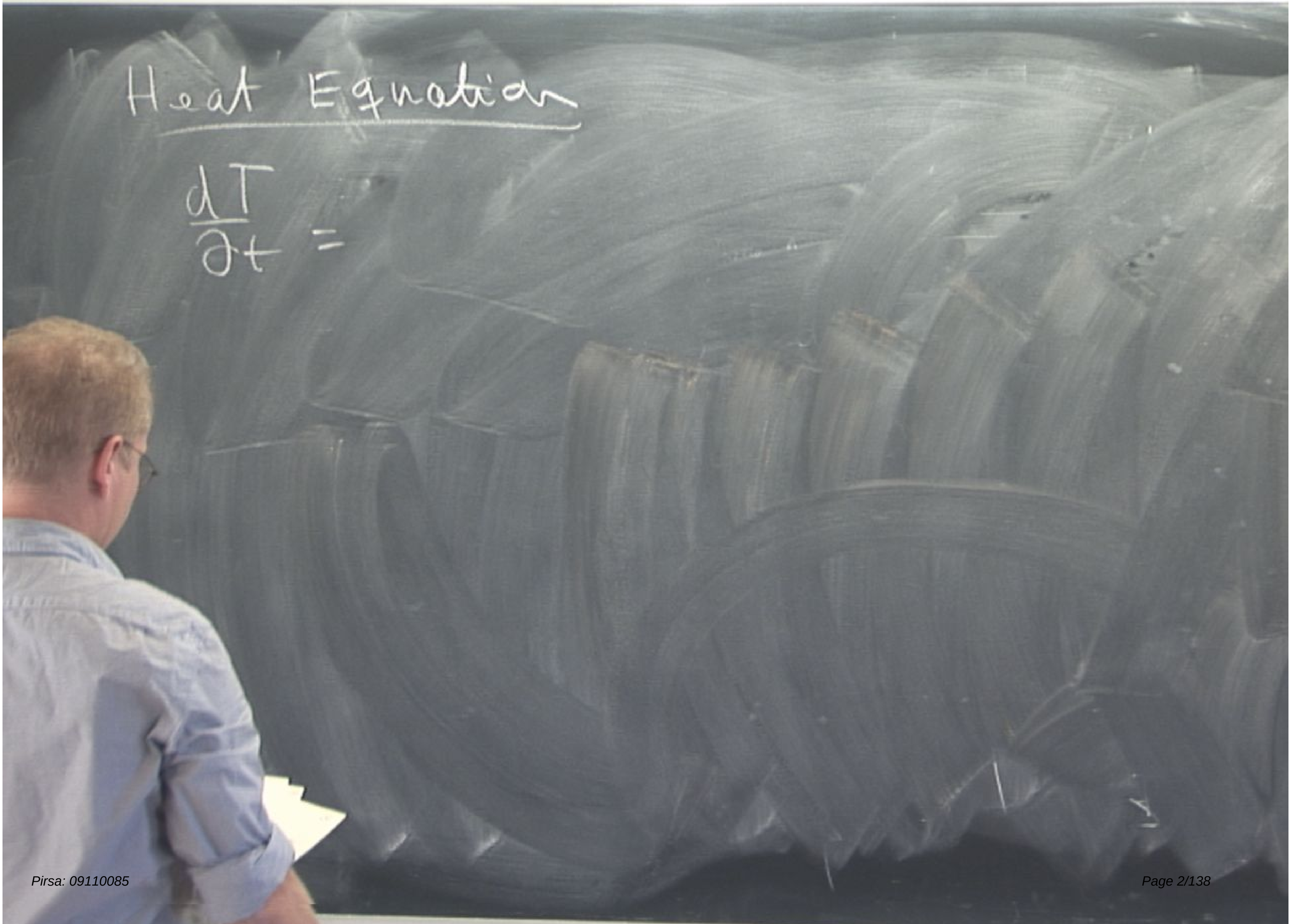
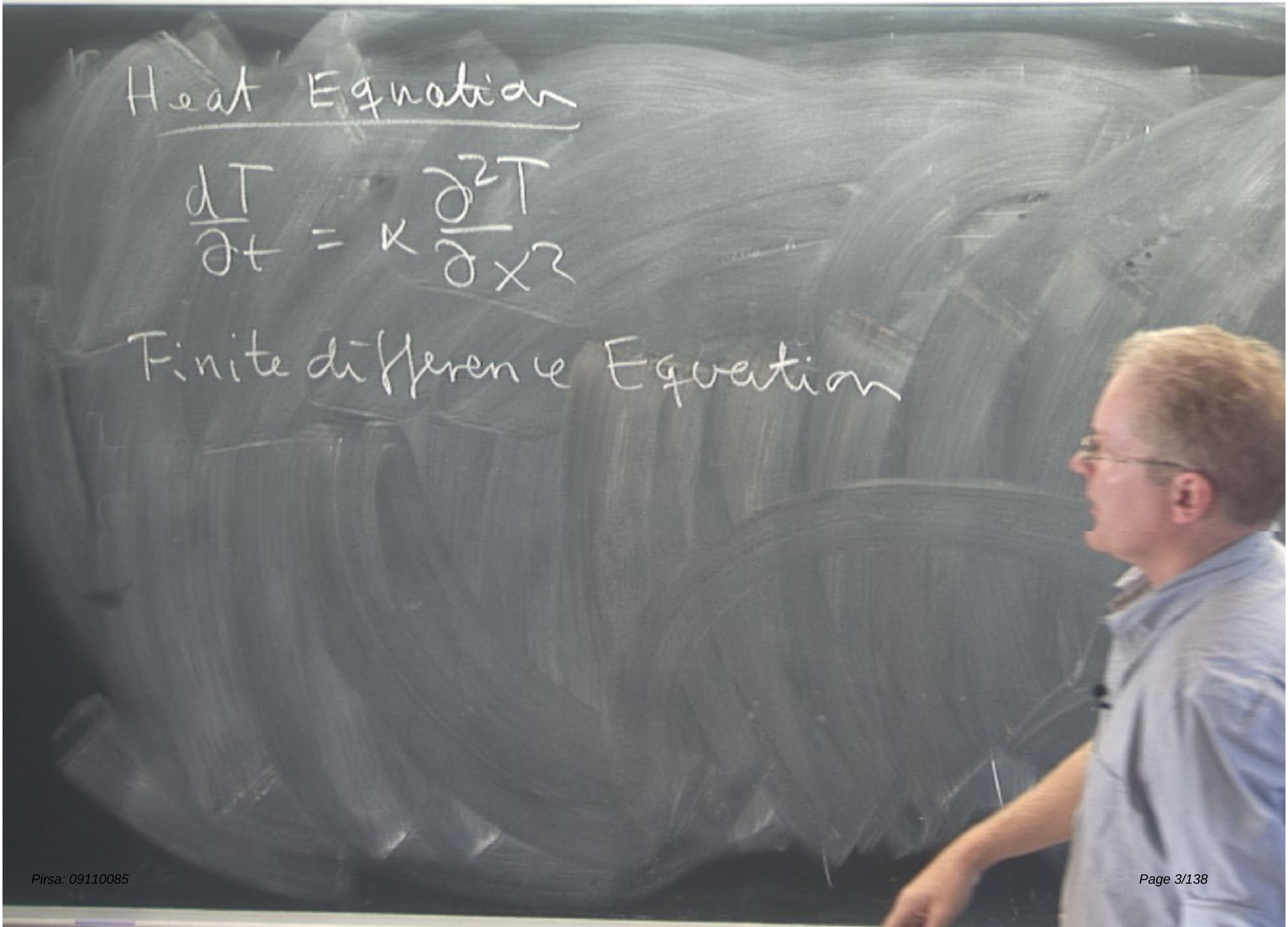
Title: Scientific Computation (PHYS 608) - Lecture 10

Date: Nov 06, 2009 10:30 AM

URL: http://pirsa.org/09110085

Abstract:


Heat Equation
$\frac{dT}{\partial t} =$


Heat Equation
$\frac{dT}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2}$
Finite difference Equation

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Finite difference Equation

$$T_{i,j+1} = T_{i,j} + \eta\left[T_{i+1,j} + T_{i-1,j} - 2T_{i,j}\right]$$

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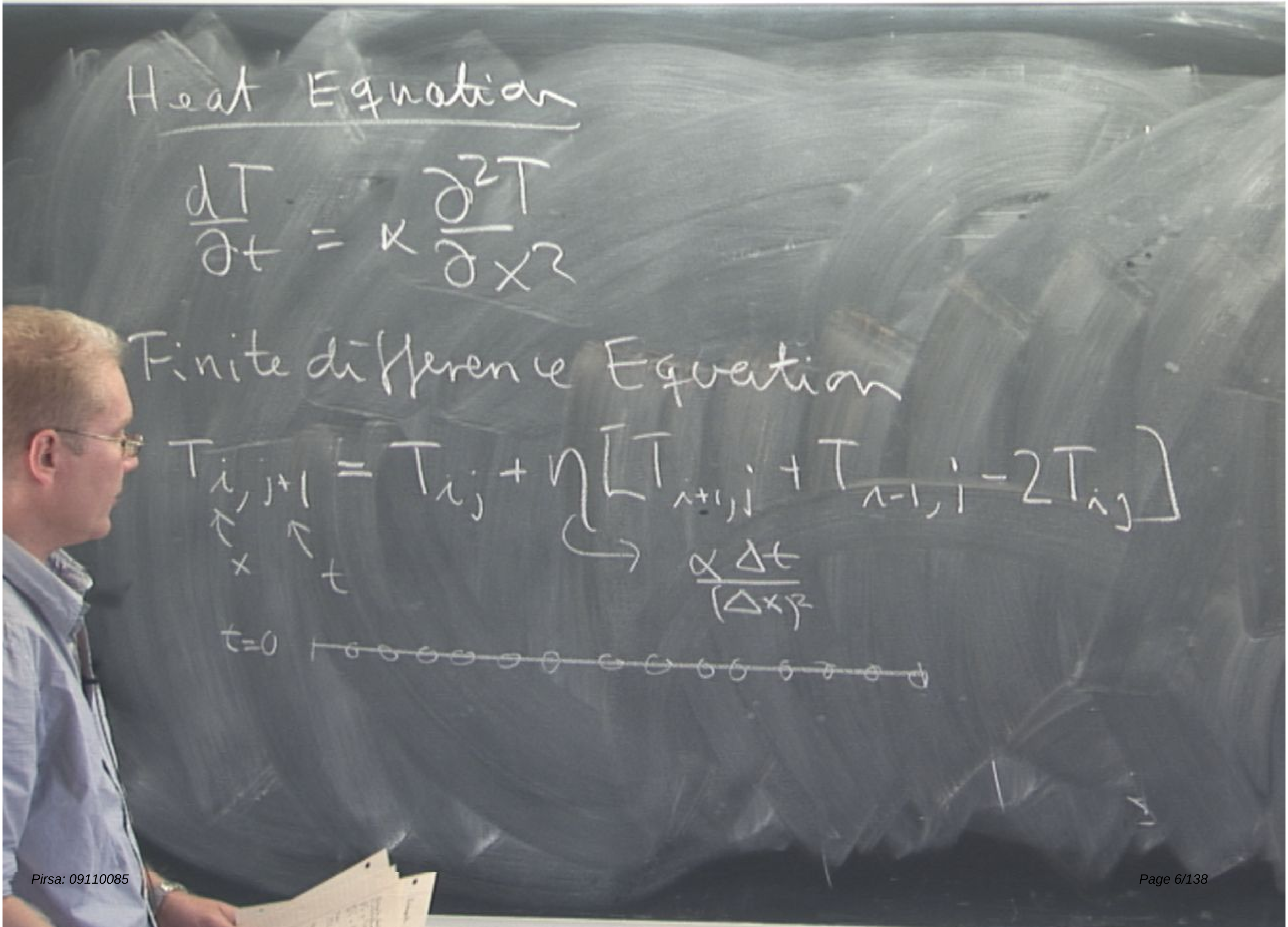
Heat Equation
$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$
Finite difference Equation
$T_{i,j+1} = T_{i,j} + \eta [T_{i+1,j} + T_{i-1,j} - 2T_{i,j}]$
x
t
$\frac{\alpha \Delta t}{(\Delta x)^2}$
t=0

Heat Equation

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Finite difference Equation

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x t

$$\eta \longrightarrow \frac{\alpha \Delta t}{(\Delta x)^2}$$

t=0

j=0

j=1

Heat Equation

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Finite difference Equation

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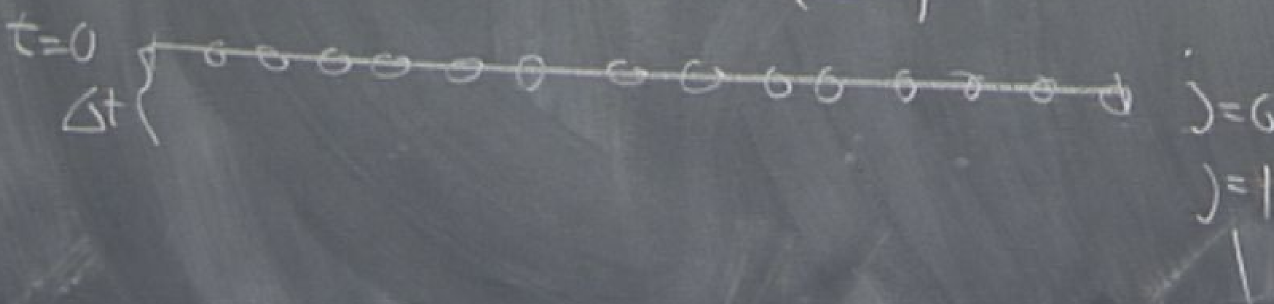
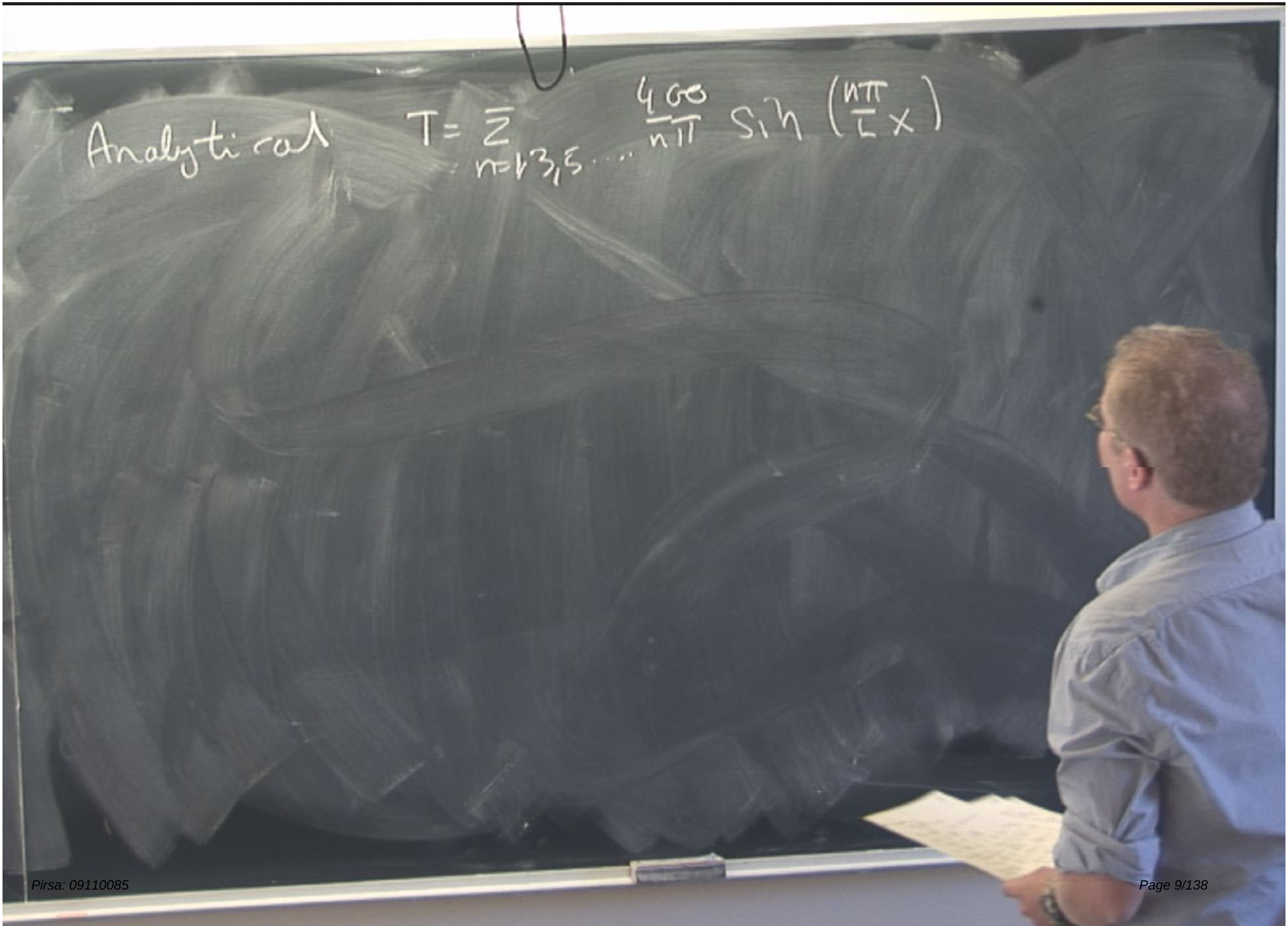
x t

$$\eta = \frac{\alpha \Delta t}{(\Delta x)^2}$$




Analytical
$T = \sum_{n=1,3,5\ldots} \frac{400}{n\pi} \sin\left(\frac{n\pi}{L}x\right)$

