Title: Perturbative cancellations in gravity theories

Date: Nov 07, 2009 03:00 PM

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Abstract: I will present recent results through four loops demonstrating that the maximally supersymmetric (N=8) generalization of gravity is surprisingly well behaved in the ultraviolet as a result of unexpected cancellations between contributing terms. These cancellations first manifest at one loop in the form of the "no-triangle property," with all-loop order implications through unitarity. I will conclude by discussing similar novel cancelations identified in pure Einstein gravity, at one loop, which suggest a possible explanation for the unexpectedly tame high energy behavior of N=8 supergravity beyond the limited UV protection of supersymmetry.

Perturbative Cancellations in Gravity Theories

John Joseph M. Carrasco

Asymptotic Safety - 30 Years Later 7 November 2009

Will present results from papers with:

Zvi Bern, Lance Dixon, Darren Forde, Harold Ita,
Henrik Johansson, David Kosower, Radu Roiban

Perturbative Cancellations in Gravity Theories

How to play an Action Hero

(while eschewing Lagrangians)

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Note: not actually real action heros

More like playing Hollywood action heros

Never see any action, and never play directly with Feynman rules.

Only work with on-shell physical quantities

But you can climb to pretty high loop orders, which looks like a lot of action...

What I want you to get out of this talk

- When organized well, perturbative calculations are straight-forward and fun: not overly laborious or painful!
- There is something deep going on between gauge and gravity theories (weak-weak unification)
- Vanilla perturbative QFT may be a more powerful framework for exploring gravity theories than people have suspected for quite some time

Outline

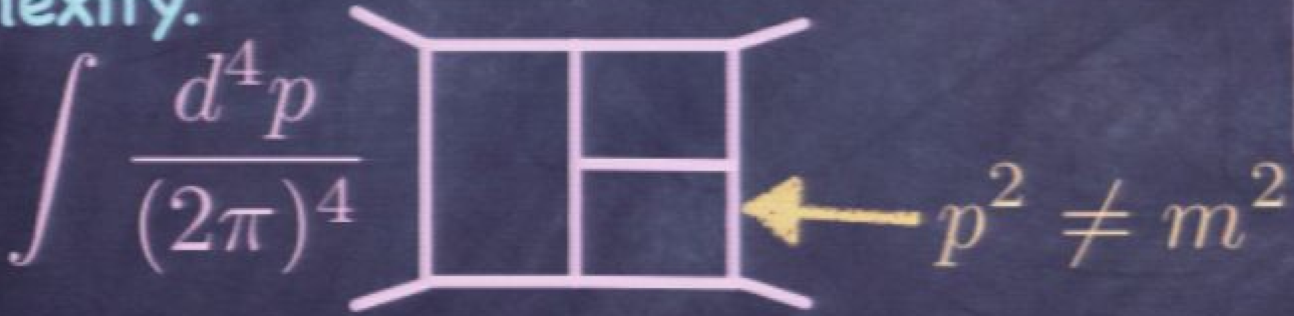
- On-shell techniques, and how they clarify relationships between gauge and gravity theories
- Tour of the remarkable cancellations observed in N=8 Supergravity
- Instructive look at pure Gravity.

Why are Feynman diagrams so difficult for high-loop or high-multiplicity processes?

painting by Nataly Meerson

Why are Feynman diagrams so difficult for high-loop or high-multiplicity processes?

- Vertices and propagators involve gauge-dependent off-shell states. An important origin of the complexity.



$$\int \frac{d^4p}{(2\pi)^4}$$

$p^2 \neq m^2$

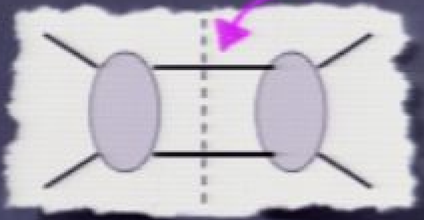
- To avoid this trouble we reorganize perturbative quantum field theory.

All steps done using gauge invariant on-shell states: On-shell Formalism. $p^2 = m^2$

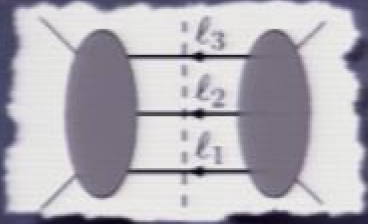
Unitarity Method

Bern, Dixon, Dunbar and Kosower

Two-particle cut:

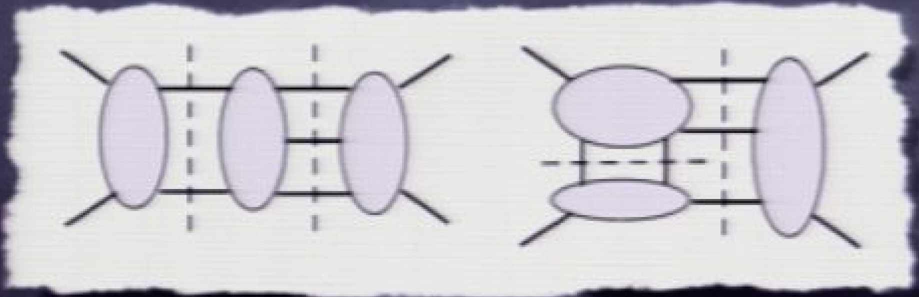

on-shell

Three-particle cut:



Systematic assembly of complete amplitudes at the integrand level from cuts for any number of particles or loops.

eneralized
nitarity as a
ractical tool:



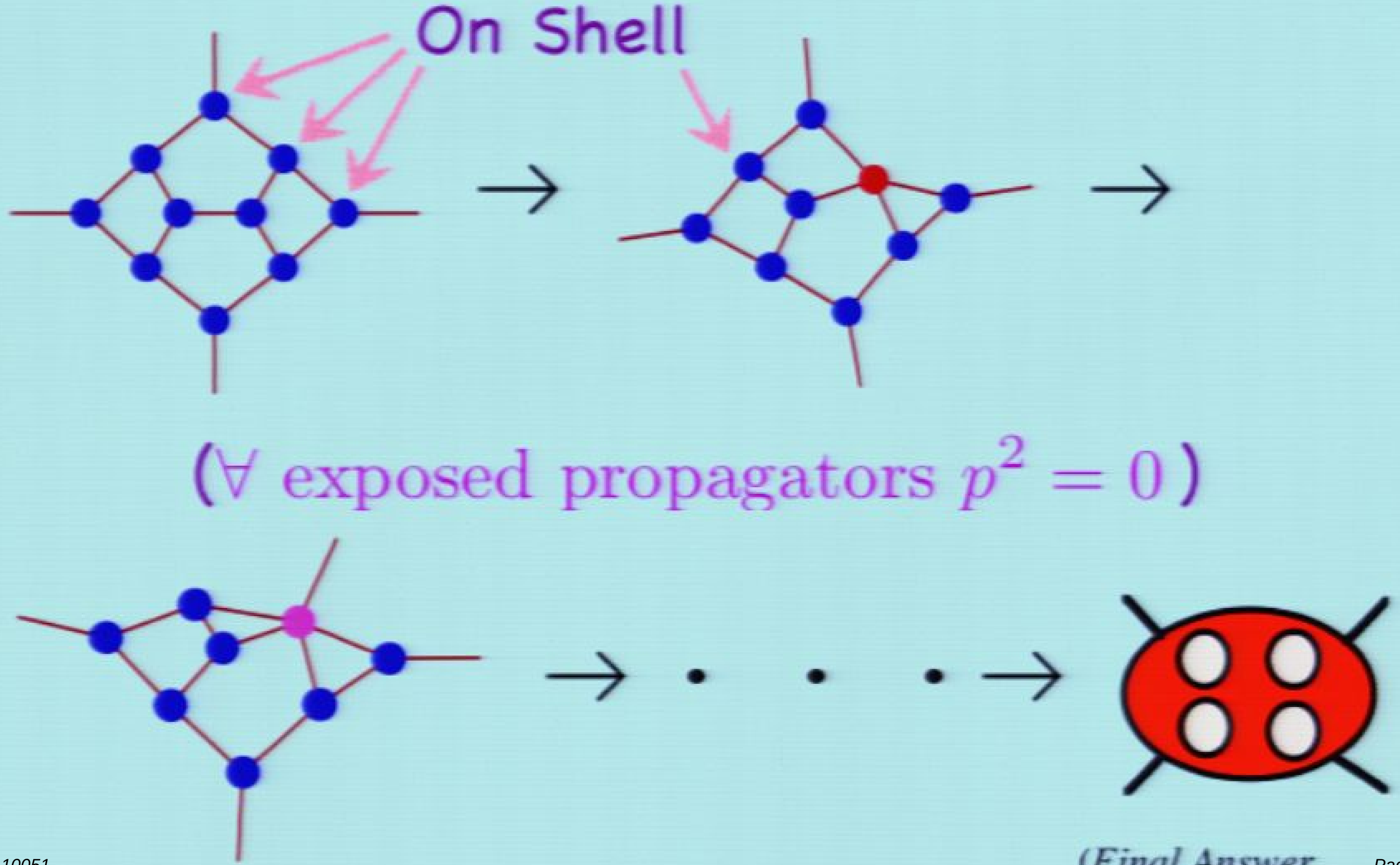
ern, Dixon and Kosower
ritto, Cachazo and Feng

Different cuts merged to give an expression with correct cuts in all channels.

Method of Maximal Cuts

Bern, JJMC, Johansson, Kosower

Systematic and complete implementation of Generalized Unitarity



On Shell

$(\forall$ exposed propagators $p^2 = 0)$

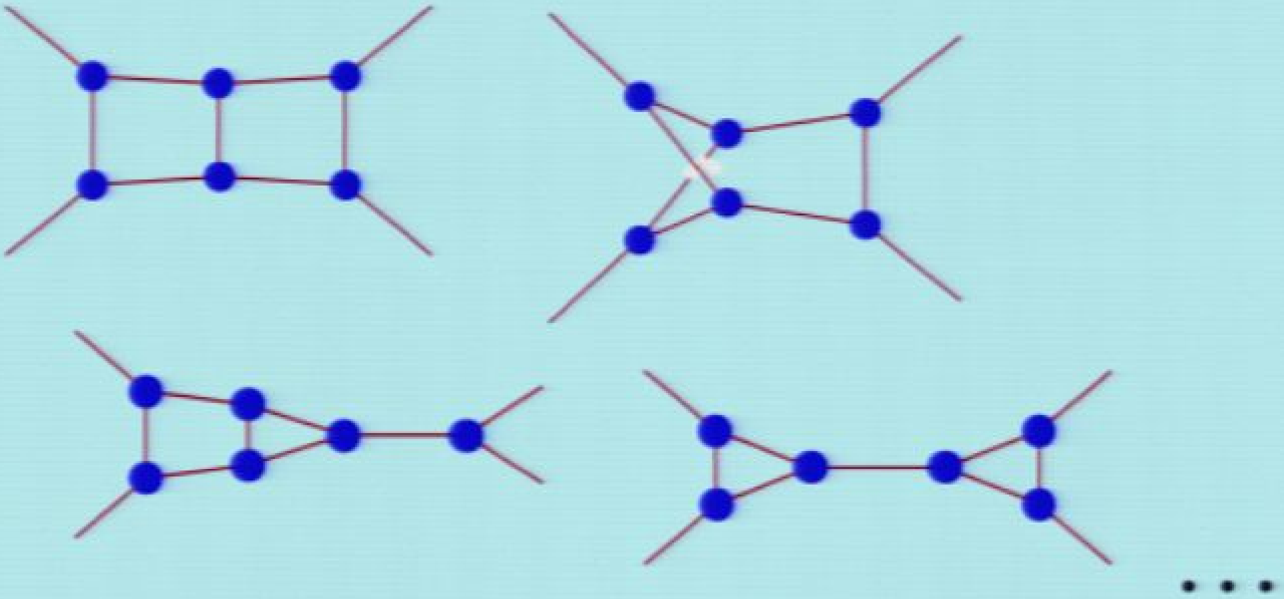
$\rightarrow \quad \rightarrow \quad \rightarrow \cdot \quad \cdot \quad \cdot \rightarrow$

(Final Answer,
no cut conditions!)

2 loops $\mathcal{N}=4$ SYM: A brief tour of the Method of Maximal Cuts.

- *Find all cubic diagrams*. (Why only cubic?)

These are all the diagrams you get from sewing 3-vertex trees: it will be the pallette of our integrands, and our first cuts!



ALL LEGS ON SHELL! ***(3-pt defined with complex momenta)***

• *Dress parents to satisfy the maximal cuts.*

Only two (isomorphically distinct) non-vanishing maximal cuts:

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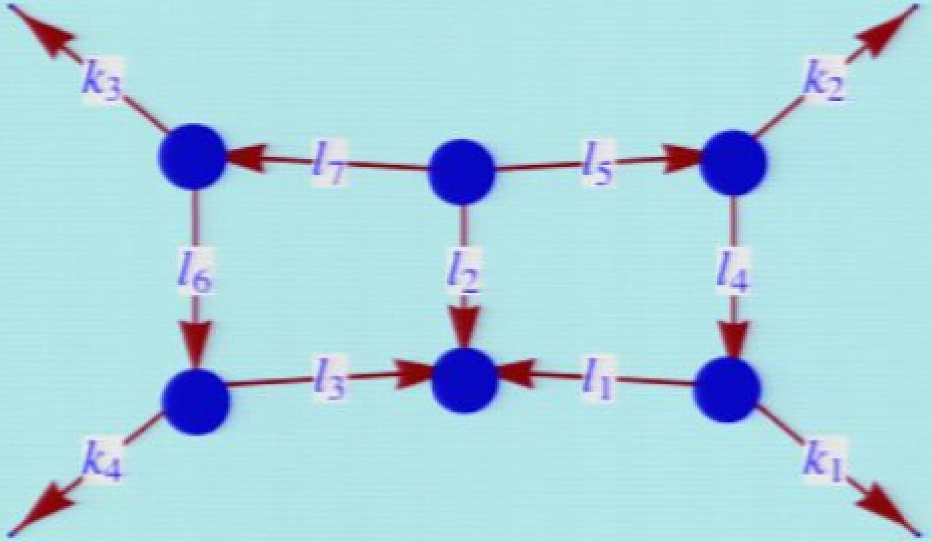
1) Planar double box

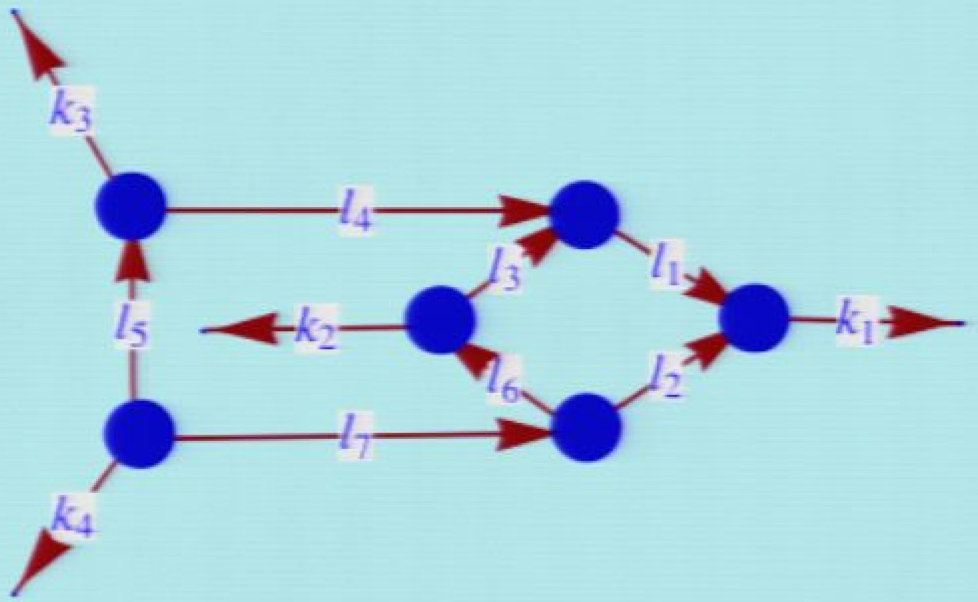
(c.f. hepta-cuts first given in: arXiv:hep-th/0506126, E. I. Buchbinder and F. Cachazo)



$$\Pi_i\, \delta\left(l_i^2\right) \sum\nolimits_{\mathcal{N}=4} (A_3(-l_1, -l_2, -l_3)\, A_3(l_1, k_1, -l_4)\, A_3(l_4, k_2, -l_5)$$
$$\times\, A_3(-l_6, l_3, k_4)\, A_3(l_6, k_3, -l_7)\, A_3(l_7, l_5, l_2))$$

$= (k_1 + k_2)^2$ (dropping overall YM 4−pt factor: $(k_1 + k_2)^2\, (k_1 + k_4)^2\, A_4(k_1, k_2, k_3, k_4)$)

Non-planar double-"box"

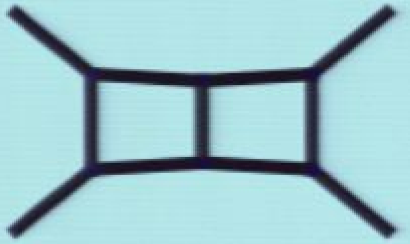


$$\Pi_i\,\delta\big(l_i^2\big)\sum\nolimits_{\mathcal{N}=4}\big(A_3(-l_1,k_1,-l_2)\,A_3(l_1,-l_3,-l_4)\,A_3(l_4,-l_5,k_3)$$
$$\times\,A_3(-l_6,l_3,k_2)\,A_3(l_6,l_2,-l_7)\,A_3(l_7,k_4,l_5)\big)$$
$$=(k_1+k_2)^2$$

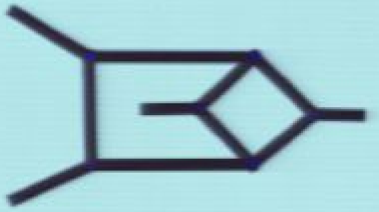
Every other maximal cut vanishes

So at the end of this step we know:

There are numerator factors $(k_1 + k_2)^2$ associated with

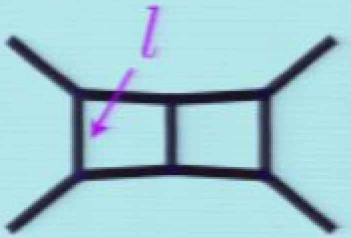


and

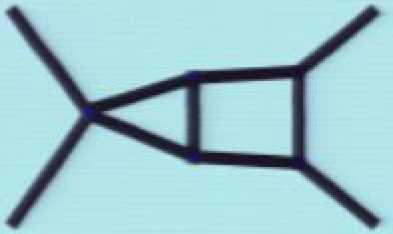


.

There might be some additional terms proportional to an l^2, (e.g. for



: something like



)

but if so we'll find them.

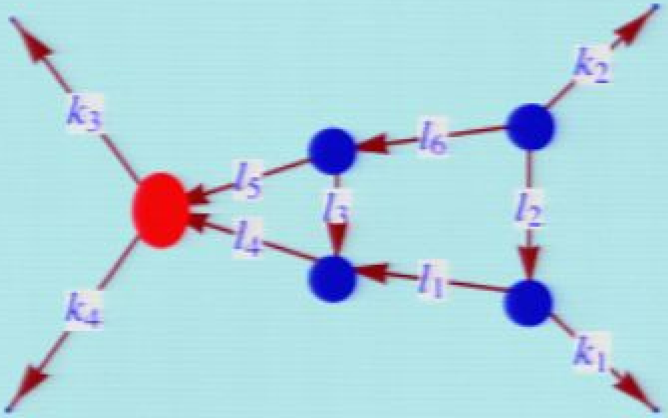
(Actually for $\mathcal{N}$=4 SYM we've already saturated the power counting, so this is really all there is to it, but lets see how the calculation plays out.)

Relax cut conditions, look for anything missing

(c.f. near-mximal cuts in planar 5 loops: arXiv:0705.1864, Z.Bern, JJMC, H. Johansson, D. Kosower)

Only 5 isomorphically distinct non (trivially) vanishing single-contact cuts.

