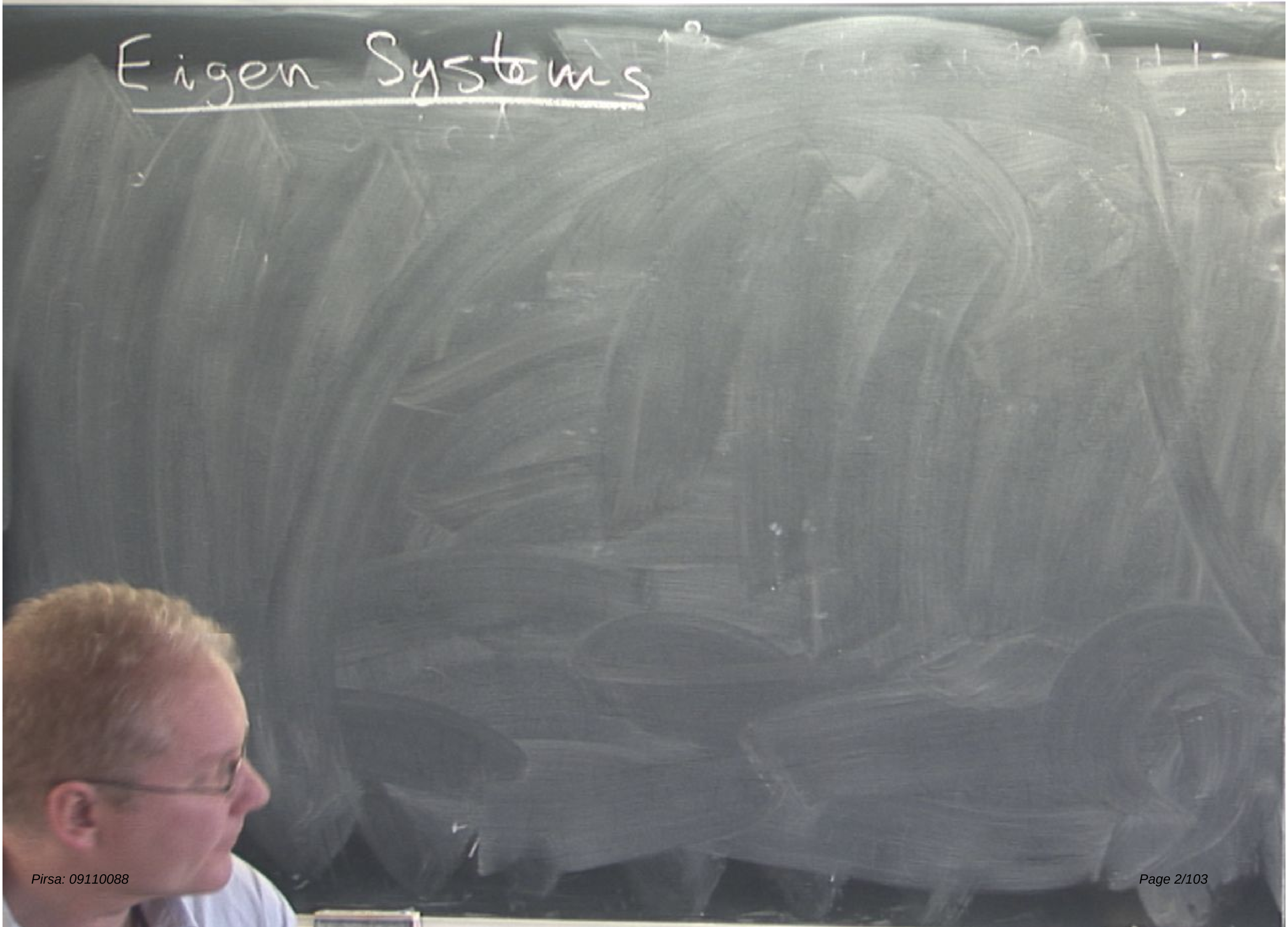
Title: Scientific Computation (PHYS 608) - Lecture 11

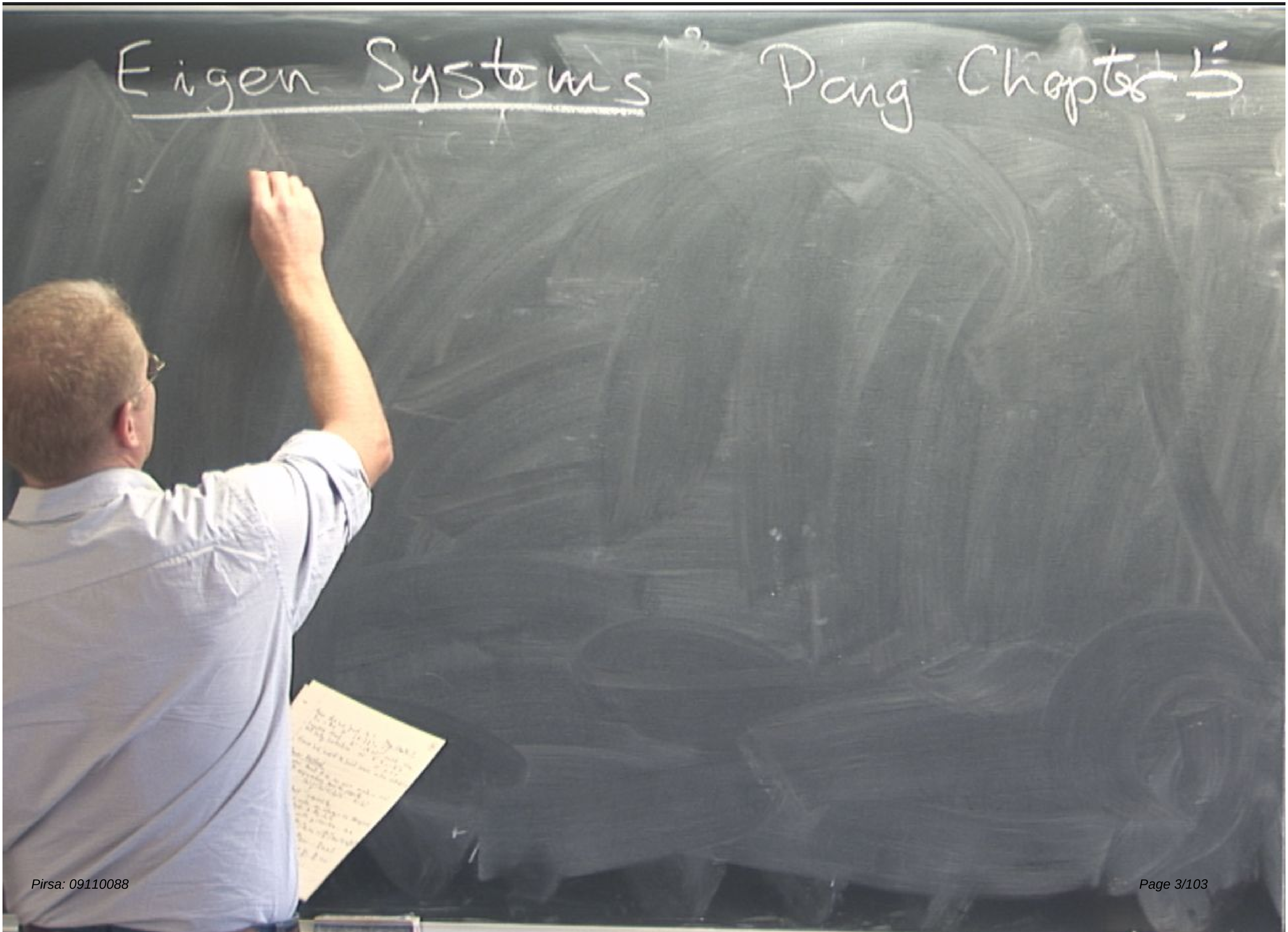
Date: Nov 09, 2009 10:30 AM

URL: http://pirsa.org/09110088

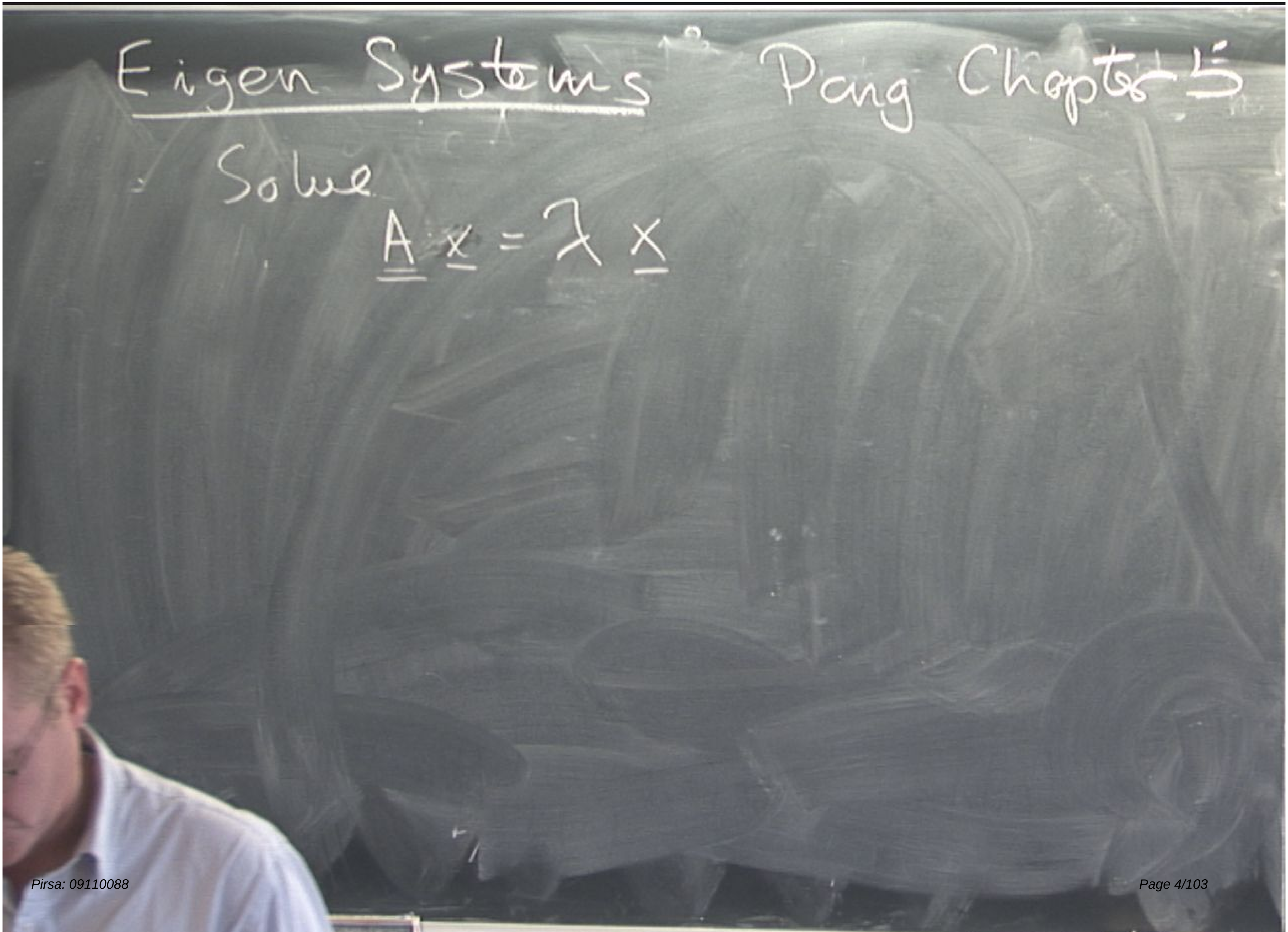
Abstract:



Eigen Systems



Eigen Systems : Pang Chapter 5

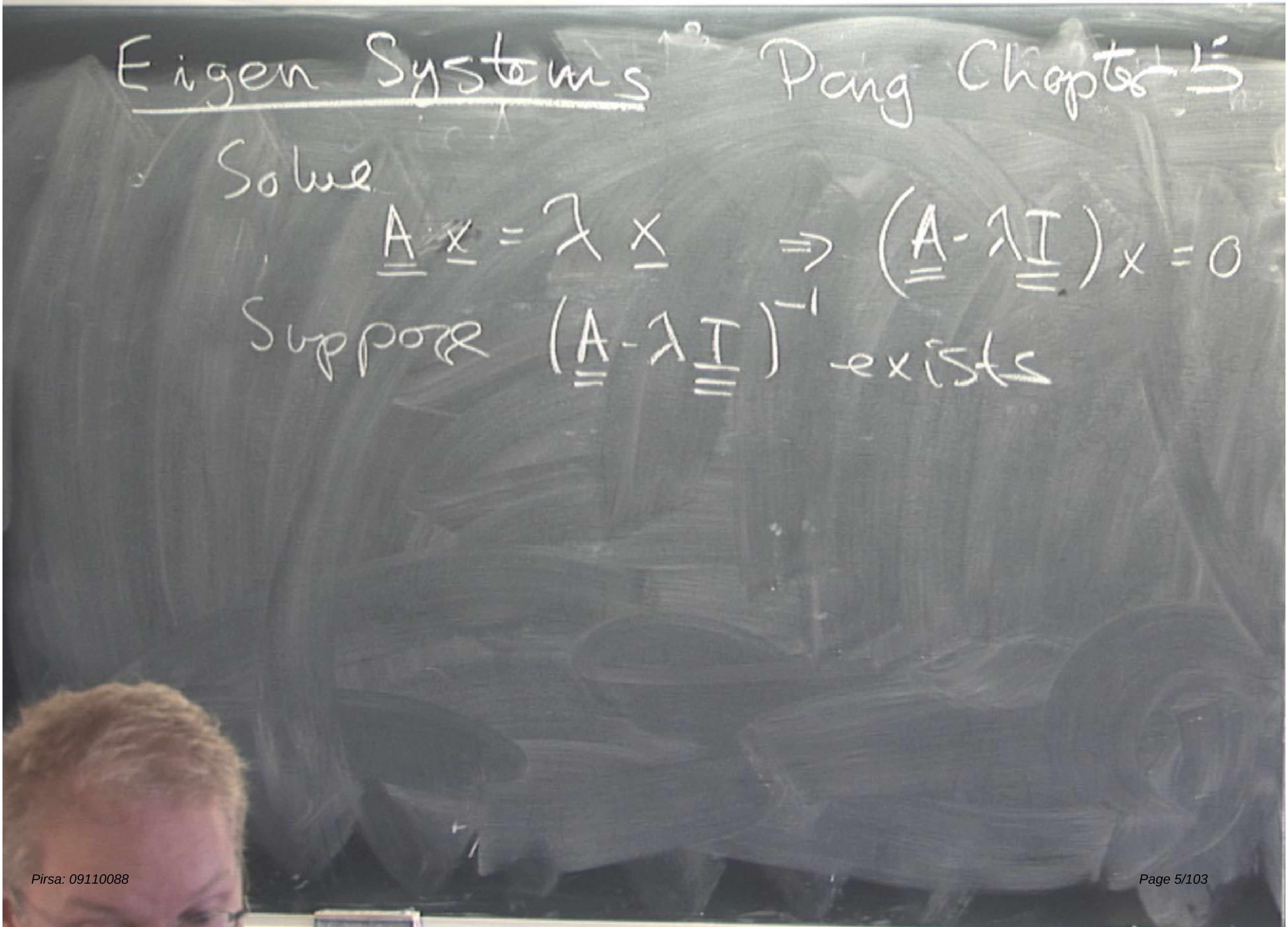


Eigen Systems

Pang Chapter 5

Solve

$$\underline{\underline{A}}\,\underline{x} = \lambda\,\underline{x}$$

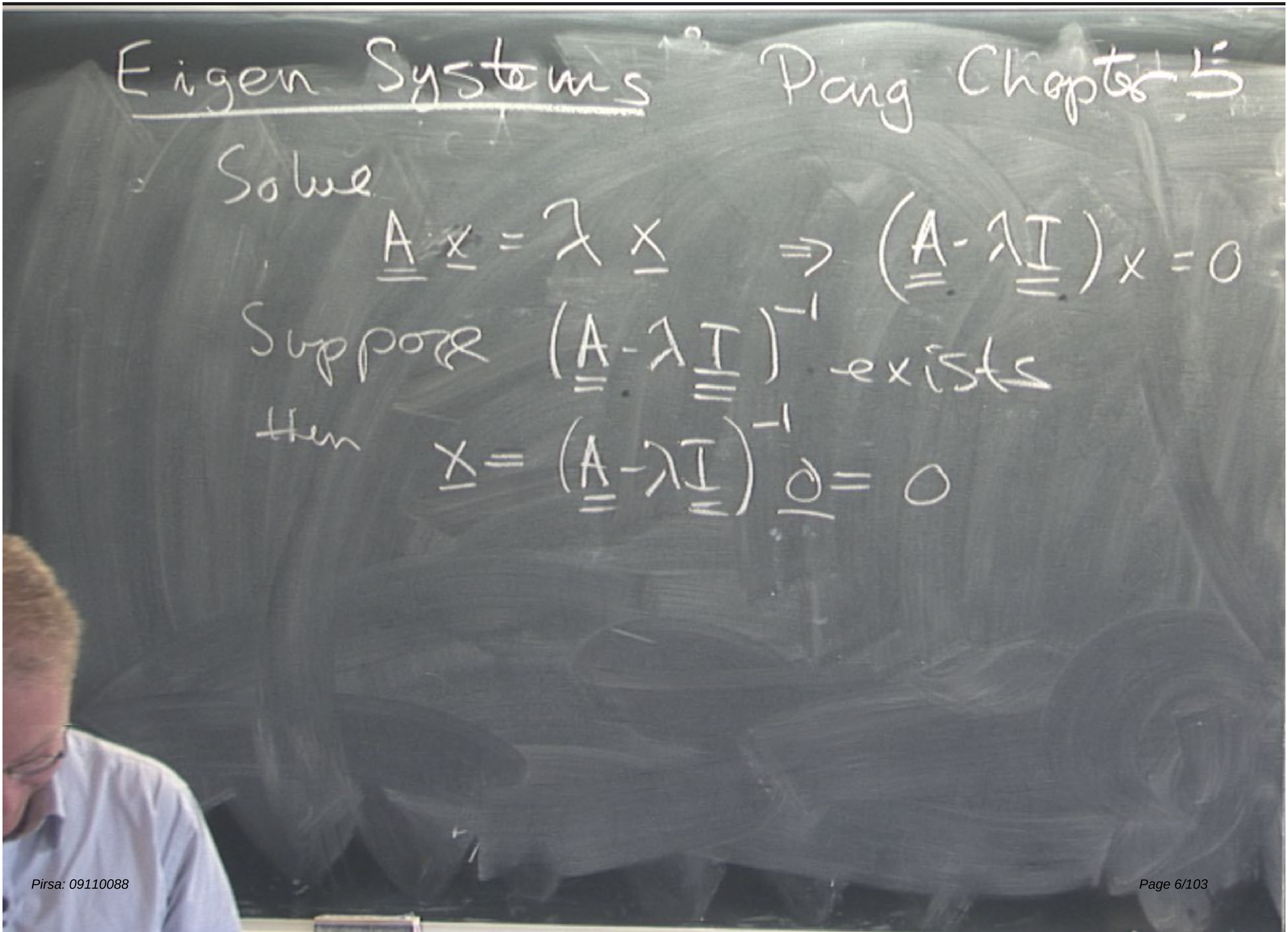


Eigen Systems   Pang Chapter 5

Solve

$$\underline{\underline{A}}\,\underline{x} = \lambda\,\underline{x} \quad \Rightarrow \quad \left(\underline{\underline{A}} - \lambda\underline{\underline{I}}\right) x = 0$$

Suppose $\left(\underline{\underline{A}} - \lambda\underline{\underline{I}}\right)^{-1}$ exists



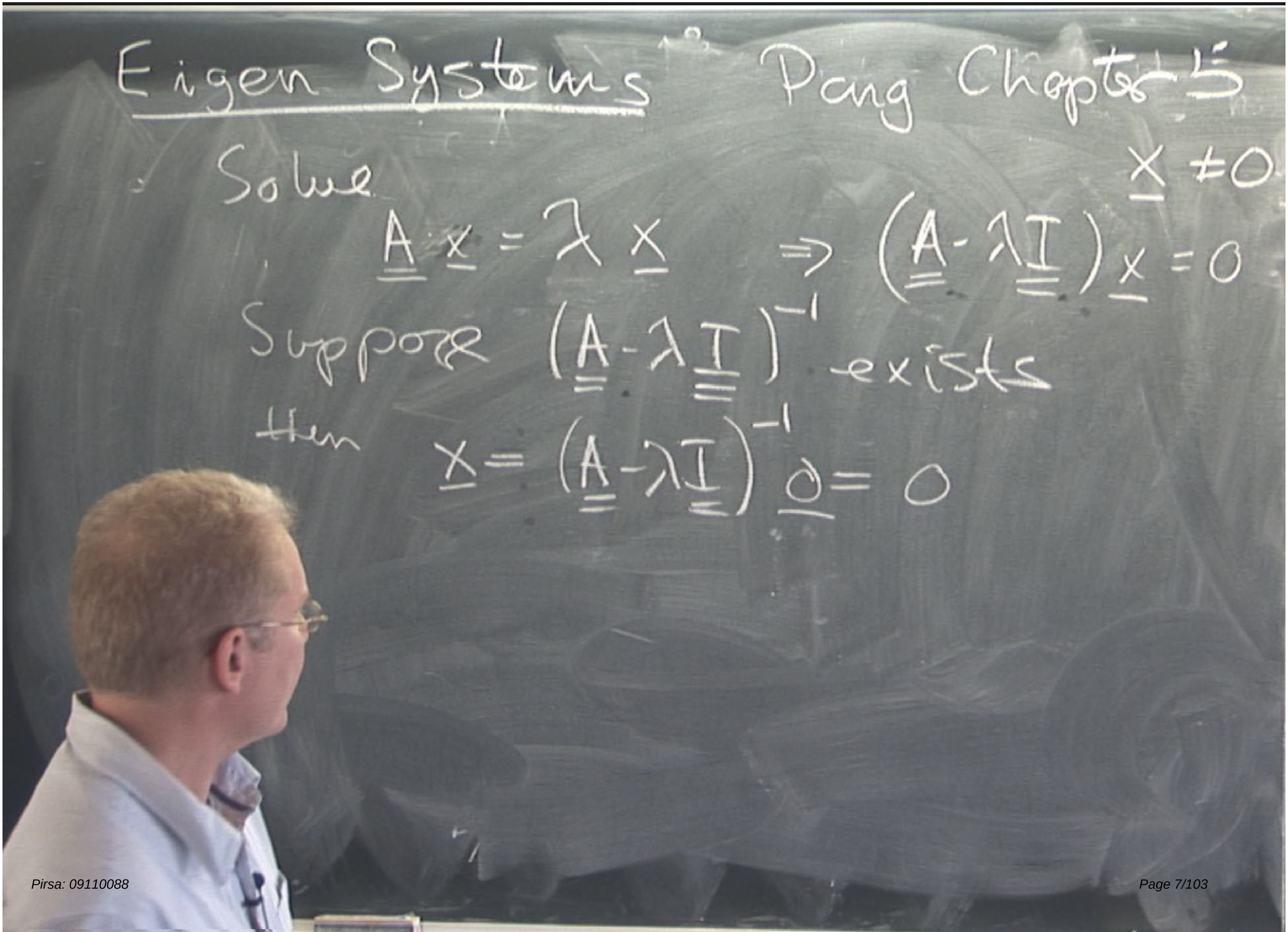
Eigen Systems Pang Chapter 5

Solve

$$\underline{\underline{A}}\,\underline{x} = \lambda\,\underline{x} \Rightarrow (\underline{\underline{A}} - \lambda\underline{\underline{I}})\,x = 0$$

Suppose $(\underline{\underline{A}} - \lambda\underline{\underline{I}})^{-1}$ exists

then $\underline{x} = (\underline{\underline{A}} - \lambda\underline{\underline{I}})^{-1}\,\underline{0} = 0$



Eigen Systems Pang Chapter 5

Solve $\underline{x} \neq 0$

$$\underline{\underline{A}}\,\underline{x} = \lambda\,\underline{x} \Rightarrow (\underline{\underline{A}} - \lambda\underline{\underline{I}})\,\underline{x} = 0$$

Suppose $(\underline{\underline{A}} - \lambda\underline{\underline{I}})^{-1}$ exists

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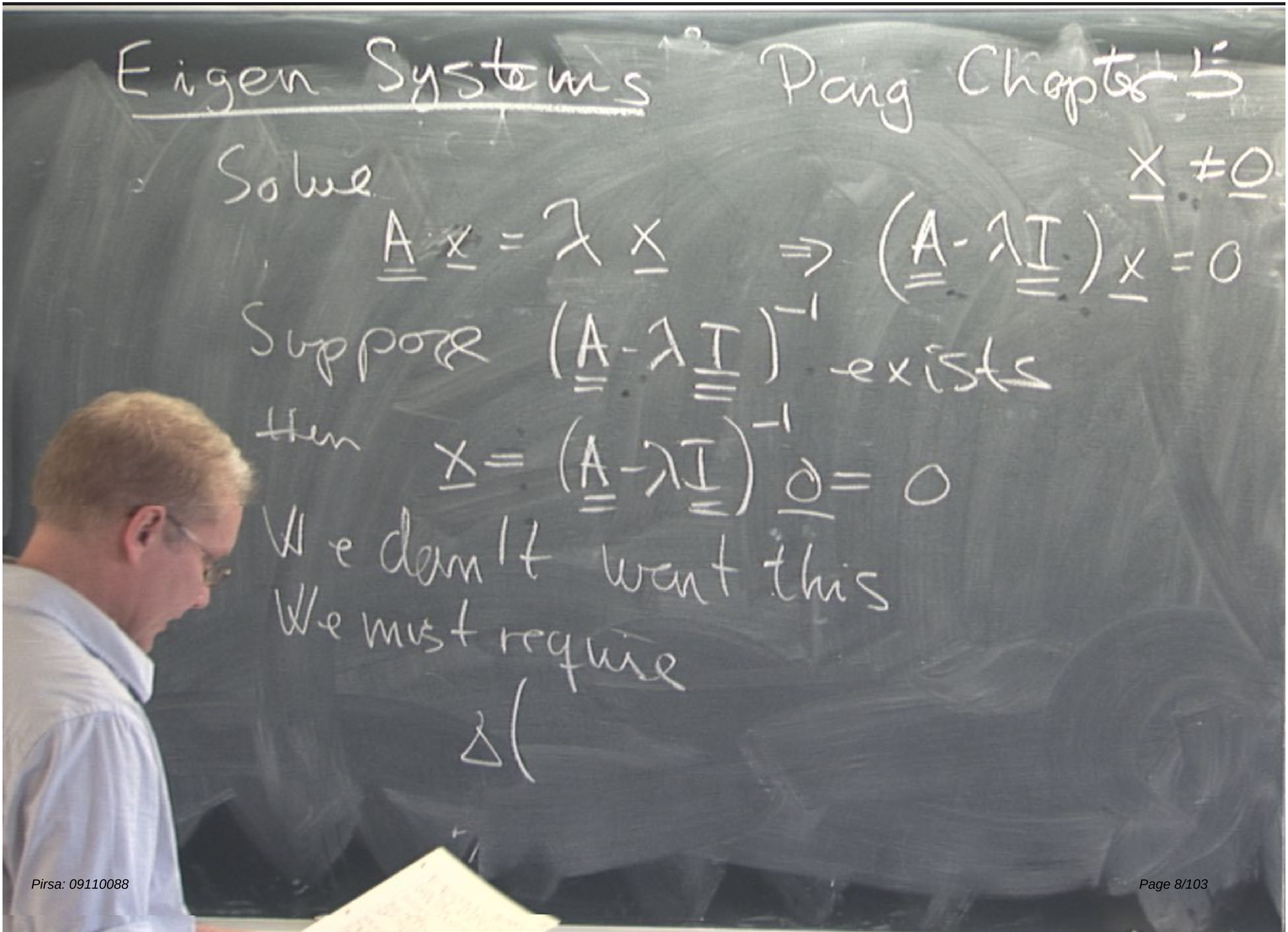

Eigen Systems   Pang Chapter 5

Solve $\underline{x} \neq \underline{0}$

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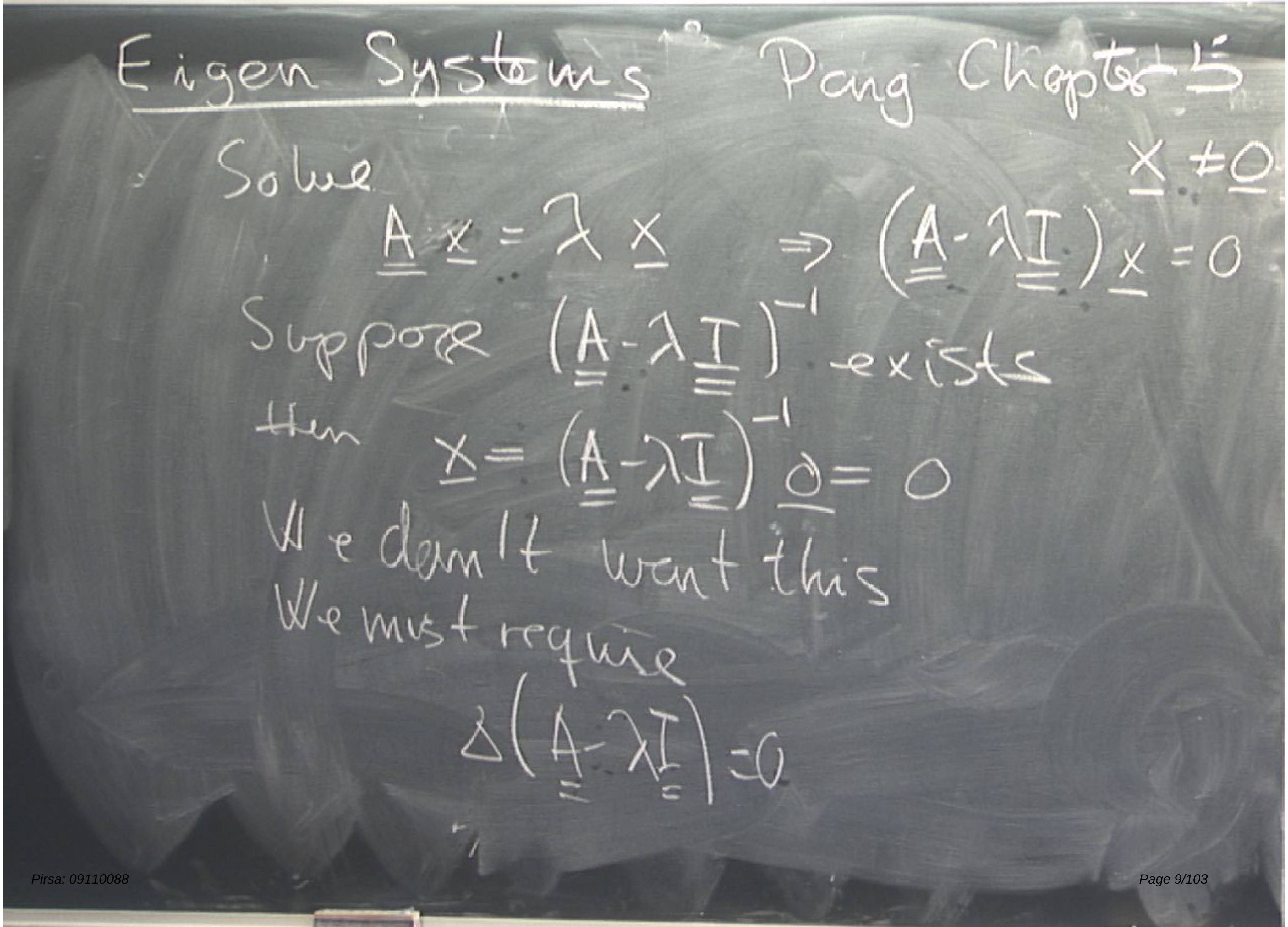
We don't want this

We must require

$\Delta($



Eigen Systems   Pang Chapter 5

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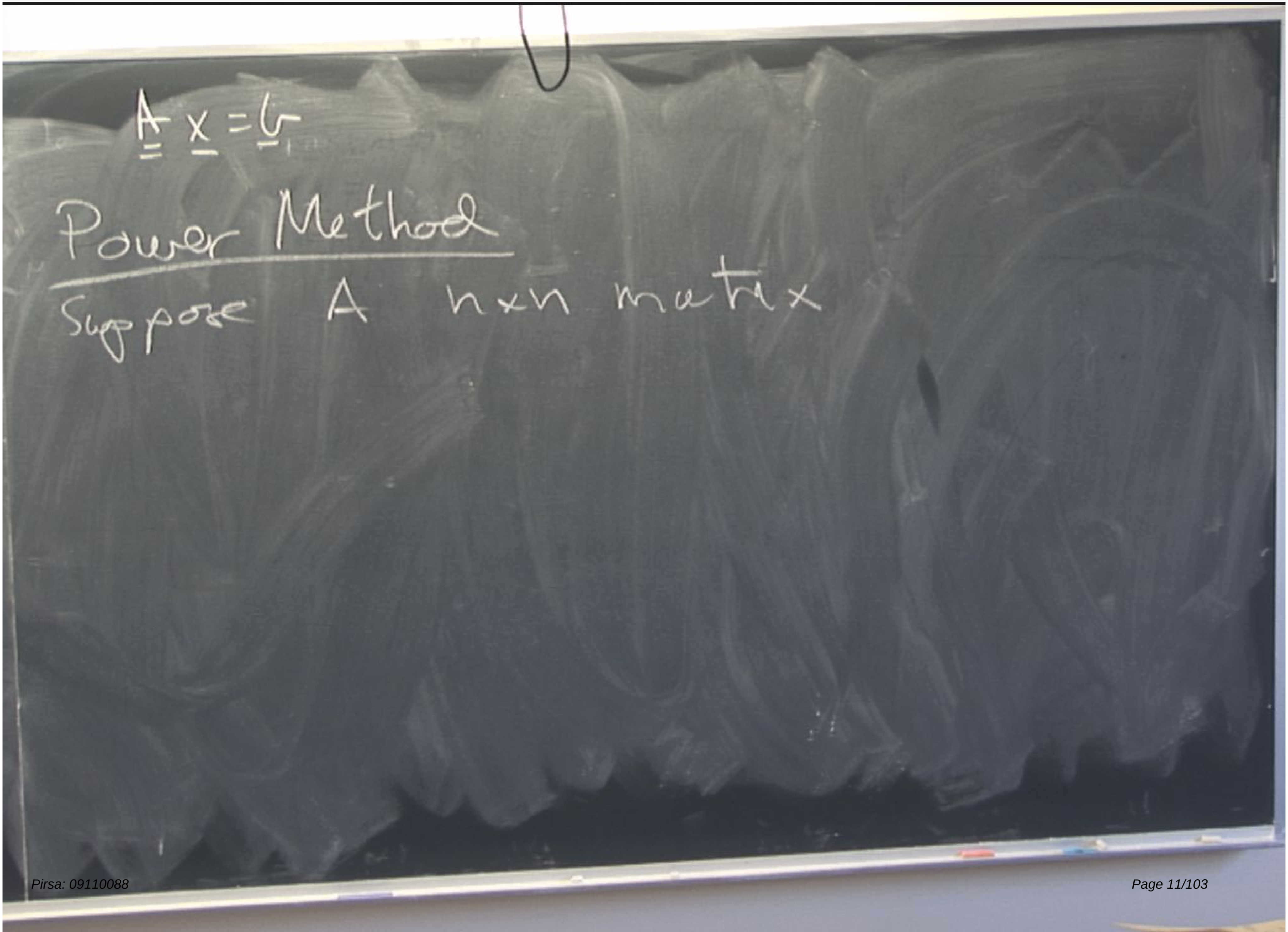
We don't want this