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Would like: Finite difference equation to approach Analytical solution

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Stability Analysis

Analytical : $T = \sum_{n=1,3,5\ldots} \frac{400}{n\pi} \sin\left(\frac{n\pi}{L} x\right) e^{-\frac{n^2\pi^2}{L^2}\alpha t}$

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Stability Analysis : Heuristic Form
von Neumann

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Assumption

Analytical : $T = \sum_{n=1,3,5\ldots} \frac{400}{n\pi} \sin\left(\frac{n\pi}{L} x\right) e^{-\frac{n^2\pi^2}{L^2}\alpha t}$

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Stability Analysis : Heuristic Form
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Assumption: $\xi(k)\, e^{ikm\Delta x}$

Analytical : $T = \sum_{n=1,3,5,\dots} \frac{400}{n\pi} \sin\left(\frac{n\pi}{L}x\right) e^{-\frac{n^2\pi^2}{L^2}\alpha t}$

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Assumption : $(\xi(k))^j e^{ikm\Delta x}$ $\sqrt{-1}$ $x = m\Delta x$

Analytical : $T = \sum_{n=1,3,5\ldots} \frac{400}{n\pi} \sin\left(\frac{n\pi}{L} x\right) e^{-\frac{n^2\pi^2}{L^2}\alpha t}$

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Assumption: $\left(\xi(k)\right)^j e^{ikm\Delta x}$ $\quad x = m\Delta x$

$i \to \sqrt{-1}$

$\xi(k) \to$ amplitude

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Stick into

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Would like: Finite difference equation to approach Analytical solution

Stability Analysis : Heuristic Form

von Neumann

Assumption: $\underbrace{(\xi(k))^j}_{\text{amplitude}} e^{\underset{\sqrt{-1}}{i}km\Delta x}$ $\qquad x = m\Delta x$

Stick into (*)

$\xi^{j+1} e^{ikm\Delta x} =$

Analytical : $T = \sum_{n=1,3,5\ldots} \frac{400}{n\pi} \sin\left(\frac{n\pi}{L}x\right) e^{-\frac{n^2\pi^2}{L^2}\alpha t}$

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$$\xi^{j+1} e^{ikm\Delta x} = \xi^j e^{ikm\Delta x} + \eta\left[\xi^j e^{ik(m+1)\Delta x} + \xi^j e^{ik(m-1)\Delta x}\right.$$

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Heat Equation

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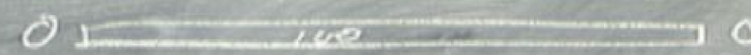

Finite difference Equation

$$\Rightarrow \quad \xi(k) = 1 + 2\eta\left(\cos(k\Delta x) - 1\right)$$

Heat Equation

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We want is $|\xi| < 1$

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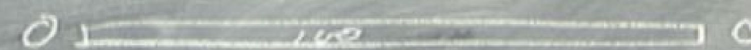

Finite difference Equation

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$$2\eta < 1 \quad \Rightarrow$$

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Finite difference Equation

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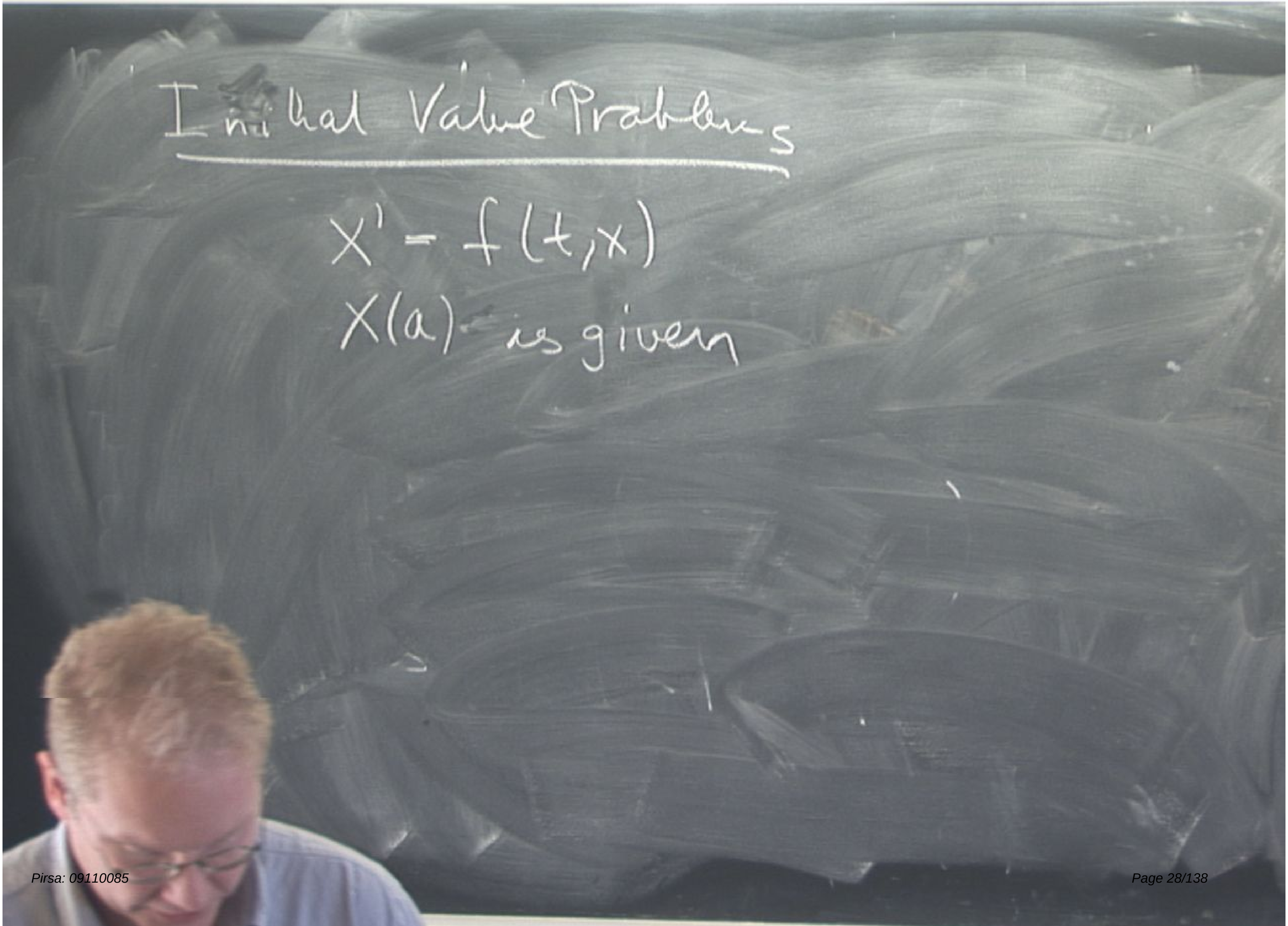
We want is $|\xi| < 1$ for all k

$$2\eta < 1 \Rightarrow \alpha \frac{\Delta t}{(\Delta x)^2} < \frac{1}{2}$$


Initial Value Problems
$x' = f(t,x)$
$x(a)$ is given

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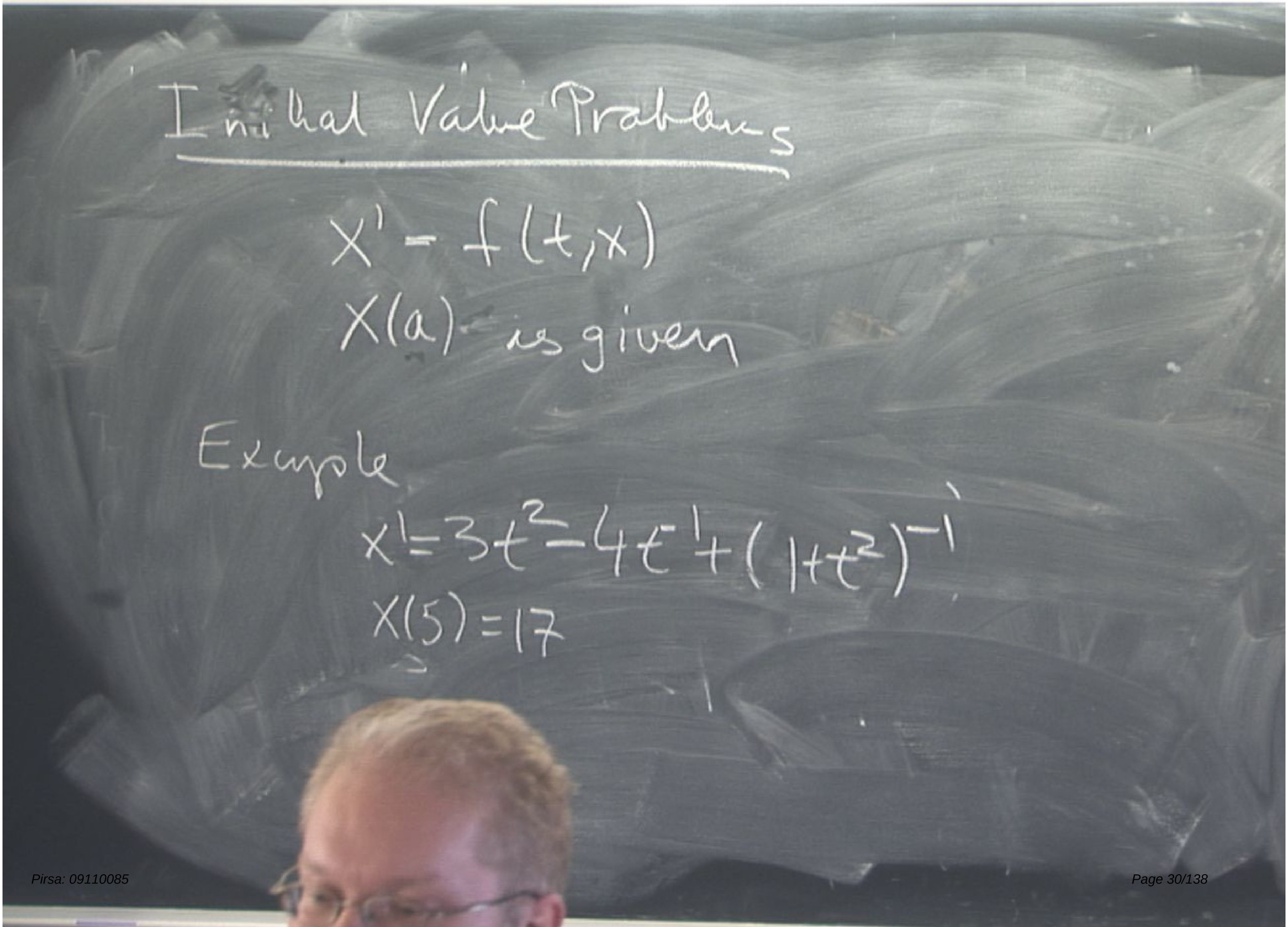
$$x' = f(t,x)$$

$x(a)$ is given

Example

$$x' = 3t^2 - 4t^{-1} + (1+t^2)^{-1}$$


Initial Value Problems
$x' = f(t,x)$
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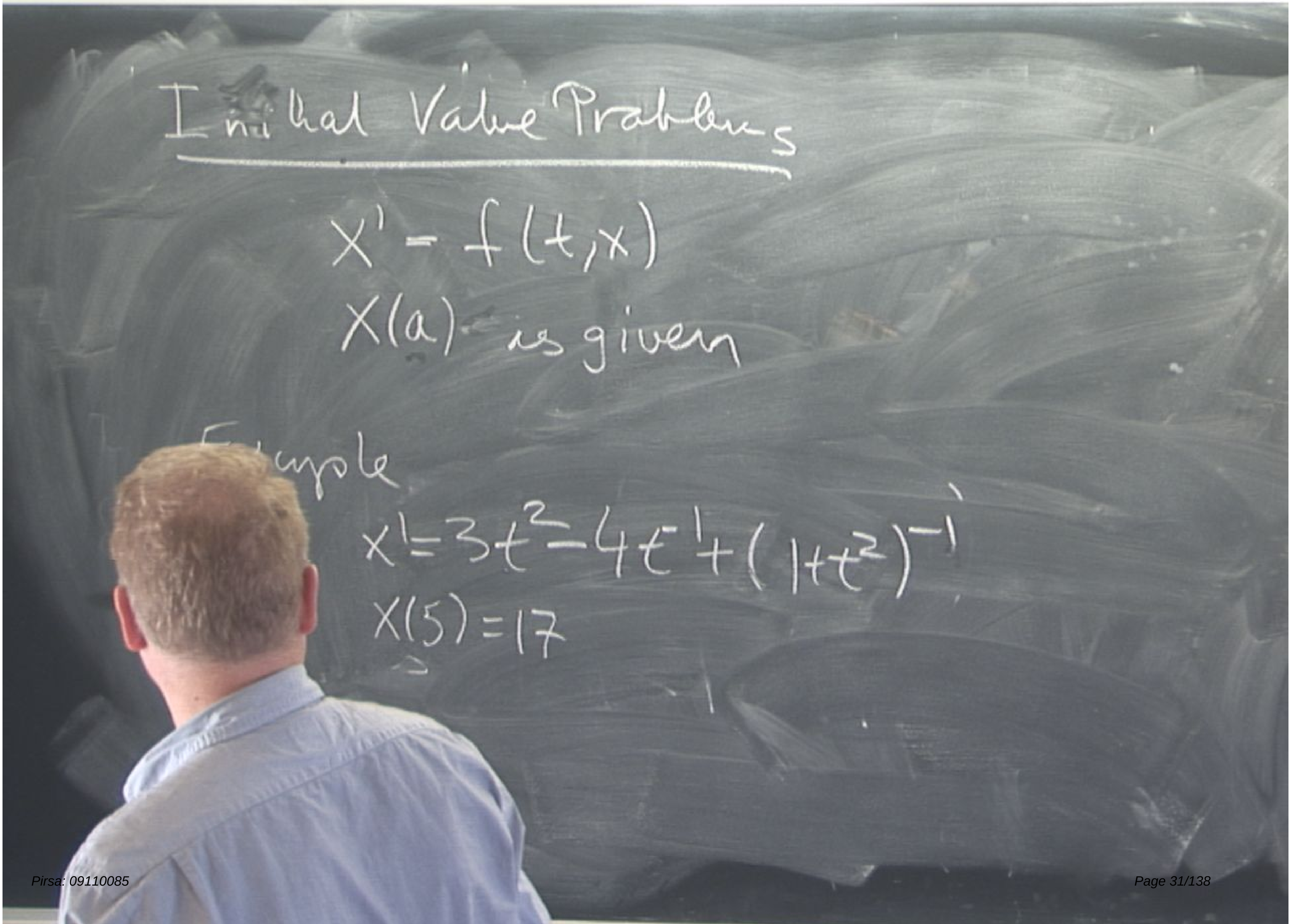
Example
$x' = 3t^2 - 4t^{-1} + (1+t^2)^{-1}$
$x(5) = 17$