Let's look at one:



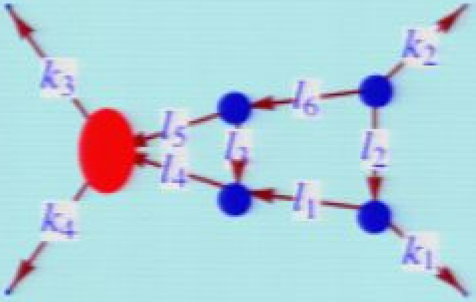
Do the cut:

$$\Pi_i\,\delta(l_i)^2 \sum_{\mathcal{N}=4} \text{trees} = \frac{(k_1+k_2)^2}{(l_5-k_3)^2}$$

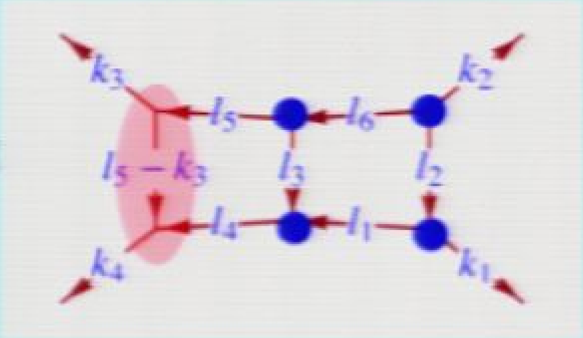
Is this accounted for by our current Ansatz or do we need to add new contributions?

raw and dress diagrams that could contribute by expanding out four-particle color ordered tree

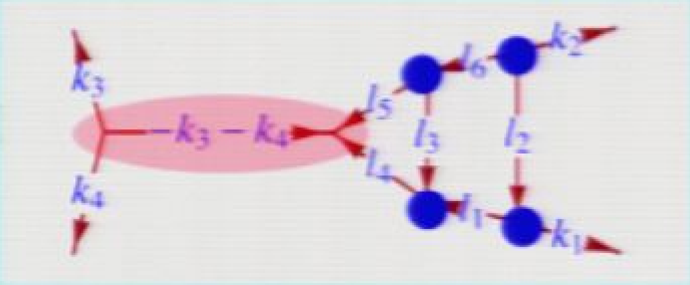
Ansatz from current dressings:



$= (k_1 + k_2)^2 \times$



$+ 0 \times$

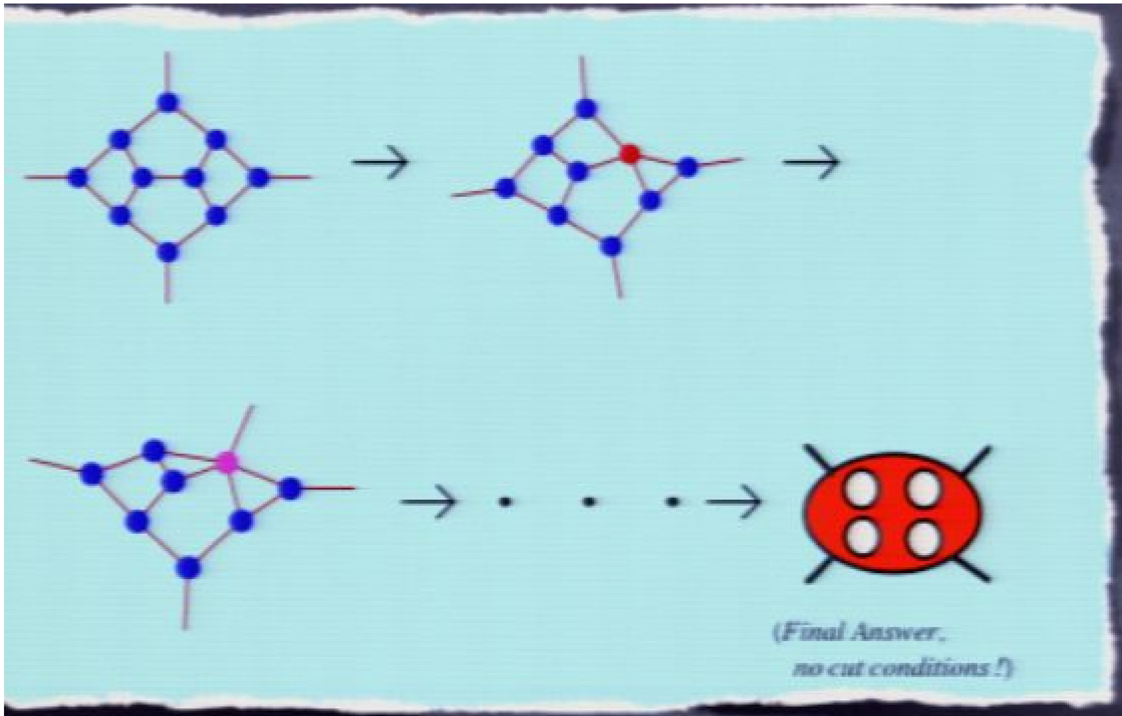


$= \left(\frac{(k_1+k_2)^2}{(l_5-k_3)^2} \right) =$ cut calc.

As the ansatz holds, there's no additional contribution to either parent dressing from this contact cut.

No additional contributions on any contact cuts for $\mathcal{N}$=4 SYM at 2 loops. There would be for QCD. Here's the prescription for incorporating contact contributions:

- Identify all contributing cuts at a given cut-level. Each cut is an equation for missing numerator dressings.
- Simultaneously solve the resulting set of linear cut equations. Solutions represent additional dressings for appropriate cubic-graphs.
- Incorporate before going on to next cut level.



(Final Answer,
no cut conditions !)

Which topologies should you assign contact terms to when given the freedom?

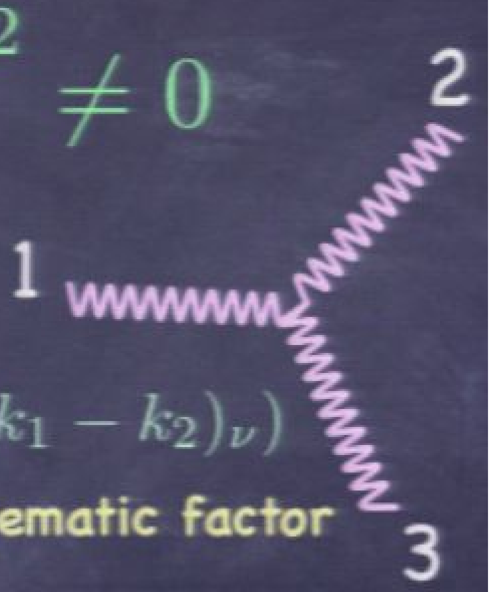
Correct answers: whichever makes your final expression the most compact? The one that makes manifest iterative structure?

↑ Hard to solve! =?= ↑ Really hard to solve

trategy that closes at four loops for N=8:
Assign to the topology P, such that P appears the east number of times in the cut, and of those, with the least symmetry.

Three Vertices
OFF SHELL

$$k_i^2 = E_i^2 - \vec{k}_i^{\,2} \neq 0$$



hree-gluon vertex:

color factor

$$V_{3\,\mu\nu\sigma}^{\;abc} = -gf^{abc}(\eta_{\mu\nu}(k_1 - k_2)_\rho + \eta_{\nu\rho}(k_1 - k_2)_\mu + \eta_{\rho\mu}(k_1 - k_2)_\nu)$$

momentum dependent kinematic factor

hree-graviton vertex:

$$G_{3\mu\alpha,\nu\beta,\sigma\gamma}(k_1, k_2, k_3) =$$

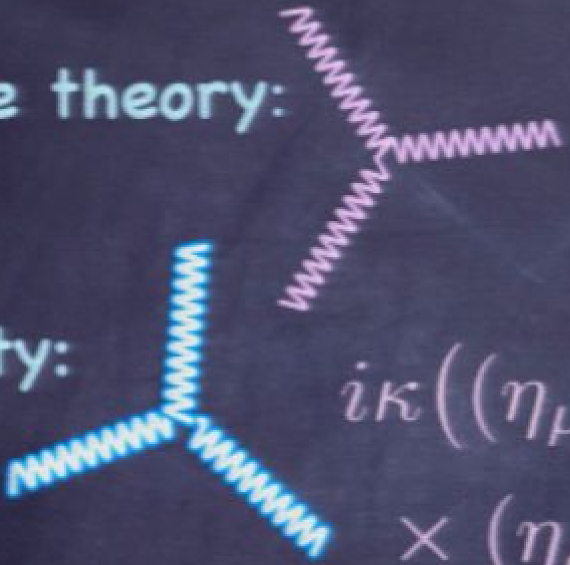
$$\begin{aligned}
&\mathrm{sym}[-\tfrac{1}{2}P_3(k_1 \cdot k_2\eta_{\mu\alpha}\eta_{\nu\beta}\eta_{\sigma\gamma}) - \tfrac{1}{2}P_6(k_{1\nu}k_{1\beta}\eta_{\mu\alpha}\eta_{\sigma\gamma}) + \tfrac{1}{2}P_3(k_1 \cdot k_2\eta_{\mu\nu}\eta_{\alpha\beta}\eta_{\sigma\gamma})\\
&\quad + P_6(k_1 \cdot k_2\eta_{\mu\alpha}\eta_{\nu\sigma}\eta_{\beta\gamma}) + 2P_3(k_{1\nu}k_{1\gamma}\eta_{\mu\alpha}\eta_{\beta\sigma}) - P_3(k_{1\beta}k_{2\mu}\eta_{\alpha\nu}\eta_{\sigma\gamma})\\
&\quad + P_3(k_{1\sigma}k_{2\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + P_6(k_{1\sigma}k_{1\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + 2P_6(k_{1\nu}k_{2\gamma}\eta_{\beta\mu}\eta_{\alpha\sigma})\\
&\quad + 2P_3(k_{1\nu}k_{2\mu}\eta_{\beta\sigma}\eta_{\gamma\alpha}) - 2P_3(k_1 \cdot k_2\eta_{\alpha\nu}\eta_{\beta\sigma}\eta_{\gamma\mu})]
\end{aligned}$$

About 100 terms in three vertex

Naïve conclusion: Gravity is a headache

Simplicity of Gravity Amplitudes $k_i^2 = 0$

On shell three vertex contains all necessary information:



Gauge theory: $-gf^{abc}(\eta_{\mu\nu}(k_1 - k_2)_\rho + \text{cyclic})$

Gravity:

$$i\kappa\left((\eta_{\mu\nu}(k_1 - k_2)_\rho + \text{cyclic}) \times (\eta_{\alpha\beta}(k_1 - k_2)_\gamma + \text{cyclic})\right)$$

"Gravity as
the square
of YM"

Any gravity scattering amplitude constructible solely from on-shell 3 vertex.

BCFW on-shell recursion for tree amplitudes.

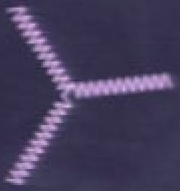
Britto, Cachazo, Feng and Witten; Brandhuber, Travaglini, Spence; Cachazo, Svrcek; Benincasa, Boucher-Veronneau, Cachazo; Arkani-Hamed and Kaplan, Hall

Unitarity method for loops.

Bern, Dixon, Dunbar and Kosower; Bern, Dixon, Kosower; Britto, Cachazo, Feng;

Unification of Color and Kinematics

Bern, JJMC, Johansson



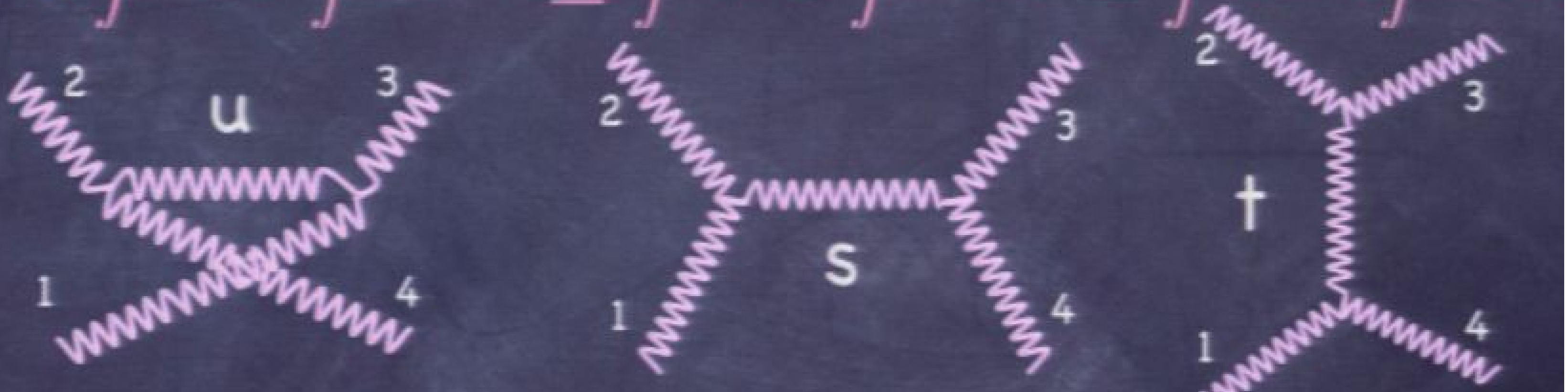
color factor

$$-gf^{abc}(\eta_{\mu\nu}(k_1-k_2)_\rho+\text{cyclic})$$

momentum dependent kinematic factor

olor factors based on a Lie Algebra: $f^{abc}=\mathrm{Tr}([T^a,T^b]T^c)$

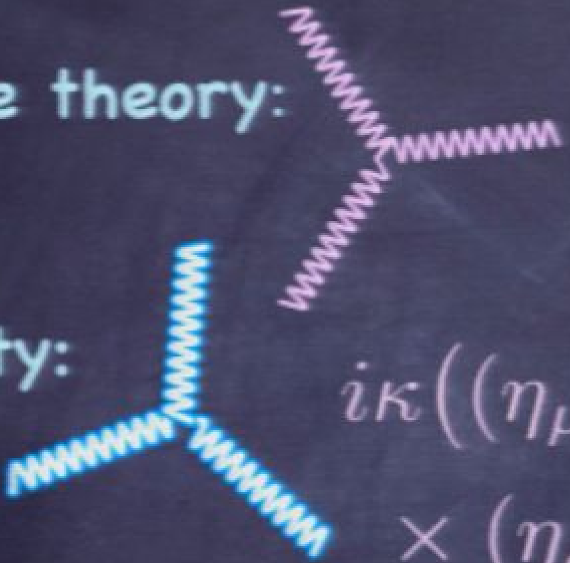
olor Jacobi Identity: $f^{a_4a_2b}f^{ba_3a_1}=f^{a_1a_2b}f^{ba_3a_4}-f^{a_1ba_4}f^{ba_2a}$



$u=(k_1+k_3)^2$ $s=(k_1+k_2)^2$ $t=(k_1+k_4)^2$

Simplicity of Gravity Amplitudes $k_i^2 = 0$

On shell three vertex contains all necessary information:



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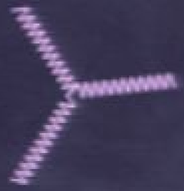
Unification of Color and Kinematics

Bern, JJMC, Johansson



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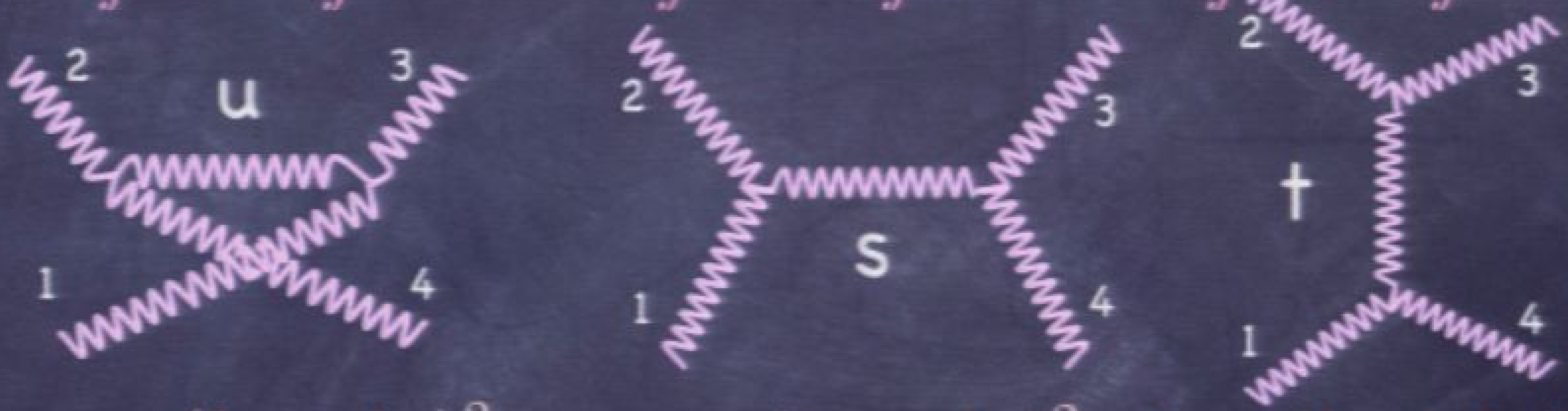

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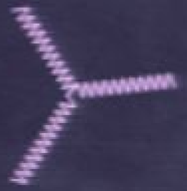

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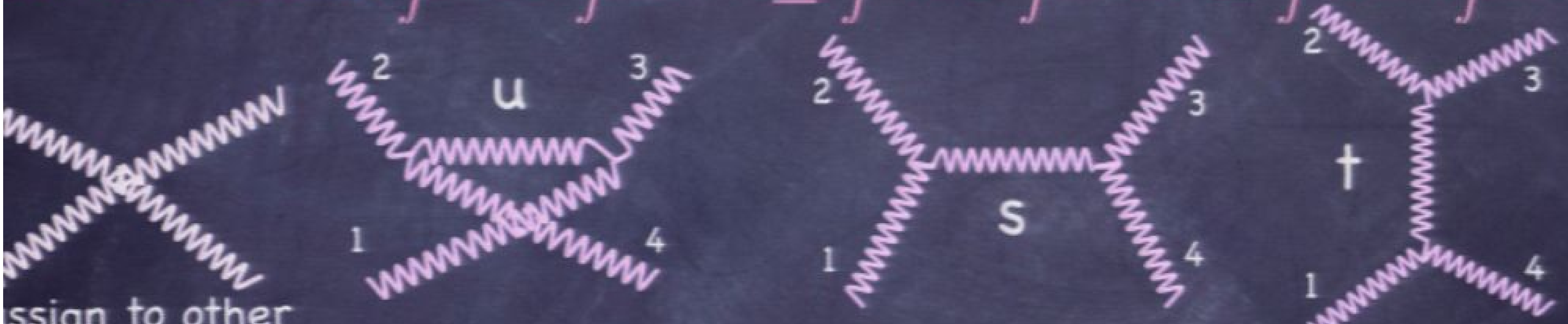


ssign to other
diags using
u/u=s/s=t/t=1

$u=(k_1+k_3)^2$ $s=(k_1+k_2)^2$ $t=(k_1+k_4)^2$

$$A_4^{\text{tree}} = g^2\Big(\frac{n_u c_u}{u} + \frac{n_s c_s}{s} + \frac{n_t c_t}{t}\Big)$$

olor factor obeys the Jacobi identity: $c_u = c_s - c_t$

kinematic factor obeys same identity: $n_u = n_s - n_t$

Bern, JJMC, Johansson

Claim: you can always write gauge tree amplitudes

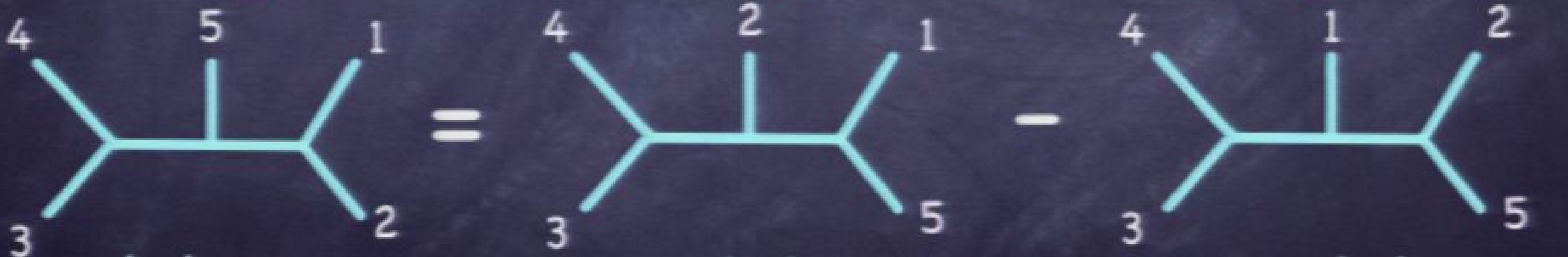
$$A_n^{\text{tree}}(1,2,3,\ldots n) = g^{n-2}\sum_i (c_i n_i \times \text{TreeDiag}_i)$$

uch that the **kinematic factors (n)** obey the
ame **Jacobi identity** as the **color factors (c)**

$$_i = c_j - c_k \Longrightarrow n_i = n_j - n_k$$

$\uparrow$ **Denoms** $\frac{1}{\left[\prod_j p_j^2\right]_i}$ **associated with color ordered diag (i)**

pt example n_3 n_5 n_8



$$_3 \equiv f^{a_3a_4b}f^{ba_5c}f^{ca_1a_2} \quad c_5 \equiv f^{a_3a_4b}f^{ba_2c}f^{ca_1a_5} \quad c_8 \equiv f^{a_3a_4b}f^{ba_1c}f^{ca_2a_5}$$

Nontrivial constraints on amplitudes -> (n-3)! indep

Recent string theory understanding: Bjerrum-Bohr, Damgaard, Vanhove; Stieberger

KLT field expressions:

Kawai, Lewellen, Tye

Bern, Dixon, Perelstein, Rozowsky

Gravity tree amplitudes

$$M_n^{\text{tree}}(\underline{1},\ldots,\underline{n-1},\underline{n}) \quad = \qquad \begin{matrix} i \in \{2,\ldots,n/2\} \\ j \in \{n/2+2,\ldots,n-2 \end{matrix}$$

$$(-1)^{n+1} \sum_{perms(2,\ldots,n-2)} \Big[A_n^{\text{tree}}(\underline{1},\ldots,\underline{n-1},\underline{n}) \sum_{perms(i,j)} f(i_1,\ldots,i_j)$$

$$\times \bar{f}(l_1,\ldots,l_{j'}) \widetilde{A}_n^{\text{tree}}(i_1,\ldots,i_j,\underline{1},\underline{n-1},l_1,\ldots,l_{j'},\underline{n}) \Big]$$

Color-ordered gauge tree amplitudes

$$f(i_1,\ldots,i_j) \; = \; s_{1,i_j} \prod_{m=1}^{j-1} \left(s_{1,i_m} + \sum_{k=m+1}^{j} g(i_m,i_k) \right),$$

$$\bar{f}(l_1,\ldots,l_{j'}) \; = \; s_{l_1,n-1} \prod_{m=2}^{j'} \left(s_{l_m,n-1} + \sum_{k=1}^{m-1} g(l_k,l_m) \right)$$

$$g(i,j) = \left\{ \begin{array}{ll} s_{i,j} & \text{if } i > j \\ 0 & \text{else} \end{array} \right\} \qquad s_{a,b} = (k_a + k_b)^2$$

KLT field expressions:

Kawai, Lewellen,Tye **Bern, Dixon, Perelstein, Rozowsky**

Gravity tree amplitudes

$$i \in \{2,\ldots,n/2\}$$
$$j \in \{n/2+2,\ldots,n-2\}$$

$$M_n^{\text{tree}}(\underline{1},\ldots,\underline{n-1},\underline{n}) \quad = $$

$$(-1)^{n+1} \sum_{perms(2,\ldots,n-2)} \Big[A_n^{\text{tree}}(\underline{1},\ldots,\underline{n-1},\underline{n}) \sum_{perms(i,j)} f(i_1,\ldots,i_j)$$

$$\times\, \overline{f}(l_1,\ldots,l_{j'})\widetilde{A}_n^{\text{tree}}(i_1,\ldots,i_j,\underline{1},\underline{n-1},l_1,\ldots,l_{j'},\underline{n})\Big]$$

Color-ordered gauge tree amplitudes

New relations allow re-expression of KLT in terms of different "basis" amplitudes: Left-right symmetric, etc.

