) = 1
y(∞) = 0
y(x) = e^(-x) + ε y_1 + ε^2 y_2 ... y_0 = e^(-x)

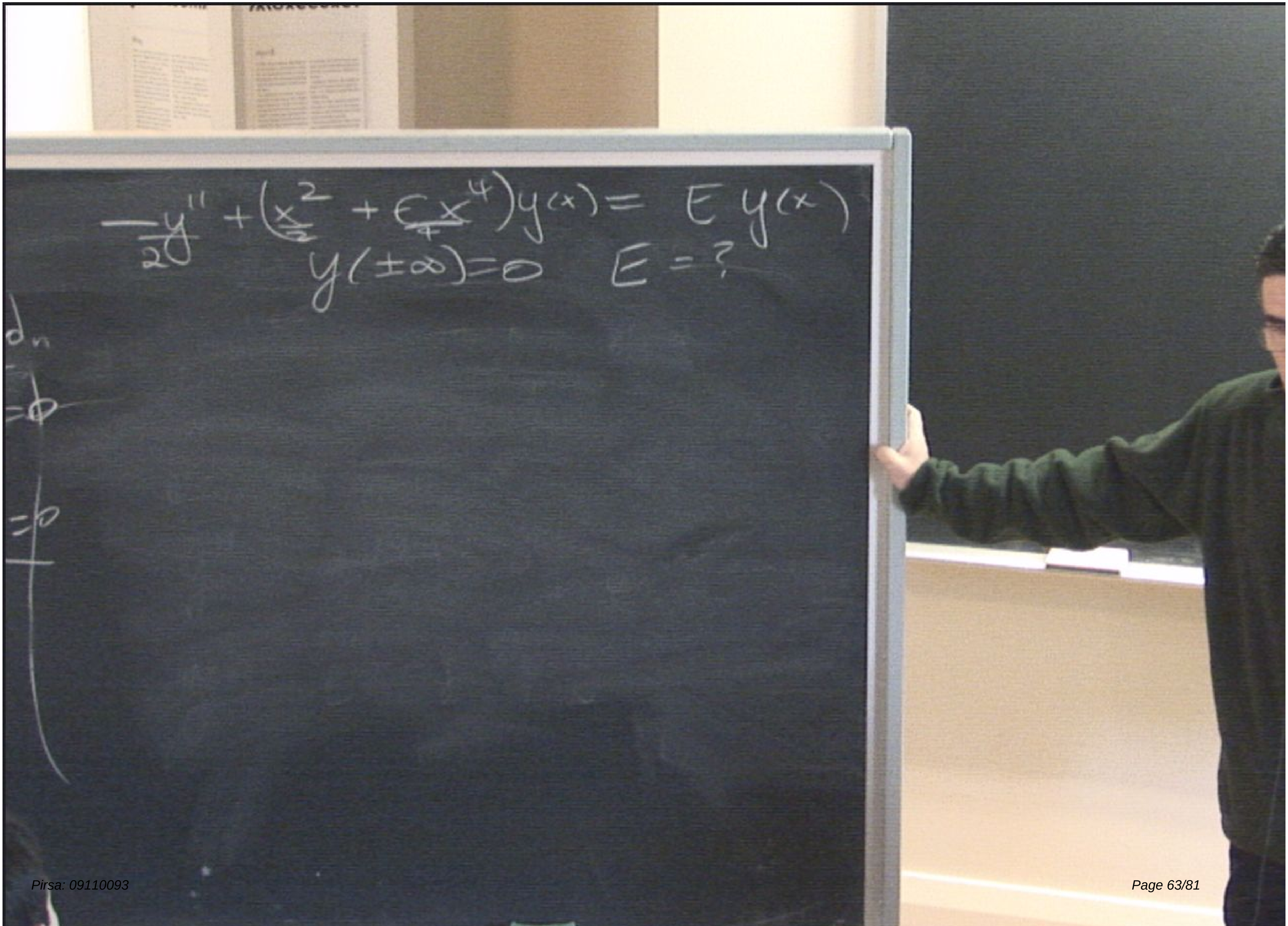

−y'' + (x² + x⁴) y(x) = E y(x)
KdV
u_t + u u_x + u_xxx = 0
+ u^ε u_x + u_xxx = 0
ε = 0
+ u_x + u_xxx = 0

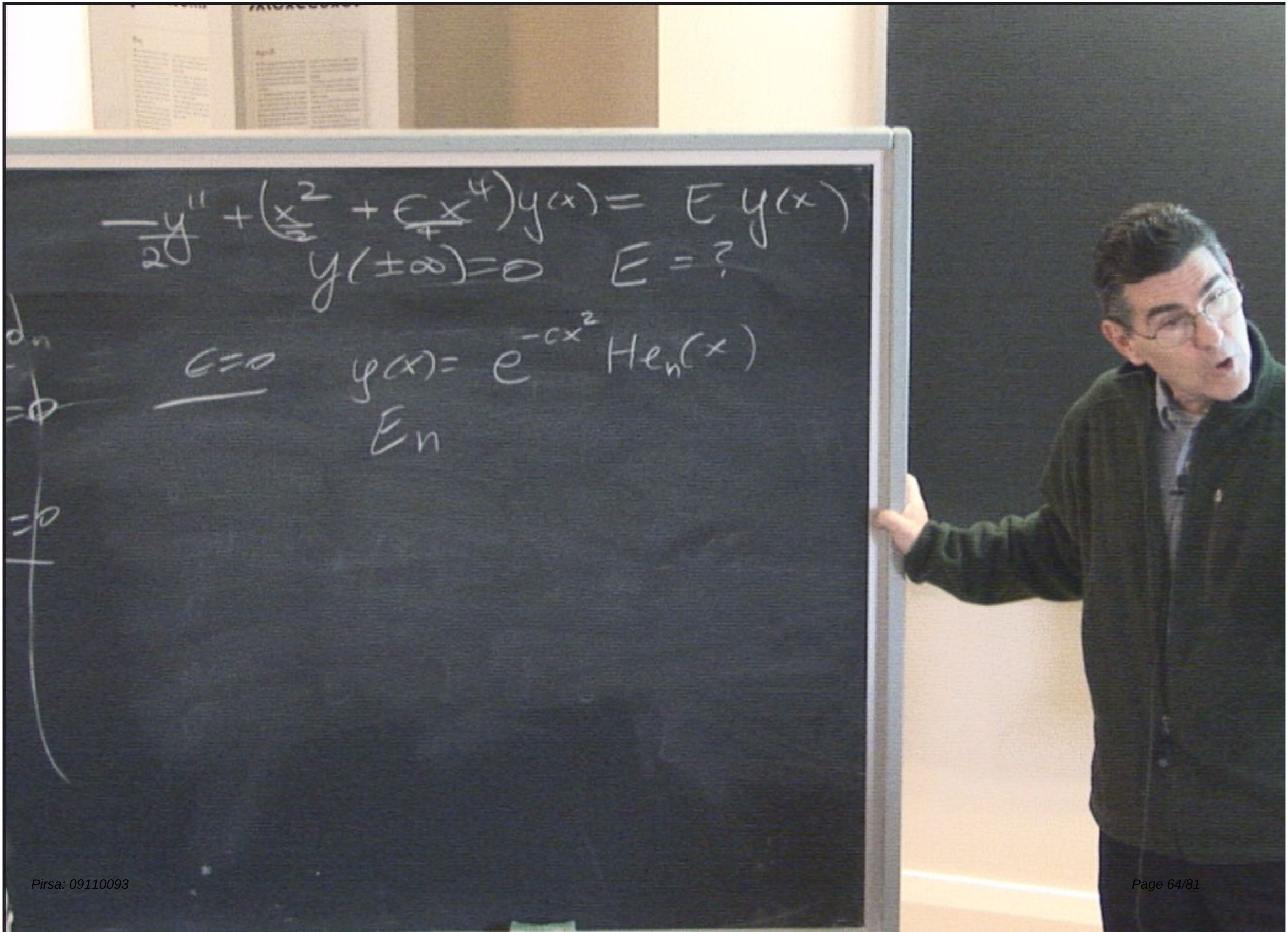

−y'' + (x² + x⁴) y(x) = E y(x)
KdV
u_t + uu_x + u_xxx = 0
t + u^ε u_x + u_xxx = 0
ε = 0
t + u_x + u_xxx = 0