We must require

$$\Delta(\underline{\underline{A}} - \lambda\underline{\underline{I}}) = 0$$

$\underline{\underline{A}}\,\underline{x} = \underline{b}$

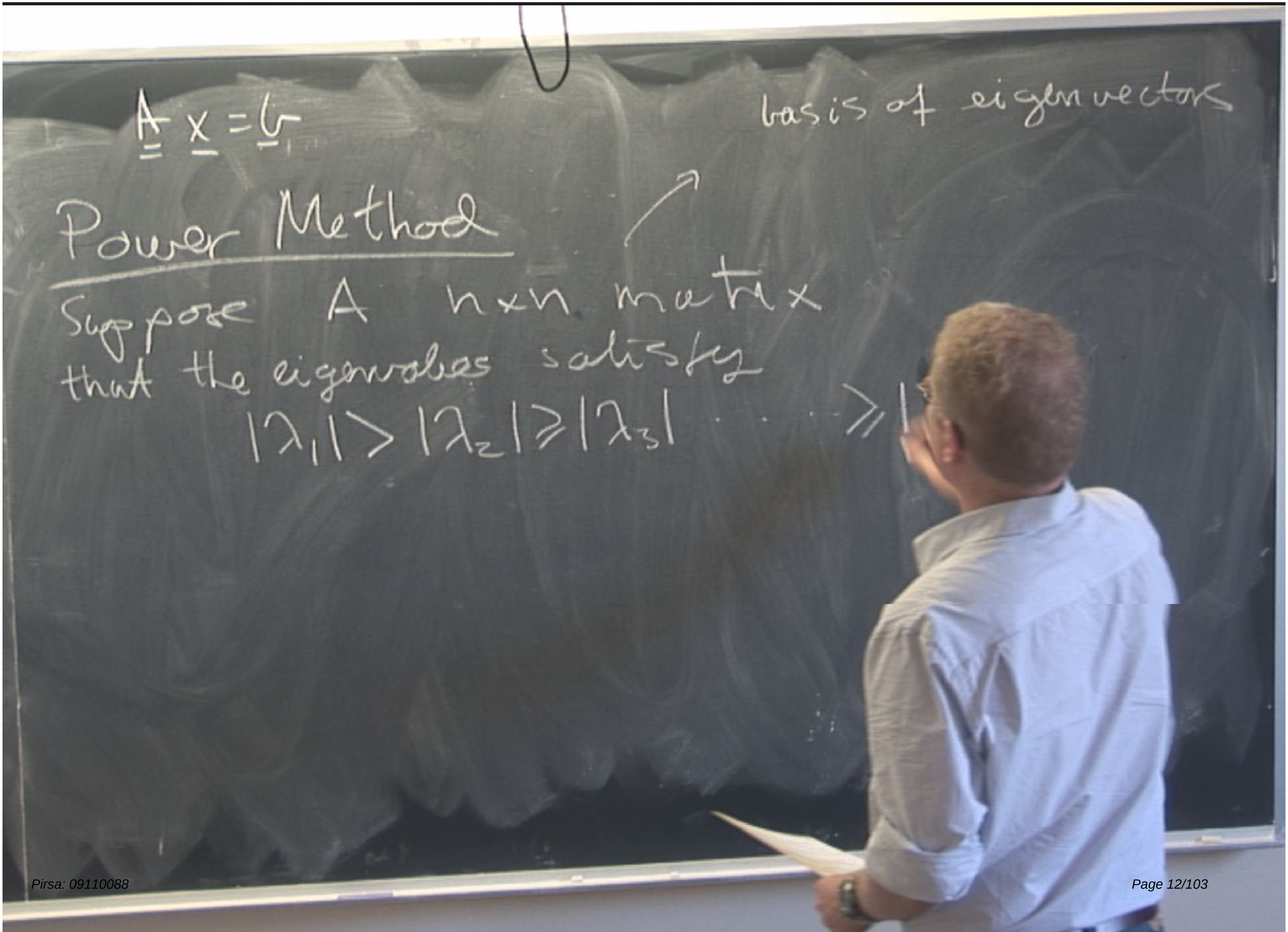
Power Method



$\underline{\underline{A}}\,\underline{x} = \underline{b}$

Power Method

Suppose A n×n matrix



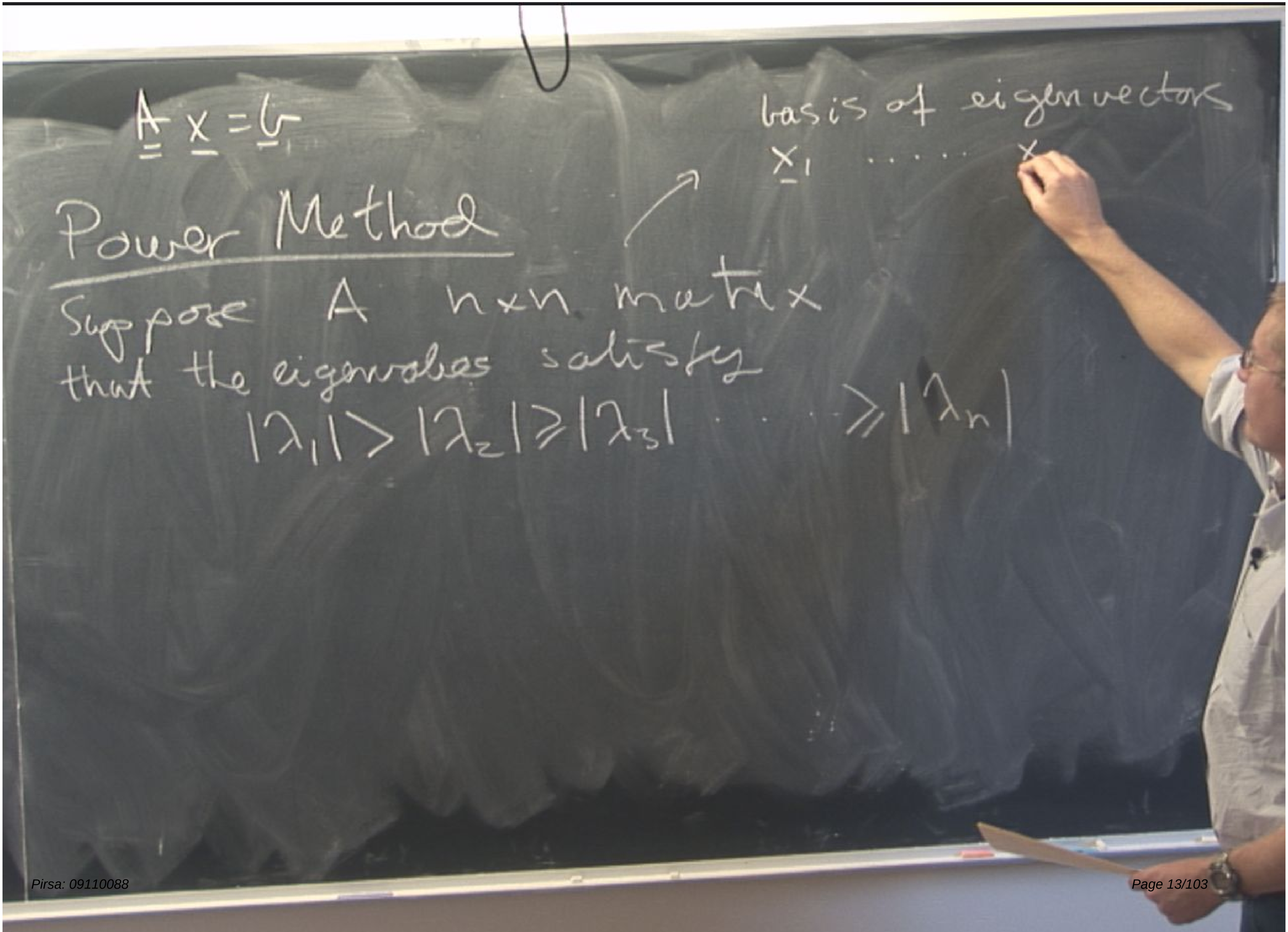
$\underline{\underline{A}}\,\underline{x} = \underline{b}$

basis of eigenvectors

Power Method

Suppose A n×n matrix that the eigenvalues satisfy

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \cdots \geq |$$



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basis of eigenvectors

$\underline{x}_1 \quad \ldots\ldots \quad x$

Power Method

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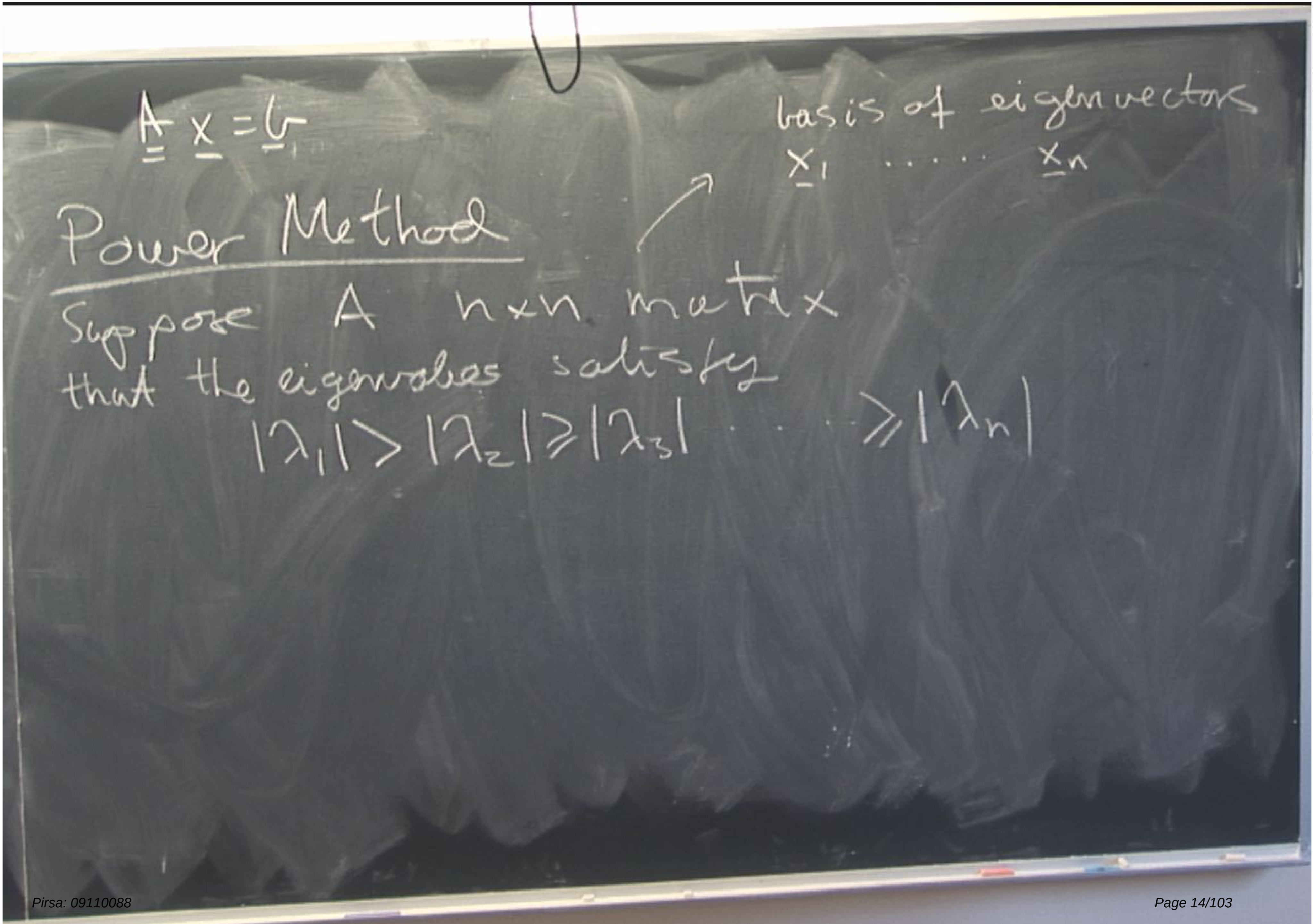

$\underline{\underline{A}}\,\underline{x} = \underline{b}$

basis of eigenvectors

$\underline{x}_1 \quad \ldots\ldots \quad \underline{x}_n$

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$\lambda_1 \quad \cdots\cdots \quad \lambda_n$

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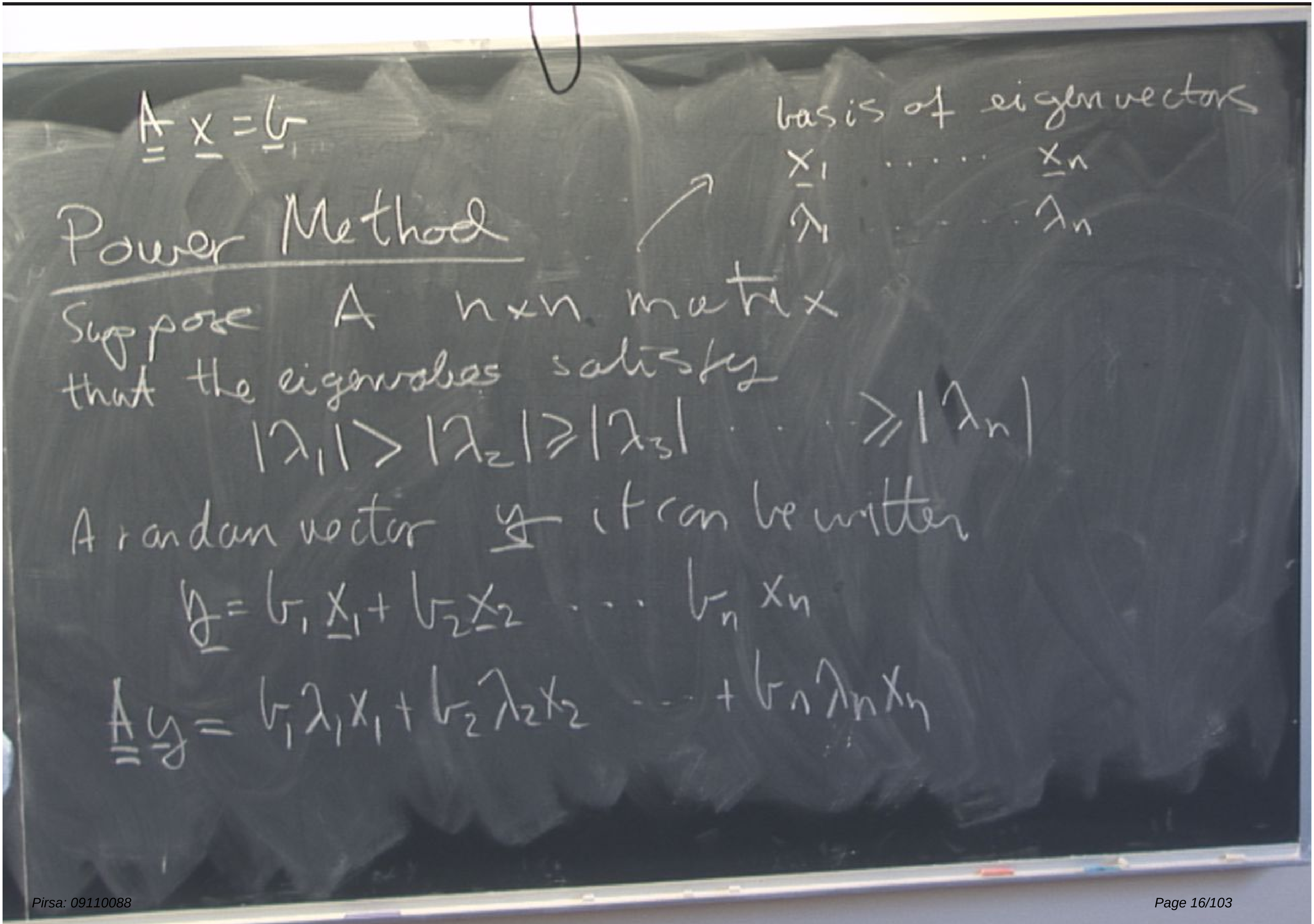
A random vector $\underline{y}$ it can be written

$$\underline{y} = b_1 \underline{x}_1 + b_2 \underline{x}_2 \cdots b_n x_n$$



$\underline{\underline{A}}\,\underline{x} = \underline{b}$

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$\lambda_1 \quad \cdots\cdots \quad \lambda_n$

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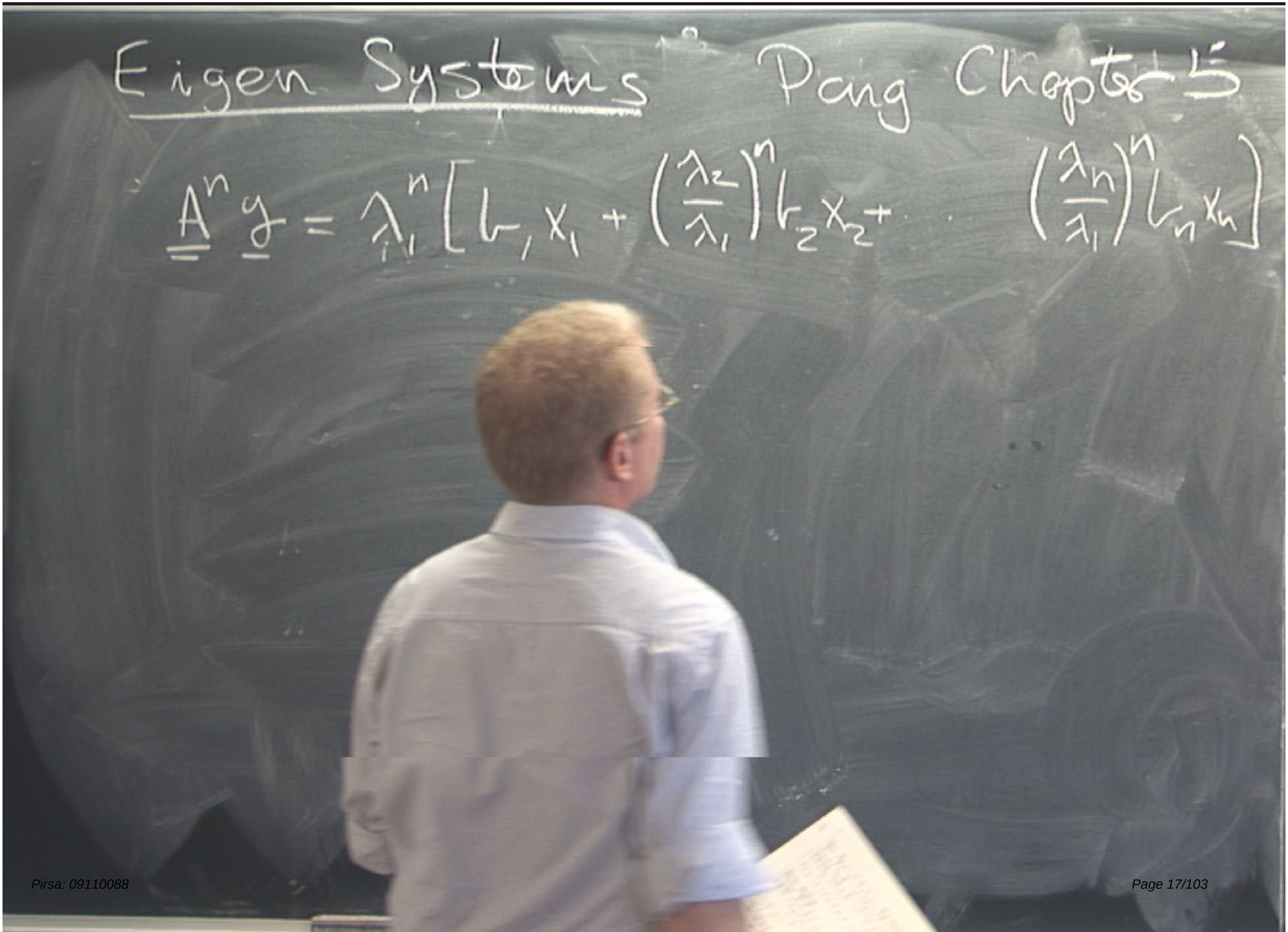
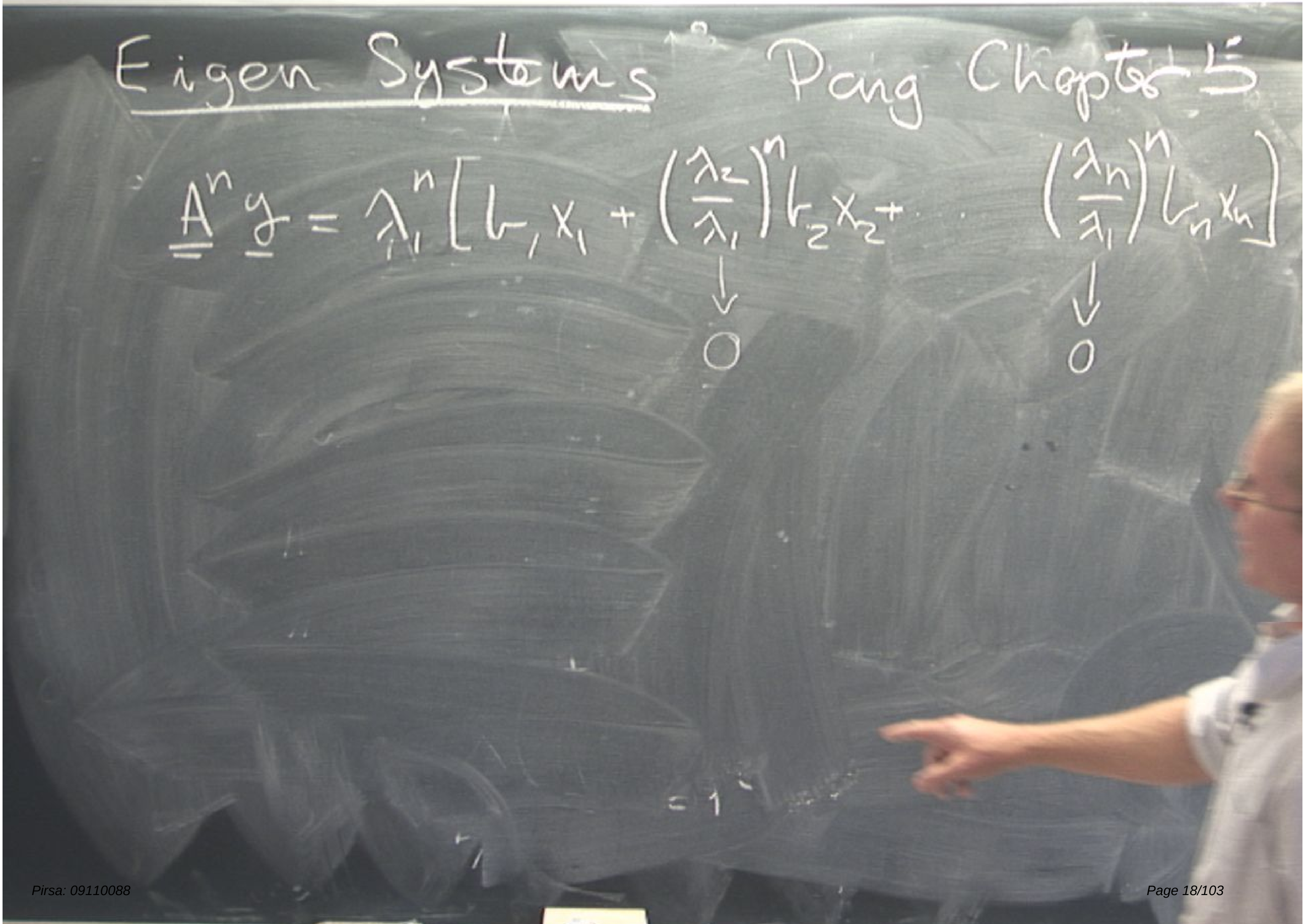
$$\underline{\underline{A}}\,\underline{y} = b_1 \lambda_1 x_1 + b_2 \lambda_2 x_2 \quad \cdots + b_n \lambda_n x_n$$



Eigen Systems   Pang Chapter 5

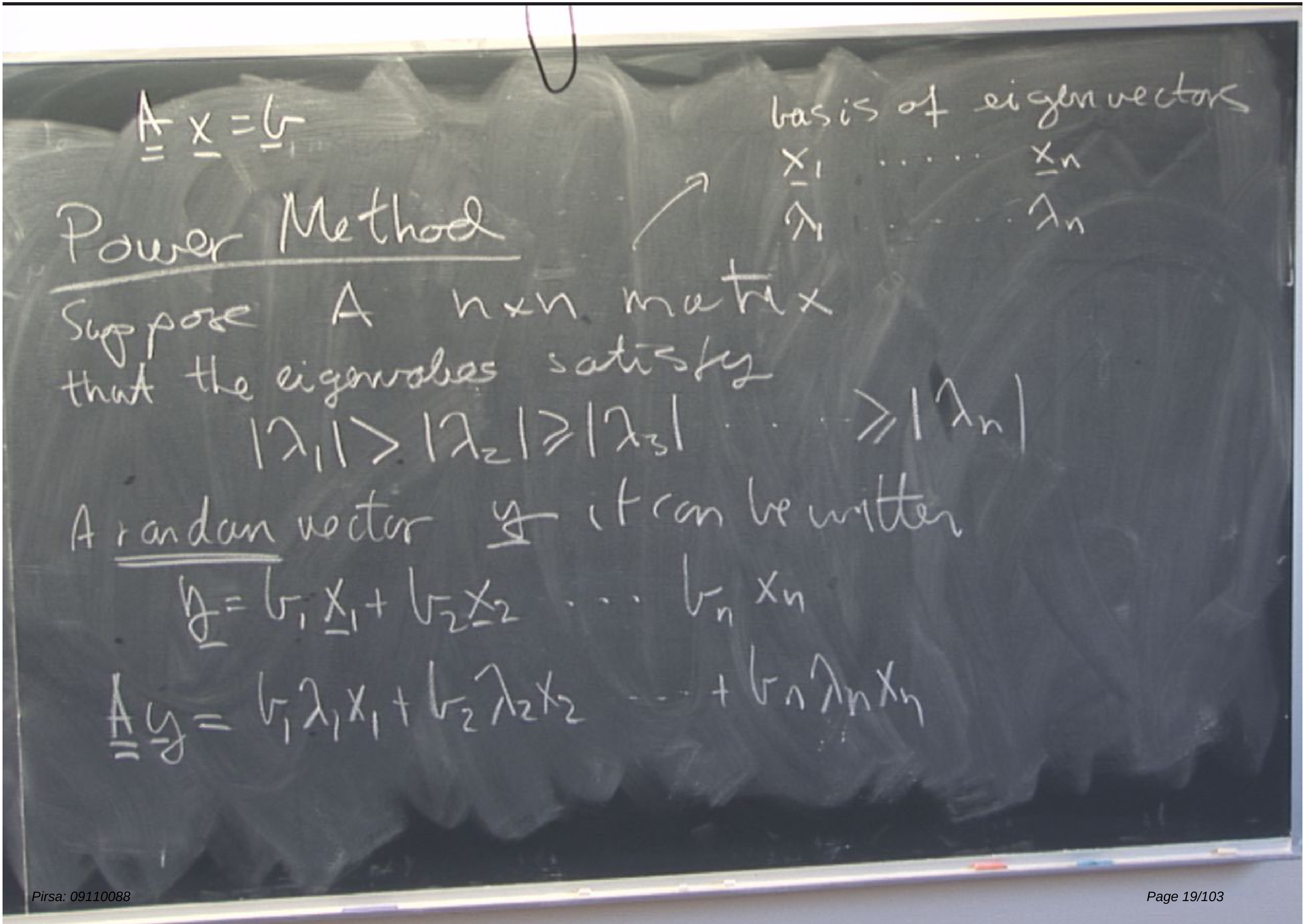
$$\underline{\underline{A}}^n \underline{y} = \lambda_1^n \left[b_1 x_1 + \left(\frac{\lambda_2}{\lambda_1}\right)^n b_2 x_2 + \cdot \quad \left(\frac{\lambda_n}{\lambda_1}\right)^n b_n x_n \right]$$



Eigen Systems Pang Chapter 5

$$\underline{\underline{A}}^n \, \underline{y} = \lambda_1^n \left[b_1 x_1 + \underset{\substack{\downarrow \\ 0}}{\left(\frac{\lambda_2}{\lambda_1}\right)^n} b_2 x_2 + \quad \underset{\substack{\downarrow \\ 0}}{\left(\frac{\lambda_n}{\lambda_1}\right)^n} b_n x_n \right]$$

$= 1$



$\underline{\underline{A}}\,\underline{x} = \underline{b}$

Power Method

basis of eigenvectors

$\underline{x}_1 \cdots\cdots \underline{x}_n$

$\lambda_1 \cdots\cdots \lambda_n$

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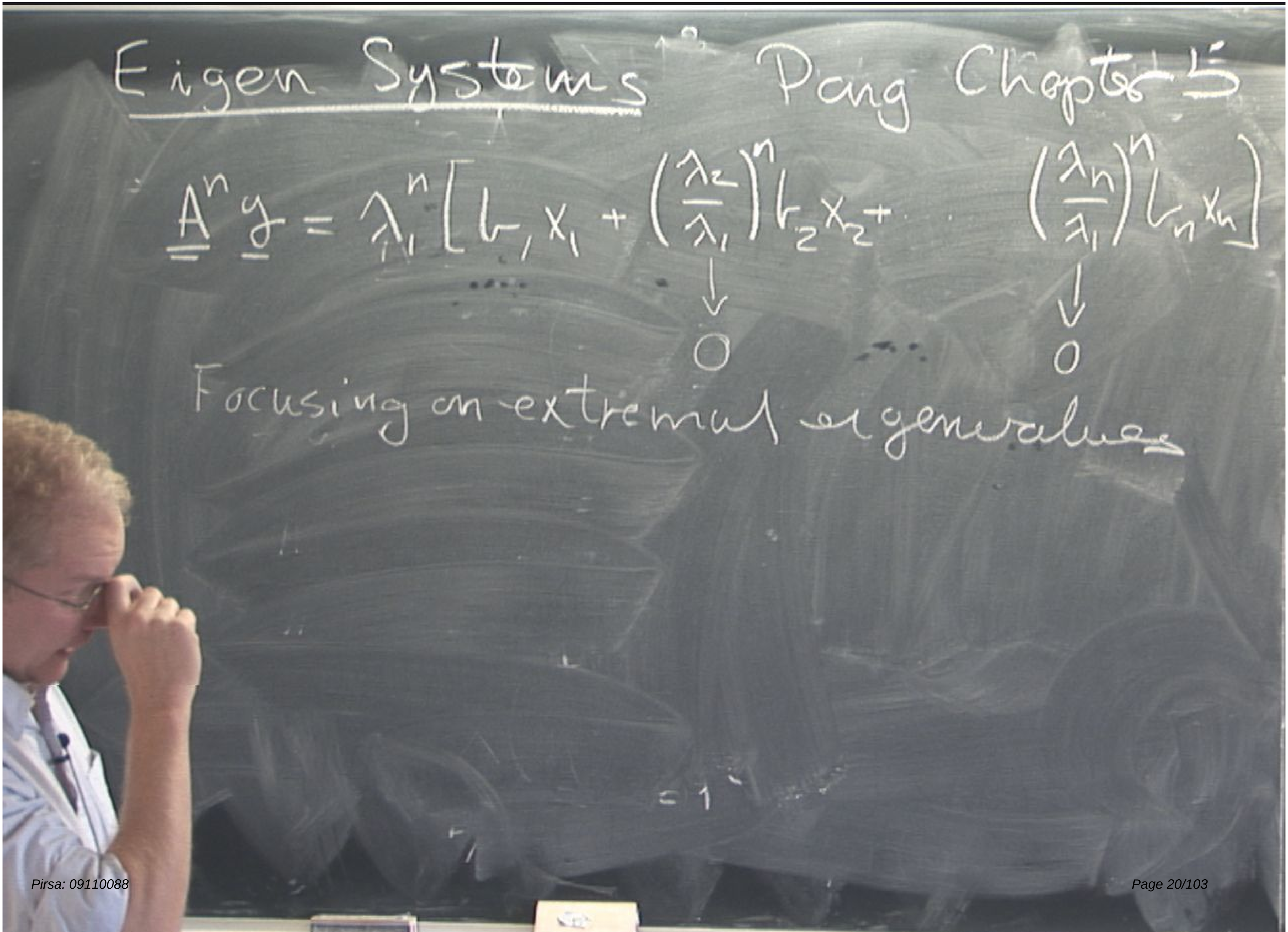
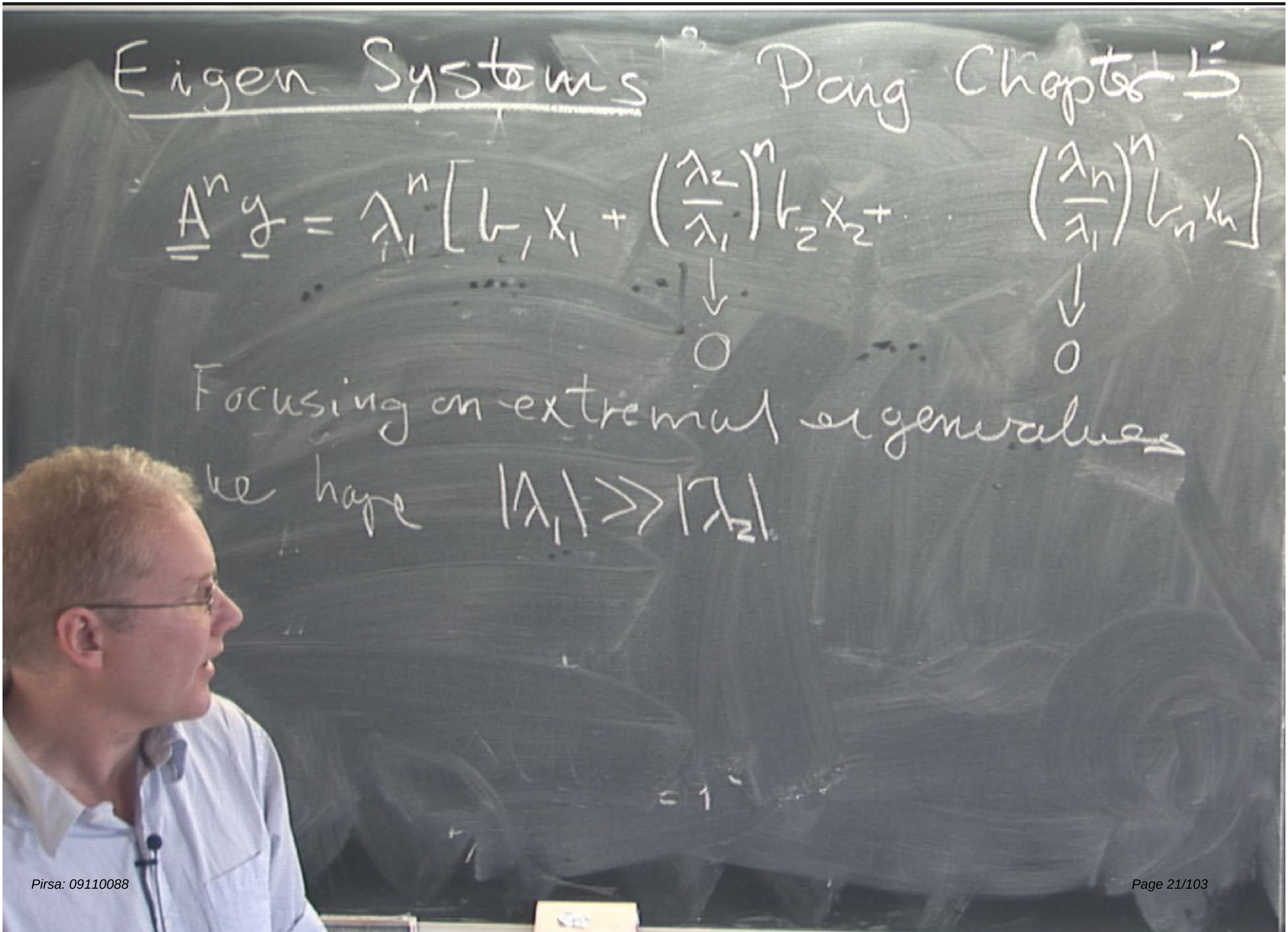