But we can do better...

Higher-Point Gravity and Gauge Theory

Bern, JJMC, Johansson

QCD:
$$\mathcal{A}_n^{\text{tree}} = i g^{n-2} \sum_i \frac{n_i c_i}{D_i}$$

sum over diagrams with only cubic vertices

Einstein Gravity:
$$\mathcal{M}_n^{\text{tree}} = i \kappa^{n-2} \sum_i \frac{n_i \tilde{n}_i}{D_i}$$

Relation extremely useful in high-loop gravity calculations.)

Remarkably simple re-expression of field theory limit of String Theory's Kawai, Lewellen and Tye relations

Physical principle involved?

Local unity of geometry and glue?

Tour of the **remarkable cancellations** observed in N=8 Supergravity

Exciting Proposition:

Perturbatively finite QFT of gravity in 4D

Why surprising if possible:

Dimensionful coupling: $\kappa \sim m_{pl}^{-1}$ → non-renormalizable

No known **structure** to make up diff btw

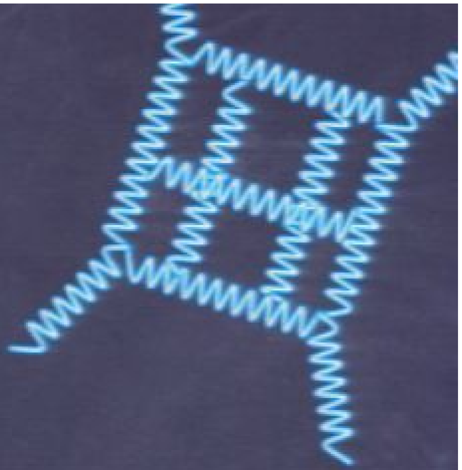
$$\frac{(\kappa\ p^{\mu}p^{\nu})\cdots}{\text{propagators}}$$ gravity

and

$$\frac{(g\ p^{\mu})\cdots}{\text{propagators}}$$ gauge

Any responsible mechanism would fundamentally impact our understanding of gravity

Evidence of spectacular cancellations
n $\mathcal{N}$=8 **Supergravity**!



Everywhere we look has the same powercounting as $\mathcal{N}$=4 super Yang-Mills

$$D_c = 4 + 6/L$$

$\mathcal{N}$=8 Supergravity

$2^8 = 256$ massless states, ~ expansion of $(x+y)^8$

$$\mathcal{N}=8: \quad 1 \leftrightarrow 8 \leftrightarrow 28 \leftrightarrow 56 \leftrightarrow 70 \leftrightarrow 56 \leftrightarrow 28 \leftrightarrow 8 \leftrightarrow 1$$

$$\text{helicity}: \quad -2 \quad -\tfrac{3}{2} \quad -1 \quad -\tfrac{1}{2} \quad 0 \quad \tfrac{1}{2} \quad 1 \quad \tfrac{3}{2} \quad 2$$

SUSY ↔

$$h^- \quad \psi_i^- \quad v_{ij}^- \quad \chi_{ijk}^- \quad s_{ijkl} \quad \chi_{ijk}^+ \quad v_{ij}^+ \quad \psi_i^+ \quad h^+$$

Cremmer and Julia

$$\mathcal{N}=4 \text{ SYM}: \quad 1 \quad 4 \quad 6 \quad 4 \quad 1$$

$2^4 = 16$ states
~ expansion of
$(x+y)^4$

$$g^- \quad \lambda_A^- \quad \phi_{AB} \quad \lambda_A^+ \quad g^+$$

all in adjoint representation

$$D_c = 4 + 6/L$$

$$\boxed{\left[\mathcal{N}=8\right] = \left[\mathcal{N}=4\right] \otimes \left[\mathcal{N}=4\right]}$$

Opinions from the 80's

If certain patterns that emerge should persist in the higher orders of perturbation theory, then ... N = 8 supergravity in four dimensions would have ultraviolet divergences starting at three loops.

Green, Schwarz, Brink, (1982)

There are no miracles... It is therefore very likely that all supergravity theories will diverge at three loops in four dimensions. ... The final word on these issues may have to await further explicit calculations.

Marcus, Sagnotti (1985)

The idea that all supergravity theories diverge has been widely accepted for over 25 years

N = 8 Supergravity No-Triangle Property

Bern, Dixon, Perelstein, Rozowsky; Bern, Bjerrum-Bohr and Dunbar; Bjerrum-Bohr, Dunbar, Ita, Perkins, Risager; **Proven by** Bjerrum-Bohr and Vanhove; Arkani-Hamed, Cachazo and Kaplan.

One-loop D = 4 theorem: Any one loop amplitude is a linear combination of scalar box, triangle and bubble integrals with rational coefficients:

Brown, Feynman; Passarino and Veltman, etc

$$A_n^{1-\mathrm{loop}} = \sum_i d_i I_4^{(i)} + \sum_j c_j I_3^{(j)} + \sum_k b_k I_2^{(k)}$$

$$\int \frac{d^4p}{(p^2)^4} \qquad \int \frac{d^4p}{(p^2)^3} \qquad \int \frac{d^4p}{(p^2)^2}$$

The "no-triangle property" of N=8 SUGRA (N=4 sYM):

one-loop reduce only to boxes -- no further.

Unordered nature of gravity is important for this property. Bjerrum-Bohr, Vanhove

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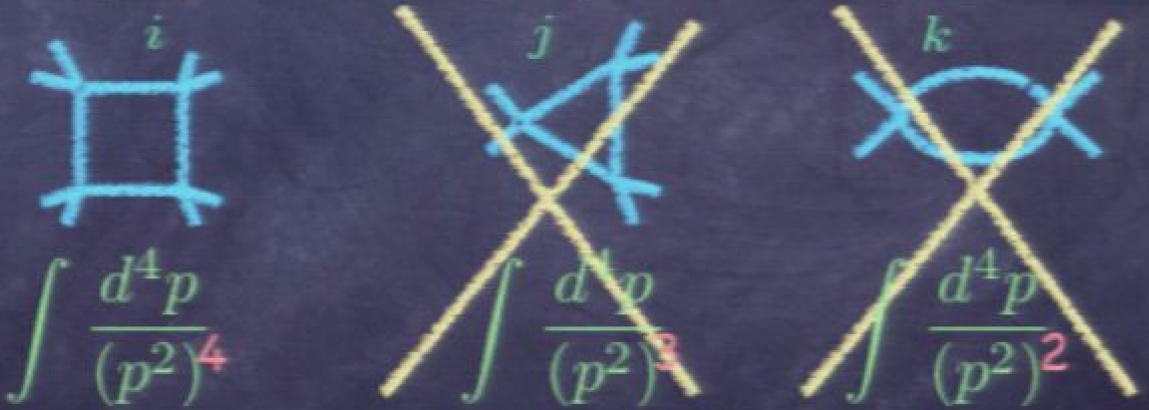

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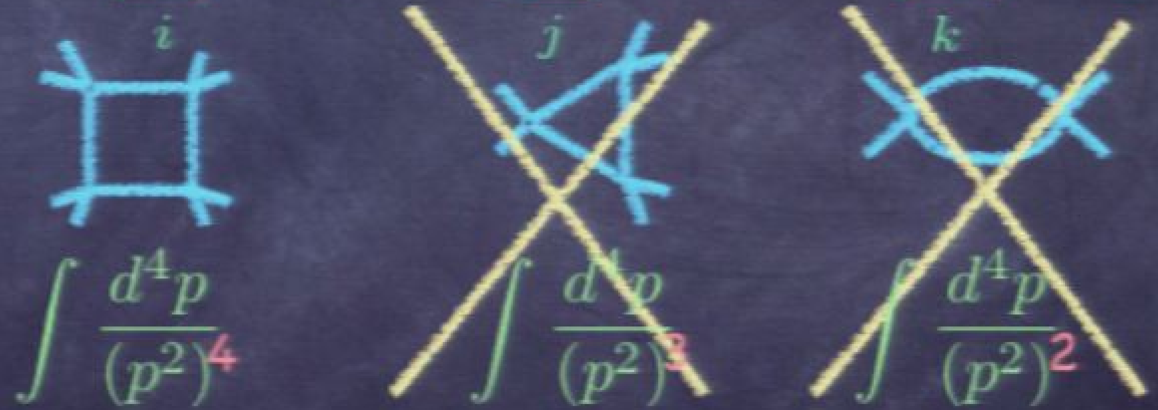

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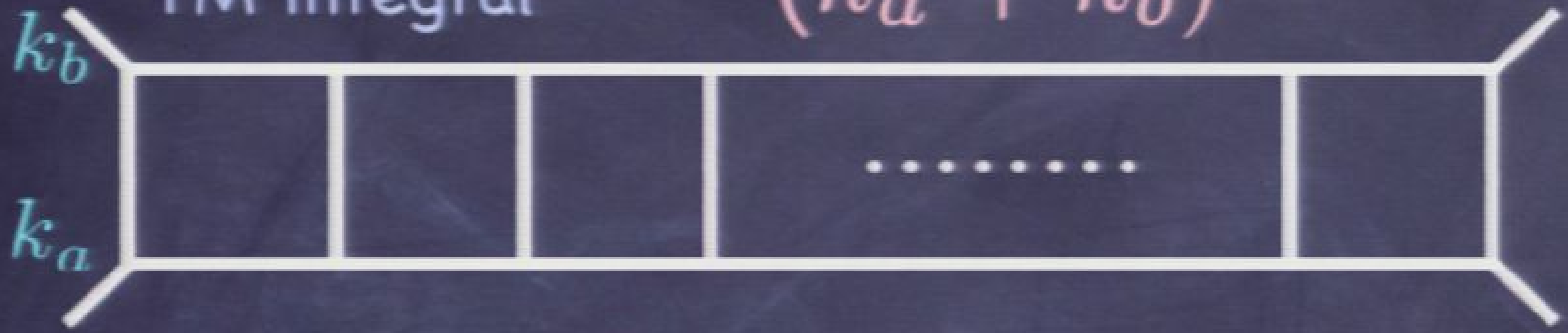
Elegant observation of Zvi, Lance, and Radu

Best behaved n-loop
YM integral

$(k_a + k_b)^{2(n-1)}$

N=4



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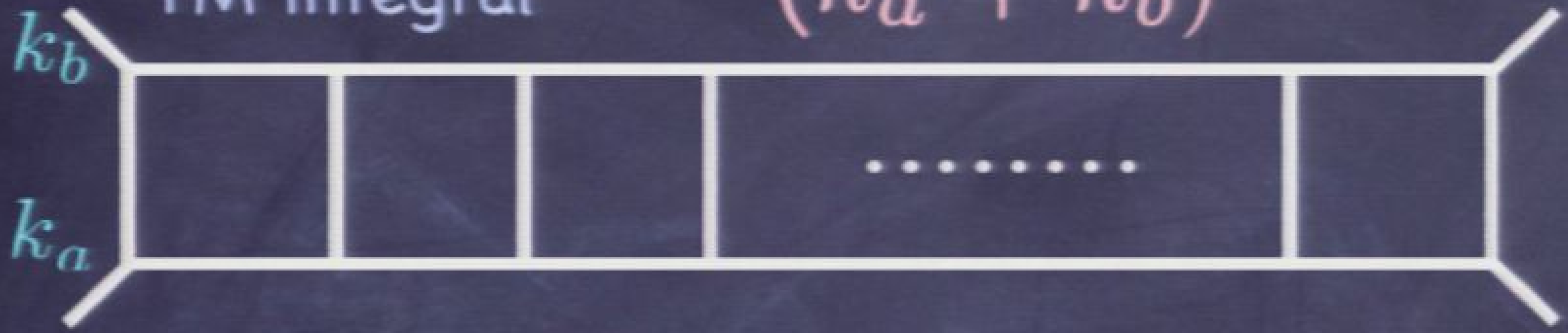

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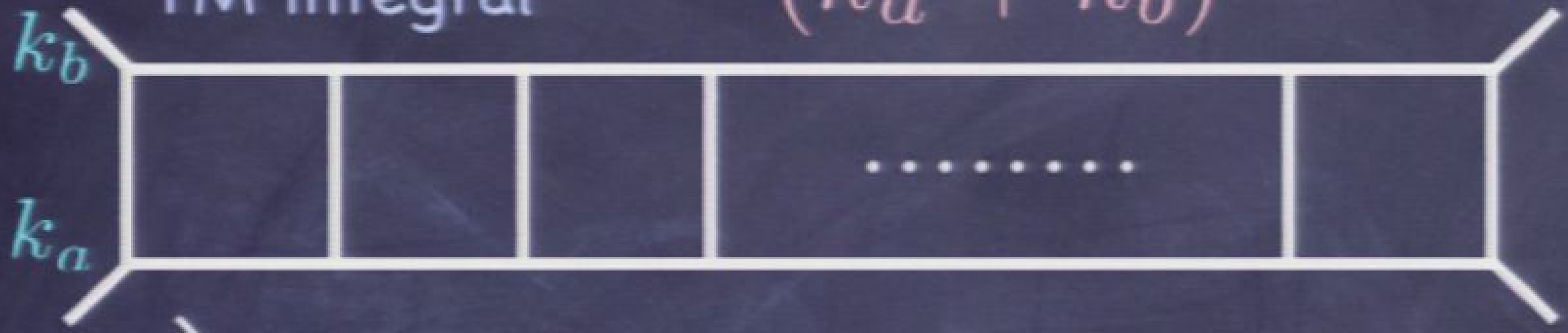
Best behaved n-loop YM integral

N=4



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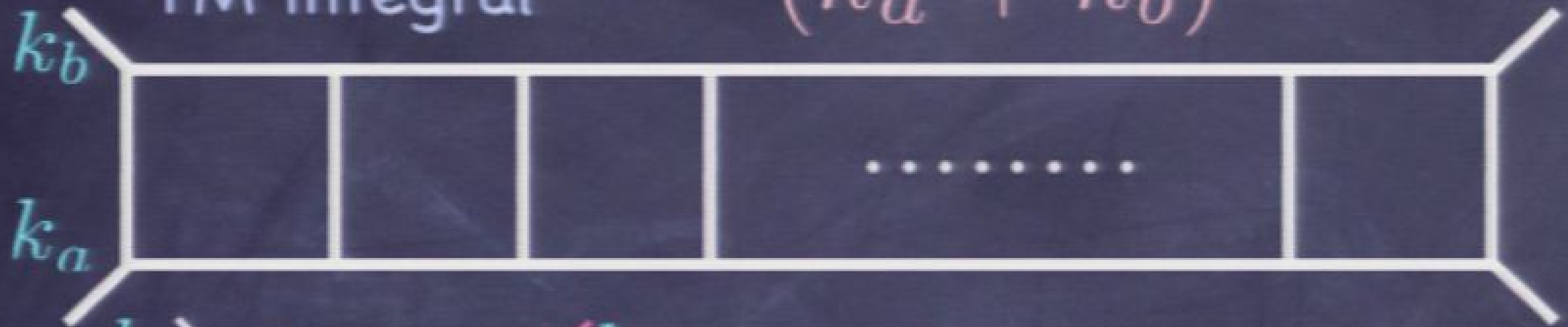

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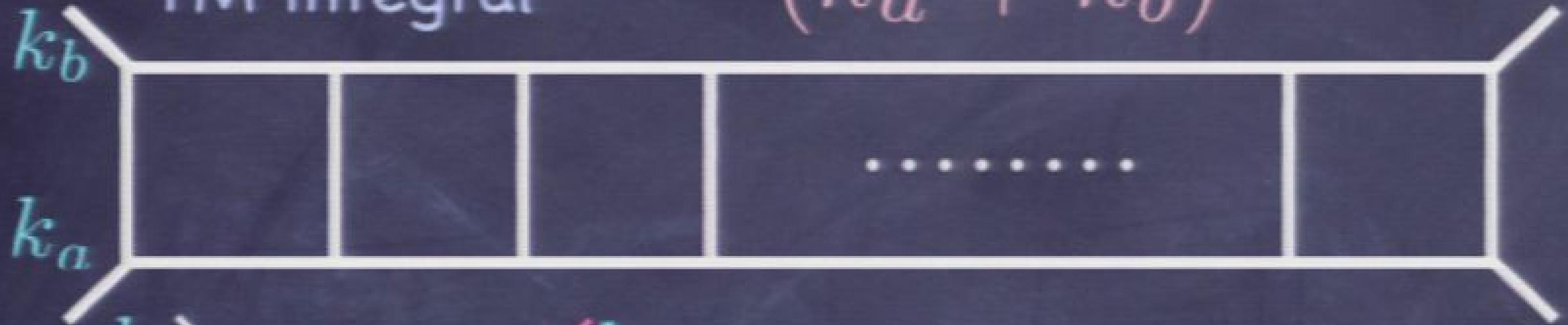
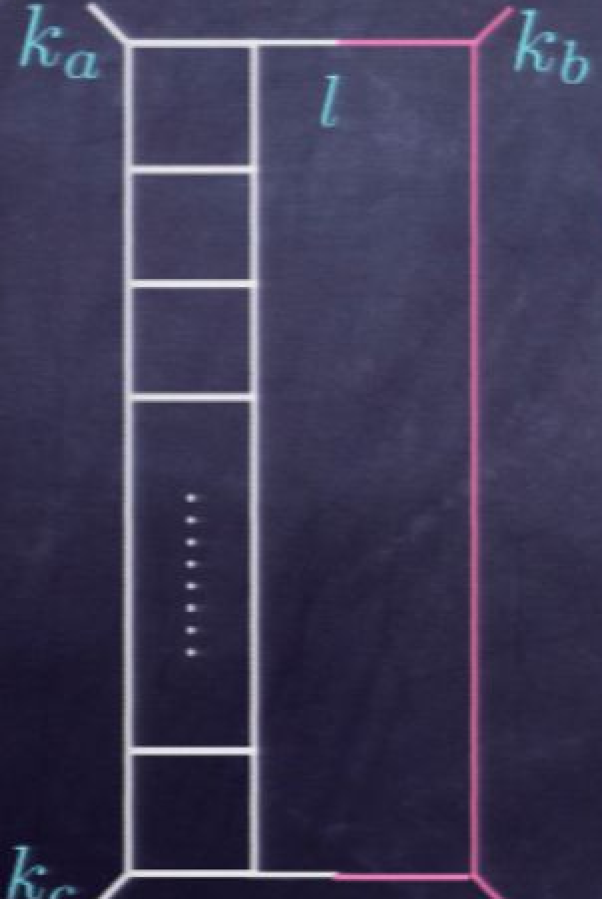