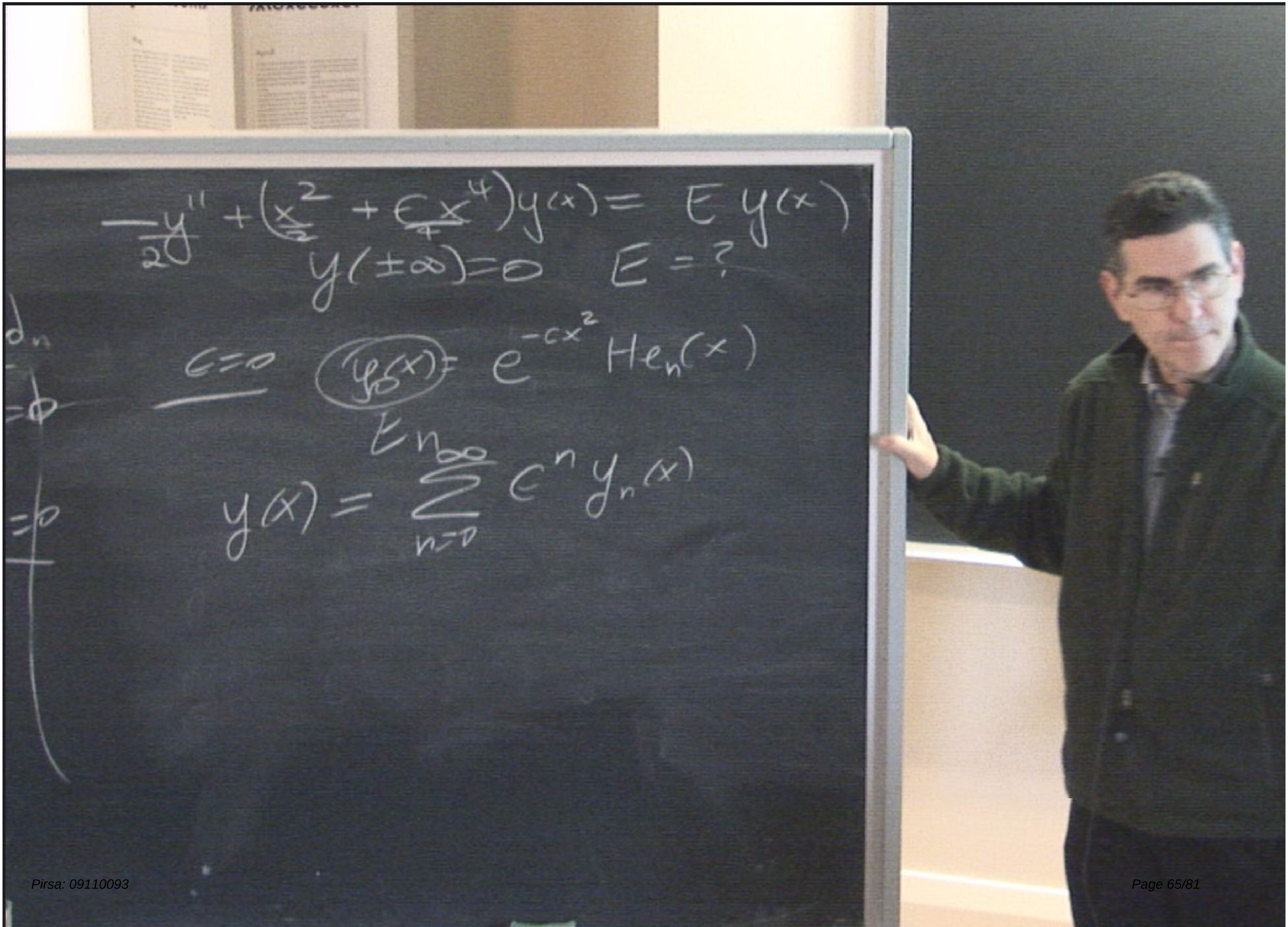

−y'' + (x² + x⁴)y(x) = E y(x)
y(±∞)=0 E=?
KdV
u_t + uu_x + u_xxx = 0
+ u^ε u_x + u_xxx = 0
ε = 0
+ u_x + u_xxx = 0