Initial Value Problems
$x' = f(t,x)$
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Initial Value Problems

$$x' = f(t,x)$$

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Example

$$\left.\begin{aligned} x' &= 3t^2 - 4t^{-1} + (1+t^2)^{-1} \\ x(5) &= 17 \end{aligned}\right\} \quad x = ?$$

[Initial] Value Problems

$$x' = f(t,x)$$

$x(a)$ is given

[E]xample

$$\left.\begin{array}{l} x' = 3t^2 - 4t^{-1} + (1+t^2)^{-1} \\ x(5) = 17 \end{array}\right\} \quad x = t^3 - 4\ln t + \tan^{-1}$$

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...ow about $x' = 3|\cos(|t|)|$

A

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...ut

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...itial Value Problems 1st order

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...ow about

$$x' = 3|\cos(|t|)|$$
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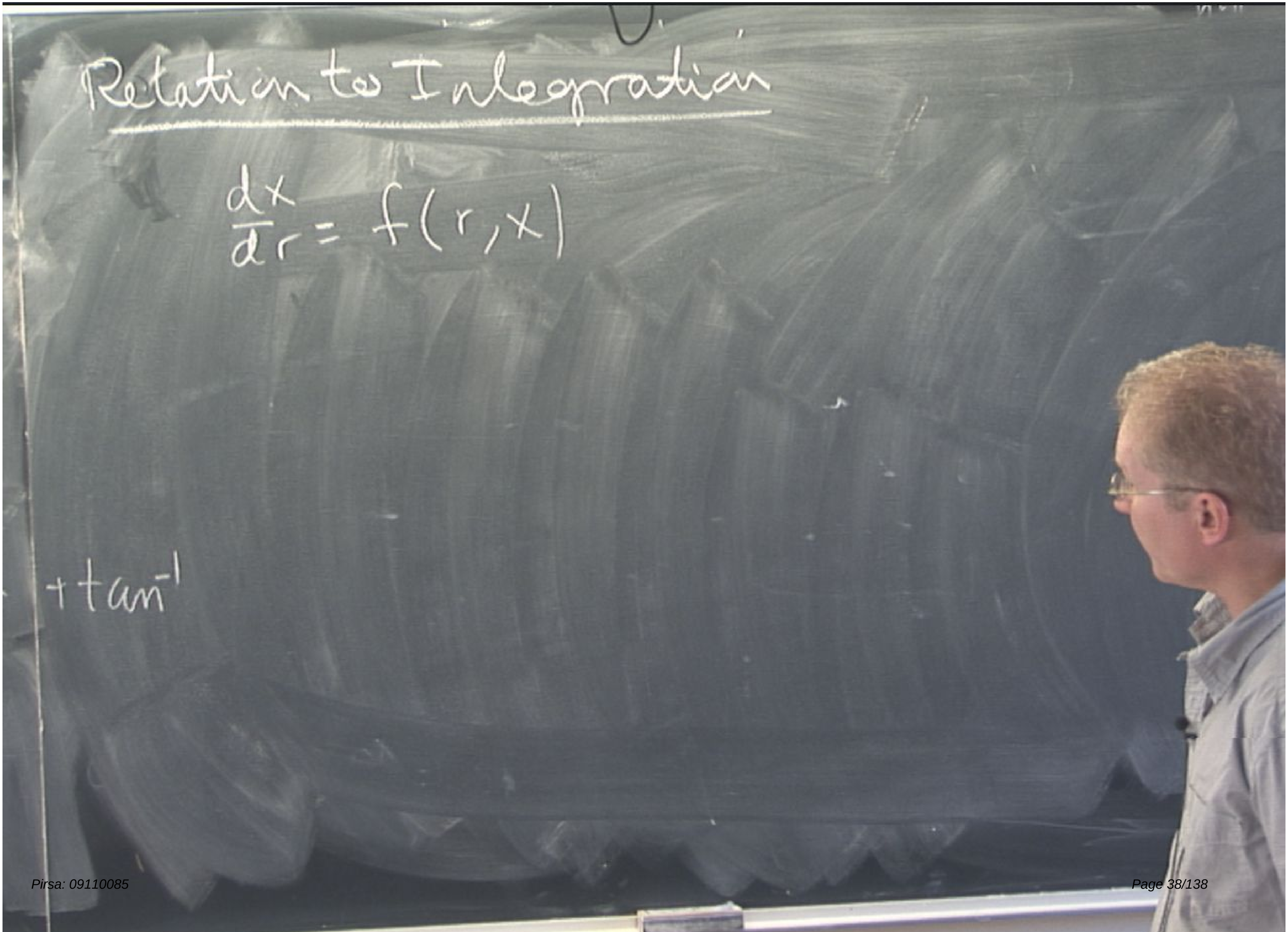
Ret...


Relation to Integration
$\frac{dx}{dr} = f(r,x)$
$+\tan^{-1}$

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$$\frac{dx}{dr} = f(r,x)$$

$$x(a) = s$$

$+\tan^{-1}$

Relation to Integration

$$\left.\begin{array}{l}\dfrac{dx}{dr} = f(r,x)\\[4pt] x(a) = s\end{array}\right\} \Rightarrow \int_t^{t+h} dx = \int$$

$+\tan^{-1}$

cation to Integration

$$\left.\begin{array}{l} \dfrac{dx}{dr} = f(r,x) \\ x(a) = s \end{array}\right\} \Rightarrow \int_t^{t+h} dx = \int_t^{t+h} f(r, x(r))\, dr$$

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$$x(t+h) = x(t) + \int_t^{t+h} f(r, x(r))\,dr$$

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← Integration

Eulers Method

$$x(t+h) = x(t)$$

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Eulers Method

$$x(t+h) = x(t) + h\,f(t,x(t))$$

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...ulers Method

$$x(t+h) \simeq x(t) + h\,f(t,x(t))$$

...his is called an explicit method

ation to Integration

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$$x(t+h) = x(t) + \underbrace{\int_t^{t+h} f(r, x(r))\, dr}_{\text{Integration}}$$

Eulers Method

$$x(t+h) = \boxed{x(t) + h f(t, x(t))}$$

his is called an explicit method

...ation to Integration

$$\left.\begin{aligned}\frac{dx}{dr} &= f(r,x)\\ x(a) &= s\end{aligned}\right\} \Rightarrow \int_t^{t+h} dx = \int_t^{t+h} f(r,x(r))\,dr$$

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← Integration

Eulers Method

$$x(t+h) = \boxed{x(t) + h f(t,x(t))}$$

only depends on $x(t)$

...his is called an explicit method

Initial Value Problems 1st order

$$x' = f(t,x)$$

$x(a)$ is given

Another derivation of Euler

Initial Value Problems   1st order

$$x' = f(t, x)$$

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Another derivation of Euler

$$\frac{dx}{dt} = \frac{x(t+h) - x(t)}{h} = f(t, x(t))$$

Initial Value Problems   1st order

$$x' = f(t,x)$$

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Another derivation of Euler

$$\frac{dx}{dt} = \frac{x(t+h)-x(t)}{h} = f(t, x(t))$$

$$x(t+h) = x(t) + h\,f(t, x(t))$$

$n^2\pi^2$

Relation to Integration

$$\left.\begin{array}{l} \frac{dx}{dr} = f(r,x) \\ x(a) = s \end{array}\right\} \Rightarrow \int_t^{t+h} dx = \int_t^{t+h} f(r,x(r))\,dr$$

$$x(t+h) = x(t) + \int_t^{t+h} f(r,x(r))\,dr$$

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Eulers Method

$$x(t+h) = \boxed{x(t) + h f(t,x(t))}$$

only depends on $x(t)$

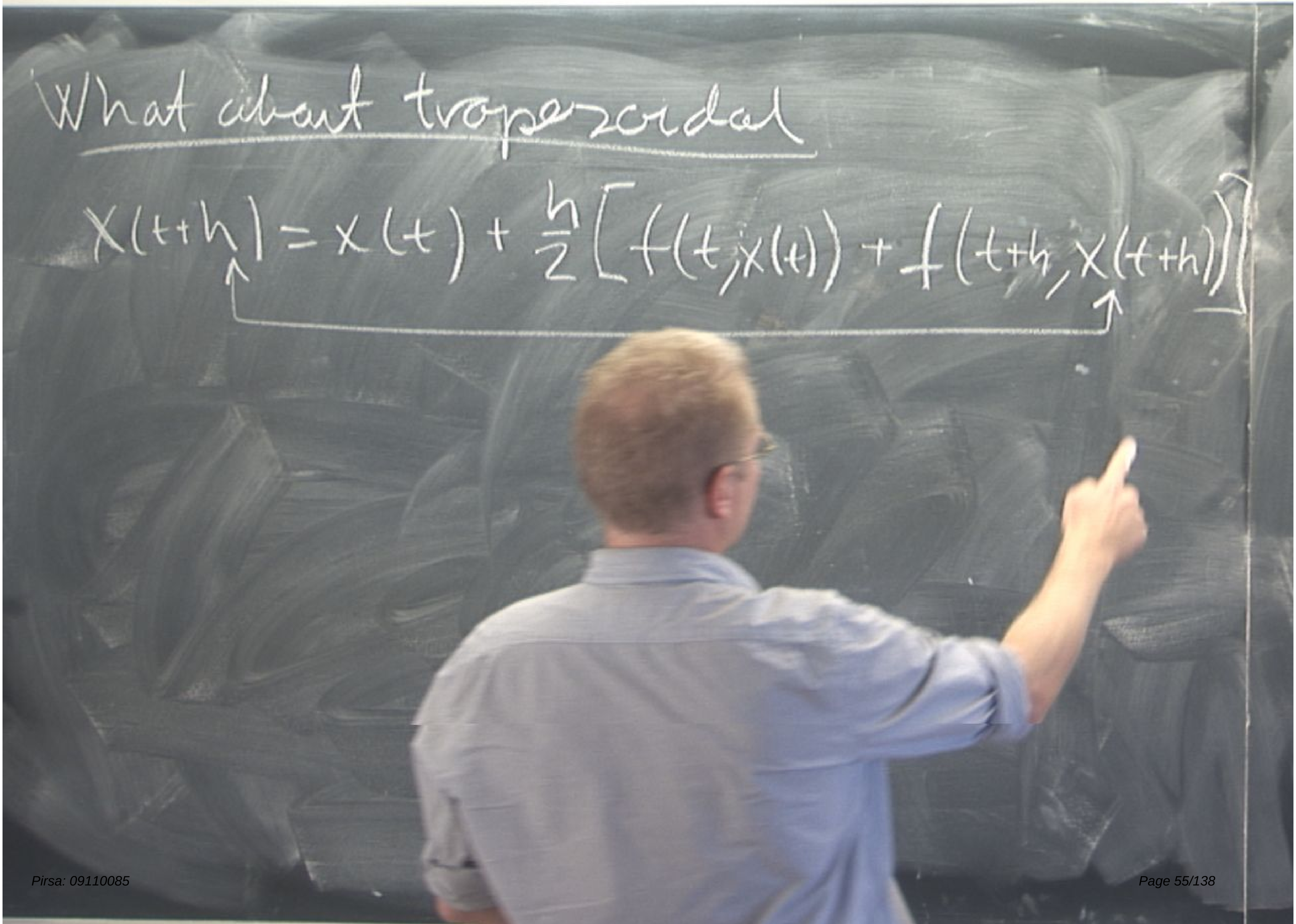
This is called an explicit method

What about trapezoidal

$$x(t+h) = x(t)$$

What about trapezoidal

$$x(t+h) = x(t) + \frac{h}{2}\left[f(t, x(t)) + f(t+h, x(t+h))\right]$$


What about tropezoidal
$x(t+h) = x(t) + \frac{h}{2}\left[f(t,x(t)) + f(t+h,x(t+h))\right]$

What about trapezoidal

$$x(t+h) = x(t) + \frac{h}{2}\left[f(t, x(t)) + f(t+h, x(t+h))\right]$$

Implicit method

What about trapezoidal

$$x(t+h) = x(t) + \frac{h}{2}\left[f(t, x(t)) + f(t+h, x(t+h)) \right]$$

Implicit method

Iterate to self consistency

$n^2\pi^2$

Picard's Method

$$x^{(1)}(t+h) = x(t) + \frac{h}{2}[f(t, x(t)] + f(t+h$$

$n^2\pi^2$

Picard's Method

$$x^{(1)}(t+h) = x(t) + \frac{h}{2}\left[f(t, x(t)) + f(t+h, x^{(0)}(t))\right]$$

$n^2\pi^2$

Picard's Method

$$x^{(1)}(t+h) = x(t) + \frac{h}{2}\left[f(t, x(t)) + f(t+h, x^{(0)}(t))\right]$$

Then

$$x^{(2)}(t+h) = x(t) + \frac{h}{2}\left[f(t, x(t)) + f(t+h,\right.$$

$n^2\pi^2$

Picard's Method

$$x^{(1)}(t+h) = x(t) + \frac{h}{2}\left[f(t,x(t)) + f(t+h, x^{(0)}(t))\right]$$

Then

$$x^{(2)}(t+h) = x(t) + \frac{h}{2}\left[f(t,x(t)) + f(t+h, x^{(1)}(t+h))\right]$$

$$\vdots$$

$$x^{(n)}(t+h) = x(t) + \frac{h}{2}\left[f(t,x(t)) + f(t+h, x^{(n-1)}(t+h))\right]$$

$n^2\pi^2$

Picard's Method

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$$x^{(n)}(t+h) = x(t) + \frac{h}{2}\left[f(t,x(t)) + f(t+h, x^{(n-1)}(t+h))\right]$$

$$\vdots$$

$$x^*(t+h) = x(t) + \frac{h}{2}\left[f(t,x(t)) + f(t+h, x^*(t+h))\right]$$

$n^2\pi^2$

Picard's Method

$$x^{(1)}(t+h) = x(t) + \frac{h}{2}\left[f(t, x(t)) + f(t+h, x^{(0)}(t))\right]$$

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$$x^{(2)}(t+h) = x(t) + \frac{h}{2}\left[f(t, x(t)) + f(t+h, x^{(1)}(t+h))\right]$$