Eigen Systems   Pang Chapter 5

$$\underline{\underline{A}}^n \underline{y} = \lambda_1^n \left[b_1 x_1 + \underset{\substack{\downarrow \\ 0}}{\left(\frac{\lambda_2}{\lambda_1}\right)^n} b_2 x_2 + \qquad \underset{\substack{\downarrow \\ 0}}{\left(\frac{\lambda_n}{\lambda_1}\right)^n} b_n x_n \right]$$

Focusing on extremal eigen*values*

$= 1$



Eigen Systems   Pang Chapter 5

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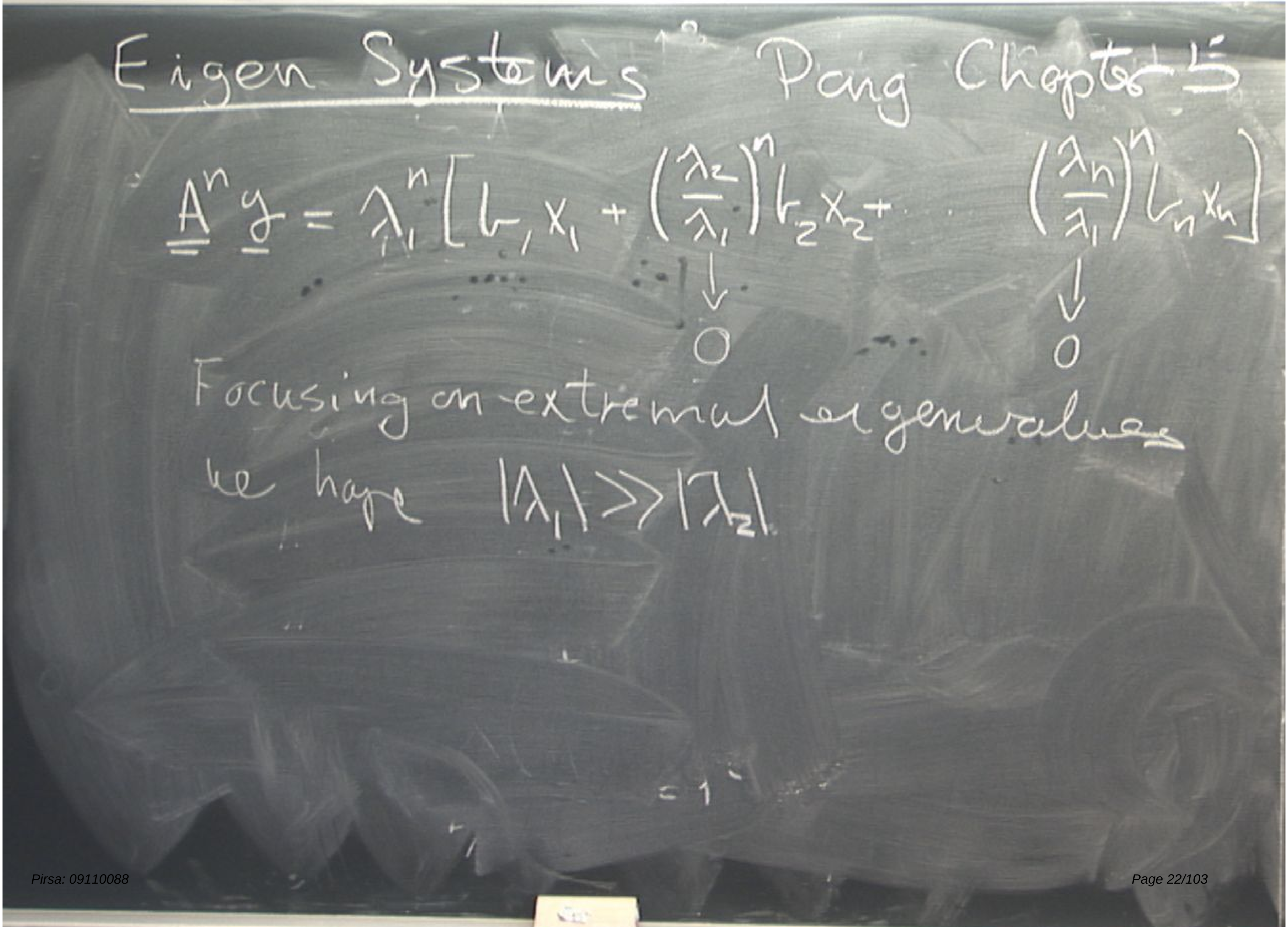
Focusing on extremal eigenvalues

we hope $|\lambda_1| \gg |\lambda_2|$



Eigen Systems   Pang Chapter 5

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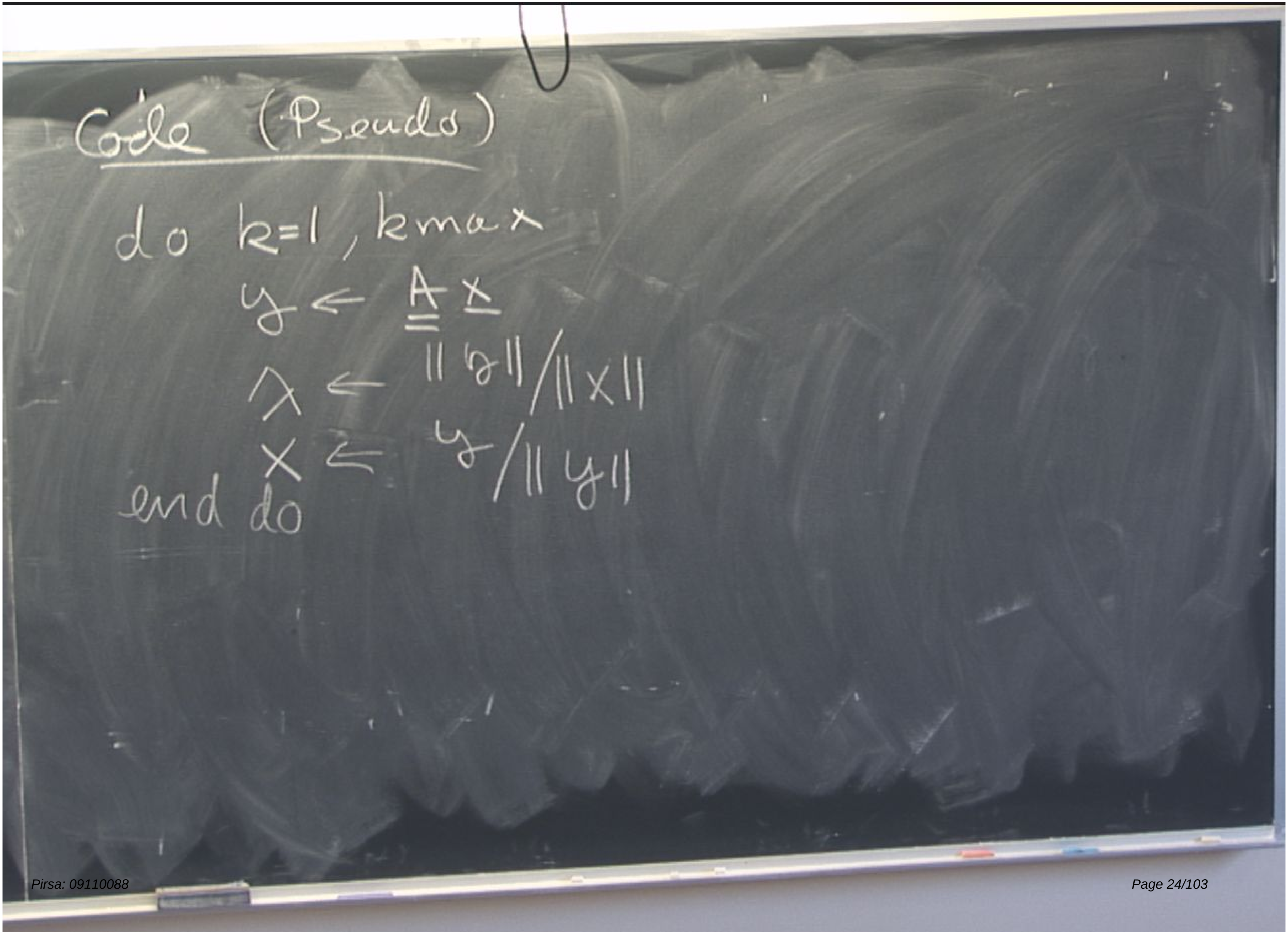
Focusing on extremal eigenvalues

we hope $|\lambda_1| >> |\lambda_2|$

Code (Pseudo)

```
do k=1, kmax
    $y \leftarrow \underline{\underline{A}}\,\underline{x}$
    $\lambda \leftarrow \|y\|/\|x\|$
```



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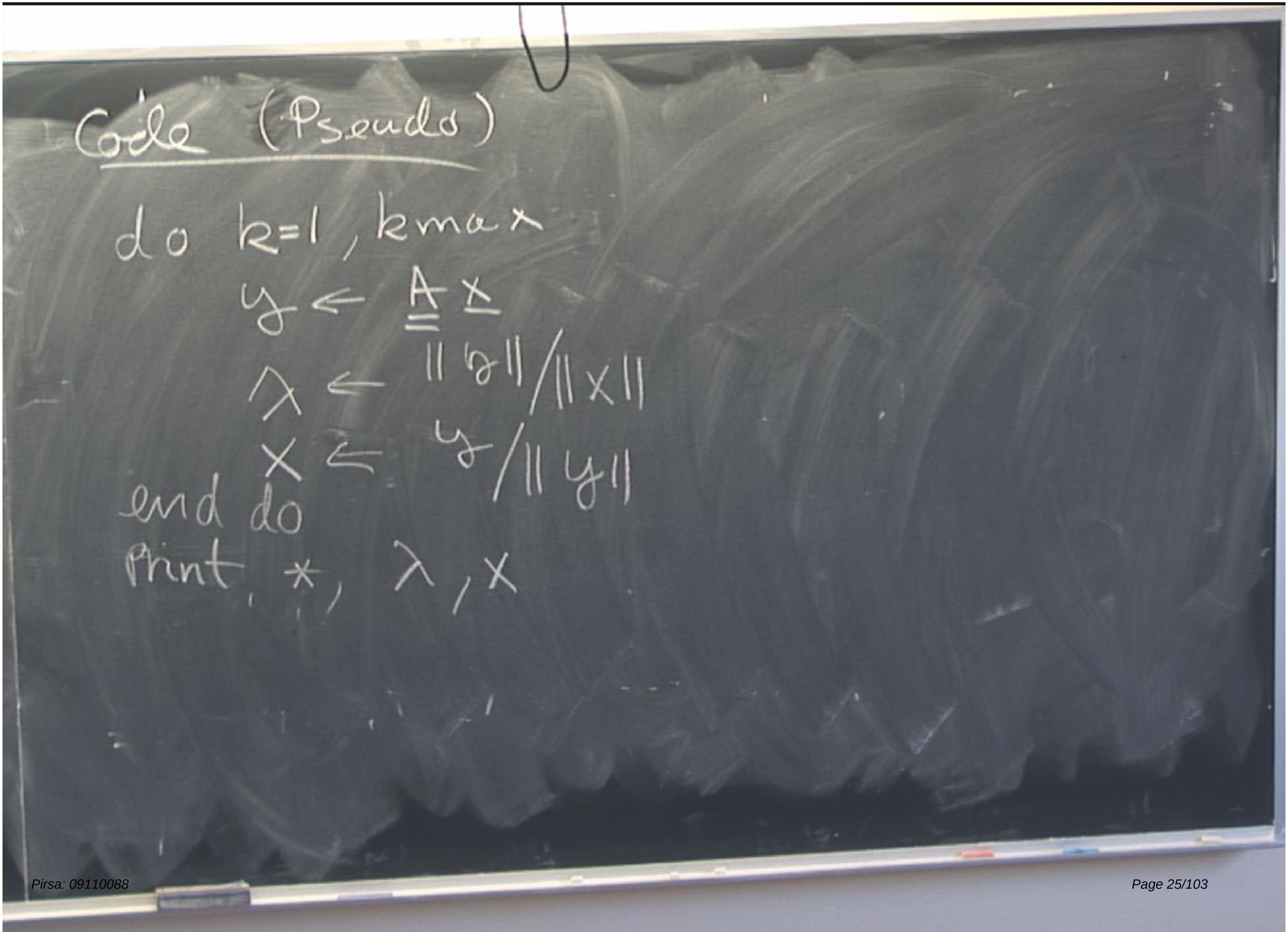

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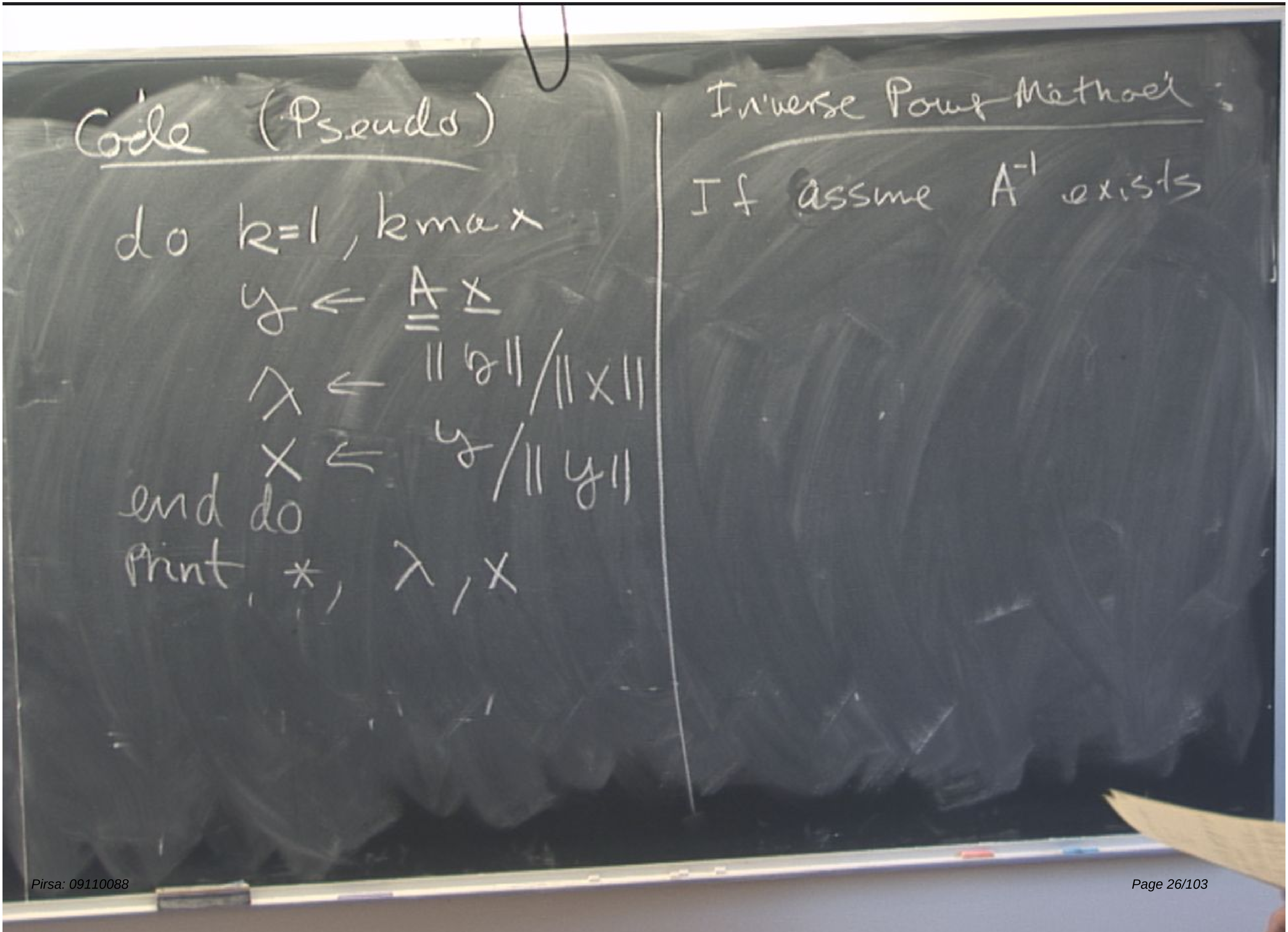
Print, *, $\lambda$, $x$
```



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Inverse Power Method:

If assme A^{-1} exists

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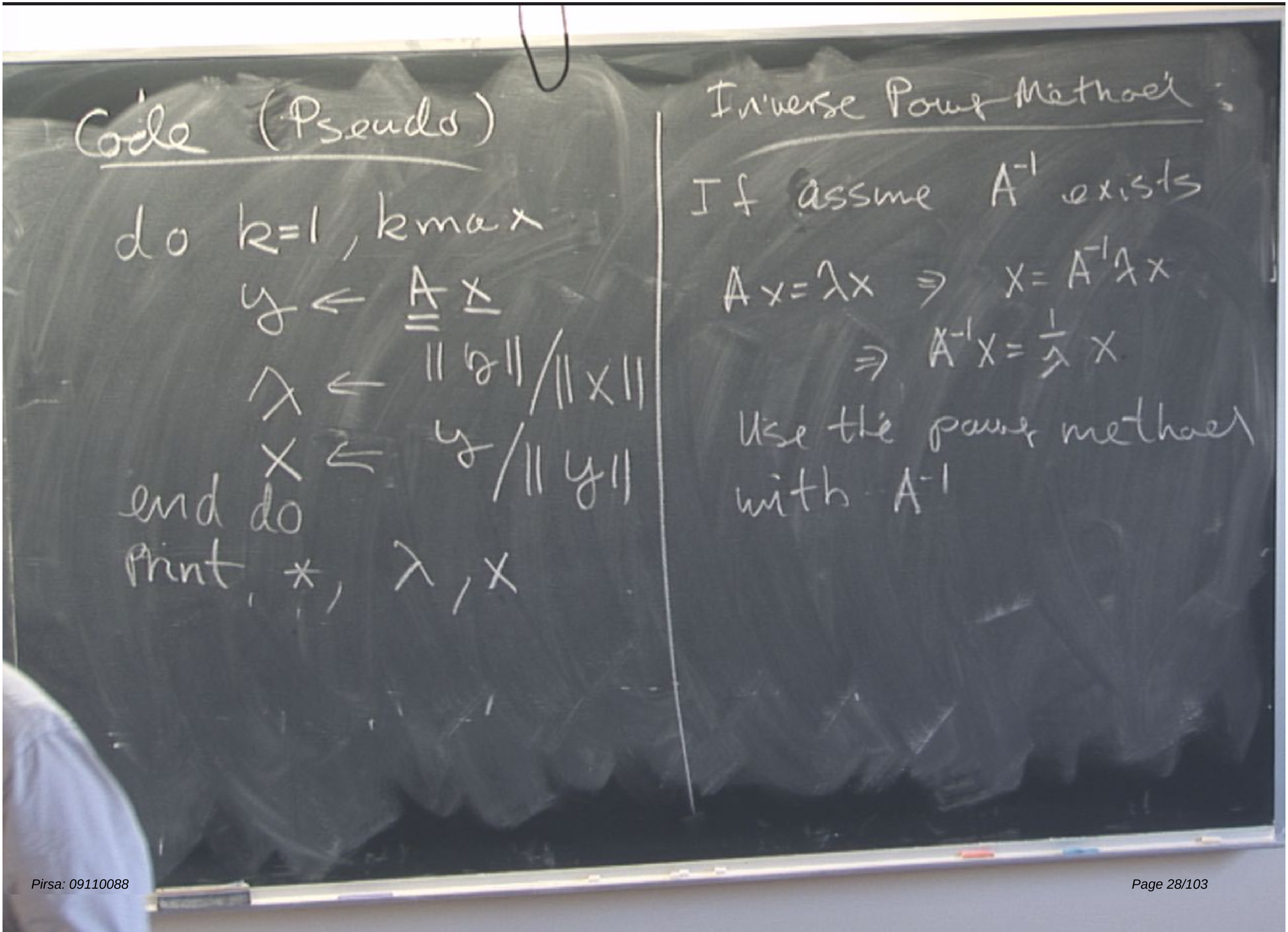
Inverse Power Method:

If assme A^{-1} exists

$$Ax = \lambda x \Rightarrow x = A^{-1}\lambda x$$

$$\Rightarrow A^{-1}x = \frac{1}{\lambda}x$$



Code (Pseudo)

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If assume A^{-1} exists

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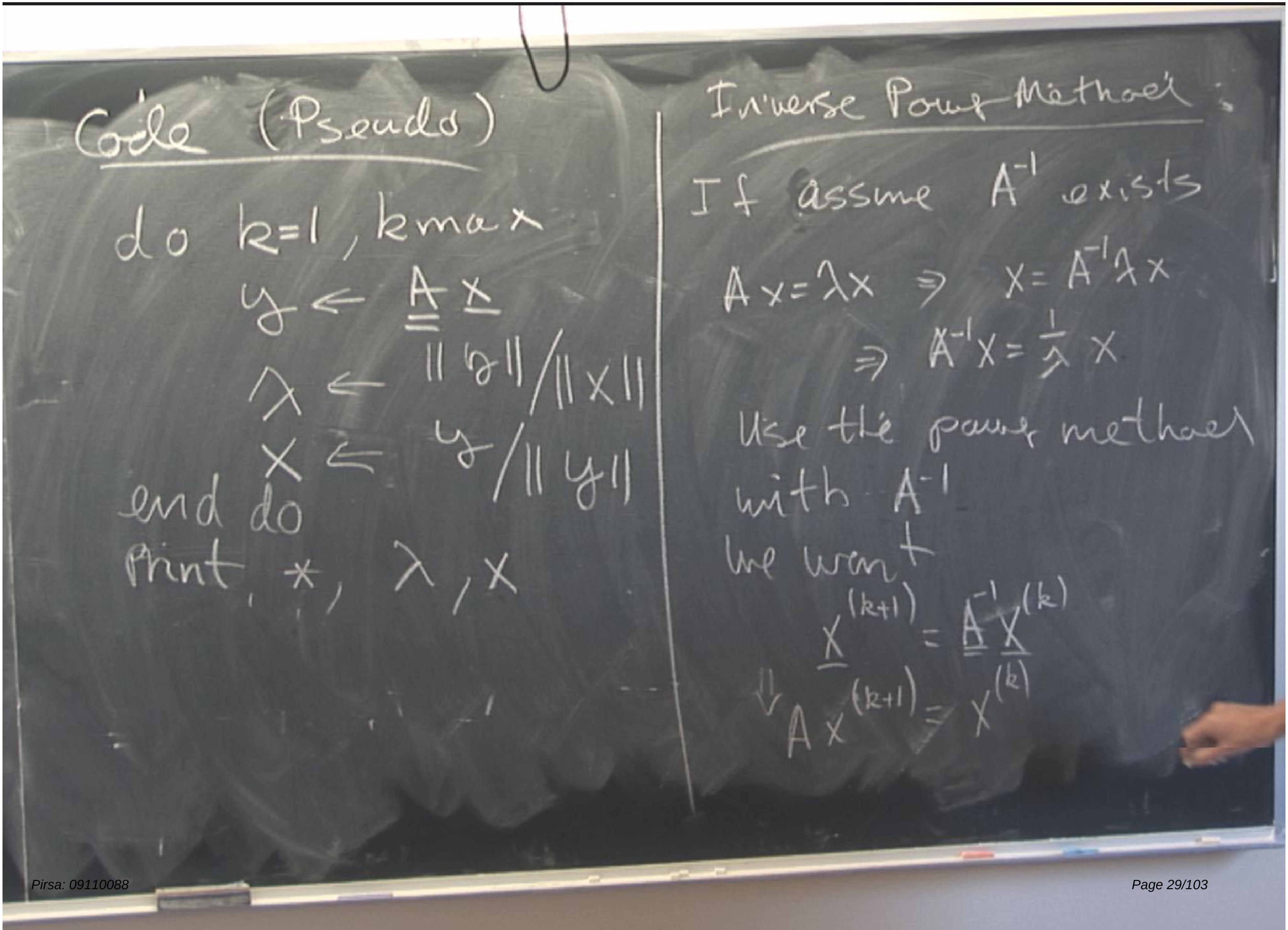
Use the power method with A^{-1}



Code (Pseudo)