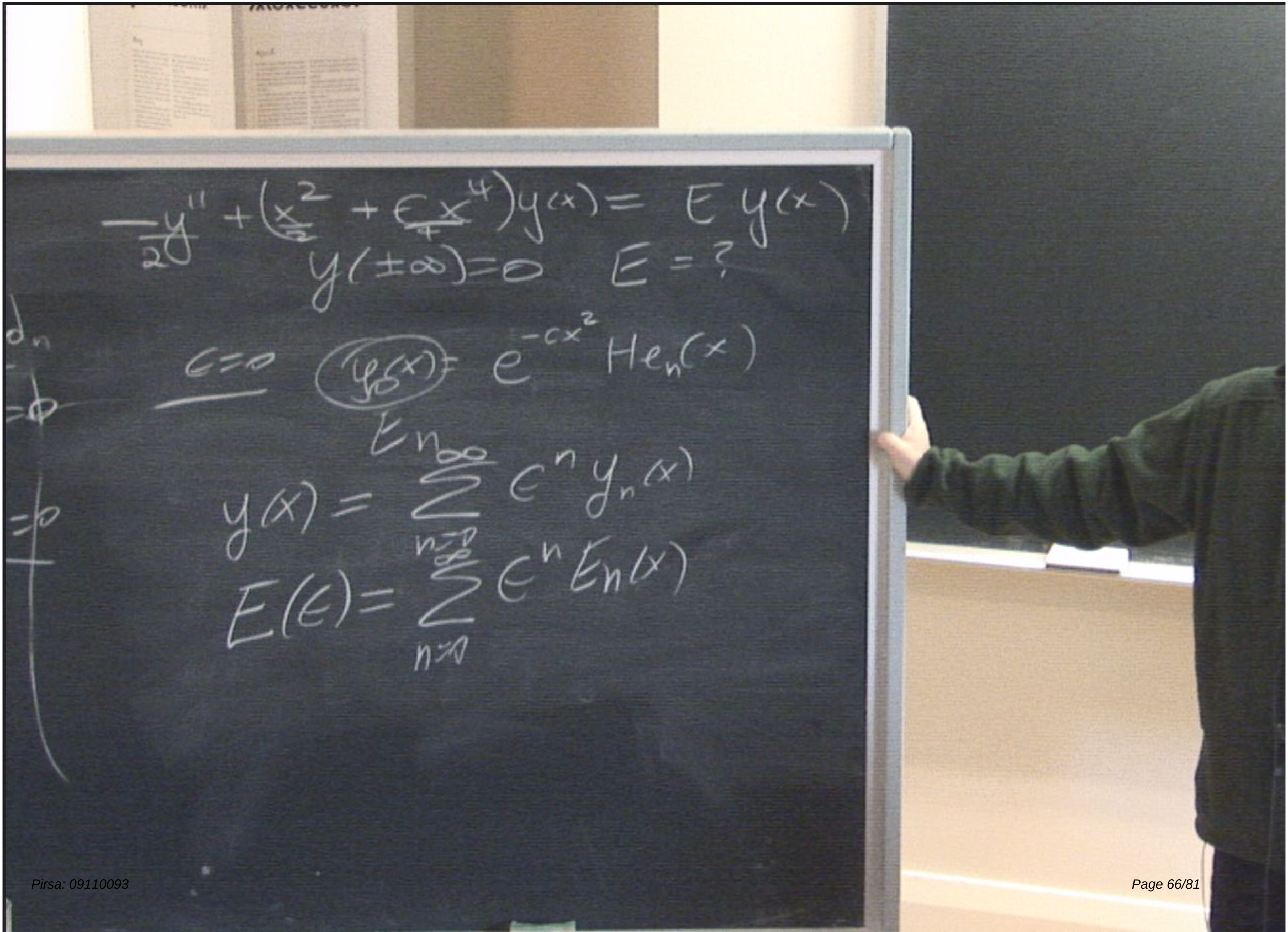

$-y'' + (x^2 + x^4)y(x) = E\, y(x)$
$y(\pm\infty) = 0 \quad E = ?$

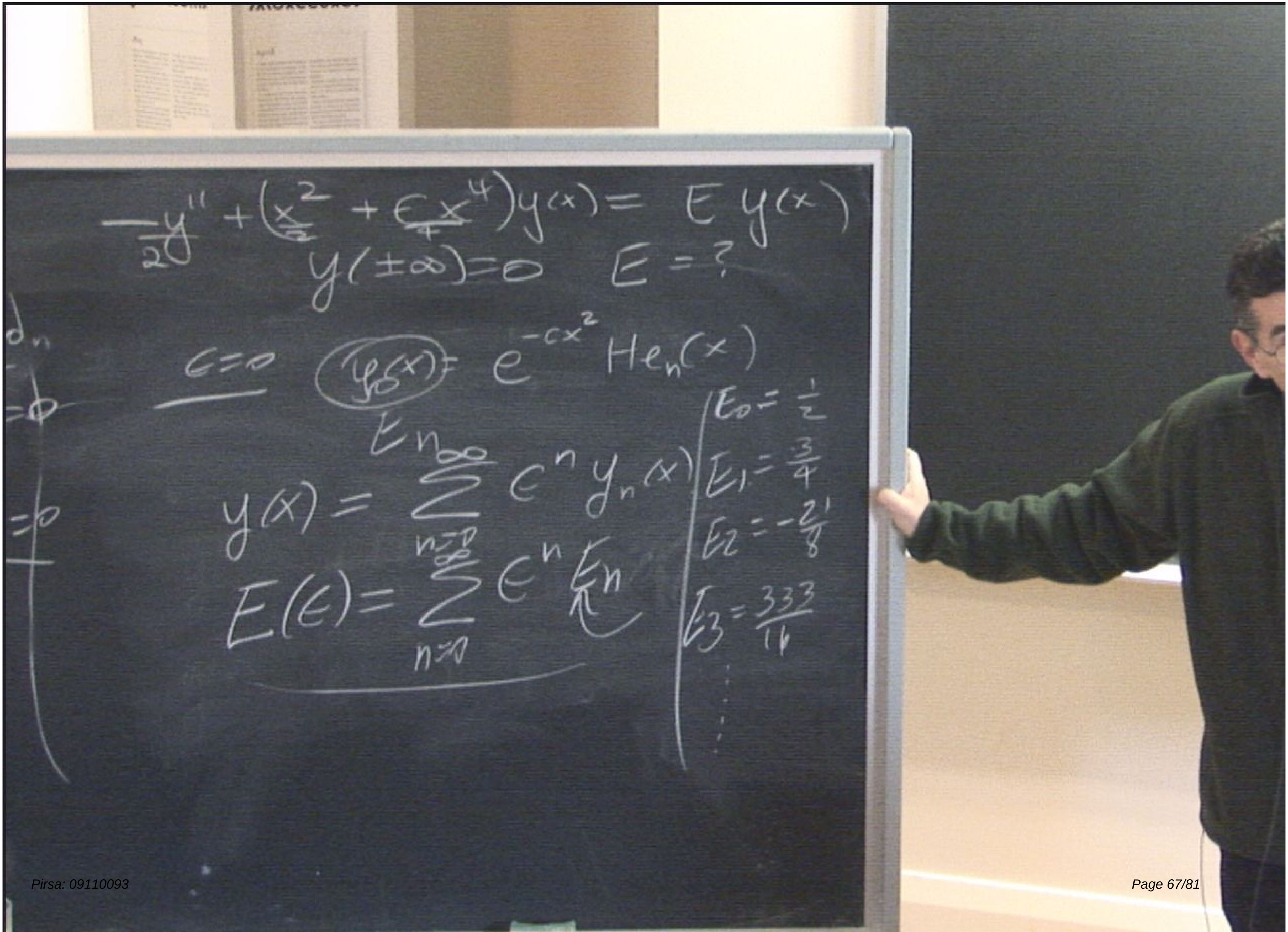

$-y'' + (x^2 + \epsilon x^4) y(x) = E\, y(x)$
$y(\pm\infty) = 0 \quad E = ?$


$-\frac{1}{2}y'' + \left(\frac{x^2}{2} + \frac{\epsilon x^4}{4}\right) y(x) = E\, y(x)$
$y(\pm\infty) = 0 \quad E = ?$

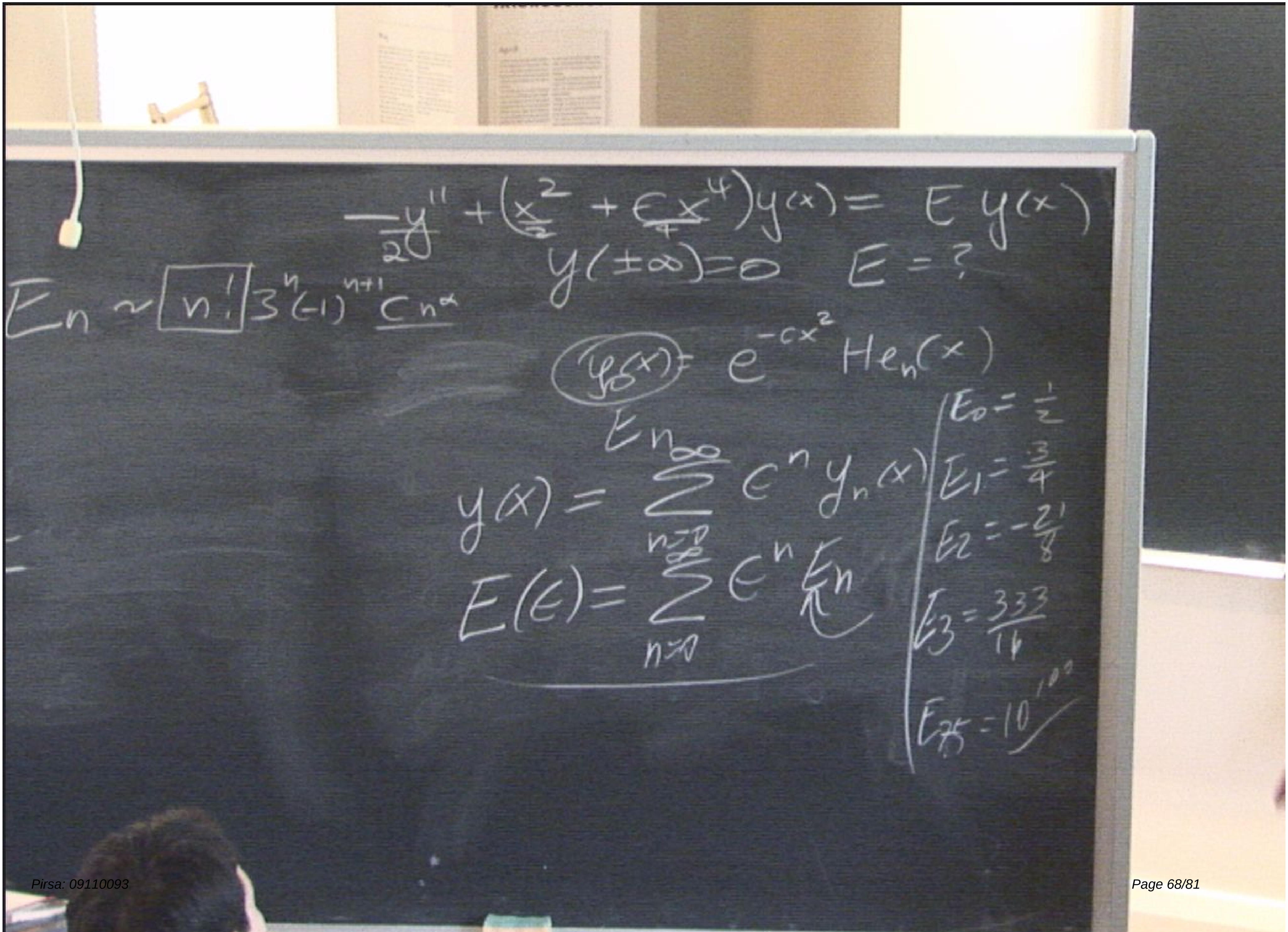

$-\frac{1}{2}y'' + \left(\frac{x^2}{2} + \frac{\epsilon x^4}{4}\right)y(x) = E\,y(x)$
$y(\pm\infty) = 0 \quad E = ?$
$\epsilon = 0 \quad y(x) = e^{-cx^2}He_n(x)$
E_n