$$(k_a + l)^{2(n-1)}(k_a + k_b)^2$$

$$\sim l^{2(n-1)}$$

Worst behaved (n+1)-loop YM integral

Best behaved n-loop SUGRA integral

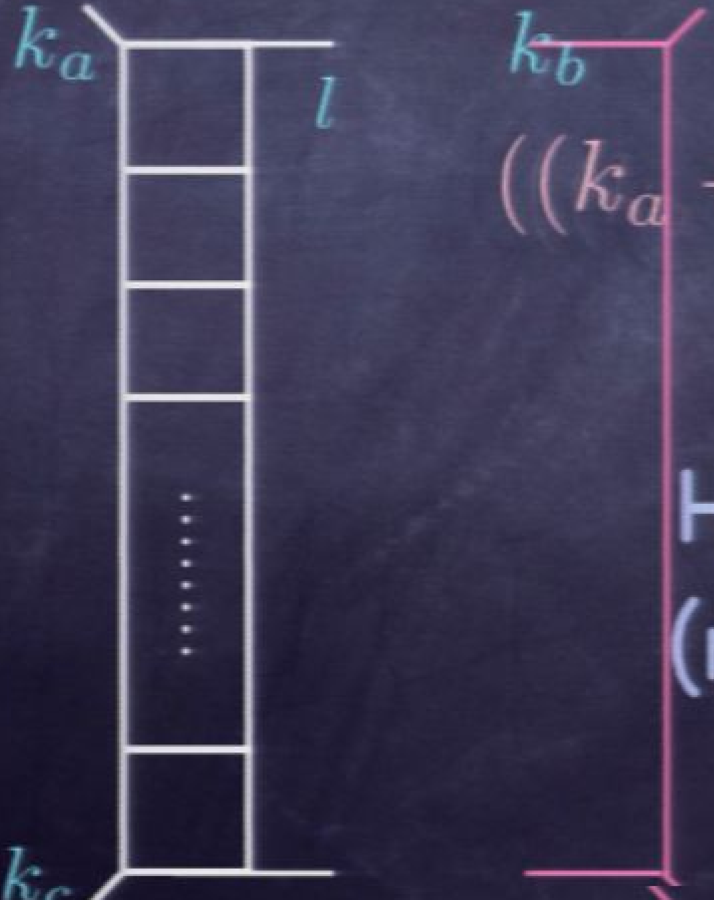
$$\left((k_a + k_b)^{2(n-1)}\right)^2$$

N=8

N=8

Best behaved n-loop SUGRA integral

$$\left((k_a+k_b)^{2(n-1)}\right)^2$$



$$\left((k_a+l)^{2(n-1)}\right)^2\left((k_a+k_b)^2\right)^2$$

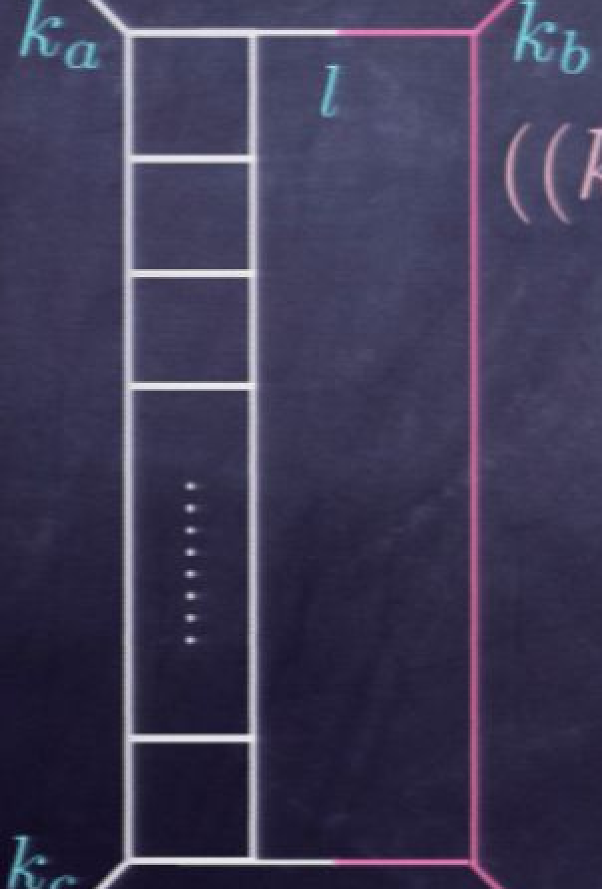
$$\sim l^{4(n-1)}$$

Horribly behaved (n+1)-loop SUGRA integral

N=8

Best behaved n-loop SUGRA integral

$$\left((k_a+k_b)^{2(n-1)}\right)^2$$



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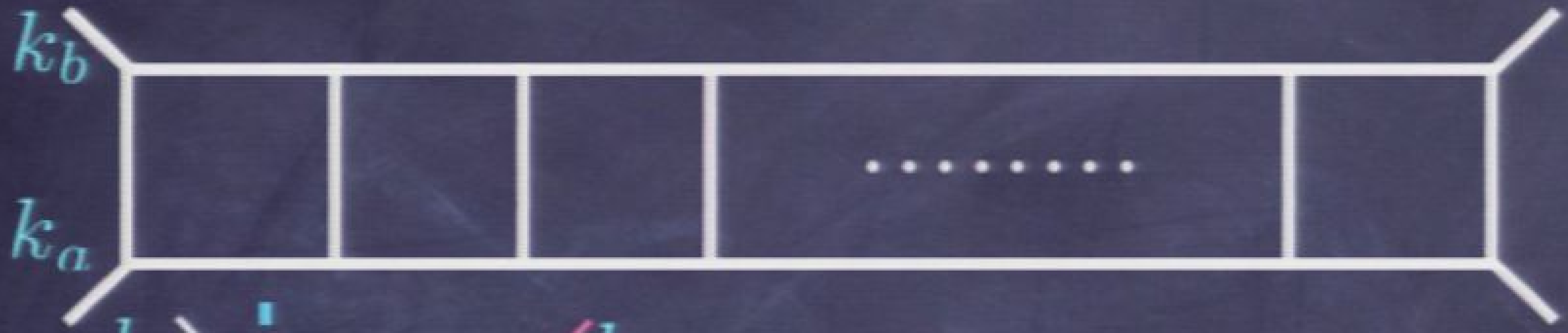

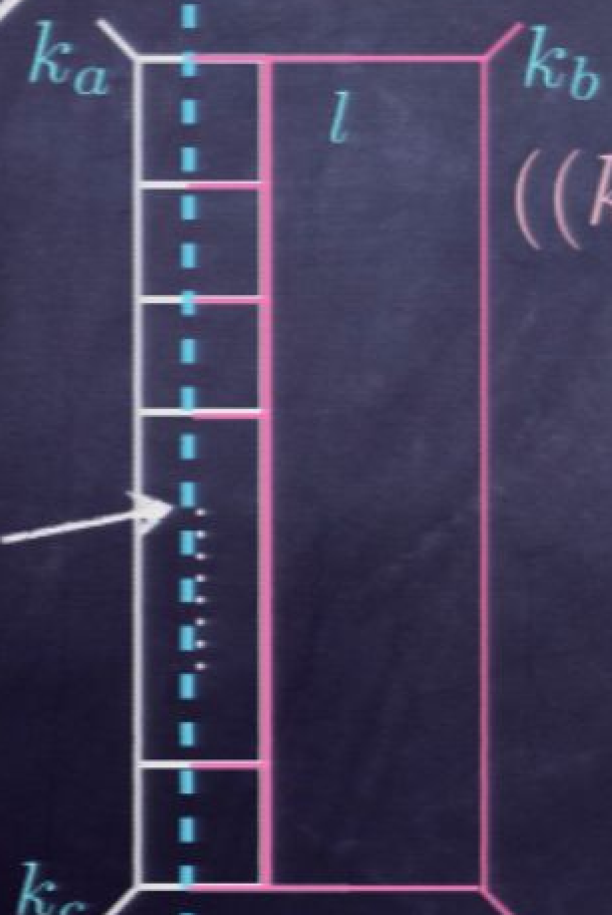

onsider
his cut

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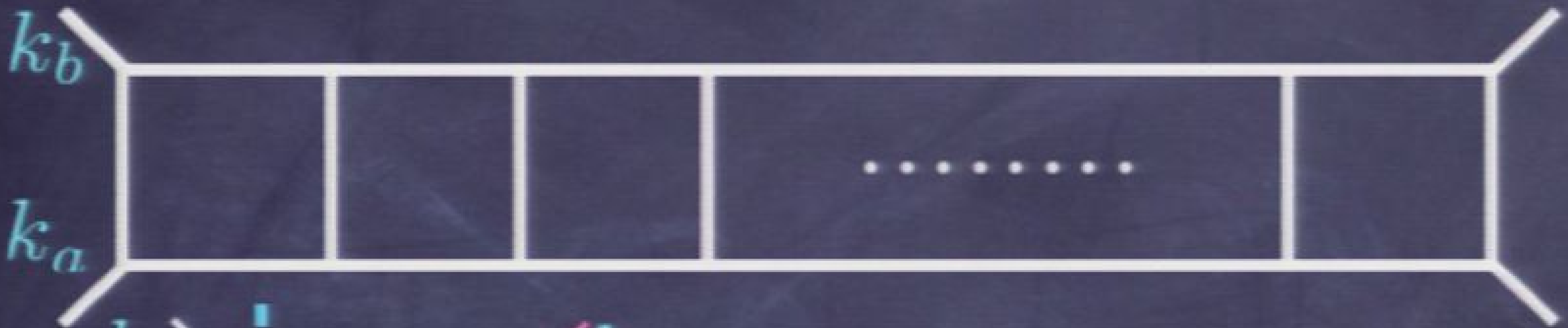

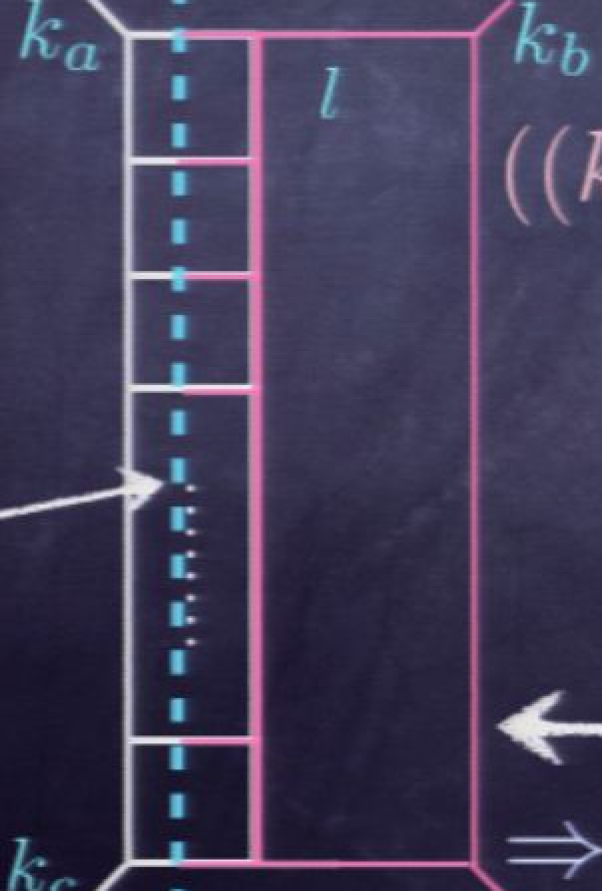

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$$\left((k_a+l)^{2(n-1)}\right)^2\left((k_a+k_b)^2\right)^2$$

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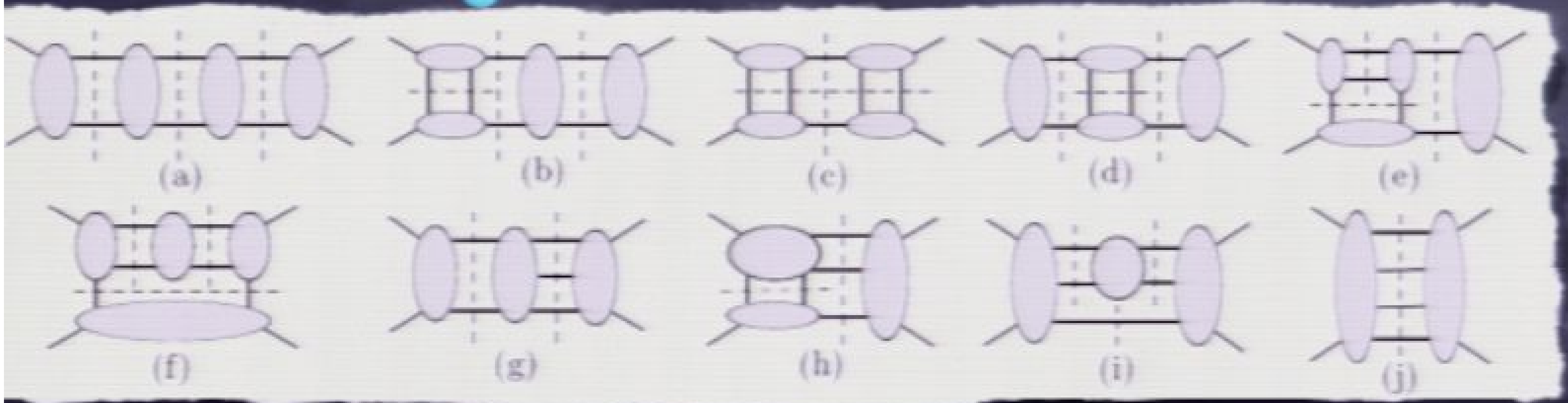
Severely violates no-triangle property of one loop

$\Rightarrow$ cancellations to $l^{2(n-1)}$

Full Three-Loop Calculation

Bern, JJMC, Dixon, Johansson, Kosower, Roiban

eed following cuts:



reduces calculation to product of tree amplitudes

or cut (g) have:

$$\sum_{=8\ \text{states}} M_4^{\text{tree}}(1,2,l_3,l_1)\times M_5^{\text{tree}}(-l_1,-l_3,q_3,q_2,q_1)\times M_5^{\text{tree}}(3,4,-q_1,-q_2,-q_3)$$

se QFT limit Kawai-Lewellen-Tye tree relations

Kawai, Lewellen,Tye

Bern, Dixon, Perelstein, Rozowsky

$$M_4^{\text{tree}}(1,2,l_3,l_1) = -is_{12}A_4^{\text{tree}}(1,2,l_3,l_1)\widetilde{A}_4^{\text{tree}}(2,1,l_3,l_1)$$

$$M_5^{\text{tree}}(-l_1,-l_3,q_3,q_2,q_1) = i\, s_{l_1q_1}s_{l_3q_3}A_5^{\text{tree}}(-l_1,-l_3,q_3,q_2,q_1)\,\widetilde{A}_5^{\text{tree}}(-l_1,q_1,q_3,-l_3,q_2) + \{l_1\leftrightarrow l_3\}\,,$$

supergravity

super-Yang-Mills

N = 8 supergravity cuts are sums of products of N = 4 super-Yang-Mills cuts.

KLT Factorizes on the Cut!

The supersum over N=8
states for a SUGRA cut

factorizes via KLT into

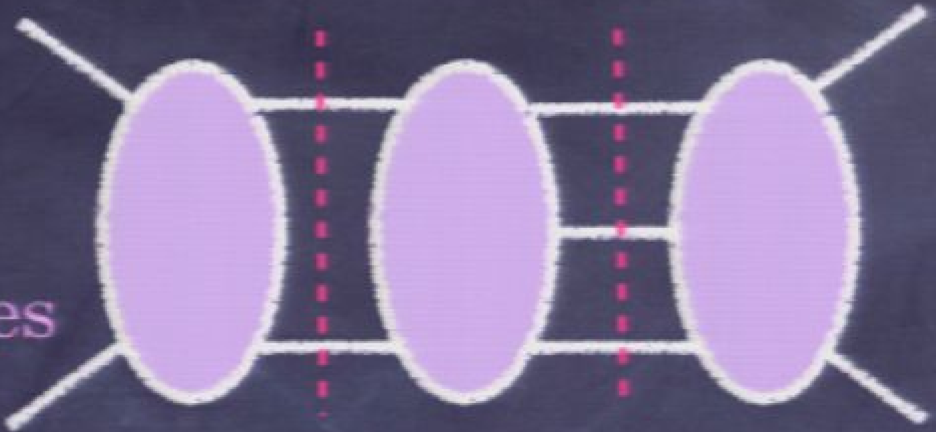
$\sum_{\text{erms}} \text{momenta} \times ($ The supersum over N=4
states for a sYM cut $) \times$

$\sim$

$($ The supersum over N=4
states for a sYM cut $)$

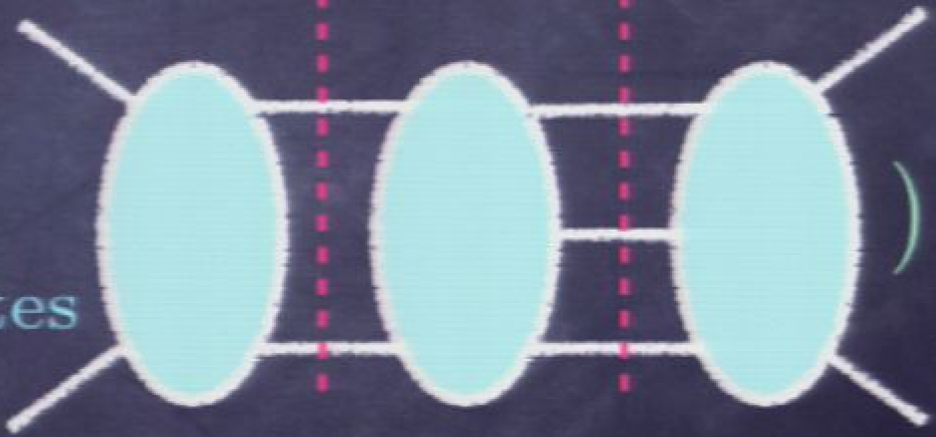
KLT Factorizes on the Cut!

$\sum\limits_{=8 \text{ states}}$



$=$

$\sum\limits_{\text{erms}} \text{momenta} \times \Bigg(\sum\limits_{\mathcal{N}=4 \text{ states}}$



$\Bigg) \times$

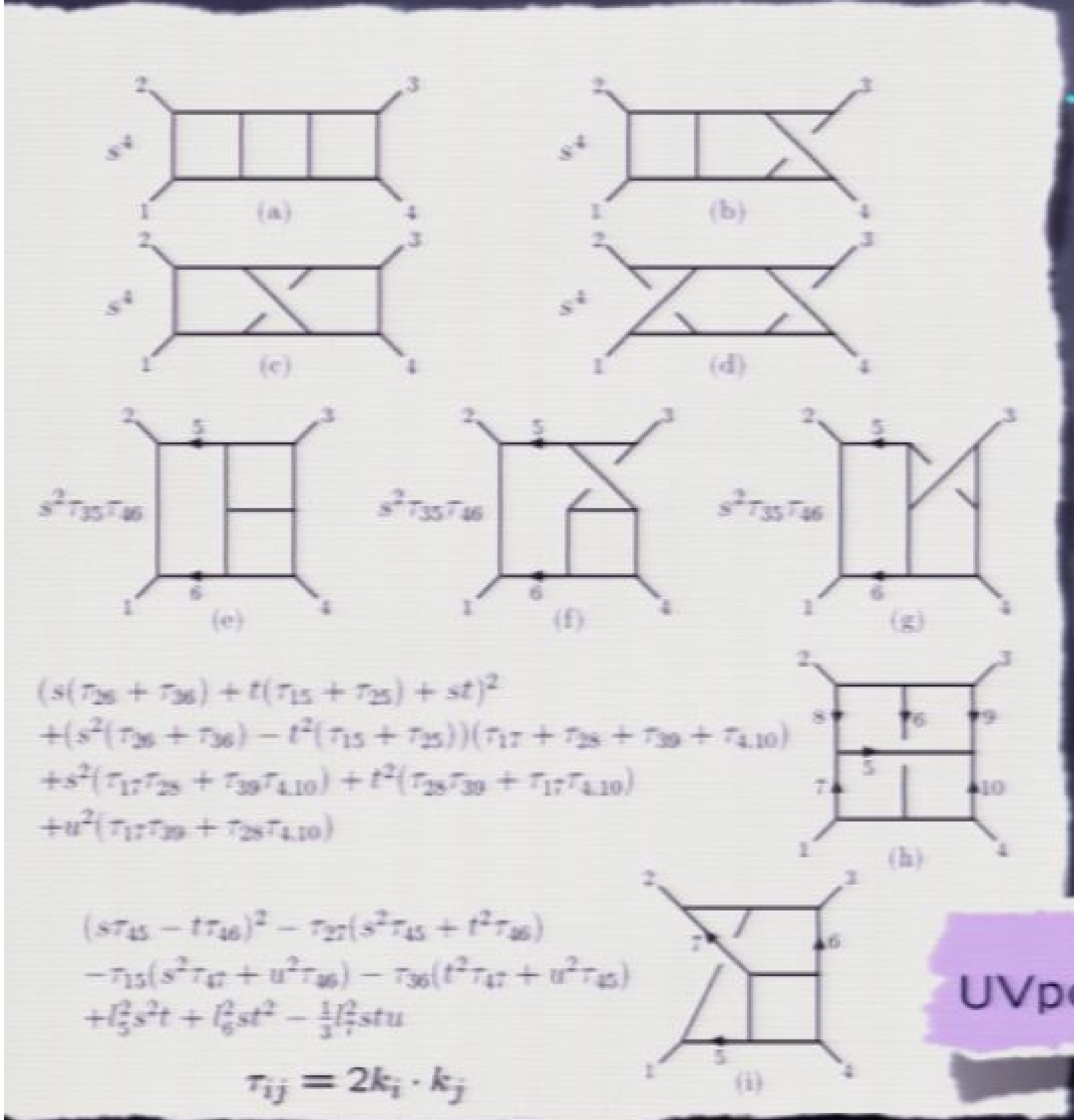
$\Bigg(\sum\limits_{\mathcal{N}=4 \text{ states}}$

$\Bigg)$

Complete Three-Loop N = 8 Supergravity Result

Bern, JJMC, Dixon, Johansson, Kosower, Roiban; hep-th/0702112

Bern, JJMC, Dixon, Johansson, Roiban arXiv:0808.4112 [hep-th]



$$(s(\tau_{26}+\tau_{36})+t(\tau_{15}+\tau_{25})+st)^2$$
$$+(s^2(\tau_{26}+\tau_{36})-t^2(\tau_{15}+\tau_{25}))(\tau_{17}+\tau_{28}+\tau_{39}+\tau_{4,10})$$
$$+s^2(\tau_{17}\tau_{28}+\tau_{39}\tau_{4,10})+t^2(\tau_{28}\tau_{39}+\tau_{17}\tau_{4,10})$$
$$+u^2(\tau_{17}\tau_{39}+\tau_{28}\tau_{4,10})$$

$$(s\tau_{45}-t\tau_{46})^2-\tau_{27}(s^2\tau_{45}+t^2\tau_{46})$$
$$-\tau_{15}(s^2\tau_{47}+u^2\tau_{46})-\tau_{36}(t^2\tau_{47}+u^2\tau_{45})$$
$$+l_5^2s^2t+l_6^2st^2-\tfrac{1}{3}l_7^2stu$$

$$\tau_{ij}=2k_i\cdot k_j$$

$$\mathcal{M}_4^{(3)} \propto \sum_{\text{ext. leg perms}} \text{9 integrals}$$

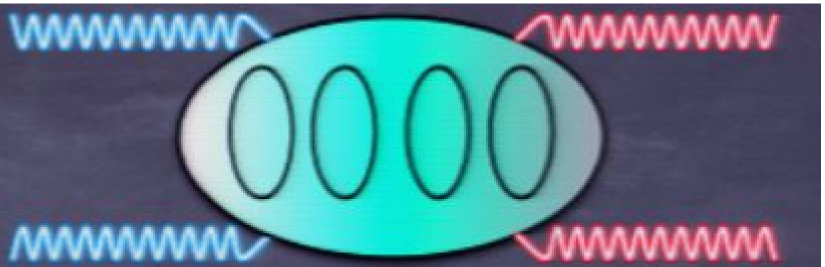
Cancellations beyond those needed for finiteness in D = 4.