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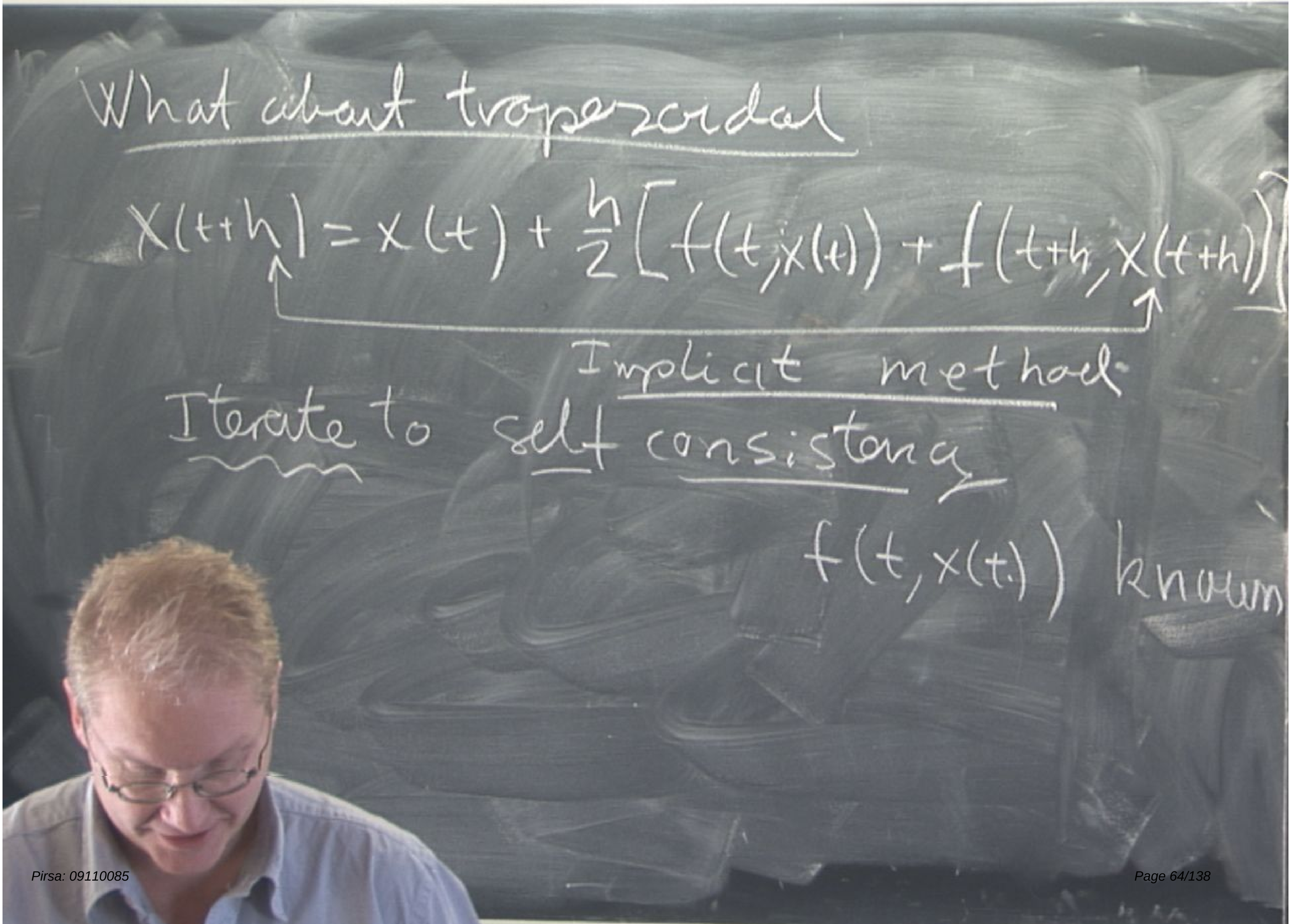
$$x^{*}(t+h) = x(t) + \frac{h}{2}\left[f(t, x(t)) + f(t+h, x^{*}(t+h))\right]$$


What about trapezoidal
$x(t+h) = x(t) + \frac{h}{2}\left[f(t,x(t)) + f(t+h, x(t+h))\right]$
Implicit method
Iterate to self consistency
$f(t,x(t))$ known

What about trapezoidal

$$x(t+h) = x(t) + \frac{h}{2}\left[f(t, x(t)) + f(t+h, x(t+h))\right]$$

Implicit method

Iterate to self consistency

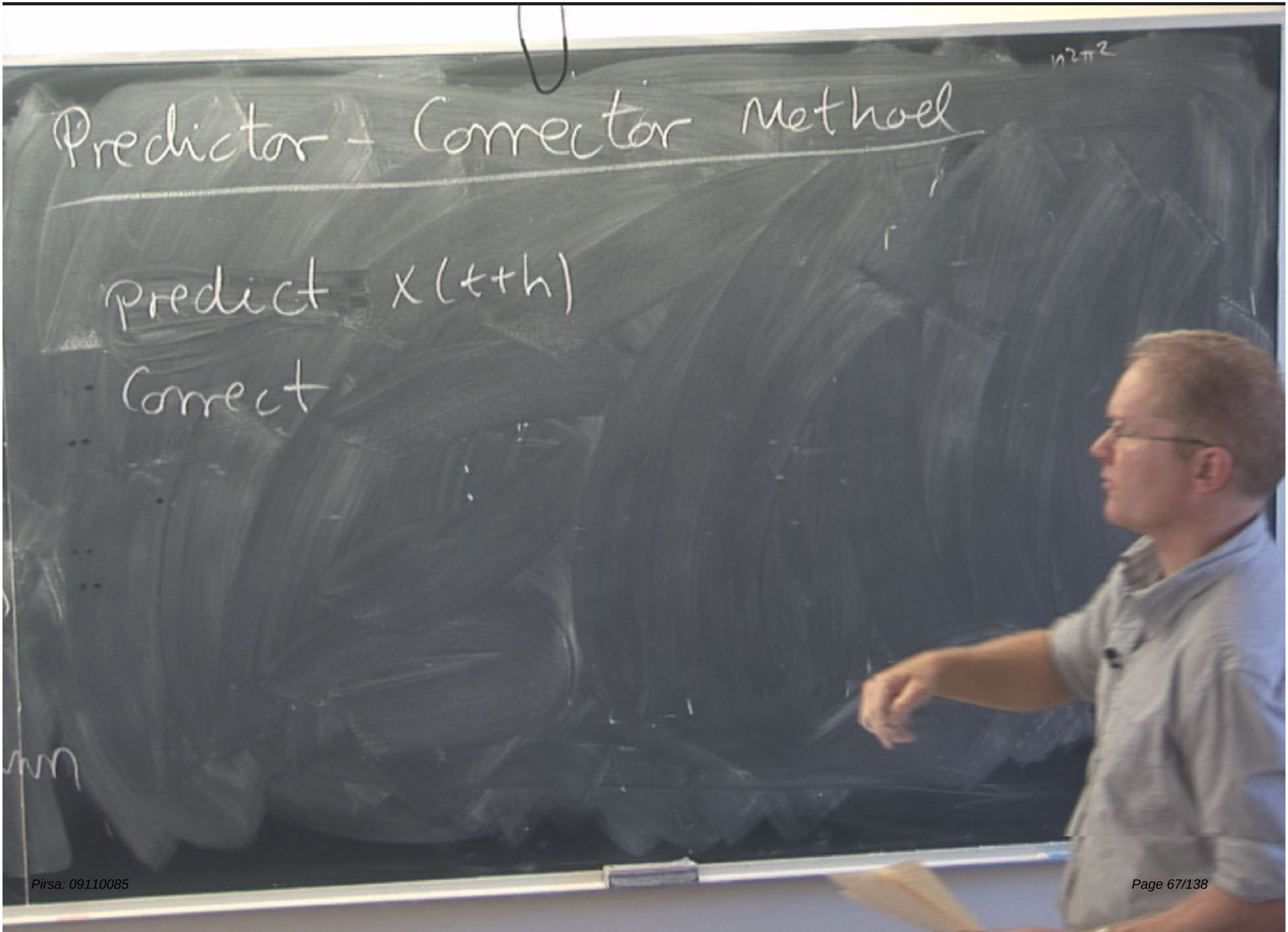
$f(t, x(t))$ known

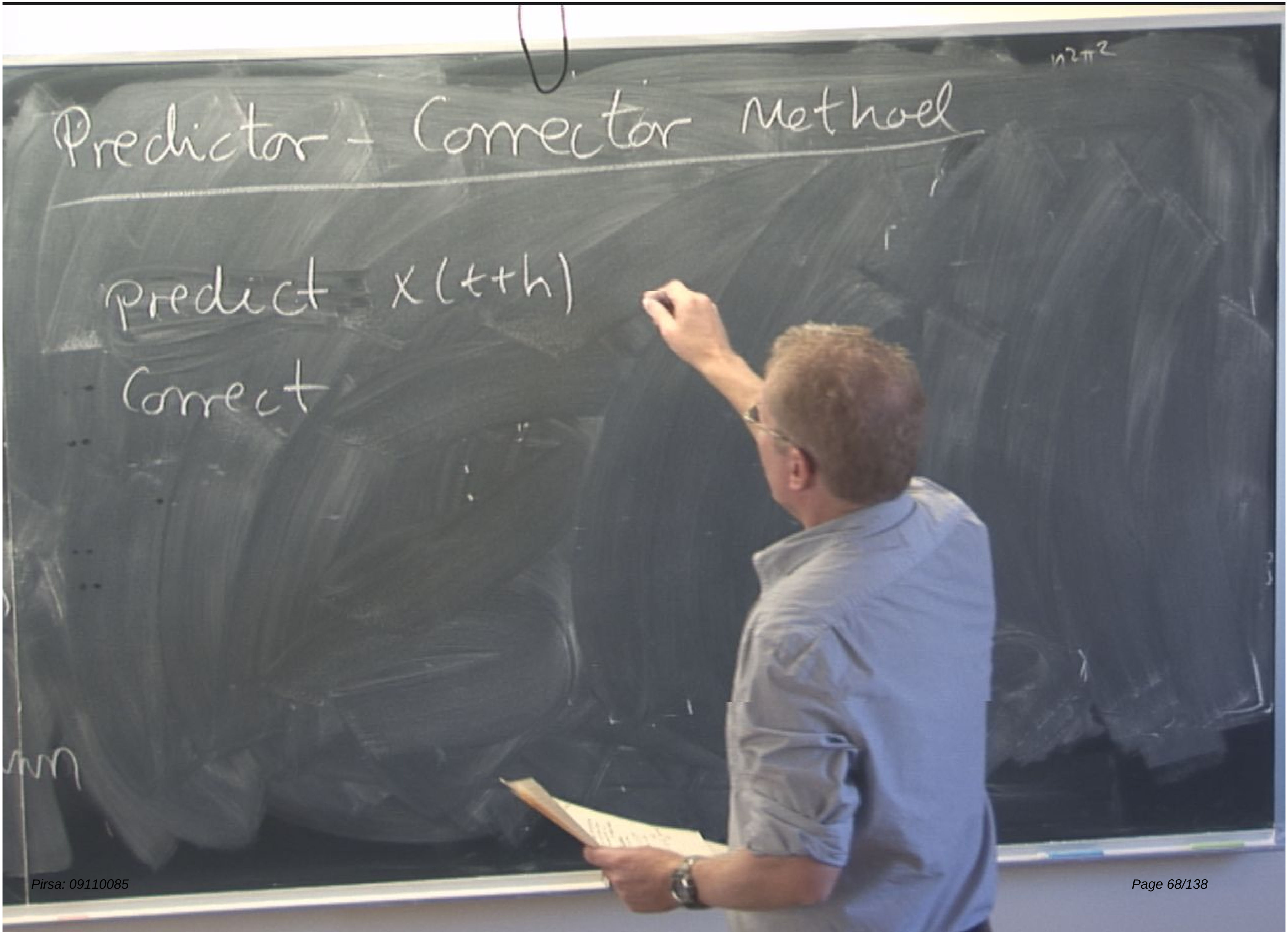
$x(t)$ is known

$x(t+h)$ is not kn

$n^2\pi^2$

Predictor - Corrector Method


Predictor - Corrector Method
Predict x(t+h)
Correct


n²π²
Predictor - Corrector Method
predict x(t+h)
Correct

Predictor - Corrector Method

$n^2\pi^2$

Euler

Predict $x(t+h) = x(t) + h f(t, x(t))$

Correct

Predictor - Corrector Method

Euler

Predict: $x(t+h) = x(t) + h f(t, x(t))$

Correct $x(t+h) = x(t) + \frac{h}{2}[f(t, x(t)) +$

$n^2\pi^2$

Predictor - Corrector Method

Euler

Predict: $x(t+h) = x(t) + h\, f(t, x(t))$

Correct $x(t+h) = x(t) + \frac{h}{2}\left[f(t, x(t)) + f(t+h, \quad) \right]$

$n^2\pi^2$

Predictor - Corrector Method

Euler

Predict: $x(t+h) = \underbrace{x(t) + h f(t, x(t))}$

Correct $x(t+h) = x(t) + \frac{h}{2}\left[f(t, x(t)) + f(t+h, \quad)\right]$

$n^2\pi^2$

Predictor - Corrector Method

Euler

Predict: $x(t+h) = x(t) + h f(t, x(t))$

Correct $x(t+h) = x(t) + \frac{h}{2}\left[f(t, x(t)) + f(t+h, \quad\quad)\right]$

Runge-Kutta of order 2

$n^2\pi^2$

Predictor - Corrector Method

Euler

Predict: $x(t+h) = \underbrace{x(t) + h f(t, x(t))}$

Correct $x(t+h) = x(t) + \frac{h}{2}\left[f(t, x(t)) + f(t+h, \quad) \right]$

Runge-Kutta of order 2

(RK 4)

$n^2\pi^2$

Predictor - Corrector Method

Euler

predict: $x(t+h) = x(t) + h f(t, x(t))$

Correct $x(t+h) = x(t) + \frac{h}{2}\left[f(t, x(t)) + f(t+h, \quad)\right]$

Runge-Kutta of order 2

(RK 4)