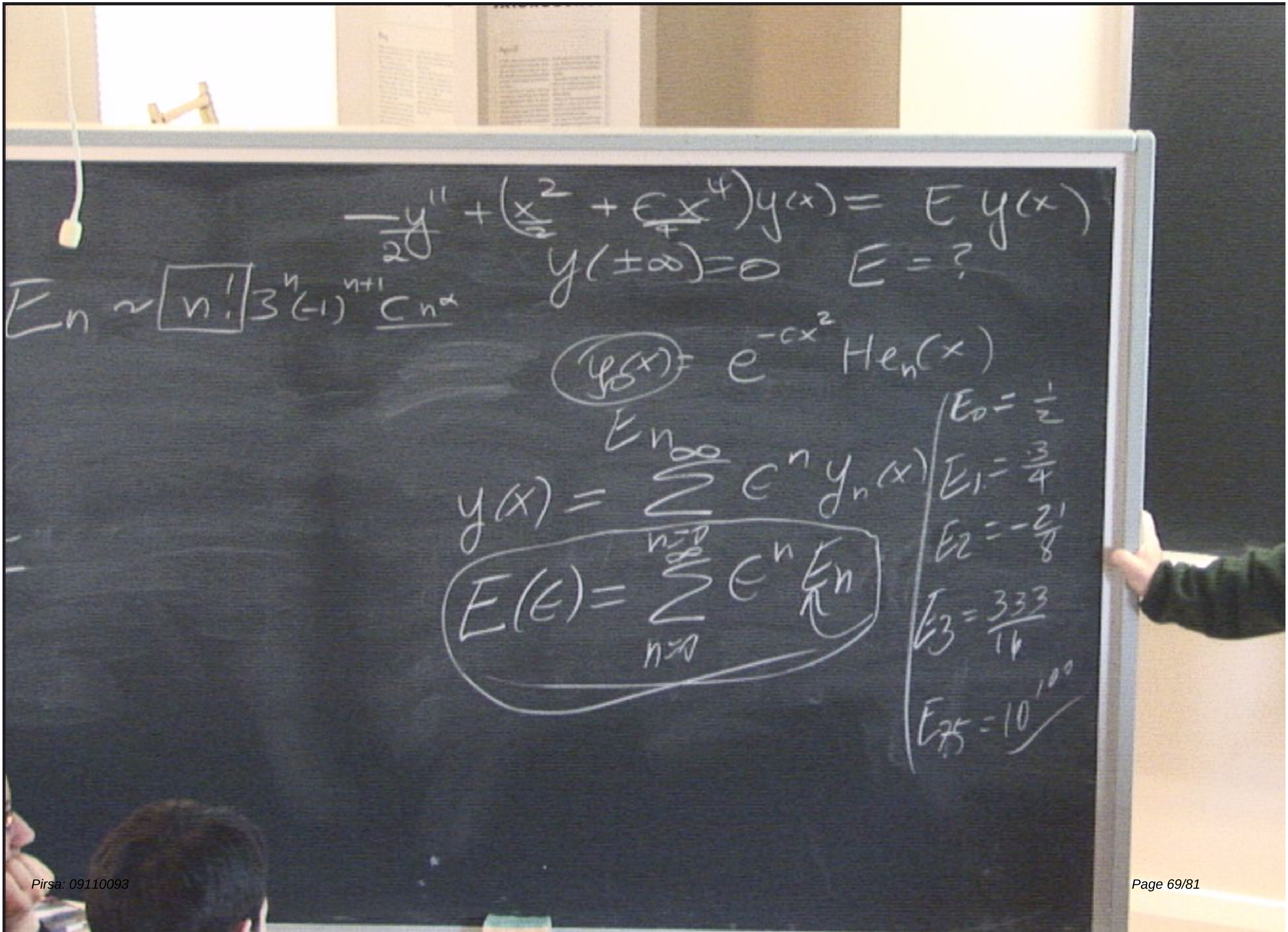

-y''/2 + (x²/2 + εx⁴/4) y(x) = E y(x)
y(±∞) = 0   E = ?
ε = 0
y₀(x) = e^{-cx²} Heₙ(x)
Eₙ
y(x) = Σ_{n=0}^{∞} εⁿ yₙ(x)


$-\frac{1}{2}y'' + \left(\frac{x^2}{2} + \frac{\epsilon x^4}{4}\right)y(x) = E\,y(x)$
$y(\pm\infty) = 0 \quad E = ?$
$\epsilon = 0 \quad \psi_0(x) = e^{-cx^2} He_n(x)$
E_n
$y(x) = \sum_{n=0}^{\infty} \epsilon^n y_n(x)$
$E(\epsilon) = \sum_{n=0}^{\infty} \epsilon^n E_n(x)$



-1/2 y'' + (x²/2 + εx⁴/4) y(x) = E y(x)
y(±∞) = 0   E = ?
ε = 0
y₀(x) = e^{-cx²} Heₙ(x)
y(x) = Σ_{n=0}^{∞} εⁿ yₙ(x)
E(ε) = Σ_{n=0}^{∞} εⁿ Eₙ
E₀ = 1/2
E₁ = 3/4
E₂ = -21/8
E₃ = 333/16


$-\frac{y''}{2}+\left(\frac{x^2}{2}+\frac{\epsilon x^4}{4}\right)y(x)=E\,y(x)$
$y(\pm\infty)=0 \quad E=?$
$E_n \sim n!\,3^n(-1)^{n+1}\,C\,n^\alpha$
$y_0(x)=e^{-cx^2}He_n(x)$
E_n
$y(x)=\sum_{n=0}^{\infty}\epsilon^n y_n(x)$
$E(\epsilon)=\sum_{n=0}^{\infty}\epsilon^n E_n$
$E_0=\frac{1}{2}$
$E_1=\frac{3}{4}$
$E_2=-\frac{21}{8}$
$E_3=\frac{333}{16}$
$E_{75}=10^{100}$



-y''/2 + (x²/2 + εx⁴/4) y(x) = E y(x)
y(±∞) = 0   E = ?
E_n ~ n! 3^n (-1)^(n+1) C n^α
y_0(x) = e^(-cx²) He_n(x)
y(x) = Σ_(n=0)^∞ ε^n y_n(x)
E(ε) = Σ_(n=0)^∞ ε^n E_n
E_0 = 1/2
E_1 = 3/4
E_2 = -21/8
E_3 = 333/16

Quantum-mechanical Eigenvalue problems

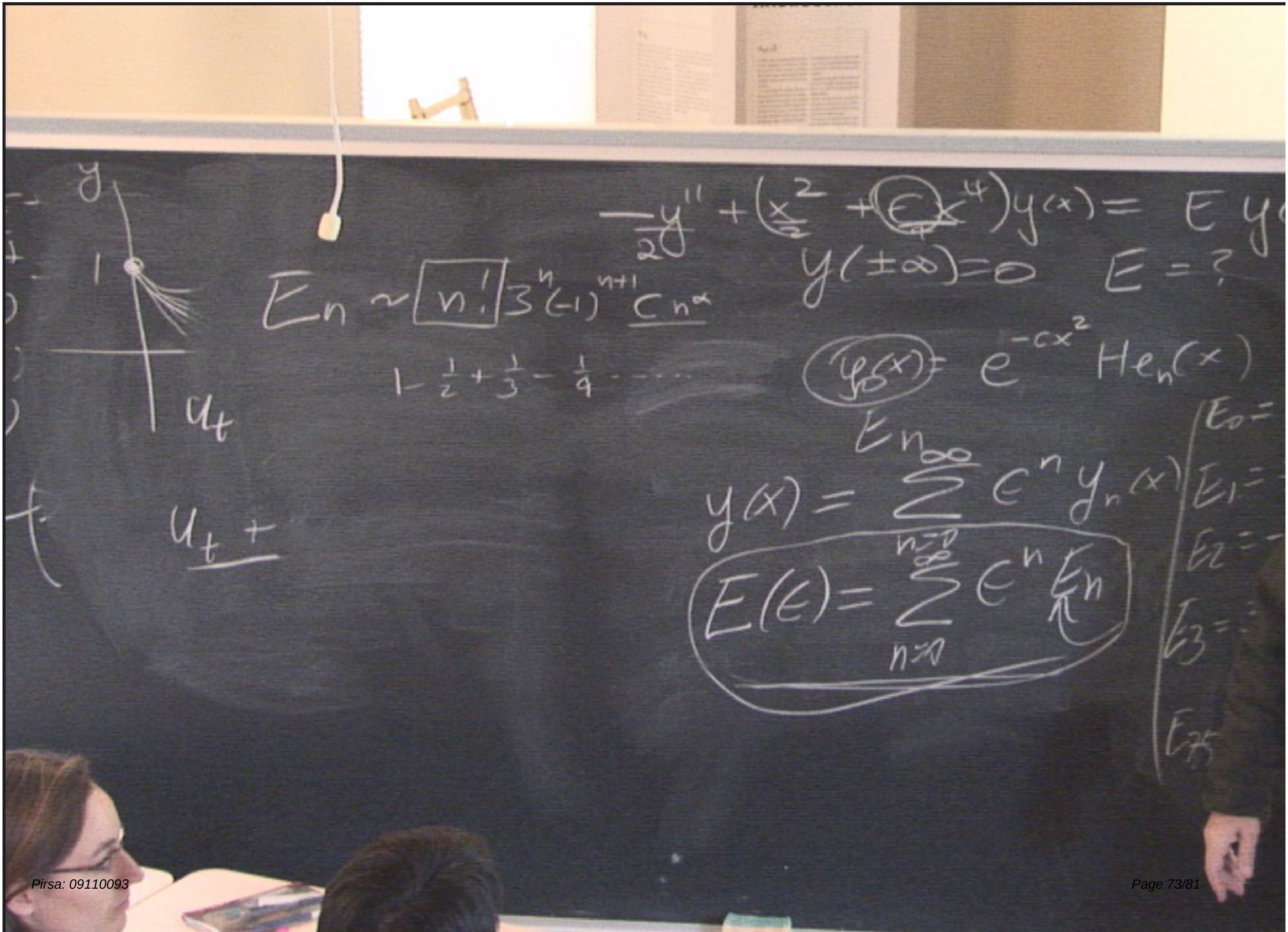
The anharmonic oscillator!