Finite for D < 6

$$\text{UVpole}_{D=6-2\epsilon} = \frac{1}{\epsilon}\frac{5\zeta_3}{(4\pi)^9}\left(\frac{\kappa}{2}\right)^8(stu)^2M_4^{\text{tree}}$$

Identical power count as N = 4 super-Yang-Mills

Why go after Four Loops?



1. To find structure responsible for cancellations, we want more data! (3-loops was only first chance to diverge from gauge-like powercounting.)

2. Bossard, Howe, Stelle predicted D = 5, L = 4 divergence from algebraic methods 0901.4661 [hep-th] (2009)

> "The algebraic formalism [...] suggests that maximal supergravity is likely to diverge at four loops in D = 5 and at five loops in D = 4, unless other infinity suppression mechanisms not involving super-symmetry or gauge invariance are at work."

Bossard, Howe & Stelle

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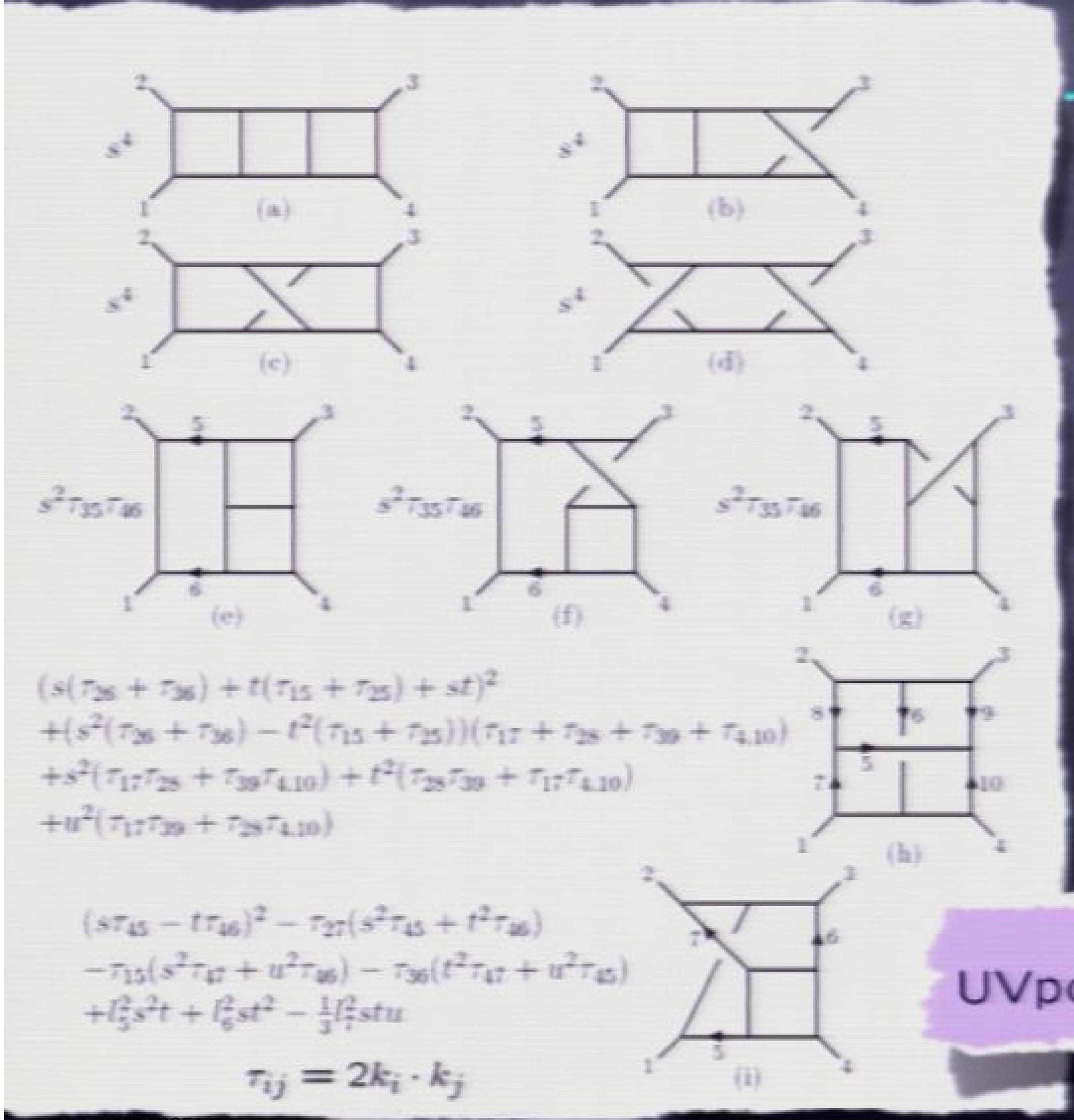

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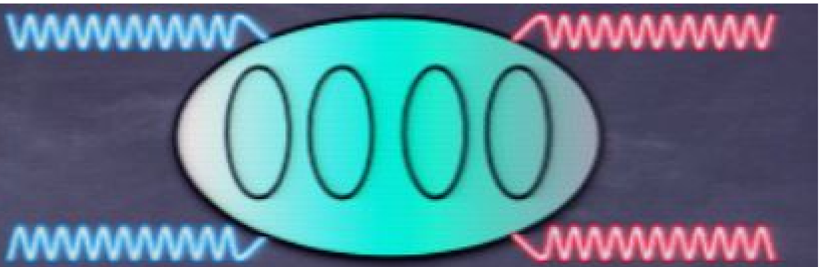

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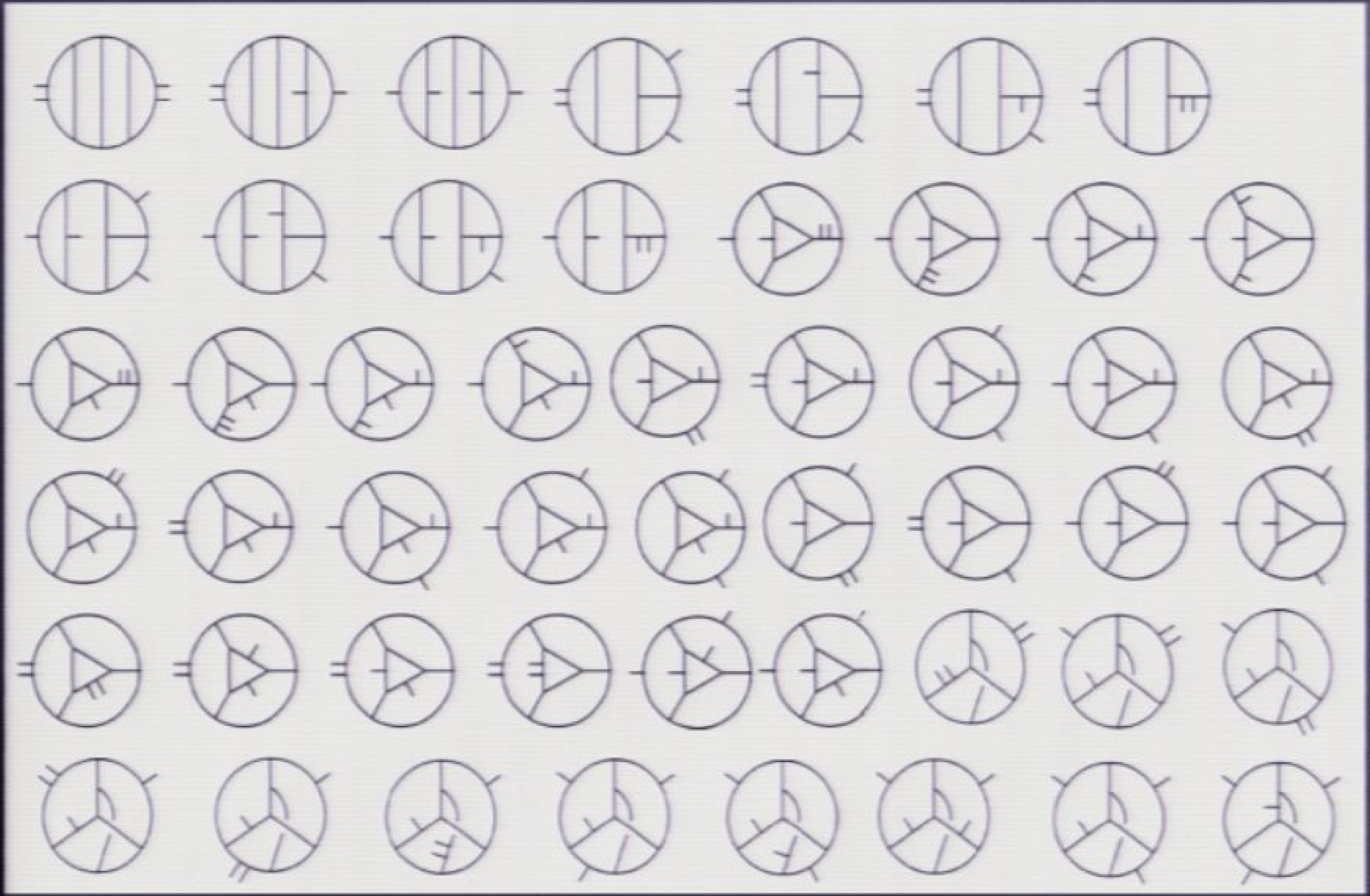
50 Integral topologies: “parent” diagrams



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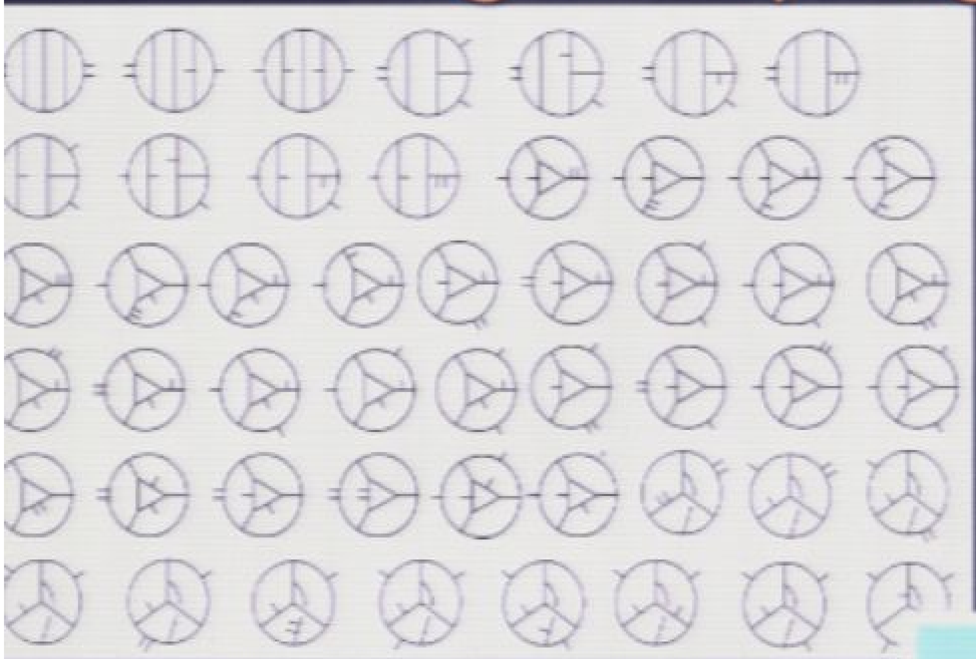
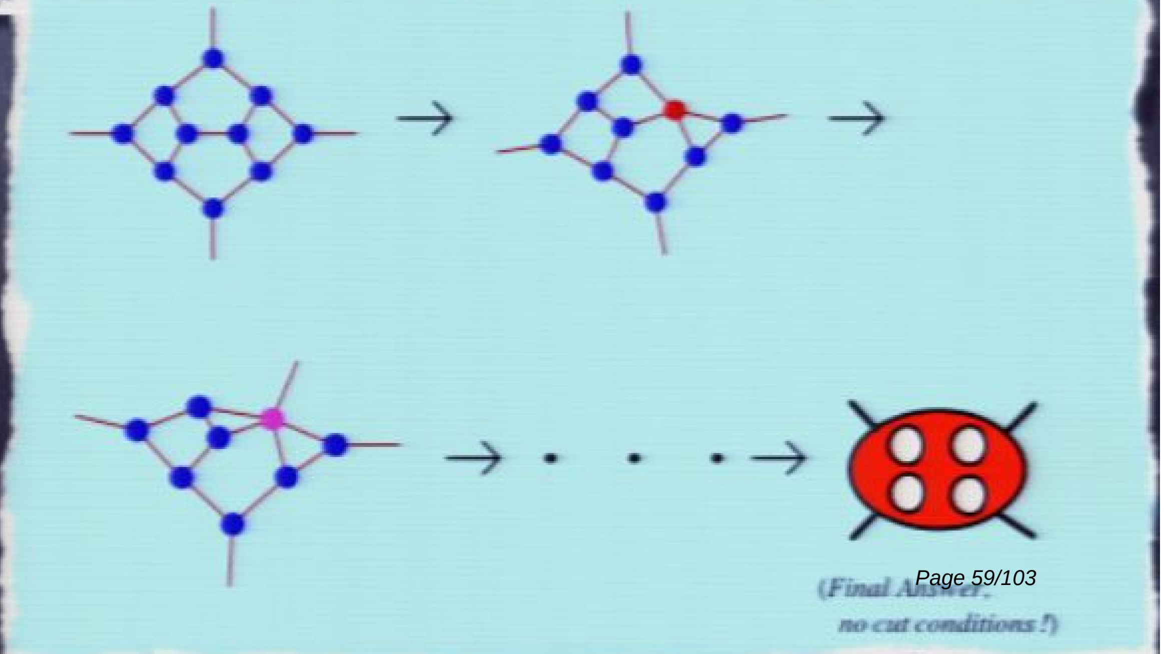
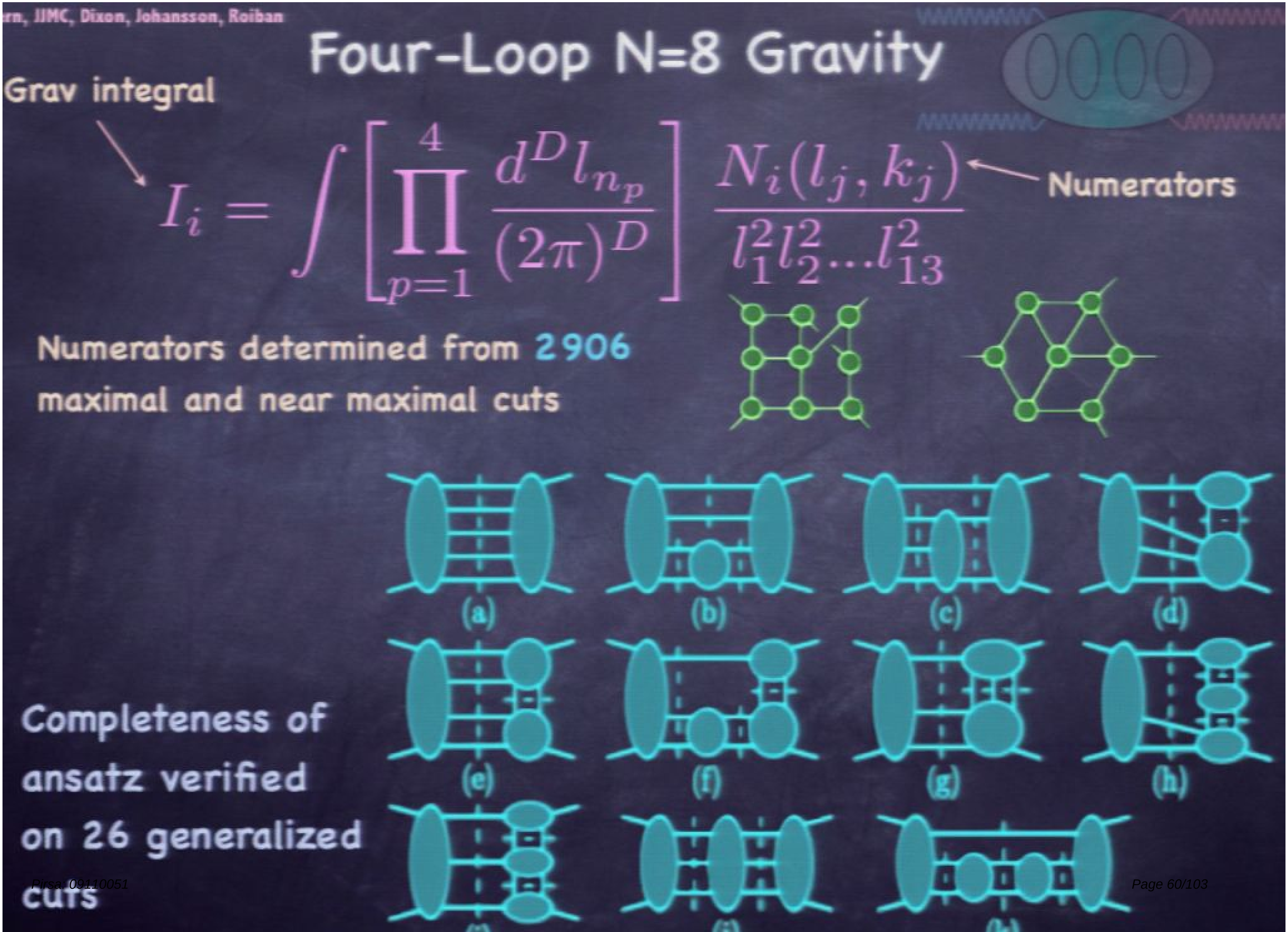




(Final Answer, no cut conditions !)



rn, JJMC, Dixon, Johansson, Roiban

Four-Loop N=8 Gravity

Grav integral

$$I_i = \int \left[\prod_{p=1}^{4} \frac{d^D l_{n_p}}{(2\pi)^D}\right] \frac{N_i(l_j, k_j)}{l_1^2 l_2^2 \ldots l_{13}^2}$$

Numerators

Numerators determined from 2906 maximal and near maximal cuts

Completeness of ansatz verified on 26 generalized cuts

UV Divergence at Four Loops

$$I_i = \int \left[\prod_{p=1}^{4} \frac{d^D l_{n_p}}{(2\pi)^D} \right] \frac{N_i(l_j, k_j)}{l_1^2 l_2^2 \ldots l_{13}^2}$$

Leading numerators $N_i \sim O(k^4 l^8)$

would have D = 4.5 divergence

k external

l internal:

too many are bad for UV

Represented by integrals which cancel in the full amplitude

Sub-leading divergence: $O(k^5 l^7)$

trivially vanishes under integration by Lorentz invariance

UV Divergence at Four Loops

lumerator
factor

$N_i \sim O(k^6 l^6)$ corresponding to D = 5 div.

Expand the integrands about small external momenta:

$$N_i^{(6)} + N_i^{(7)} \frac{K_n \cdot l_j}{l_j^2} + N_i^{(8)} \left(\frac{K_n^2}{l_j^2} + \frac{K_n \cdot l_j \; K_q \cdot l_p}{l_j^2 l_p^2} \right)$$

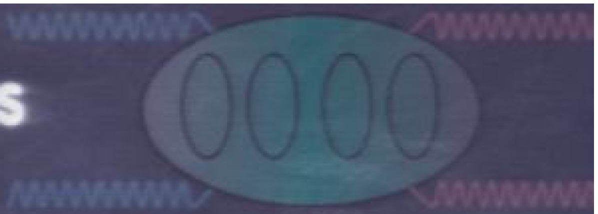
(K_i annotates sums
ver external momenta)

Marcus & Sagnotti UV extraction method

- Many ways of expanding the contributing ntegrands in terms of independent momenta.
- Each must be equivalent order by order in small external momenta.
- Equating expansions results in novel integral dentities

lumerator factor

UV Divergence at Four Loops



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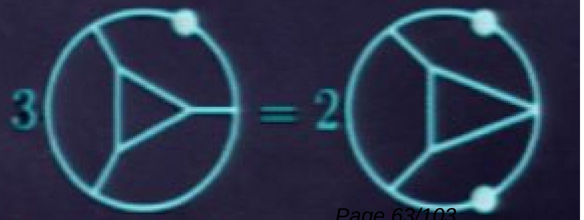
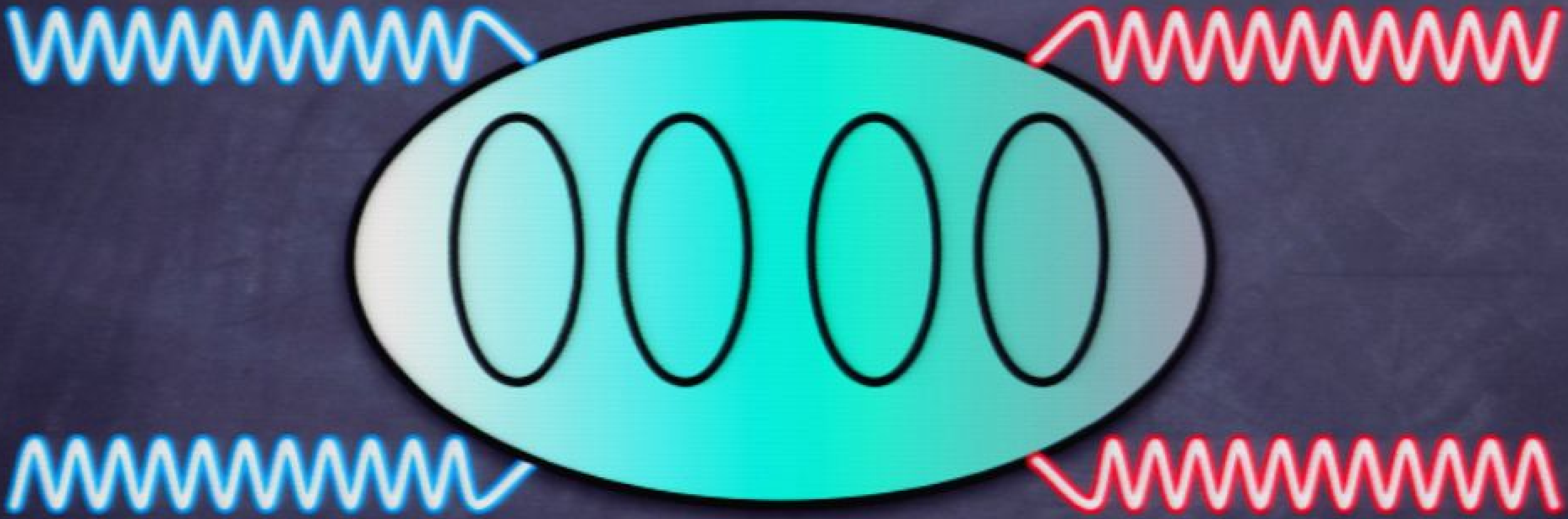


Verified by explicit integration!