— Multistep Methods

$n^2\pi^2$

Predictor - Corrector Method

Euler

$$\text{predict: } x(t+h) = \underbrace{x(t) + h\, f(t, x(t))}$$

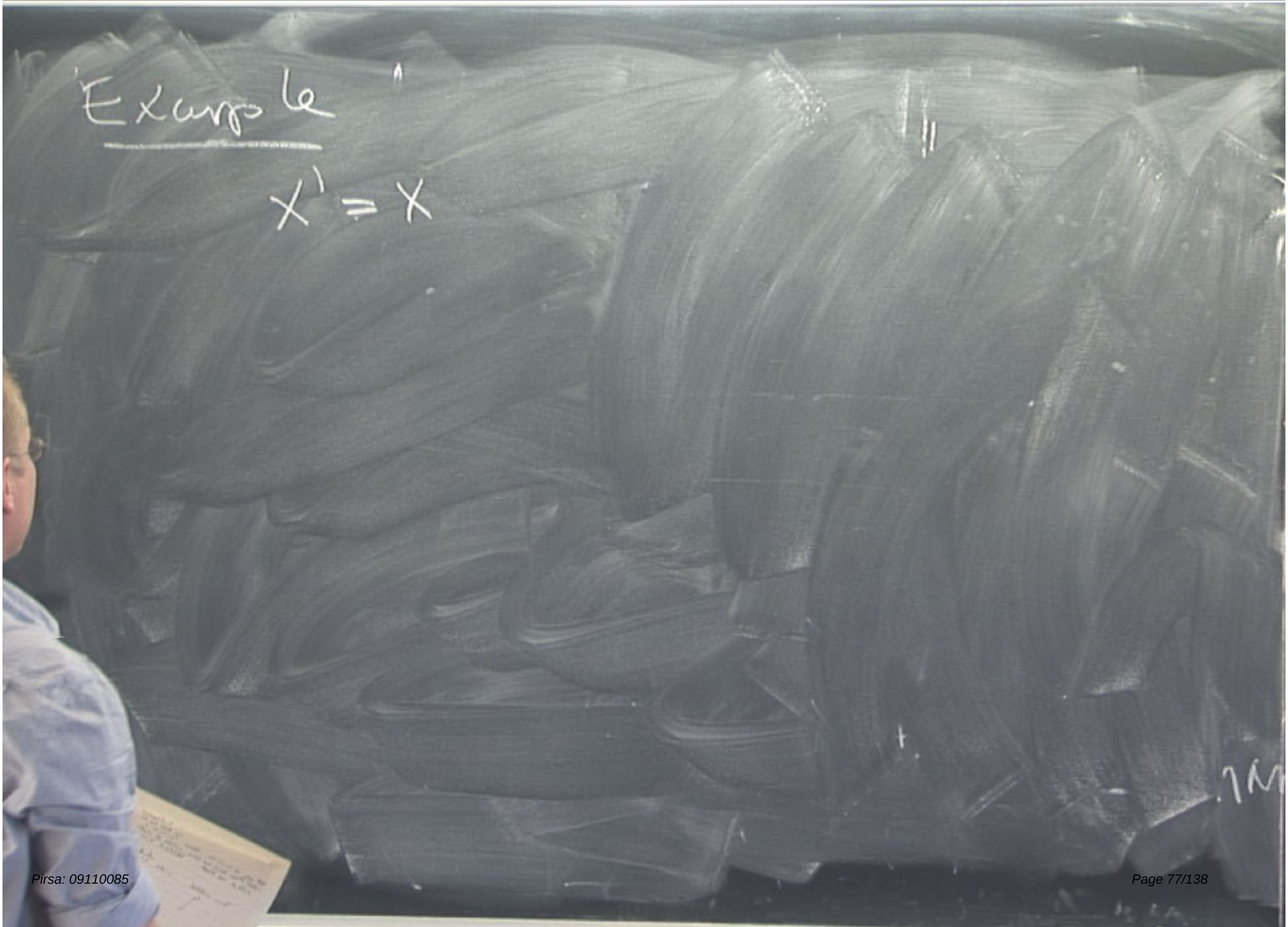
$$\text{Correct } x(t+h) = x(t) + \frac{h}{2}\left[f(t, x(t)) + f(t+h, \quad) \right]$$

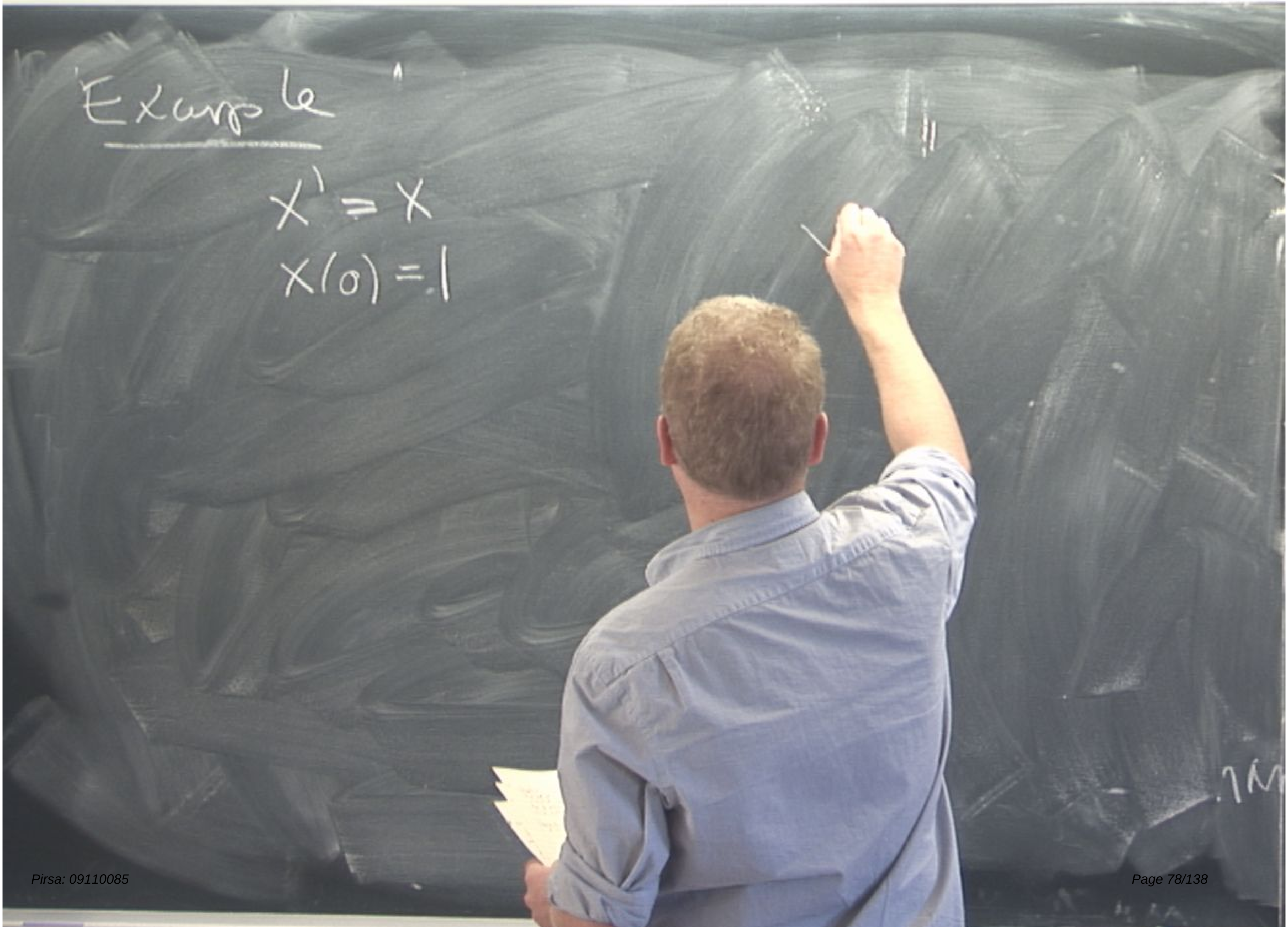
Runge-Kutta of order 2 ← single step

(RK 4)

— Multistep Methods

Single Step Methods


Example
x' = x


Example
x' = x
x(0) = 1

Example

$$x' = x$$
$$x(0) = 1$$

$$x = e^t$$

Example

$$x' = x$$

$$x(0) = 1$$

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Example

$x' = x$

$x(0) = 1$

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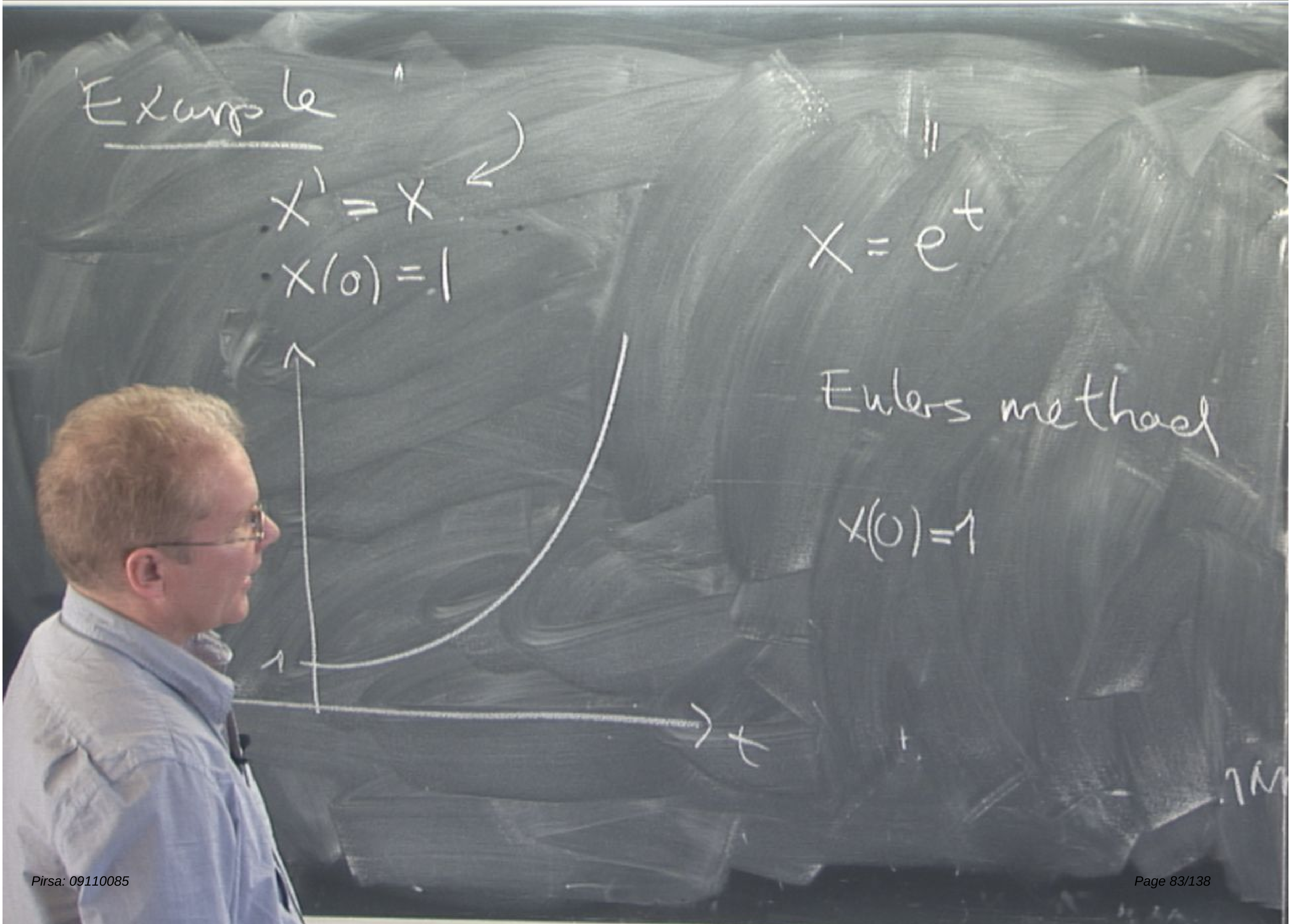
Example