Outline of Course

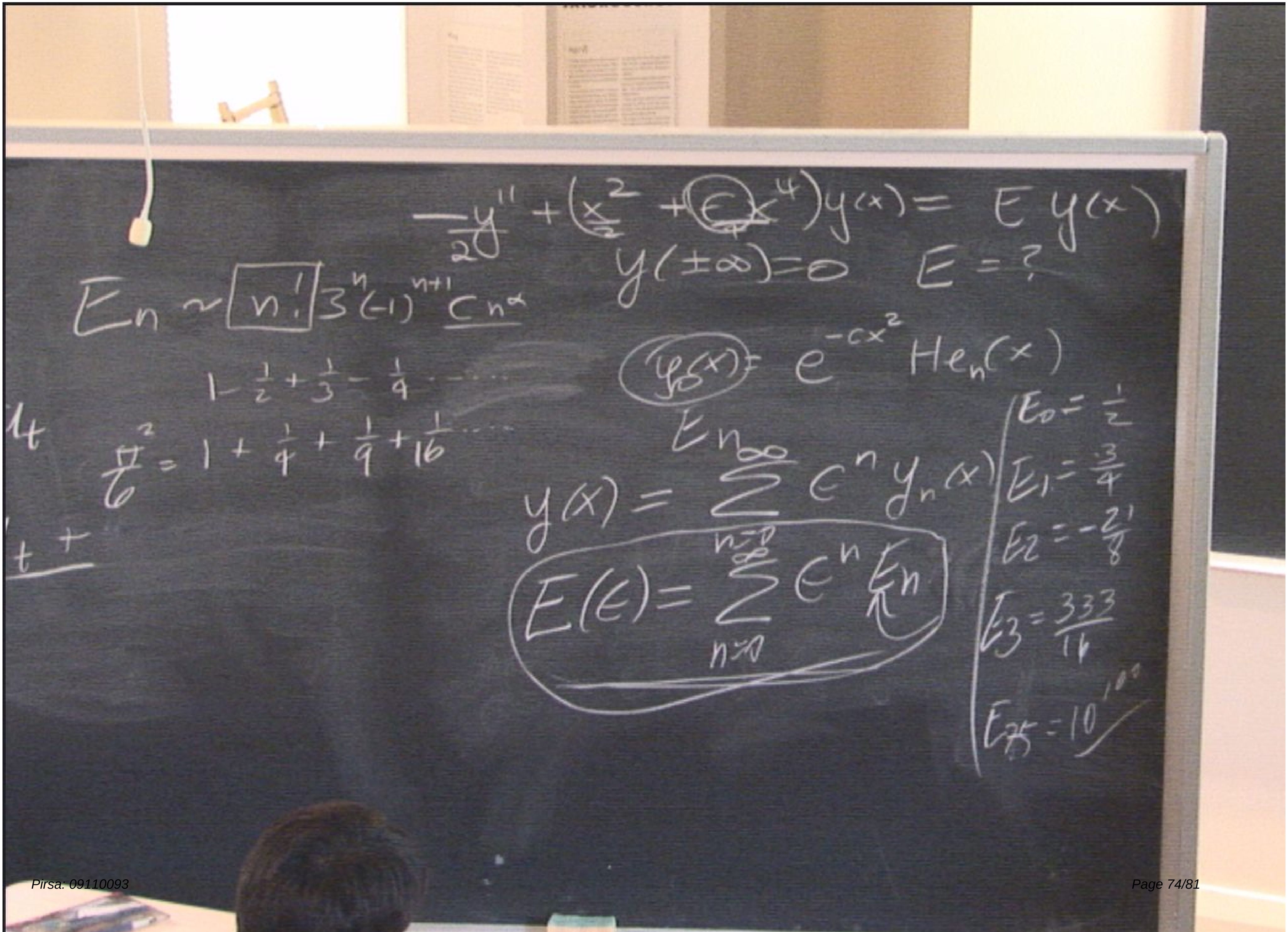
- Beginning
- Middle
- End
- (applause)