```
do k=1, kmax
    y ← $\underline{\underline{A}}\,\underline{x}$
    λ ← $\|y\| / \|x\|$
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Print, *, λ, x
```

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Use the power method with A^{-1}

We want

$$\underline{x}^{(k+1)} = \underline{\underline{A}}^{-1}\underline{x}^{(k)}$$

$$\Downarrow\ A x^{(k+1)} = x^{(k)}$$

Code (Pseudo)

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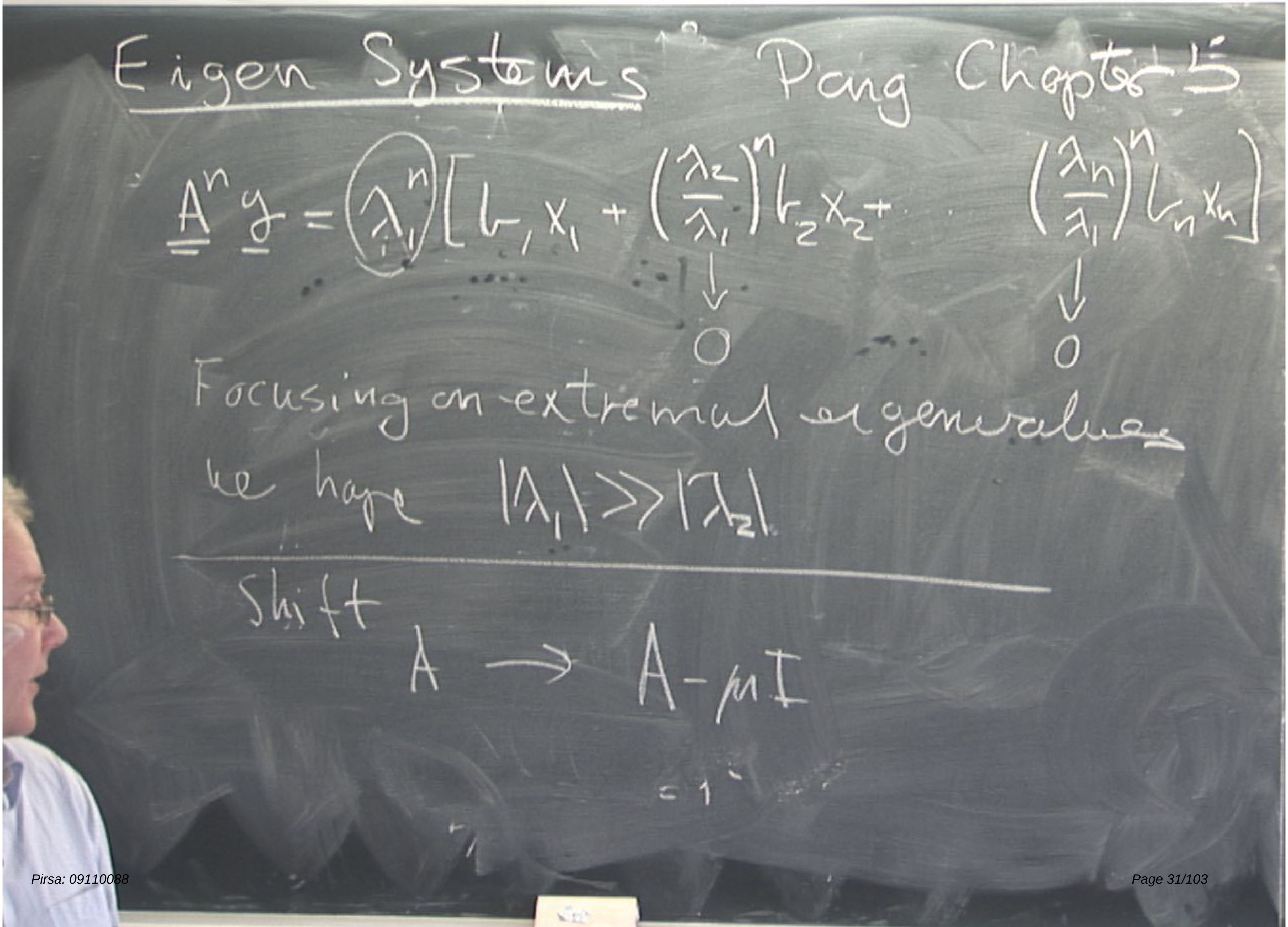
$$\underline{x}^{(k+1)} = \underline{\underline{A}}^{-1}\underline{x}^{(k)}$$

$$\Downarrow\ A x^{(k+1)} = x^{(k)} \leftarrow \text{solve linear algebra}$$



Eigen Systems   Pang Chapter 5

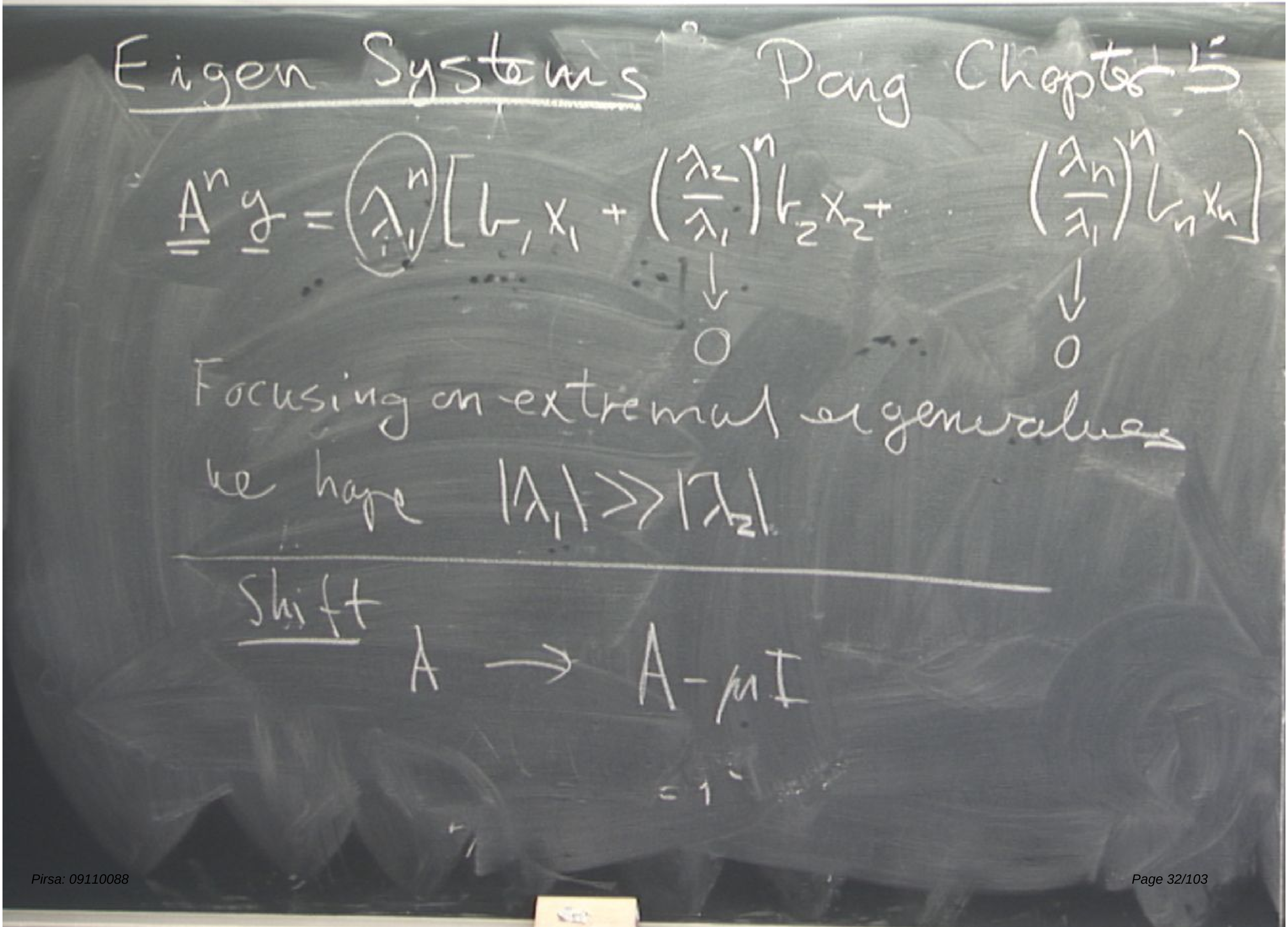
$$\underline{\underline{A}}^n \underline{y} = \boxed{\lambda_1^n}\left[b_1 x_1 + \underset{\substack{\downarrow \\ 0}}{\left(\frac{\lambda_2}{\lambda_1}\right)^n} b_2 x_2 + \cdots \underset{\substack{\downarrow \\ 0}}{\left(\frac{\lambda_n}{\lambda_1}\right)^n} b_n x_n\right]$$

Focusing on extremal eigenvalues

We have $|\lambda_1| \gg |\lambda_2|$

Shift

$$A \longrightarrow A - \mu I$$



Eigen Systems   Pang Chapter 5

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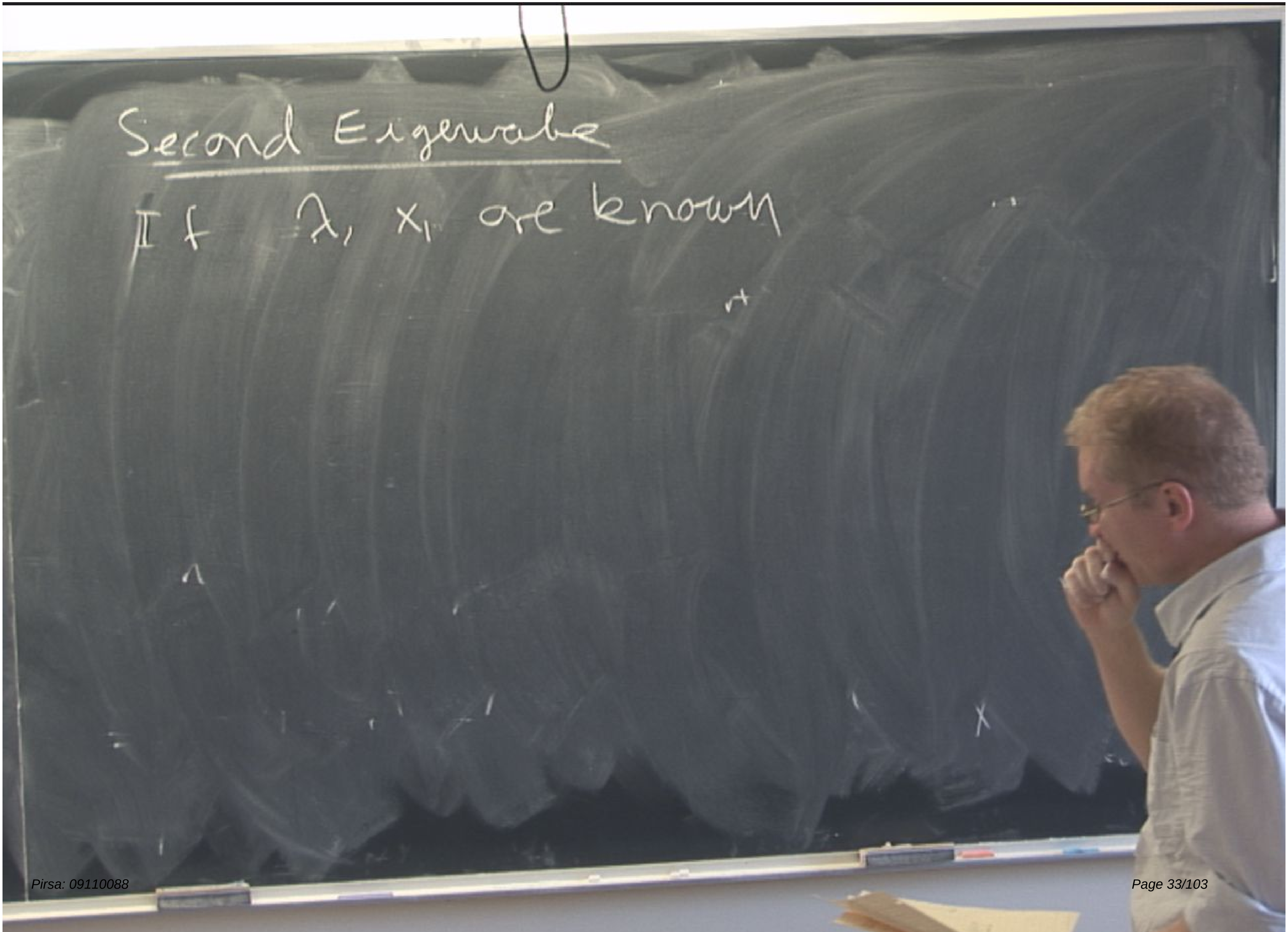
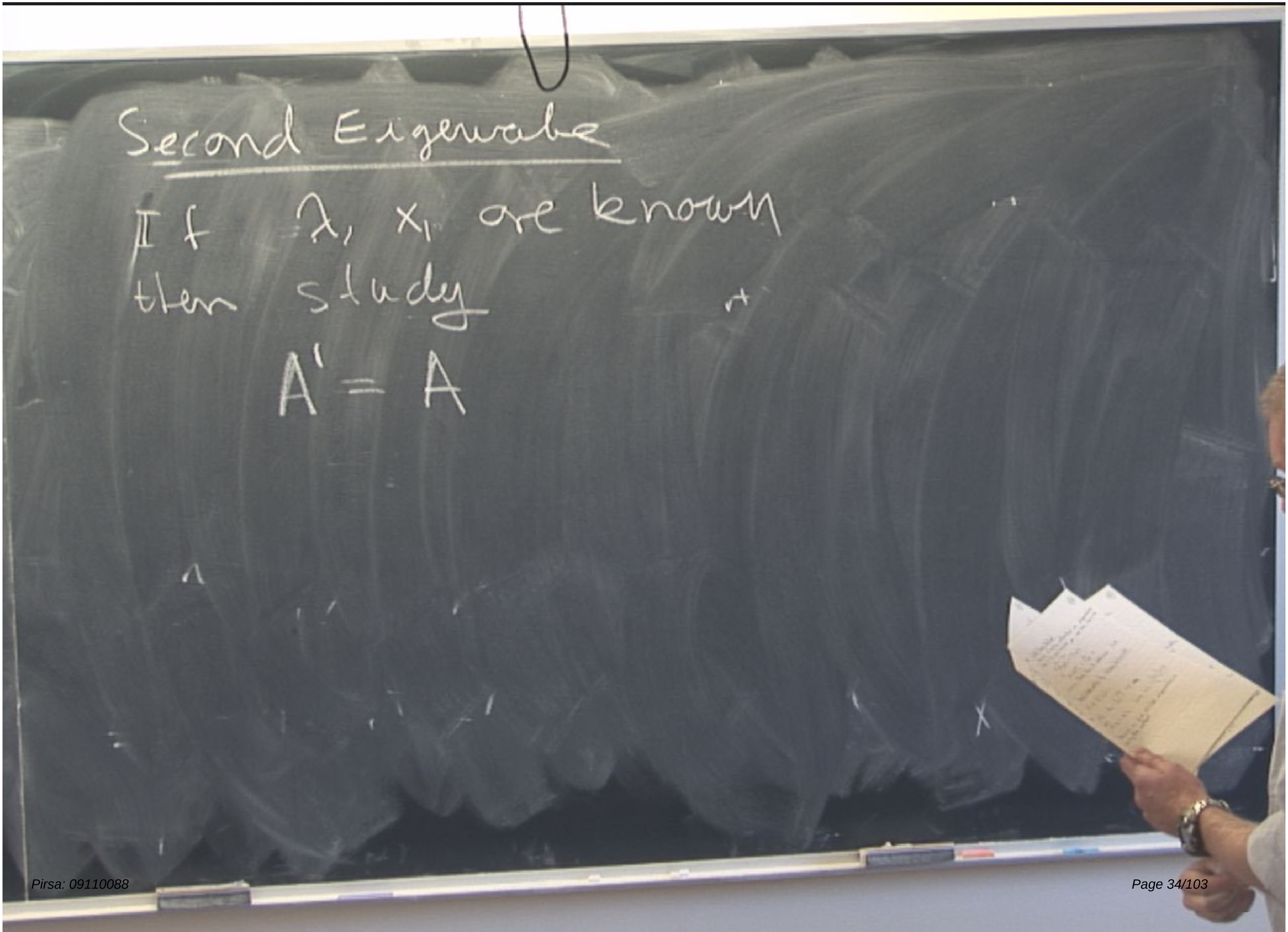
we hope $|\lambda_1| \gg |\lambda_2|$

Shift $A \longrightarrow A - \mu I$



Second Eigenvalue

If λ_1, x_1 are known

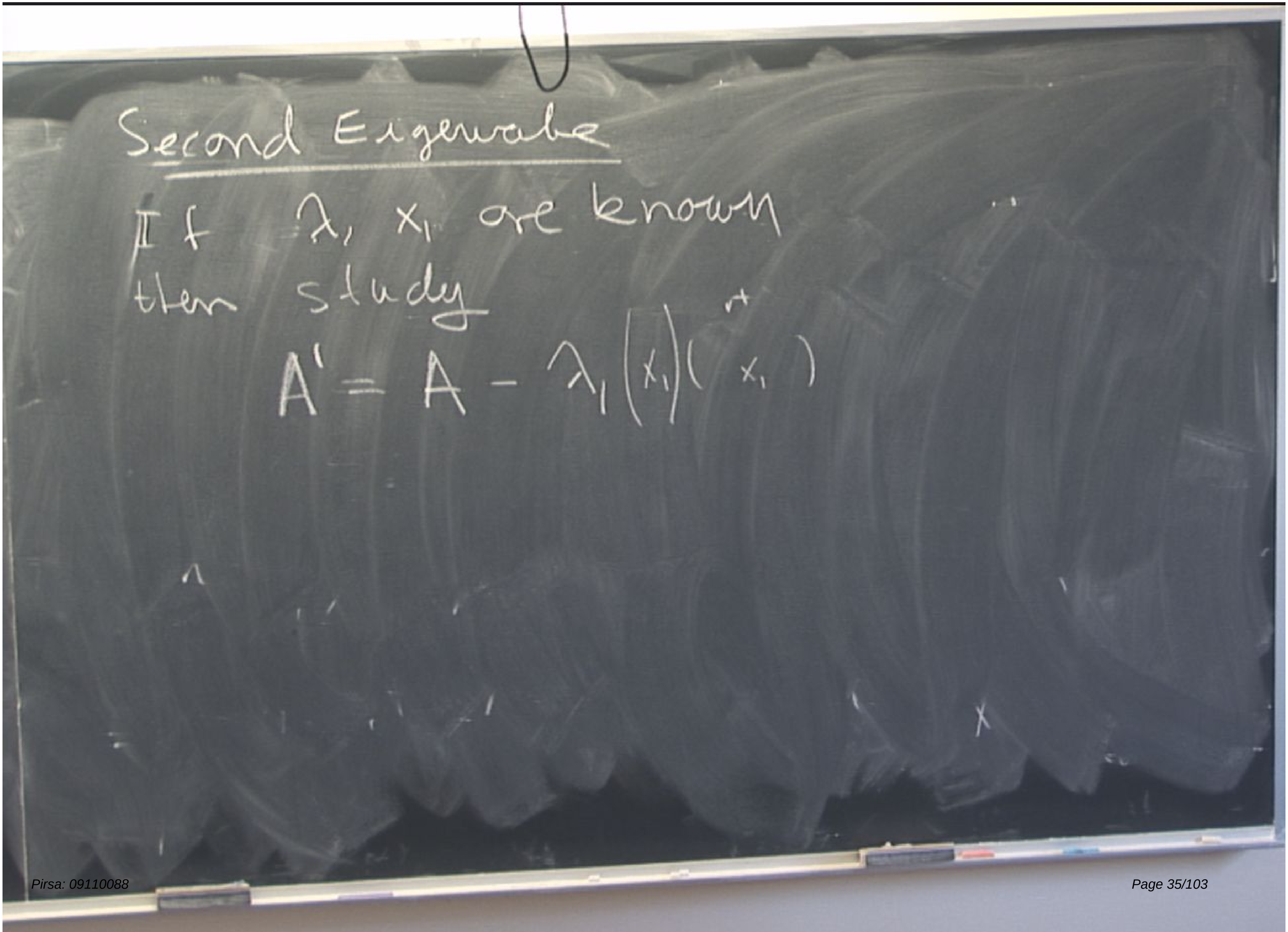


Second Eigenvalue

If λ_1, x_1 are known

then study

$$A' = A$$



Second Eigenvalue

If λ_1, x_1 are known

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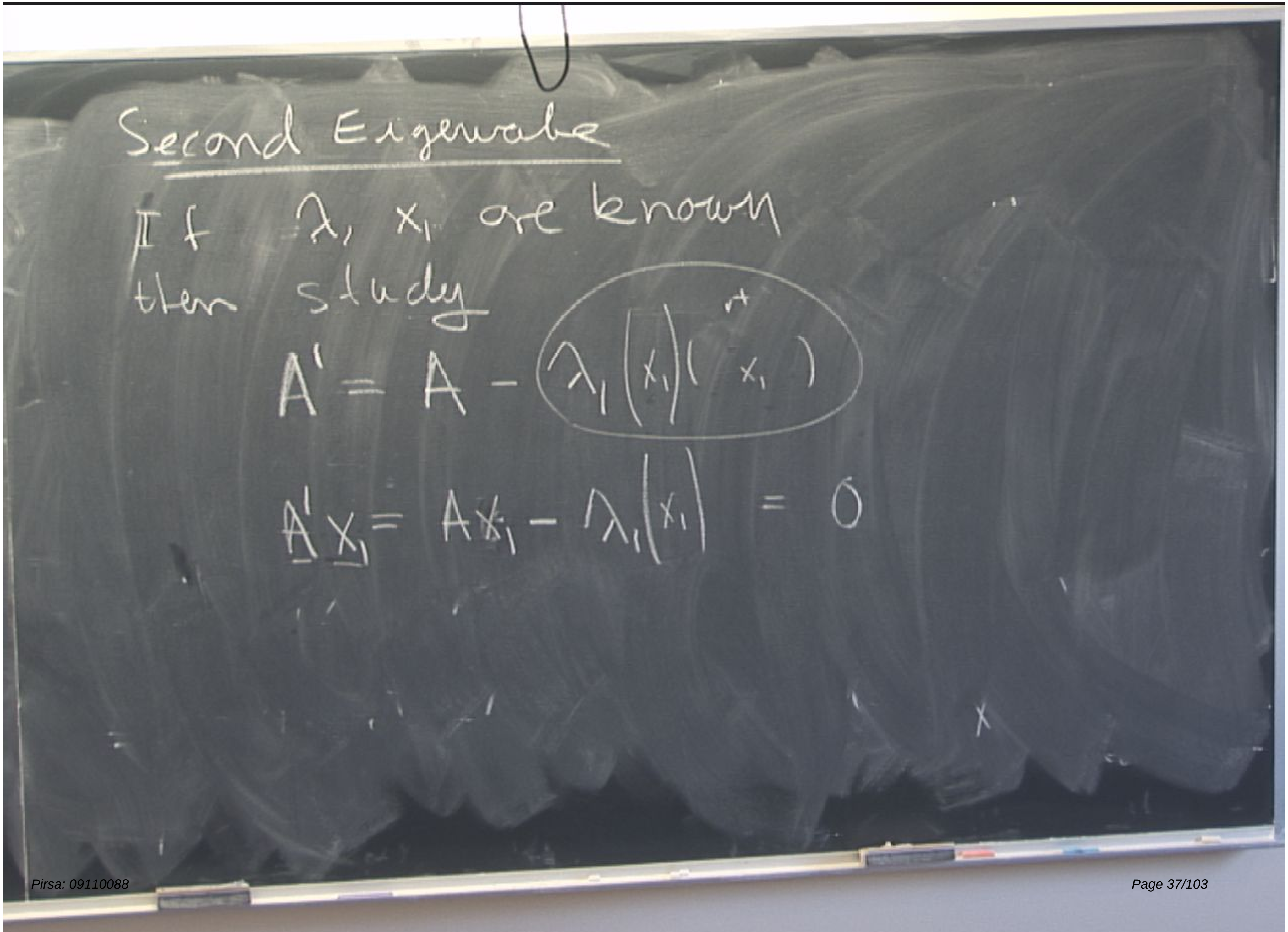
$$A' = A - \lambda_1 \left(x_1\right)\left(x_1^{\dagger}\right)$$

Second Eigenvalue

If λ_1, x_1 are known
then study

$$A' = A - \lambda_1 \left(x_1 \right) \left(\; x_1^{\dagger} \; \right)$$

$$A' x_1 = A \cancel{x}_1 - \lambda_1 \left(x_1 \right) \; = \; 0$$



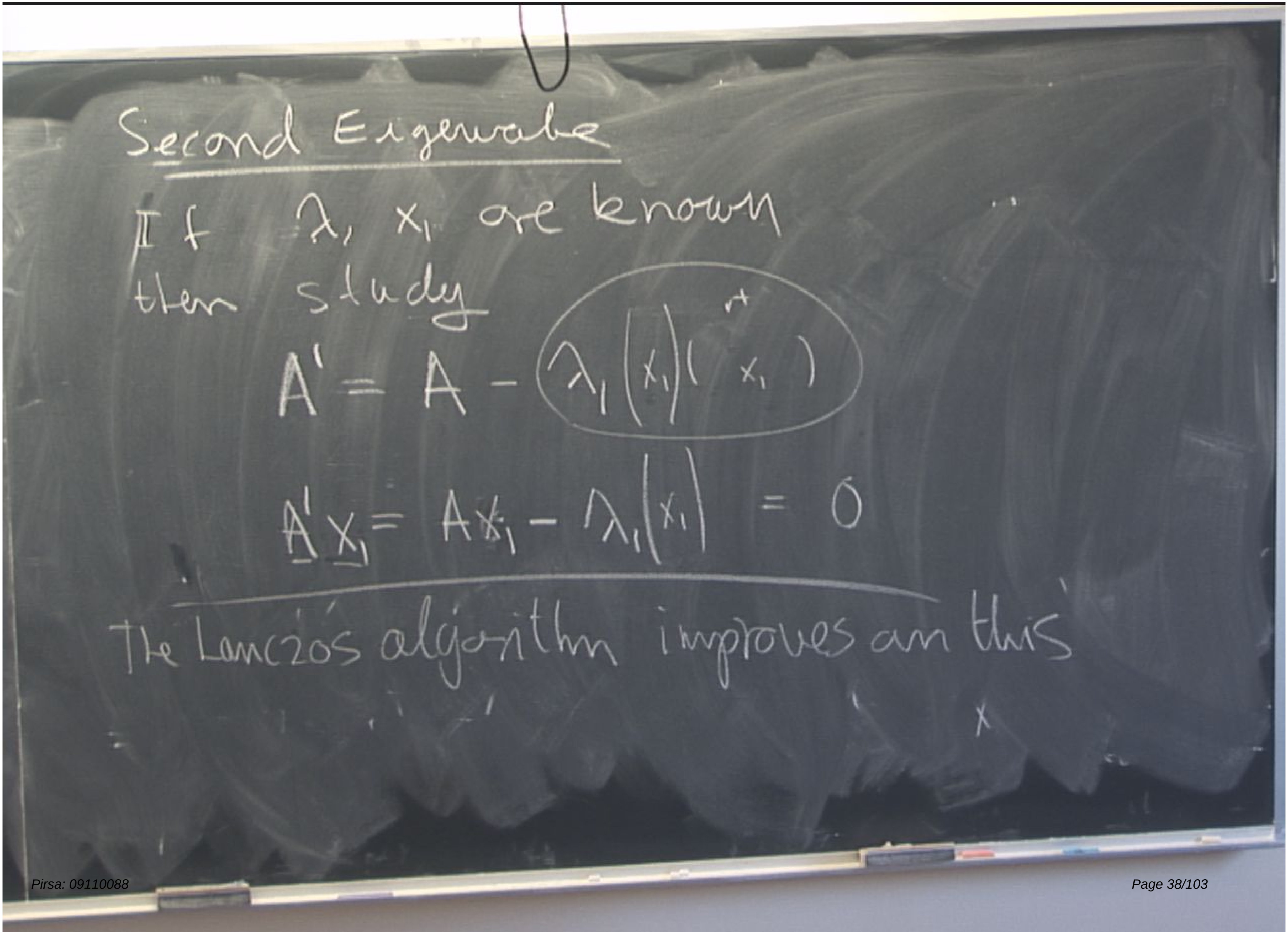
Second Eigenvalue

If λ_1, x_1 are known

then study

$$A' = A - \boxed{\lambda_1 \left(x_1\right)\left(x_1^T\right)}$$

$$\underline{A}'\underline{x_1} = Ax_1 - \lambda_1\left(x_1\right) = 0$$



Second Eigenvalue

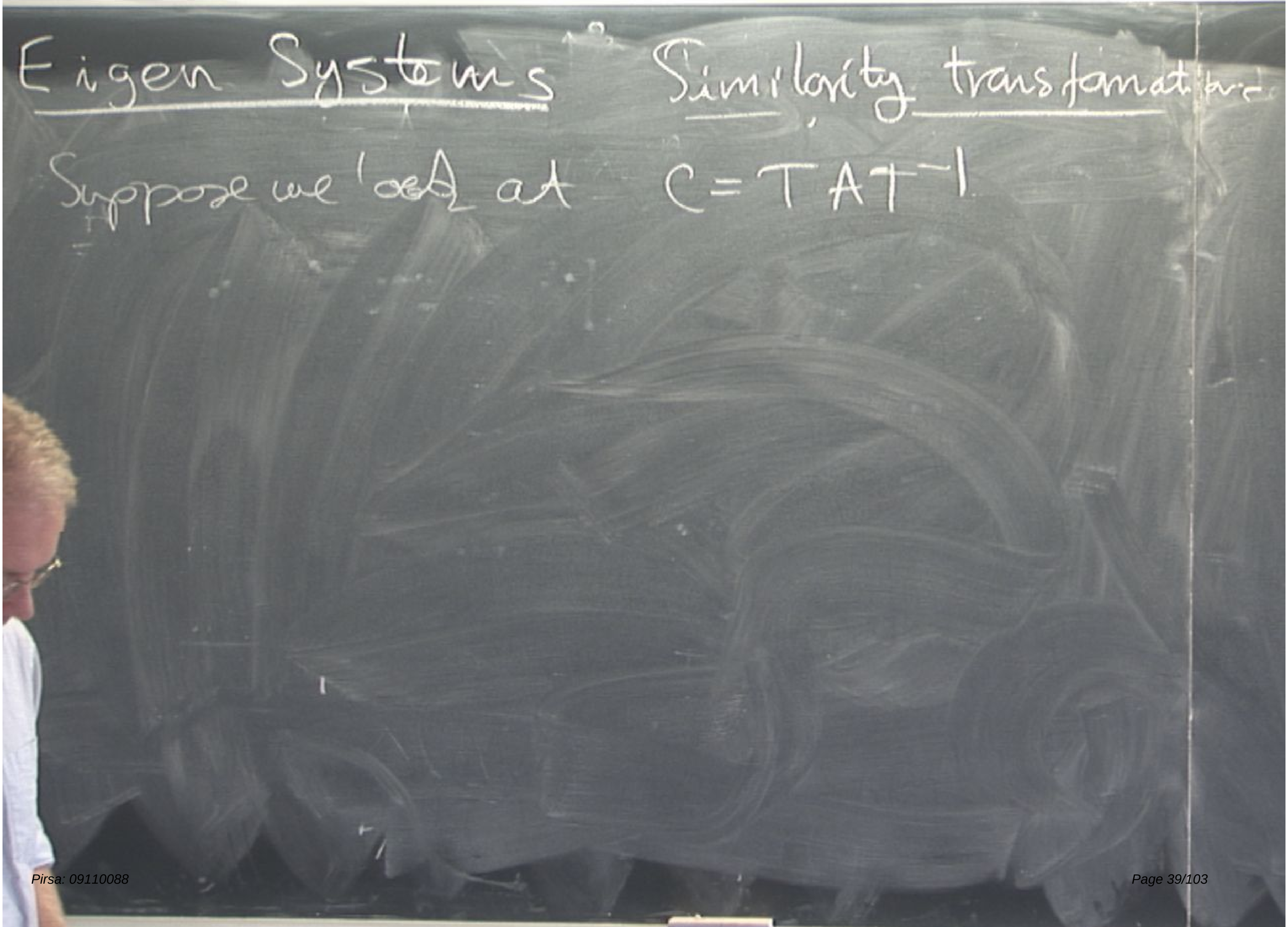
If λ_1, x_1 are known

then study

$$A' = A - \boxed{\lambda_1 (x_1)(x_1^*)}$$

$$\underline{A}' \underline{x}_1 = A x_1 - \lambda_1 (x_1) = 0$$

The Lanczos algorithm improves on this



Eigen Systems

Similarity transformations

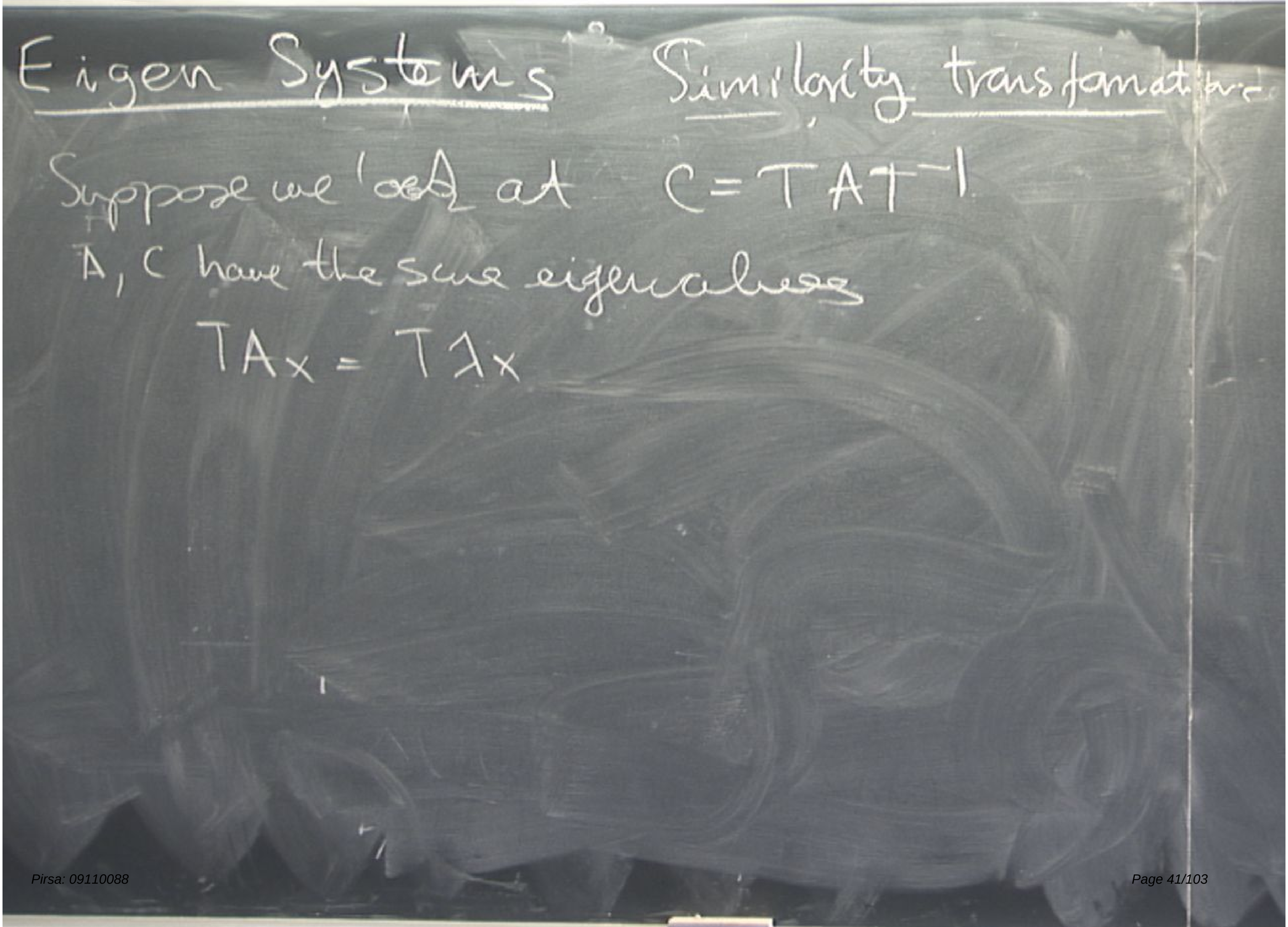
Suppose we look at

$C = TAT^{-1}$

Eigen Systems: Similarity transformations

Suppose we look at $C = TAT^{-1}$

A, C have the same eigenvalues

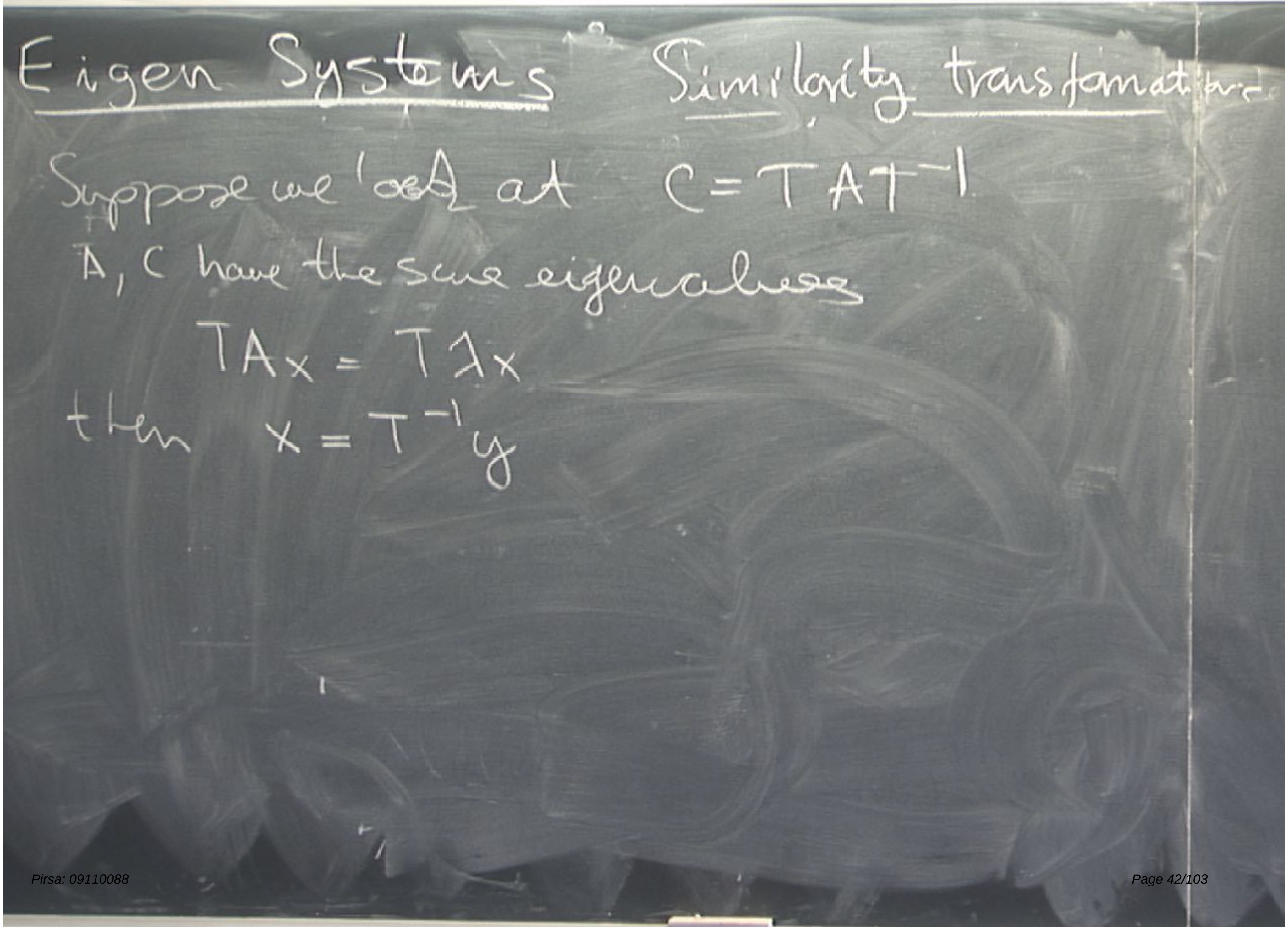


Eigen Systems Similarity transformations

Suppose we look at $C = TAT^{-1}$

A, C have the same eigenvalues

$$TAx = T\lambda x$$



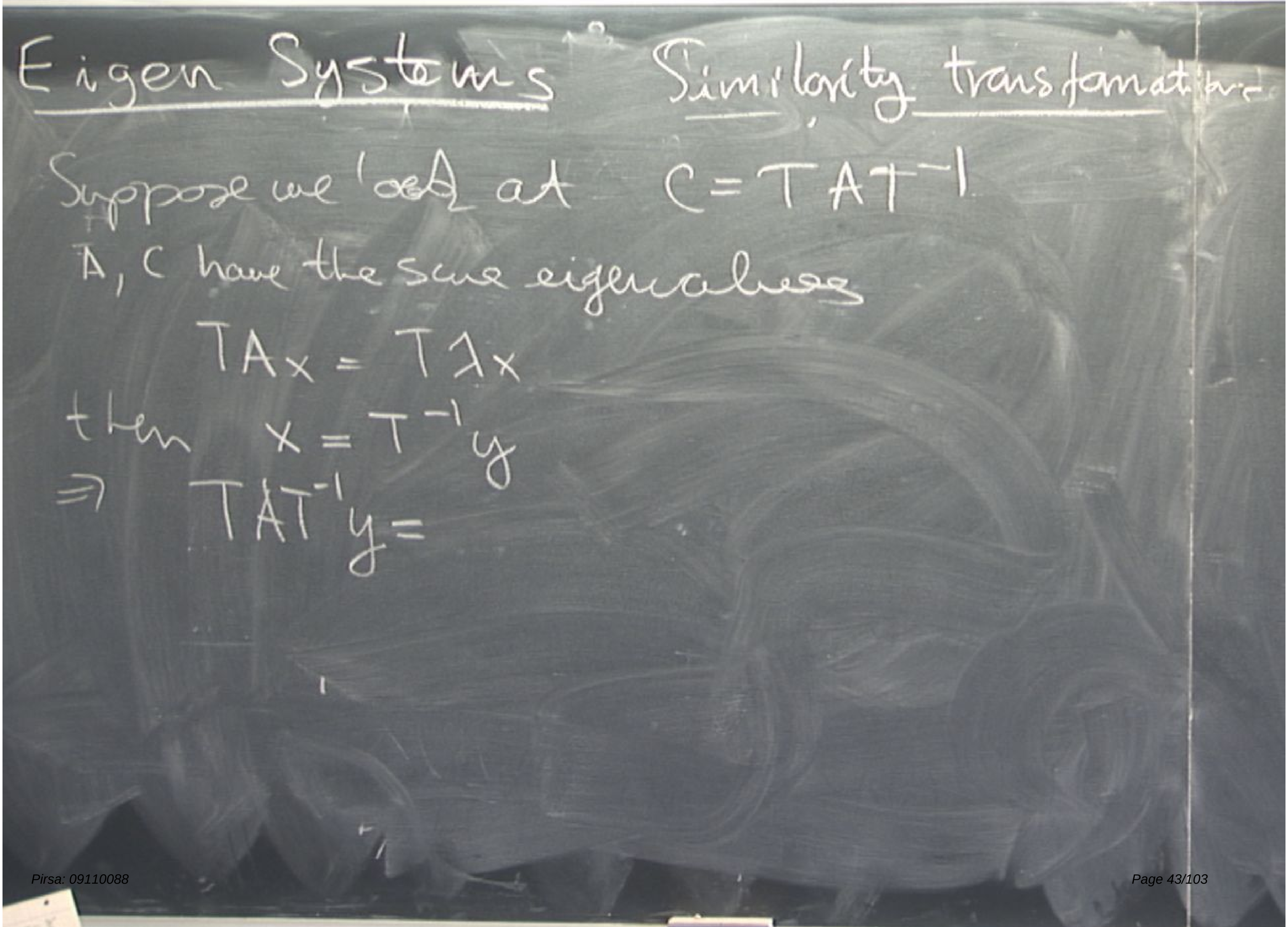
Eigen Systems ; Similarity transformations

Suppose we look at $C = TAT^{-1}$

A, C have the same eigenvalues

$$TAx = T\lambda x$$

then $x = T^{-1}y$



Eigen Systems   Similarity transformation

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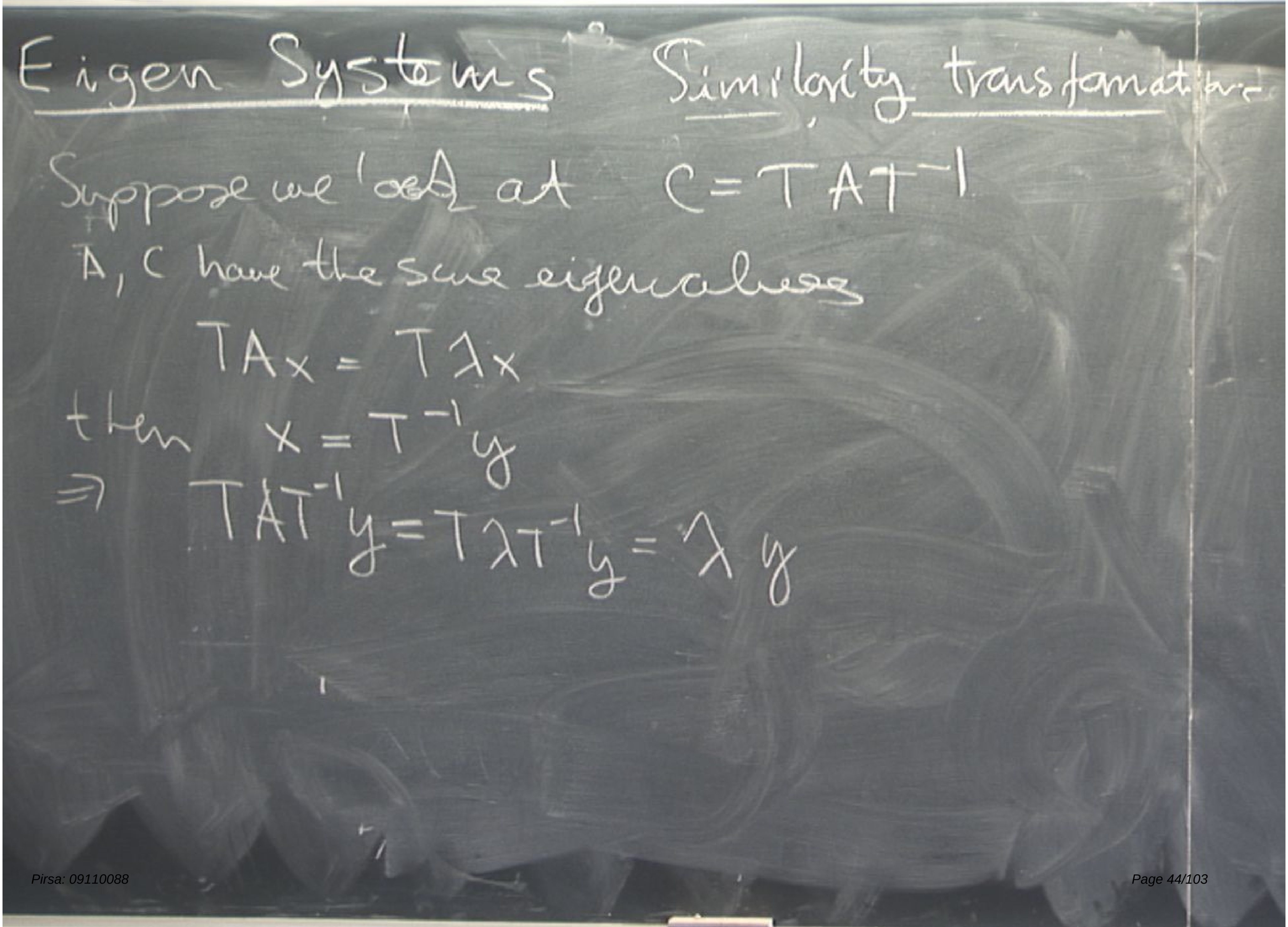
then $x = T^{-1}y$

$$\Rightarrow \quad TAT^{-1}y =$$



Eigen Systems

Similarity transformation

Suppose we look at $C = TAT^{-1}$

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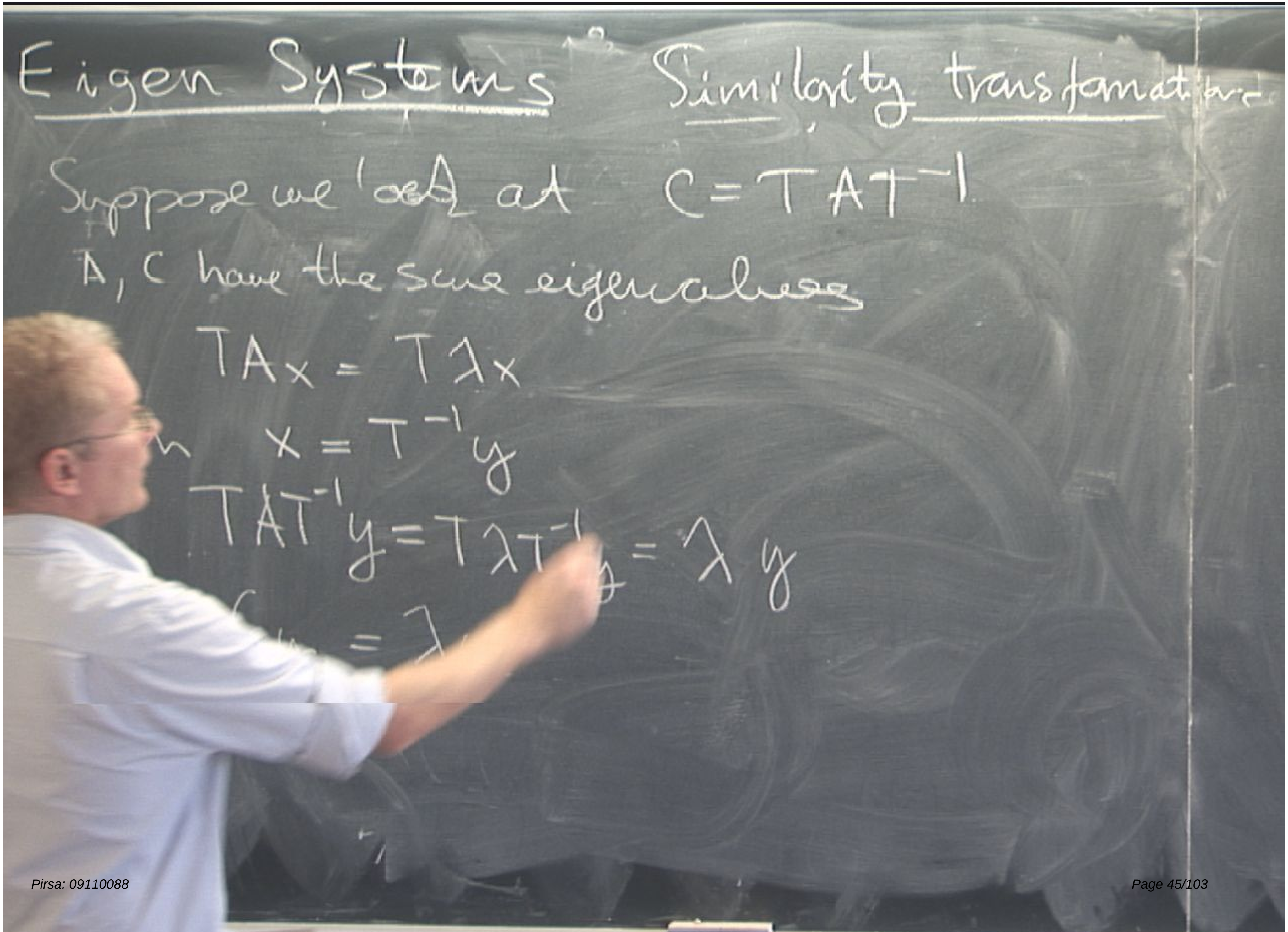
then $x = T^{-1}y$

$$\Rightarrow \quad TAT^{-1}y = T\lambda T^{-1}y = \lambda y$$



Eigen Systems

Similarity transformations

Suppose we look at $C = TAT^{-1}$

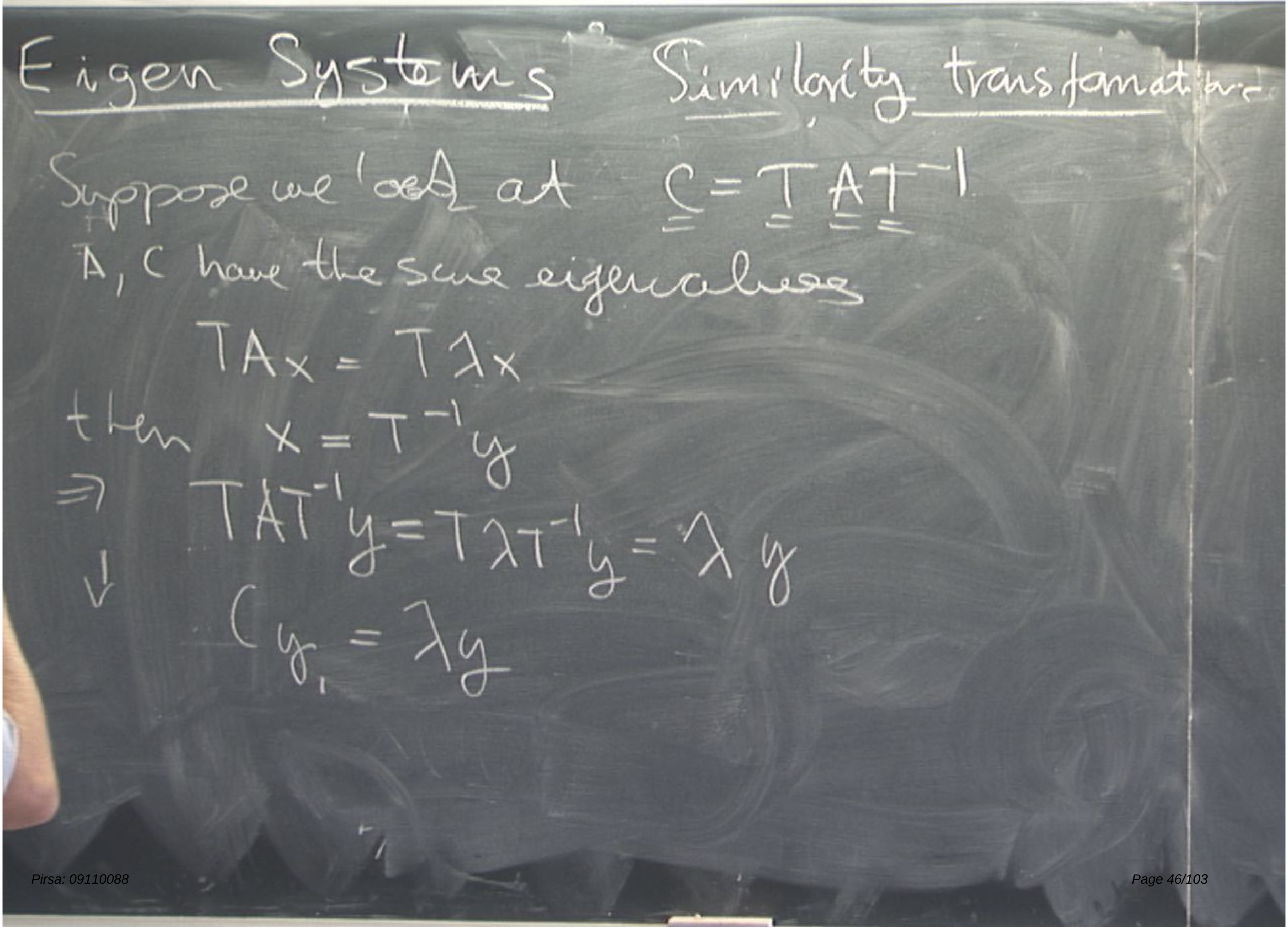
A, C have the same eigenvalues

$$TAx = T\lambda x$$

n $x = T^{-1}y$

$$TAT^{-1}y = T\lambda T^{-1}y = \lambda y$$

C [illegible] $= \lambda$



Eigen Systems: Similarity transformations

Suppose we look at $\underline{C} = \underline{T}\,\underline{A}\,\underline{T}^{-1}$

A, C have the same eigenvalues

$$TAx = T\lambda x$$

then $x = T^{-1}y$

$$\Rightarrow \quad TAT^{-1}y = T\lambda T^{-1}y = \lambda y$$

$$\Downarrow \quad Cy_1 = \lambda y$$

Eigen Systems   Similarity transformations

Suppose we look at $\underline{\underline{C}} = \underline{\underline{T}}\,\underline{\underline{A}}\,\underline{\underline{T}}^{-1}$