Four Loop Four Point $\mathcal{N}$=8 SUGRA



is finite in D=5!

actually finite for D < 5.5

Four loops actually finite for $D < 5.5$

onstrains potential supersymmetry explanation of
hree loop result by Bossard, Howe, Stelle

The cancellations are stronger at 4 loops than at 3 loops, which is in turn stronger than at 2 loops. Surprising from traditional SUSY viewpoint.

Story's not over: there exists structure yet to be found.

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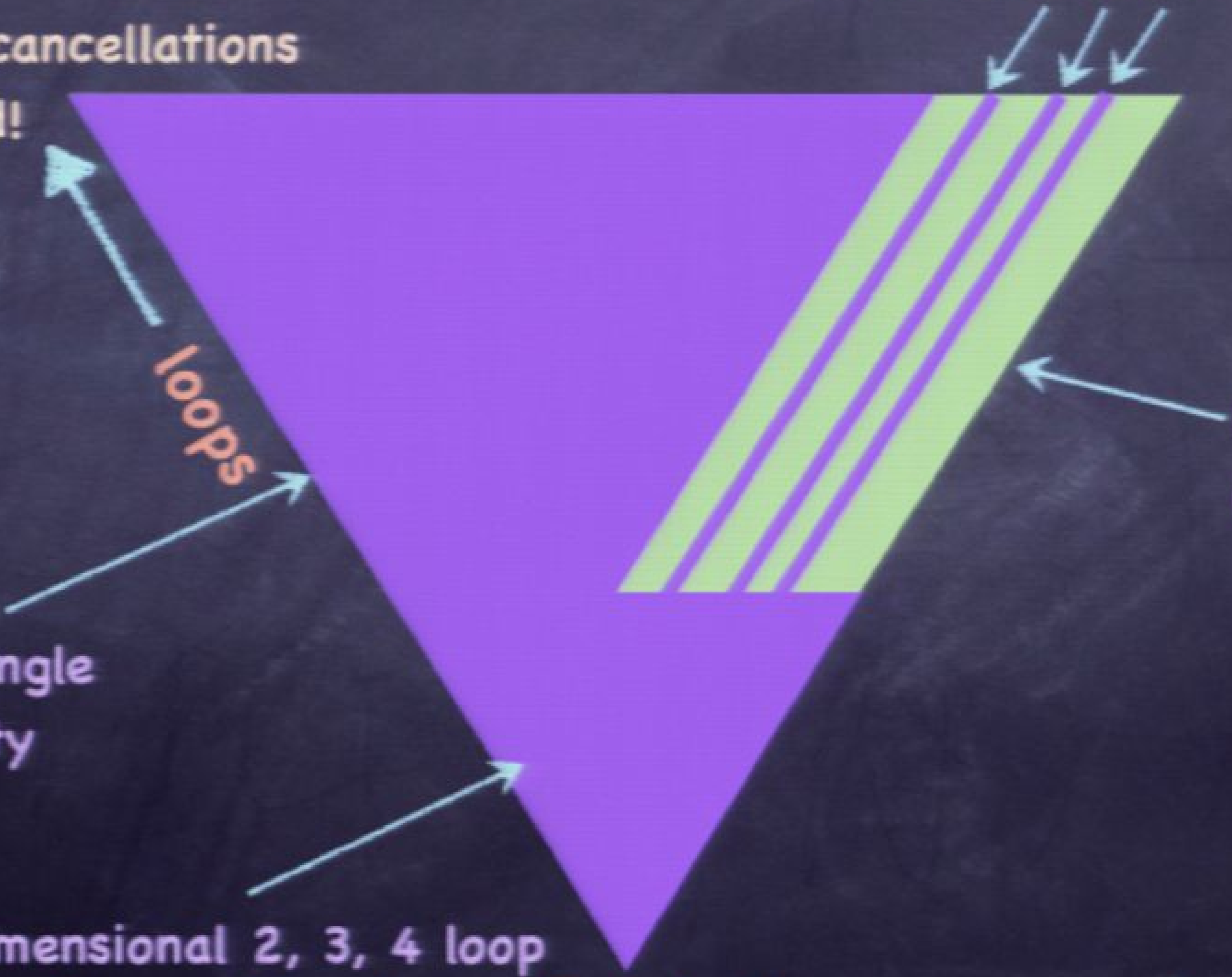
http://ftp.aip.org/epaps/phys_rev_lett/E-PRLTAO-103-025932/

http://www.aip.org/pubservs/epaps.html.

Schematic Illustration of N=8 SUGRA Status

- 🟪 Same power count as N=4 super-Yang-Mills: $D_c = 4 + 6/L$
- 🟩 UV behavior unknown



feeding 2, 3, 4 loop calculations into iterated cuts.

All-loop UV cancellations demonstrated!

behavior unproven

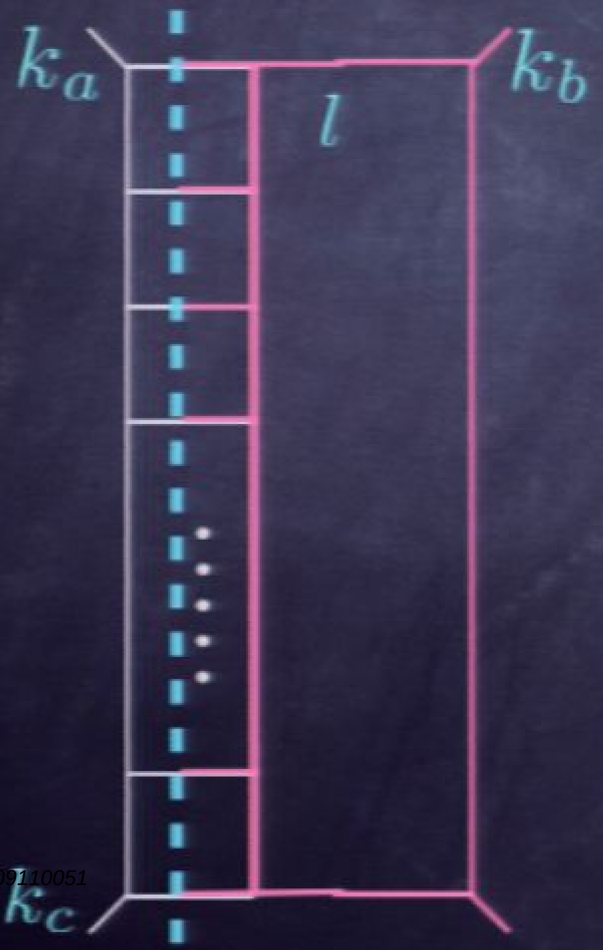
No triangle property

explicit D-dimensional 2, 3, 4 loop

Number of 4pt diagrams

Instructive look at one loop pure Gravity.

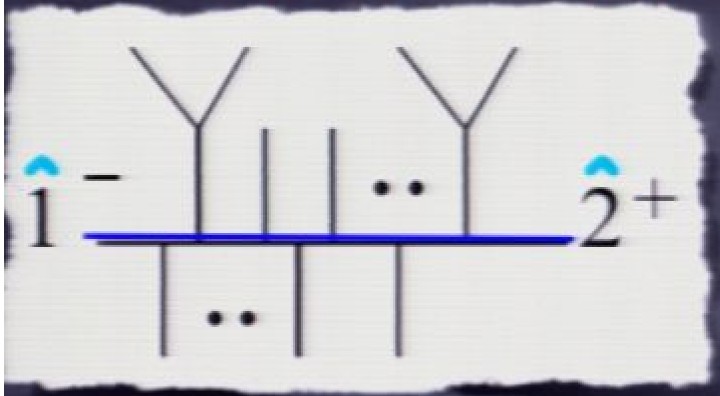
There does not appear to be a supersymmetry explanation for observed all-loop cancellations.



If not SUSY then what?

Origin of Cancellations?

First consider behavior of tree level diagram



with m propagators and m+1 vertices between legs 1 and 2, under the **following shift:**

$$k_1^\mu \to k_1^\mu + zq \qquad k_2^\mu \to k_2^\mu - zq$$
$$q^2 = 0 \qquad k_1 \cdot q = k_2 \cdot q = 0$$

Yang-Mills scaling: $z^{m+1} \times \frac{1}{z^m} \times \frac{1}{z^2} \sim \frac{1}{z}$ **well behaved**

$z \longrightarrow \infty$

vertices

propagators

polarizations

gravity scaling: $z^{2(m+1)} \times \frac{1}{z^m} \times \frac{1}{z^4} \sim z^{(m-2)}$ **poorly behaved**

Summing over all Feynman diagrams, correct gravity scaling is: $M_4^{\text{tree}}(z) \sim \frac{1}{z^2}$

Remarkable tree-level cancellations.

Better than gauge theory!

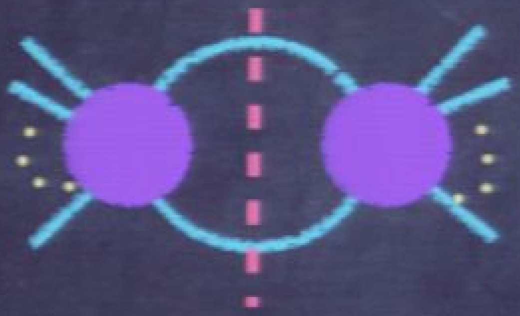
$z^{(m-2)}$ CANCELS TO $\frac{1}{z^2}$

Bedford, Brandhuber, Spence, Travaglini;
Cachazo and Svrcek;
Benincasa, Boucher-Veronneau, Cachazo

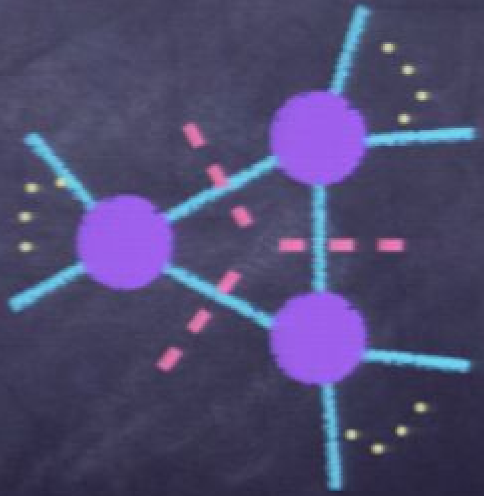
Looking at one loop

Powerful new one-loop integration method due to Darren Forde.

Integration by considering particular complex shifts of "cut" topologies



Allows us to link one-loop cancellations to tree-level cancellations.



We can understand role of maximal SUSY by looking at one-loop pure gravity!

Loop Cancellations in Pure Gravity

Bern, JJMC, Forde, Ita, Johansson

cancellations specific to SUSY

N=8 Supergravity:

$$(l^\mu)^{n-4} = (l^\mu)^{2n} \times (l^\mu)^{-(n-4)} \times (l^\mu)^{-8}$$

max power loop momenta

cancellations generic to gravity

Pure Gravity:

$$(l^\mu)^{n+4} = (l^\mu)^{2n} \times (l^\mu)^{-(n-4)}$$

Most of the one-loop cancellations observed in N = 8 supergravity leading to "no-triangle property" are already present in non-SUSY gravity.

Loop Cancellations in Pure Gravity

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Speculation:

This continues to higher loops: majority of the observed N = 8 multi-loop cancellations are generic to all gravity theories!

All-loop finiteness of N = 8 supergravity (if finite!!) would follow from powerful generic cancellations present in pure Einstein gravity + finite UV protection of SUSY

Summary:

On-shell methods very powerful way of exploring perturbative QFT

A four-loop calculation can teach us something very interesting about trees:

$$-iM_n^{\text{tree}}(1,2,3,\ldots,n) = \sum_i \frac{n_i \tilde{n}_i}{(\prod_j p_j^2)_i}$$

Vanilla perturbative QFT, due to the relative ease of calculation, may have a lot to teach us about fundamental physical principles involving gravity.

N=8 Supergravity might be finite ...

OPEN QUESTION!

A gift from my wife's parents when they realized I really was committed to a career in physics

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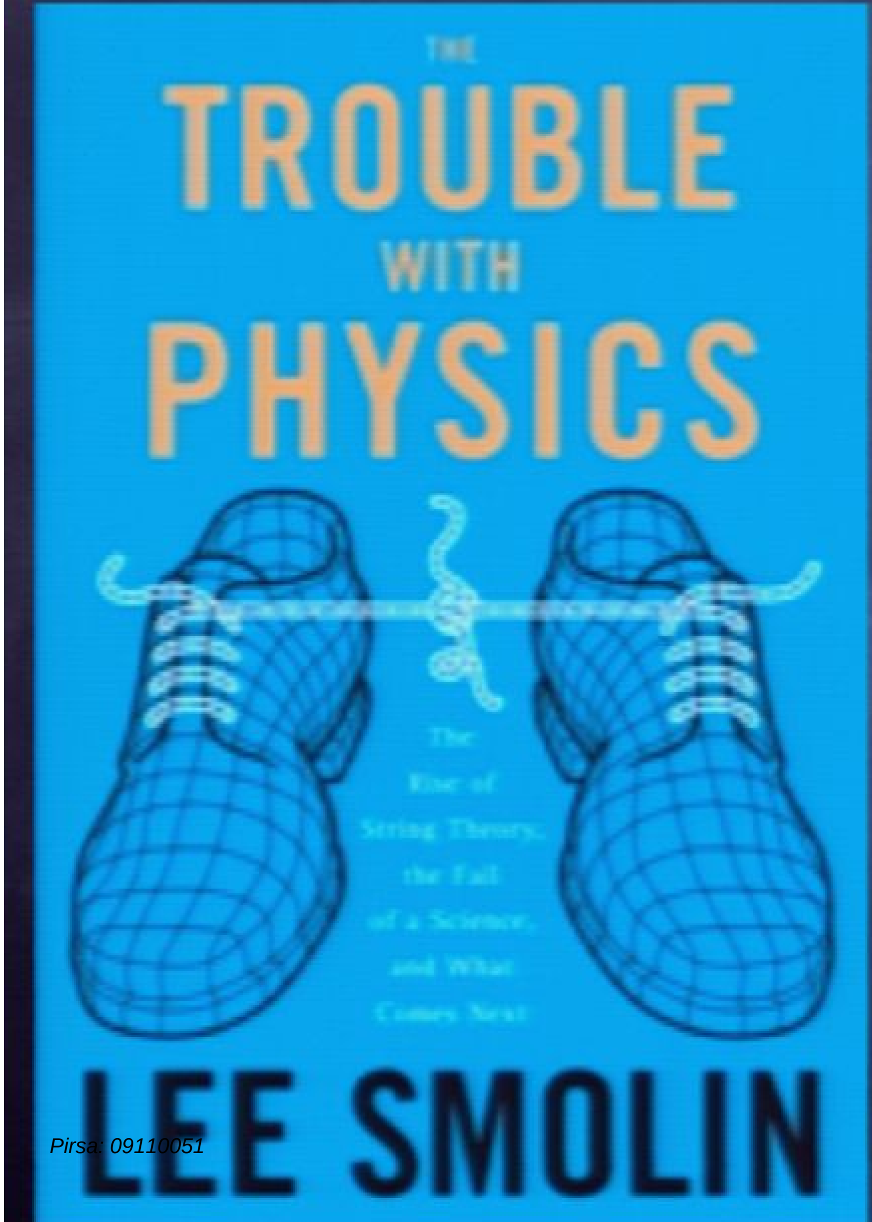

They told me they just liked the title.

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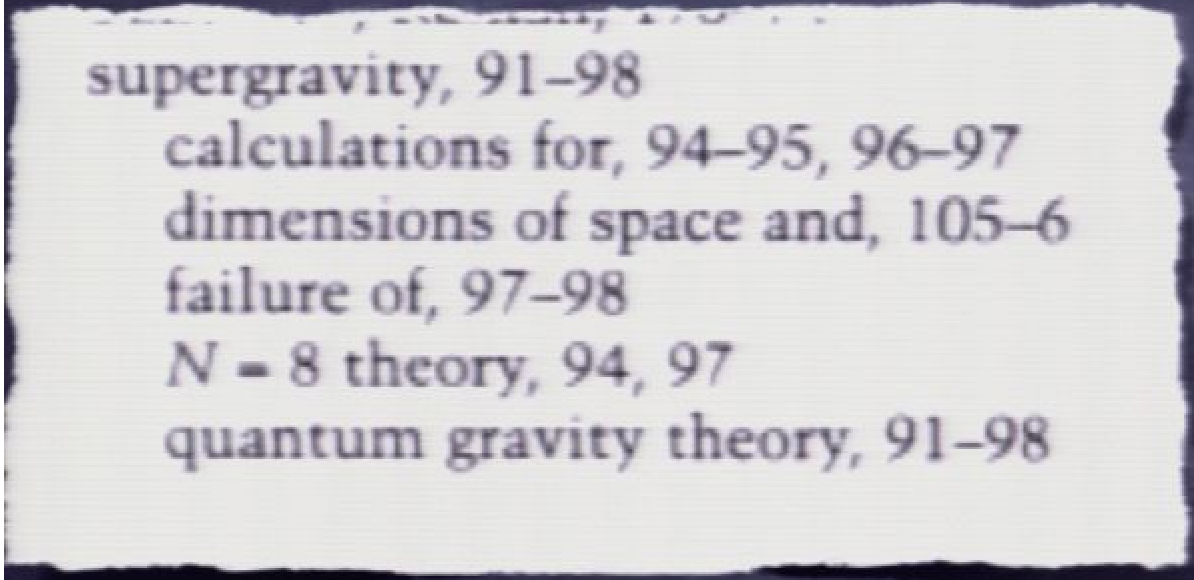

supergravity, 91–98
  calculations for, 94–95, 96–97
  dimensions of space and, 105–6
  failure of, 97–98
  $N = 8$ theory, 94, 97
  quantum gravity theory, 91–98

Concluding Quote

"Although [supergravity] was indeed a proposal for a new unification, it was one that could be expressed, and checked, only in the context of **mind-crushingly boring calculations**. "

–Lee Smolin

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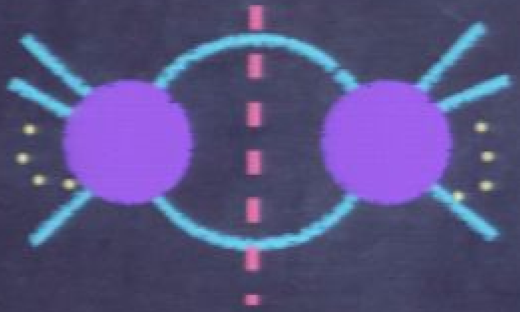

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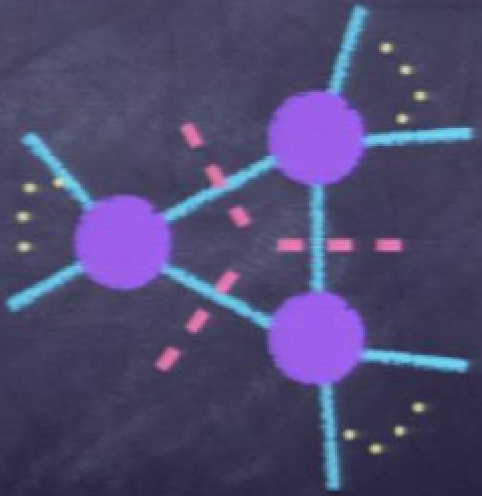

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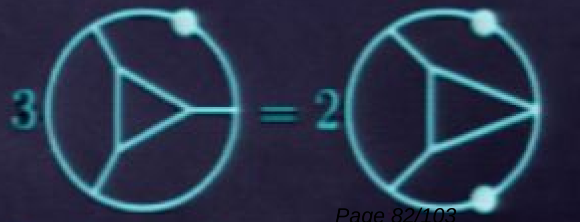

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UV Divergence at Four Loops

$$I_i = \int \left[\prod_{p=1}^{4} \frac{d^D l_{n_p}}{(2\pi)^D} \right] \frac{N_i(l_j, k_j)}{l_1^2 l_2^2 \ldots l_{13}^2}$$