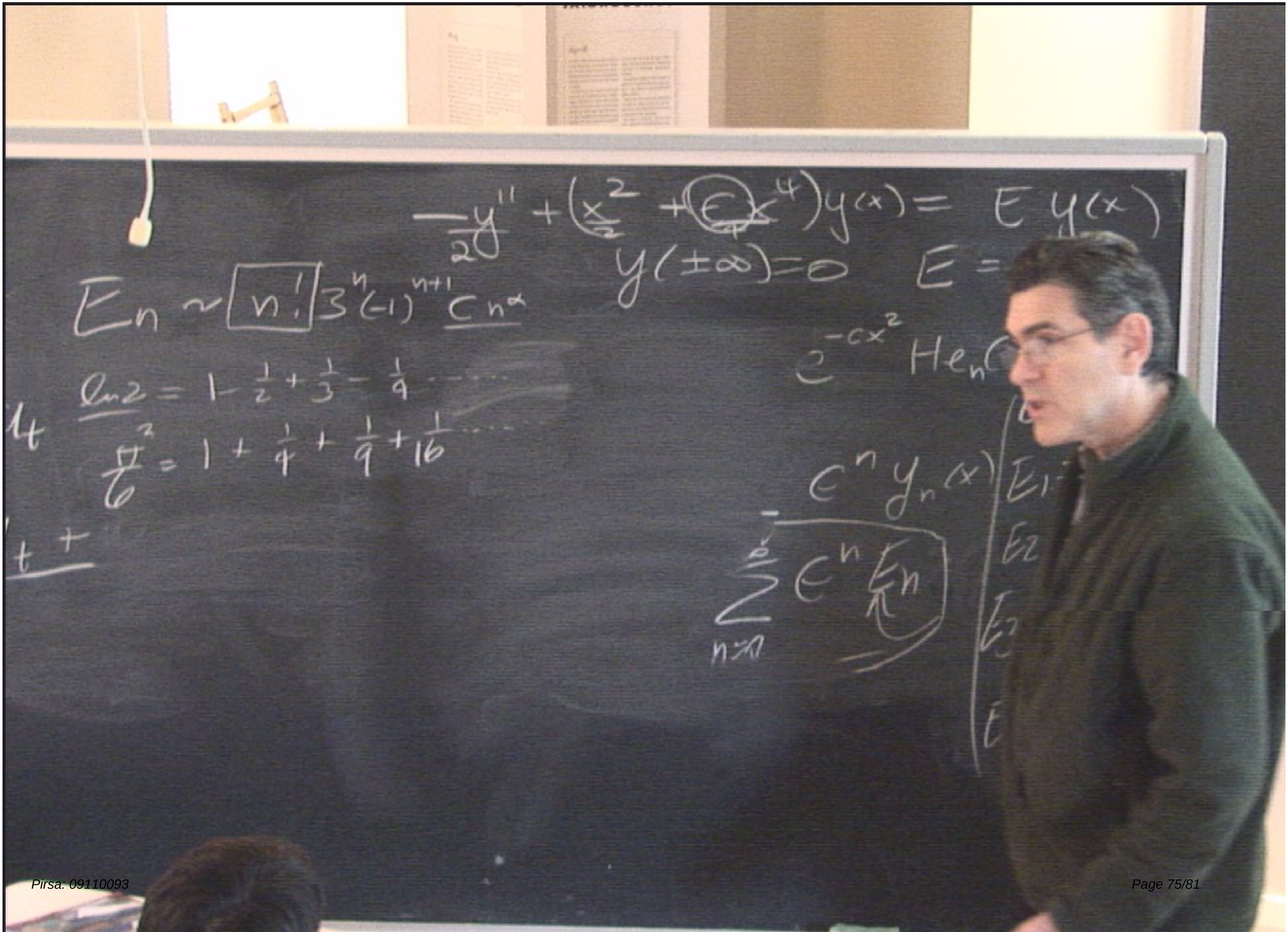
Just kidding…

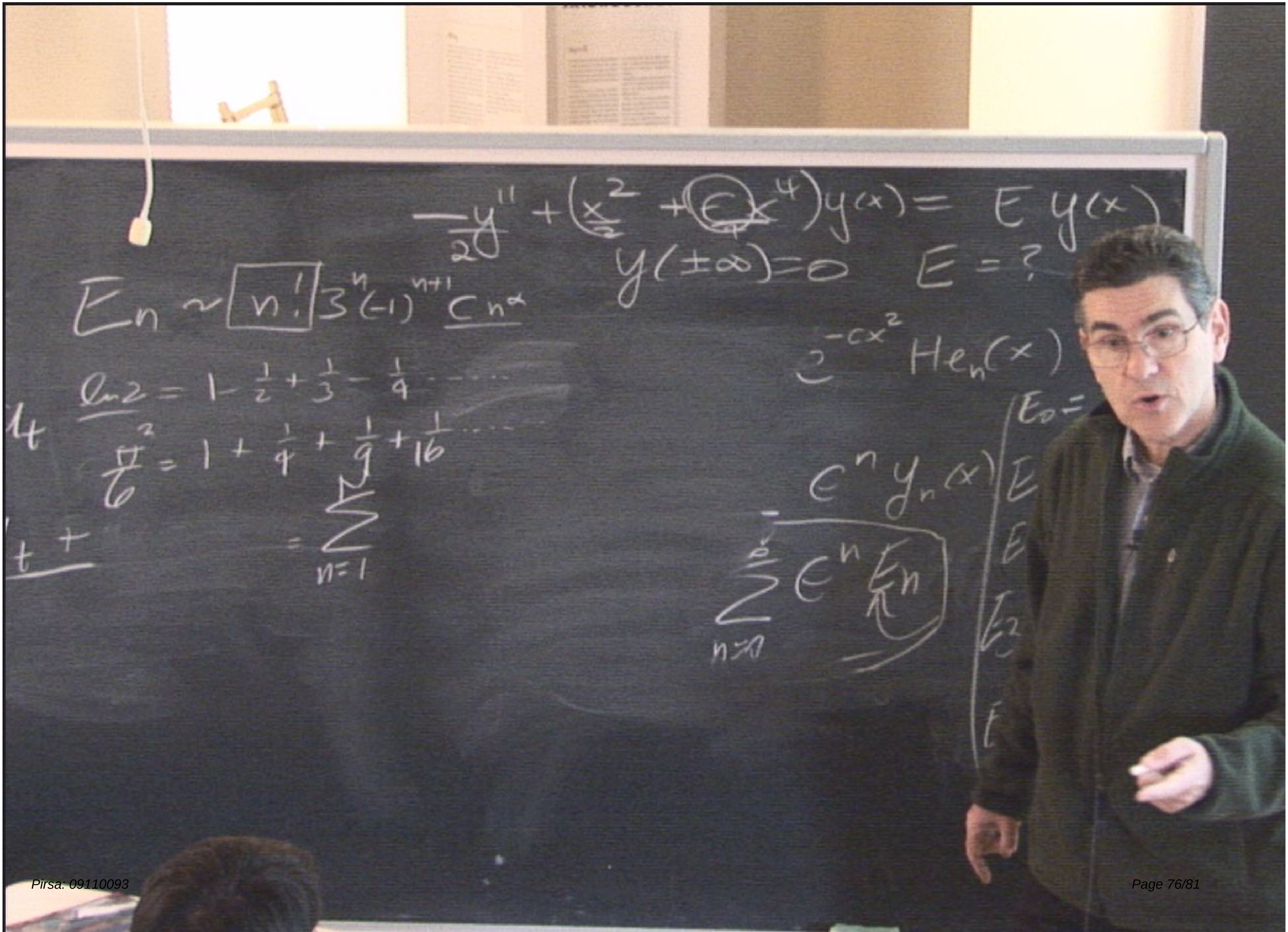
(1) Acceleration of convergence
(2) Shanks, Richardson, and so on
(3) Fourier series and Gibbs phenomenon
(4) Summation of divergent series…
(5) Pade, continued fractions, etc.

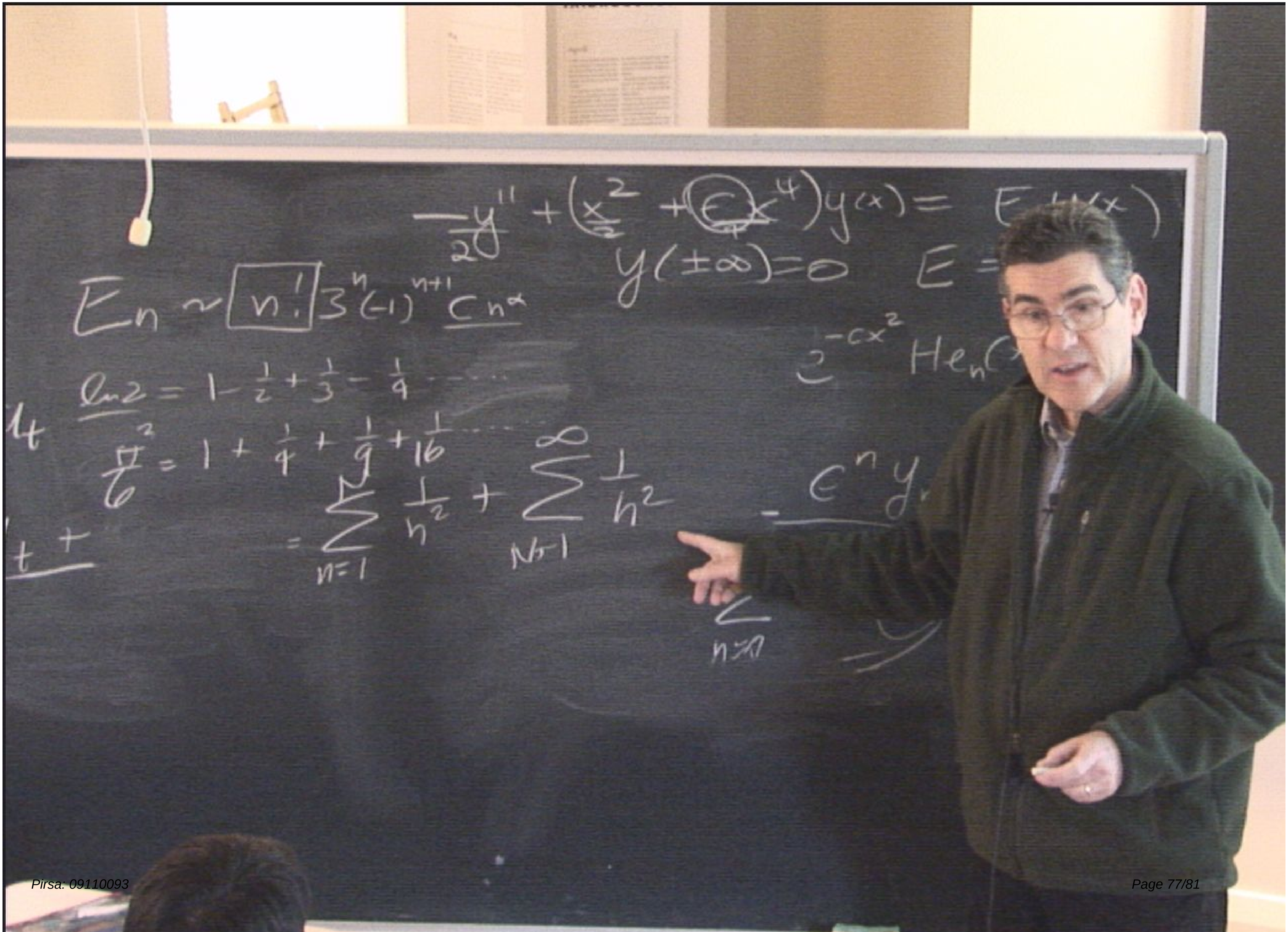


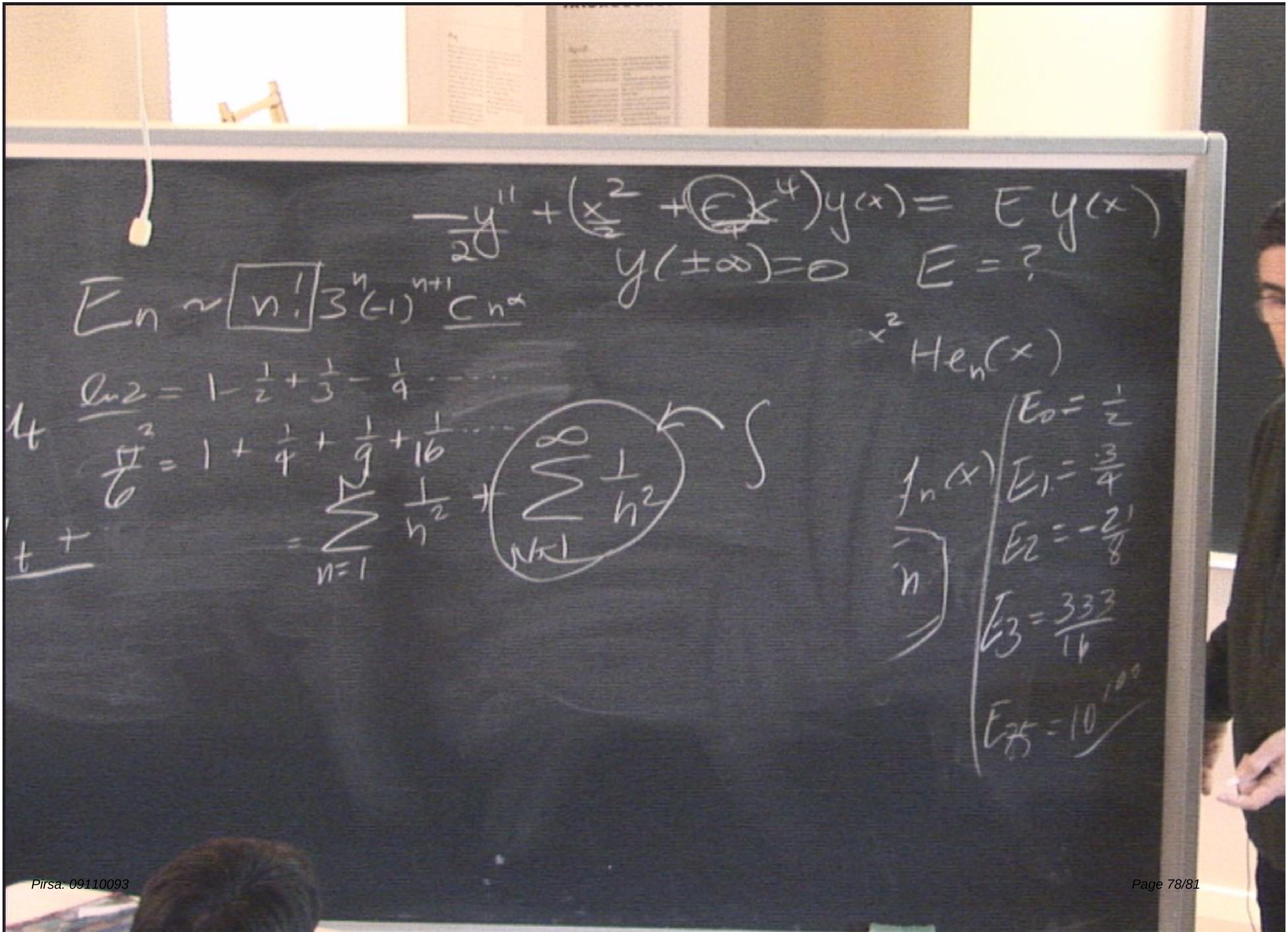
-y''/2 + (x^2/2 + εx^4/4) y(x) = E y
y(±∞) = 0   E = ?
E_n ~ n! 3^n (-1)^(n+1) C n^α
1 - 1/2 + 1/3 - 1/4 ......
y_0(x) = e^(-cx^2) He_n(x)
E_n
y(x) = Σ_(n=0)^∞ ε^n y_n(x)
E(ε) = Σ_(n=0)^∞ ε^n E_n
E_0 =
E_1 =
E_2 =
E_3 =
E_75
u_t
u_t +