A, C have the same eigenvalues   T invertible

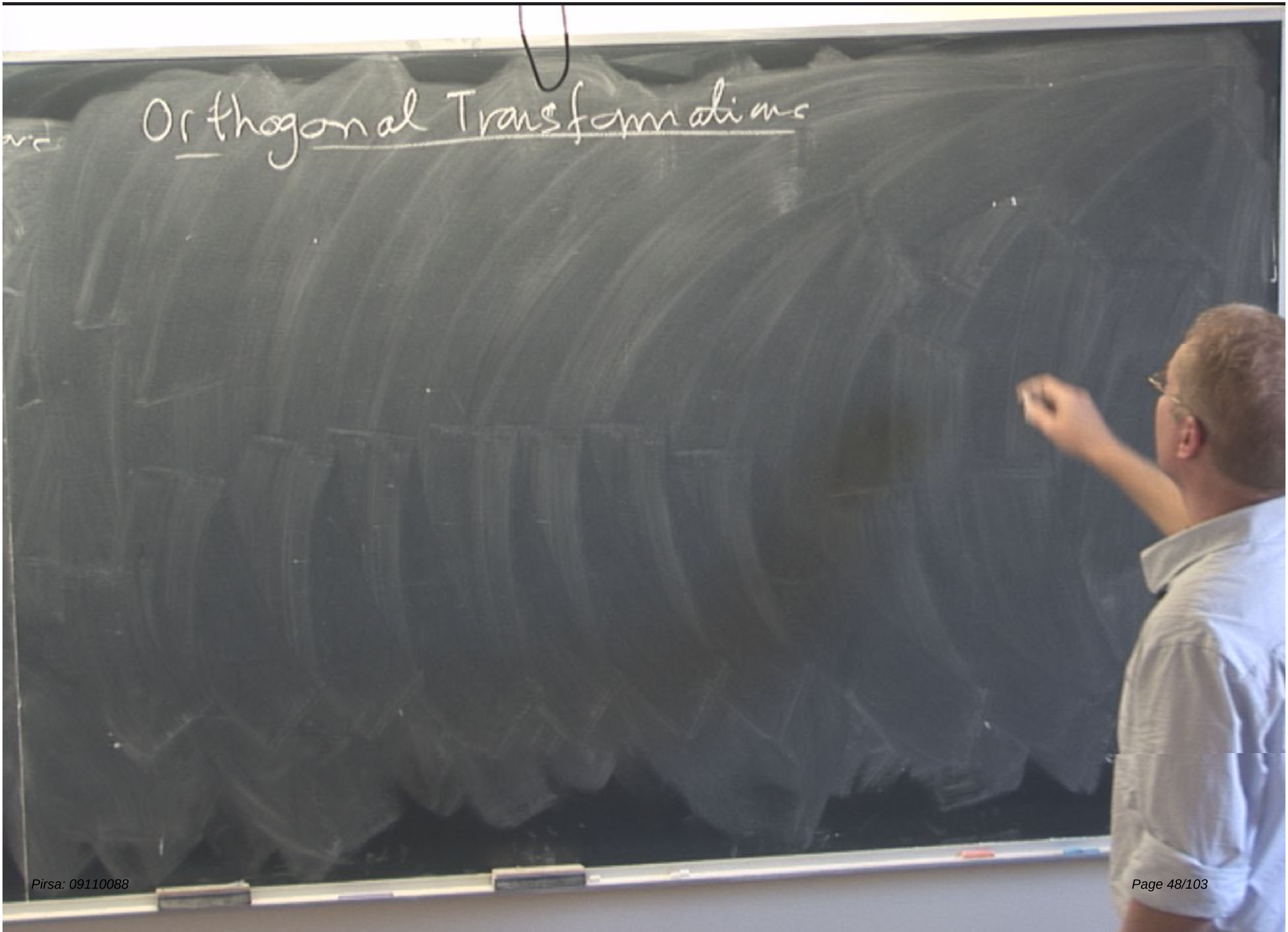
$$TAx = T\lambda x$$

then $x = T^{-1}y$

$$\Rightarrow \quad TAT^{-1}y = T\lambda T^{-1}y = \lambda y$$

$$\downarrow$$

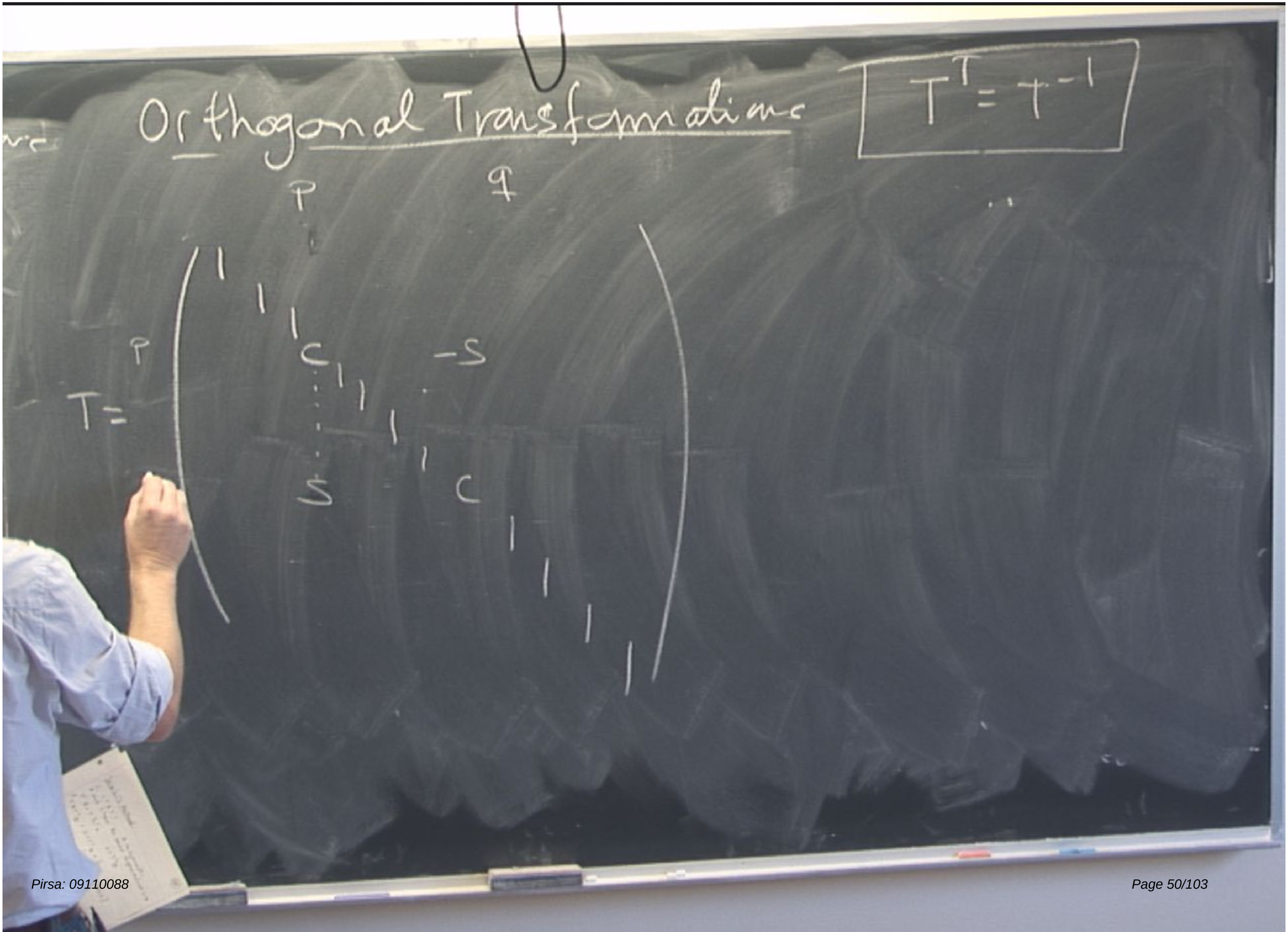
$$Cy_1 = \lambda y$$



Orthogonal Transformations

Orthogonal Transformations

$T^T = T^{-1}$



Orthogonal Transformations

$$\boxed{T^T = T^{-1}}$$

$$T = \left(\begin{array}{c|ccccccccccccc}
 & & & & p & & & & & q & & & & \\
\hline
 & 1 & & & & & & & & & & & & \\
 & & 1 & & & & & & & & & & & \\
 & & & 1 & & & & & & & & & & \\
p & & & & c & & & & & -s & & & & \\
 & & & & \vdots & 1 & & & & & & & & \\
 & & & & \vdots & & 1 & & & & & & & \\
 & & & & \vdots & & & 1 & & & & & & \\
 & & & & \vdots & & & & 1 & & & & & \\
 & & & & s & & & & & c & & & & \\
 & & & & & & & & & & 1 & & & \\
 & & & & & & & & & & & 1 & & \\
 & & & & & & & & & & & & 1 & \\
 & & & & & & & & & & & & & 1
\end{array}\right)$$

Orthogonal Transformations

$$\boxed{T^T = T^{-1}}$$

$$T = \left(\begin{array}{c|cccccccccccc}
 & & & & p & & & & q & & & & \\
 & 1 & & & & & & & & & & & \\
 & & 1 & & & & & & & & & & \\
 & & & 1 & & & & & & & & & \\
p & & & & c & & & & -s & & & & \\
 & & & & \vdots & 1 & & & \vdots & & & & \\
 & & & & \vdots & & 1 & & \vdots & & & & \\
 & & & & \vdots & & & 1 & \vdots & & & & \\
q & & & & s & & & & c & & & & \\
 & & & & & & & & & 1 & & & \\
 & & & & & & & & & & 1 & & \\
 & & & & & & & & & & & 1 & \\
 & & & & & & & & & & & & 1
\end{array}\right)$$

$$\overset{\sin(\theta)}{c^2 + s^2 = 1}$$

$$\hookrightarrow \cos(\theta)$$

Orthogonal Transformations

$$\boxed{T^T = T^{-1}}$$

$$T = \begin{array}{c|ccccccccccc}
 & & & & p & & & & q & & & \\
 & 1 & & & & & & & & & & \\
 & & 1 & & & & & & & & & \\
 & & & 1 & & & & & & & & \\
p & & & & c & & & & -s & & & \\
 & & & & \vdots & 1 & & & \vdots & & & \\
 & & & & \vdots & & 1 & & \vdots & & & \\
 & & & & \vdots & & & 1 & \vdots & & & \\
q & & & & s & \cdots & & \cdots & c & & & \\
 & & & & & & & & & 1 & & \\
 & & & & & & & & & & 1 & \\
 & & & & & & & & & & & 1
\end{array}$$

$$c^2 + s^2 = 1$$

$s \to \sin(\theta)$

$c \to \cos(\theta)$

Determine c, s

Eigen Systems

Similarity transformat

$$\begin{pmatrix} 1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & c & \cdots & & & & -s & \\ & & \vdots & 1 & & & & & \\ & & & & 1 & & & & \\ & & & & & 1 & & & \\ & & & & & & 1 & & \\ & & s & & & & & c & \\ & & & & & & & & 1 \end{pmatrix} A = \begin{pmatrix} & & & & & & & & \end{pmatrix} \swarrow A'$$

Eigen Systems

Similarity transformat

$$\begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & c & \cdots & -s & \\ & & \vdots & 1 & & \\ & & \vdots & & 1 & \\ & & s & & c & \\ & & & & & 1 \end{pmatrix} A = \begin{array}{c} p \\ \\ q \end{array} \begin{pmatrix} \cdots\cdots\cdots\cdots \\ \\ \cdots\cdots\cdots\cdots \end{pmatrix} \leftarrow A'$$

$$A' \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}$$

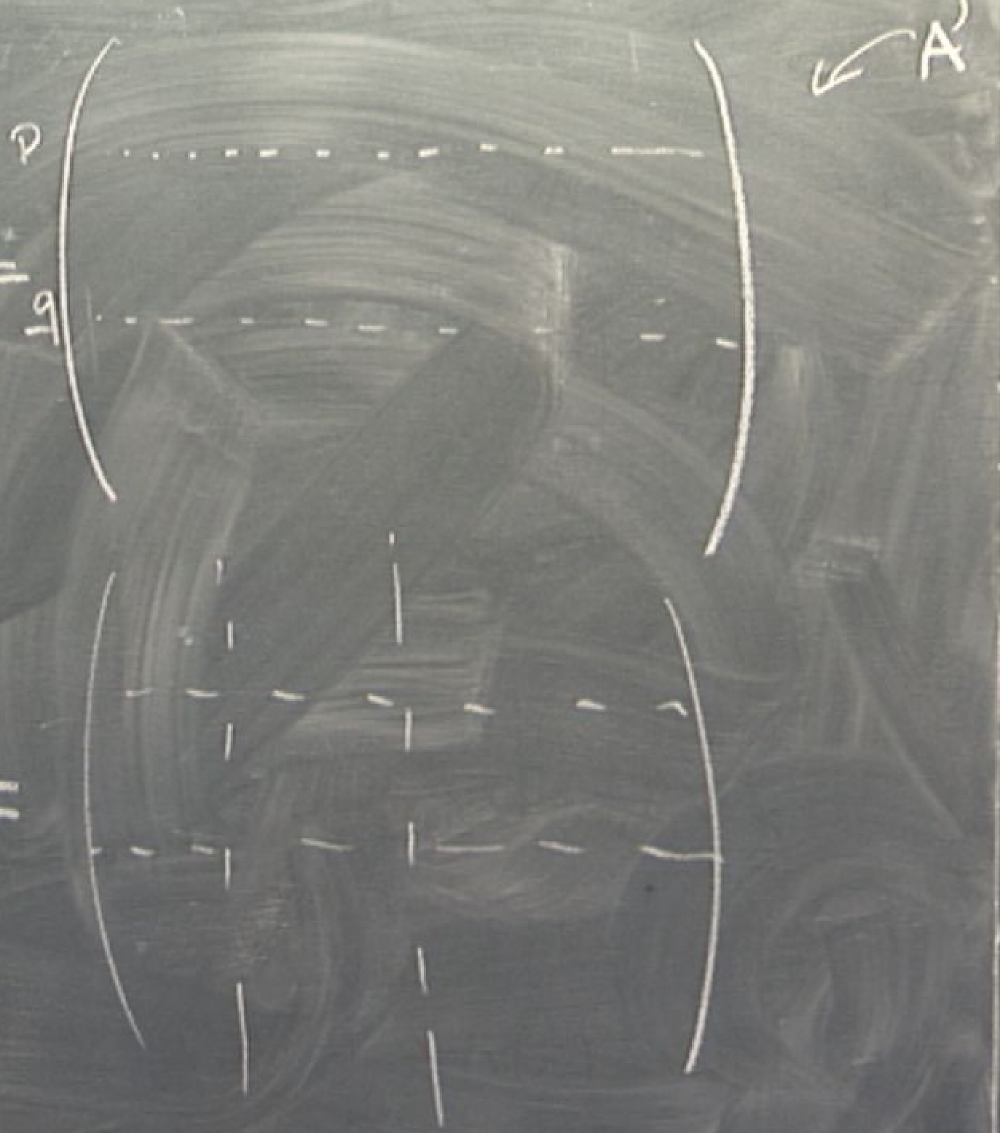
Eigen Systems

Similarity transformat

$$\begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & c & \cdots & -s & \\ & & \vdots & 1 & & \\ & & \vdots & & 1 & \\ & & s & & c & \\ & & & & & 1 \end{pmatrix} A =$$

$$\Downarrow$$

$$A' \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & c & & s & \\ & & & 1 & & \\ & & -s & & c & \\ & & & & & 1 \end{pmatrix} =$$



Eigen Systems

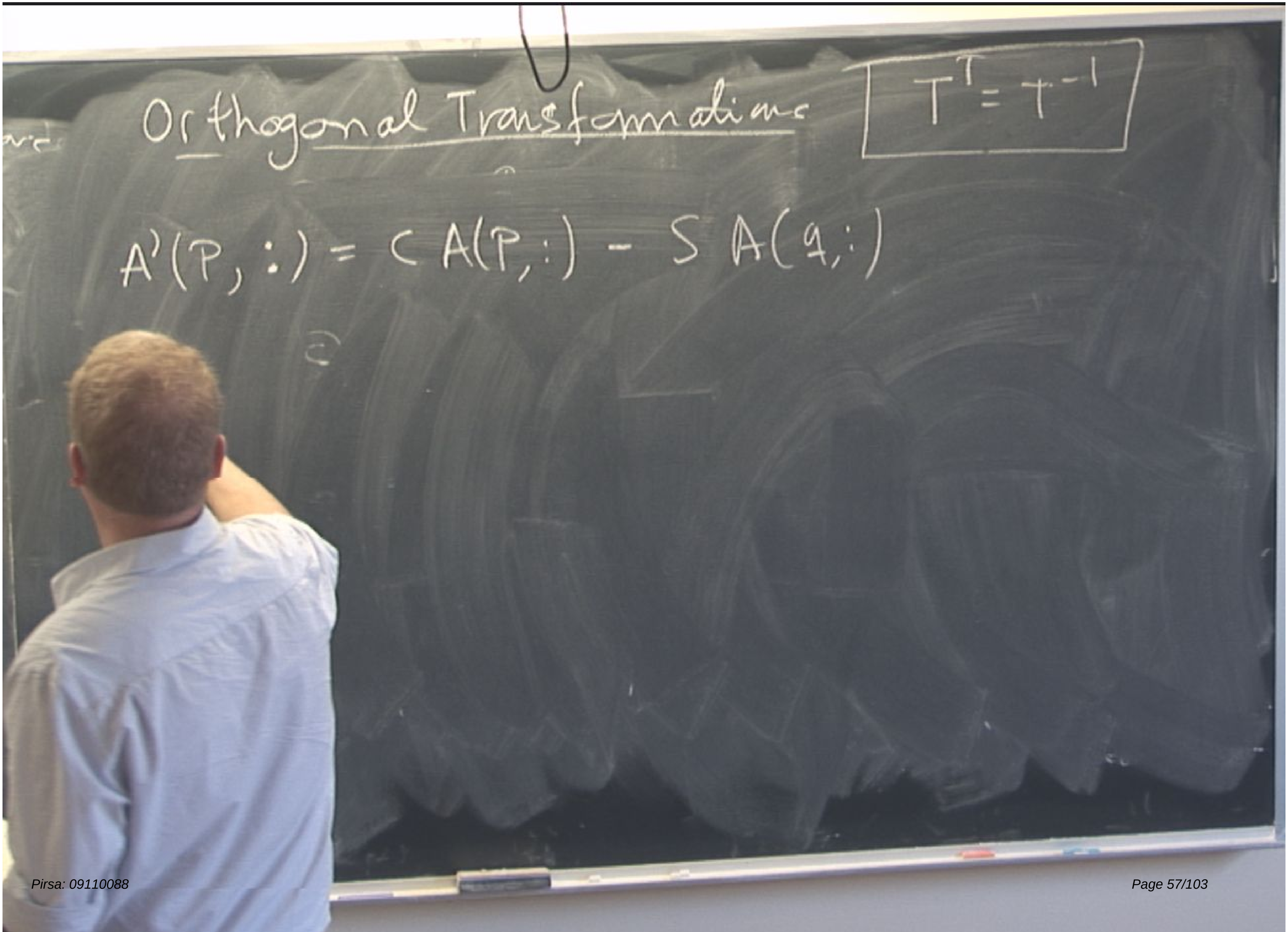
Similarity transformat

$\swarrow A'$

$$\begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & c & \cdots & -s & \\ & & \vdots & 1 & & \\ & & \vdots & & 1 & \\ & & s & & c & \\ & & & & & 1 \end{pmatrix} A = \begin{array}{c} \\ p \\ \\ q \\ \\ \end{array} \begin{pmatrix} & & & \\ \cdots & \cdots & \cdots & \cdots \\ & & & \\ \cdots & \cdots & \cdots & \cdots \\ & & & \end{pmatrix}$$

$$\Downarrow$$

$$A' \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & c & & s & \\ & & & 1 & & \\ & & & & 1 & \\ & & -s & & c & \\ & & & & & 1 \end{pmatrix} = \begin{array}{cccccc} & & p & & q & \\ & & \vdots & & \vdots & \\ p & \cdots & \cdot & \cdots & \cdot & \cdots \\ & & \vdots & & \vdots & \\ q & \cdots & \cdot & \cdots & \cdot & \cdots \\ & & \vdots & & \vdots & \end{array}$$