$$x' = x$$
$$x(0) = 1$$

$$x = e^t$$

Eulers method


Example
x' = x
x(0) = 1
1
t
x = e^t
Eulers method
x(0)=1

Example

$$x' = x$$

$$x(0) = 1$$

$$x = e^t$$

Eulers method

$$x(0) = 1$$

Example

$$x' = x$$

$$x(0) = 1$$

$$x = e^t$$

Eulers method

$$x(0) = 1$$

Example

$$x' = x$$
$$x(0) = 1$$

$$x = e^t$$

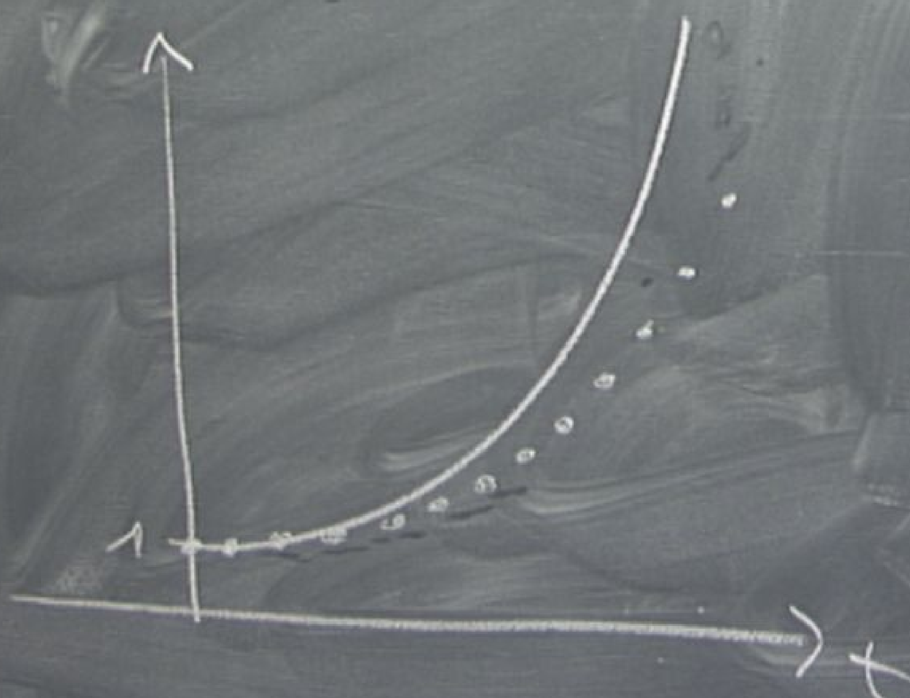
Eulers method

$$x(0) = 1$$

Example

$$x' = x$$

$$x(0) = 1$$



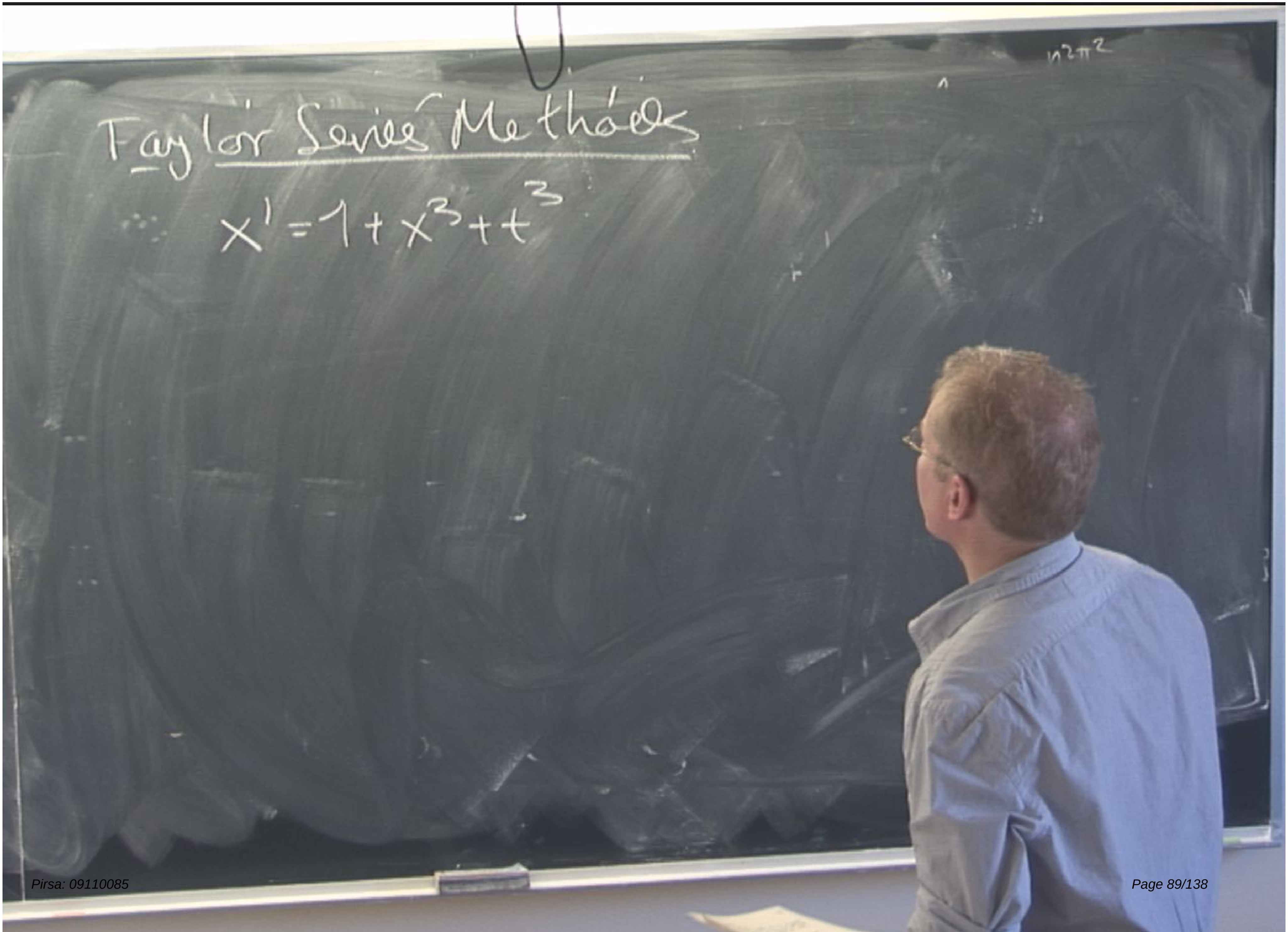
$$x = e^t$$

Eulers method

$$x(0) = 1$$

$n^2\pi^2$

Taylor Series Methods


Taylor Series Methods
$x' = 1 + x^3 + t^3$
$n^2\pi^2$

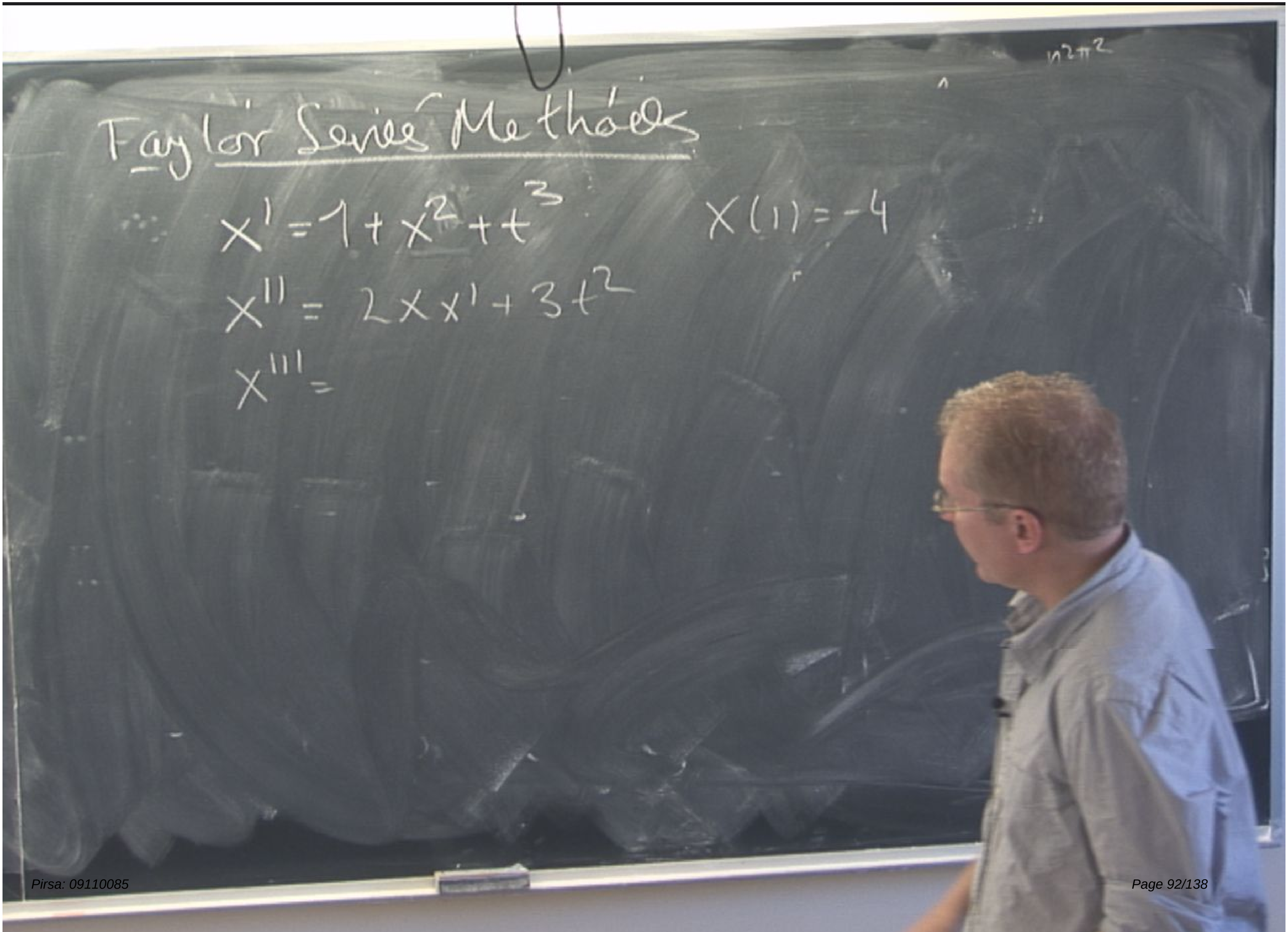
Taylor Series Methods

$x' = 1 + x^3 + t^3$ $\quad x(1) = -4$

Taylor Series Methods

$$x' = 1 + x^3 + t^3 \qquad x(1) = -4$$

$$x''$$


n²π²
Taylor Series Methods
x' = 1 + x² + t³
x(1) = -4
x'' = 2xx' + 3t²
x''' =

$n^2\pi^2$

Taylor Series Methods

$$x' = 1 + x^2 + t^3 \qquad x(1) = -4$$

$$x'' = 2x\,x' + 3t^2$$

$$x''' = 2x\,x'' + 2x'x'$$

$n^2\pi^2$

Taylor Series Methods

$x' = 1 + x^2 + t^3 \qquad x(1) = -4$

$x'' = 2xx' + 3t^2$

$x''' = 2xx'' + 2x'x' + 6t$

$x^{(4)} = 2xx''' + 6x'x'' + 6$

Taylor Series Methods

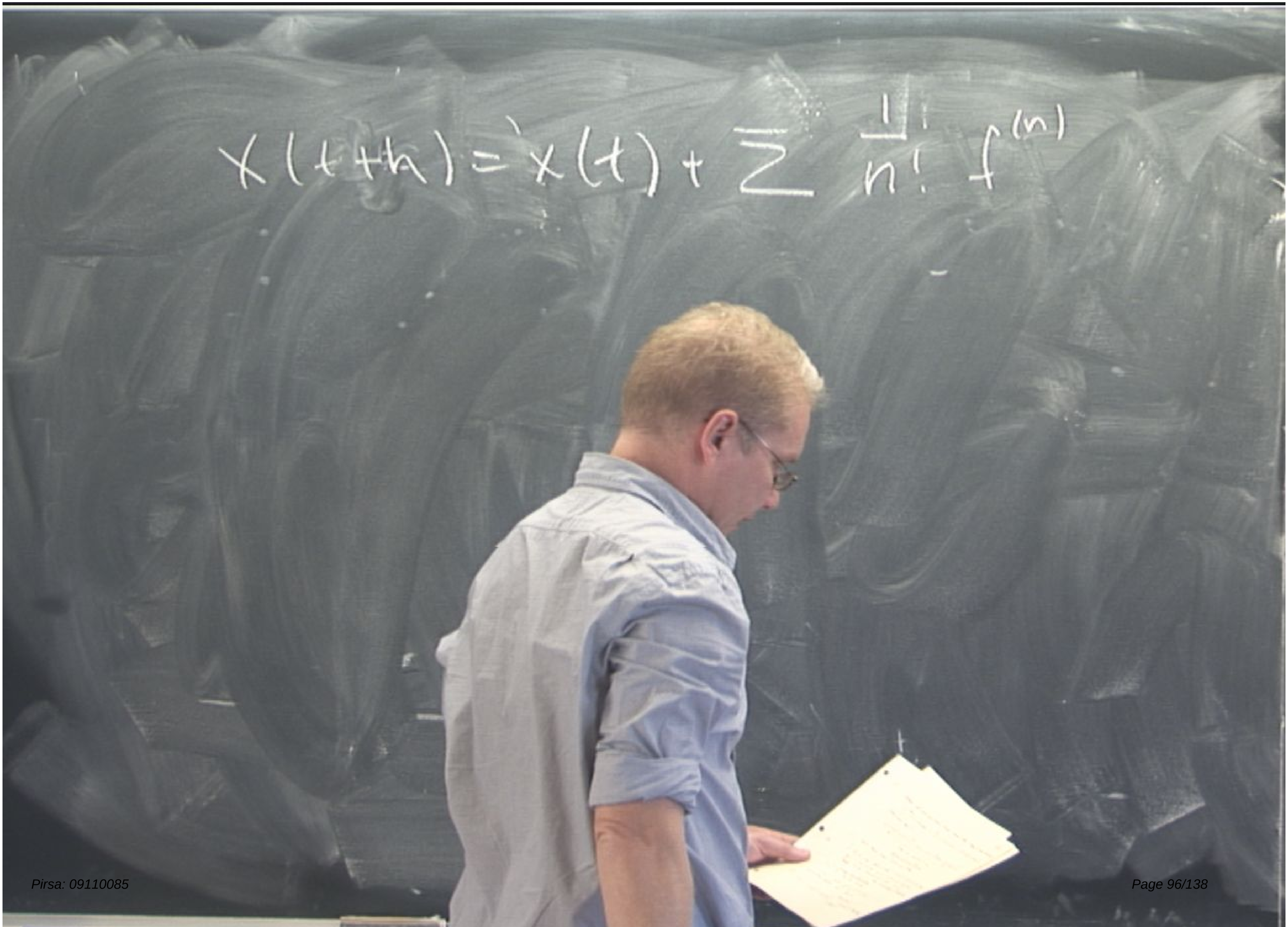
$$x' = 1 + x^2 + t^3 \qquad x(1) = -4$$

$$x'' = 2xx' + 3t^2$$

$$x''' = 2xx'' + 2x'x' + 6t$$

$$x^{(4)} = 2xx''' + 6x'x'' + 6$$

$n^2\pi^2$


$x(t+h) = x(t) + \sum \frac{1}{n!} f^{(n)}$

$$x(t+h) = x(t) + \sum_{n=1}^{4} \frac{h^n}{n!} x^{(n)}(t)$$

$$x(t+h) = x(t) + \sum_{n=1}^{4} \frac{h^n}{n!} x^{(n)}(t)$$

do $k=1, M$

x^1

$$x(t+h) = x(t) + \sum_{n=1}^{4} \frac{h^n}{n!} x^{(n)}(t)$$

do $k=1, M$

$x' = 1 + x^2 + t^3$

$x'' = 2xx' + 3t^2$

$x''' = \ldots$

$x^{(4)} = \ldots$

$X = X$

$$x(t+h) = x(t) + \sum_{n=1}^{4} \frac{h^n}{n!} x^{(n)}(t)$$

do $k=1, M$

$$x' = 1 + x^2 + t^3$$

$$x'' = 2xx' + 3t^2$$

$$x''' = \dots$$

$$x^{(4)} = \dots$$

$$x = x + h[$$

$$x(t+h) = x(t) + \sum_{n=1}^{4} \frac{h^n}{n!} x^{(n)}(t)$$

do $k=1, M$

$$x' = 1 + x^2 + t^3$$

$$x'' = 2xx' + 3t^2$$

$$x''' = \ldots$$

$$x^{(4)} = \ldots$$

$$x = x + h\left[x' + \frac{h}{2}\left[x'' + \frac{h}{3}\left[x''' + \frac{h}{4} x^{(4)}\right]\right]\right]$$

$$x(t+h) = x(t) + \sum_{n=1}^{4} \frac{h^n}{n!} x^{(n)}(t)$$

$t = a$

do $k = 1, M$

$x' = 1 + x^2 + t^3$

$x'' = 2xx' + 3t^2$

$x''' = \ldots$

$x^{(4)} = \ldots$

$x = x + h\left[x' + \frac{h}{2}\left[x'' + \frac{h}{3}\left[x''' + \frac{h}{4} x^{(4)}\right]\right]\right]$

$t = a + kh$

end do

$$x(t+h) = x(t) + \sum_{n=1}^{4} \frac{h^n}{n!} x^{(n)}(t)$$

$t = a$

do $k = 1, M$

$$x' = 1 + x^2 + t^3$$

$$x'' = 2xx' + 3t^2$$

$$x''' = \dots$$

$$x^{(4)} = \dots$$

$$x = x + h\left[x' + \frac{h}{2}\left[x'' + \frac{h}{3}\left[x''' + \frac{h}{4} x^{(4)}\right]\right]\right]$$

$t = a + kh$

end do

$$x(t+h) = x(t) + \sum_{n=1}^{4} \frac{h^n}{n!} x^{(n)}(t)$$

$t = a$

do $k=1, M$

$x' = 1 + x^2 + t^3$

$x'' = 2xx' + 3t^2$

$x''' = \ldots$

$x^{(4)} = \ldots$

$x = x + h\left[x' + \frac{h}{2}\left[x'' + \frac{h}{3}\left[x''' + \frac{h}{4} x^{(4)}\right]\right]\right]$

$t = a + kh$

end do

R–K 2nd order

$n^2\pi^2$

R–K 2nd order

$$y_{n+1} = y_n + \int_{t_n}^{t_{n+1}} f(t,y)\, dt$$

$n^2\pi^2$

R–K 2nd order

$$y_{n+1} = y_n + \int_{t_n}^{t_{n+1}} f(t,y)\,dt$$

Idea ; Expand about midpoint of interval

$$f(t,y)$$

R–K 2nd order

$$y_{n+1} = y_n + \int_{t_n}^{t_{n+1}} f(t,y)\,dt$$

Idea ; Expand about midpoint of interval

$$f(t,y) = f\left(t_{n+1/2},\, y_{n+1/2}\right) + \left(t - t_{n+1/2}\right)\frac{df}{dt}$$

$n^2\pi^2$

R–K 2nd order

$$y_{n+1} = y_n + \int_{t_n}^{t_{n+1}} f(t,y)\,dt$$

Idea ; Expand about midpoint of interval

$$f(t,y) = f\left(t_{n+1/2}, y_{n+1/2}\right) + \left(t - t_{n+1/2}\right) \frac{df}{dt}\Big|_{\ } + O(h$$

R–K 2nd order

$$y_{n+1} = y_n + \int_{t_n}^{t_{n+1}} f(t,y)\,dt$$

Idea: Expand about midpoint of interval

$$f(t,y) = f\left(t_{n+1/2},\, y_{n+1/2}\right) + \left(t - t_{n+1/2}\right) \frac{df}{dt}\Big|_{t_n}$$

$$\Rightarrow \int_{t_n}^{t_{n+1}} f(t,y)\,dt \simeq h\, f\left(t_{n+1/2},\, y_{n+1/2}\right)$$