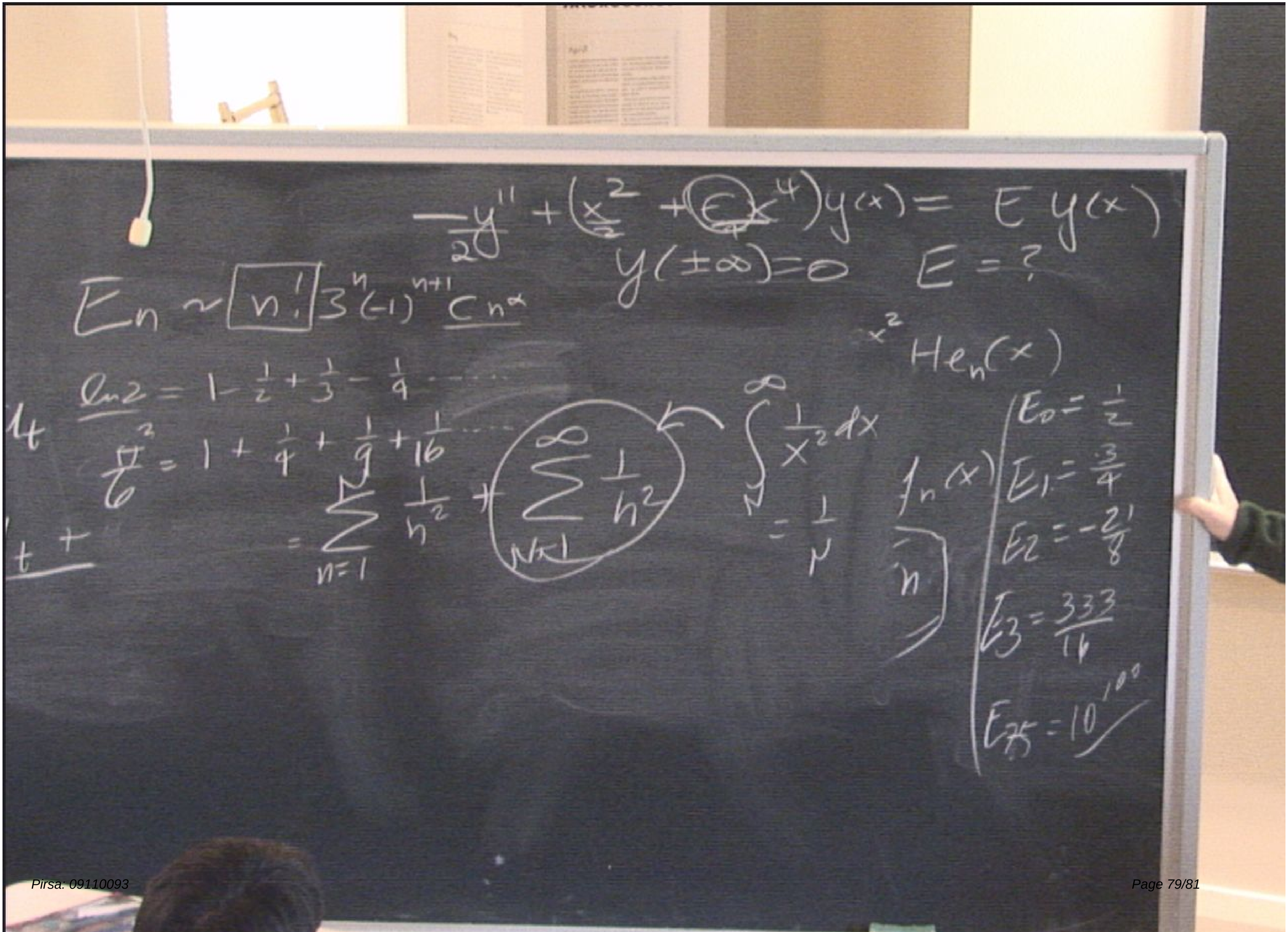

$-\frac{1}{2}y'' + \left(\frac{x^2}{2} + \frac{\epsilon x^4}{4}\right)y(x) = E\,y(x)$
$y(\pm\infty) = 0 \quad E = ?$
$E_n \sim n!\,3^n(-1)^{n+1}\,C n^\alpha$
$\psi_n(x) = e^{-cx^2} He_n(x)$
$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \cdots$
$\frac{\pi^2}{6} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} \cdots$
$y(x) = \sum_{n=0}^{\infty} \epsilon^n y_n(x)$
$E(\epsilon) = \sum_{n=0}^{\infty} \epsilon^n E_n$
$E_0 = \frac{1}{2}$
$E_1 = \frac{3}{4}$
$E_2 = -\frac{21}{8}$
$E_3 = \frac{333}{16}$
$E_{75} = 10^{100}$