Orthogonal Transformations

$$\boxed{T^T = T^{-1}}$$

$$A'(p, :) = c\,A(p, :) - s\,A(q, :)$$

$$=$$

Orthogonal Transformations

$$\boxed{T^T = T^{-1}}$$

$$A'(p,:) = c\,A(p,:) - s\,A(q,:)$$

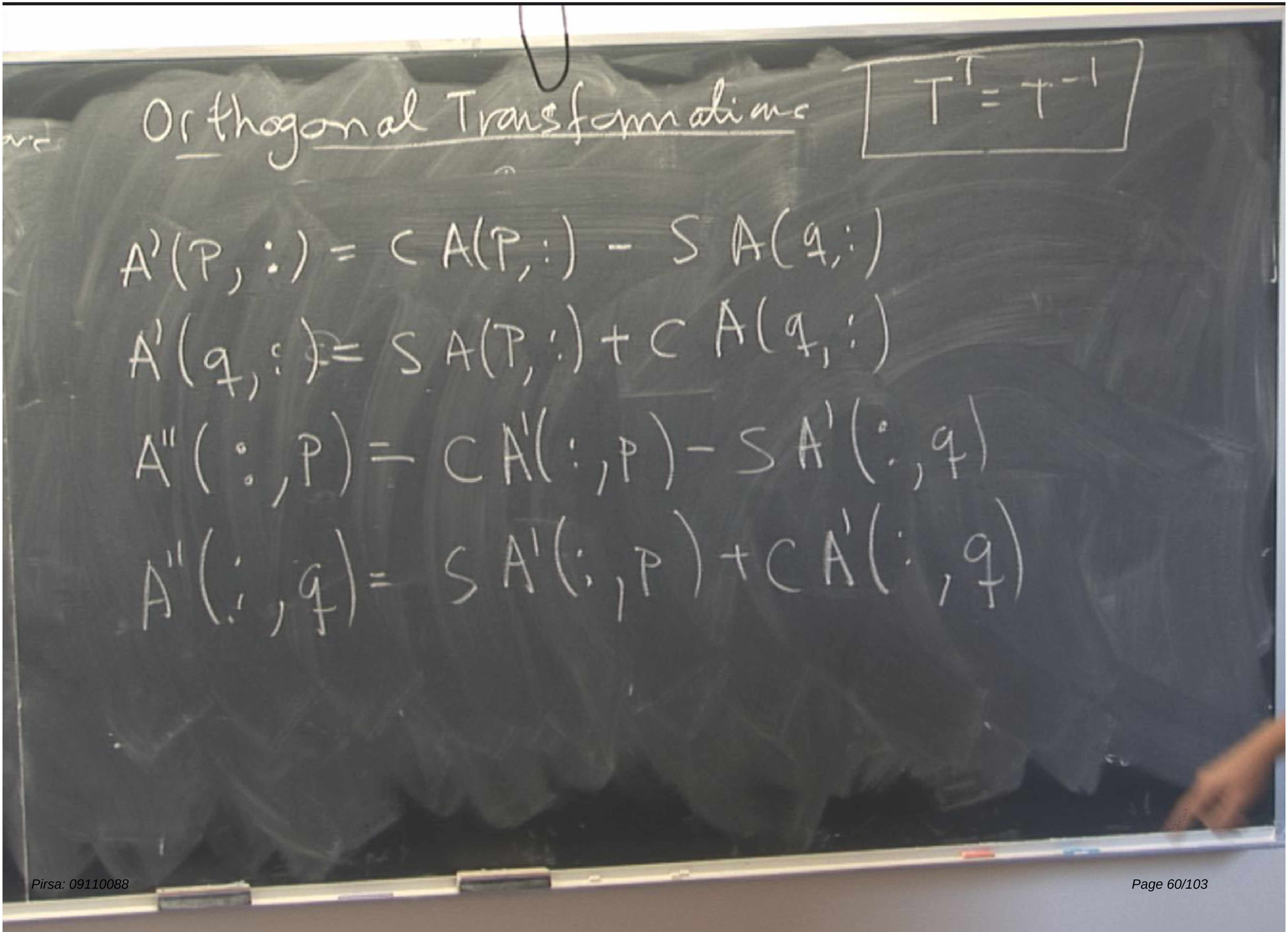
$$A'(q,:) = s\,A(p,:) + c\,A(q,:)$$

Orthogonal Transformations

$$\boxed{T^T = T^{-1}}$$

$$A'(P,:) = c\,A(P,:) - S\,A(q,:)$$

$$A'(q,:) = S\,A(P,:) + c\,A(q,:)$$



Orthogonal Transformations

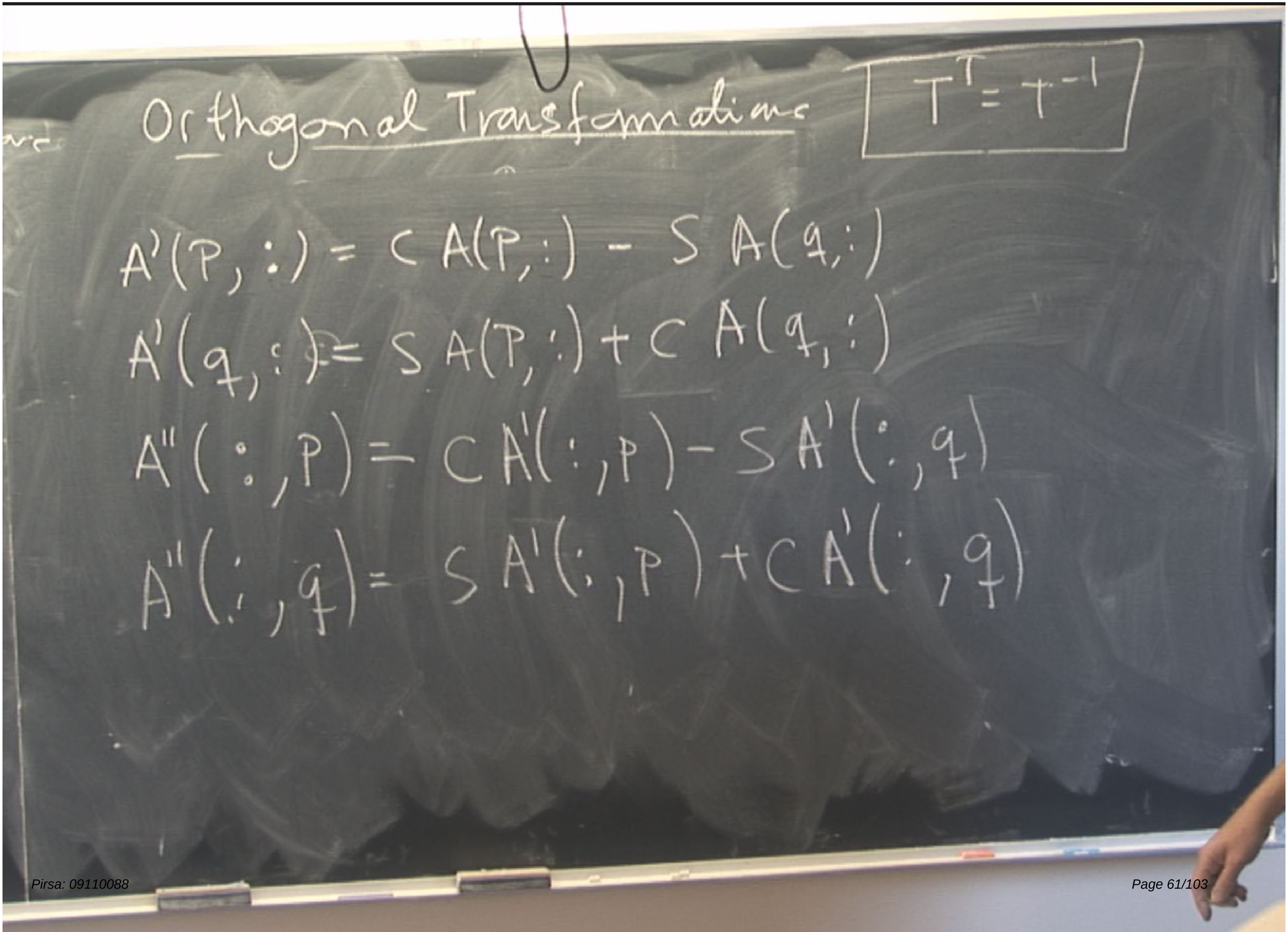
$$\boxed{T^T = T^{-1}}$$

$$A'(p,:) = c\,A(p,:) - s\,A(q,:)$$

$$A'(q,:) = s\,A(p,:) + c\,A(q,:)$$

$$A''(:,p) = c\,A'(:,p) - s\,A'(:,q)$$

$$A''(:,q) = s\,A'(:,p) + c\,A'(:,q)$$



Orthogonal Transformations

$$\boxed{T^T = T^{-1}}$$

$$A'(p,:) = c\,A(p,:) - s\,A(q,:)$$

$$A'(q,:) = s\,A(p,:) + c\,A(q,:)$$

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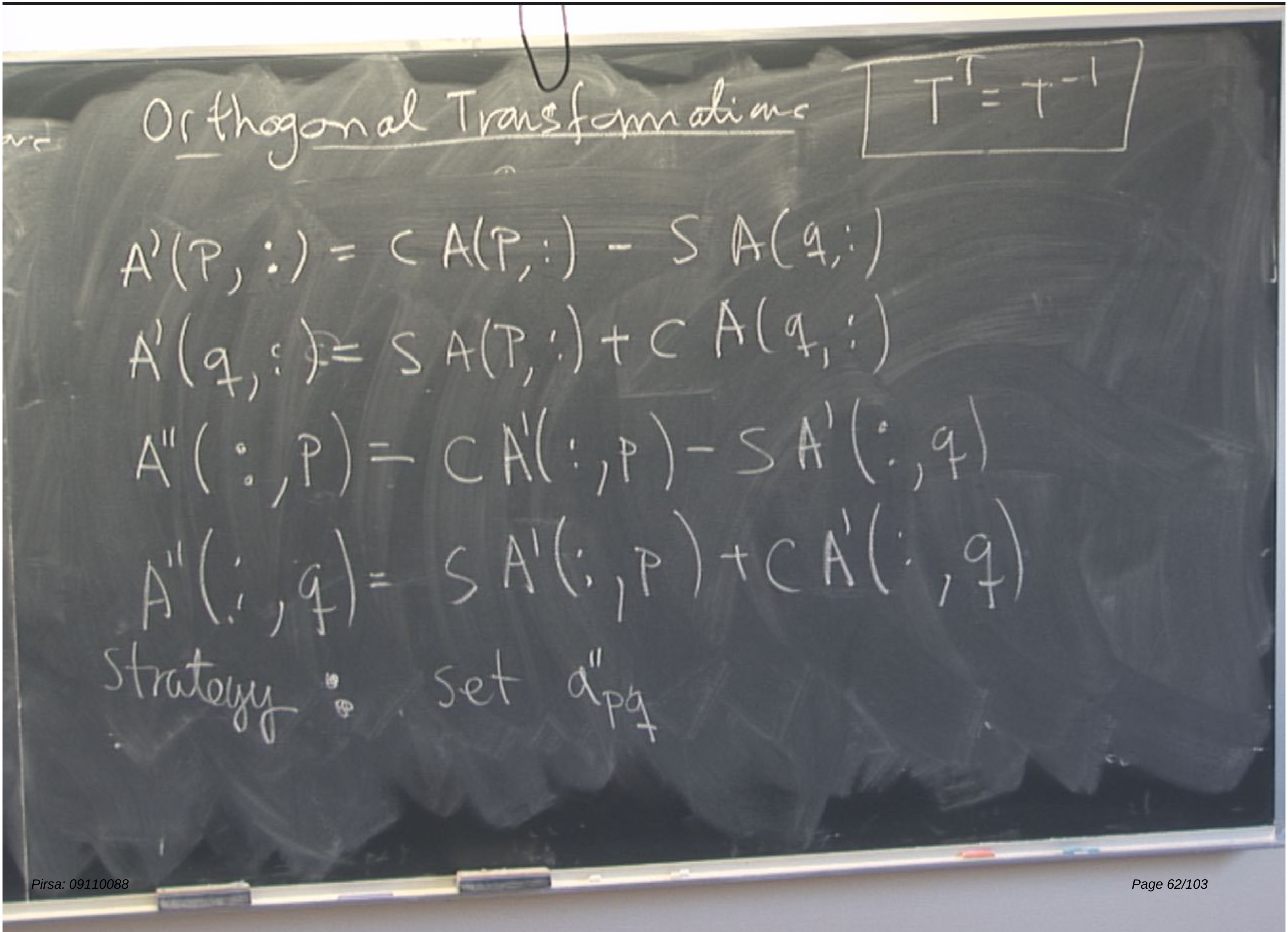
$$A''(:,q) = s\,A'(:,p) + c\,A'(:,q)$$



Orthogonal Transformations

$$\boxed{T^T = T^{-1}}$$

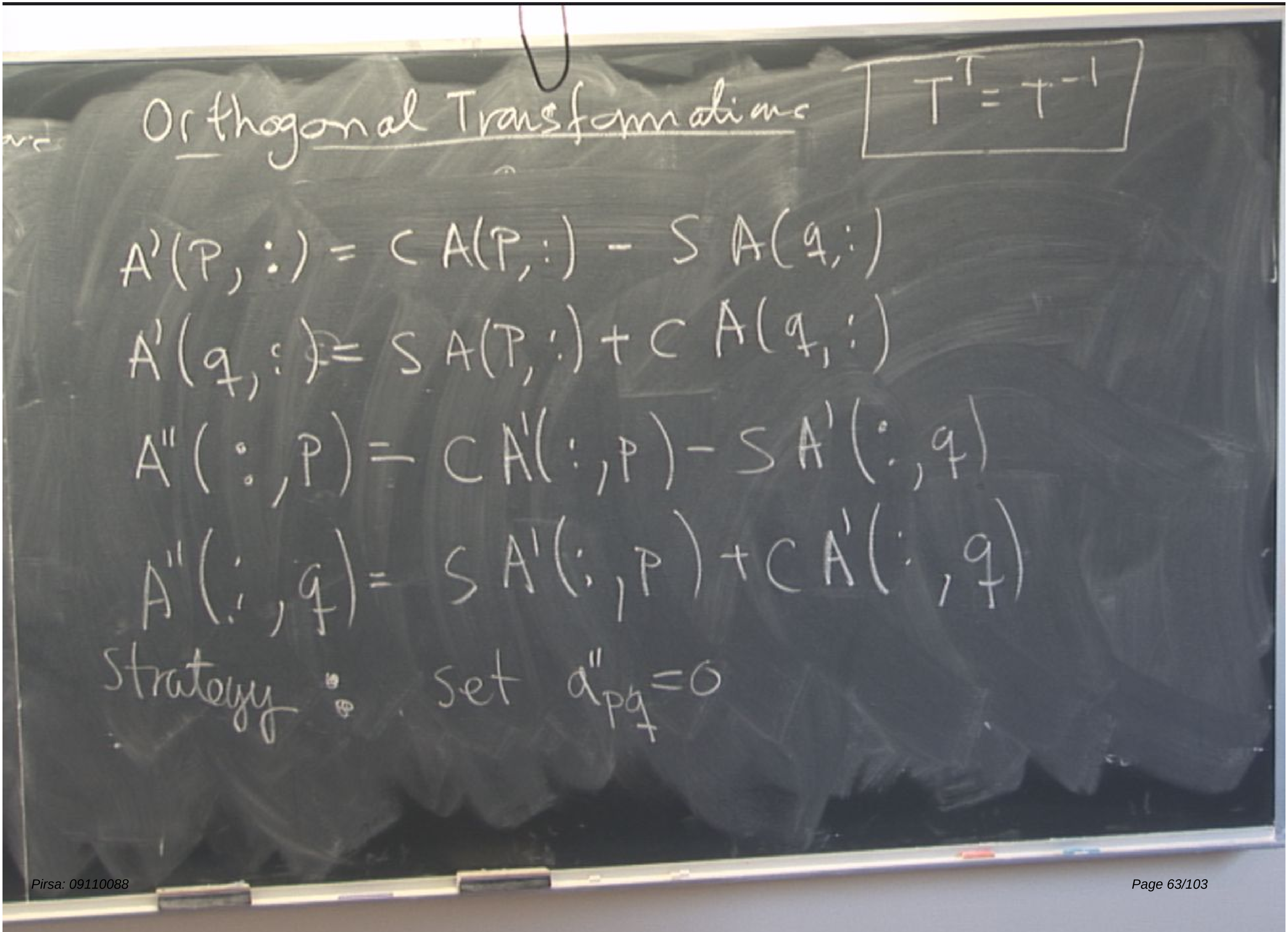
$$A'(p,:) = c\,A(p,:) - s\,A(q,:)$$

$$A'(q,:) = s\,A(p,:) + c\,A(q,:)$$

$$A''(:,p) = c\,A'(:,p) - s\,A'(:,q)$$

$$A''(:,q) = s\,A'(:,p) + c\,A'(:,q)$$

Strategy: set a''_{pq}



Orthogonal Transformations

$$\boxed{T^T = T^{-1}}$$

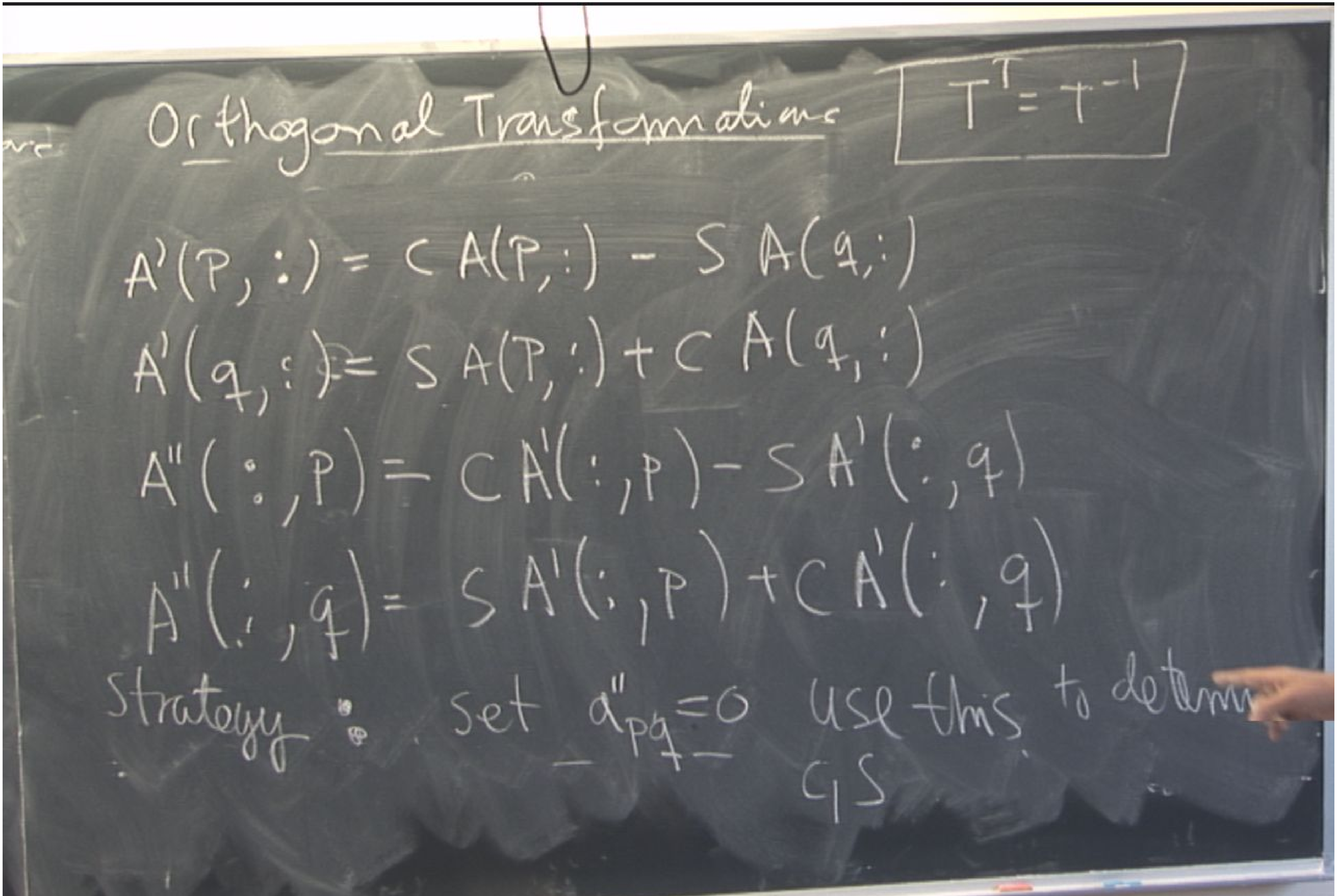
$$A'(p,:) = c\,A(p,:) - s\,A(q,:)$$

$$A'(q,:) = s\,A(p,:) + c\,A(q,:)$$

$$A''(:,p) = c\,A'(:,p) - s\,A'(:,q)$$

$$A''(:,q) = s\,A'(:,p) + c\,A'(:,q)$$

Strategy: set $a''_{pq} = 0$



Orthogonal Transformations

$$\boxed{T^T = T^{-1}}$$

$$A'(p,:) = c\,A(p,:) - s\,A(q,:)$$

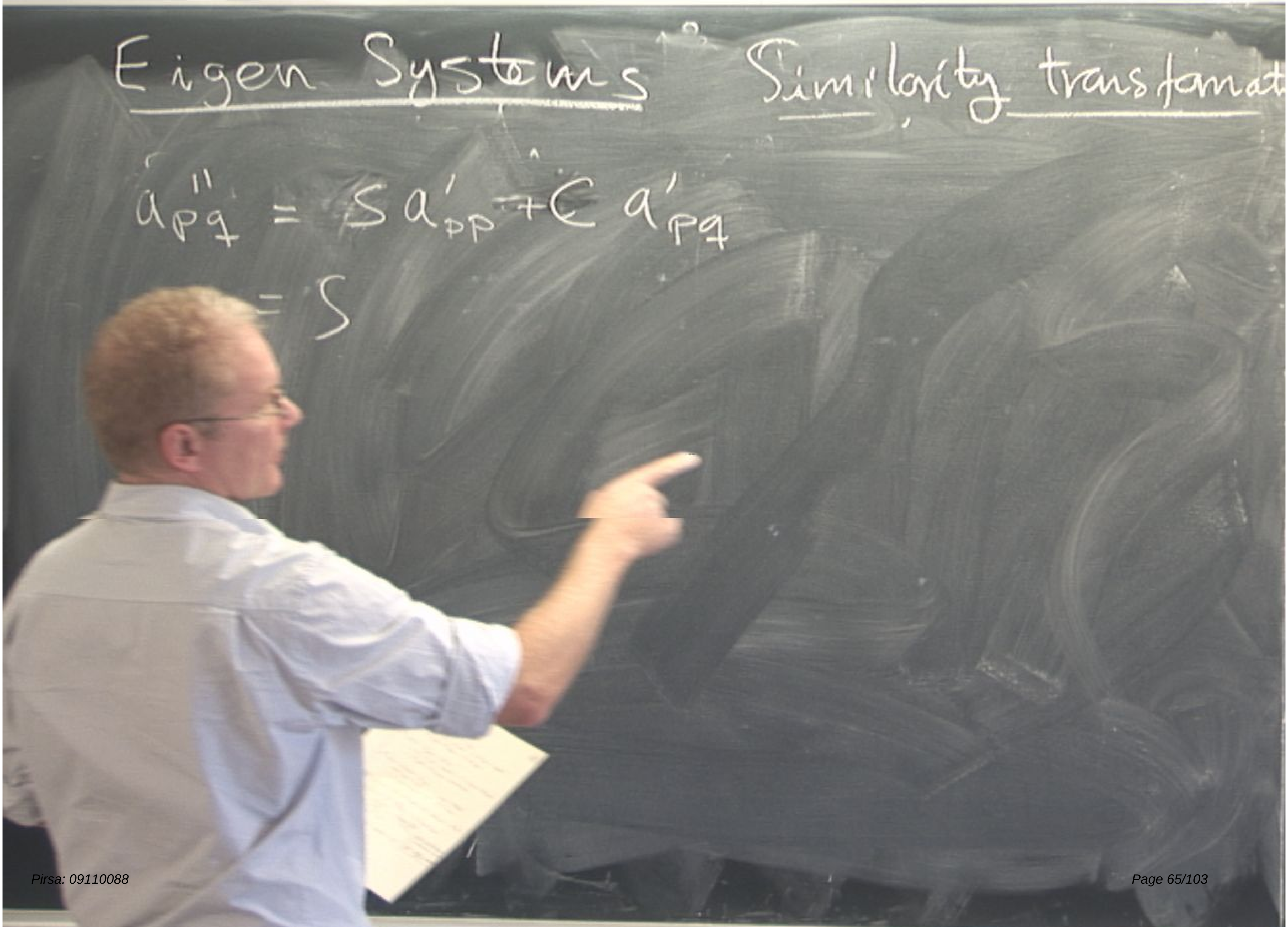
$$A'(q,:) = s\,A(p,:) + c\,A(q,:)$$

$$A''(:,p) = c\,A'(:,p) - s\,A'(:,q)$$

$$A''(:,q) = s\,A'(:,p) + c\,A'(:,q)$$

Strategy: set $\underline{a''_{pq} = 0}$ use this to determ

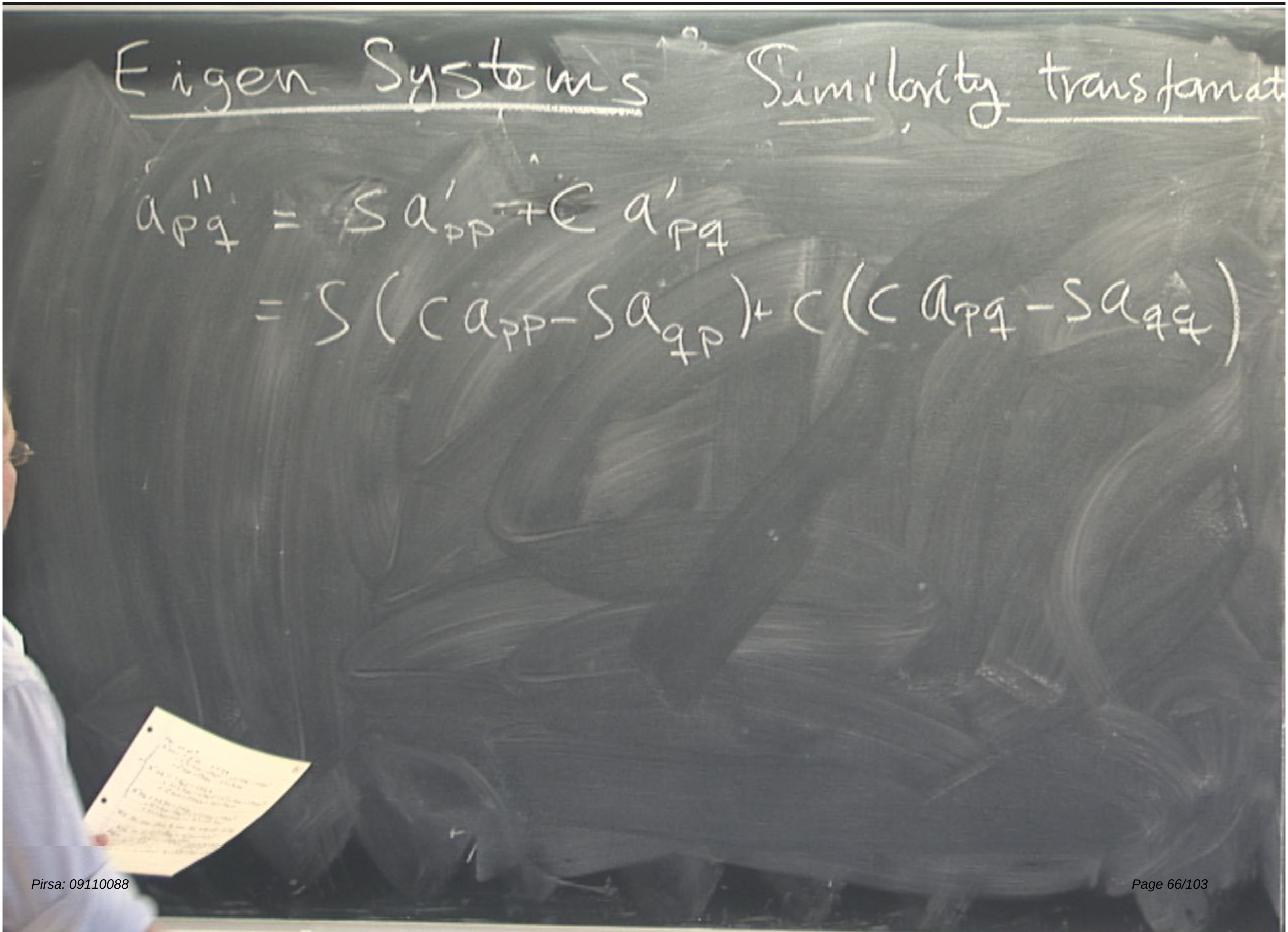
c, s



Eigen Systems: Similarity transformat

$$a''_{pq} = S a'_{pp} + C a'_{pq}$$

$$= S$$

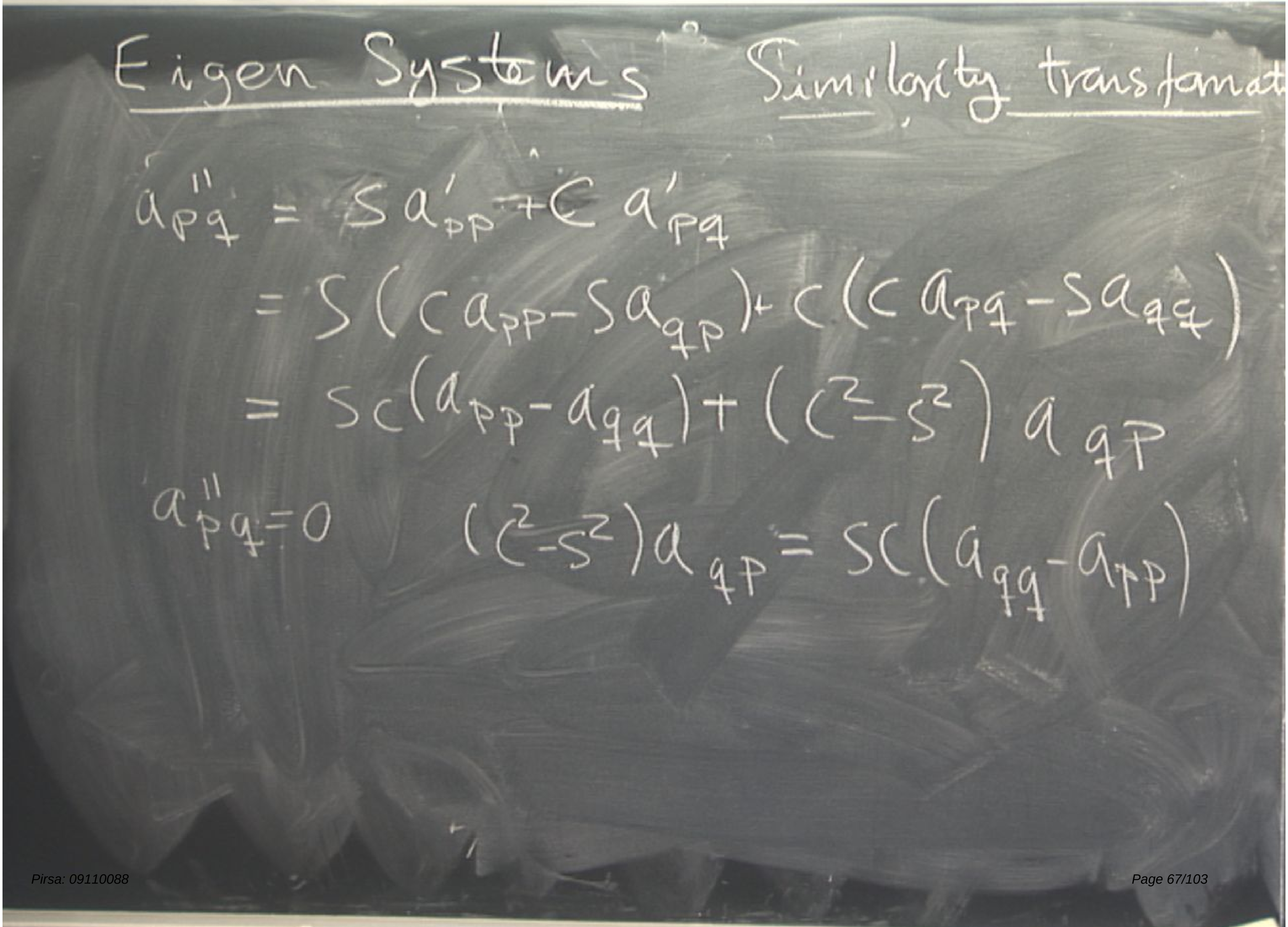


Eigen Systems

Similarity transformat

$$a''_{pq} = s\,a'_{pp} + c\,a'_{pq}$$

$$= s\,(c\,a_{pp} - s\,a_{qp}) + c\,(c\,a_{pq} - s\,a_{qq})$$



Eigen Systems

Similarity transformat

$$a''_{pq} = s a'_{pp} + c a'_{pq}$$

$$= s(c a_{pp} - s a_{qp}) + c(c a_{pq} - s a_{qq})$$

$$= sc(a_{pp} - a_{qq}) + (c^2 - s^2) a_{qp}$$

$a''_{pq} = 0$ $\quad (c^2 - s^2) a_{qp} = sc(a_{qq} - a_{pp})$

Eigen Systems Similarity transformat

$$a''_{pq} = s a'_{pp} + c a'_{pq}$$

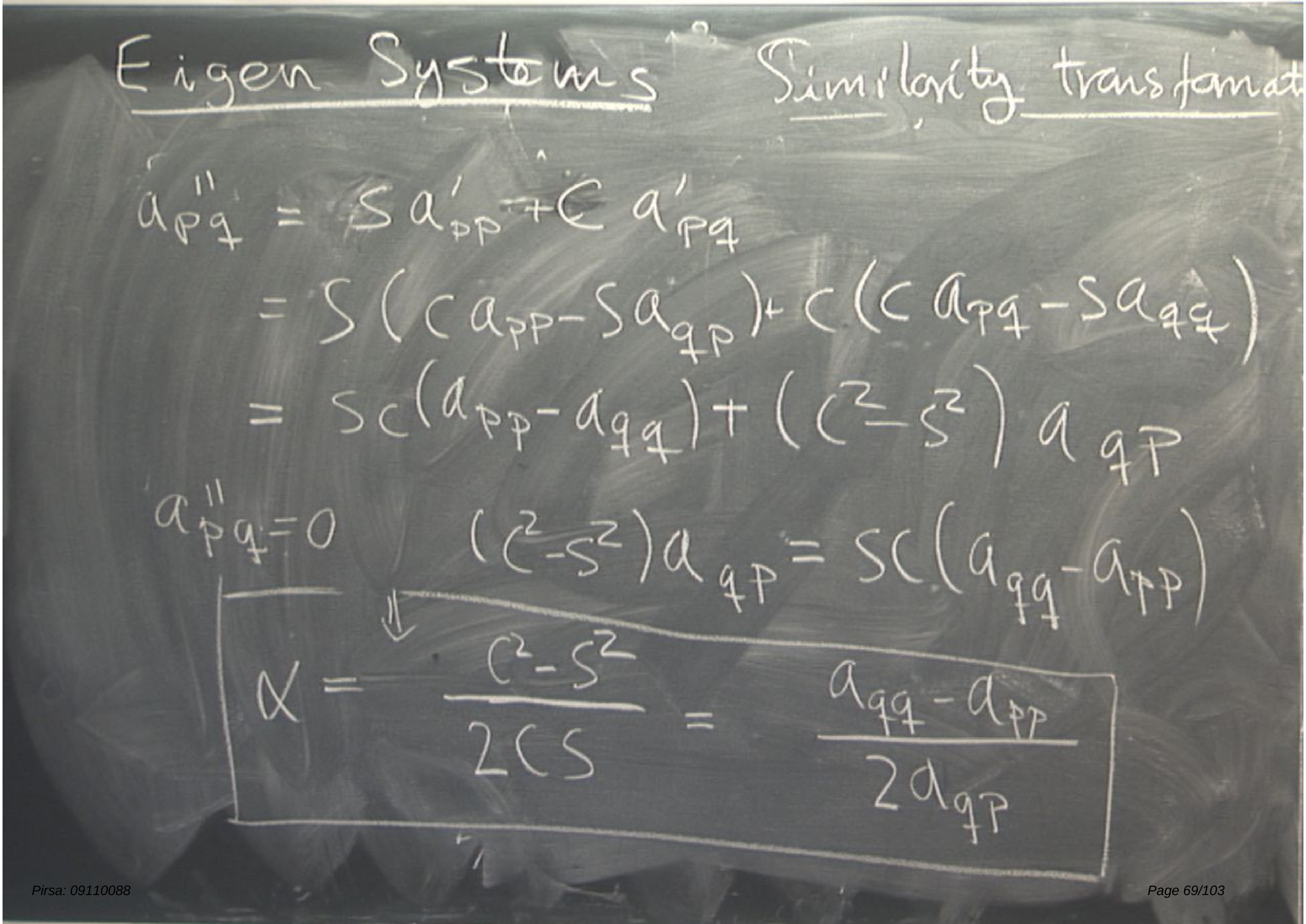
$$= s(c a_{pp} - s a_{qp}) + c(c a_{pq} - s a_{qq})$$

$$= sc(a_{pp} - a_{qq}) + (c^2 - s^2) a_{qp}$$

$a''_{pq} = 0$ $\quad (c^2 - s^2) a_{qp} = sc(a_{qq} - a_{pp})$

$\Downarrow$

$$\frac{c^2 - s^2}{cs} = \frac{a_{qq} - a_{pp}}{a_{qp}}$$



Eigen Systems: Similarity transformat

$$a''_{pq} = S a'_{pp} + C a'_{pq}$$

$$= S(C a_{pp} - S a_{qp}) + C(C a_{pq} - S a_{qq})$$

$$= SC(a_{pp} - a_{qq}) + (C^2 - S^2) a_{qp}$$

$$a''_{pq} = 0 \qquad (C^2 - S^2) a_{qp} = SC(a_{qq} - a_{pp})$$

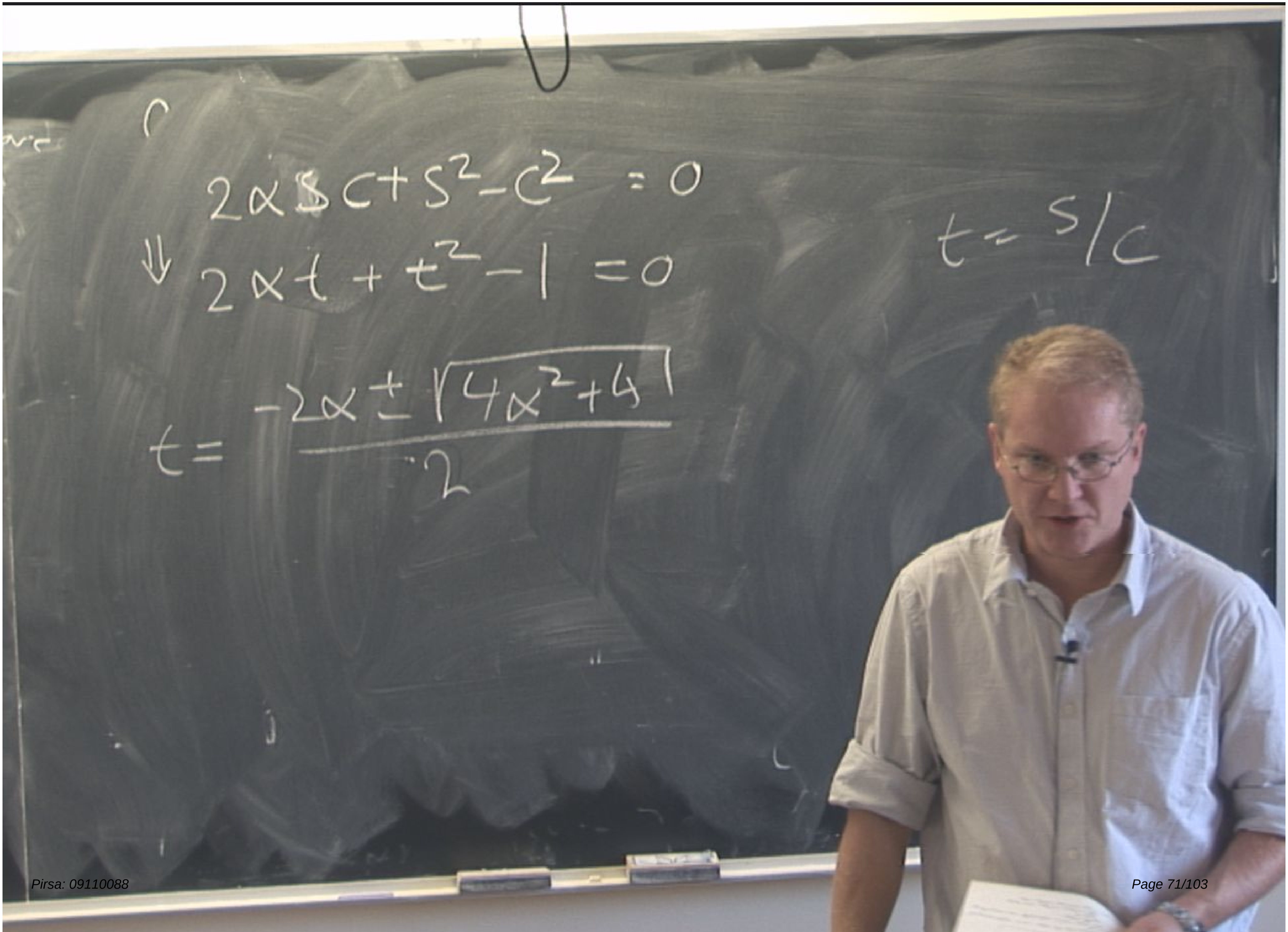
$$\Downarrow$$

$$\boxed{\alpha = \frac{C^2 - S^2}{2CS} = \frac{a_{qq} - a_{pp}}{2a_{qp}}}$$

$$2\alpha sc + s^2 - c^2 = 0$$

$$\Downarrow \quad 2\alpha t + t^2 - 1 = 0$$

$$t = s/c$$



$$2\alpha sc + s^2 - c^2 = 0$$

$$\Downarrow \quad 2\alpha t + t^2 - 1 = 0$$

$$t = \frac{-2\alpha \pm \sqrt{4\alpha^2 + 4}}{2}$$

$$t = s/c$$

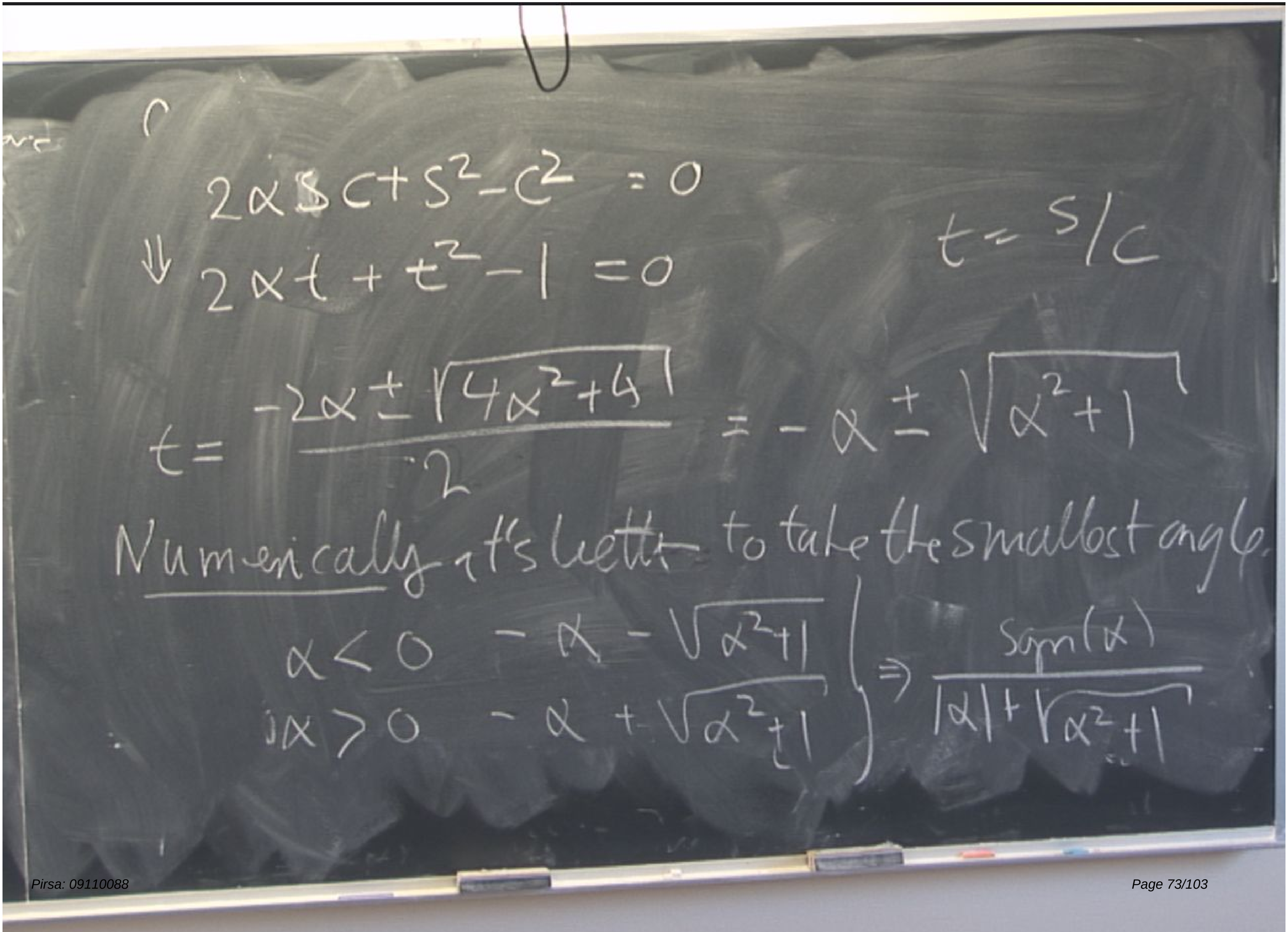
$$2\alpha sc + s^2 - c^2 = 0$$

$$\Downarrow \quad 2\alpha t + t^2 - 1 = 0$$

$$t = s/c$$

$$t = \frac{-2\alpha \pm \sqrt{4\alpha^2 + 4}}{2} = -\alpha \pm \sqrt{\alpha^2 + 1}$$

<u>Numerically</u>, it's better to take the smallest angle.



$$2\alpha sc + s^2 - c^2 = 0$$

$$\Downarrow \quad 2\alpha t + t^2 - 1 = 0 \qquad t = s/c$$

$$t = \frac{-2\alpha \pm \sqrt{4\alpha^2+4}}{2} = -\alpha \pm \sqrt{\alpha^2+1}$$

Numerically it's better to take the smallest angle.

$$\left.\begin{array}{ll} \alpha < 0 & -\alpha - \sqrt{\alpha^2+1} \\ \alpha > 0 & -\alpha + \sqrt{\alpha^2+1} \end{array}\right\} \Rightarrow \frac{\operatorname{sgn}(\alpha)}{|\alpha| + \sqrt{\alpha^2+1}}$$

$$2\alpha sc + s^2 - c^2 = 0$$

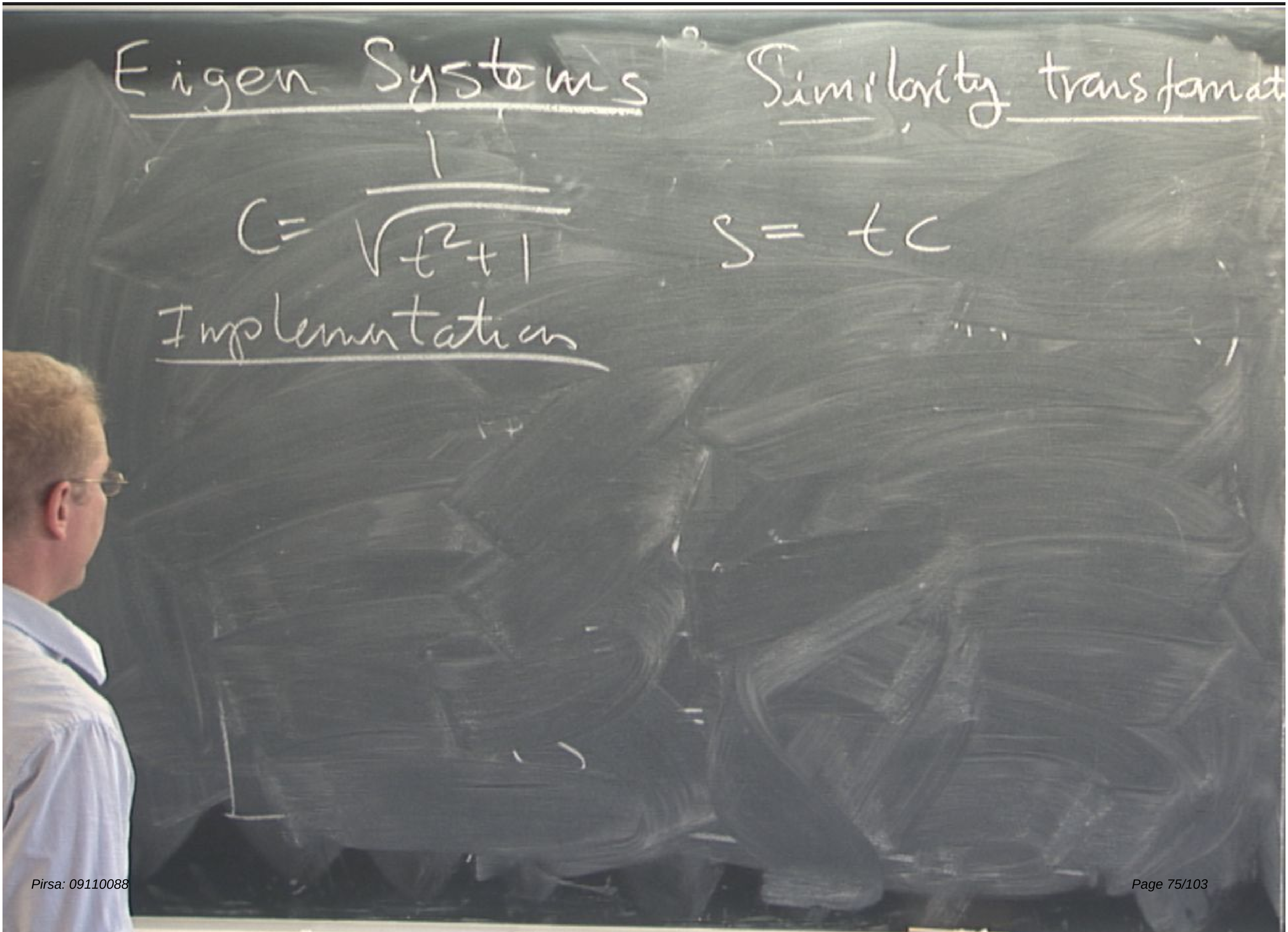
$$\Downarrow$$

$$2\alpha t + t^2 - 1 = 0 \qquad t = s/c$$