R–K 2nd order

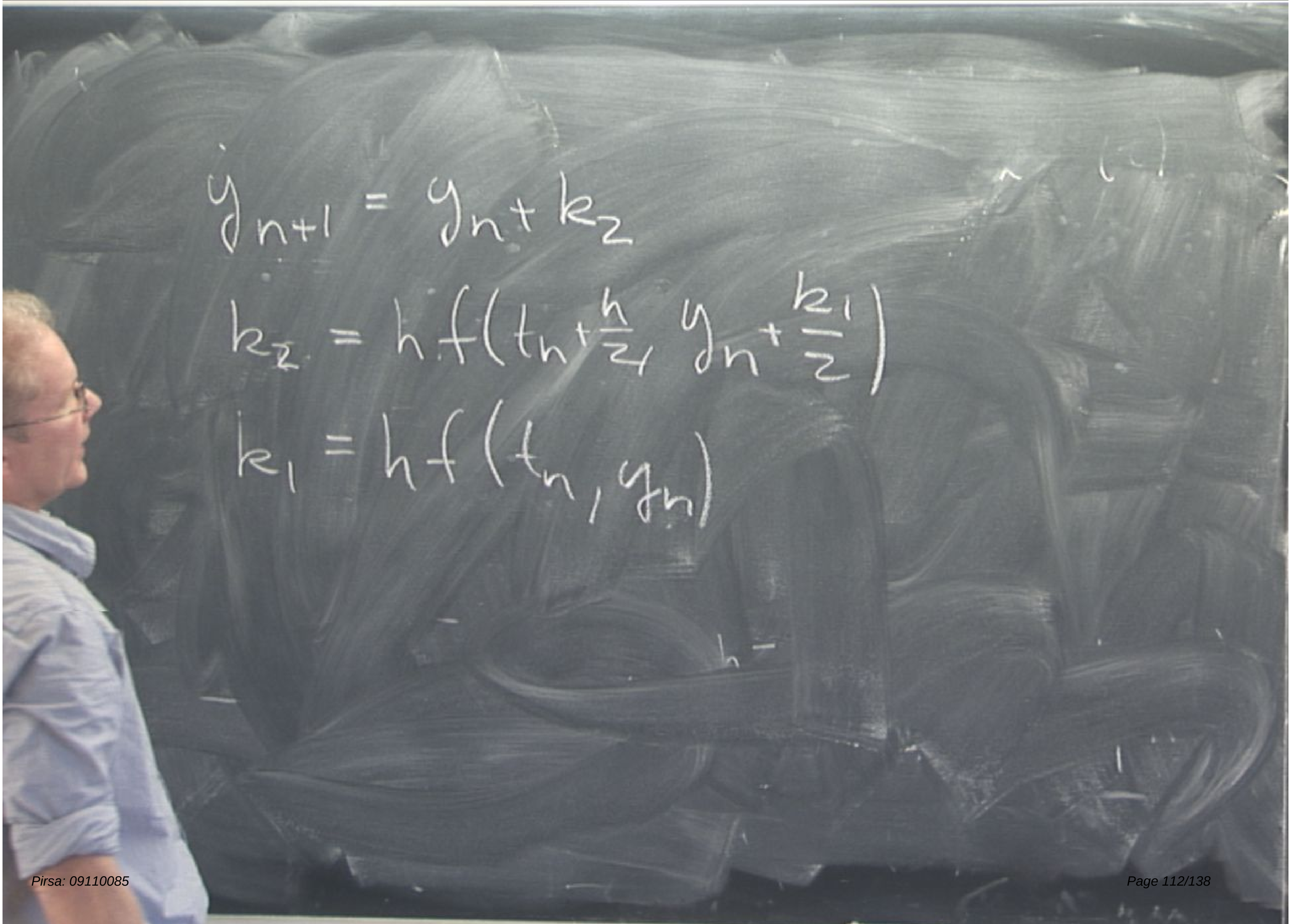
$$y_{n+1} = y_n + \int_{t_n}^{t_{n+1}} f(t,y)\,dt$$

Idea: Expand about midpoint of interval

$$f(t,y) = f(t_{n+1/2}, y_{n+1/2}) + (t - t_{n+1/2}) \frac{df}{dt}\Big|_{t_{n+1/2}} + \mathcal{O}(h$$

$$\Rightarrow \int_{t_n}^{t_{n+1}} f(t,y)\,dt \simeq h\, f(t_{n+1/2}, y_{n+1/2})$$

Enter to get $y_{n+1/2} = y_n + \frac{h}{2} f(t_n, y_n)$


$y_{n+1} = y_n + k_2$
$k_2 = hf(t_n + \frac{h}{2}, y_n + \frac{k_1}{2})$
$k_1 = hf(t_n, y_n)$

$$y_{n+1} = y_n + k_2$$

$$k_2 = hf\left(t_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_1 = hf(t_n, y_n)$$

4th order

$$\mathcal{O}(h^5)$$

$$y_{n+1} = y_n + k_2$$

$$k_2 = hf\left(t_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_1 = hf(t_n, y_n)$$

4th order

$O(h^5)$

h error $O(h^6)$

$$y_{n+1} = y_n + k_2$$

$$k_2 = hf\left(t_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_1 = hf(t_n, y_n)$$

4th order

$O(h^5)$

h error $O(h^6)$

flock seagulls
herd of sheep
pride lions
School of fish

errors

$$y_{n+1} = y_n + k_2$$

$$k_2 = hf(t_n + \tfrac{h}{2}, y_n$$

$$k_1 = hf(t_n, y_n)$$

4th order

$$O(h^5)$$

h –

numerror $O(h^6)$

flock seagulls
herd of sheep
pride lions
School of fish
cancellation errors

$$y_{n+1} = y_n + k_2$$

$$k_2 = hf(t_n + \tfrac{h}{2}, y_n$$

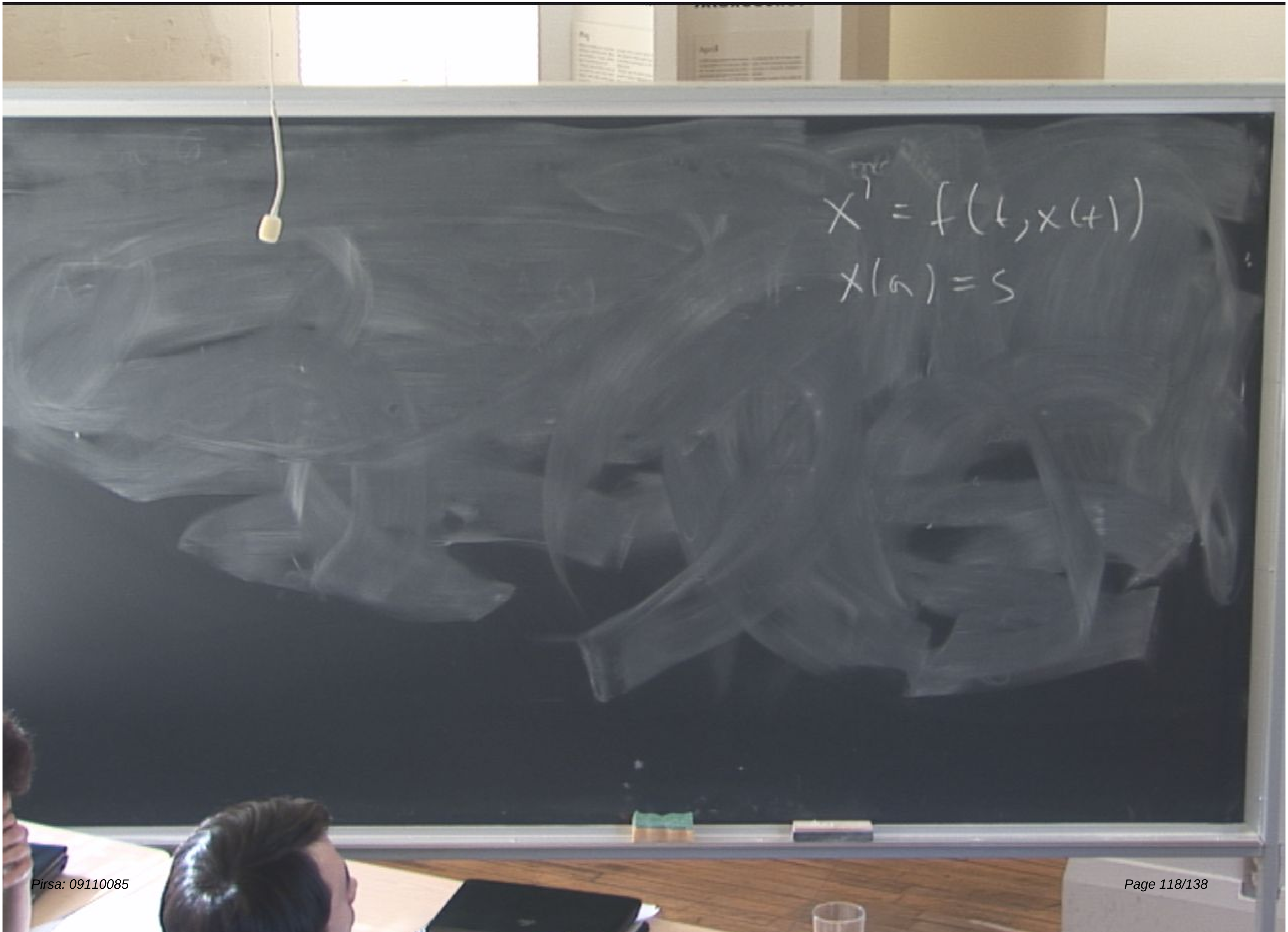
$$k_1 = hf(t_n, y_n)$$

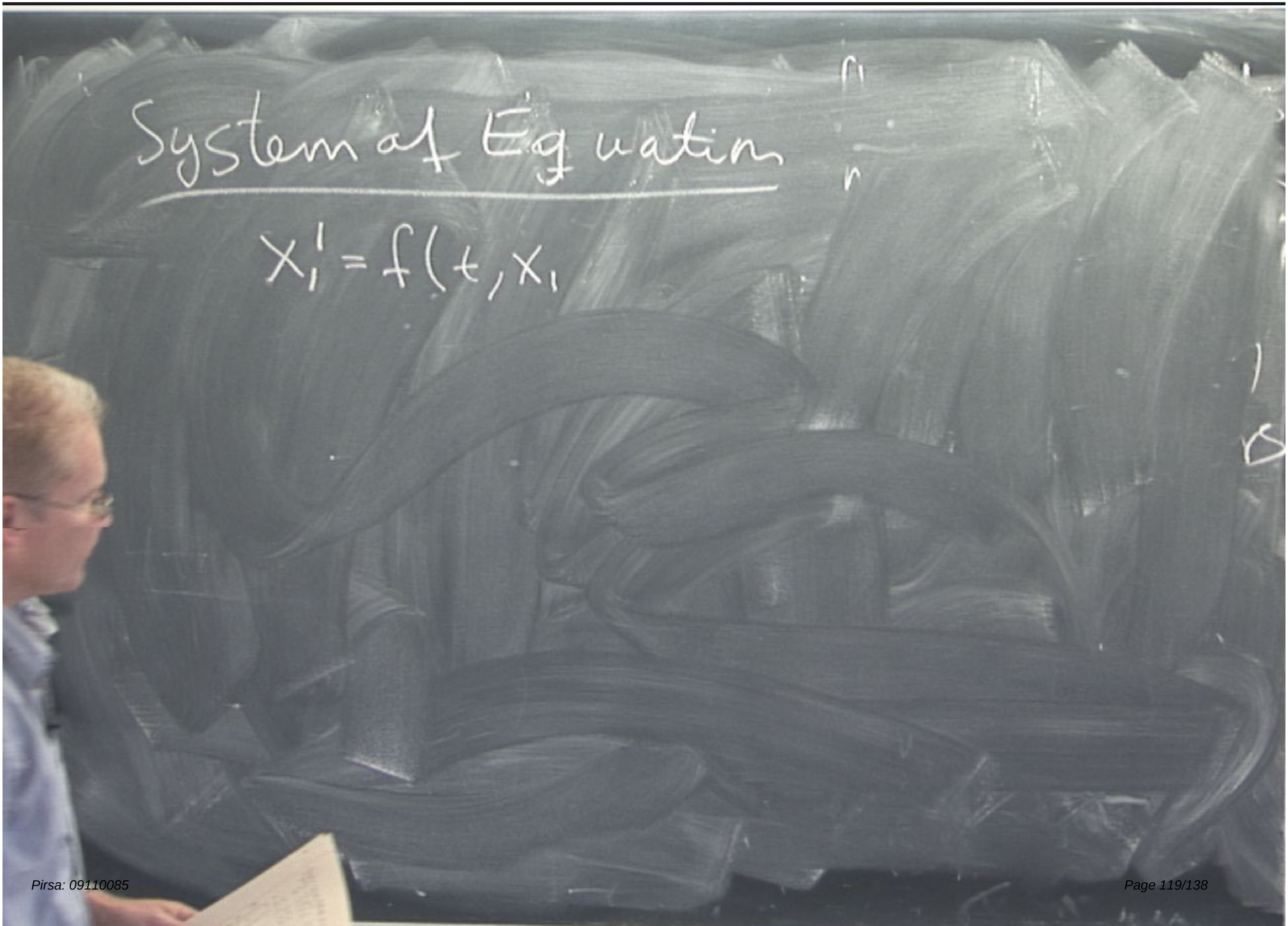
4th order

$$O(h^5)$$

h^-

h error $O(h^6)$


$x' = f(t, x(t))$
$x(a) = s$


System of Equations
$x_1' = f(t, x_1$

System of Equations

$$x_1' = f_1(t, x_1, x_2 \cdots x_n)$$
$$x_2' = f_2(t, x_1, x_2 \cdots x_n)$$
$$x_n' = f_n(t, x_1, x_2 \cdots x_n)$$

System of Equations

$$x_1' = f_1(t, x_1\ x_2 \cdots x_n)$$
$$x_2' = f_2(t, x_1\ x_2 \cdots x_n)$$
$$x_n' = f_n(t, x_1\ x_2 \cdots x_n)$$
$$x_1(a) = S_1 \quad x_2(a) = S_2 \cdots x_n(a) = S_n$$

$n^2\pi^2$

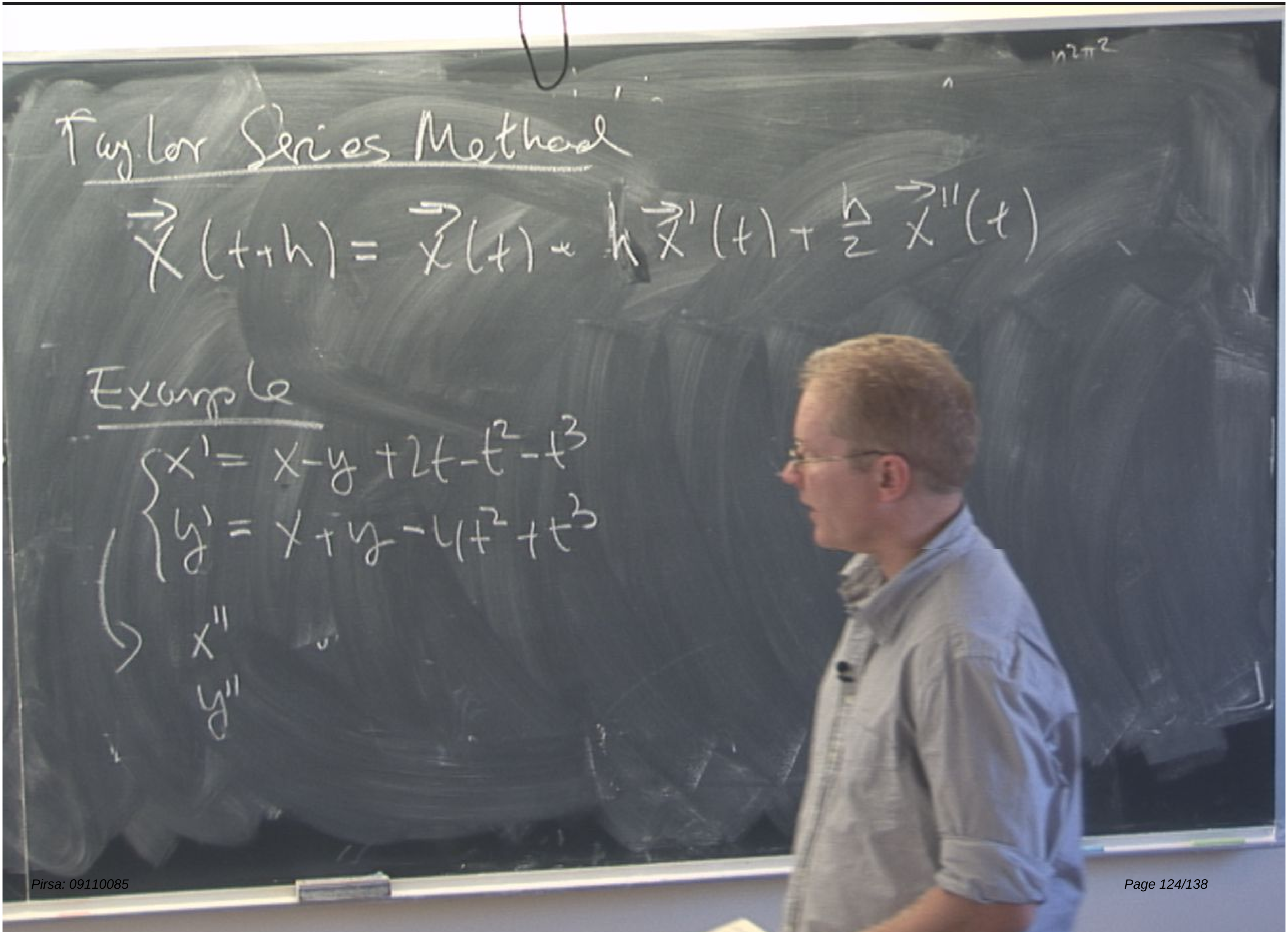
Taylor Series Method

$$x(t+h) = x(t) + h\,x'(t) + \frac{h}{2}\,x''(t)$$

$n^2\pi^2$

Taylor Series Method

$$\vec{x}(t+h) = \vec{x}(t) + h\vec{x}'(t) + \frac{h}{2}\vec{x}''(t)$$


Taylor Series Method
$\vec{x}(t+h) = \vec{x}(t) + h\vec{x}'(t) + \frac{h}{2}\vec{x}''(t)$
Example
$x' = x - y + 2t - t^2 - t^3$
$y' = x + y - 4t^2 + t^3$
x''
y''