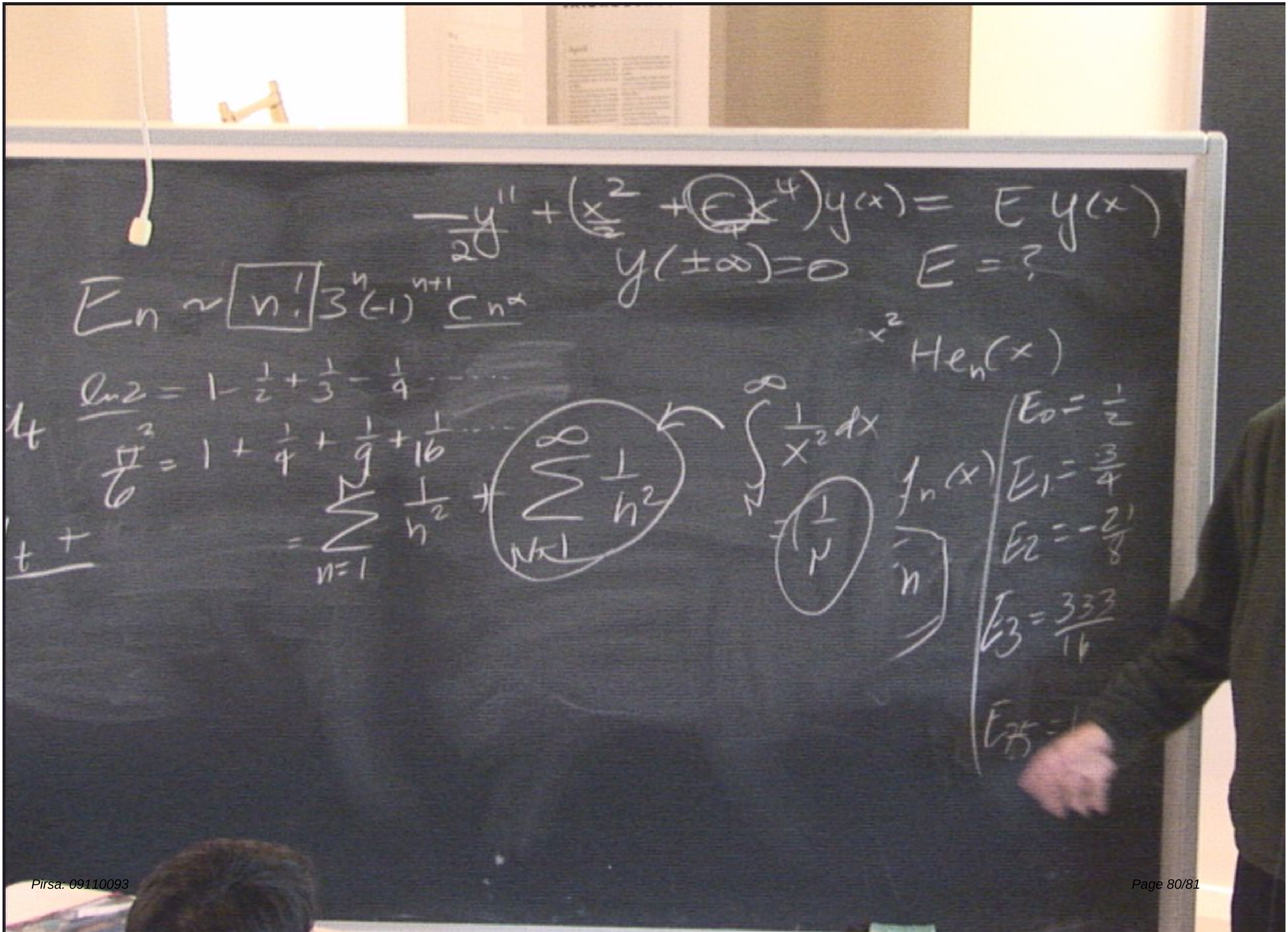

-1/2 y'' + (x²/2 + ε x⁴/4) y(x) = E y(x)
y(±∞) = 0
En ~ n! 3ⁿ (-1)ⁿ⁺¹ C n^α
ln 2 = 1 - 1/2 + 1/3 - 1/4
π²/6 = 1 + 1/4 + 1/9 + 1/16
e^(-cx²) Heₙ
εⁿ yₙ(x)
Σ εⁿ Eₙ


$-\frac{1}{2}y'' + (\frac{x^2}{2} + \epsilon x^4)y(x) = E\,y(x)$
$y(\pm\infty) = 0 \quad E = ?$
$E_n \sim n!\, 3^n (-1)^{n+1} C n^\alpha$
$e^{-cx^2} He_n(x)$
$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \cdots$
$\frac{\pi^2}{6} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} \cdots$
$= \sum_{n=1}$
$\epsilon^n y_n(x)$
$\sum_{n=0} \epsilon^n E_n$

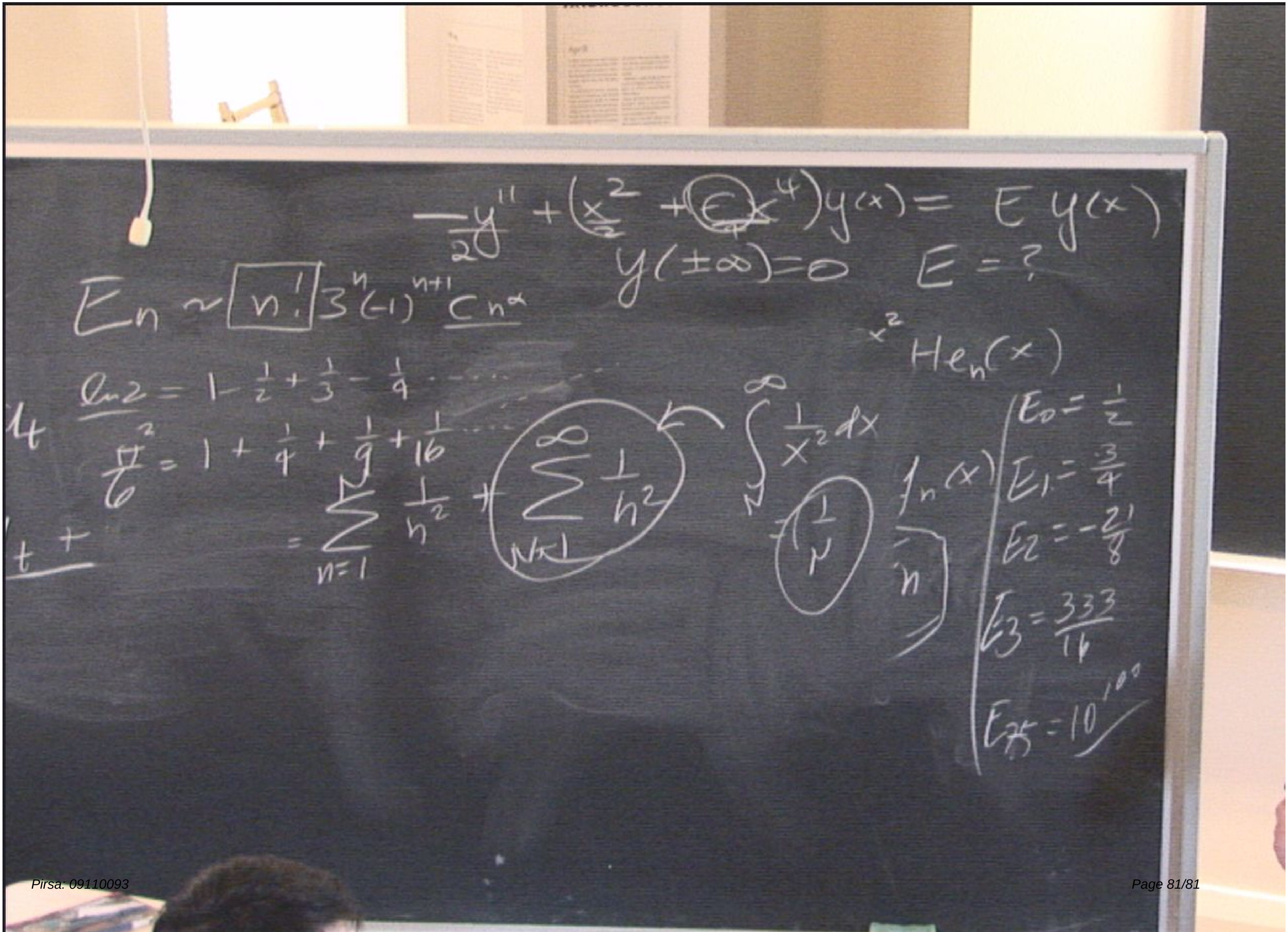

y(±∞)=0
ln2 = 1 - 1/2 + 1/3 - 1/4 ......
π²/6 = 1 + 1/4 + 1/9 + 1/16 ......


-y''/2 + (x^2/2 + εx^4/4) y(x) = E y(x)
y(±∞) = 0
E = ?
E_n ~ n! 3^n (-1)^(n+1) C n^α
ln 2 = 1 - 1/2 + 1/3 - 1/4
π^2/6 = 1 + 1/4 + 1/9 + 1/16
He_n(x)
E_0 = 1/2
E_1 = 3/4
E_2 = -21/8
E_3 = 333/16
E_75 = 10^100


-y''/2 + (x²/2 + εx⁴/4) y(x) = E y(x)
y(±∞) = 0   E = ?
Eₙ ~ n! 3ⁿ (-1)ⁿ⁺¹ c n^α
ln 2 = 1 - 1/2 + 1/3 - 1/4 ......
π²/6 = 1 + 1/4 + 1/9 + 1/16 ......
He_n(x)
E₀ = 1/2
E₁ = 3/4
E₂ = -21/8
E₃ = 333/16
E₇₅ = 10¹⁰⁰



-y''/2 + (x²/2 + εx⁴/4) y(x) = E y(x)
y(±∞) = 0   E = ?
Eₙ ~ n! 3ⁿ (-1)ⁿ⁺¹ C n^α
Heₙ(x)
ln 2 = 1 - 1/2 + 1/3 - 1/4 ......
π²/6 = 1 + 1/4 + 1/9 + 1/16 ......
E₀ = 1/2
E₁ = 3/4
E₂ = -21/8
E₃ = 333/16


−y''/2 + (x²/2 + εx⁴) y(x) = E y(x)
y(±∞) = 0   E = ?
Eₙ ~ n! 3ⁿ (−1)ⁿ⁺¹ C n^α
ln2 = 1 − 1/2 + 1/3 − 1/4 ......
π²/6 = 1 + 1/4 + 1/9 + 1/16 ......
Hₑₙ(x)
E₀ = 1/2
E₁ = 3/4
E₂ = −21/8
E₃ = 333/16
E₇₅ = 10¹⁰⁰