Leading numerators $N_i \sim O(k^4 l^8)$

would have **D = 4.5** divergence

k external
l internal:
too many are
bad for UV

Represented by integrals which **cancel** in the full amplitude

Sub-leading divergence: $O(k^5 l^7)$

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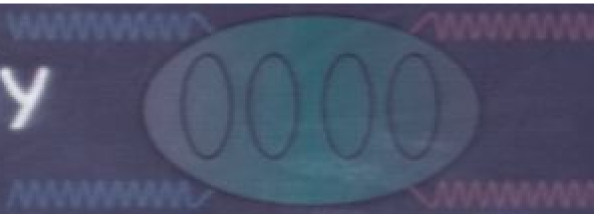
Four-Loop N=8 Gravity



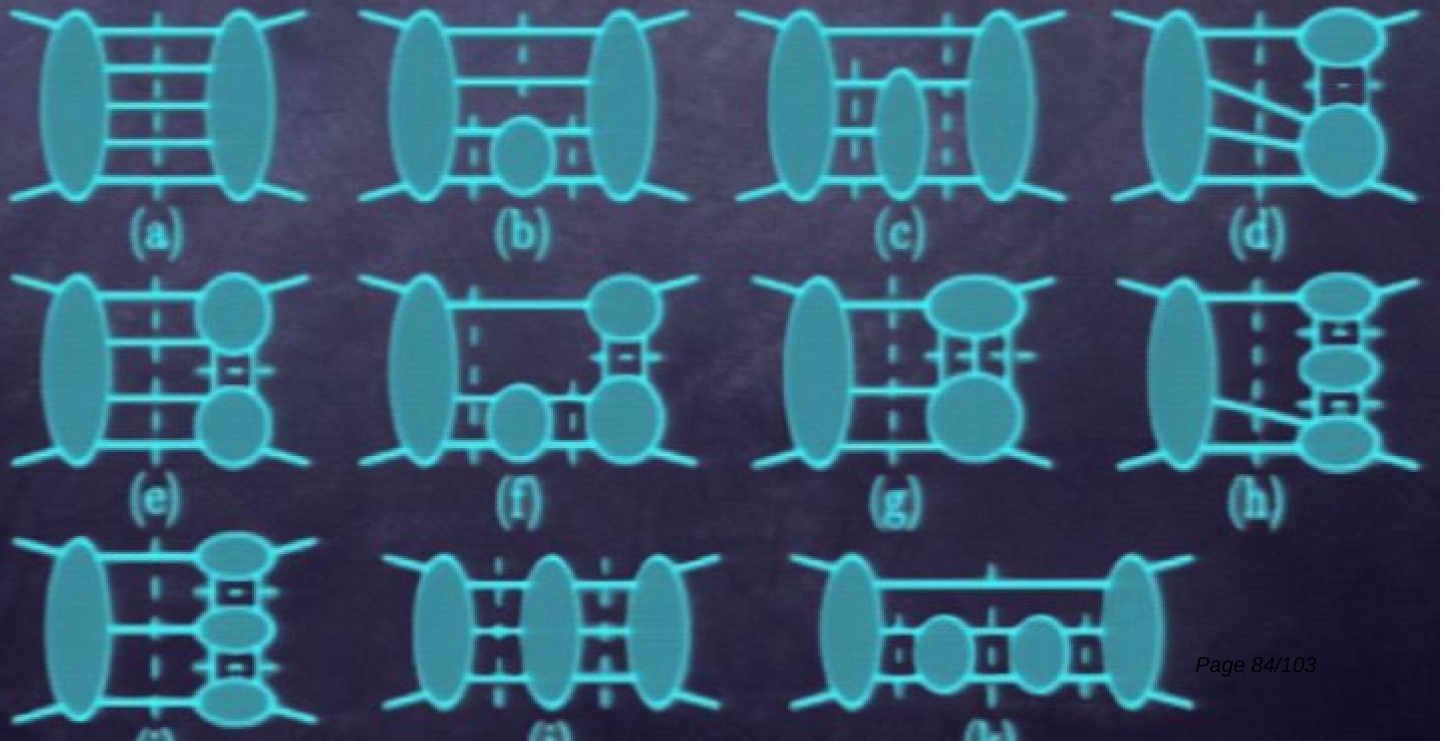
Grav integral

$$I_i = \int \left[\prod_{p=1}^{4} \frac{d^D l_{n_p}}{(2\pi)^D} \right] \frac{N_i(l_j, k_j)}{l_1^2 l_2^2 ... l_{13}^2}$$

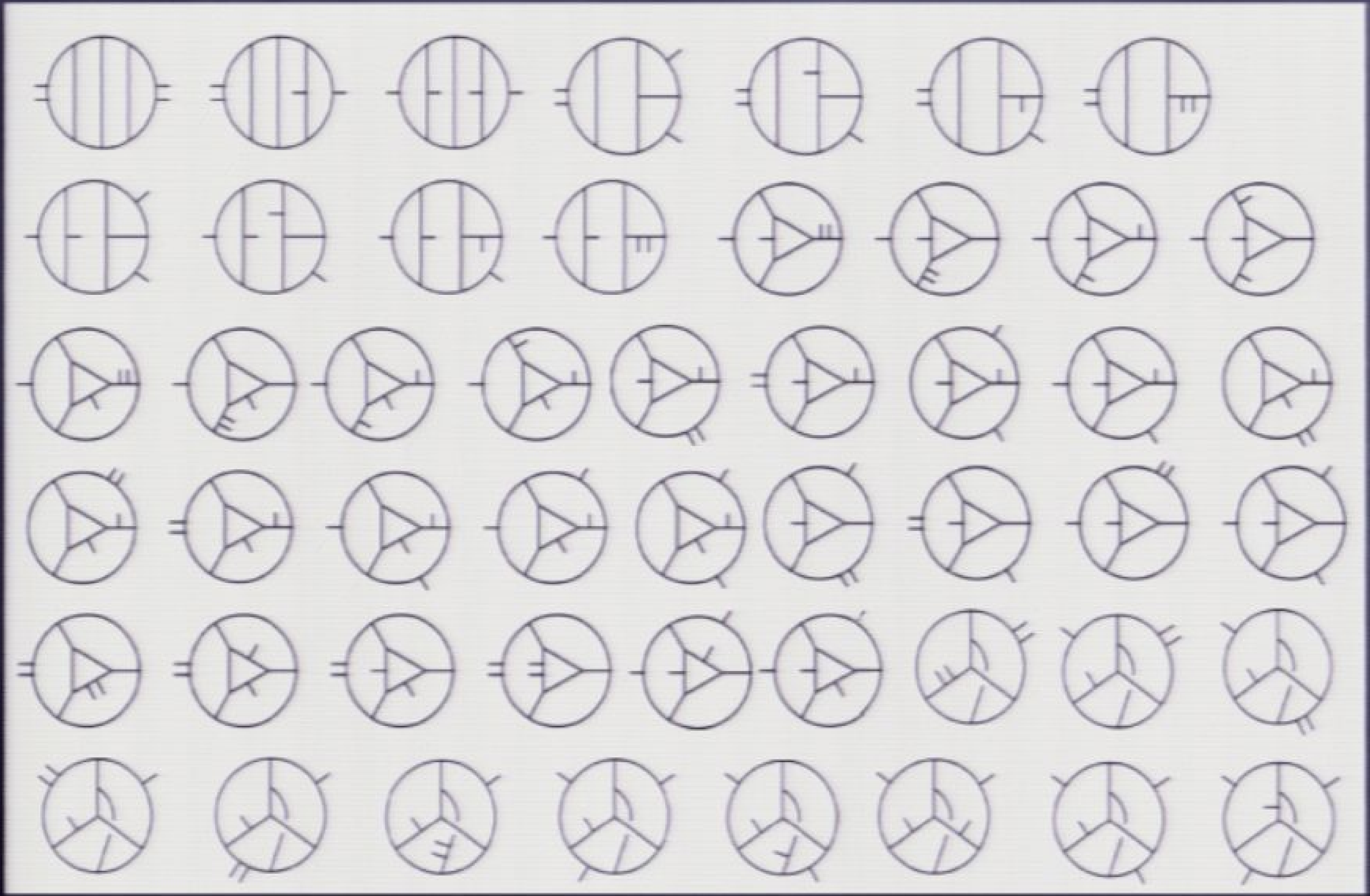
Numerators

Numerators determined from 2906 maximal and near maximal cuts

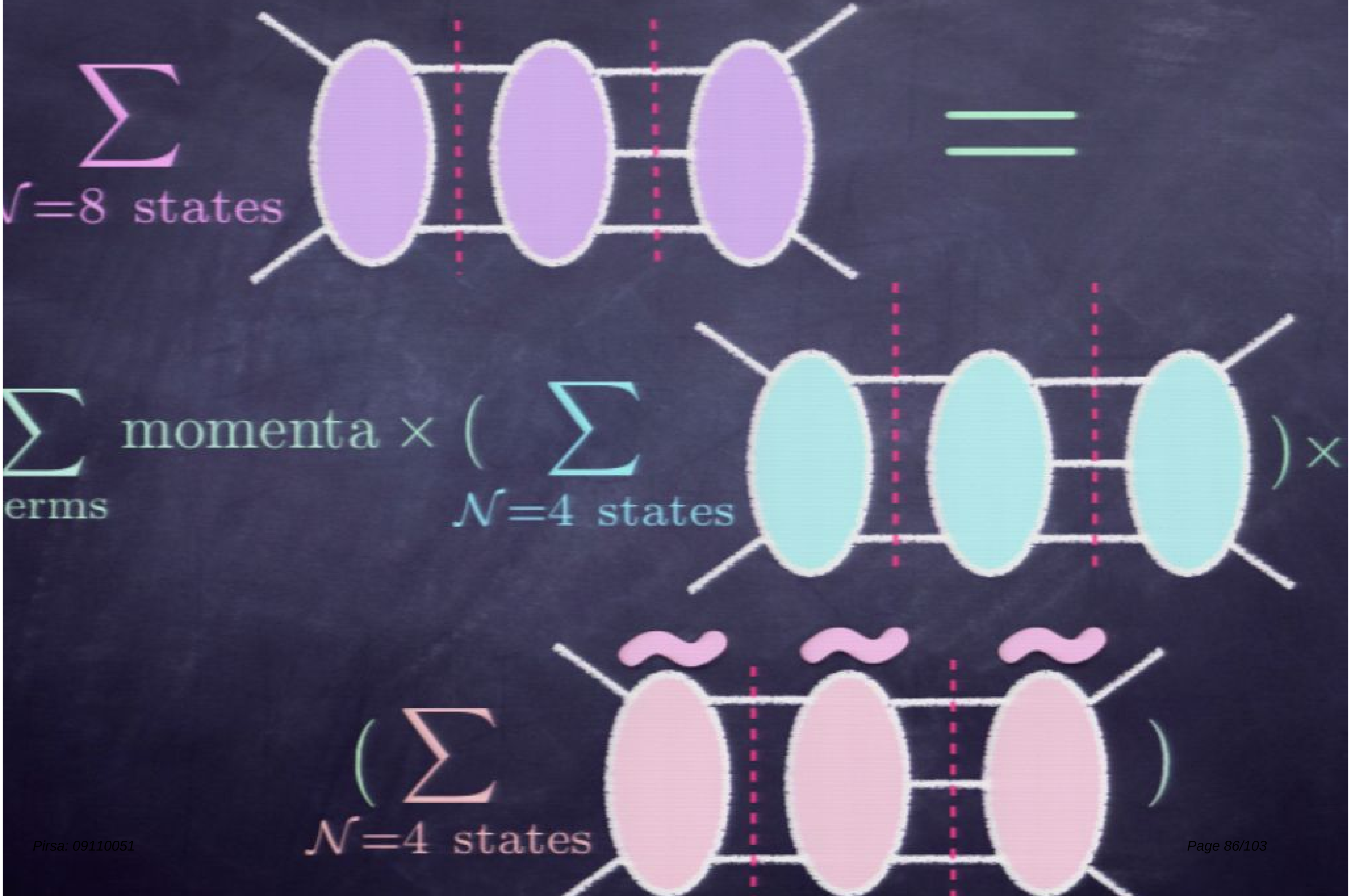
Completeness of ansatz verified on 26 generalized cuts



50 Integral topologies: “parent” diagrams



KLT Factorizes on the Cut!



$$\sum_{\mathcal{}=8\ \text{states}} \quad =$$

$$\sum_{\text{erms}} \text{momenta} \times \Big(\sum_{\mathcal{N}=4\ \text{states}} \Big) \times$$

$$\Big(\sum_{\mathcal{N}=4\ \text{states}} \Big)$$

KLT Factorizes on the Cut!

The supersum over N=8
states for a SUGRA cut

factorizes via KLT into

$\sum_{\text{erms}}$ momenta × (The supersum over N=4 states for a sYM cut) ×

~

(The supersum over N=4 states for a sYM cut)

$\mathcal{N}$=8 Supergravity

$2^8 = 256$ massless states, ~ expansion of $(x+y)^8$

$$\mathcal{N}=8: \quad 1 \leftrightarrow 8 \leftrightarrow 28 \leftrightarrow 56 \leftrightarrow 70 \leftrightarrow 56 \leftrightarrow 28 \leftrightarrow 8 \leftrightarrow 1$$

$$\text{helicity}: \quad -2 \quad -\tfrac{3}{2} \quad -1 \quad -\tfrac{1}{2} \quad 0 \quad \tfrac{1}{2} \quad 1 \quad \tfrac{3}{2} \quad 2$$

SUSY ⟷

$$h^- \quad \psi_i^- \quad v_{ij}^- \quad \chi_{ijk}^- \quad s_{ijkl} \quad \chi_{ijk}^+ \quad v_{ij}^+ \quad \psi_i^+ \quad h^+$$

Cremmer and Julia

$$\mathcal{N}=4 \text{ SYM}: \quad 1 \quad 4 \quad 6 \quad 4 \quad 1$$

$2^4 = 16$ states
~ expansion of
$(x+y)^4$

$$g^- \quad \lambda_A^- \quad \phi_{AB} \quad \lambda_A^+ \quad g^+$$

all in adjoint representation

$$D_c = 4 + 6/L$$

$$\boxed{\left[\mathcal{N}=8\right] = \left[\mathcal{N}=4\right] \otimes \left[\mathcal{N}=4\right]}$$

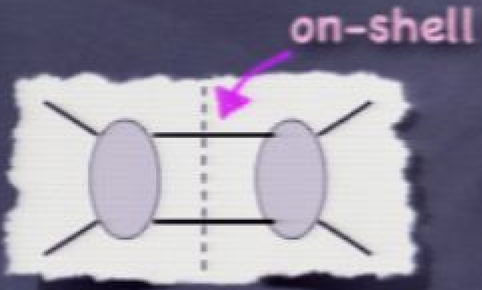
No additional contributions on any contact cuts for $\mathcal{N}$=4 SYM at 2 loops. There would be for QCD. Here's the prescription for incorporating contact contributions:

- Identify all contributing cuts at a given cut-level. Each cut is an equation for missing numerator dressings.
- Simultaneously solve the resulting set of linear cut equations. Solutions represent additional dressings for appropriate cubic-graphs.
- Incorporate before going on to next cut level.

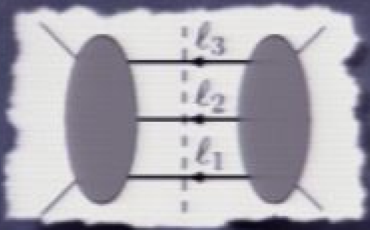
Unitarity Method

Bern, Dixon, Dunbar and Kosower

Two-particle cut:

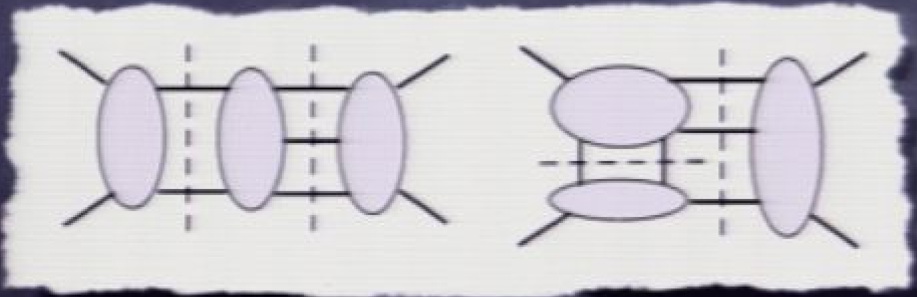

on-shell

Three-particle cut:



Systematic assembly of complete amplitudes at the integrand level from cuts for any number of particles or loops.

eneralized
nitarity as a
ractical tool:

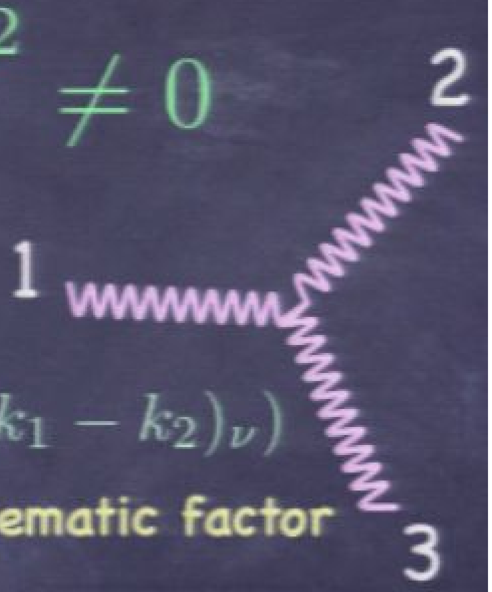


ern, Dixon and Kosower
ritto, Cachazo and Feng

Different cuts merged to give an expression with correct cuts in all channels.

Three Vertices
OFF SHELL

$$k_i^2 = E_i^2 - \vec{k}_i^{\,2} \neq 0$$



hree-gluon vertex:

color factor

$$V_{3\,\mu\nu\sigma}^{\;abc} = -g f^{abc}(\eta_{\mu\nu}(k_1 - k_2)_\rho + \eta_{\nu\rho}(k_1 - k_2)_\mu + \eta_{\rho\mu}(k_1 - k_2)_\nu)$$

momentum dependent kinematic factor

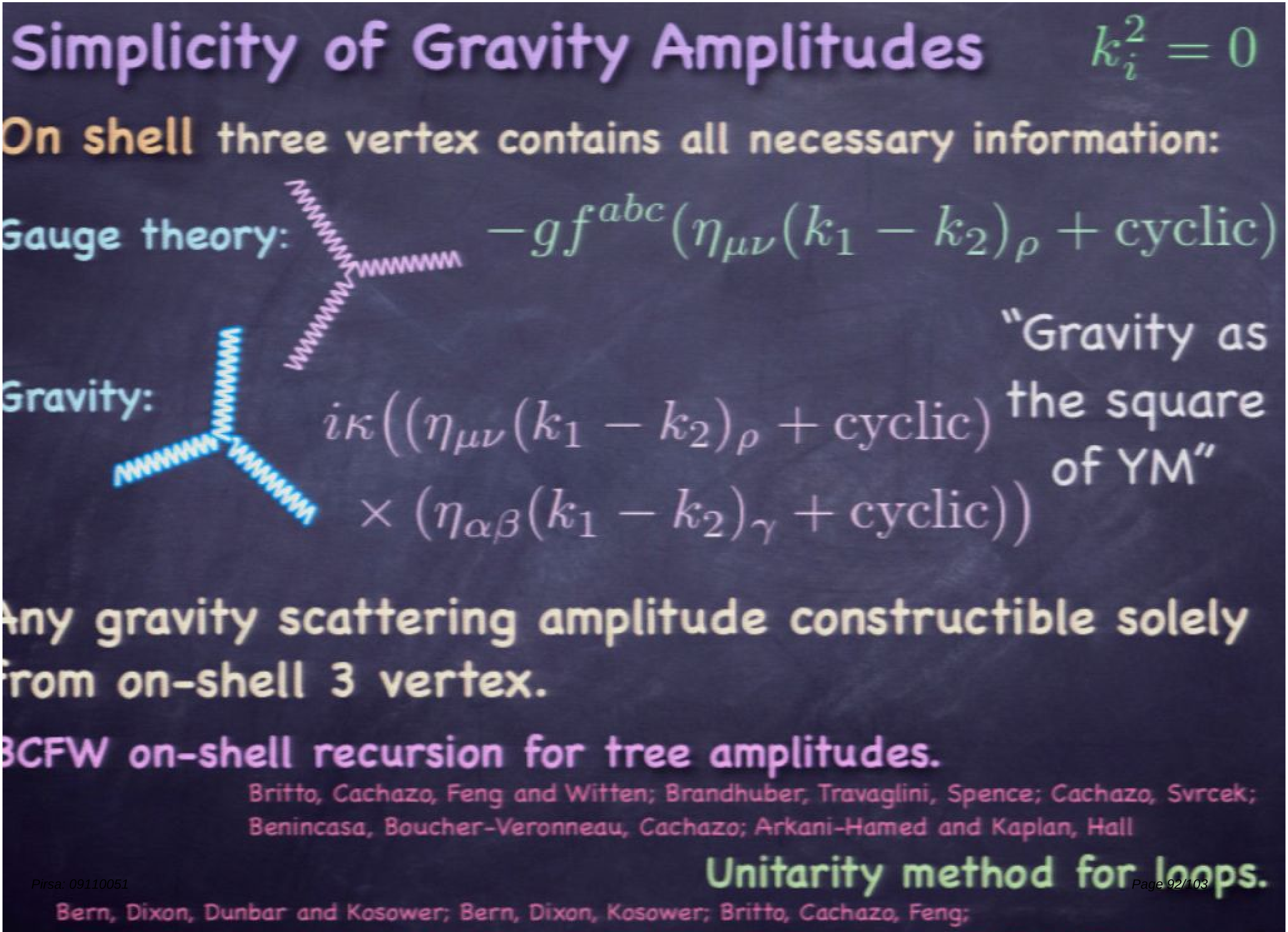
hree-graviton vertex:

$$G_{3\mu\alpha,\nu\beta,\sigma\gamma}(k_1, k_2, k_3) =$$

$$\begin{aligned} &\mathrm{sym}[-\tfrac{1}{2}P_3(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\nu\beta}\eta_{\sigma\gamma}) - \tfrac{1}{2}P_6(k_{1\nu}k_{1\beta}\eta_{\mu\alpha}\eta_{\sigma\gamma}) + \tfrac{1}{2}P_3(k_1\cdot k_2\eta_{\mu\nu}\eta_{\alpha\beta}\eta_{\sigma\gamma}) \\ &\quad + P_6(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\nu\sigma}\eta_{\beta\gamma}) + 2P_3(k_{1\nu}k_{1\gamma}\eta_{\mu\alpha}\eta_{\beta\sigma}) - P_3(k_{1\beta}k_{2\mu}\eta_{\alpha\nu}\eta_{\sigma\gamma}) \\ &\quad + P_3(k_{1\sigma}k_{2\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + P_6(k_{1\sigma}k_{1\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + 2P_6(k_{1\nu}k_{2\gamma}\eta_{\beta\mu}\eta_{\alpha\sigma}) \\ &\quad + 2P_3(k_{1\nu}k_{2\mu}\eta_{\beta\sigma}\eta_{\gamma\alpha}) - 2P_3(k_1\cdot k_2\eta_{\alpha\nu}\eta_{\beta\sigma}\eta_{\gamma\mu})] \end{aligned}$$

About 100 terms in three vertex

Naïve conclusion: Gravity is a headache



Simplicity of Gravity Amplitudes

$k_i^2 = 0$

On shell three vertex contains all necessary information:

Gauge theory: $-gf^{abc}(\eta_{\mu\nu}(k_1 - k_2)_\rho + \text{cyclic})$

Gravity:

$$i\kappa((\eta_{\mu\nu}(k_1 - k_2)_\rho + \text{cyclic}) \times (\eta_{\alpha\beta}(k_1 - k_2)_\gamma + \text{cyclic}))$$

"Gravity as the square of YM"

Any gravity scattering amplitude constructible solely from on-shell 3 vertex.