Taylor Series Method

$$\vec{x}(t+h) = \vec{x}(t) + h\,\vec{x}'(t) + \frac{h^2}{2}\vec{x}''(t)$$

Example

$$\begin{cases} x' = x - y + 2t - t^2 - t^3 \\ y' = x + y - 4t^2 + t^3 \end{cases}$$

$$\begin{cases} x'' \\ y'' \end{cases}$$

$$\begin{cases} x''' \\ y''' \end{cases}$$

$n^2\pi^2$

Taylor Series Method

$$\vec{x}(t+h) = \vec{x}(t) + h\,\vec{x}'(t) + \frac{h^2}{2}\,\vec{x}''(t)$$

Example

$$\begin{cases} x' = x - y + 2t - t^2 - t^3 \\ y' = x + y - 4t^2 + t^3 \end{cases} = \vec{x}'$$

$$\begin{cases} x'' \\ y'' \end{cases} = \vec{x}''$$

$$\begin{cases} x''' \\ y''' \end{cases} = \vec{x}'''$$

$n^2\pi^2$

Taylor Series Method

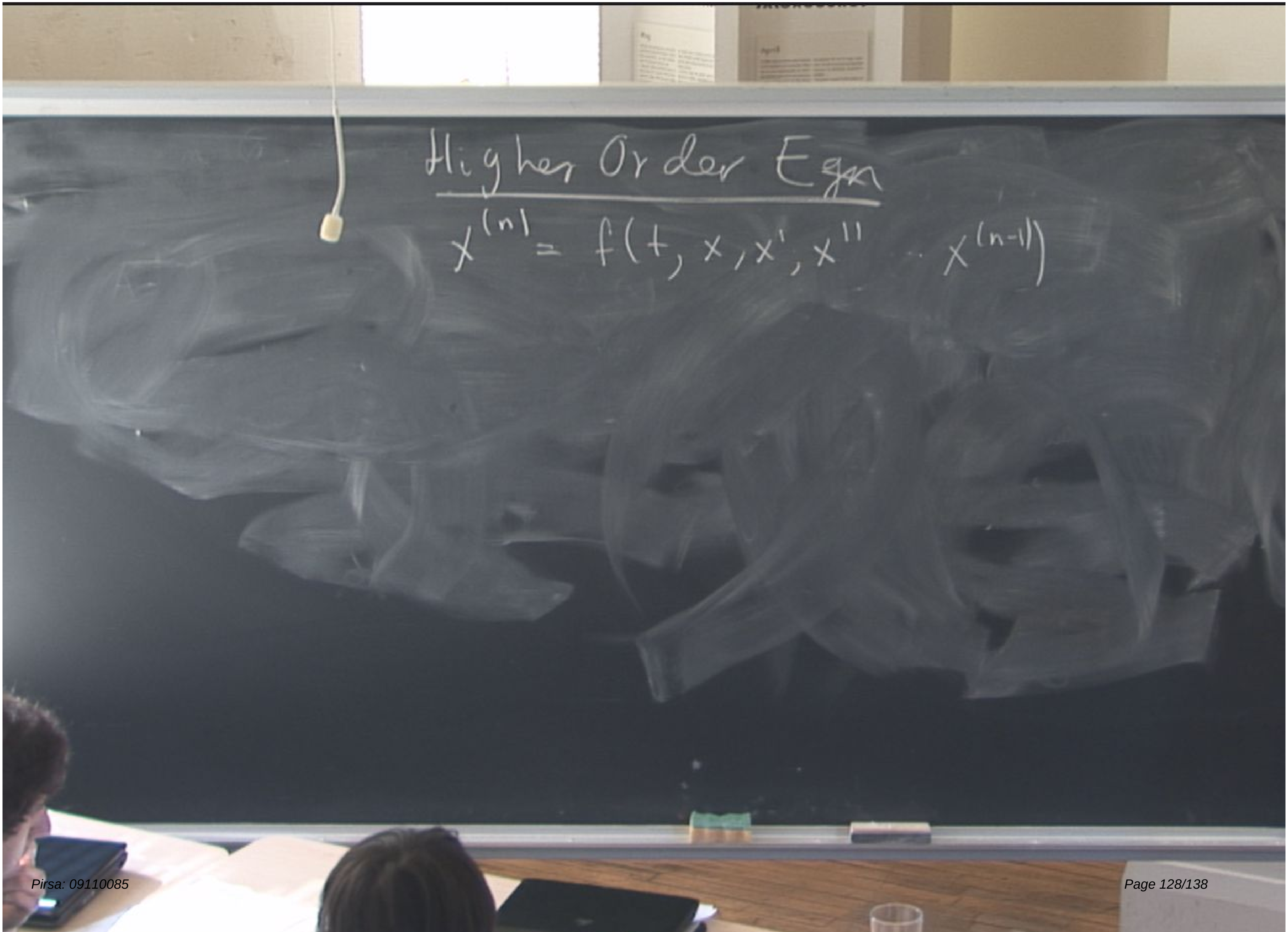
$$\vec{x}(t+h) = \vec{x}(t) + h\vec{x}'(t) + \frac{h}{2}\vec{x}''(t)$$

Example

$$\begin{cases} x' = x - y + 2t - t^2 - t^3 \\ y' = x + y - 4t^2 + t^3 \end{cases} \quad \vec{x}' = \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$\begin{cases} x'' \\ y'' \end{cases} \quad \vec{x}''$$

$$\begin{cases} x''' \\ y''' \end{cases} \quad \vec{x}'''$$


Higher Order Eqn
$x^{(n)} = f(t, x, x', x'' \dots x^{(n-1)})$

Higher Order Eqn

$$x^{(n)} = f(t, x, x', x'' \dots x^{(n-1)})$$

$x(a)$, $x'(a)$, $x''(a)$ $\dots$ $x^{(n-1)}(a)$ given

Higher Order Eqn

$$\begin{cases} x^{(n)} = f(t, x, x', x'', \dots x^{(n-1)}) \\ x(a), x'(a), x''(a) \dots x^{(n-1)}(a) \text{ given} \end{cases}$$

convert into a system of 1st order Eqn

Higher Order Eqn

$$\begin{cases} x^{(n)} = f(t, x, x', x'' \ldots x^{(n-1)}) \\ x(a), x'(a), x''(a) \ldots x^{(n-1)}(a) \text{ given} \end{cases}$$

convert into a system of 1st order Eqn

$x_1 = x \qquad x_1' = x_2$

$x_2 = x' \qquad x_2' = x_3$

$\vdots$

$x_n = x^{(n-1)}$

Higher Order Eqn

$$\begin{cases} x^{(n)} = f(t, x, x', x'', \dots x^{(n-1)}) \\ x(a), x'(a), x''(a) \dots x^{(n-1)}(a) \text{ given} \end{cases}$$

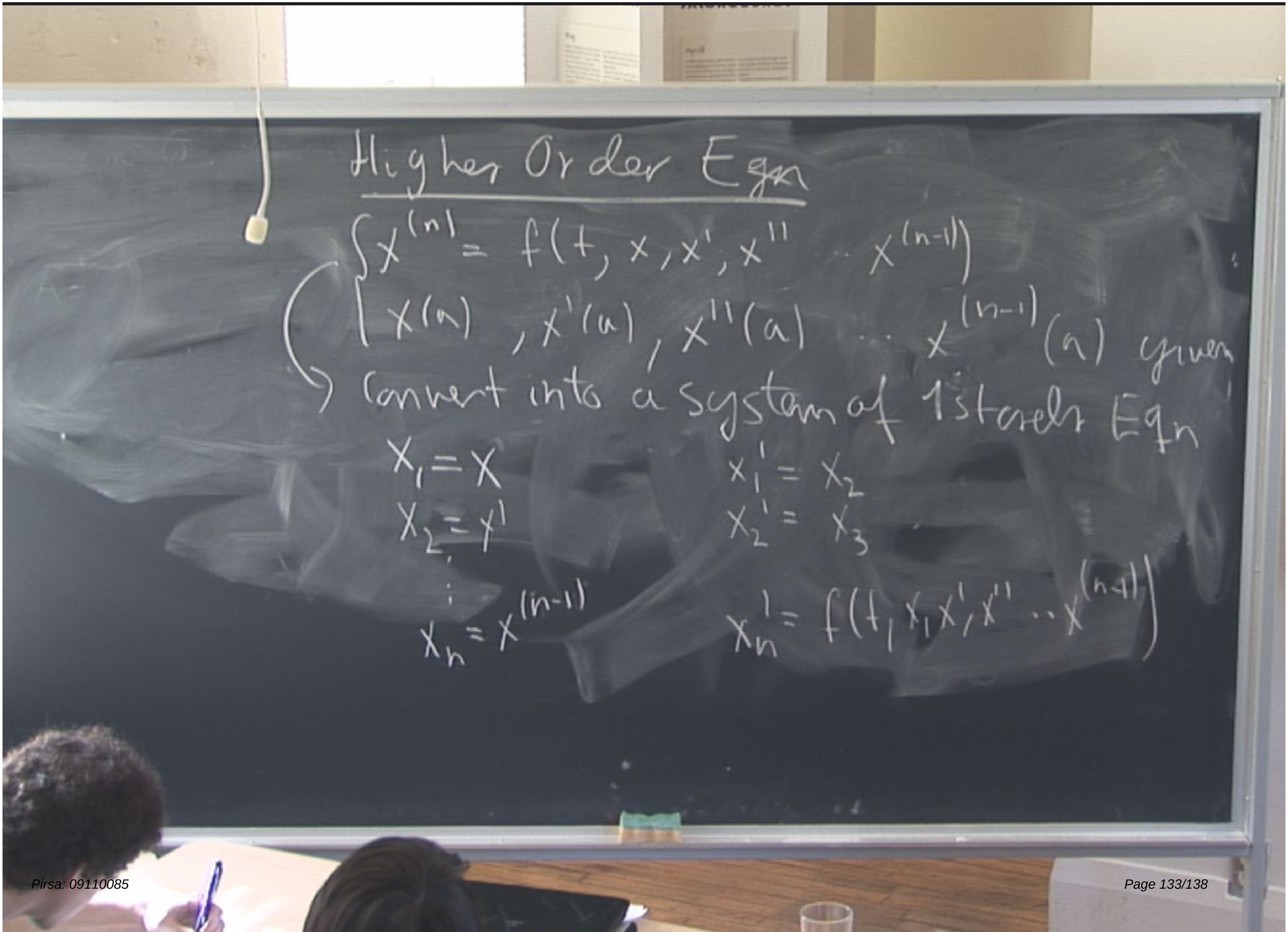
convert into a system of 1st order Eqn

$$x_1 = x \qquad x_1' = x_2$$

$$x_2 = x' \qquad x_2' = x_3$$

$$\vdots$$

$$x_n = x^{(n-1)} \qquad x_n' =$$


Higher Order Eqn
$x^{(n)} = f(t, x, x', x'' \dots x^{(n-1)})$
$x(a), x'(a), x''(a) \dots x^{(n-1)}(a)$ given
convert into a system of 1st order Eqn
$X_1 = X$
$X_2 = x'$
$\vdots$
$X_n = x^{(n-1)}$
$x_1' = X_2$
$X_2' = X_3$
$X_n' = f(t, x, x', x'' \dots x^{(n-1)})$

For constant coefficients

$$\frac{d^n y}{dt^n} = a_{n-1} y^{(n-1)} + a_{n-2} y^{(n-2)} \cdots + a_0 y$$

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$$\frac{d^n y}{dt^n} = a_{n-1} y^{(n-1)} + a_{n-2} y^{(n-2)} \cdots + a_0 y$$

$$\frac{d}{dt}\begin{pmatrix} y \\ y^{(1)} \\ \vdots \\ y^{(n-1)} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & & \cdots \\ & & & & \end{pmatrix}$$

For constant coefficiente

$$\frac{d^n y}{dt^n} = a_{n-1} y^{(n-1)} + a_{n-2} y^{(n-2)} \cdots + a_0 y$$

$$\frac{d}{dt}\begin{pmatrix} y \\ y^{(1)} \\ \vdots \\ y^{(n-1)} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & & \cdots \\ & & & & \\ a_0 & a_1 & \cdots & \cdots & a_{n-1} \end{pmatrix} \begin{pmatrix} y \\ y^{(1)} \\ \vdots \\ y^{(n-1)} \end{pmatrix}$$

For constant coefficients

$$\frac{d^n y}{dt^n} = a_{n-1} y^{(n-1)} + a_{n-2} y^{(n-2)} \cdots + a_0 y$$

$$\frac{d}{dt}\begin{pmatrix} y \\ y^{(1)} \\ \vdots \\ y^{(n-1)} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & & \cdots \\ & & & & \\ a_0 & a_1 & \cdots & \cdots & a_{n-1} \end{pmatrix} \begin{pmatrix} y \\ y^{(1)} \\ \vdots \\ y^{(n-1)} \end{pmatrix}$$

$$\vec{y} = e^{\underline{A}t} \vec{y}_0$$

For constant coefficients

$$\frac{d^n y}{dt^n} = a_{n-1} y^{(n-1)} + a_{n-2} y^{(n-2)} \cdots + a_0 y$$

$$\frac{d}{dt}\begin{pmatrix} y \\ y^{(1)} \\ \vdots \\ y^{(n-1)} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & & \cdots \\ & & & \ddots & \\ a_0 & a_1 & \cdots & \cdots & a_{n-1} \end{pmatrix} \begin{pmatrix} y \\ y^{(1)} \\ \vdots \\ y^{(n-1)} \end{pmatrix}$$

$$\vec{y} = e^{\underline{A}t} \vec{y}_0$$