$$t = \frac{-2\alpha \pm \sqrt{4\alpha^2+4}}{2} = -\alpha \pm \sqrt{\alpha^2+1}$$

<u>Numerically</u>, it's better to take the smallest angle.

$$\left.\begin{array}{ll} \alpha < 0 & -\alpha - \sqrt{\alpha^2+1} \\ \alpha > 0 & -\alpha + \sqrt{\alpha^2+1} \end{array}\right\} \Rightarrow \boxed{\frac{\mathrm{sgn}(\alpha)}{|\alpha| + \sqrt{\alpha^2+1}} = t}$$



Eigen Systems

Similarity transformat

$$C = \frac{1}{\sqrt{t^2+1}} \qquad s = tc$$

Implementation

Eigen Systems

Similarity transformat

$$c = \frac{1}{\sqrt{t^2+1}} \qquad s = tc$$

Implementation

1 sweep of the off diagonal elements

Eigen Systems

Similarity transformat

$$c = \frac{1}{\sqrt{t^2+1}} \qquad s = tc$$

Implementation

1 sweep of the off diagonal elements

We need more than 1 sweep

Eigen Systems

Similarity transformat

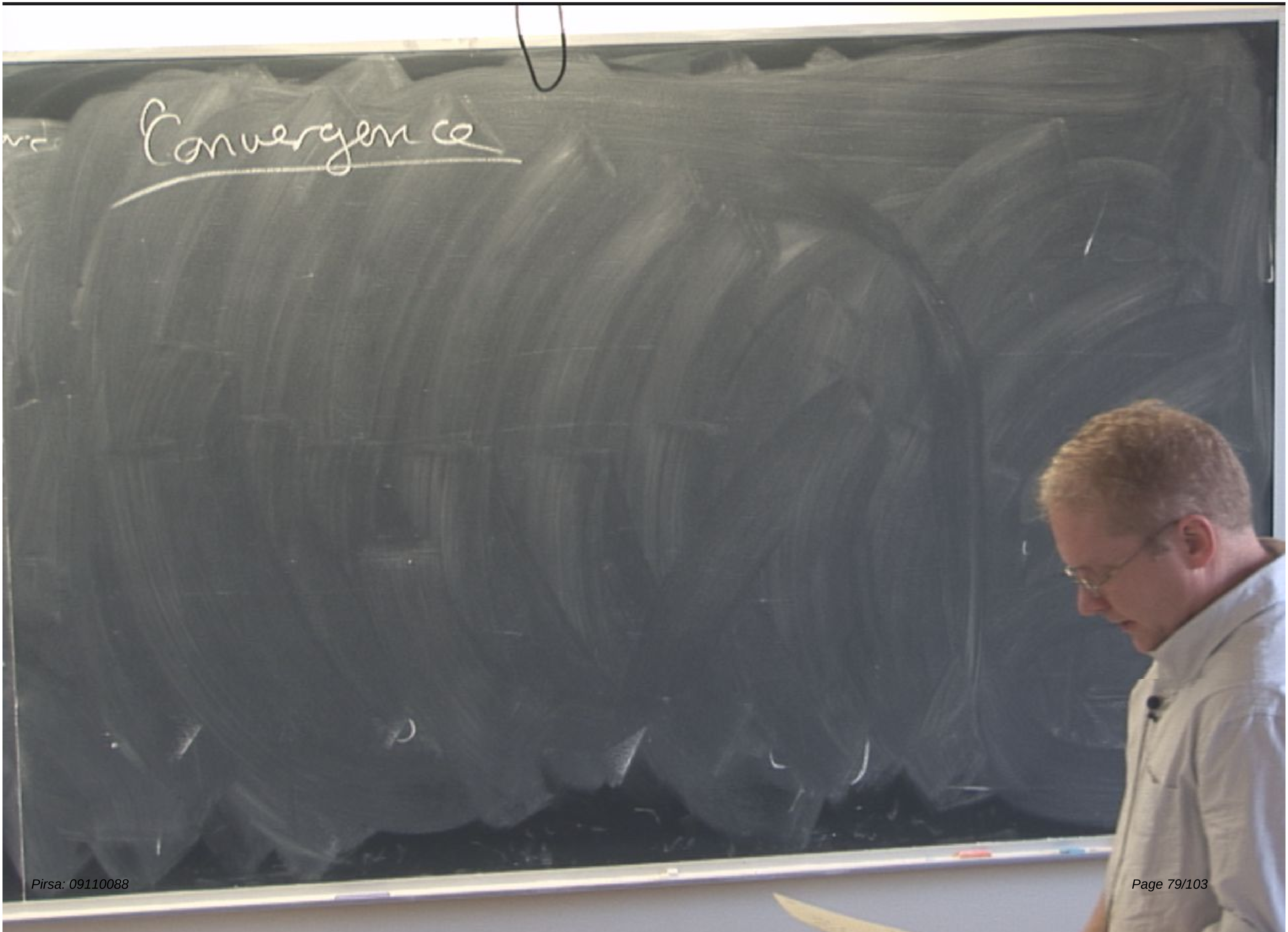
$$C = \frac{1}{\sqrt{t^2+1}} \qquad s = tc$$

Implementation

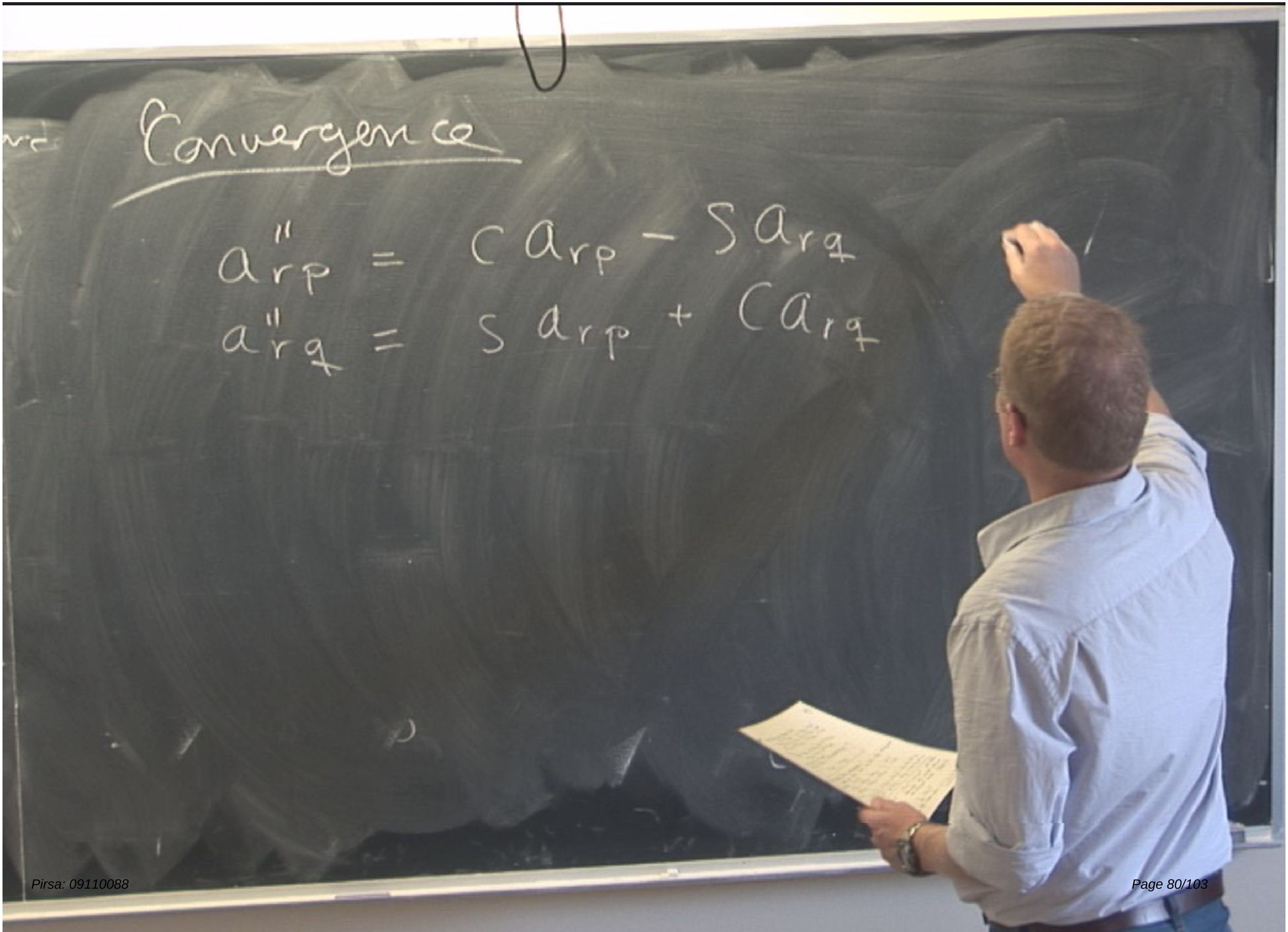
1 sweep of the off diagonal elements

We need more than 1 sweep

10 sweeps



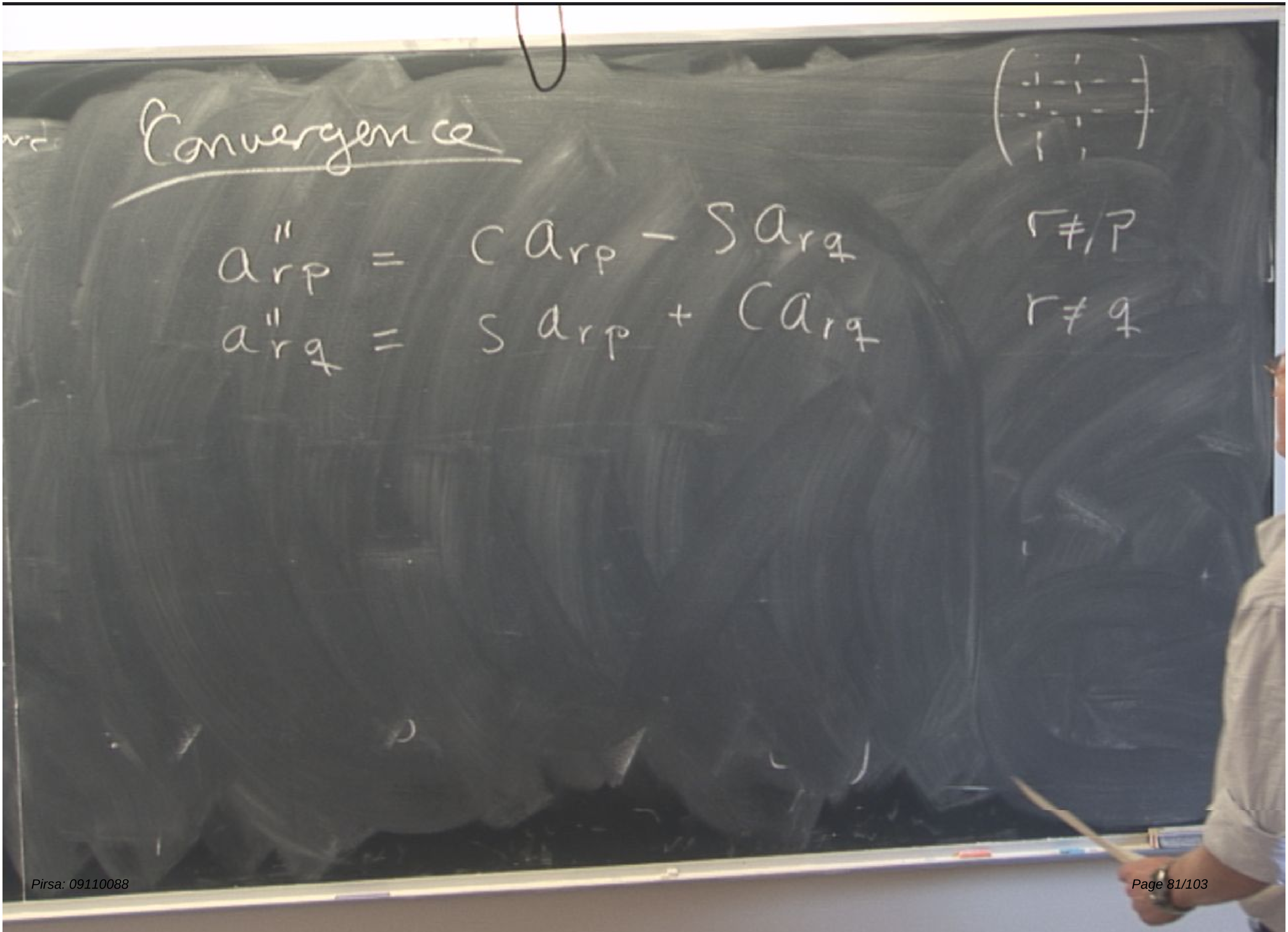
Convergence



Convergence

$$a''_{rp} = c\,a_{rp} - s\,a_{rq}$$

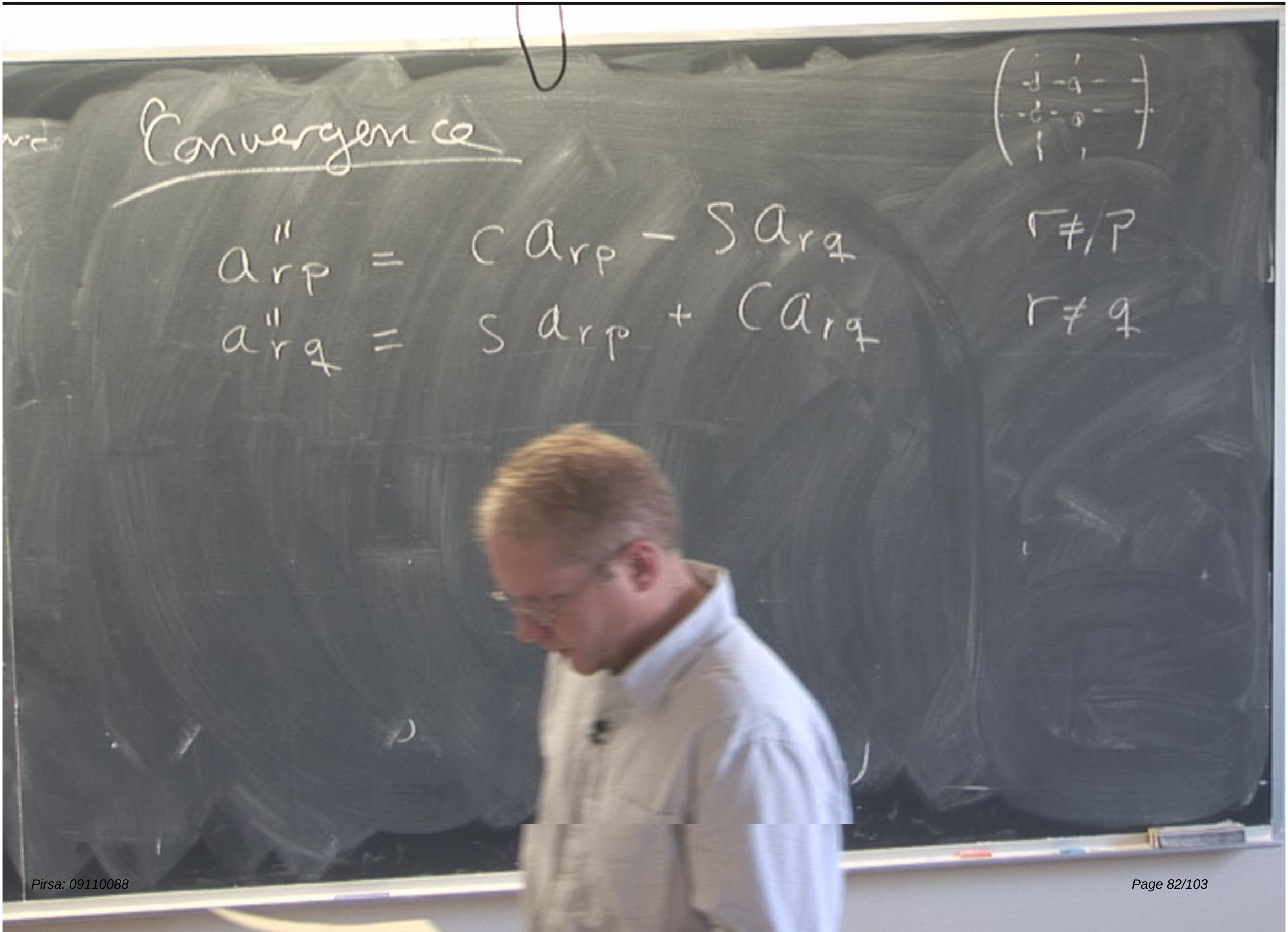
$$a''_{rq} = s\,a_{rp} + c\,a_{rq}$$



Convergence

$$a''_{rp} = c\,a_{rp} - s\,a_{rq} \qquad r \neq p$$

$$a''_{rq} = s\,a_{rp} + c\,a_{rq} \qquad r \neq q$$

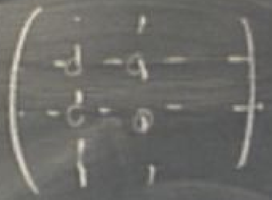


Convergence

$$a''_{rp} = c\,a_{rp} - s\,a_{rq} \qquad r \neq p$$

$$a''_{rq} = s\,a_{rp} + c\,a_{rq} \qquad r \neq q$$

Convergence



$$a''_{rp} = c\,a_{rp} - S\,a_{rq} \qquad r \neq p$$

$$a''_{rq} = s\,a_{rp} + C\,a_{rq} \qquad r \neq q$$

$$|a''_{rp}|^2 = C^2 a_{rp}^2 + S^2 a_{rq}^2 - 2SC\,a_{rp} a_{rq}$$

$$|a''_{rq}|^2 = S^2 a_{rp}^2 + C^2 a_{rq}^2 + 2SC\,a_{rp} a_{rq}$$

Convergence

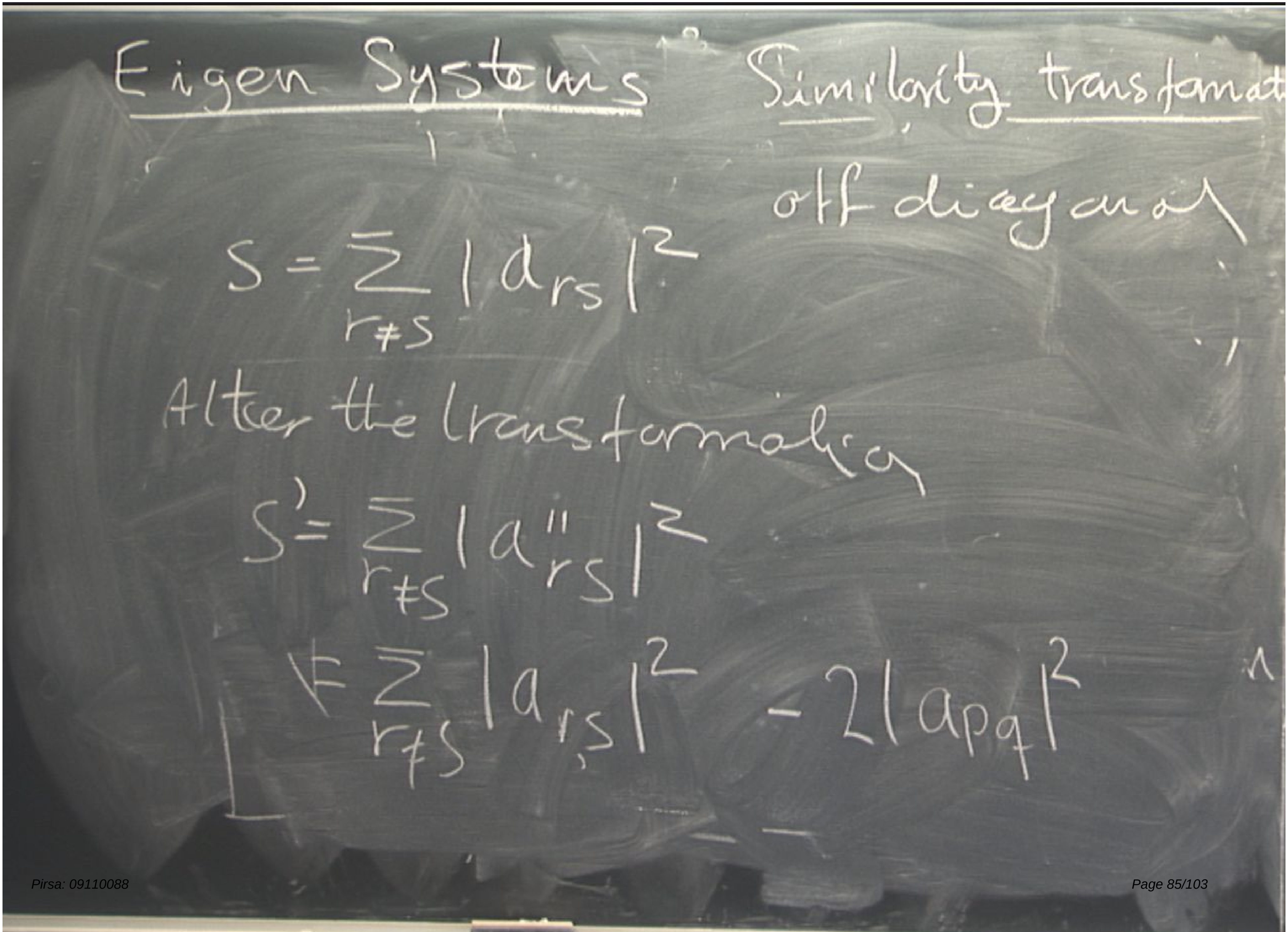
$$a''_{rp} = c\,a_{rp} - s\,a_{rq} \qquad r \neq p$$

$$a''_{rq} = s\,a_{rp} + c\,a_{rq} \qquad r \neq q$$

$$|a''_{rp}|^2 = c^2 a_{rp}^2 + s^2 a_{rq}^2 - 2sc\,a_{rp}a_{rq}$$

$$|a''_{rq}|^2 = s^2 a_{rp}^2 + c^2 a_{rq}^2 + 2sc\,a_{rp}a_{rq}$$

$$|a''_{rp}|^2 + |a''_{rq}|^2 = |a_{rp}|^2 + |a_{rq}|^2 \qquad \begin{matrix} r \neq p \\ r \neq q \end{matrix}$$



Eigen Systems

Similarity transformat

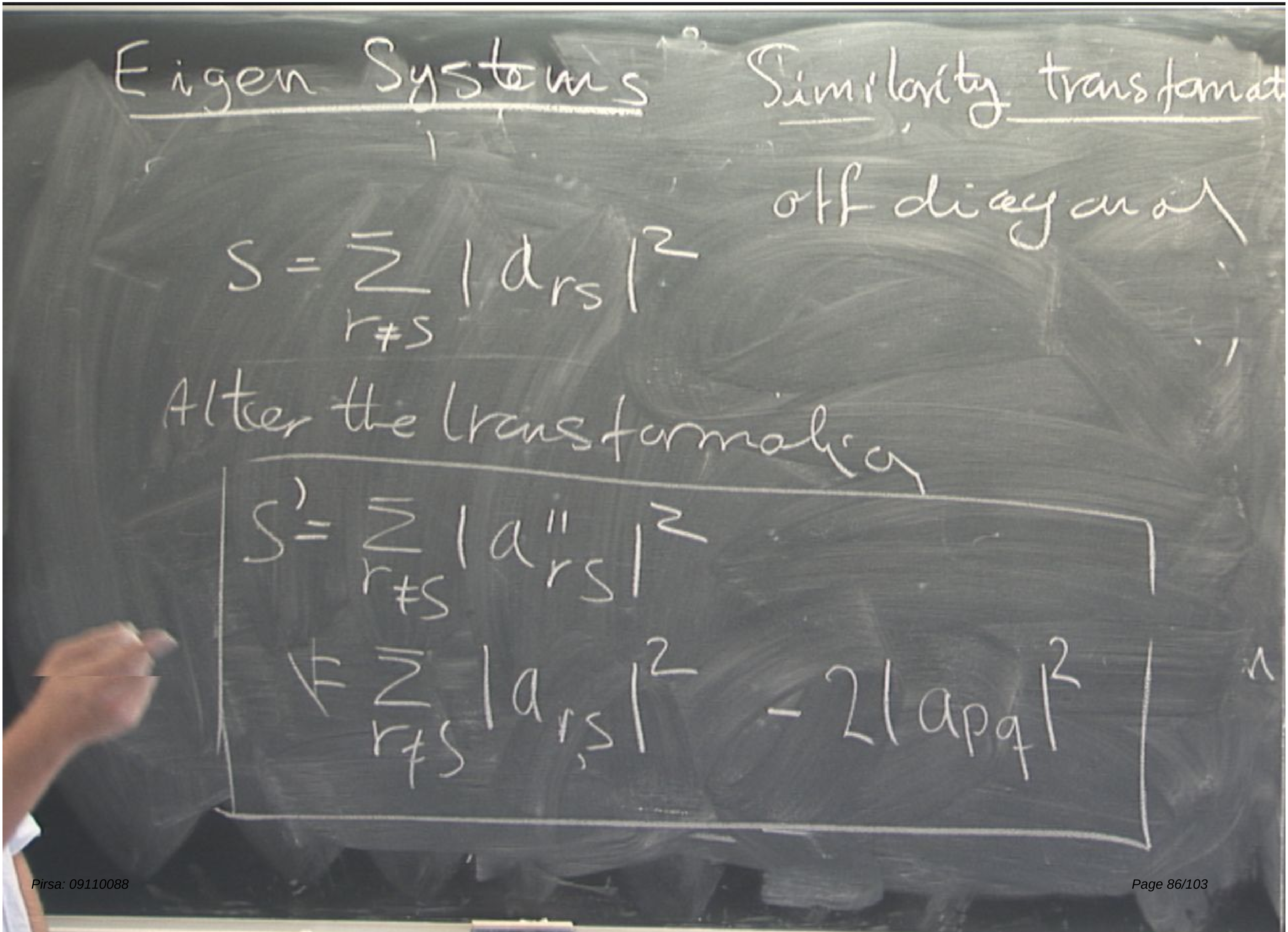
off diagonal

$$S = \sum_{r\neq s} |a_{rs}|^2$$

Alter the transformation

$$S' = \sum_{r\neq s} |a''_{rs}|^2$$

$$= \sum_{r\neq s} |a_{r,s}|^2 - 2|a_{pq}|^2$$



Eigen Systems

Similarity transformat

off diagonal

$$S = \sum_{r \neq s} |a_{rs}|^2$$

Alter the transformation

$$\boxed{\begin{aligned} S' &= \sum_{r \neq s} |a''_{rs}|^2 \\ &= \sum_{r \neq s} |a_{r,s}|^2 - 2|a_{pq}|^2 \end{aligned}}$$

Eigen Systems

Similarity transformat

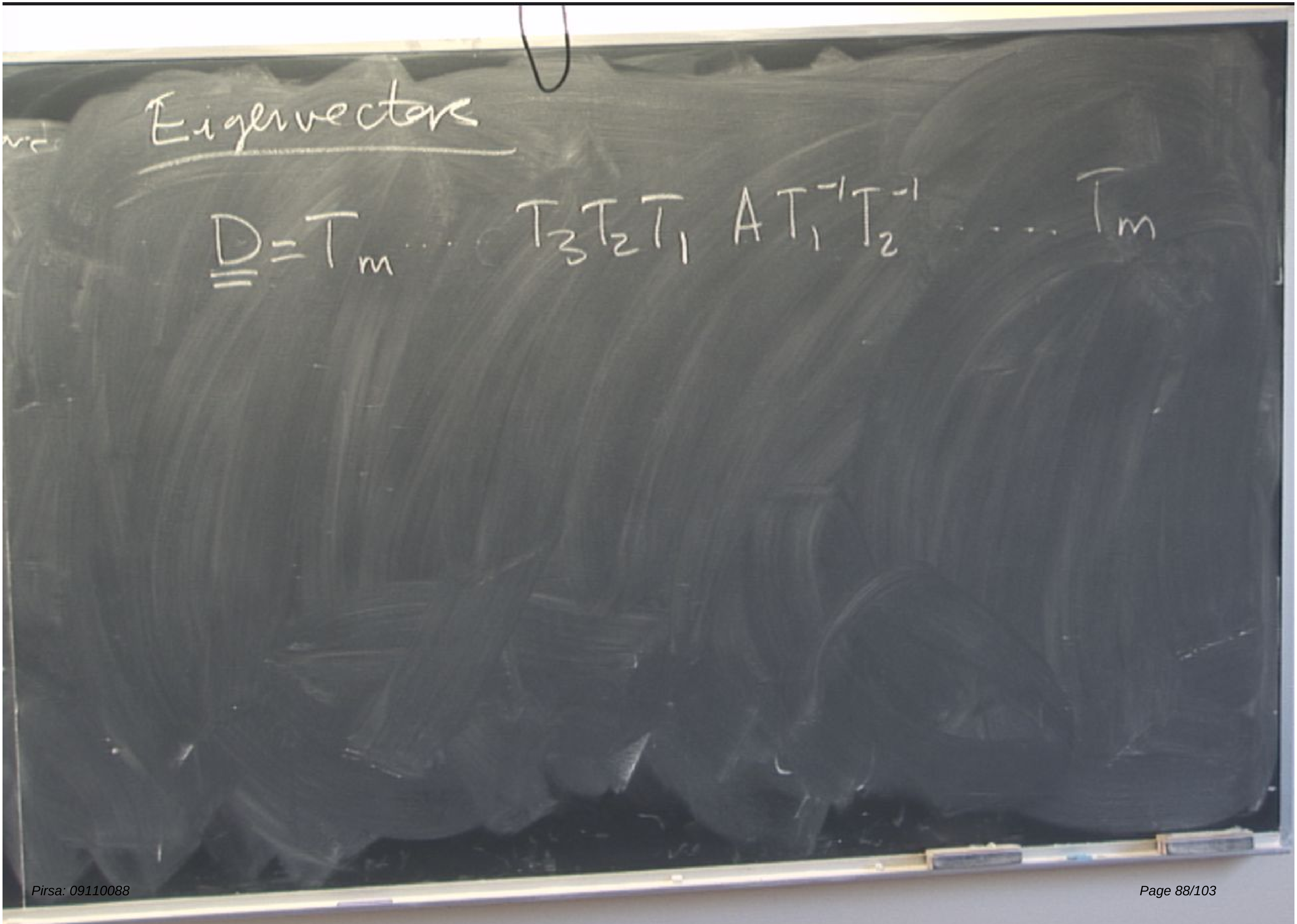
off diagonal

$$S = \sum_{r \neq s} |a_{rs}|^2$$

guaranteed, convergence

Alter the transformation

$$\boxed{\begin{aligned} S' &= \sum_{r \neq s} |a''_{rs}|^2 \\ &= \sum_{r \neq s} |a_{r,s}|^2 - 2|a_{pq}|^2 \end{aligned}}$$



Eigenvectors

$$\underline{\underline{D}} = T_m \cdots T_3 T_2 T_1 \, A \, T_1^{-1} T_2^{-1} \cdots T_m$$

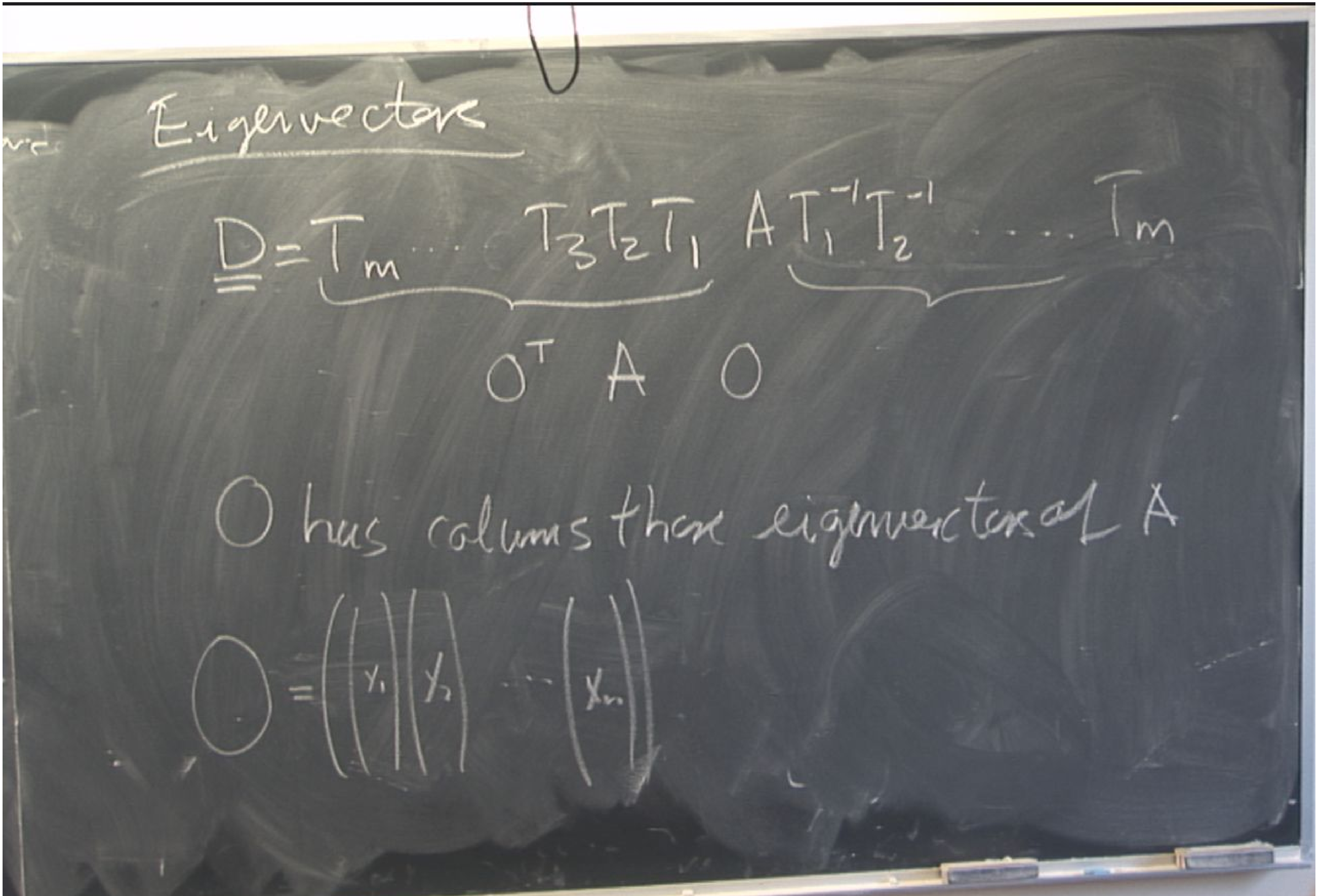
Eigenvectors

$$\underline{\underline{D}} = \underbrace{T_m \cdots T_3 T_2 T_1}_{O^T} A \underbrace{T_1^{-1} T_2^{-1} \cdots T_m}_{O}$$

$$O^T \; A \; O$$

O has columns thare eigenvectors of A

$$O = \left(\begin{pmatrix} \\ x_1 \\ \\ \end{pmatrix} \begin{pmatrix} \\ x_2 \\ \\ \end{pmatrix} \cdots \begin{pmatrix} \\ x_n \\ \\ \end{pmatrix} \right)$$

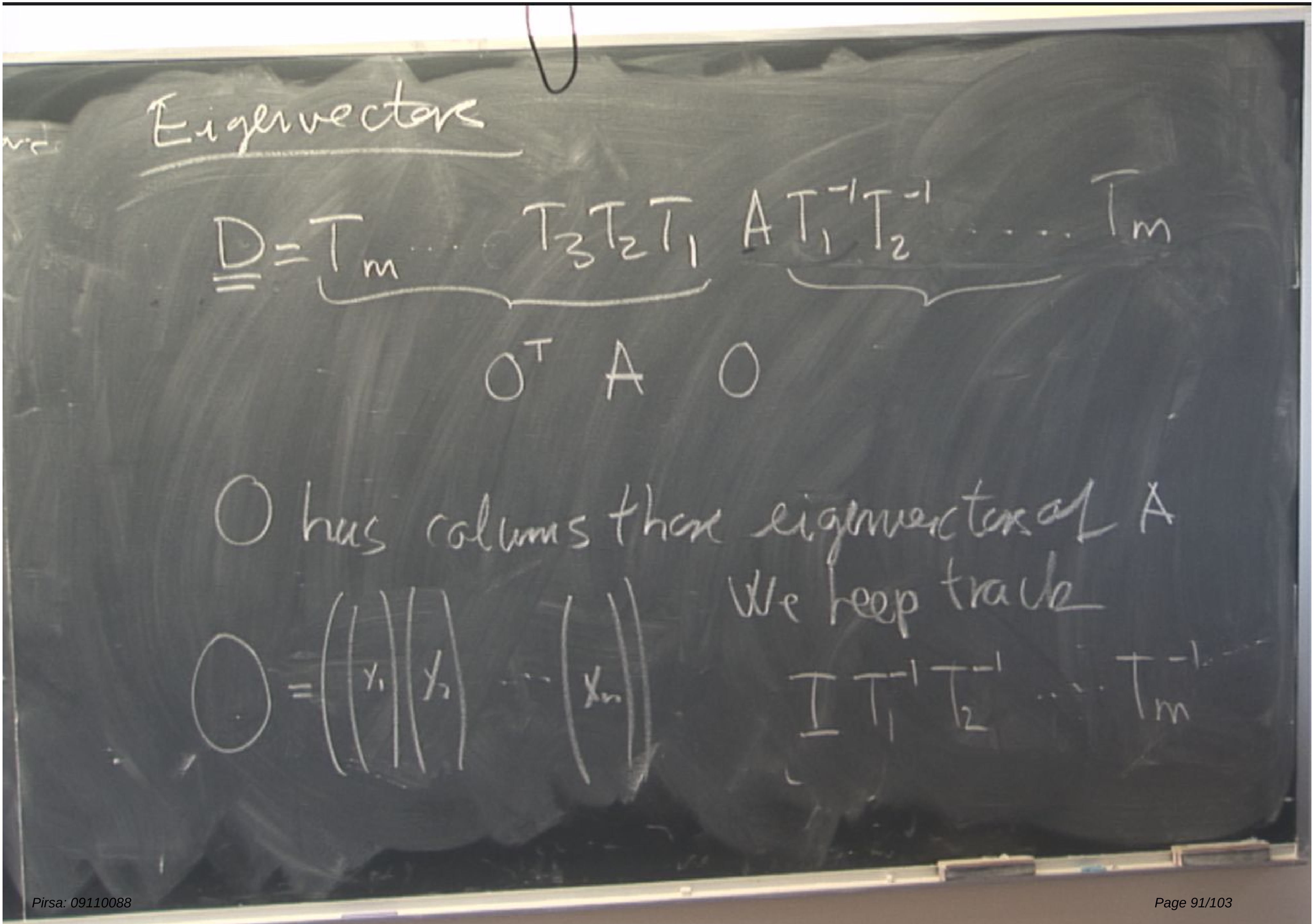


Eigenvectors

$$\underline{\underline{D}} = \underbrace{T_m \cdots T_3 T_2 T_1}_{O^T} \; A \; \underbrace{T_1^{-1} T_2^{-1} \cdots T_m}_{O}$$

O has columns the eigenvectors of A

$$O = \begin{pmatrix} \begin{pmatrix} x_1 \end{pmatrix} \begin{pmatrix} x_2 \end{pmatrix} \cdots \begin{pmatrix} x_n \end{pmatrix} \end{pmatrix}$$



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We keep track

$$I \, T_1^{-1} T_2^{-1} \cdots T_m^{-1}$$

Eigenvectors

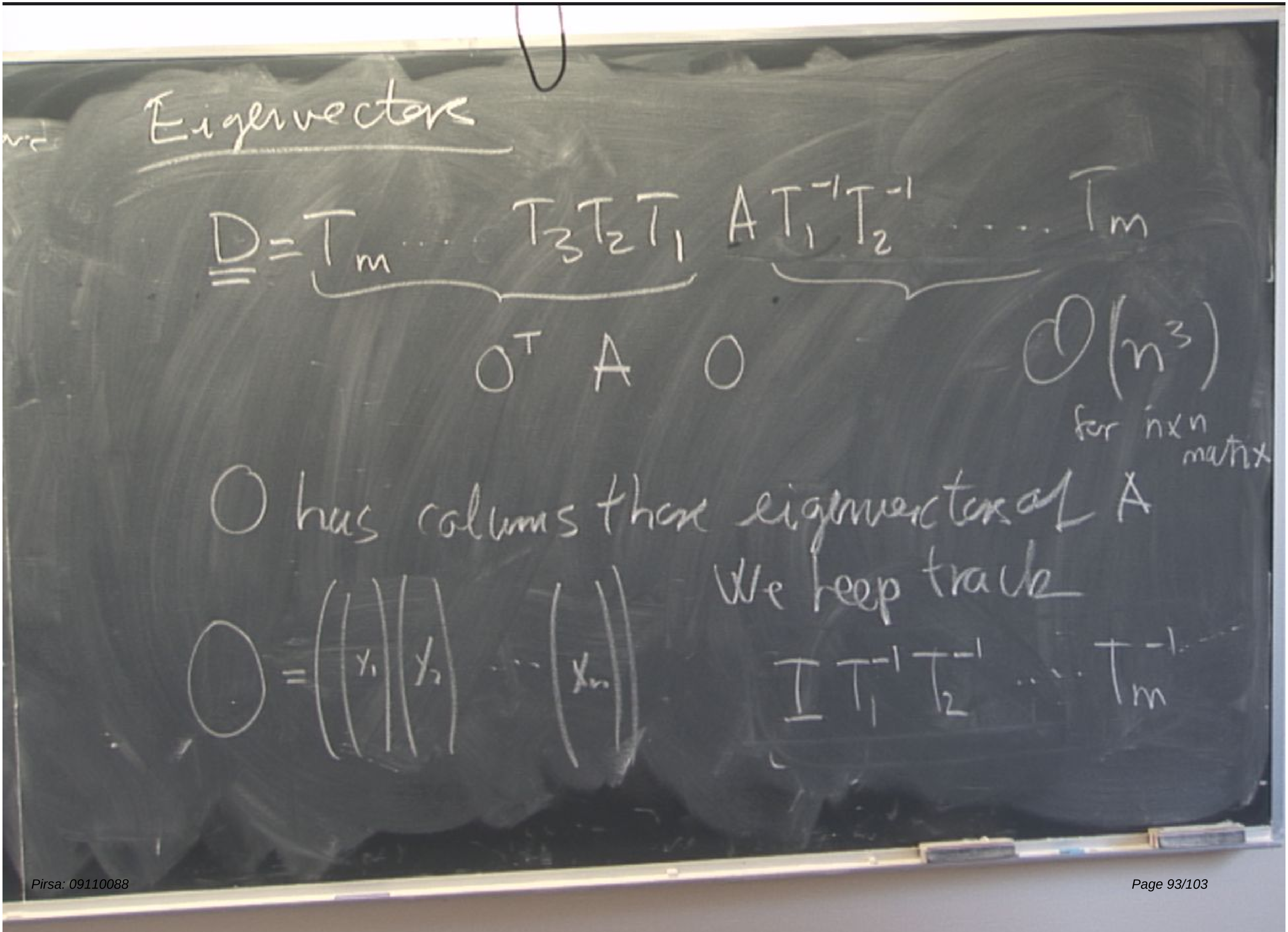
$$\underline{\underline{D}} = \underbrace{T_m \cdots T_3 T_2 T_1}_{} A \underbrace{T_1^{-1} T_2^{-1} \cdots T_m}_{}$$

$$O^T \; A \; O \qquad \mathcal{O}(n^3)$$