BCFW on-shell recursion for tree amplitudes.

Britto, Cachazo, Feng and Witten; Brandhuber, Travaglini, Spence; Cachazo, Svrcek; Benincasa, Boucher-Veronneau, Cachazo; Arkani-Hamed and Kaplan, Hall

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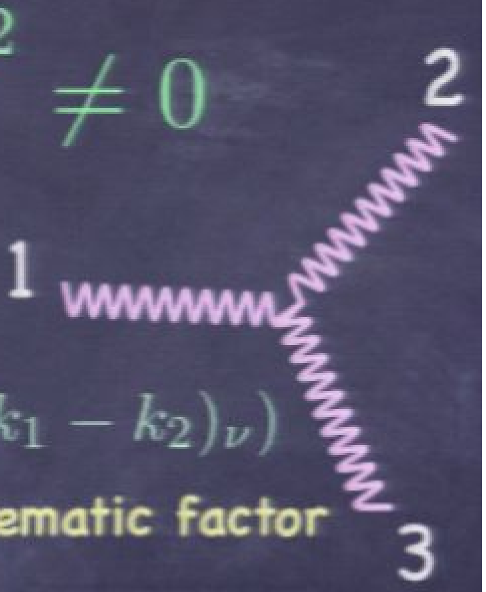

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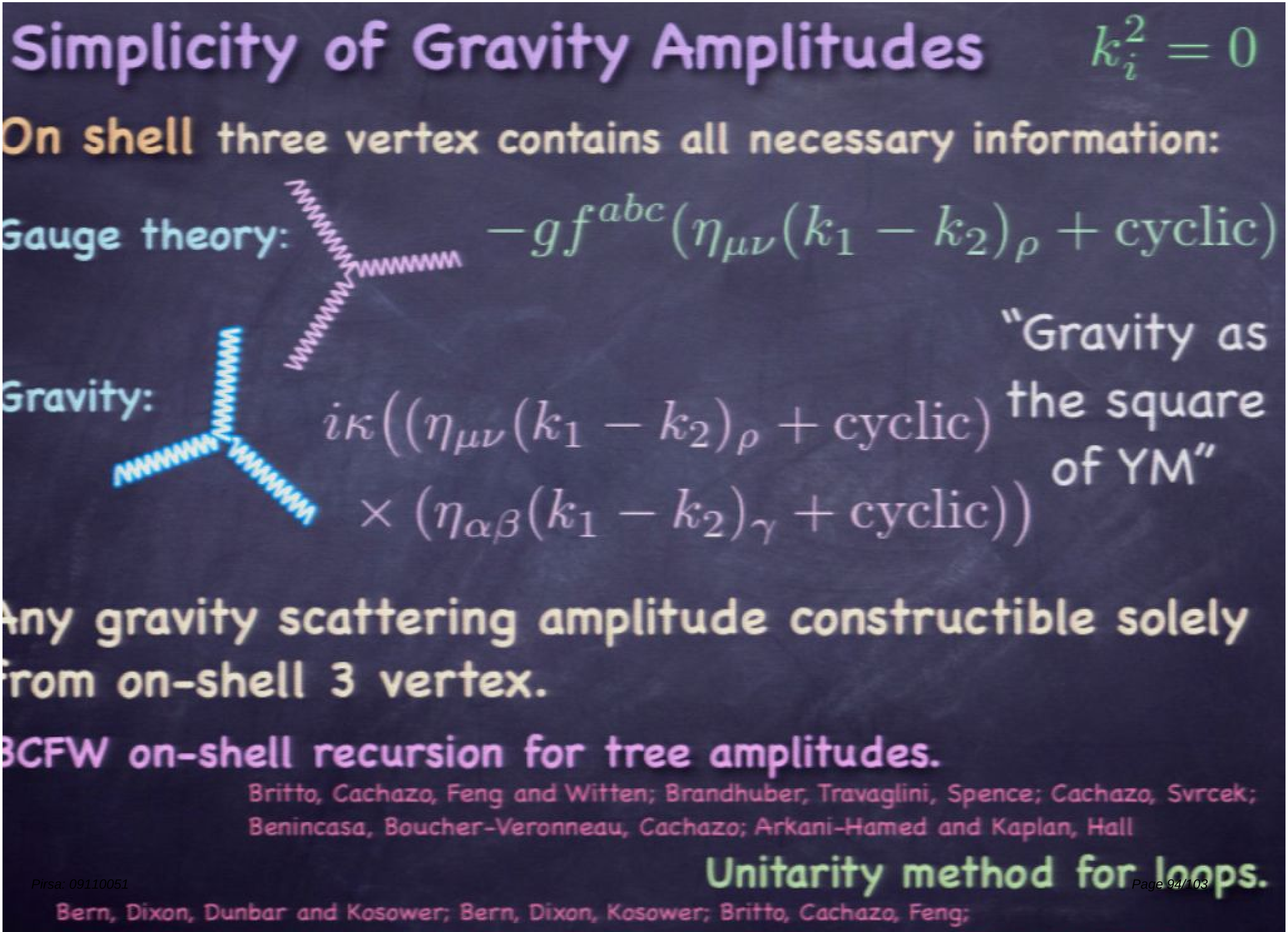

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Unification of Color and Kinematics

Bern, JJMC, Johansson

color factor

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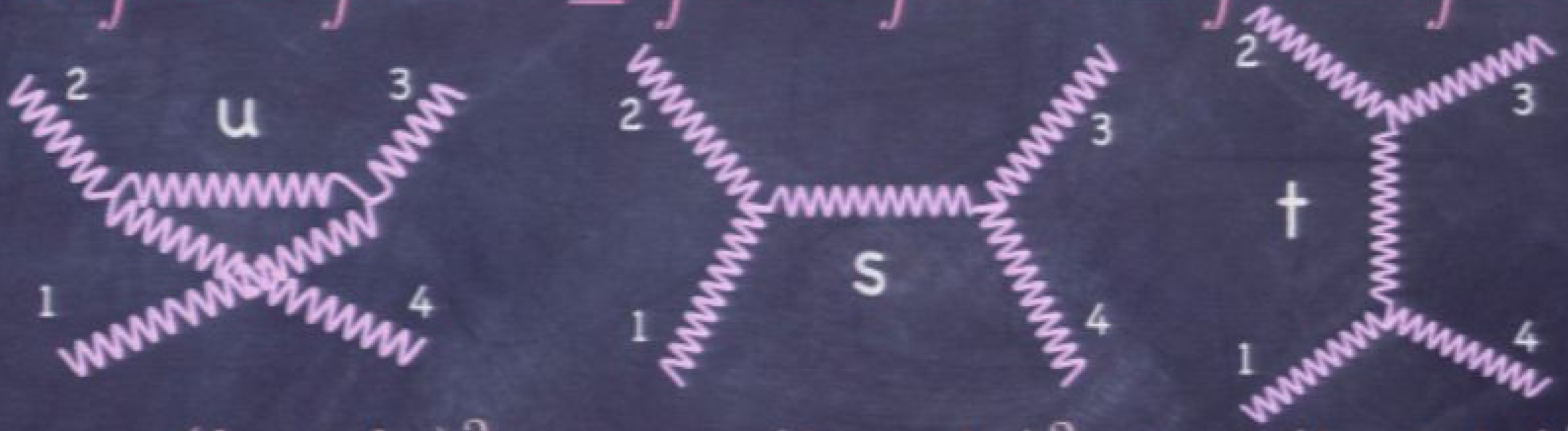
momentum dependent kinematic factor

olor factors based on a Lie Algebra: $f^{abc} = \mathrm{Tr}([T^a,T^b]T^c)$

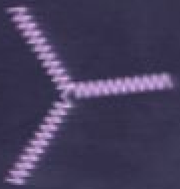
olor Jacobi Identity: $f^{a_4a_2b}f^{ba_3a_1} = f^{a_1a_2b}f^{ba_3a_4} - f^{a_1ba_4}f^{ba_2a}$



$u = (k_1+k_3)^2$ $s = (k_1+k_2)^2$ $t = (k_1+k_4)^2$

Unification of Color and Kinematics

Bern, JJMC, Johansson



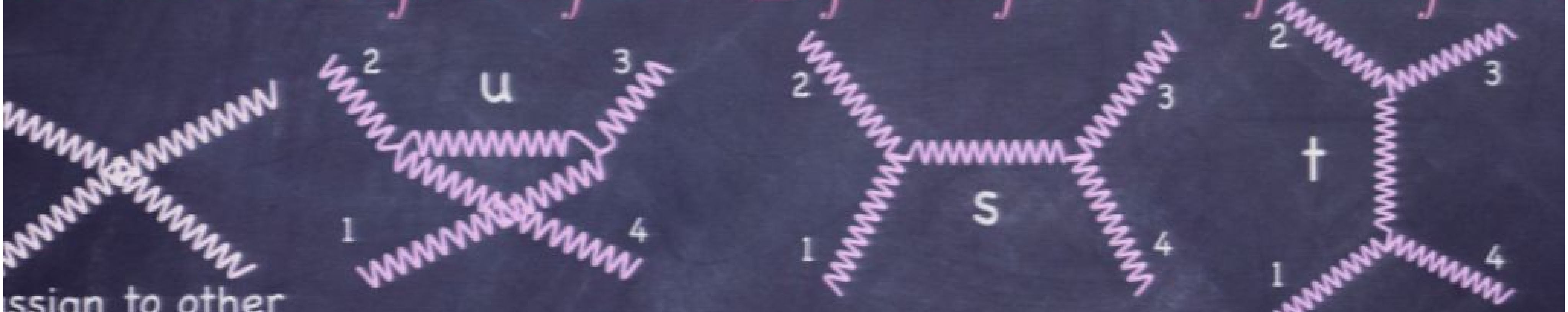
color factor

$$-gf^{abc}(\eta_{\mu\nu}(k_1 - k_2)_\rho + \text{cyclic})$$

momentum dependent kinematic factor

olor factors based on a Lie Algebra: $f^{abc} = \mathrm{Tr}([T^a, T^b]T^c)$

olor Jacobi Identity: $f^{a_4a_2b}f^{ba_3a_1} = f^{a_1a_2b}f^{ba_3a_4} - f^{a_1ba_4}f^{ba_2a}$



ssign to other
diags using
u/u=s/s=t/t=1

$u = (k_1 + k_3)^2$ $s = (k_1 + k_2)^2$ $t = (k_1 + k_4)^2$

$$A_4^{\text{tree}} = g^2\left(\frac{n_u c_u}{u} + \frac{n_s c_s}{s} + \frac{n_t c_t}{t}\right)$$

olor factor obeys the Jacobi identity: $c_u = c_s - c_t$

kinematic factor obeys same identity: $n_u = n_s - n_t$

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Claim: you can always write gauge tree amplitudes

$$A_n^{\text{tree}}(1,2,3,\ldots n) = g^{n-2}\sum_i (c_i n_i \times \text{TreeDiag}_i)$$

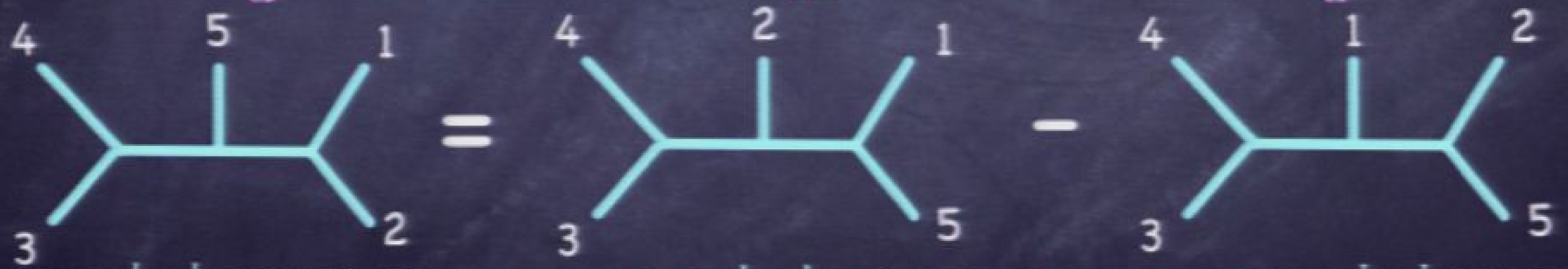
uch that the kinematic factors (n) obey the
ame Jacobi identity as the color factors (c)

$${}_i = c_j - c_k \Longrightarrow n_i = n_j - n_k$$

$$\uparrow \quad \frac{1}{\left[\prod_j p_j^2\right]_i}$$

Denoms associated with color ordered diag (i)

pt example n_3 n_5 n_8



$${}_3 \equiv f^{a_3a_4b}f^{ba_5c}f^{ca_1a_2} \quad c_5 \equiv f^{a_3a_4b}f^{ba_2c}f^{ca_1a_5} \quad c_8 \equiv f^{a_3a_4b}f^{ba_1c}f^{ca_2a_5}$$

Nontrivial constraints on amplitudes -> (n-3)! indep

Recent string theory understanding: Bjerrum-Bohr, Damgaard, Vanhove; Stieberger

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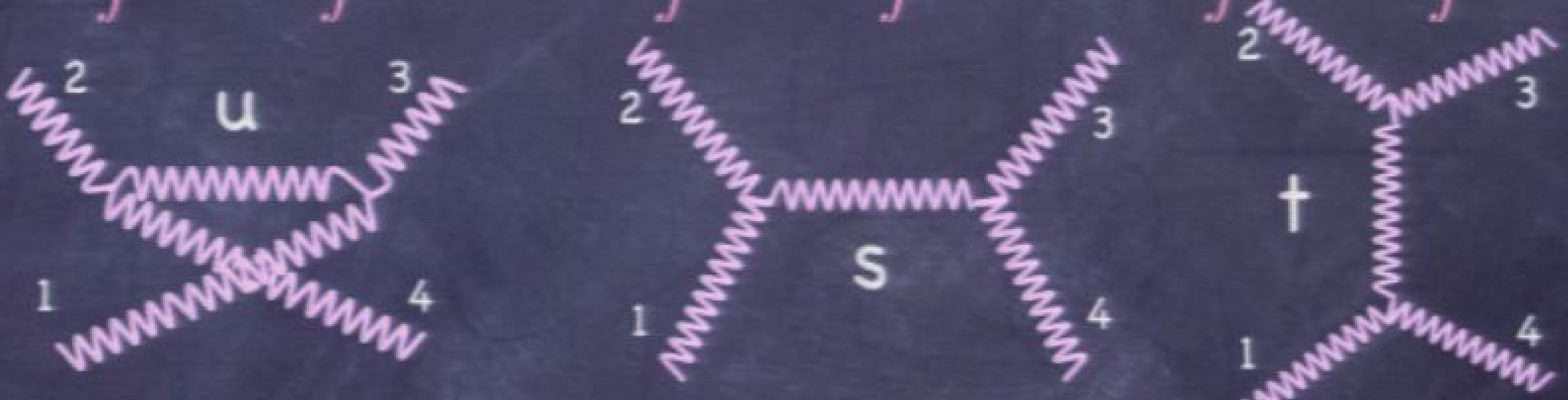

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kinematic factor obeys same identity:

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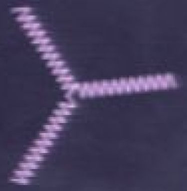

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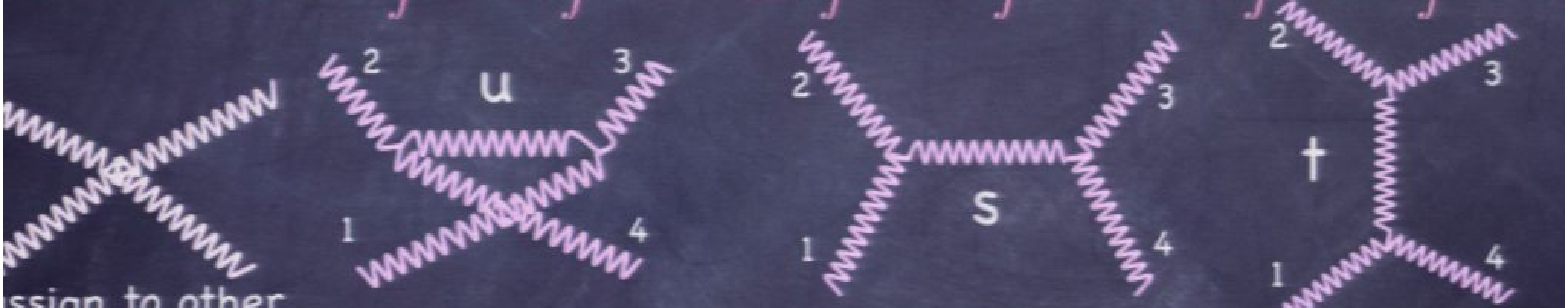

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KLT field expressions:

Kawai, Lewellen,Tye

Bern, Dixon, Perelstein, Rozowsky

Gravity tree amplitudes

$$M_n^{\text{tree}}(\underline{1},\ldots,\underline{n-1},\underline{n}) = \qquad i\in\{2,\ldots,n/2\} \quad j\in\{n/2+2,\ldots,n-2$$

$$(-1)^{n+1}\sum_{perms(2,\ldots,n-2)}\Big[A_n^{\text{tree}}(\underline{1},\ldots,\underline{n-1},\underline{n})\sum_{perms(i,j)} f(i_1,\ldots,i_j)$$

$$\times\,\overline{f}(l_1,\ldots,l_{j'})\widetilde{A}_n^{\text{tree}}(i_1,\ldots,i_j,\underline{1},\underline{n-1},l_1,\ldots,l_{j'},\underline{n})\Big]$$

Color-ordered gauge tree amplitudes

New relations allow re-expression of KLT in terms of different "basis" amplitudes: Left-right symmetric, etc.

But we can do better...

Higher-Point Gravity and Gauge Theory

Bern, JJMC, Johansson

QCD:

$$A_n^{\text{tree}} = i g^{n-2} \sum_i \frac{n_i c_i}{D_i}$$

sum over diagrams with only cubic vertices

Einstein Gravity:

$$\mathcal{M}_n^{\text{tree}} = i \kappa^{n-2} \sum_i \frac{n_i \tilde{n}_i}{D_i}$$

Relation extremely useful in high-loop gravity calculations.)

Remarkably simple re-expression of field theory limit of String Theory's Kawai, Lewellen and Tye relations

Physical principle involved?

Local unity of geometry and glue?

Exciting Proposition:

Perturbatively finite QFT of gravity in 4D

Vhy surprising if possible:

Dimensionful coupling: $\kappa \sim m_{pl}^{-1}$ → non-renormalizable

No known **structure** to make up diff btw

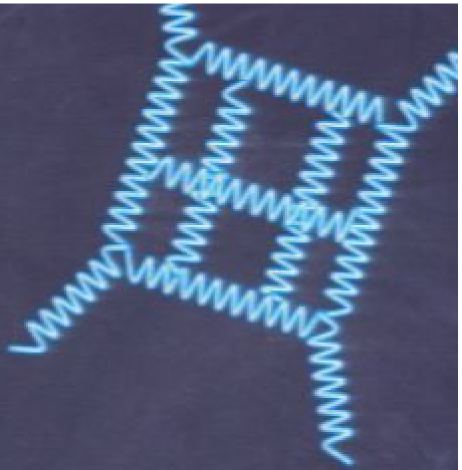
$$\frac{(\kappa\ p^\mu p^\nu)\cdots}{\text{propagators}}$$ gravity

and

$$\frac{(g\ p^\mu)\cdots}{\text{propagators}}$$ gauge

Any responsible mechanism would fundamentally impact our understanding of gravity

Evidence of spectacular cancellations
n $\mathcal{N}$=8 **Supergravity!**



Everywhere we look has the same powercounting as $\mathcal{N}$=4 super Yang-Mills

$$D_c = 4 + 6/L$$