O has columns those eigenvectors of A

We keep track

$$O = \left(\left(y_1 \right) \left(y_2 \right) \cdots \left(x_n \right) \right) \qquad I \, T_1^{-1} T_2^{-1} \cdots T_m^{-1}$$



Eigenvectors

$$\underline{\underline{D}} = \underbrace{T_m \cdots T_3 T_2 T_1}_{O^T} \, A \, \underbrace{T_1^{-1} T_2^{-1} \cdots T_m}_{O}$$

$$O^T A \; O$$

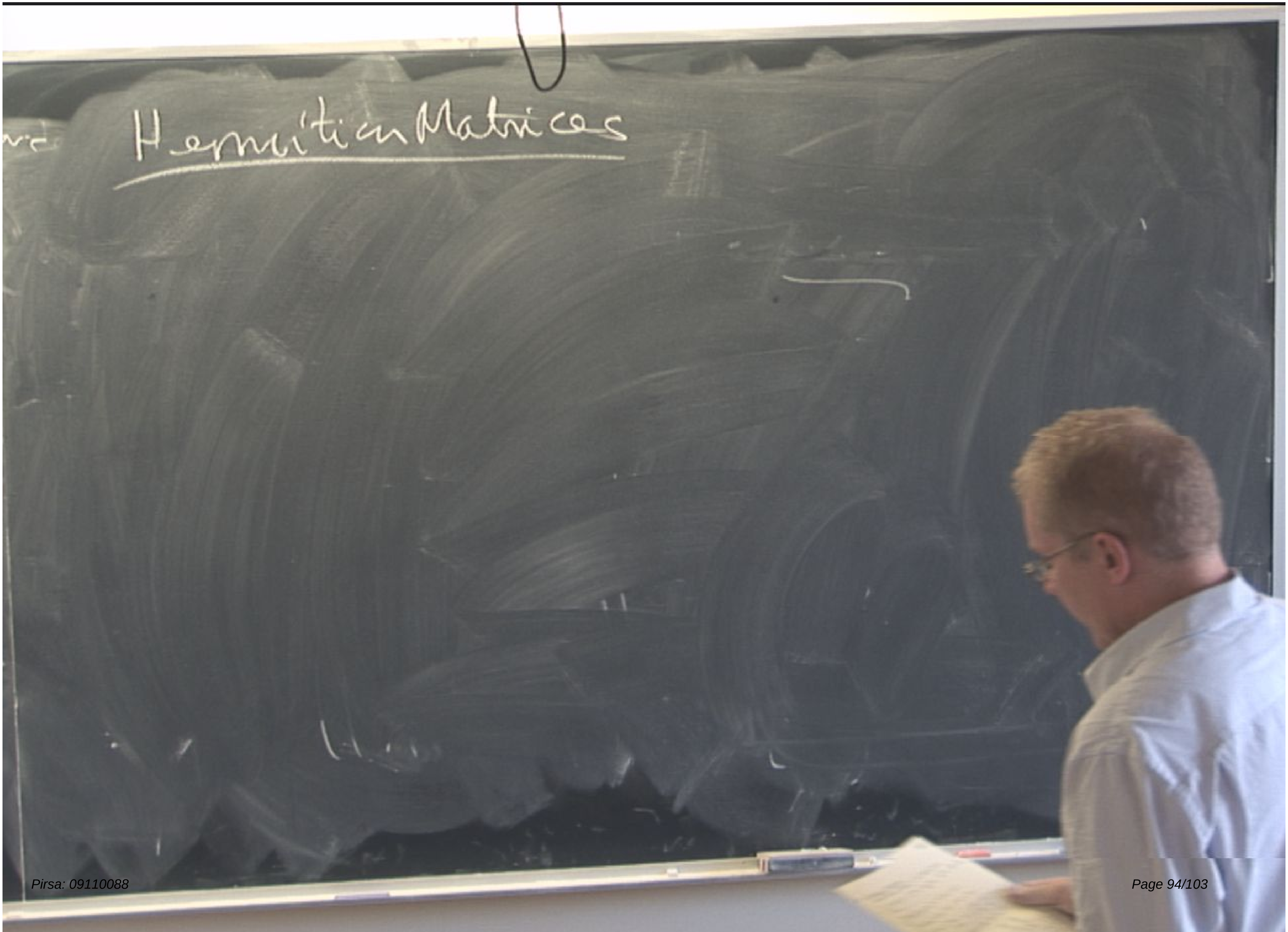
$\mathcal{O}(n^3)$ for $n \times n$ matrix

O has columns those eigenvectors of A

$$O = \left(\begin{array}{c|c|c|c} x_1 & x_2 & \cdots & x_n \end{array} \right)$$

We keep track

$$I \, T_1^{-1} T_2^{-1} \cdots T_m^{-1}$$



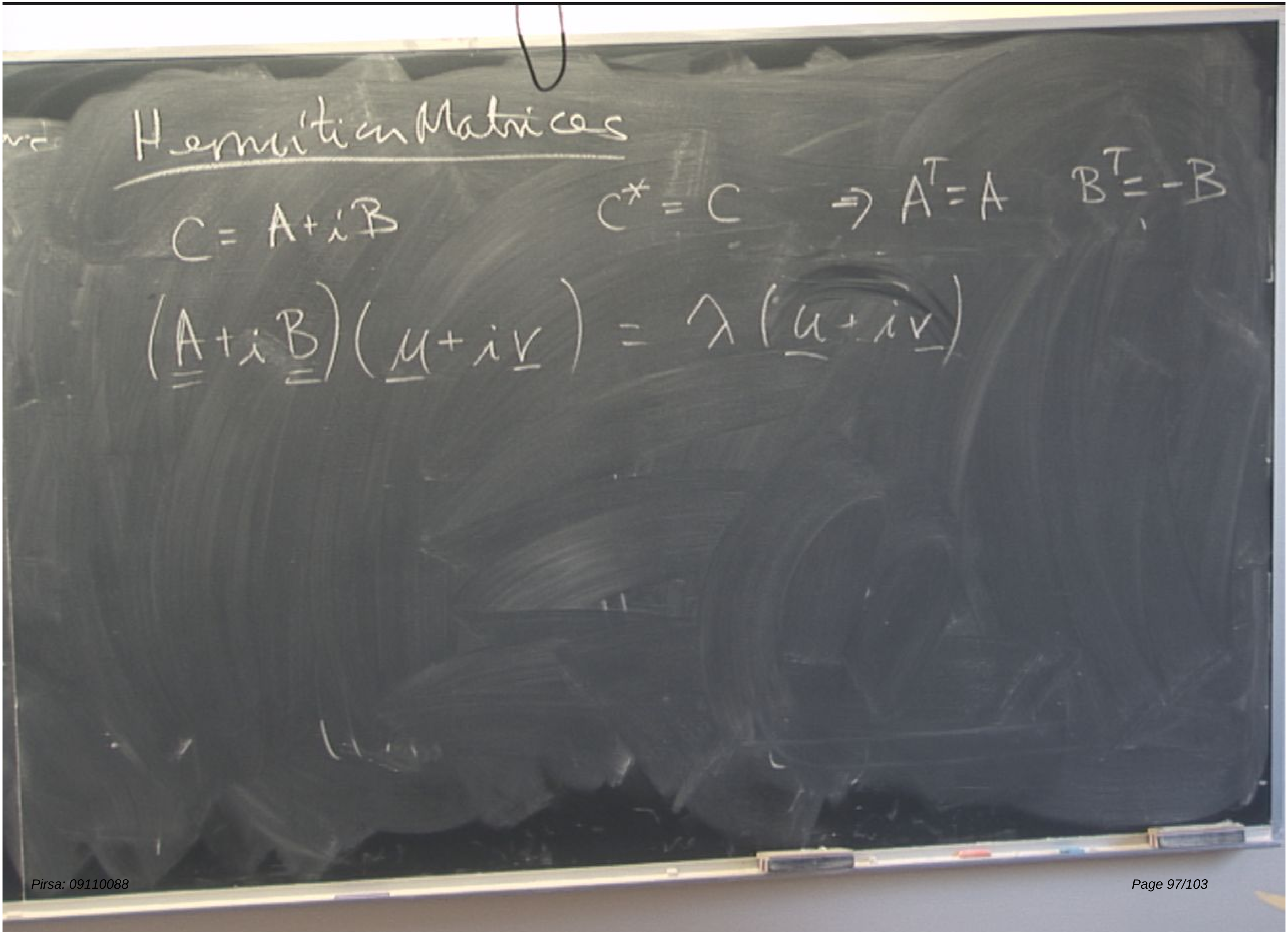
Hermitian Matrices

Hermitian Matrices

$$C = A + iB$$

Hermitian Matrices

$C = A + iB$ $\quad C^* = C \quad \Rightarrow A^T = A \quad B^T = -B$



Hermitian Matrices

$C = A + iB \qquad C^* = C \quad \Rightarrow A^T = A \qquad B^T = -B$

$$(\underline{\underline{A}} + i\underline{\underline{B}})(\underline{u} + i\underline{v}) = \lambda(\underline{u} + i\underline{v})$$

Hermitian Matrices

$$C = A + iB \qquad C^{*} = C \quad \Rightarrow A^{T} = A \quad B^{T} = -B$$

$$\left(\underline{\underline{A}} + i\underline{\underline{B}}\right)\left(\underline{u} + i\underline{v}\right) = \lambda\left(\underline{u} + i\underline{v}\right)$$

$$\Downarrow$$

$$\underline{\underline{A}}\,\underline{u} + i\underline{\underline{B}}\,\underline{u} + i\underline{\underline{A}}\,\underline{v} - \underline{\underline{B}}\,\underline{v} = \lambda\underline{u} + i\lambda\underline{v}$$

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$$C = A + iB \qquad C^* = C \quad \Rightarrow A^T = A \quad B^T = -B$$

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$$\Downarrow$$

$$\begin{aligned} \underline{\underline{A}}\,\underline{u} - \underline{\underline{B}}\,\underline{v} &= \lambda\underline{u} \\ \underline{\underline{B}}\,\underline{u} + \underline{\underline{A}}\,\underline{v} &= \lambda\underline{v} \end{aligned} \Rightarrow \left(\qquad \right)$$

Hermitian Matrices

$$C = A + iB \qquad C^* = C \Rightarrow A^T = A \quad B^T = -B$$

$$(\underline{\underline{A}} + i\underline{\underline{B}})(\underline{u} + i\underline{v}) = \lambda(\underline{u} + i\underline{v})$$

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$$\underline{\underline{A}}\,\underline{u} + i\underline{\underline{B}}\,\underline{u} + i\underline{\underline{A}}\,\underline{v} - \underline{\underline{B}}\,\underline{v} = \lambda\underline{u} + i\lambda\underline{v}$$

$$\Downarrow$$

$$\underline{\underline{A}}\,\underline{u} - \underline{\underline{B}}\,\underline{v} = \lambda\underline{u}$$

$$\underline{\underline{B}}\,\underline{u} + \underline{\underline{A}}\,\underline{v} = \lambda\underline{v}$$

$$\Rightarrow \underbrace{\begin{pmatrix} A & -B \\ B & A \end{pmatrix}}_{\uparrow \text{ real symmetric}} \begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} u \\ v \end{pmatrix}$$

Eigen Systems   Similarity transformat

if $\begin{pmatrix} u \\ v \end{pmatrix}$ is eigenvector then so is:
$\begin{pmatrix} -v \\ u \end{pmatrix}$

all eigenvalues occur twice

stems

Similarity transformation

Hermiti

$C = A + i$

is eigenvector then so is.

$(\underline{\underline{A}} + i\underline{\underline{B}})($

$\Downarrow$

$\underline{\underline{A}}\,\underline{u} + i\underline{\underline{B}}\,\underline{u}$

$\Downarrow$

$\underline{\underline{A}}\,\underline{u} - \underline{\underline{B}}\,\underline{v}$

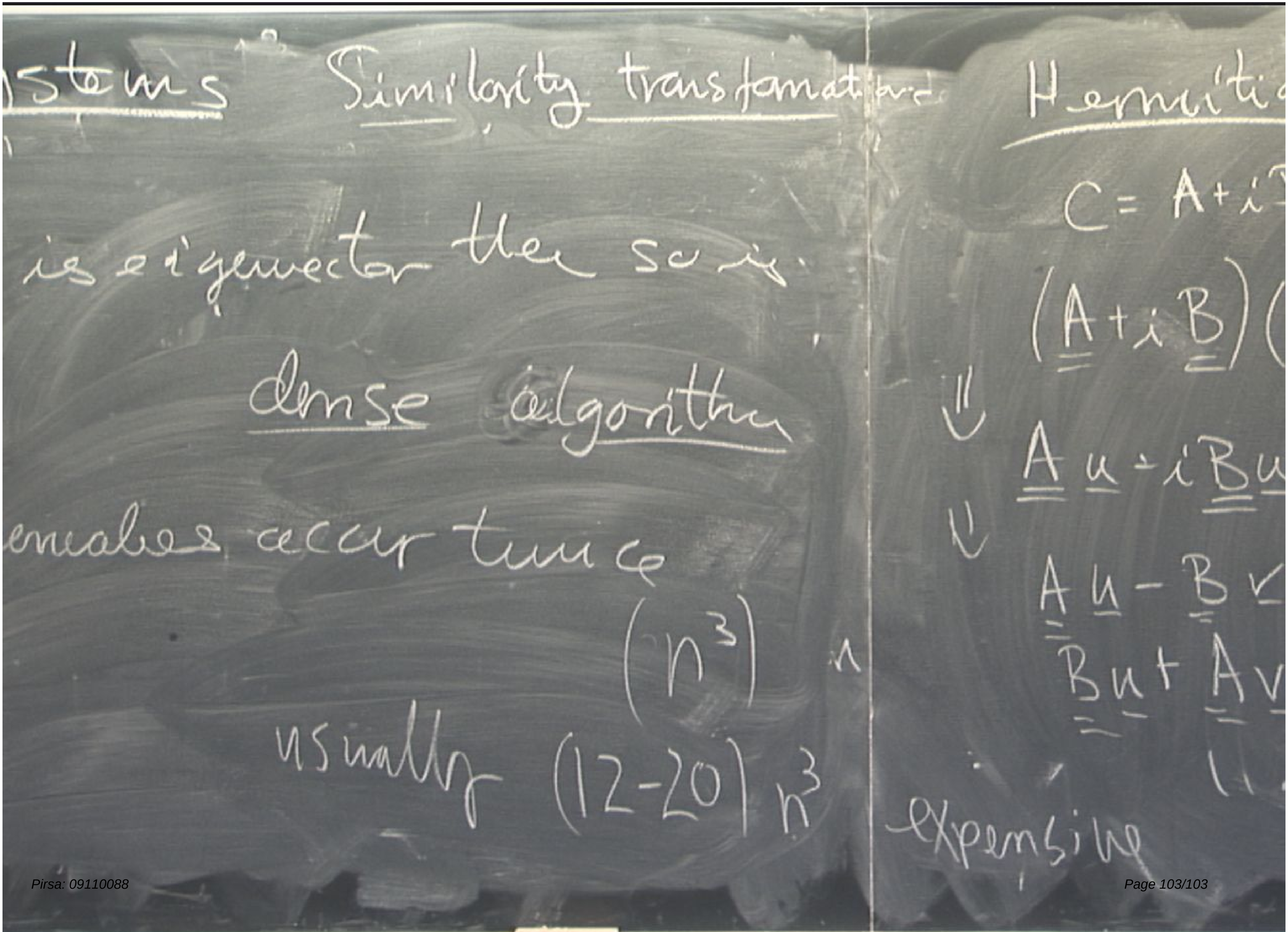
$\underline{\underline{B}}\,\underline{u} + \underline{\underline{A}}\,\underline{v}$

envalues occur twice

(n^3) ∧

usually $(12\text{-}20)\, n^3$

expensive



stems

Similarity transformat

Hermiti

is eigenvector then so is

dense algorithm

envalues occur twice

(n^3) ^

usually $(12-20)\,n^3$

$C = A + i$

$(\underline{\underline{A}} + i\underline{\underline{B}})($

$\Downarrow$

$\underline{\underline{A}}\,\underline{u} + i\underline{\underline{B}}\,\underline{u}$

$\Downarrow$

$\underline{\underline{A}}\,\underline{u} - \underline{\underline{B}}\,\underline{v}$

$\underline{\underline{B}}\,\underline{u} + \underline{\underline{A}}\,\underline{v}$

expensive